Solving differential equations with MAPLE

Functions and graphic representation

A single variable function can be defined as follows:

```
> f:=x->sin(x)/x;
```

$$f := x \rightarrow \frac{\sin(x)}{x}$$

```
> f(3*Pi/2),f(1.5);
```

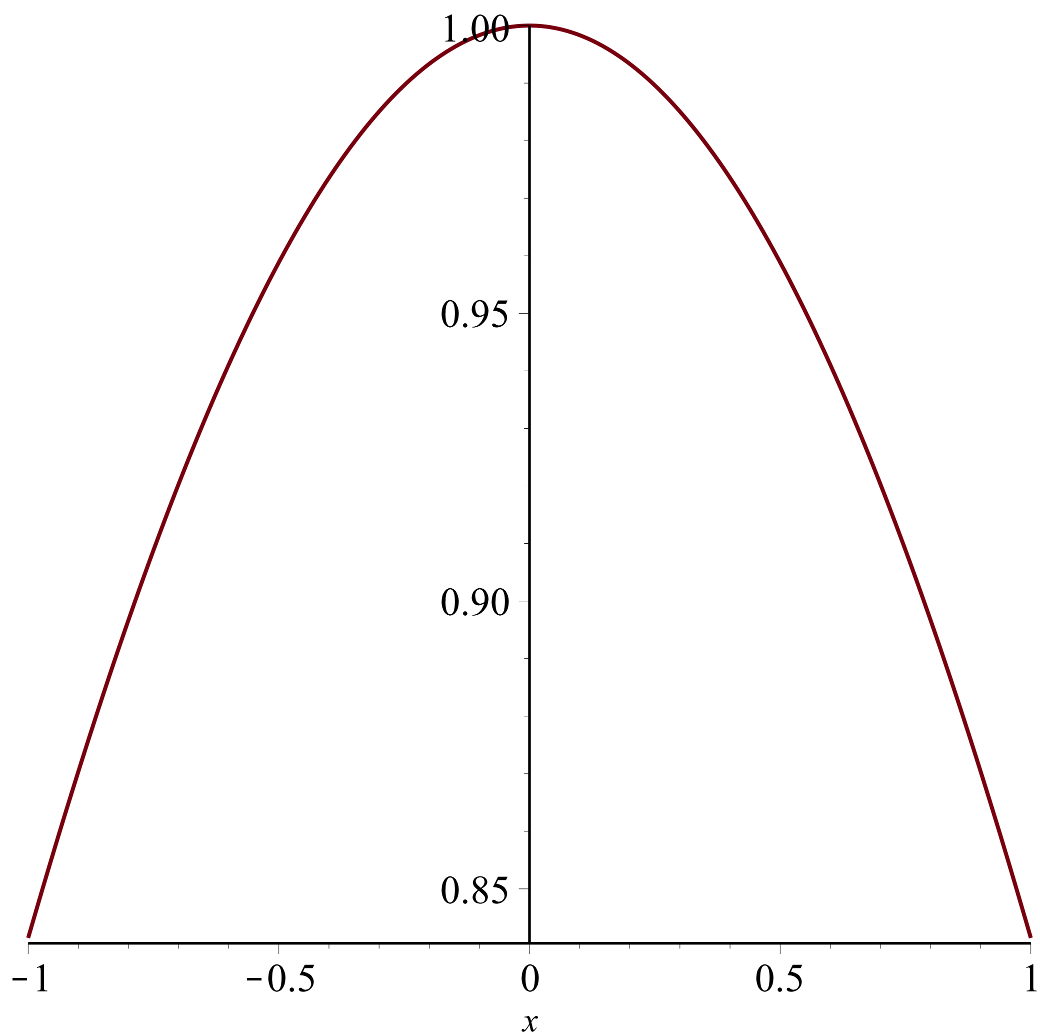
$$-\frac{2}{3\,\pi},\ 0.6649966577$$

```
> f(a+b);
```

$$\frac{\sin(a+b)}{a+b}$$

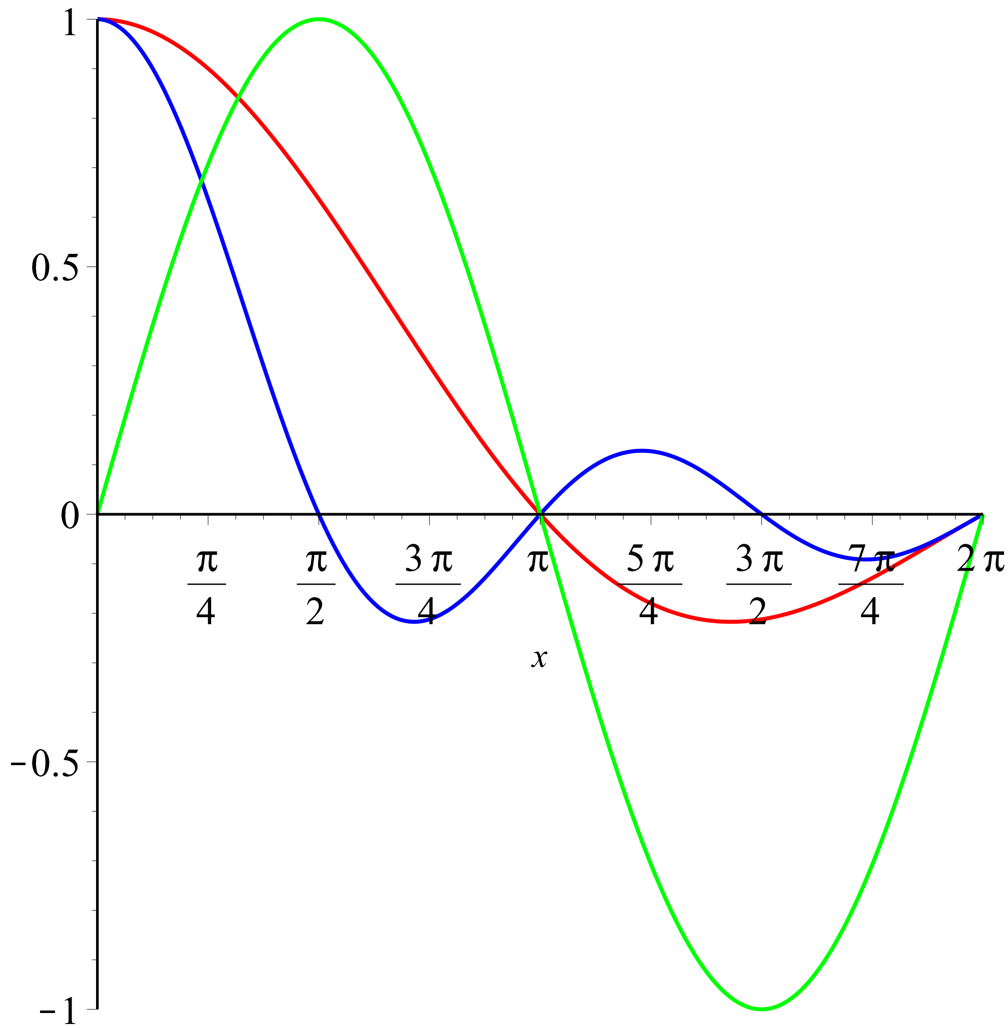
For graphical representation we need to load **plots** package using **with** command

```
> with(plots):
> plot(f(x),x=-1..1);
```



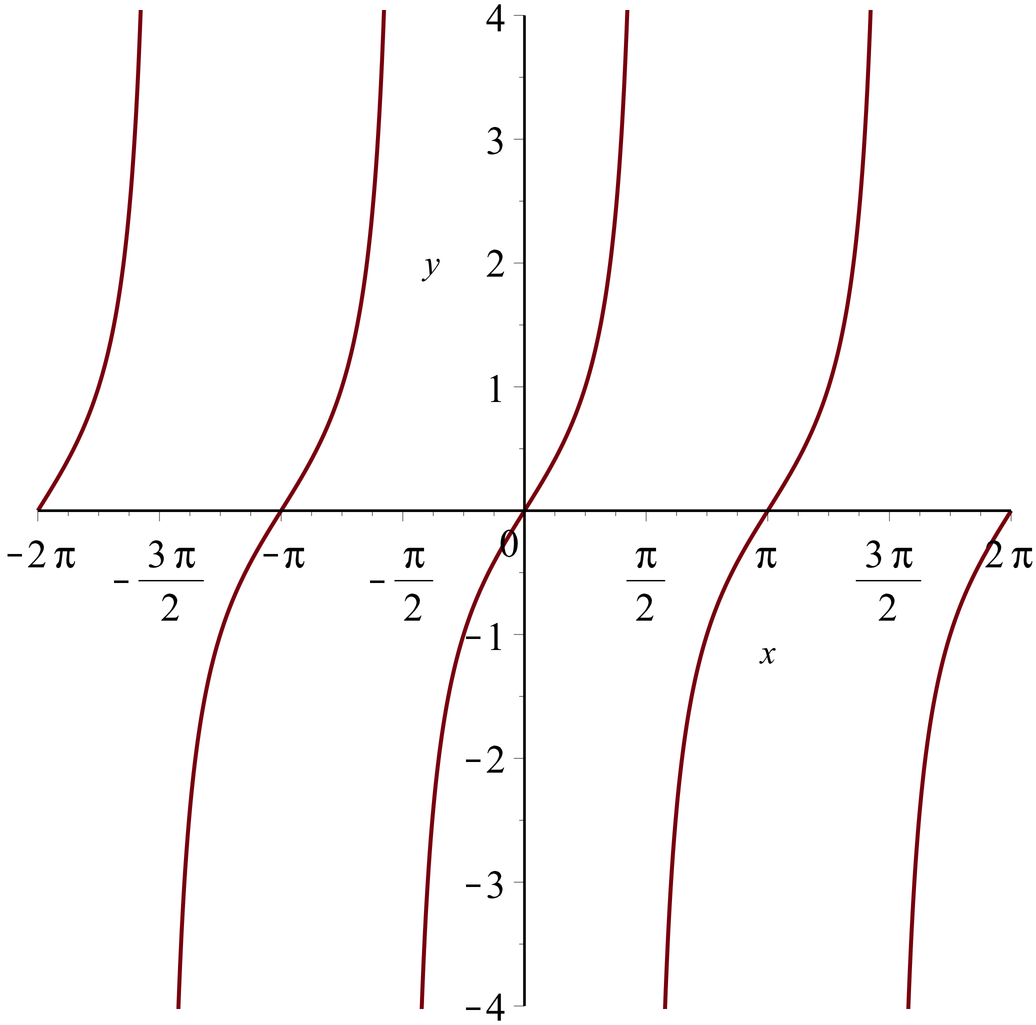
We can represent more than one function in the same window:

```
> plot([f(x),f(2*x),sin(x)],x=0..2*Pi,color=[red,blue,green]);
```



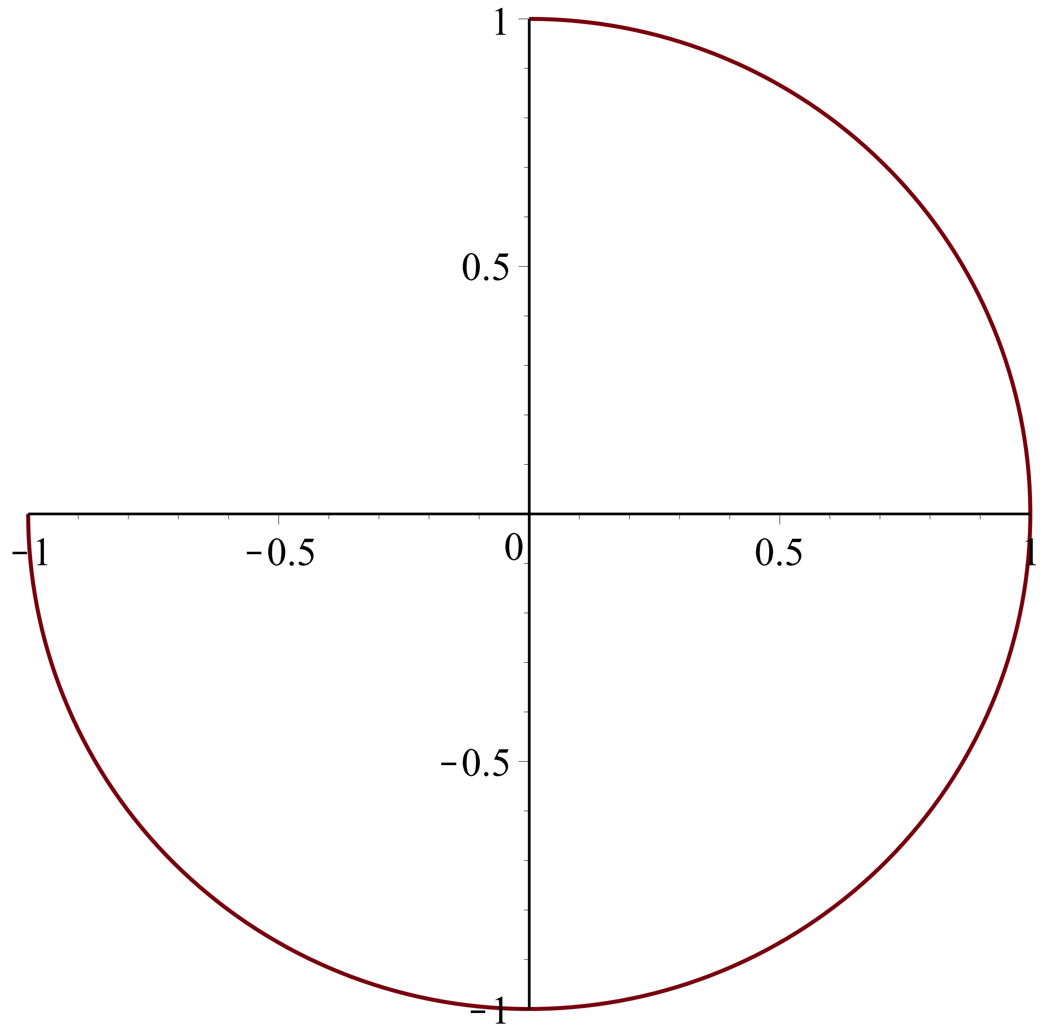
In the case of discontinuous points we need to use the option **discont** = **true**:

```
> plot(tan(x), x = -2*Pi..2*Pi, y = -4..4, discont = true);
```



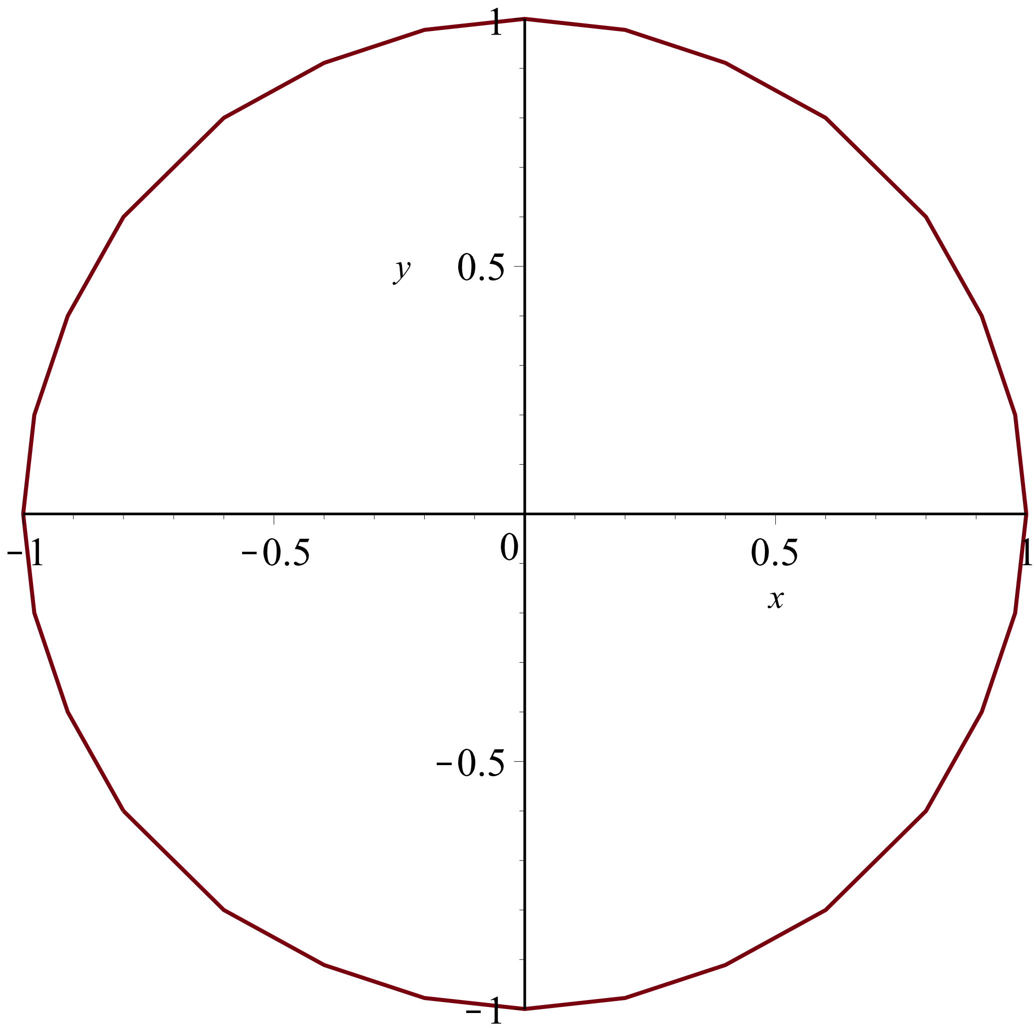
If a curve is given in a parametric form (for example: $x(t) = \sin(t)$, $y(t) = \cos(t)$, $t = 0..\pi$) we use the instruction:

```
> plot([sin(t),cos(t),t=0..3/2*Pi]);
```



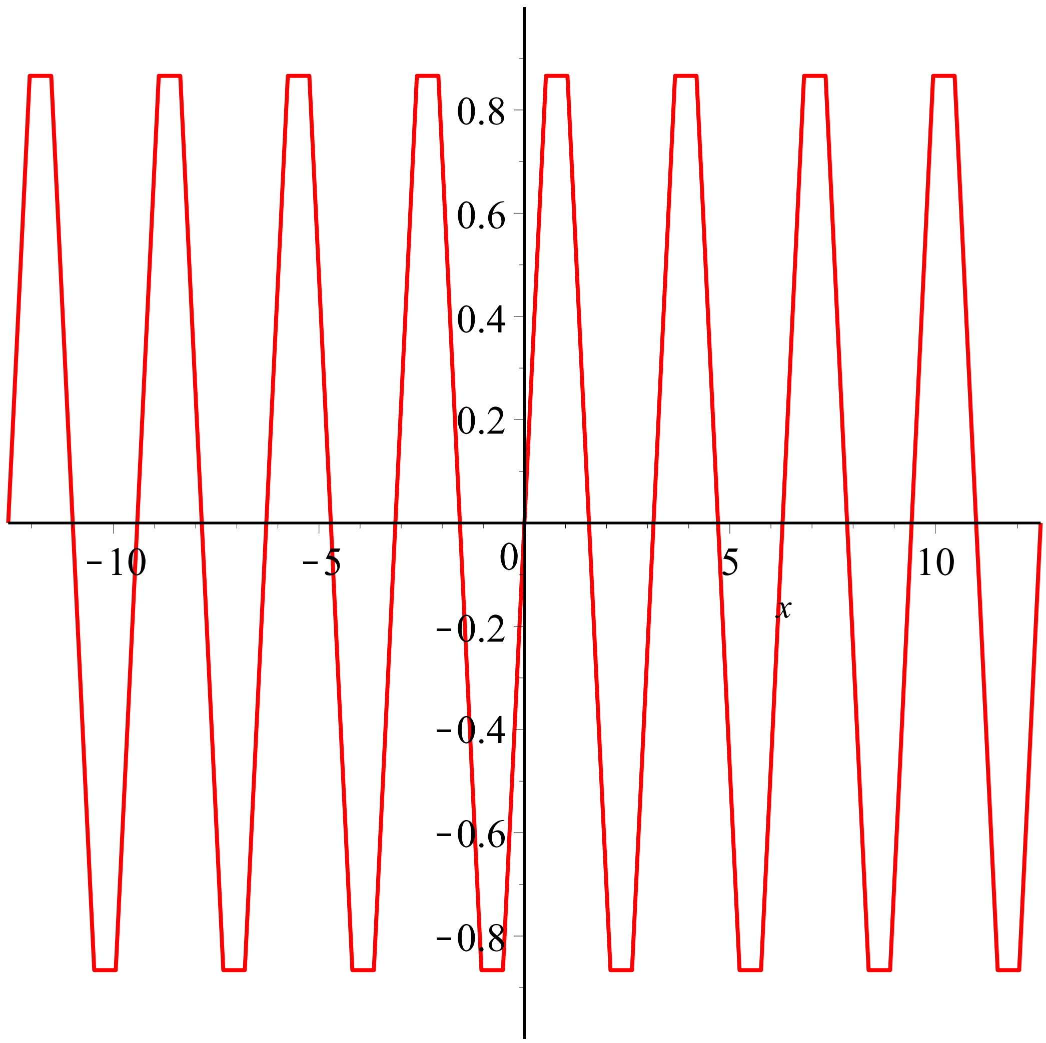
In the case of a curve given by the implicit equation we use the instruction **implicitplot**:

```
> implicitplot(x^2+y^2=1,x=-10..10,y=-10..10,numpoints=10000);
```



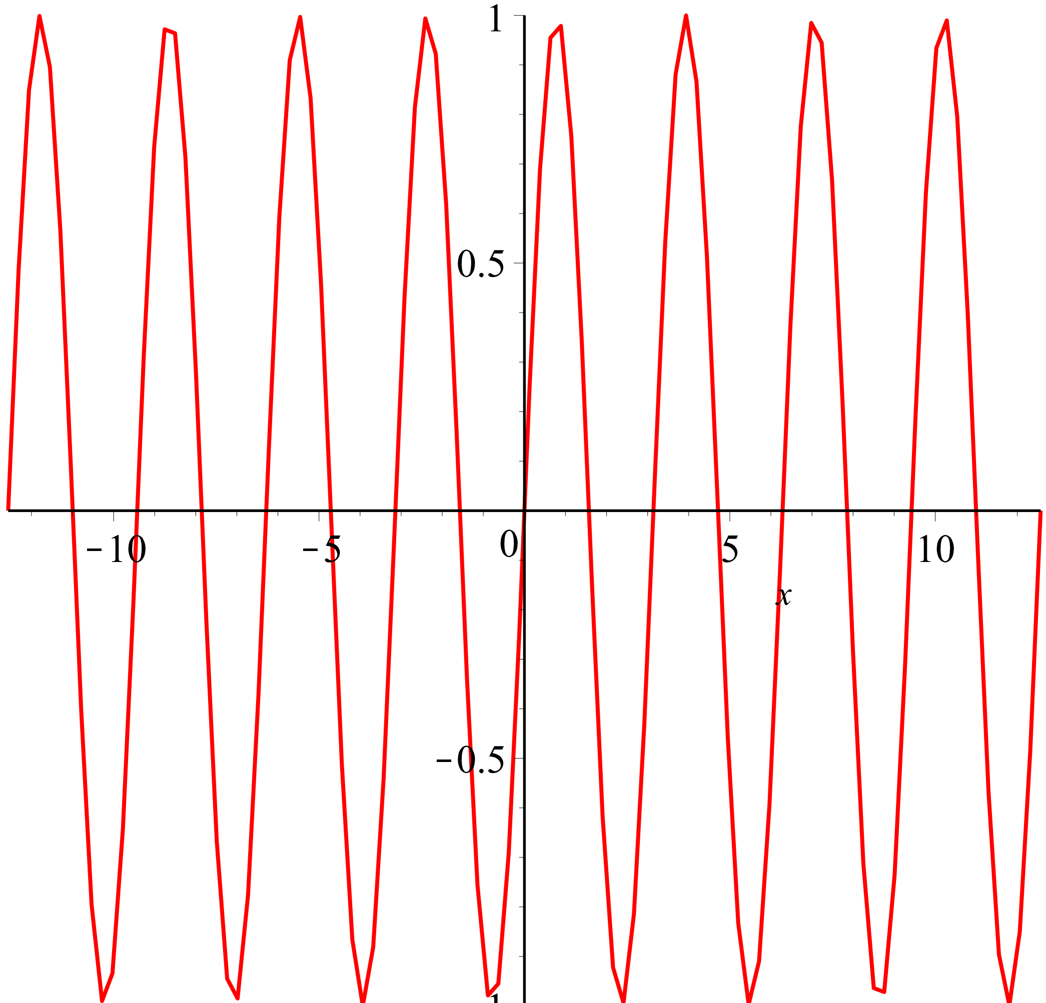
In the case we need to visualize the parameter dependence of a function we can use the command **animate** (right click on the image, select *Animation* and *Play*)

```
> animate(sin(x*t),x=-4*Pi..4*Pi,t=0..2,color=red);
```



If we need more precision we can increase the number of point and frames:

```
> animate(sin(x*t),x=-4*Pi..4*Pi,t=0..2,color=red,numpoints=
  100,frames=100);
```



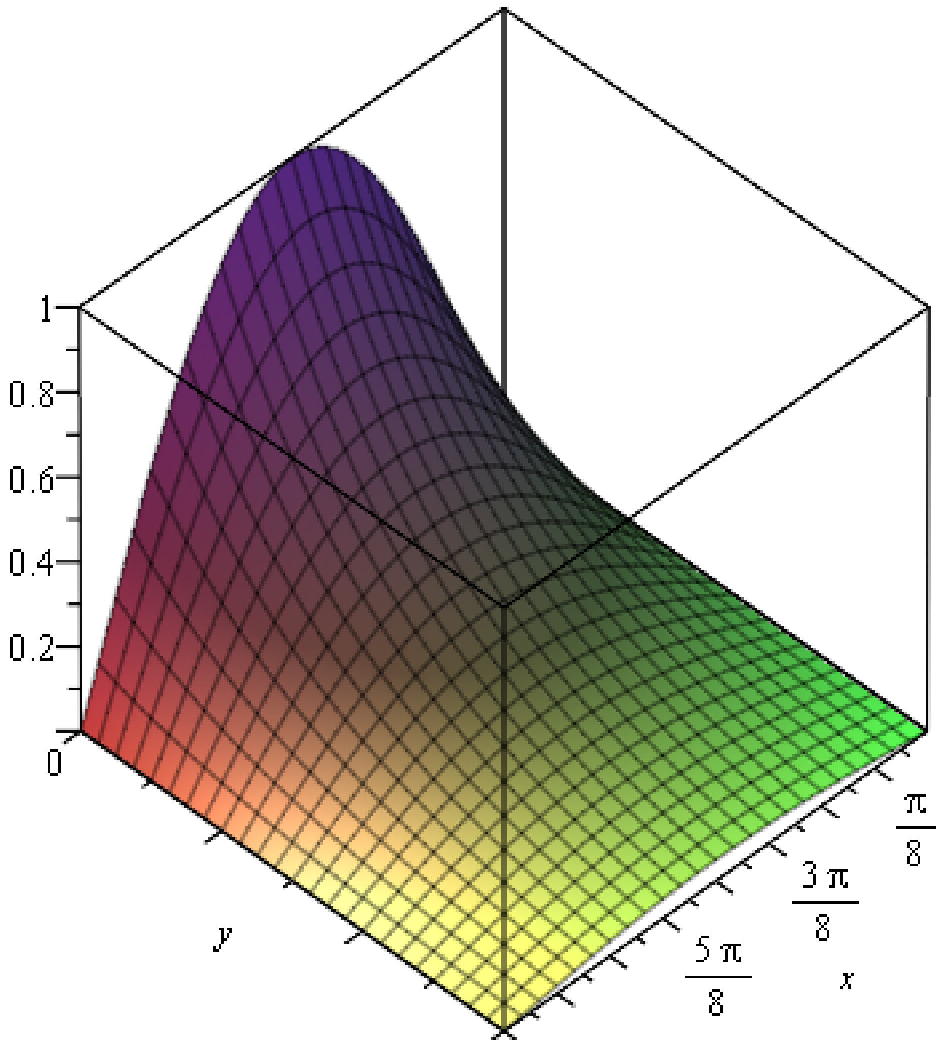
A function with more than one variable can be defined as follows:

```
> g:=(x,y)->sin(x)*exp(-y);
```

$$g := (x, y) \rightarrow \sin(x)\, e^{-y}$$

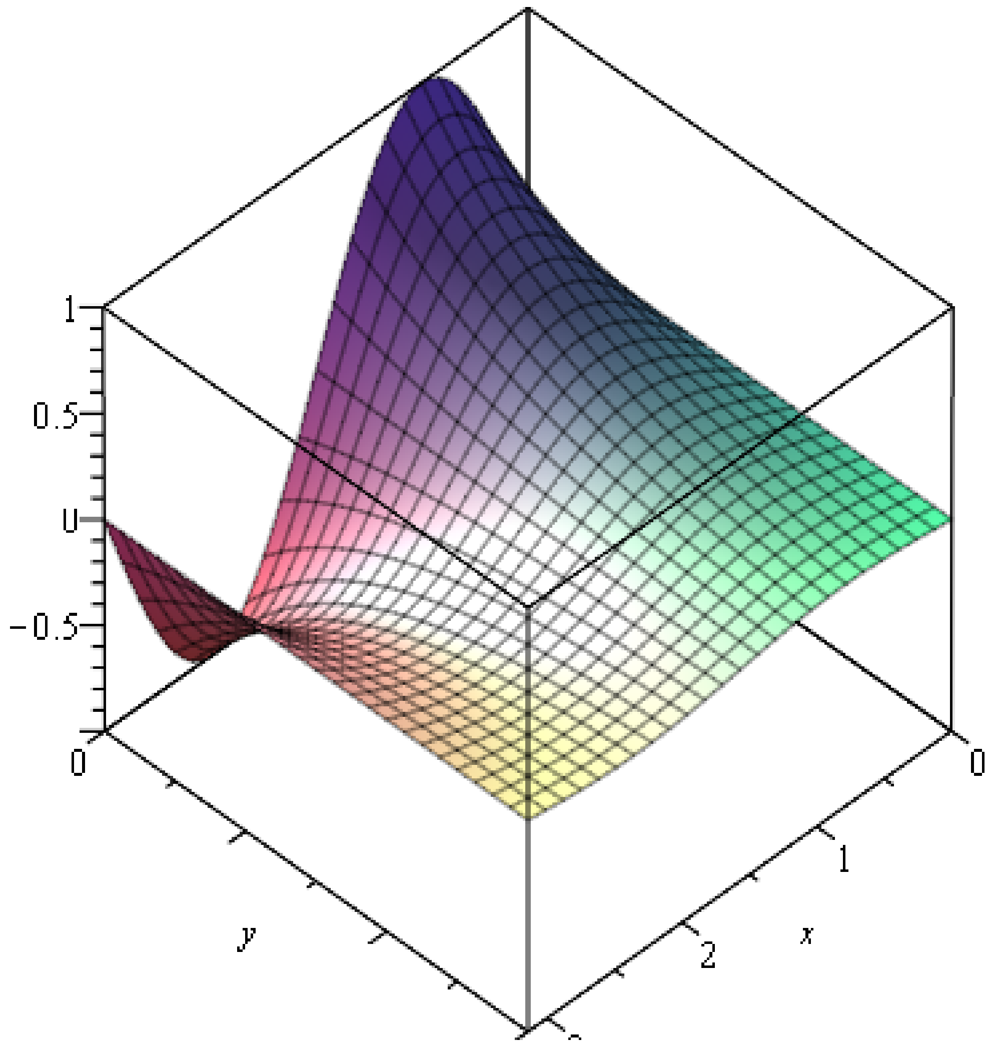
For 3d graphical representation we use the command **plot3d**:

```
> plot3d(g(x,y),x=0..Pi,y=0..3,axes=boxed);
```



The animation of 3D graphs can be made using the instruction **animate3d**:

```
> animate3d(g(t*x,y),x=0..Pi,y=0..3,t=0..2);
```



The derivation of the functions

The derivation of the functions can be made in two ways: using **`diff`** command or using the derivation operator **D**:

```
> f:=x->exp(x^2)+3;
```

$$f := x \rightarrow e^{x^2} + 3$$

The **diff** command execute the derivation of the given expresion with respect to the specified variable. The derivation operator **D** returns the derivate as a function.

```
> diff(f(x),x);
```

$$2\,x\,e^{x^2}$$

```
> diff(f(x),x)
```

$$2\,x\,e^{x^2}$$

```
> eval(diff(f(x),x),x=1)
```

$$2\,e$$

the second order derivate is given by

```
> diff(f(x),x,x);
```

$$2\,e^{x^2} + 4\,x^2\,e^{x^2}$$

also we can use the option **t$n** to get n-order derivative

```
> diff(f(x),x$2);
```

$$2\,e^{x^2}+4\,x^2\,e^{x^2}$$

```
> diff(f(x),x$3);
```

$$12\,x\,e^{x^2}+8\,x^3\,e^{x^2}$$

```
> diff(x^2+y^2,y);
```

$$2\,y$$

Using the derivation operator:

```
> D(f)(x);
```

$$2\,x\,e^{x^2}$$

```
> D(f)(1);
```

$$2\,e$$

```
> (D@D)(f)(x);
```

$$2\,e^{x^2}+4\,x^2\,e^{x^2}$$

```
> (D@D)(f)(1);
```

$$6\,e$$

```
> (D@@2)(f)(x);
```

$$2\,e^{x^2}+4\,x^2\,e^{x^2}$$

```
> (D@D@D)(f)(x);
```

$$12\,x\,e^{x^2}+8\,x^3\,e^{x^2}$$

```
> (D@@10)(f)(x);
```

$$30240\,e^{x^2}+302400\,x^2\,e^{x^2}+403200\,x^4\,e^{x^2}+161280\,x^6\,e^{x^2}+23040\,x^8\,e^{x^2}+1024\,x^{10}\,e^{x^2}$$

Initialization of the solving ODE package

```
> restart:
```
clears the memory of all previously saved values and variables

```
> with(DEtools):
```
load the differential equations package

```
> with(plots):
```
load the graphical package

Define and solve a first order differential equation

Let consider the differential equation $\frac{d}{dx}\,y(x)=k\,y(x)$ where k is a real coeficient. The differential equation can be introduse in MAPLE as follows:

```
> diff_eq1:=diff(y(x),x) = k*y(x);
```

$$diff\_eq1 := \frac{d}{dx}\,y(x)=k\,y(x)$$

To obtain the general solution of the equation use **dsolve** command

```
> dsolve(diff_eq1,y(x));
```

$$y(x) = \_C1\ \mathrm{e}^{kx}$$

The general solution is seen as an expresion. Notice that the undetermined constant is called *_C1*
How can we manipulate this expression?
We can use the function definition command:

```
> sol:=(x,k,c)->c*exp(k*x);
```

$$sol := (x, k, c) \rightarrow c\ \mathrm{e}^{kx}$$

If the expresion of the solution is too complicated we can use the command **rhs** (*right hand side*) and **unapply** in order to obtain the solution as a function

```
> right_hand_expr:=rhs(dsolve(diff_eq1,y(x)));
```

$$right\_hand\_expr := \_C1\ \mathrm{e}^{kx}$$

Using the **unapply** command we transform the expresion sol1 into a function specifying the variables:

```
> sol1:=unapply(right_hand_expr,x,k,_C1);
```

$$sol1 := (x, k, \_C1) \rightarrow \_C1\ \mathrm{e}^{kx}$$

and we get the same result.

The graphics of ODE solutions

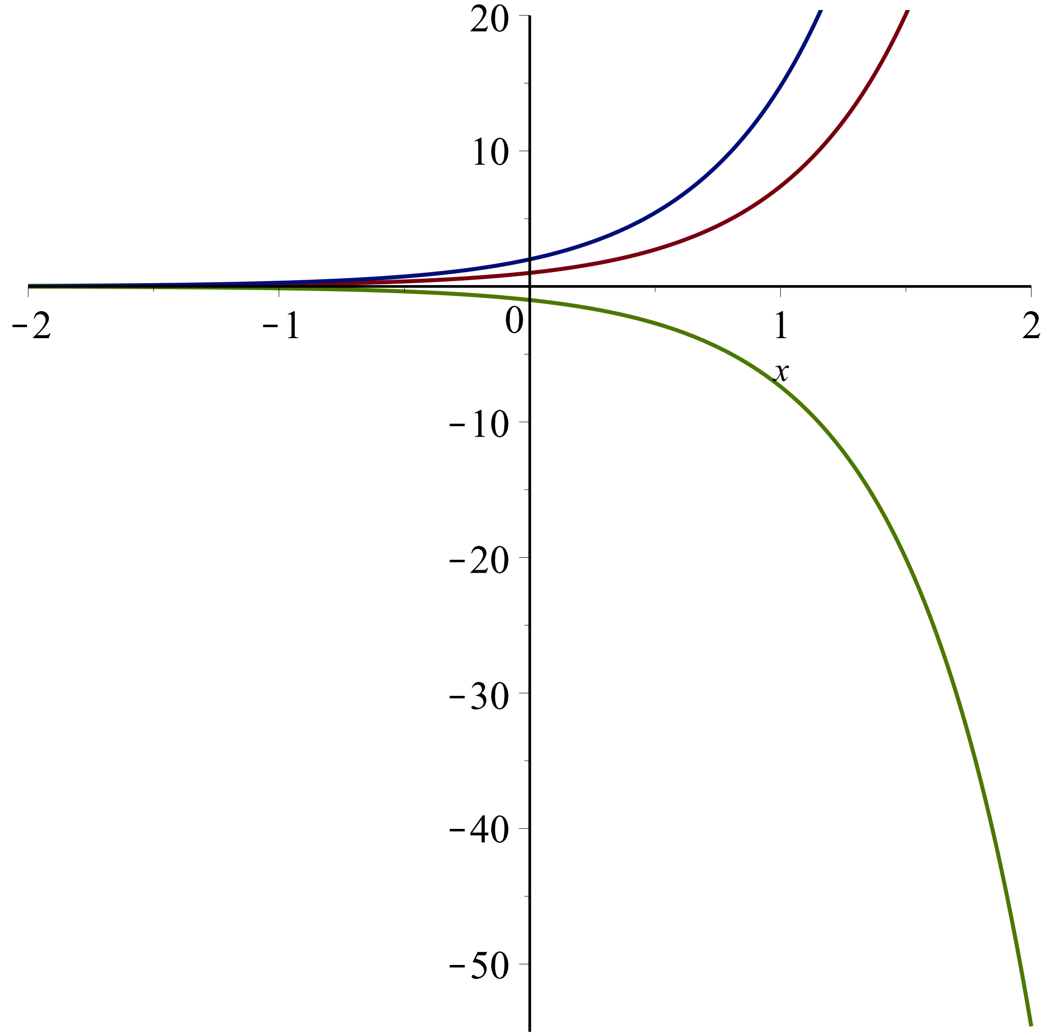
Let suppose that $k := 2$. Then the corresponding general solution is:

```
> y:=(x,c)->sol(x,2,c);
```

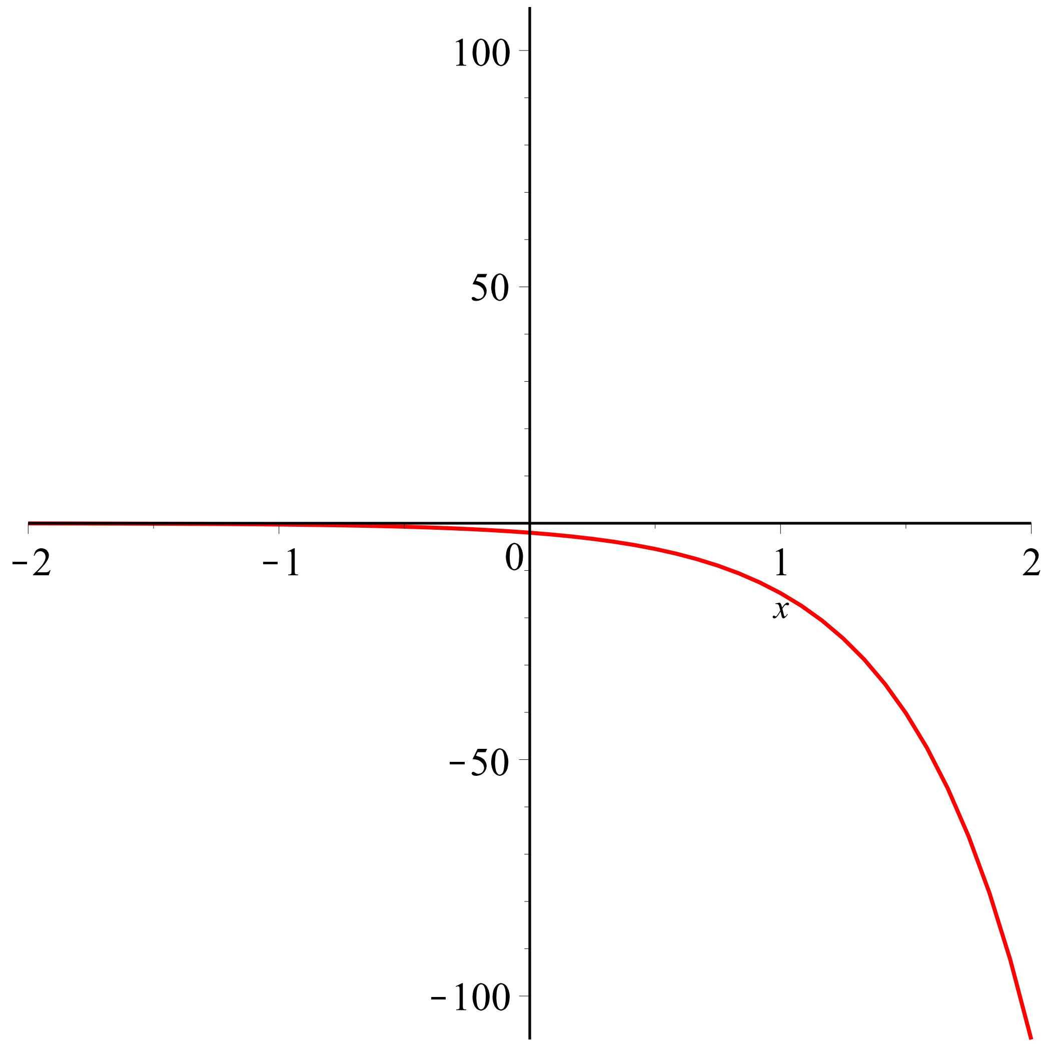
$$y := (x, c) \rightarrow sol(x, 2, c)$$

To draw the solutions curves you just assign some values for the constant c. For example take c:=1 c:=2 and c:=-1

```
> plot([y(x,1),y(x,2),y(x,-1)],x=-2..2);
```

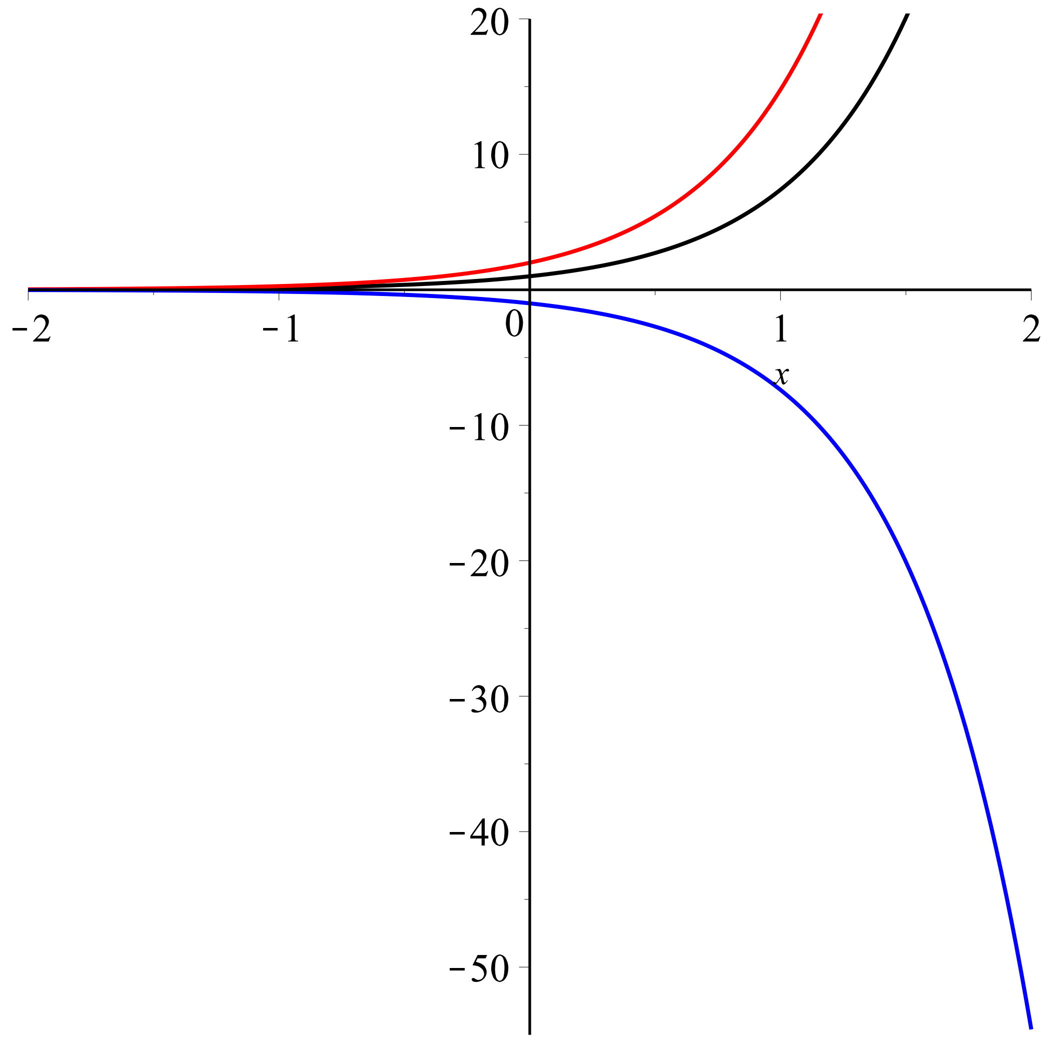



```
> animate( y(x,c),x=-2..2,c=-2..2,frames=50);
```



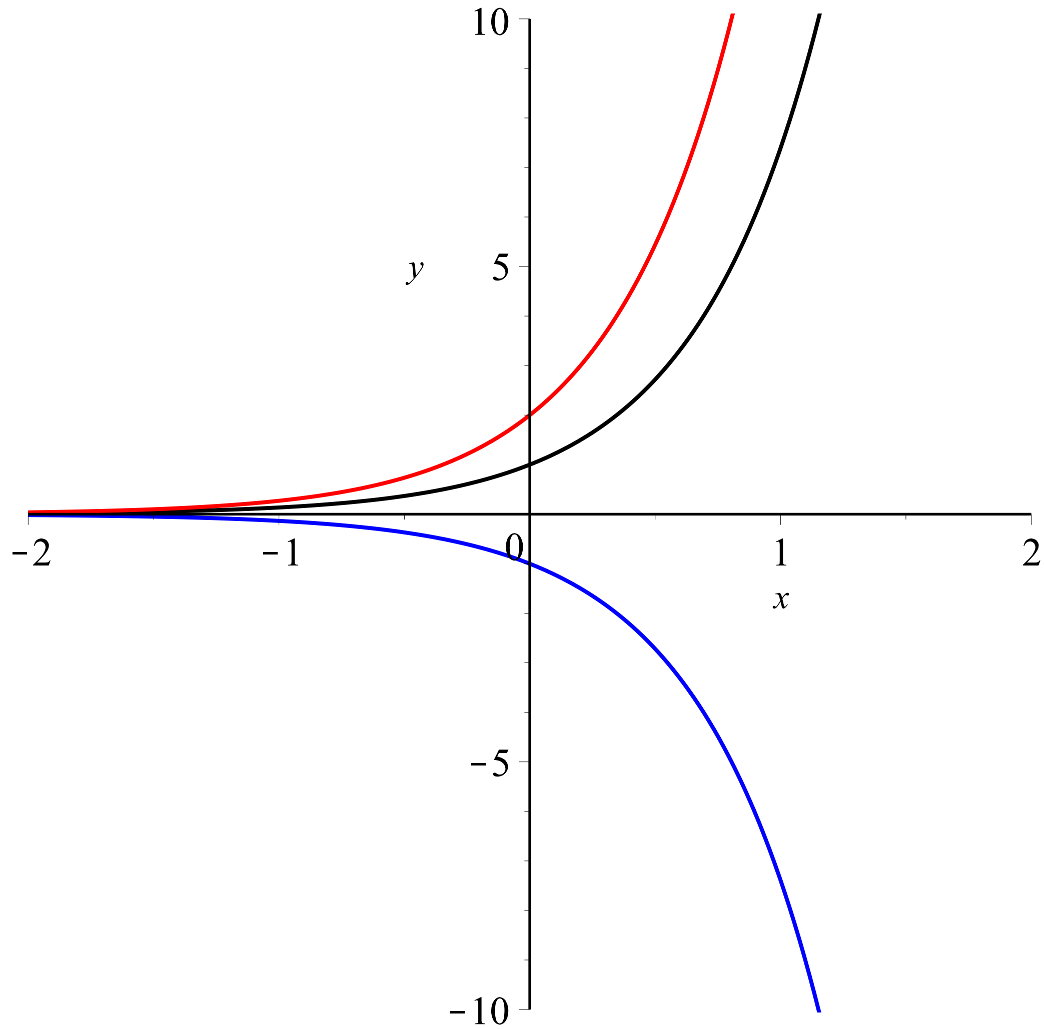
If you want to obtain the solutions with some specified colors use the command:

```
> plot([y(x,1),y(x,2),y(x,-1)],x=-2..2,color=[black,red,blue]);
```



Also you specify the window of the graphic:

```
> plot([y(x,1),y(x,2),y(x,-1)],x=-2..2,y=-10..10,color=[black,
  red,blue]);
```

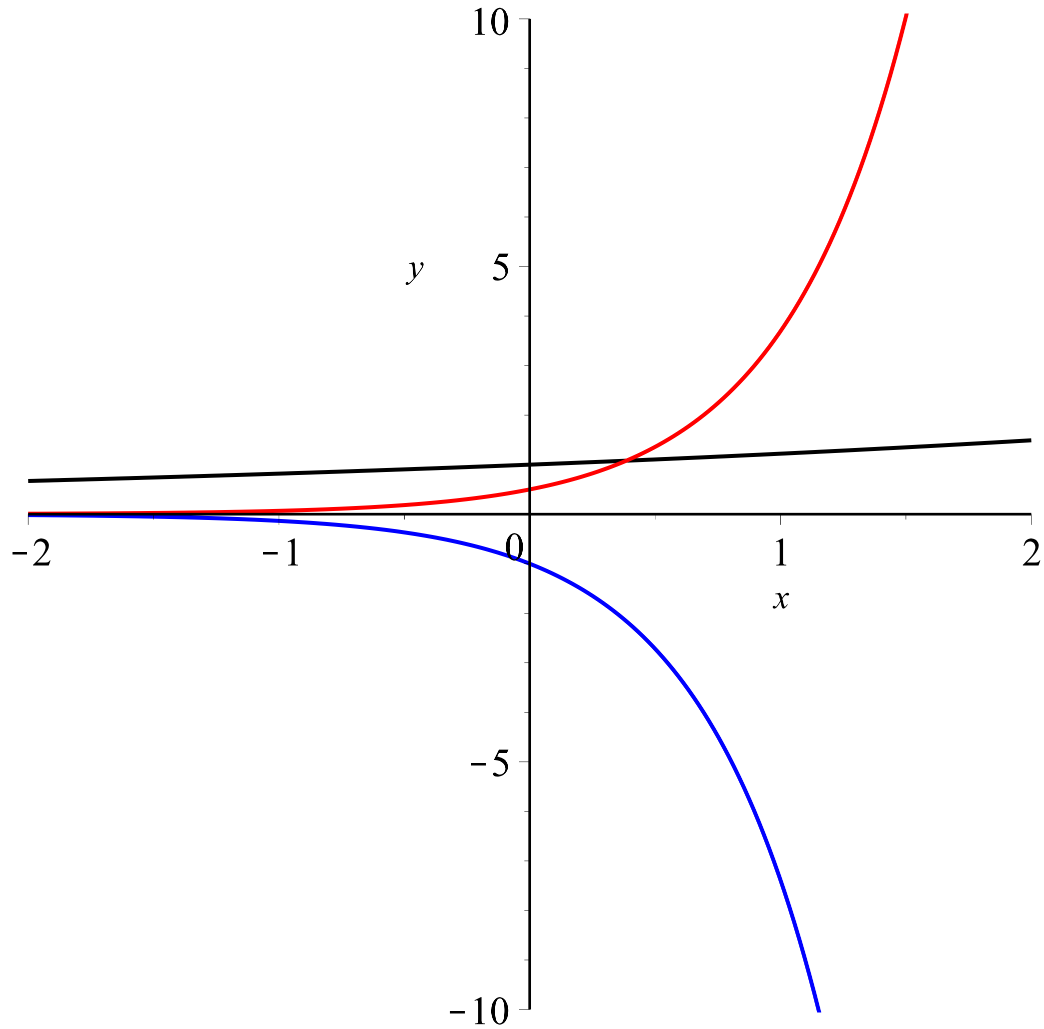


Using this way of manipulation for the solution you can see also how the solutiond depends on the k parameter. Let us consider c:=1 and assign some values for th parameter k.

```
> y1:=(x,k)->sol(x,k,1);
```

$$y1 := (x, k) \rightarrow sol(x, k, 1)$$

```
> plot([y1(x,0.2),y(x,0.5),y(x,-1)],x=-2..2,y=-10..10,color=
  [black,red,blue]);
```



Solving an IVP

Suppose that we want to solve the IVP $\frac{d}{dx} y(x) = k\, y(x)$ with the initial condition $y(0) = 1$

```
> restart:with(DEtools):
> diff_eq:=diff(y(x),x) = k*y(x);
```

$$diff\_eq := \frac{d}{dx} y(x) = k\, y(x)$$

```
> in_cond:=y(0)=1;
```

$$in\_cond := y(0) = 1$$

```
> dsolve({diff_eq,in_cond},y(x));
```

$$y(x) = e^{k x}$$

Let consider the case $k = 2$

```
> k:=2;
```

$$k := 2$$

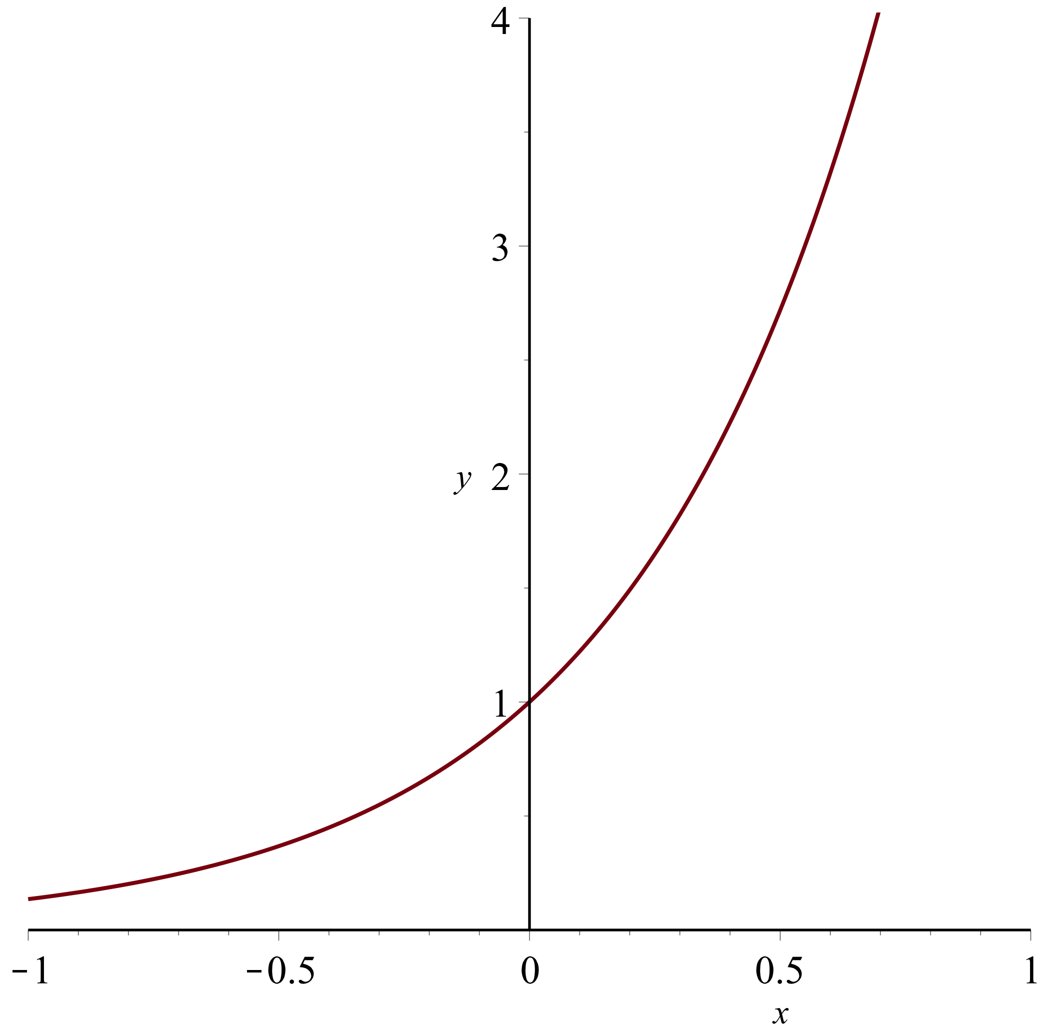
```
> sol:=dsolve({diff_eq,in_cond},y(x));
```

$$sol := y(x) = e^{2 x}$$

```
> yy:=unapply(rhs(sol),x);
```

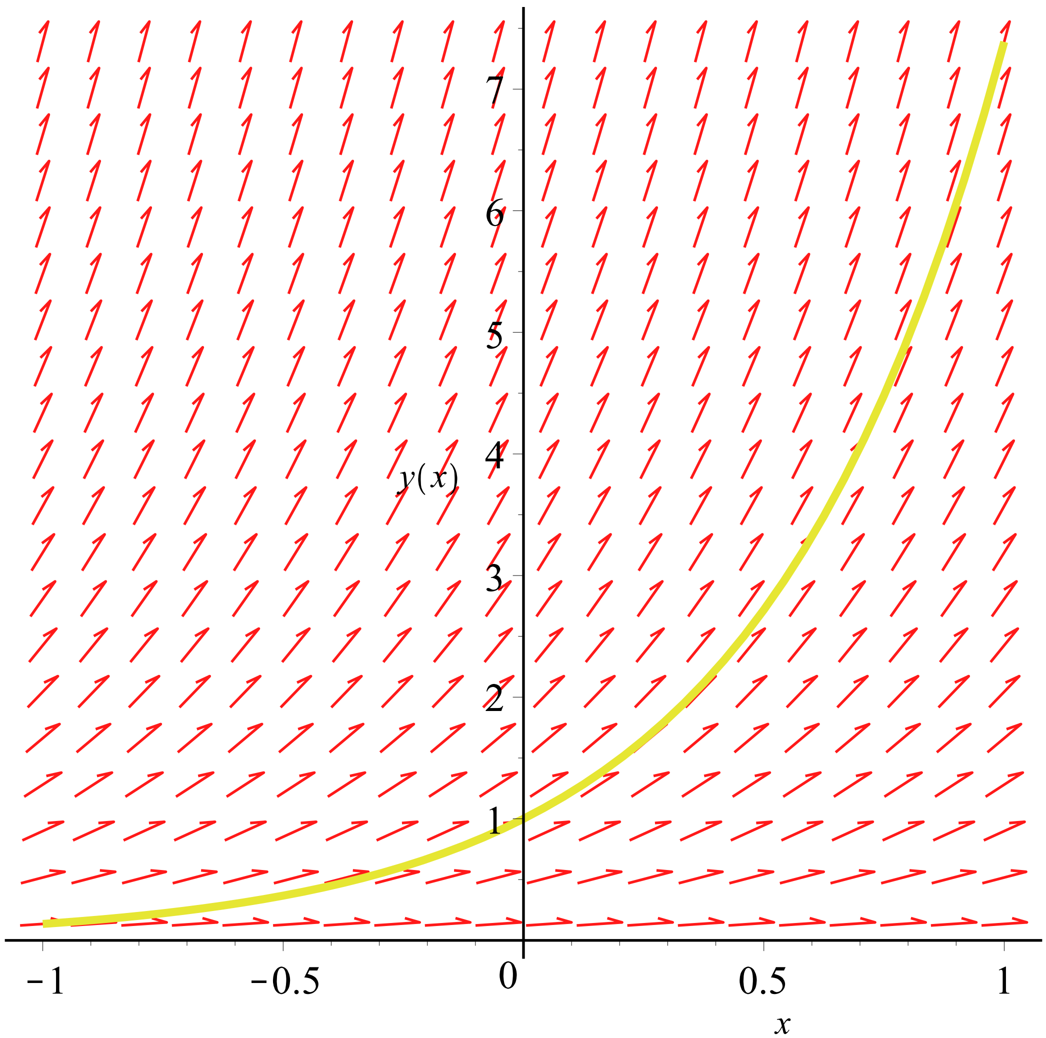
$$yy := x \to e^{2x}$$

```
> plot(yy(x),x=-1..1,y=0..4);
```



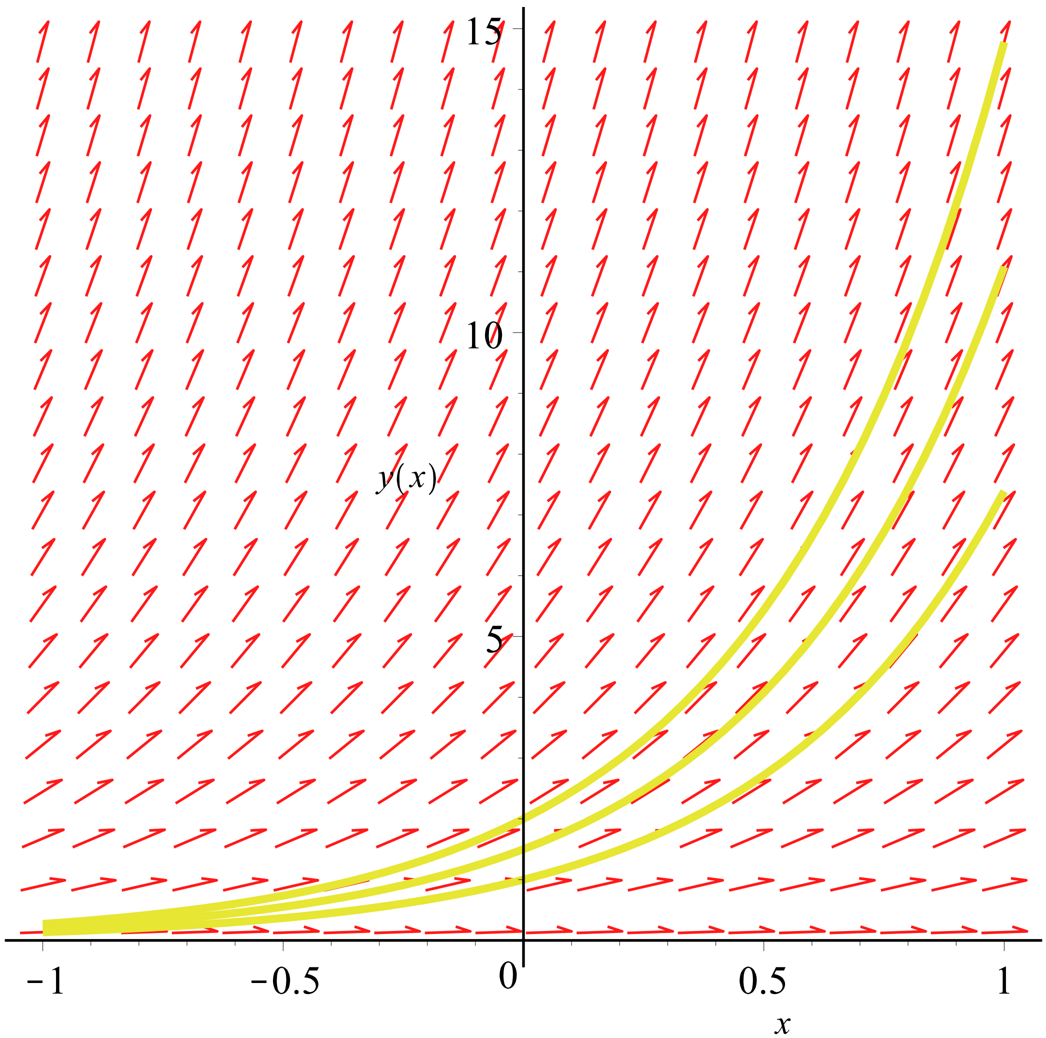
You can obtain the graph the IVP directly using the command DEplot:

```
> DEplot(diff_eq,y(x),x=-1..1,[[in_cond]]);
```



In this graph is also represented the direction field of the equation. If you want the graphs of the solutions for different initial condition ($y(0)=1$, $y(0)=1.5$, $y(0)=2$) you can use the same command and specify the list of initial conditions:

```
> DEplot(diff_eq,y(x),x=-1..1,[[y(0)=1],[y(0)=1.5],[y(0)=2]]);
```



First Order Solvable Differential Equations

Separable Differential Equations

$$\frac{d}{dx}\, y(x) = f(x)\, g(y(x))$$

```
> restart:with(DEtools):
> sep_eq:=diff(y(x),x)=f(x)*g(y(x));
```

$$sep\_eq := \frac{d}{dx}\, y(x) = f(x)\, g(y(x))$$

```
> dsolve(sep_eq,y(x));
```

$$\int f(x)\, dx - \left(\int^{y(x)} \frac{1}{g(\_a)}\, d\_a\right) + \_C1 = 0$$

Examples:

a) $y' = 2\,x\,(1 + y^2)$

b) $(x^2-1)\,y' + 2\,x\,y^2 = 0$
c) $y' = e^{x+y}$

```
> eq_a:=diff(y(x),x)=2*x*(1+(y(x))^2);
```

$$eq\_a := \frac{d}{dx}\,y(x) = 2\,x\,\left(1+y(x)^2\right)$$

```
> dsolve(eq_a,y(x));
```

$$y(x) = \tan\left(x^2 + 2\,\_C1\right)$$

```
> eq_b:=(x^2-1)*diff(y(x),x)+2*x*(y(x))^2=0;
```

$$eq\_b := \left(x^2-1\right)\left(\frac{d}{dx}\,y(x)\right) + 2\,y(x)^2\,x = 0$$

```
> dsolve(eq_b,y(x));
```

$$y(x) = \frac{1}{\ln(x-1)+\ln(x+1)+\_C1}$$

```
> eq_c:=diff(y(x),x)=exp(x+y(x));
```

$$eq\_c := \frac{d}{dx}\,y(x) = e^{x+y(x)}$$

```
> dsolve(eq_c,y(x));
```

$$y(x) = \ln\left(-\frac{1}{e^x + \_C1}\right)$$

Homogeneous (in the Euler sense) Differential Equations

$$\frac{d}{dx}\,y(x) = F\left(\frac{y(x)}{x}\right)$$

```
> hom_eq:=diff(y(x),x)=F(y(x)/x);
```

$$hom\_eq := \frac{d}{dx}\,y(x) = F\left(\frac{y(x)}{x}\right)$$

```
> dsolve(hom_eq,y(x));
```

$$y(x) = RootOf\left(-\left(\int^{\_Z} \frac{1}{F(\_a)-\_a}\,d\_a\right) + \ln(x) + \_C1\right)x$$

Examples:
a) $2\,x^2\,y' = x^2 + y^2$
b) $x\,y' = \sqrt{x^2-y^2} + y$
c) $2\,x^3\,y' = y^3 + x^2\,y$

```
> eq_a:=2*x^2*diff(y(x),x)=x^2+y(x)^2;
```

$$eq\_a := 2\left(\frac{d}{dx}\,y(x)\right)x^2 = x^2 + y(x)^2$$

```
> dsolve(eq_a,y(x));
```

$$y(x) = \frac{x\,(\ln(x) + \_C1 - 2)}{\ln(x) + \_C1}$$

```
> eq_b:=x*diff(y(x),x)=sqrt(x^2-y(x)^2)+y(x);
```

$$eq\_b := x\left(\frac{d}{dx}\,y(x)\right) = \sqrt{x^2 - y(x)^2} + y(x)$$

```
> ans:=dsolve(eq_b,y(x));
```

$$ans := -\arctan\left(\frac{y(x)}{\sqrt{x^2 - y(x)^2}}\right) + \ln(x) - \_C1 = 0$$

```
> solve(ans,y(x));
```

$$-\tan(-\ln(x) + \_C1)\sqrt{\frac{x^2}{\tan(-\ln(x) + \_C1)^2 + 1}}$$

```
> eq_c:=2*x^3*diff(y(x),x)=y(x)^3+x^2*y(x);
```

$$eq\_c := 2\,x^3\left(\frac{d}{dx}\,y(x)\right) = y(x)^3 + y(x)\,x^2$$

```
> dsolve(eq_c,y(x));
```

$$y(x) = \frac{x}{\sqrt{\_C1\,x + 1}},\ y(x) = -\frac{x}{\sqrt{\_C1\,x + 1}}$$

First Order Linear Differential Equations

$$\frac{d}{dx}\,y(x) + P(x)\,y(x) = Q(x)$$

```
> lin_eq:=diff(y(x),x)+P(x)*y(x)=Q(x);
```

$$lin\_eq := \frac{d}{dx}\,y(x) + P(x)\,y(x) = Q(x)$$

```
> dsolve(lin_eq,y(x));
```

$$y(x) = \left(\int Q(x)\,e^{\int P(x)\,dx}\,dx + \_C1\right)e^{\int (-P(x))\,dx}$$

Examples:

a) $y' + y\tan(x) = \frac{1}{\cos(x)}$

b) $y' + \frac{y(x)}{x} = 3\,x$

c) $x\,y' + y = e^x$

```
> eq_a:=diff(y(x),x)+tan(x)*y(x)=1/cos(x);
```

$$eq\_a := \frac{d}{dx}\,y(x) + \tan(x)\,y(x) = \frac{1}{\cos(x)}$$

```
> dsolve(eq_a,y(x));
```

$$y(x) = \cos(x)\tan(x) + \cos(x)\,\_C1$$

```
> eq_b:=diff(y(x),x)+1/x*y(x)=3*x;
```

$$eq\_b := \frac{d}{dx}\,y(x) + \frac{y(x)}{x} = 3\,x$$

```
> dsolve(eq_b,y(x));
```

$$y(x) = \frac{x^3 + \_C1}{x}$$

```
> eq_c:=x*diff(y(x),x)+y(x)=exp(x);
```

$$eq\_c := x\left(\frac{d}{dx}\,y(x)\right) + y(x) = e^x$$

```
> dsolve(eq_c,y(x));
```

$$y(x) = \frac{e^x + \_C1}{x}$$

Solving a second order ODE

```
> restart:
> with(DEtools):
> with(plots):
```

Consider the linear differential equation with the constant coeficients y '' + 3 y ' + 2 y = 1 + x^2

```
> deq1:=diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=1+x^2;
```

$$deq1 := \frac{d^2}{dx^2}\,y(x) + 3\left(\frac{d}{dx}\,y(x)\right) + 2\,y(x) = x^2 + 1$$

To obtain the general solution we use the dsolve command

```
> dsolve(deq1,y(x));
```

$$y(x) = \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{9}{4} - e^{-2x}\,\_C1 + e^{-x}\,\_C2$$

If we want to study the solution we can use the same technique as in the previous section in order to draw the solution graph.

```
> sol:=dsolve(deq1,y(x));
```

$$sol := y(x) = \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{9}{4} - e^{-2x}\,\_C1 + e^{-x}\,\_C2$$

```
> right_hand:=rhs(sol);
```

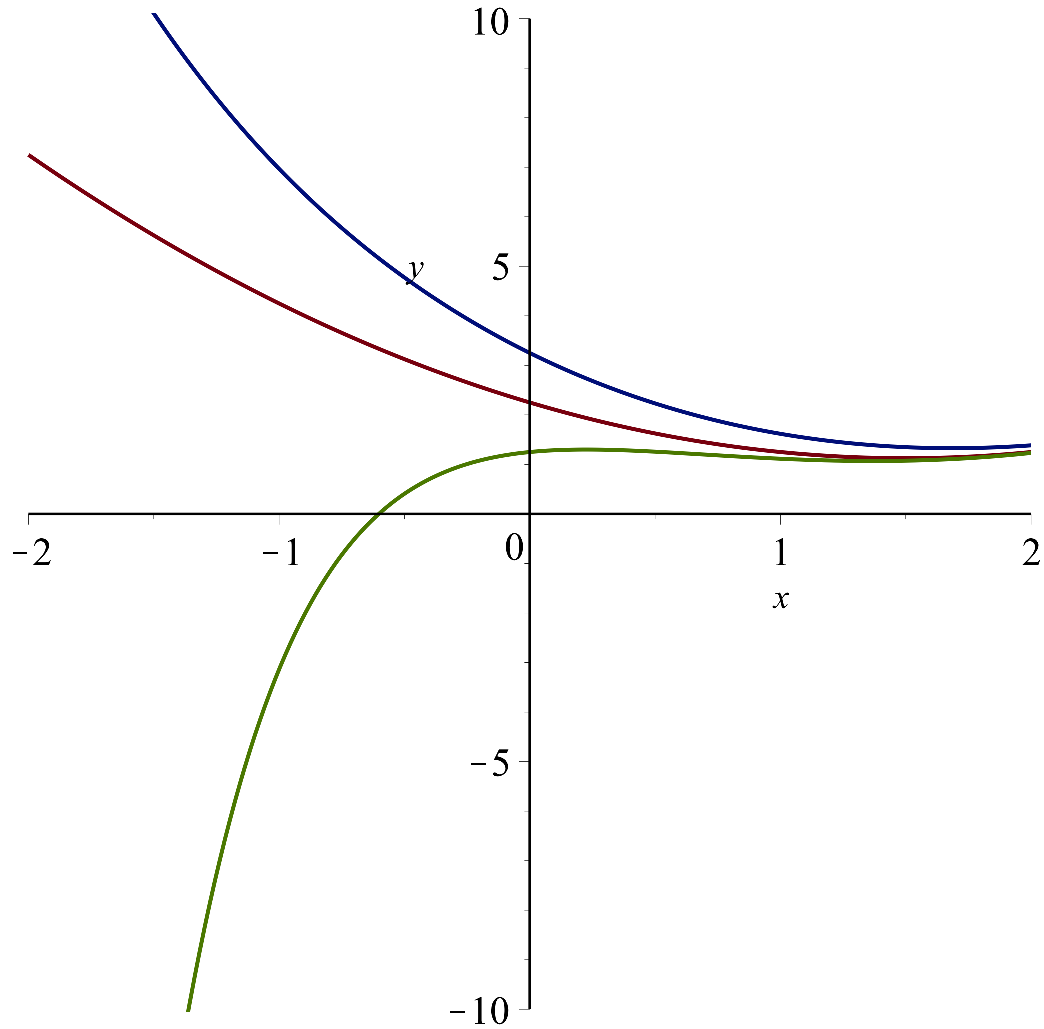
$$right\_hand := \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{9}{4} - e^{-2x}\,\_C1 + e^{-x}\,\_C2$$

```
> y_sol:=unapply(right_hand,x,_C1,_C2);
```

$$y\_sol := (x, \_C1, \_C2) \rightarrow \frac{1}{2}\,x^2 - \frac{3}{2}\,x + \frac{9}{4} - e^{-2x}\,\_C1 + e^{-x}\,\_C2$$

Now we are able to one ore more than one solution graphs using the **plot** command.

```
> plot([y_sol(x,0,0),y_sol(x,0,1),y_sol(x,1,0)],x=-2..2,y=-10.
  .10);
```



In the case of initial value problem we have two initial conditions, for example lets take $y(0) = 1$ and $y'(0) = 0$.

```
> in_cond:=y(0)=1,D(y)(0)=0;
```

$$in\_cond := y(0) = 1, D(y)(0) = 0$$

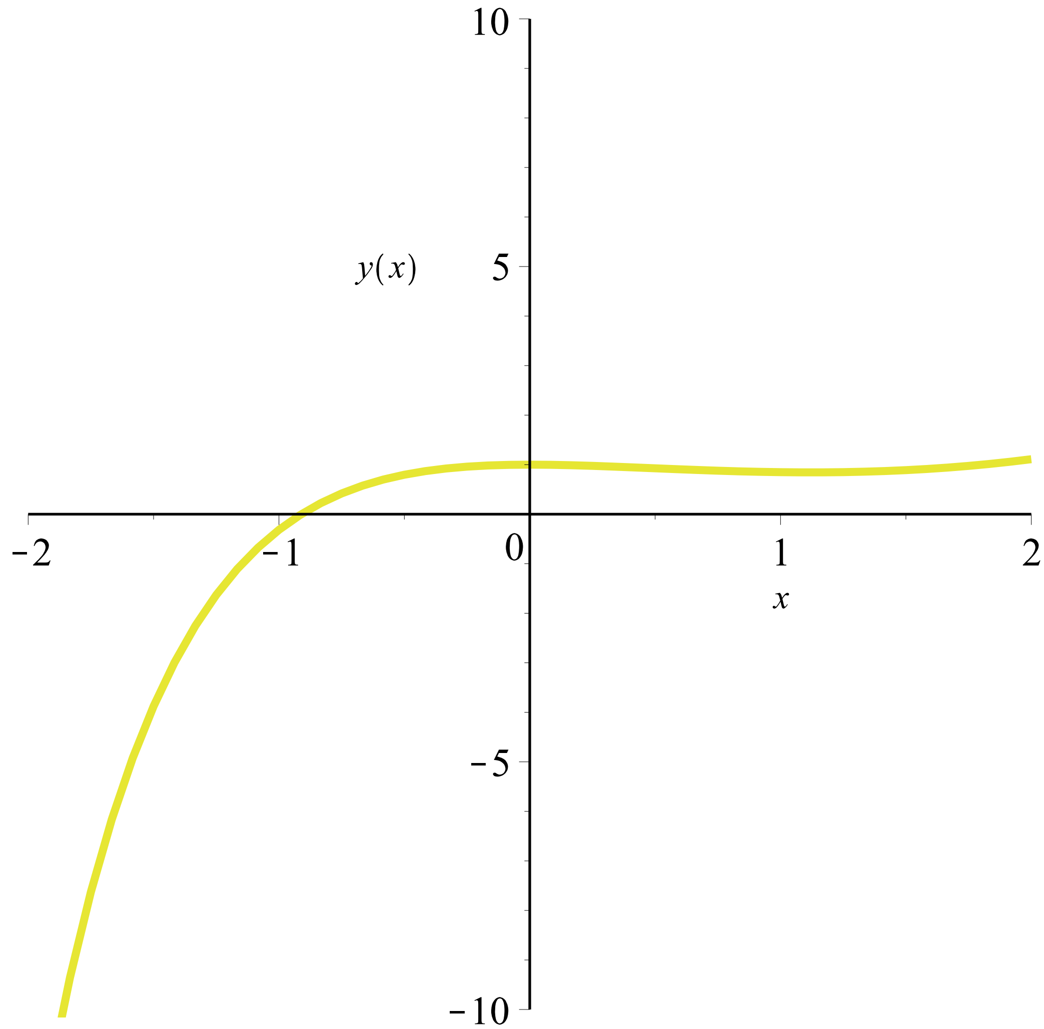
To obtain the corresponding solution we use **dsolve** command in the following form:

```
> dsolve({deq1,in_cond},y(x));
```

$$y(x) = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{4} - \frac{1}{4}e^{-2x} - e^{-x}$$

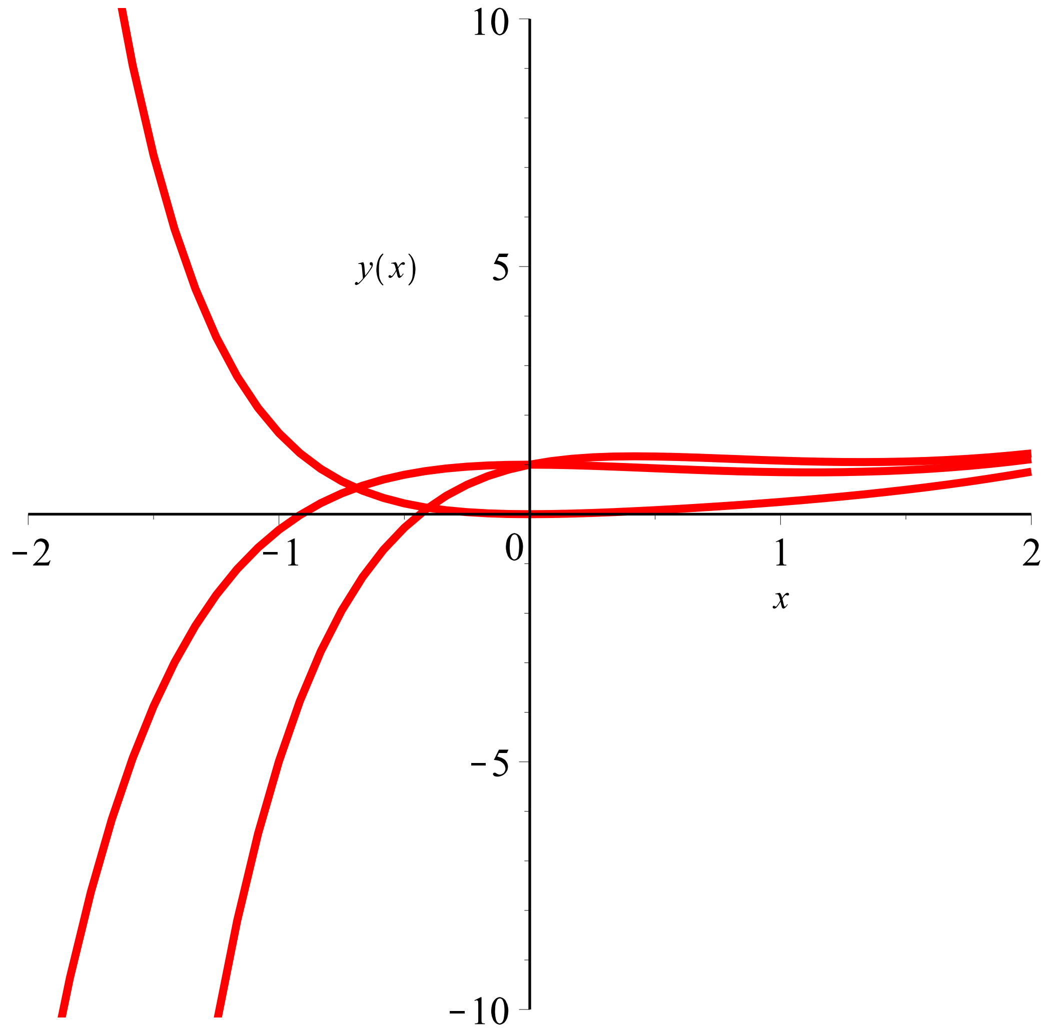
Now we can use the previous technique (**rhs** and **unapply** comands) to construct the solution as a function and after that to represent its graph or we can obtain this graph directly using **DEplot** command.

```
> DEplot(deq1,y(x),x=-2..2,y=-10..10,[[in_cond]]);
```



If we need to draw more than one solution corresponding to different initial value problem we can use the same **DEplot** command specifying the list of initial conditions:

```
> DEplot(deq1,y(x),x=-2..2,[[y(0)=1,D(y)(0)=0],[y(0)=1,D(y)(0)=
  1],[y(0)=0,D(y)(0)=0]],y=-10..10,linecolor=red);
```



The general second order linear DE, y '' + P(x) y ' + Q(x) y = f(x):
Note: Maple is unable to solve most second-order DE's explicitly. For information on numerically solving DE's, see Numerical Solutions with dsolve.
Consider the differential equation y'' + x y'(x) + y(x) = sin(x). Try to use the **dsolve** command.

```
> deq2:=diff(y(x),x$2)+x*diff(y(x),x)+y(x)=sin(x);
```

$$deq2 := \frac{d^2}{dx^2}\, y(x) + x\left(\frac{d}{dx}\, y(x)\right) + y(x) = \sin(x)$$

```
> dsolve(deq2,y(x));
```

$$y(x) = e^{-\frac{1}{2}x^2}\operatorname{erf}\left(\frac{1}{2}\,I\sqrt{2}\,x\right)\_C1 + e^{-\frac{1}{2}x^2}\_C2 + \frac{1}{4}\,I\sqrt{\pi}\,e^{\frac{1}{2}}\sqrt{2}\left(\operatorname{erf}\left(\frac{1}{2}\,I\sqrt{2}\,x - \frac{1}{2}\sqrt{2}\right) + \operatorname{erf}\left(\frac{1}{2}\,I\sqrt{2}\,x + \frac{1}{2}\sqrt{2}\right)\right)e^{-\frac{1}{2}x^2}$$

```
> ?erf
```

```
> in_cond2:=y(0)=1,D(y)(0)=1;
```

$$in\_cond2 := y(0) = 1, D(y)(0) = 1$$

```
> dsolve({deq2,in_cond2},y(x));
```

$$y(x) = -\frac{1}{2} \mathrm{I}\, e^{-\frac{1}{2} x^2} \operatorname{erf}\left(\frac{1}{2} \mathrm{I} \sqrt{2}\, x\right) \sqrt{\pi} \left(e^{\frac{1}{2}} e^{-\frac{1}{2}} + 1\right) \sqrt{2} + e^{-\frac{1}{2} x^2}$$
$$+ \frac{1}{4} \mathrm{I} \sqrt{\pi}\, e^{\frac{1}{2}} \sqrt{2} \left(\operatorname{erf}\left(\frac{1}{2} \mathrm{I} \sqrt{2}\, x - \frac{1}{2} \sqrt{2}\right) + \operatorname{erf}\left(\frac{1}{2} \mathrm{I} \sqrt{2}\, x + \frac{1}{2} \sqrt{2}\right)\right) e^{-\frac{1}{2} x^2}$$

Maple expresses the solution in terms of a modified Bessel function I and the error function **`erf`**.

We can obtain the numerical solution using in the dsolve command the option '**`type=numeric`**' and the **`odeplot`** comand to draw the corresponding graph.

```
> n_sol:=dsolve({deq2,in_cond2},y(x),type=numeric):
> odeplot(n_sol);
```