

# Mathematical Models. Modeling with Difference Equations

## Mathematical models

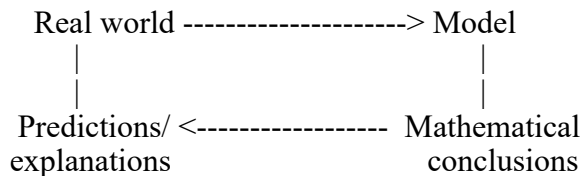
In modeling our world, we are often interested in predicting the value of a variable at some time in the future.

Model:

- a miniature representation of something
- a pattern of something to be made
- an example for imitation or emulation
- a description or analogy used to help visualize something that cannot be directly observed

**Mathematical Model:** a representation in mathematical terms of the behaviour of a real devices or phenomena.

Often a mathematical model can help us understand a behaviour better or aid us in planning for the future. Let's think of a mathematical model as a mathematical construct designed to study a particular real-world system or behaviour of interest. The model allow us to reach mathematical conclusion about the behaviour. These coclusions can be interpreted to help a decision maker plan for the future.



What means a good model:

- Does the structure of the model resemble the system being modeled?
- Why is the selected model appropriate to use in a given application?
- How well does the model perform?
- What is the accuracy of the model output?

## Discrete models, continuous models

In discrete models, the state variables is measured at a countable number of points in time  
 $x_n$  - the size of the variable after  $n$  time intervals (seasons)

In continuous models, the state variables is measured continously in time  
 $x(t)$  - the size of the variable at the moment  $t$  in time

Variation of the variable

discrete case:  $x_{n+1} - x_n$

$\Delta x_n = x_{n+1} - x_n$  the first order difference operator

continuous case:  $\frac{x(t + \Delta t) - x(t)}{\Delta t}$  average speed of the change

$\lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = x'(t)$  the instantaneous speed of change

Discrete models:

$$\begin{cases} x_{n+1} = f(x_n) \\ x_0 \text{ given} \end{cases}$$

$$\begin{cases} x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}) \\ x_0, \dots, x_k \text{ given} \end{cases}$$

Continuous models

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases}$$

## Modeling with difference equations

### Example 1: Testing for proportionality

**Definition:** Two variables  $y$  and  $x$  are proportional if one is always a constant multiple of the other, that is  $y = kx$  for some nonzero constant  $k$ .

Consider a spring-mass system and measure the stretch of the spring as a function of the mass placed on the spring. We have the following data:

**Mass    Elongation**

50        1.000

100       1.875

150       2.750

200       3.250

250       4.375

300       4.875

350       5.675

400       6.500

450       7.250

500       8.000

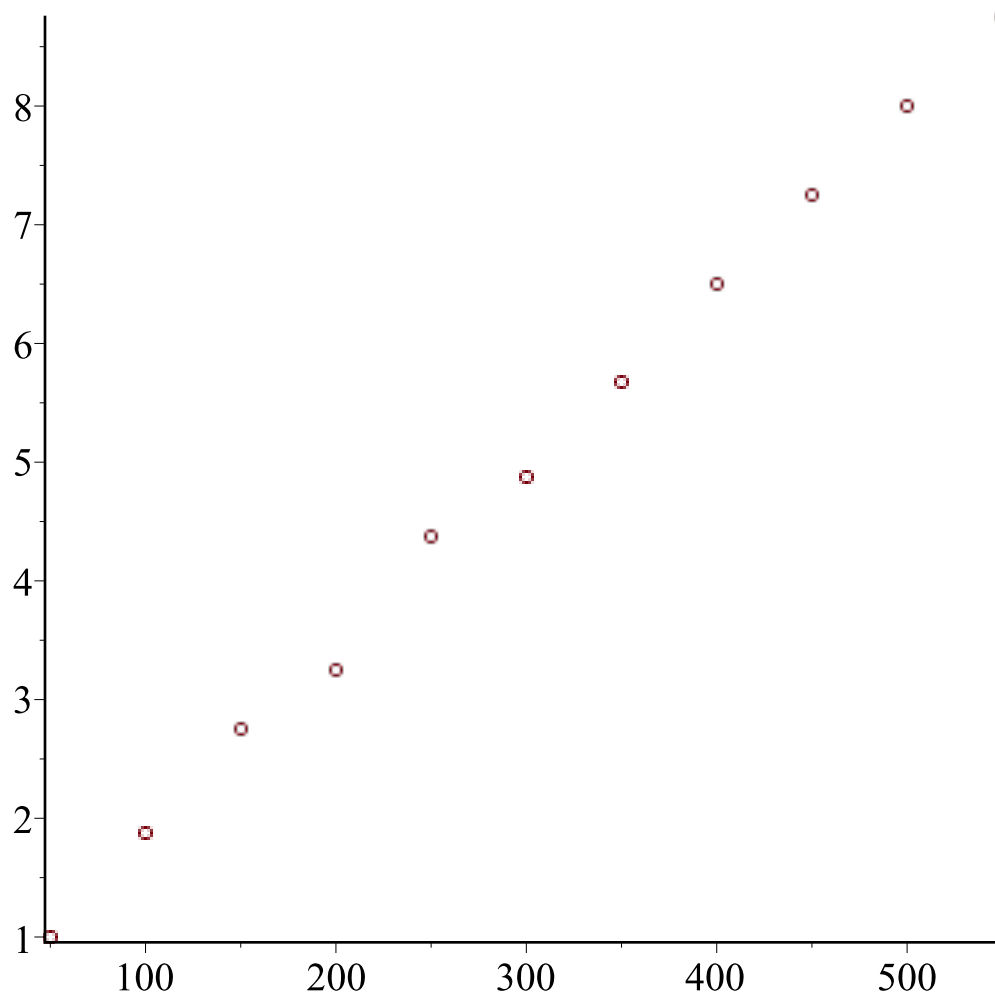
550       8.750

Let introduce the mass data as a sequence

```

> mass:=[50,100,150,200,250,300,350,400,450,500,550];
      mass := [50, 100, 150, 200, 250, 300, 350, 400, 450, 500, 550]
> mass[4];
      200
or we construct the mass sequence using the seq command
> mass:=[seq(i*50,i=1..11)];
      mass := [50, 100, 150, 200, 250, 300, 350, 400, 450, 500, 550]
> mass[7];
      350
> elong:=[1,1.875,2.75,3.25,4.375,4.875,5.675,6.5,7.25,8,8.75];
      elong := [1, 1.875, 2.75, 3.25, 4.375, 4.875, 5.675, 6.5, 7.25, 8, 8.75]
> elong[11];
      8.75
> [mass[i],elong[i]]$i=1..11;
[50, 1], [100, 1.875], [150, 2.75], [200, 3.25], [250, 4.375], [300, 4.875], [350, 5.675],
[400, 6.5], [450, 7.25], [500, 8], [550, 8.75]
> plot([mass[i],elong[i]]$i=1..11,style=point,symbol=circle);

```



We want to find (aproximate) the proportionality constant  $k$  such that

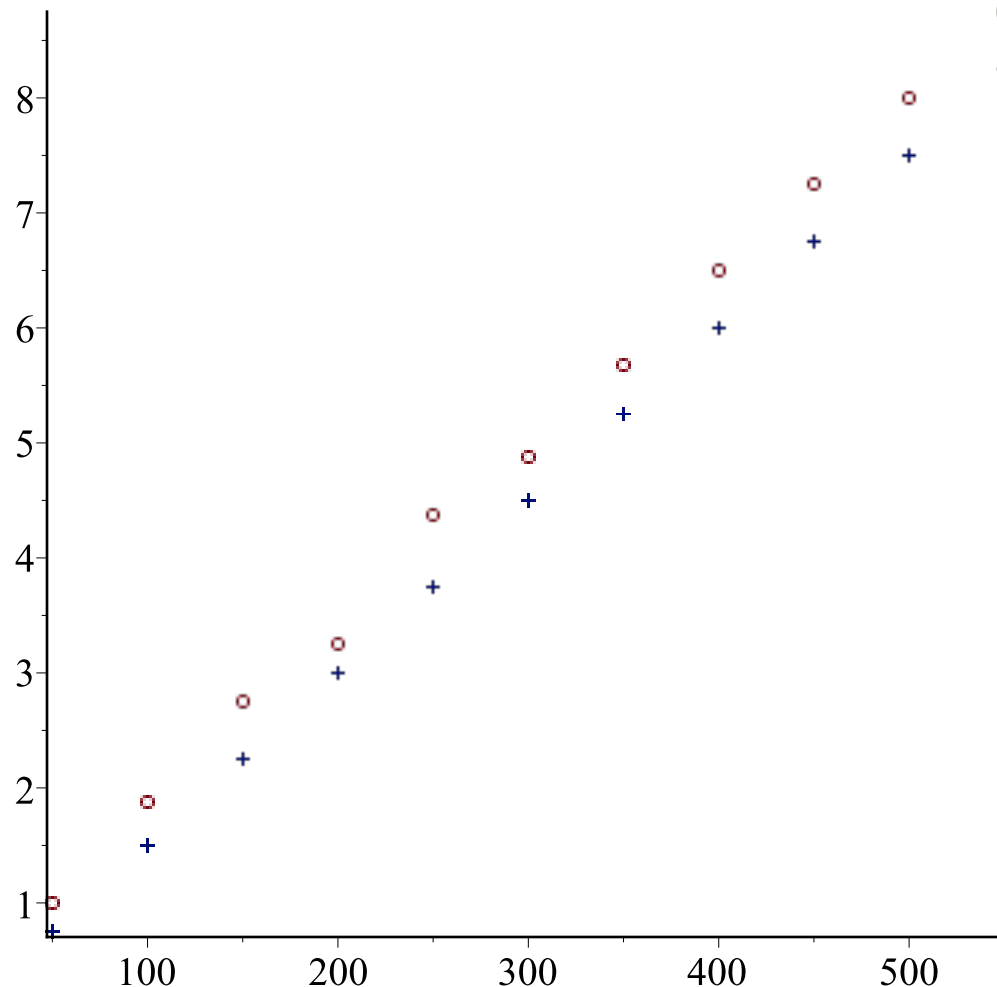
$$elong_i = k \text{ mass}_i$$

To estimate the slope  $k$  we need to use two data  $P_1(m_1, e_1)$  and  $P_2(m_2, e_2)$  and we have

$$\text{slope} = k = \frac{e_2 - e_1}{m_2 - m_1}$$

Lets pick the second point (100,1.875) and sixth (300,4.875)

```
> k:=(elong[6]-elong[2])/(mass[6]-mass[2]);
      k:= 0.01500000000
> real_data:=[mass[i],elong[i]]$i=1..11];
real_data:= [[50, 1], [100, 1.875], [150, 2.75], [200, 3.25], [250, 4.375], [300, 4.875],
             [350, 5.675], [400, 6.5], [450, 7.25], [500, 8], [550, 8.75]]
> est_data:=[mass[i],k*mass[i]]$i=1..11];
est_data:= [[50, 0.7500000000], [100, 1.500000000], [150, 2.250000000], [200,
             3.000000000], [250, 3.750000000], [300, 4.500000000], [350, 5.250000000], [400,
             6.000000000], [450, 6.750000000], [500, 7.500000000], [550, 8.250000000]]
> plot([real_data,est_data],style=point,symbol=[circle,cross]);
```



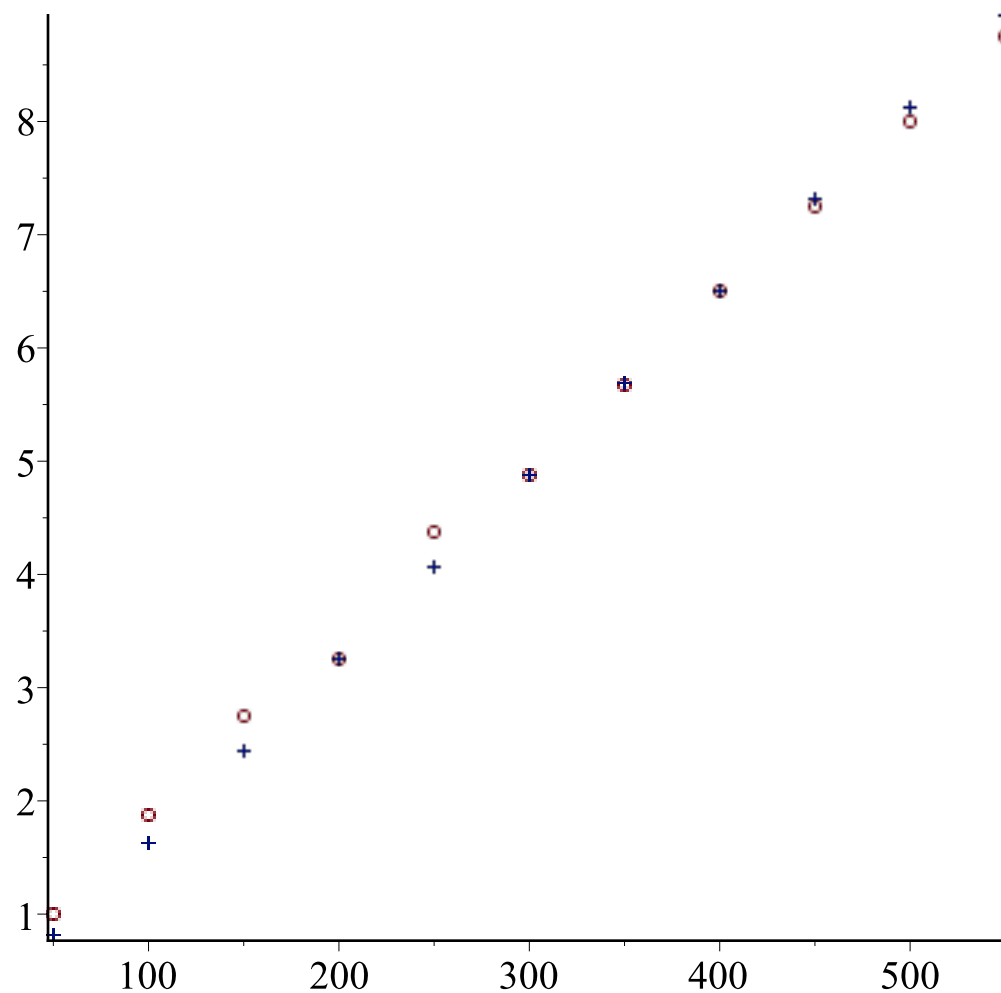
**Exercise:** Try to find a better choice.

```
> k:=(elong[6]-elong[4])/(mass[6]-mass[4]);
```

```

k:= 0.01625000000
> est_data:=[mass[i],k*mass[i]]$i=1..11];
> plot([real_data,est_data],style=point,symbol=[circle,cross]);
est_data:= [[50, 0.812500000], [100, 1.625000000], [150, 2.437500000], [200,
3.250000000], [250, 4.062500000], [300, 4.875000000], [350, 5.687500000], [400,
6.500000000], [450, 7.312500000], [500, 8.125000000], [550, 8.937500000]]

```



## Modeling Change

The law of modeling change is

$$future\_value = present\_value + change$$

Often, we wish to predict the future on what we know now, in the present, and add the change that has been carefully observed. In such cases, we begin by studying the change itself according to the formula:

$$change = future\_value - present\_value$$

By collecting data over a period of time and plotting that data, we often can discern the patterns to model the trend of change. If the behavior is taking over discrete time periods, that leads us to a difference equation. If the behavior is taking place continuously with respect to time then the construct lead us to a differential equation.

**Definition:** For a sequence of numbers  $a_n$  the *nth first difference* is

$$\Delta a_n = a_{n+1} - a_n$$

The first difference represents the rise or fall between two consecutive values of the sequence, that is the vertical change in the graph of the sequence.

### Example 2: A Savings Certificate

Consider the value of a savings certificate initially worth 1000. The interest paid each month is 1% per month. The following sequence of numbers represents the value of the certificate month by month:

1000, 1010, 1020.10, 1030.301, ...

The first differences are

$$\Delta a_0 = a_1 - a_0 = 1010 - 1000 = 10$$

$$\Delta a_1 = a_2 - a_1 = 1020.20 - 1010 = 10.10$$

$$\Delta a_2 = a_3 - a_2 = 1030.30 - 1020.10 = 10.201$$

Note that the first difference represent the *change in the sequence* during one time period. Also, we can see that the law of change is given by the relation

$$\Delta a_n = a_{n+1} - a_n = \frac{1}{100} a_n$$

or

$$a_{n+1} = a_n + \frac{1}{100} a_n$$

this expression is called a *difference equation* and

$$a_{n+1} = 1.01 a_n, \quad a_0 = 1000$$

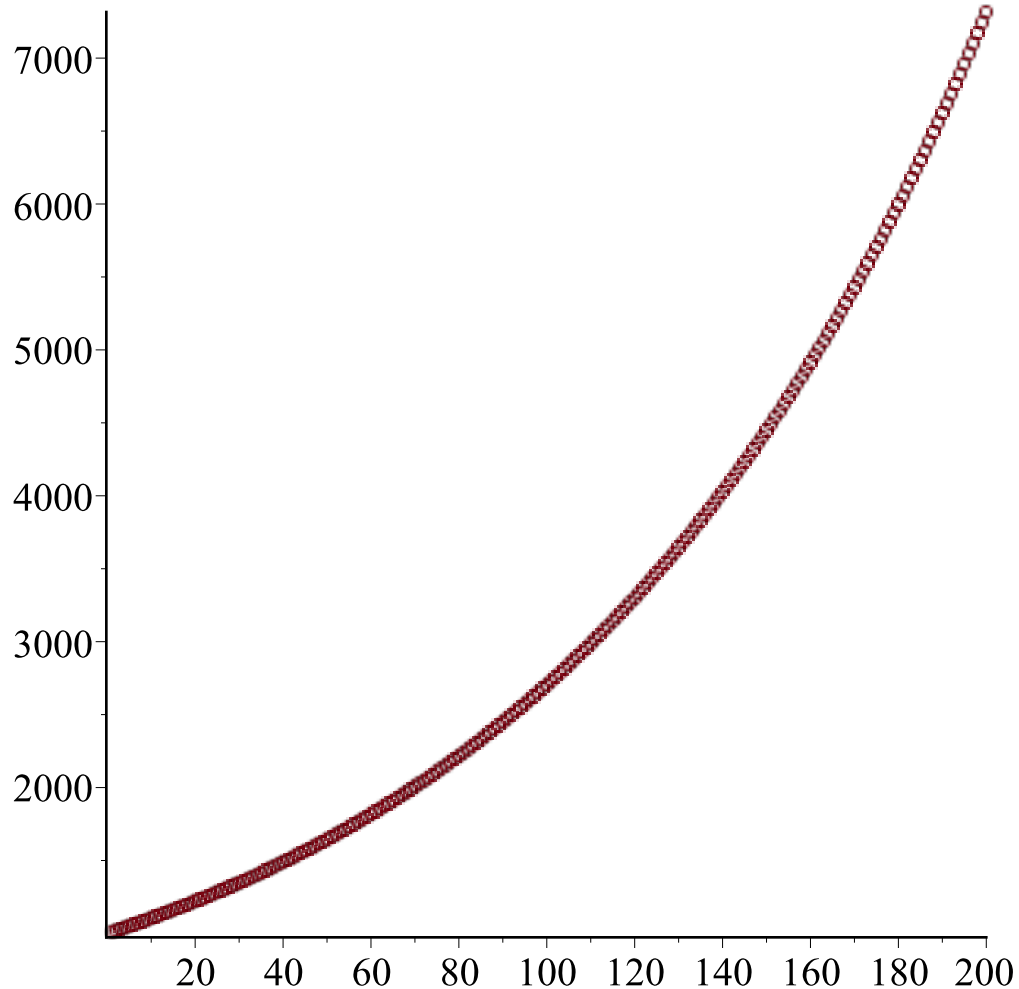
gives us the (*discrete*) *dynamical system* model.

Lets find the certificate value after the 60 month

```
> a[0]:=1000;
                                     a0 := 1000
=
> for i from 0 to 200 do
    a[i+1]:=1.01*a[i]
end do:
> a[60];
                                     1816.696698
=
> seq(a[i], i=0..60);
1000, 1010.00, 1020.1000, 1030.301000, 1040.604010, 1051.010050, 1061.520150,
1072.135352, 1082.856706, 1093.685273, 1104.622126, 1115.668347, 1126.825030,
```

1138.093280, 1149.474213, 1160.968955, 1172.578645, 1184.304431, 1196.147475,  
 1208.108950, 1220.190040, 1232.391940, 1244.715859, 1257.163018, 1269.734648,  
 1282.431994, 1295.256314, 1308.208877, 1321.290966, 1334.503876, 1347.848915,  
 1361.327404, 1374.940678, 1388.690085, 1402.576986, 1416.602756, 1430.768784,  
 1445.076472, 1459.527237, 1474.122509, 1488.863734, 1503.752371, 1518.789895,  
 1533.977794, 1549.317572, 1564.810748, 1580.458855, 1596.263444, 1612.226078,  
 1628.348339, 1644.631822, 1661.078140, 1677.688921, 1694.465810, 1711.410468,  
 1728.524573, 1745.809819, 1763.267917, 1780.900596, 1798.709602, 1816.696698

```
> plot([n,a[n]]$n=1..200,style=point,symbol=circle);
```



In general, discrete mathematical model are given by  
 $change = \Delta a_n = f(\text{terms\_of\_sequence}, \text{external\_terms})$

Modeling the change in this way becomes a problem of determining or approximating function  $f$  that represent the change.

### Example 3: Mortgaging a Home

6 years ago you purchased a home by financing 80000 for 20 years paying monthly payments of 880.87 with a monthly interest of 1%. You made 72 payments and you wish to know the

remaining value of the mortgage.

The change in the amount for each period increases by the amount of interest and decreases by the amount of the payment. So, we have

$$\Delta b_n = b_{n+1} - b_n = \frac{b_n}{100} - 880.87$$

Solving for  $b_{n+1}$  and incorporate the initial condition we get the following dynamical system

$$b_{n+1} = b_n + \frac{b_n}{100} - 880.87, \quad b_0 = 80000$$

```
> b[0]:=80000;
```

$b_0 := 80000$

```
> paym:=880.87;
```

$paym := 880.87$

```
> for i from 0 to 71 do  
    b[i+1]:=1.01*b[i]-paym  
end do;
```

### Approximating Change with Difference Equations

In most examples, describing the change mathematically will not be as precise procedure as in the cases of previous examples. Typically, we must plot the change, observe a pattern and approximate the change to complete the expression

$$change = \Delta a_n = \text{some\_function\_f}$$

#### Example 4: Growth of a Yeast Culture

The following data was collected from an experiment measuring the growth of yeast culture:

Time in hours	Observed yeast biomass
0	9.6
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3

First we plot the change of the biomass with respect to the biomass

```
> p:=[9.6,18.3,29,47.2,71.1,119.1,174.6,257.3];
```

$p := [9.6, 18.3, 29, 47.2, 71.1, 119.1, 174.6, 257.3]$

```
> for i from 1 to 7 do  
    delta_p[i]:=p[i+1]-p[i]  
end do;
```

$$\text{delta\_}p_1 := 8.7$$

$$\text{delta\_}p_2 := 10.7$$

$$\text{delta\_}p_3 := 18.2$$

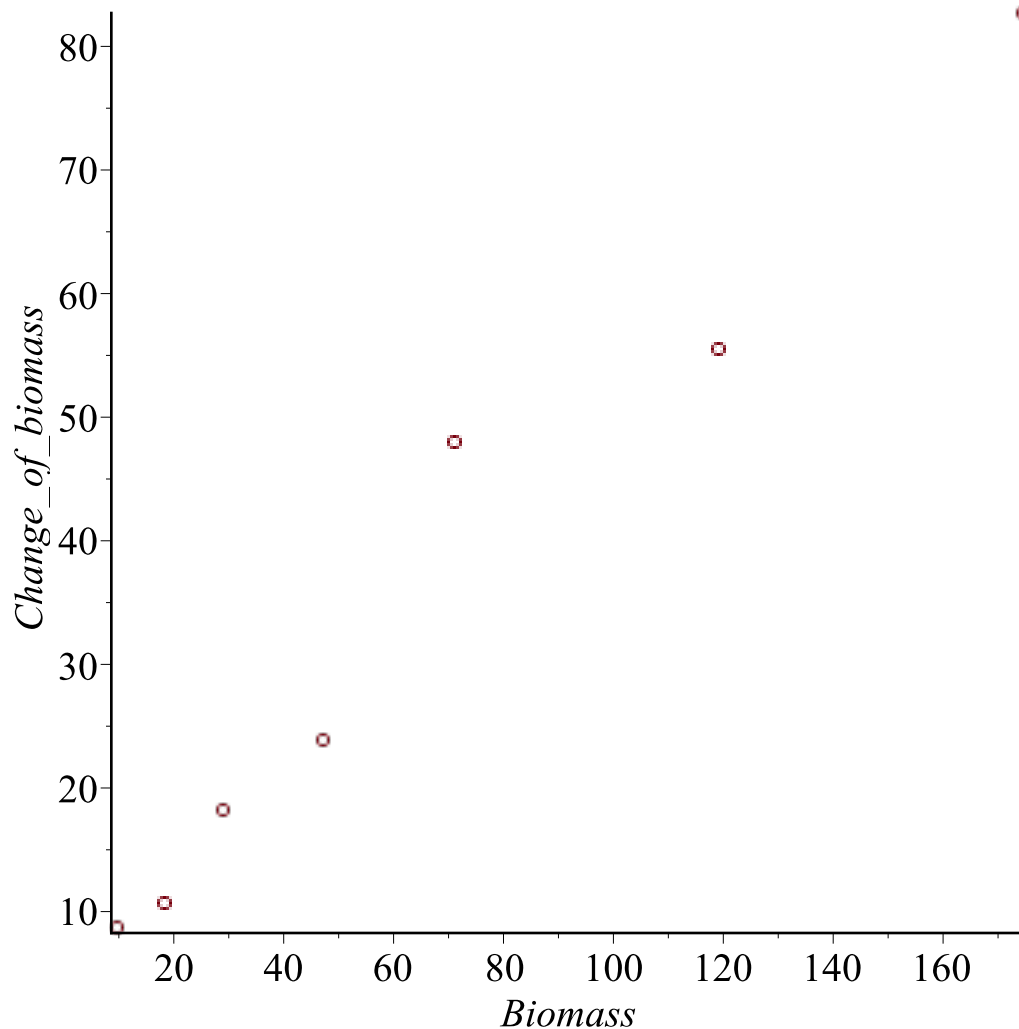
$$\text{delta\_}p_4 := 23.9$$

$$\text{delta\_}p_5 := 48.0$$

$$\text{delta\_}p_6 := 55.5$$

$$\text{delta\_}p_7 := 82.7$$

```
> plot([p[n],delta_p[n]]$n=1..7,style=point,symbol=circle,
labels=[Biomass, Change_of_biomass],labeldirections=
[HORIZONTAL, VERTICAL]);
```



Analyzing the graph it seems to have some proportionality between biomass and the change of biomass, so

$$\Delta p_n = k p_n$$

Now we have to find the proportionality constant. We apply the same method as in Example 1:

Testing the proportionality. We use two data to find an approximating value for k (we use 4th and

6th point of the graph).

$$k = \frac{\Delta p_6 - \Delta p_4}{p_6 - p_4}$$

```
[> k:=(delta_p[6]-delta_p[4])/(p[6]-p[4]);  
      k:=0.4394993046
```

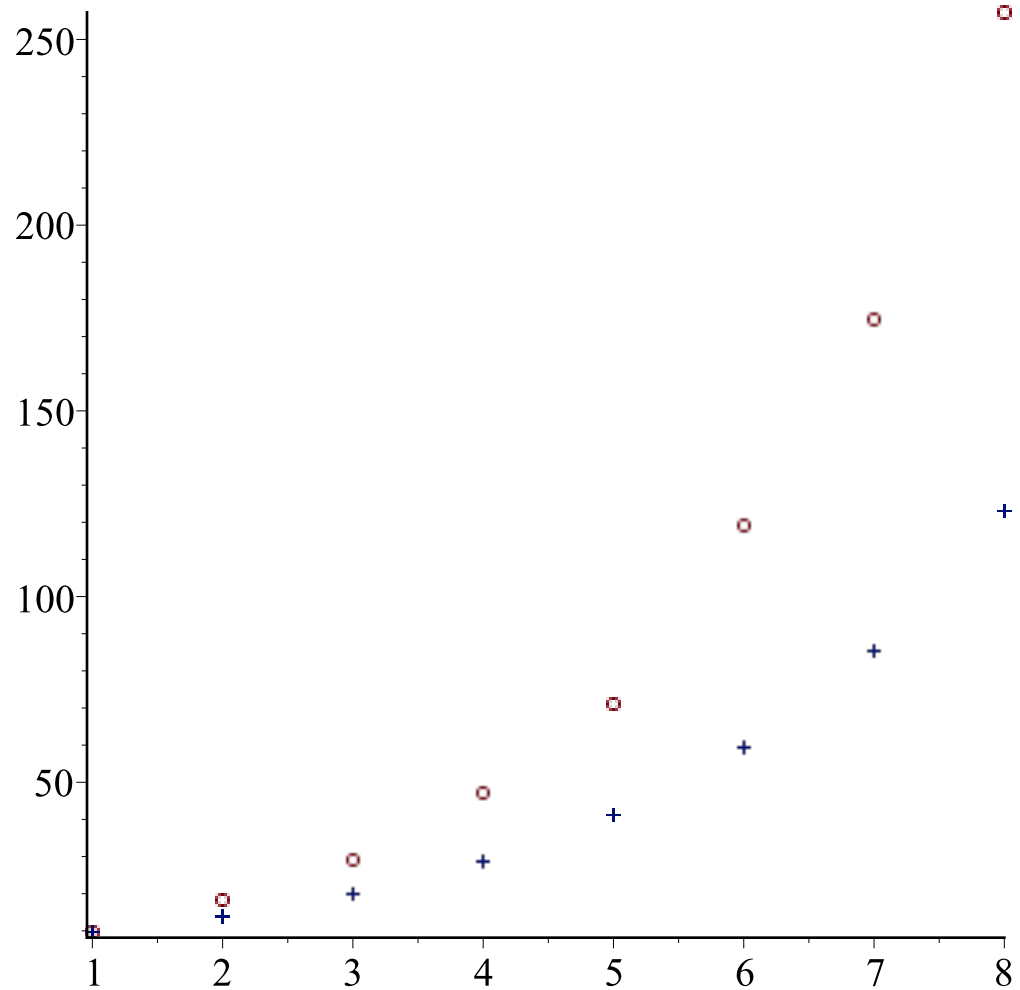
Thus we get the following growth model

$$p_{n+1} = (1 + k) p_n, \quad p_0 = 9.6$$

```
[> p_est[1]:=p[1];  
      p_est_1:=9.6  
=> for i from 1 to 7 do  
    p_est[i+1]:=(1+k)*p_est[i]  
  end do;  
      p_est_2:=13.81919333  
      p_est_3:=19.89271919  
      p_est_4:=28.63555545  
      p_est_5:=41.22086217  
      p_est_6:=59.33740245  
      p_est_7:=85.41614959  
      p_est_8:=122.9564880
```

Lets compare graphically the real date with the estimated data

```
[> real_data:=[[n,p[n]]$n=1..8]:  
=> est_data:=[[n,p_est[n]]$n=1..8]:  
=> plot([real_data,est_data],style=point,symbol=[circle,cross]);
```



Analyzing the graph we notice a considerable difference between estimated data and real data, which implies that our assumption, the change of biomass is proportional with the biomass, is not very good. In order to improve the model we need to increase the number of observations in the experiment. Let consider the following data:

Time in hours	Observed yeast biomass
0	9.6
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441
10	513.3
11	559.7

12	594.8
13	629.4
14	640.8
15	651.1
16	655.9
17	659.6
18	661.8

```

> p:=[9.6,18.3,29,47.2,71.1,119.1,174.6,257.3,350.7,441,513.3,
      559.7,594.8,629.4,640.8,651.1,655.9,659.6,661.8];
p := [9.6, 18.3, 29, 47.2, 71.1, 119.1, 174.6, 257.3, 350.7, 441, 513.3, 559.7, 594.8, 629.4,
      640.8, 651.1, 655.9, 659.6, 661.8]
> for i from 1 to 18 do
      delta_p[i]:=p[i+1]-p[i]
    end do;
      delta_p1 := 8.7
      delta_p2 := 10.7
      delta_p3 := 18.2
      delta_p4 := 23.9
      delta_p5 := 48.0
      delta_p6 := 55.5
      delta_p7 := 82.7
      delta_p8 := 93.4
      delta_p9 := 90.3
      delta_p10 := 72.3
      delta_p11 := 46.4
      delta_p12 := 35.1
      delta_p13 := 34.6
      delta_p14 := 11.4
      delta_p15 := 10.3
      delta_p16 := 4.8
      delta_p17 := 3.7
      delta_p18 := 2.2

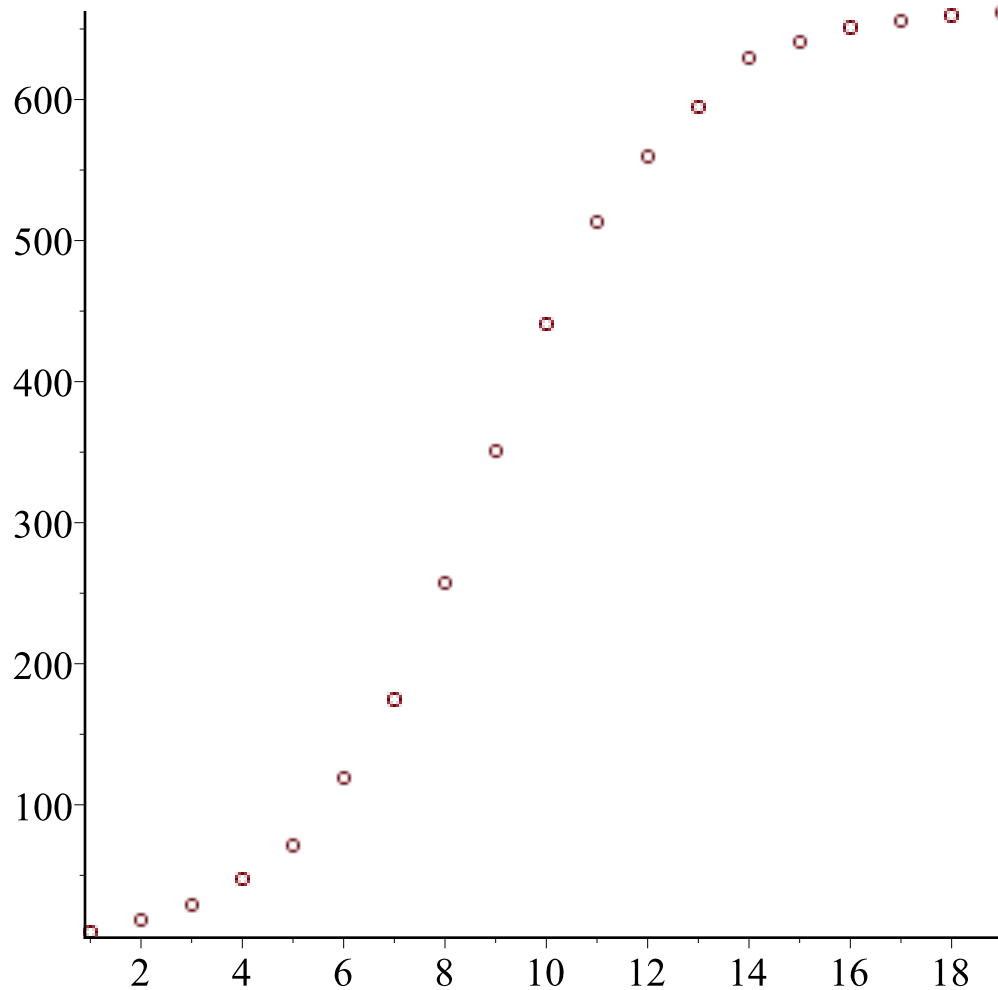
```

From the first difference sequence we notice that the change decreases as the biomass increases, this happens because of the restricted area in which the yeast culture grows. Let's plot the real data

```

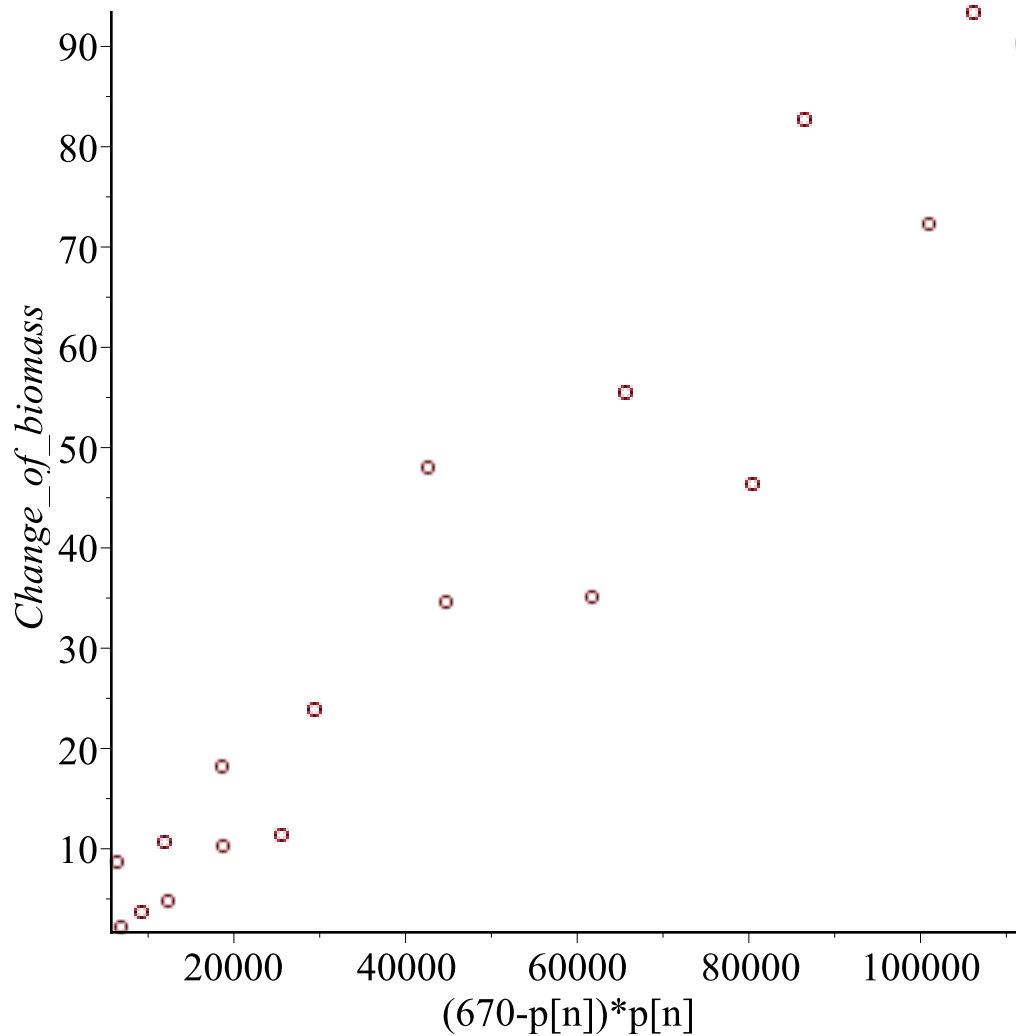
real_data := [[n,p[n]]$n=1..19]:
> plot(real_data, style=point, symbol=circle);

```



From the graph of population versus time, the population appears to be approaching to a limiting value (carrying capacity). Based on our graph we estimate that the carrying capacity is 670. As  $p_n$  approaches to 670 the change slows considerable, this shows us that the change depends on  $670 - p_n$  and, also, on the biomass  $p_n$ . Lets plot the change of biomass versus  $(670 - p_n) p_n$

```
> plot([(670-p[n])*p[n],delta_p[n]]$n=1..18),style=point,
      symbol=circle,labels=["(670-p[n])*p[n]", Change_of_biomass],
      labeldirections=[HORIZONTAL, VERTICAL]);
```



It seems to have some proportionality here, so

$$\Delta p_n = k (670 - p_n) p_n$$

Now, we find the value of  $k$  using two data (11th and 13th point of the graph)

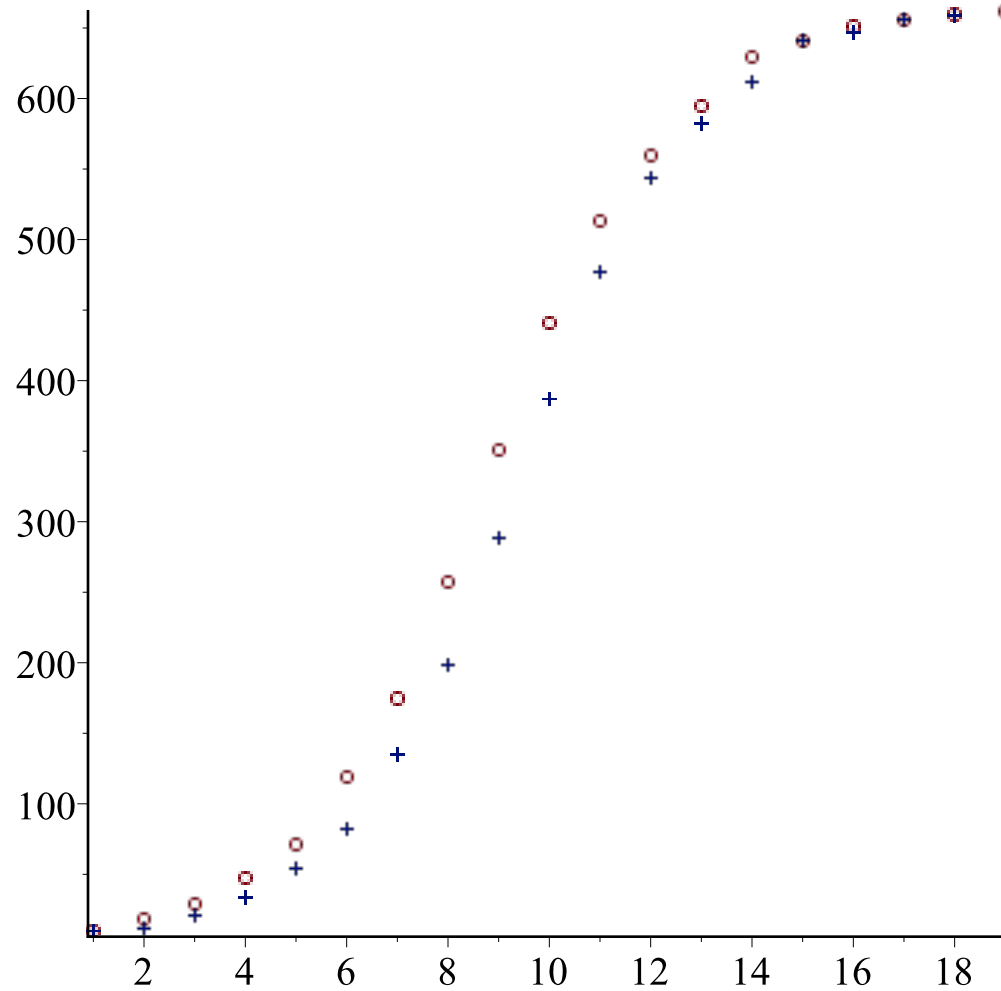
```
> k:=(delta_p[13]-delta_p[11])/((670-p[13])*p[13]-(670-p[11])*p[11]);
      k := 0.0003304845379
```

Now we have the mathematical model

$$p_{n+1} = p_n + k (670 - p_n) p_n, p_0 = 9.6$$

```
> p_est[1]:=p[1];
      p_est_1 := 9.6
> for i from 1 to 19 do
    p_est[i+1]:=p[i]+k*(670-p_est[i])*p_est[i]
end do:
> est_data:=[n,p_est[n]]$n=1..18):
```

```
> plot([real_data,est_data],style=point,symbol=[circle,cross]);
```



In this case, we get a better estimated data compared with the real data.