Relaxed Recentered Log-Barrier Function Based Nonlinear Model Predictive Control

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Abstract— **This paper investigates the use of relaxed recentered logarithmic barrier functions in the context of nonlinear model predictive control. These functions are a variation of the regular log-barrier functions that are introduced in the objective function of an optimization problem as a penalty for the deviation from the constraint set. The resulting MPC scheme has been studied in the case of linear dynamics. Several interesting results on the global nominal asymptotic stability of the corresponding closed-loop system and constraint satisfaction guarantees have been obtained. Extending them to the case of nonlinear dynamics is non-trivial, and we show in this paper that these properties can still hold locally. The theoretical results are demonstrated by the numerical implementation of a nonlinear benchmark system with four states and two inputs.**

I. Introduction

Model predictive control (MPC) is an advanced control method that has been extensively studied for its capability of handling complex constrained dynamical systems [1]. As an optimization-based control technique, the basic idea of MPC is to solve a finite-horizon open-loop optimal control in a receding horizon fashion and apply the first element of the optimal control input to the process [2], [3]. These online optimal control problems are formulated based on the predicted dynamics of the system to be controlled, both a user-defined cost objective and potential constraints on the system states and inputs. Various theoretical results exist concerning the closed-loop stability properties of the resulting feedback law for both linear and nonlinear systems [4], [5].

In practice, there always exists a gap between the actual implementation of MPC and its theoretical concepts, which poses significant challenges for real-world applications [1]. For example, solving the online optimization problem to optimality at frequencies in the kHz range on embedded platforms can be difficult, or even impossible [6]–[9]. Moreover, based on the current state measurement, the resulting online optimal control problem is reformulated at every sampling time. Due to the presence of disturbances, sensor outliers, or observer errors, these problems might become infeasible, leading to a complete crash of the respective control algorithm [10].

To deal with this issue, considerable research has been spent on simplifying the optimization problems involved in MPC by leveraging ideas from classical mathematical optimization. One of those is replacing constraints with logarithmic barrier functions, as is done in interior point methods (see [11]). Such modifications lead to equality-only constrained, or even unconstrained, optimization problems that may be in some sense "easier to solve" but that still present some flaws, like the non-definition of these barrier functions outside the constraint set, which can lead to infeasible problems. This is usually solved by introducing soft constraints (see [12]–[14]), although no theoretical guarantees can be given on the convergence of the resulting optimization problem.

An alternative is to *relax* the barrier function by extending it outside the constraint set in a twice continuously differentiable fashion. These *relaxed (recentered) logarithmic barrier functions* (that we will call RRLB functions) have been heavily studied in the context of stabilization with MPC for linear dynamical systems in [10], [15]. In this simple case, it was proven that globally asymptotically stabilizing schemes can be designed with the help of these functions, in particular by leveraging appropriate terminal costs and/or sets. What's more, constraint satisfaction guarantees can be derived locally around the reference state.

The goal of this paper is to extend the RRLB based MPC to the more general case of nonlinear dynamics. Compared to the classical nonlinear MPC (NMPC) scheme, the resulting RRLB based NMPC leads to unconstrained online problems. As a result, a fixed memory allocation is necessary, which makes it more suitable for embedded systems. Moreover, in the real-time iteration scheme, the RRLB based formulation does not need to consider the changes of the active set while the closed-loop stability can be guaranteed. In Section II, we recall the nominal MPC scheme and the definition of RRLB functions, and then, Section III presents the general RRLB based MPC scheme and recalls the results obtained in the linear case. The main contribution of this paper in Section IV shows that we can ensure asymptotic stability, but only in a neighborhood of the reference state in Theorem 3. Moreover, Theorem 4 implies that constraint satisfaction guarantees can still hold in a smaller neighborhood. Finally, Section V illustrates the effectiveness of the proposed RRLB based NMPC scheme by implementing it to a benchmark continuous stirred tank reactor and comparing it with the regular MPC scheme.

Notation: we use the notation $\mathrm{diag}(a_1, \ldots, a_n)$ to denote a diagonal matrix with diagonal elements $a_1, \ldots, a_n$; $\|\cdot\|$ denotes the Euclidean norm; $\mathbb{S}^n_{++}(\mathbb{S}^n_{+})$ denote the set of symmetric positive (semi-) definite the matrices in $\mathbb{R}^{n\times n}$; a

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minimizer of an equality constrained optimization problem is called a regular KKT point, if the Linear Independence Constraint Qualifications (LICQ) and Second Order Sufficient Conditions (SOSC) are satisfied [16].

II. Preliminaries

A. Model Predictive Control

This paper considers discrete-time nonlinear dynamical systems with states and controls $x^t \in \mathbb{R}^{n_x}$, $u^t \in \mathbb{R}^{n_u}$ and evolution equation given by

$$\forall\, t \in \mathbb{Z}_{\geq 0}, \quad x^{t+1} = f(x^t, u^t) \tag{1}$$

with f a general nonlinear function. We focus here on the task of stabilizing such systems to a steady state that is constant in time. To accomplish this task, we are going to use Model Predictive Control (MPC), which in essence solves the nonlinear program given by

$$\begin{aligned} V_N(x^t) = \min_{\mathbf{x},\mathbf{u}} \quad & J(\mathbf{x},\mathbf{u}) := \sum_{k=0}^{N-1} l(x_k, u_k) \; + \; F(x_N), \\ \text{subject to} \quad & x_0 = x^t, \\ & x_{k+1} = f(x_k, u_k),\; k = 0, \dots, N-1, \\ & x_k \in \mathcal{X},\; k = 0, \dots, N-1, \\ & u_k \in \mathcal{U},\; k = 0, \dots, N-1 \end{aligned} \tag{2}$$

in a receding horizon fashion, and then apply to the system the control input given by the optimal solution at stage 0, i.e., u_0^*. In this paper, we are going to focus on quadratic stage and terminal costs:

$$l(x,u) = x^\top Q x + u^\top R u, \quad F(x) = x^\top P x$$

with $Q, P \in \mathbb{S}_{++}^{n_x}$, $R \in \mathbb{S}_{++}^{n_u}$. Moreover, the state and control constraints are polytopes given by

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid C_x x \leq d_x\}, \tag{3a}$$

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid C_u u \leq d_u\} \tag{3b}$$

with matrices $C_x \in \mathbb{R}^{m_x \times n_x}$, $C_u \in \mathbb{R}^{m_u \times n_u}$ and vectors $d_x \in \mathbb{R}^{m_x}$, $d_u \in \mathbb{R}^{m_u}$. Finally, x^t denotes the initial condition at the current time instant and $V_N(x^t)$ denotes the cost-to-go function with prediction horizon of length $N \in \mathbb{Z}_{>0}$. Notice that here we use superscripts to denote the time instant of the closed-loop trajectories, and subscripts to denote open-loop predictions.

Throughout this paper, we make the following assumptions:

A1 The dynamics f are twice Lipschitz-continuously differentiable in a neighborhood of the origin,

A2 the origin is a steady state, i.e., $f(0,0) = 0$, $0 \in \operatorname{int}(\mathcal{X})$, $0 \in \operatorname{int}(\mathcal{U})$, where $\operatorname{int}(S)$ denotes the interior of a set S.

Moreover, we define the matrices

$$A = \frac{\partial f(0,0)}{\partial x}, \quad B = \frac{\partial f(0,0)}{\partial u},$$

and assume that

A3 the pair (A, B) is (asymptotically) stabilizable.

This assumption implies that there exists a stabilizing cost K such that the matrix $A_K := A + BK$ only has eigenvalues in the open unit disk.

B. Relaxed Recentered Log-Barrier Function

Based on the polytopic state and control constraint sets (3), we define the relaxed recentered logarithmic barrier functions as presented in [15] by

$$B_x(x) = \sum_{i=1}^{m_x} (1 + w_{x,i}) B_{x,i}(x), \tag{4a}$$

$$B_u(u) = \sum_{i=1}^{m_u} (1 + w_{u,i}) B_{u,i}(u) \tag{4b}$$

with

$$B_{x,i}(x) = \begin{cases} \log(d_{x,i}) - \log(d_{x,i} - \operatorname{row}_i(C_x)x) \\ \qquad\qquad \text{if } d_{x,i} - \operatorname{row}_i(C_x)x > \delta\,, \\ \log(d_{x,i}) + \beta(d_{x,i} - \operatorname{row}_i(C_x)x;\, \delta) \\ \qquad\qquad \text{otherwise,} \end{cases}$$

and

$$B_{u,i}(u) = \begin{cases} \log(d_{u,i}) - \log(d_{u,i} - \operatorname{row}_i(C_u)u) \\ \qquad\qquad \text{if } d_{u,i} - \operatorname{row}_i(C_u)u > \delta\,, \\ \log(d_{u,i}) + \beta(d_{u,i} - \operatorname{row}_i(C_u)u;\, \delta) \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Here, δ is called the relaxation parameter and β is the function that ensures B_x and B_u be twice continuously differentiable. The simplest example for such a function β is:

$$\beta(z;\, \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)\,. \tag{5}$$

The weights $w_{x,i}$ and $w_{u,i}$ are chosen such that

$$B_x(0) = 0, B_u(0) = 0, \nabla B_x(0) = 0 \text{ and } \nabla B_u(0) = 0\,.$$

These equations only have solutions if

$$0 < \delta \leq \min\{d_{x,1}, \dots, d_{x,m_x}, d_{u,1}, \dots, d_{u,m_u}\}\,,$$

which is always possible if $0 \in \mathcal{X}$ and $0 \in \mathcal{U}$.

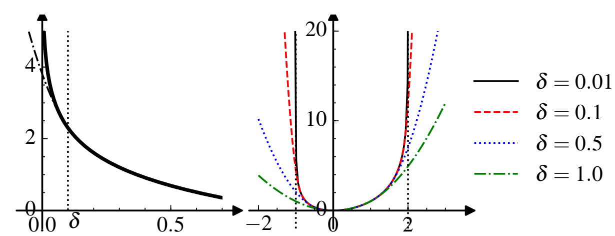

Fig. 1: *(Left)*: Principle of relaxed log barrier function based on quadratic relaxation. (*Right*): Regular weight recentered log barrier function (solid line) and RRLB function for $\delta \in \{0.01, 0.1, 0.5, 1\}$ for the constraint $z \in \mathbb{R},\ -1 \leq z \leq 2$

An illustration of this relaxation procedure can be found in Figure 1. More details on these RRLB functions can be found in [15]. Note that B_x, B_u are positive definite functions that can be upper bounded by coercive quadratic functions. In the following, we use the notation

$$M_x = \nabla^2 B_x(0) \quad \text{and} \quad M_u = \nabla^2 B_u(0)$$

to denote the Hessians of B_x and B_u at the origin.

III. RRLB-based Model Predictive Control

In this section, we define the RRLB-based nonlinear MPC problem as follows:

$$\begin{aligned}\widetilde{V}_N(x^t) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k,u_k) \ + \ \tilde{F}(x_N) \\ \text{subject to} \quad & x_0 = x^t, \\ & x_{k+1} = f(x_k,u_k),\ k=0,\ldots,N-1\,.\end{aligned} \tag{6}$$

with modified stage and terminal cost given by

$$\tilde{l}(x,u) = l(x,u) + \epsilon B_x(x) + \epsilon B_u(u),\ \tilde{F}(x) = x^\top \tilde{P} x\,. \tag{7}$$

Here, ϵ is the barrier parameter and $\tilde{P}$ is assumed to be given by the solution to the following modified Discrete Algebraic Riccati Equation (DARE):

$$\begin{aligned}\tilde{P} = {} & Q + \epsilon M_x + A^\top \tilde{P} A \\ & - A^\top \tilde{P} B (R + \epsilon M_u + B^\top \tilde{P} B)^{-1} B^\top \tilde{P} A\,.\end{aligned} \tag{8}$$

Compared to the classical NMPC formulation (2), (6) rewrites the state and control input constraints into the objective by using the relaxed recentered log-barrier (RRLB) functions (4a). Moreover, we only consider here terminal costs and no terminal constraints because this RRLB formulation (presented in [10]) is the one that makes the least assumptions on the system dynamics.

In the case of linear time-invariant dynamics, i.e.

$$\forall\, t \in \mathbb{Z}_{\geq 0},\ \ x^{t+1} = Ax^t + Bu^t,$$

the RRLB-based MPC formulation (6) becomes

$$\begin{aligned}\widetilde{V}_N(x^t) = \min_{\mathbf{x},\mathbf{u}} \quad & \sum_{k=0}^{N-1} \tilde{l}(x_k,u_k) + \tilde{F}(x_N) \\ \text{subject to} \quad & x_0 = x^t, \\ & x_{k+1} = Ax_k + Bu_k,\ k=0,\ldots,N-1.\end{aligned} \tag{9}$$

Accordingly, the following results hold:

Theorem 1 (Theorem 5 in [10]) *Let assumptions **A1**–**A3** hold and if the matrix $\tilde{P} \in \mathbb{S}^{n_x}_{++}$ in the terminal cost $\tilde{F}$ is given by the solution of* (8)*, the controlled dynamical system yielded by* (9) *is globally asymptotically stable.*

Theorem 2 (Lemma 5 in [10]) *Let the assumptions in Theorem 1 hold, then there exists a neighborhood $\mathcal{X}_N$ of the origin such that for any initial state $x^0 \in \mathcal{X}_N$, all the constraints are satisfied along the closed-loop trajectories. Furthermore, the set $\mathcal{X}_N$ can be given analytically by:*

$$\mathcal{X}_N = \left\{x \in \mathcal{X} \mid \widetilde{V}_N(x) - x^\top P x \leq \epsilon \min\{\beta_x, \beta_u\}\right\},$$

where P is the solution to the DARE (8) *with $\epsilon = 0$, and*

$$\beta_x = \min_{i,x} \left\{B_x(x) \mid \mathrm{row}_i(C_x x) = d_{x,i}\right\},$$

$$\beta_u = \min_{i,u} \left\{B_u(u) \mid \mathrm{row}_i(C_u u) = d_{u,i}\right\}\,.$$

Theorem 1 establishes the closed-loop stability of RRLB-based linear MPC, while Theorem 2 presents local constraint satisfaction guarantees. In the following sections, we will investigate nonlinear MPC design by using RRLB functions and analyze the corresponding closed-loop performance.

IV. Closed-Loop Analysis of RRLB-Based Nonlinear MPC

Throughout the rest of this paper, we write

$$\begin{aligned}&\tilde{\mathbf{x}}(x^t) = \left\{x^t, \tilde{x}_1(x^t), \ldots, \tilde{x}_{N-1}(x^t), \tilde{x}_N(x^t)\right\} \\ \text{and } \ &\tilde{\mathbf{u}}(x^t) = \left\{\tilde{u}_0(x^t), \tilde{u}_1(x^t), \ldots, \tilde{u}_{N-1}(x^t)\right\}\end{aligned}$$

for the optimal sequences of states and controls given by solving (6) for a current state $x^t \in \mathbb{R}^{n_x}$. We further drop the "(x^t)" when no confusion is possible.

A. Nominal Closed-Loop Stability

To prove the asymptotic stability of the controlled system yielded by (6), we first need the following lemma on the regularity of the solution of (6).

Lemma 1 *If **A1** holds and for a given initial state $\bar{x}$, the solution of* (6) *is a regular KKT point, then in a neighborhood of $\bar{x}$ we have:*

$$\|\tilde{u}_k(\bar{x})\| = \mathcal{O}(\|\bar{x}\|),\ \|\tilde{x}_{k+1}(\bar{x})\| = \mathcal{O}(\|\bar{x}\|)$$

for all $k = 0, 1, ..., N-1$.

We can use the proof of the Theorem 4.2 in [17] to show this Lemma. A sketch of the proof can be found in Appendix A.

Now, we can establish the closed-loop stability of an MPC controller based on (6).

Theorem 3 *Let assumptions **A1**–**A3** hold. Moreover, let the matrix $\tilde{P}$ defining the terminal costs be the unique positive definite solution to the following Lyapunov equation:*

$$\tilde{P} = A_K^\top \tilde{P} A_K + \mu \tilde{Q}_K \quad \textit{with} \ \ \mu > 1$$

$$\textit{and} \ \ \tilde{Q}_K = \underbrace{Q + \epsilon M_x}_{=:\tilde{Q}} + K^\top (\underbrace{R + \epsilon M_u}_{=:\tilde{R}}) K\,,$$

*where K is the stabilizing control based on assumption **A3**. Then, for all initial states in a neighborhood of the origin, the controlled system yielded by the RRLB nonlinear MPC* (6) *is locally asymptotically stable.*

In practice, we can compute the matrix K by solving the DARE (8) with $\mu \approx 1$. The factor μ is very important in theory but in practice could be taken very close, or even equal to one.

Proof. To prove local asymptotic stability, we will show that $\widetilde{V}_N$ is a Lyapunov function in a neighborhood of the origin. First, since the RRLB functions are recentered, we can write the Taylor expansion of the stage costs $\tilde{l}$ as:

$$\begin{aligned}&\tilde{l}(x,u) = x^\top \tilde{Q} x + u^\top \tilde{R} u + \mathcal{O}(\|x\|^3 + \|u\|^3) \\ \Rightarrow \quad &\tilde{l}(x, Kx) = x^\top \tilde{Q}_K x + \mathcal{O}(\|x\|^3)\,.\end{aligned}$$

By the assumptions, we also have $\forall x \in \mathbb{R}^{n_x}$,

$$\tilde{F}(A_K x) + \mu x^\top \tilde{Q}_K x - \tilde{F}(x) = 0 \tag{10}$$

hold, and analogously as discussed in [18, Chapter 2.5.5], there exists a neighborhood of the origin in which:

$$\tilde{F}(f(x,Kx)) + x^\top \tilde{Q}_K x - \tilde{F}(x) \leq 0 \tag{11}$$

$$\Rightarrow \tilde{F}(f(x,Kx)) - \tilde{F}(x) + \tilde{l}(x,Kx) = \mathcal{O}(\|x\|^3)\,. \tag{12}$$

According to Lemma 1, the predicted states are all Lipschitz with respect to the initial state, so we can choose an even smaller neighborhood such that $\tilde{x}_N(x_t)$ is also in it.

Now using the optimal sequence of states and controls of the RRLB MPC starting at x^t, we can construct new feasible sequences for the problem starting at $x^{t+1} = \tilde{x}_1(x^t)$ as

$$\mathbf{x}' = (\tilde{x}_1, \dots, \tilde{x}_N, f(\tilde{x}_N, K\tilde{x}_N))$$
$$\text{and } \mathbf{u}' = (\tilde{u}_1, \dots, \tilde{u}_{N-1}, K\tilde{x}_N).$$

Then we obtain:

$$\begin{aligned}\widetilde{V}_N(x^{t+1}) &\le \tilde{J}(\mathbf{x}', \mathbf{u}') = \tilde{J}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) - \tilde{l}(x^t, \tilde{u}_0) \\ &\quad + \tilde{F}(f(\tilde{x}_N, K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N, K\tilde{x}_N) \\ &= \widetilde{V}_N(x^t) - \mathcal{O}(\|x^t\|)^2 + \mathcal{O}(\|x^t\|^3) \\ &= \widetilde{V}_N(x^t) - \mathcal{O}(\|x^t\|)^2,\end{aligned}$$

which establishes the descent property for $\widetilde{V}_N$. Finally, it is easy to show that $\widetilde{V}_N$ is lower and upper bounded by coercive quadratic functions (since Q, R are symmetric, positive definite and B_x, B_u are positive definite and locally twice-Lipschitz continuously differentiable, i.e., both are upper bounded by quadratics), so that $\widetilde{V}_N$ is a Lyapunov function in a small neighborhood at the origin. □

One can see that the proof of Theorem 3 is the same as the closed-loop stability proof of the nominal MPC scheme without state and control constraints [18], [19], i.e., $\mathcal{X} = \mathbb{R}^{n_x}$ and $\mathcal{U} = \mathbb{R}^{n_u}$ in (2). In this case, the MPC formulation (6) is recursively feasible such that one can use the optimal solution at the current time instant to construct a feasible solution for the shifted problem. Under the regularity assumption, the closed-loop stability is thus shown based on the fact that $\widetilde{V}_N$ is a local Lyapunov function.

Remark 1 *In practice, one needs an efficient online solver to deal with* (6). *A commonly used approach is the Newton-type method, which has been implemented in many state-of-the-art open-source toolkits such as* `ACADO` *[20],* `acados` *[21],* `PolyMPC` *[22]. In their implementations, a single iteration of a Newton-type approach could be applied to deal with* (6) *suboptimally, which is a so-called real-time iteration (RTI), while the closed-loop performance can be still guaranteed. Based on our results, one can establish the closed-loop stability of the RRLB NMPC based RTI scheme by following the analysis in [23].*

B. Constraint Satisfaction Guarantees

Now we aim at proving a nonlinear counterpart of Theorem 2. We follow the same construction as discussed in [10] but now the error terms that appeared in the previous section will become problematic.

Lemma 2 *Consider the RRLB nonlinear MPC* (6) *and suppose that assumptions **A1–A3** hold. Let further x^0 be an initial state in the neighborhood given by Theorem 3. Let's denote by $\{x^0, x^1, \dots\}$ and $\{u^0, u^1, \dots\}$ the closed-loop state and control trajectories obtained by applying the controls $u^t = \tilde{u}_0(x^t)$. There exists a function $\alpha : \mathbb{R}^{n_x} \to \mathbb{R}$ such that $\forall t \in \mathbb{Z}_{\ge 0}$:*

$$B_x(x^t), B_u(u^t) \le \frac{1}{\epsilon}\alpha(x^0).$$

Proof. In the proof of Theorem 3 we showed that $\forall t \in \mathbb{Z}_{\ge 0}$,

$$\begin{aligned}&\widetilde{V}_N(x^{t+1}) - \widetilde{V}_N(x^t) \\ &\le -\tilde{l}(x^t, \tilde{u}_0(x^t)) + \tilde{l}(\tilde{x}_N(x^t), K\tilde{x}_N(x^t)) \\ &\qquad - \tilde{F}(\tilde{x}_N(x^t)) + \tilde{F}(f(\tilde{x}_N(x^t), K\tilde{x}_N(x^t)))\,.\end{aligned}$$

Let us define the error term

$$\begin{aligned}\eta(x) := &-\tilde{l}(\tilde{x}_N(x), K\tilde{x}_N(x)) + \tilde{F}(\tilde{x}_N(x)) \\ &- \tilde{F}(f(\tilde{x}_N(x), K\tilde{x}_N(x))).\end{aligned}$$

Then, if we sum the terms in the previous inequality, we obtain a telescopic sum that we can compute using the fact that the system is asymptotically stable, i.e.,

$$\lim_{t\to\infty} \widetilde{V}(x^t) = 0 \;\Rightarrow\; \widetilde{V}_N(x^t) \ge \sum_{t=0}^{\infty} \tilde{l}(x^t, u^t) + \eta(x^t).$$

Based on the modified stage cost $\tilde{l}(x^t, u^t)$ and noticing that

$$\begin{aligned}\sum_{t=0}^{\infty} l(x^t, u^t) &\ge V_\infty(x^t) \\ &= x^{t\top} P x^t + \mathcal{O}(\|x^t\|^3)\end{aligned}$$

by the same arguments as in [10], we can define α by

$$\alpha(x) := \widetilde{V}_N(x) - x^\top P x + \mathcal{O}(\|x\|^3) - \sum_{t=0}^{\infty} \eta(x^t) \qquad (13)$$

to obtain

$$\alpha(x^0) \ge \epsilon \sum_{t=0} B_x(x^t) + \epsilon \sum_{t=0} B_u(u^t).$$

Then, since the RRLB functions are all positive definite, we can easily have

$$B_x(x^t), B_u(u^t) \le \frac{1}{\epsilon}\alpha(x^0),$$

which concludes the proof. □

Unlike the analog expression of α in the linear case ($\alpha(x) = \widetilde{V}_N(x) - x^\top P x$), there are some error terms (the cubic term and the sum of the $\eta(x^t)$ terms in (13)) that are not actually calculable. This comes from the generality of the considered dynamics.

Lemma 3 (Lemma 1 in [10]) *Let us define the bounds*

$$\beta_x = \min_{i,x}\{B_x(x) \mid \mathrm{row}_i(C_x)x = d_{x,i}\},$$
$$\beta_u = \min_{i,u}\{B_u(u) \mid \mathrm{row}_i(C_u)u = d_{u,i}\}\,.$$

Then for all relaxation parameters $\delta > 0$, we have

$$\{x \in \mathbb{R}^{n_x} \mid B_x(x) \le \beta_x\} \subseteq \mathcal{X},$$
$$\text{and} \quad \{u \in \mathbb{R}^{n_u} \mid B_u(u) \le \beta_u\} \subseteq \mathcal{U}.$$

Theorem 4 *Let the assumptions in Lemma 2 hold, for any initial state $x_t \in \mathcal{X}_N$ with*

$$\mathcal{X}_N := \{x \in \mathbb{R}^{n_x} \mid \alpha(x)) \le \epsilon \min\{\beta_x, \beta_u\}\}\,,$$

there is no state or control constraint violation along the closed loop trajectories.

Proof. Lemma 2 implies that for any $x^0 \in \mathcal{X}_N$ and $\forall t \in \mathbb{Z}_{\geq 0}$,

$$\epsilon B_x(x^t) \leq \alpha(x^0) \leq \epsilon\beta_x \implies B_x(x^t) \leq \beta_x,$$
$$\epsilon B_u(u^t) \leq \alpha(x^0) \leq \epsilon\beta_u \implies B_u(u^t) \leq \beta_u,$$

and thus, by Lemma 3, $x^t \in \mathcal{X}$, $u^t \in \mathcal{U}$. □

V. Numerical Experiments

This section demonstrates the theoretical results numerically by applying RRLB MPC to a nonlinear benchmark system: the Continuously Stirred Tank Reactor (CSTR)[1].

A. Experimental Setup

We use the CSTR system description in [24, Chapter 1.2]. The continuous dynamics are given by

$$\dot{c}_A = u_1(c_{A0} - c_A) - k_1(\vartheta)c_A - k_3(\vartheta)c_A^2 \tag{14a}$$
$$\dot{c}_B = -u_1 c_B + k_1(\vartheta)c_A - k_2(\vartheta)c_B \tag{14b}$$
$$\begin{aligned}\dot{\vartheta} &= u_1(\vartheta_0 - \vartheta) + \frac{k_w A_R}{\rho C_p V_R}(\vartheta_K - \vartheta) \\ &\quad - \frac{1}{\rho C_p}\left[k_1(\vartheta)c_A H_1 + k_2(\vartheta)c_B H_2 \right. \\ &\quad \left. + k_3(\vartheta)c_A^2 H_3\right]\end{aligned} \tag{14c}$$
$$\dot{\vartheta}_K = \frac{1}{m_K C_{PK}}\left[u_2 + k_w A_R(\vartheta - \vartheta_K)\right], \tag{14d}$$

with $k_i(\vartheta) = k_{i,0}\exp\left(\frac{E_i}{\vartheta + 273.15}\right)$ for $i = 1, 2, 3$, involves four states $x = (c_A, c_B, \vartheta, \vartheta_K)$ and two control inputs $u = (u_1, u_2)$. These continuous dynamics are discretized with a 10-step RK4 integrator. We use the same steady-state-control (x^*, u^*) pair, input constraints, prediction horizon of $200s$ divided into 10 control intervals, and quadratic costs Q, R as presented in [24, Chapter 1.2]. The only change we made was adding state constraints:

$$c_A \in [0\,\frac{\text{mol}}{\text{L}},\ 10\,\frac{\text{mol}}{\text{L}}],\ c_B \in [0\,\frac{\text{mol}}{\text{L}},\ 10\,\frac{\text{mol}}{\text{L}}],$$
$$\vartheta \in [98\,°\text{C},\ 150\,°\text{C}],\ \vartheta_K \in [92\,°\text{C},\ 150\,°\text{C}].$$

We implemented this MPC problem using the regular MPC scheme (2) and the RRLB MPC scheme (6). Both implementations are based on the RTI method offered by the `acados` library (see [21]) with the `HPIPM` solver (see [25]). For the RRLB MPC scheme, we set the relaxation parameter as

$$\delta = \frac{1}{2}\min\{d_{x,1}, \dots, d_{x,m_x}, d_{u,1}, \dots, d_{u,m_u}\}.$$

and chose the barrier parameter $\epsilon = 30.0$ by a simple grid-search.

B. Illustration of Local Asymptotic Stability

To demonstrate Theorem 3, we implemented our RRLB NMPC scheme with randomly choosing six different initial states inside the state constraint set, and Figure 2 illustrates the convergence of the closed-loop trajectory.

This figure reports the evolution of the distance between the state and the reference state throughout the iterations. We actually did the experiment on a larger number of initial states and only reported six of them to maintain visual clarity. Convergence (in terms of distance to the reference point within a tolerance of 10^{-4}) was measured for all of them. In our experiments, we actually found no initial states that do not lead to convergence to the reference state, the closed-loop trajectories always converge.

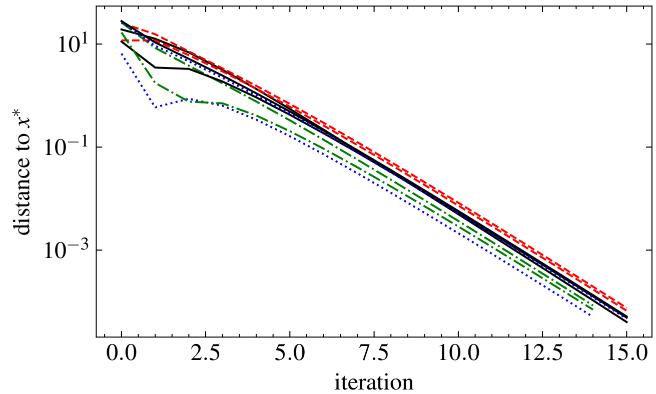

Fig. 2: Evolution of the distance to the reference state.

C. Comparison of RRLB and Regular MPC

For this experiment, we chose a single initial state and ran a closed-loop simulation with both regular MPC and RRLB MPC.

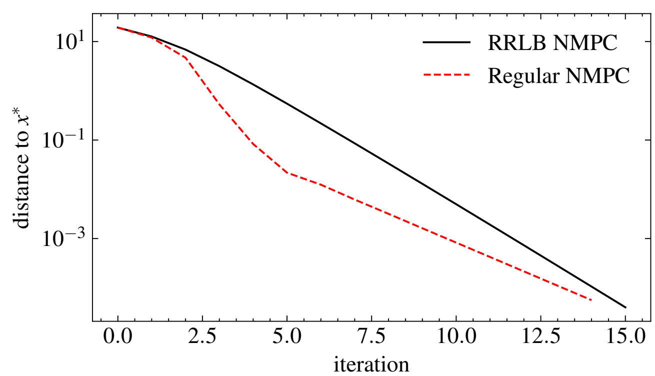

Fig. 3: Evolution of the distance to the reference state for regular and RRLB MPC

Figure 3 shows that the convergence of RRLB NMPC is slower but smoother because the control inputs are not actually saturated. This comes from the high barrier parameter ϵ necessary to ensure constraint satisfaction during the first iteration.

D. Illustration of the Constraint Satisfaction Guarantee

To illustrate the constraint satisfaction property of RRLB NMPC controllers, we ran the controller in closed loop for a grid of different initial states and for different barrier parameters ϵ. Since the initial states are 4-dimensional and hard to visualize in their entirety, we have fixed

$$\vartheta = 130.28, \quad \vartheta_K = 94.0,$$

and only created a 2D grid of 30×30 values for (c_A, c_B).

[1]The code of our implementation is available at `https://github.com/tudoroancea/paper_rrlb_mpc`

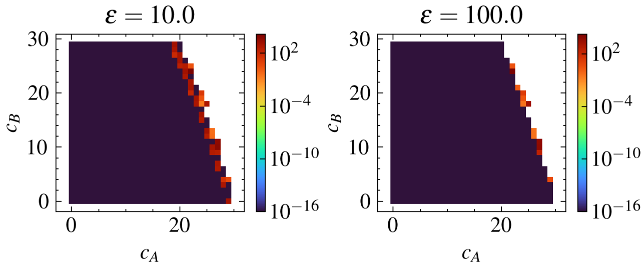

Fig. 4: Accumulated constraint violation along the closed-loop trajectory

Figure 4 displays how the constraint violation decreases and goes to 0 when ϵ increases. This is because the constraint violation are more heavily penalized when ϵ is bigger. This figure, thus, shows (a small part of) the neighborhood given by Theorem 4 and could hint its potential monotonicity with respect to ϵ, i.e., if we call this neighborhood $\mathcal{X}_N(\epsilon)$ (to indicate the dependence on ϵ) then $\epsilon < \epsilon' \Rightarrow \mathcal{X}_N(\epsilon) \subseteq \mathcal{X}_N(\epsilon')$. A rigorous proof of this observation will be a future research direction on this subject.

VI. Summary and Outlook

This paper shows that in the case of nonlinear dynamics, the novel RRLB MPC schemes can still yield locally asymptotically stable systems even without strictly enforcing state and control constraints. We also proved the existence of a neighborhood around the reference state where the constraints are never violated. These two proofs provide an advancement on the subject of RRLB MPC and could be the groundwork for new MPC implementations. Future advancements could include theoretical lines of research through the investigation of the monotonicity properties of the neighborhood involved in our proofs with respect to the horizon size or the barrier parameter, as well as numerical endeavors such as the development of tailored optimization methods to solve the RRLB problem being more efficient in embedded system.

Appendix

A. Proof of Lemma 1

We illustrate the main idea of the proof by defining the k-step model by

$$x_k = \xi_k(x_0, \mathbf{u}) = f(\dots f(f(x_0, u_0), u_1)\dots, u_{k-1})$$

such that (6) can be equivalently rewritten as

$$\widetilde{V}_N(x_t) := \min_{\mathbf{u}} \ \bar{J}(x_0, \mathbf{u}) := \sum_{k=0}^{N-1} \tilde{l}(\xi_k(x_0, \mathbf{u}), u_k) + \tilde{F}(\xi_N(x_0, \mathbf{u}))$$

$$\text{subject to } \ x_0 = x^t, \tag{15}$$

which is the so-called single-shooting formulation in the context of numerical optimal control [18, Chapter 8]. Based on the assumptions, we can have $\nabla_{\mathbf{u}} \bar{J}(\bar{x}, \tilde{\mathbf{u}}) = 0$ and $\nabla^2_{\mathbf{uu}} \bar{J}(\bar{x}, \tilde{\mathbf{u}}) \in \mathbb{S}^n_{++}$, such that each $\tilde{u}_k$ and $\tilde{x}_k$ are continuously differentiable in a closed neighborhood of the origin. Therefore, by continuity, their gradient is bounded on this neighborhood, and they are therefore Lipschitz-continuous.

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