Study of the properties of the Relaxed Recentered Log-Barrier function based Nonlinear Model Predictive Control (RRLB NMPC)

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***Abstract*—**

I. Introduction

Nowadays, model predictive control is a very important control framework that has been extensively studied in the last few decades for its capability of handling complex constrained multi-input multi-output systems which are omnipresent in real-life contexts. The theory and mechanisms of linear model predictive control is now well understood and there are numerous examples of successful industrial applications, the most obvious one being chemical plants. Nonlinear model predictive control (NMPC) theory is however much less mature and is still an active field of research.

Great effort has been put in the last decade to develop practical implementations of NMPC that are real-time capable, i.e. that can be deployed on embedded platforms which have to solve the optimization problem online at frequencies in the kHz range. One of the most important of such methods is certainly Real Time Iteration (RTI) that was first introduced by M. Diehl in [1] and which solves the nonlinear optimization problem by solving a single quadratic program approximation at each call. Theoretical studies have been pursued to prove the actual legitimacy of this method in [2] and it is today implemented in well known NMPC software packages like acados (see original paper [3]).

At the same time, researchers also focused on simplifying the optimization problems involved in NMPC by leveraging ideas coming from classical mathematical optimization. One of those is the one of replacing constraints by a logarithmic barrier functions, similarly as interior point methods (see [4]). Such modifications lead to equality-only-constrained or even unconstrained optimization problems that could be in some sense "easier to solve", but that still present some flaws, like the non-definition of these barrier functions outside the constraint set, which can lead to infeasible problems. This is usually solved by introducing soft constraints, however no theoretical guarantees can be given on the convergence of the resulting optimization problem. Another solution consists in *relaxing* the barrier function by extending it outside the constraint set in a twice continuously differentiable fashion. These *relaxed (recentered) logarithmic barrier functions* (RRLB functions) have been introduced in [] and heavily studied in the context of stabilization with Linear MPC in [5] and [6]. In this simple case, it was proven that globally asymptotically stabilizing schemes can be desgined with the help of these functions, in particular by leveraging appropriate terminal costs and/or sets. What's more, constraint satisfaction guarantees can be derived locally around the reference state.

The goal of this paper is to extend part of this results in the more general case of nonlinear dynamics. In particular, we are going to use the fourth scheme described in [6], that only uses quadratic terminal costs and no terminal constraints, show that we can ensure asymptotic stability but only in a neighborhood of the reference state (Section III-A), and show that constraint satisfaction guarantees ... (Section III-B). We also aim at providing a numerical illustration of the theoretical results by using the RTI method and actually comparing a regular MPC scheme with its relaxed barrier function based counterpart (Section IV).

In this paper we will call $\mathcal{C}^k$ the set of all k-times continuously differentiable functions, and we will call $D_x f(x,u)$ and $\nabla_x^2 f(x,u)$ the jacobian and the hessian with respect to x of a function f defined on $R^{n_x} \times \mathbb{R}^{n_u}$.

II. Background material

A. The RRLB MPC

In this paper we consider the control of general nonlinear $\mathcal{C}^2$ time-invariant discrete-time dynamical systems of the form

$$x(k+1) = f(x(k), u(k)),\ x(k) \in \mathbb{R}^{n_x}, u(k) \in \mathbb{R}^{n_u}, \quad (1)$$

that we are also going to abbreviate $x^+ = f(x,u)$ and that we want to stabilize around a steady state that we can take without loss of generality as the origin. This means that $f(0,0) = 0$. We further suppose that our the states and control inputs are constrained in the form of compact polytopic constraints:

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^{n_x} \mid C_x x \le d_x\}, \\ \mathcal{U} &= \{u \in \mathbb{R}^{n_u} \mid C_u u \le d_u\}, \end{aligned} \quad (2)$$

where $C_x, C_u \in \mathbb{R}^{q_x \times n_x}, \mathbb{R}^{q_u \times n_u}$ are matrices and $d_x, d_u \in \mathbb{R}^{q_x}, \mathbb{R}^{q_u}$ are vectors. These constraints can be translated into *(weight) recentered log-barrier functions* that are added in the objective function of the MPC to keep the states and control inputs inside the feasible set.

We can however relax these functions as proposed in [6] and define *relaxed recentered log-barrier functions* (or ***RRLB***

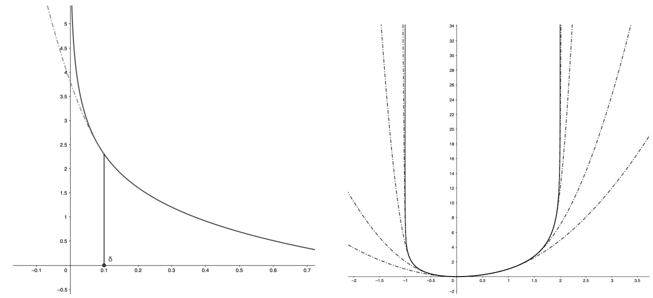

Fig. 1: *(Left)*: Principle of relaxed log barrier function based on quadratic relaxation. (*Right*): Regular weight recentered log barrier function (solid line) and RRLB function for $\delta \in \{0.01, 0.1, 0.5, 1\}$ for the constraint $z \in \mathbb{R},\ -1 \leq z \leq 2$

functions) B_x and B_u. We recall below the definitions only for B_x (since they are similar for B_u):

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x), \tag{3}$$

$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,i}) - \log(d_{x,i} - \mathrm{row}_i(C_x)x) \\ \quad \text{if } d_{x,i} - \mathrm{row}_i(C_x)x > \delta \\ \beta(d_{x,i} - \mathrm{row}_i(C_x)x; \delta) \\ \quad \text{otherwise,} \end{cases} \tag{4}$$

Here δ is called the *relaxation parameter* and β is the function that extends the usual log-barrier functions to the whole space in a twice continuously differntiable fashion. The simplest example for such a function β is:

$$\beta(z; \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)\,. \tag{5}$$

The weights $w_{x,i}, i = 1, \ldots, q_x$ are chosen such that $B_x(0) = 0$ and $\nabla B_x(0) = 0$. These equations only have solutions if $0 < \delta \leq \min\{d_{x,1}, \ldots, d_{x,q_x}, d_{u,1}, \ldots, d_{u,q_u}\}$, which is always possible if $0 \in \mathcal{X}$ and $0 \in \mathcal{U}$. An illustration of this relaxation procedure can be found in figure 1 found in [6].

Now if we consider the following regular nonlinear MPC:

$$\begin{aligned} V(x) = \min_{\mathbf{x},\mathbf{u}} \quad & J(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} l(x_k, u_k) \ + \ F(x_N), \\ \text{s.t.} \quad & x_0 = x, \\ & x_{k+1} = f(x_k, u_k),\ k = 0, \ldots, N-1, \\ & x_k \in \mathcal{X},\ k = 0, \ldots, N, \\ & u_k \in \mathcal{U},\ k = 0, \ldots, N-1, \end{aligned} \tag{6}$$

where the stage costs $l(x, u) = x^T Q x + u^T R u$ and the terminal cost $F(x) = x^T P x$ are positive definite quadratic functions, we can define the nonlinear *RRLB MPC*:

$$\begin{aligned} \tilde{V}(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k, u_k) \ + \ \tilde{F}(x_N) \\ \text{s.t.} \quad & x_0 = x, \\ & x_{k+1} = f(x_k, u_k),\ k = 0, \ldots, N-1, \end{aligned} \tag{7}$$

where ϵ is the barrier parameter and the new stage and terminal costs are defined by $\tilde{l}(x, u) = l(x, u) + \epsilon B_x(x) + \epsilon B_u(u)$ and $\tilde{F}(x) = x^T \tilde{P} x$ (for a matrix $\tilde{P}$ that will be determined in such a way that the control law given by the MPC yields an asymptotically stable system).

B. RRLB MPC in the case of linear dynamics

When we consider the RRLB MPC (7) in the case of linear dynamics $x^+ = f(x, u) = Ax + Bu$, we know the two following theorems to be true:

Theorem II.1 (Theorem 5 in [6])**.** *Suppose that the pair (A, B) is stabilizable and that the matrix P is chosen as the unique positive definite solution to the following modified DARE:*

$$P = A^T P A - A^T P B (R + \epsilon M_u + B^T P B)^{-1} B^T P A + Q + \epsilon M_x \tag{8}$$

where $M_x = \nabla^2 B_x(0)$ and $M_u = \nabla^2 B_u(0)$ are the hessian of the RRLB functions at the origin.

Then for any initial state $x(0) \in \mathbb{R}^{n_x}$, the control law given by the MPC yields a globally asymptotically stable system.

Theorem II.2 (Lemma 5 in [6])**.** *In the same conditions as in the previous theorem, there is a neighborhood $\mathcal{X}_N(\delta)$ of the origin such that for any initial state $x(0) \in \mathcal{X}_N(\delta)$, all the constraint will be satisfied along the closed-loop trajectories. Furthermore, the set $\mathcal{X}_N(\delta)$ is given explicitly by:*

$$\begin{aligned} \mathcal{X}_N(\delta) = \{x \in \mathcal{X} \mid \tilde{V}(x(0)) - x(0)^T P_{\mathrm{LQR}} x(0) \\ \leq \min\{\beta_x(\delta), \beta_u(\delta)\}\} \end{aligned} \tag{9}$$

where P_{LQR} is the solution to the classical DARE and

$$\beta_x(\delta) = \min_{i,x} \{B_x(x) \mid \mathrm{row}_i(C_x x) = d_{x,i}\}\,, \tag{10}$$

$$\beta_u(\delta) = \min_{i,u} \{B_u(u) \mid \mathrm{row}_i(C_u u) = d_{u,i}\} \tag{11}$$

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Additional results were proven in the linear case, such as the existence of a procedure to find δ such that the maximum constraint violation along the closed-loop trajectories is bounded by a pre-defined tolerance. In the nonlinear case, however, we will not be able to prove such powerful results because of the lack of knowledge on the error terms that appear in the dynamics.

III. Theoretical properties of nonlinear RRLB MPC

In this section, for an initial state $x \in \mathbb{R}^{n_x}$ we will denote by $\tilde{\mathbf{x}}(x) = \{\tilde{x}_0(x) = x, \tilde{x}_1(x), \ldots, \tilde{x}_N(x)\}$ and $\tilde{\mathbf{u}}(x) =$

$\{\tilde{u}_0(x), \tilde{u}_1(x), \ldots, \tilde{u}_{N-1}(x)\}$ the optimal sequences of states and controls found by the RRLB MPC. When no confusion is possible we will drop the "(x)".

A. Nominal asymptotic stability

Lemma III.1. *Consider the RRLB MPC* (7) *that can be written as the unconstrained problem*

$$\tilde{V}(x) = \min_{\mathbf{u}} \quad \hat{J}(x, \mathbf{u})$$

where $\hat{J}(x, \mathbf{u}) = \tilde{l}(x, u_0) + \tilde{l}(f(x, u_0), u_1) + \cdots + \tilde{F}(f(f(...), u_{N-1}))$. *If for a certain initial state $\bar{x}$ we suppose that $D_{\mathbf{u}}\hat{J}(\bar{x}, \tilde{\mathbf{u}}(\bar{x})) = 0$ and $\nabla^2_{\mathbf{uu}}\hat{J}(\bar{x}, \tilde{\mathbf{u}}(\bar{x})) \succ 0$ (the matrix is positive definite) then in a neighborhood of $\bar{x}$ we have:*

- $\forall k = 0, \ldots, N-1, \quad \|\tilde{u}_k(x)\| = O(\|x\|)$
- $\forall k = 1, \ldots, N, \quad \|\tilde{x}_k(x)\| = f(f(\ldots, u_{k-2}), u_{k-1}) = O(\|x\|)$

Proof. We can use the proof of the Theorem 4.2 in [7] to argue that each $\tilde{x}_k$ and $\tilde{u}_k$ is continuously differentiable in a closed neighborhood of the origin. Therefore, by continuity their gradient is bounded on this neighborhood and they are Lipschitz. □

Theorem III.2. *Let's consider the problem* (7) *and assume the following:*

1) *If we denote the objective function as $J(x, \mathbf{u})$ as in Lemma III.1, we have $D_{\mathbf{u}}J(0, \mathbf{u}(0)) = D_{\mathbf{u}}J(0, 0) = 0$ and $\nabla^2_{\mathbf{uu}}J(0, \mathbf{u}(0)) = \nabla^2_{\mathbf{uu}}J(0, 0) \succ 0$*
2) *When linearizing the system dynamics around the origin and letting $A = D_x f(0, 0)$, $B = D_u f(0, 0)$, we suppose that the pair (A, B) is stabilizable. This implies in particular that there exists a stabilizing cost K, i.e. a matrix such that $A_K := A + BK$ only has eigenvalues in the unit disk.*
3) *The matrix $\tilde{P}$ defining the terminal costs is the unique positive definite solution to the following Lyapunov equation:*
$$\tilde{P} = A_K^T \tilde{P} A_K + \mu \tilde{Q}_K \tag{12}$$
where $\mu > 1$ and $\tilde{Q}_K = \underbrace{Q + \epsilon M_x}_{=:\tilde{Q}} + K^T(\underbrace{R + \epsilon M_u}_{=:\tilde{R}})K$.

Then the dynamical system $x^+ = f(x, \tilde{u}_0(x))$ is locally asymptotically stable, i.e. for all initial state in a neighborhood of the origin, the control law given by the RRLB MPC yields an asymptotically stable system.

Remark. To identify the matrix K, we can approximate $\mu \approx 1$ and solve the following classical DARE:

$$\begin{cases} K = -(\tilde{R} + B^T \tilde{P} B)^{-1} B^T \tilde{P} A, \\ \tilde{P} = A_K^T \tilde{P} A_K + \tilde{Q}_K \,. \end{cases} \tag{13}$$

The factor μ is very important in theory but in practice could be taken very close to 1 or actually equal to 1.

Proof. To prove local asymptotic stability we will show that $\tilde{V}$ is a Lyapunov function in a neighborhood of the origin. First, using the recentering of the RRLB functions, we can write the Taylor expansion of the stage costs $\tilde{l}$:

$$\begin{aligned} \tilde{l}(x, u) &= x^T[\nabla^2_{xx}\tilde{l}(0,0)]x + u^T[\nabla^2_{uu}\tilde{l}(0,0)]u + O(\|x\|^3 + \|u\|^3) \\ &= x^T \tilde{Q} x + u^T \tilde{R} u + O(\|x\|^3 + \|u\|^3) \\ &\implies \tilde{l}(x, Kx) = x^T \tilde{Q}_K x + O(\|x\|^3)\,. \end{aligned}$$

By the second assumption, we also have $\forall x \in \mathbb{R}^{n_x}$:

$$\tilde{F}(A_K x) + \mu x^T Q_K x - \tilde{F}(x) = 0, \tag{14}$$

and by the same reasoning as in the paragraph 2.5.5 of [8], there exists a neighborhood of the origin in which:

$$\tilde{F}(f(x, Kx)) + x^T Q_K x - \tilde{F}(x) \leq 0 \tag{15}$$

$$\Longrightarrow \tilde{F}(f(x, Kx)) - \tilde{F}(x) + \tilde{l}(x, Kx) = O(\|x\|^3)\,. \tag{16}$$

What's more, by lemma III.1, the predicted states are all Lipschitz with respect to the initial state, so we can choose an even smaller neighborhood such that $\tilde{x}_N(x)$ is also in it.

Now using the optimal sequence of states and controls of the RRLB MPC starting at x, we can construct new feasible sequences for the problem starting at $\tilde{x}_1(x)$ as $\mathbf{x}' = (\tilde{x}_1, \ldots, \tilde{x}_N, f(\tilde{x}_N, K\tilde{x}_N))$ and $\mathbf{u}' = (\tilde{u}_1, \ldots, \tilde{u}_{N-1}, K\tilde{x}_N)$. Then we obtain:

$$\begin{aligned} \tilde{V}(\tilde{x}_1) \leq \tilde{J}(\mathbf{x}', \mathbf{u}') &= \tilde{J}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) - \tilde{l}(x, \tilde{u}_0) \\ &\quad + \tilde{F}(f(\tilde{x}_N, K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N, K\tilde{x}_N) \\ &= \tilde{V}(x) - O(\|x\|)^2 + O(\|x\|^3), \end{aligned}$$

which proves the descent property for $\tilde{V}$. Finally, it is easy to show that $\tilde{V}$ is lower and upper bounded by coercive quadratic functions (since Q, R are positive definite and B_x, B_u are positive definite and upper bounded by quadratics), so that $\tilde{V}$ is indeed a Lyapunov function in a small neighborhood around the origin. □

B. Constraints satisfaction guarantees

Lemma III.3. *Consider the RRLB MPC* (7) *and an initial state $x(0)$ in the neighborhood given by Theorem III.2. Let's denote by $\{x(0), x(1), \ldots\}$ and $\{u(0), u(1), \ldots\}$ the closed-loop state and control trajectories (given by $u(k) = \tilde{u}_0(x(k)),\ x(k+1) = f(x(k), u(k))$), and define*

$$\alpha(x(0)) := \frac{1}{\epsilon}\left(\tilde{V}(x(0)) - x(0)^T P_{\mathrm{LQR}} x(0) - \sum_{k=0}^{\infty} \eta(x(k))\right)$$

with P_{LQR} is the solution to the classical DARE, and $\eta(x) = \tilde{l}(\tilde{x}_N(x), K\tilde{x}_N(x)) + \tilde{F}(\tilde{x}_N(x)) - \tilde{F}(f(\tilde{x}_N(x), K\tilde{x}_N(x)))$. Then $\forall k \geq 0$:

$$B_x(x(k)), B_u(u(k)) \leq \alpha(x(0))$$

Proof. In the proof of theorem III.2 we showed that $\forall k \geq 0$:

$$\begin{aligned} \tilde{V}(x(k+1)) - \tilde{V}(x(k)) &\leq \\ &- \tilde{l}(x(k), u(k)) + \tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) \\ &- \tilde{F}(\tilde{x}_N(x(k))) + \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k)))), \end{aligned}$$

so by summing for $k = 0, 1, \ldots$, we get a telescopic sum that we can compute using the fact that the system is asymptotically stable so $\lim_{k\to\infty} \tilde{V}(x(k)) = 0$:

$$\begin{aligned}\tilde{V}(x(0)) &\geq \sum_{k=0}^{\infty} \tilde{l}(x(k), u(k)) - \tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) \\ &\qquad + \tilde{F}(\tilde{x}_N(x(k))) - \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k)))) \\ &= \sum_{k=0}^{\infty} l(x(k), u(k)) + \epsilon B_x(x(k)) + \epsilon B_u(u(k)) + \eta(x(k)) \\ &\geq x(0)^T P_{\mathrm{LQR}} x(0) + \sum_{k=0}^{\infty} \eta(x(k)) \\ &\qquad + \epsilon \sum_{k=0}^{\infty} B_x(x(k)) + \epsilon \sum_{k=0}^{\infty} B_u(u(k))\end{aligned}$$

Even if we don't know a closed form for $\eta(x(k))$, we know that $\sum_{k=0}^{\infty} \eta(x(k))$ must be finite because it is bounded by $\tilde{V}(x(0))$. Now since the RRLB functions are all positive definite, we can easily conclude. ☐

Lemma III.4 (Lemma 1 in [6])**.** *If we define the bounds*

$$\beta_x = \min_{i,x} \{B_x(x) \mid \mathrm{row}_i(C_x)x = d_{x,i}\},$$
$$\beta_u = \min_{i,u} \{B_u(u) \mid \mathrm{row}_i(C_u)u = d_{u,i}\},$$

then then for all relaxation parameters δ:

$$\{x \mid B_x(x) \leq \beta_x\} \subseteq \mathcal{X}, \quad \{u \mid B_u(u) \leq \beta_u\} \subseteq \mathcal{U} \tag{17}$$

Theorem III.5. *In the same setting as lemma III.3, for any initial state $x(0)$ in the set*

$$\mathcal{X}_N(\delta) := \{x \in \mathbb{R}^{n_x} \mid \alpha(x(0)) \leq \epsilon \min\{\beta_x, \beta_u\}\}$$

there is no state or control constraint violation along the closed loop trajectories.

Proof. Lemma III.3 implies that for any $x(0) \in \mathcal{X}_N(\delta)$ and for all $k \geq 0$, $\epsilon B_x(x(k)) \leq \alpha(x(0)) \leq \epsilon\beta_x$ so $B_x(x(k)) \leq \beta_x$ and $x(k) \in \mathcal{X}$. The same reasoning applies on the controls. ☐

IV. Numerical experiments

Appendix

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