Laboratory 3: Stability of the equilibrium points

1. Let's consider the difference equation:

$$x_{n+1} = f(x_n)$$

   Determine, in each case, the equilibrium points and study their stability using the stability theorem in the first approximation. Make some numerical simulations.

   (a) $x_{n+1} = -\frac{1}{2}x_n$

   (b) $x_{n+1} = x_n^2 - 4$

   (c) $x_{n+1} = \dfrac{2x_n}{1+x_n}$

2. Consider the following mosquito model

$$x_{n+1} = \left(ax_n + bx_{n-1} \cdot e^{-x_{n-1}}\right) \cdot e^{-x_n}$$

   where $a \in (0;1)$, $b \in [0;+\infty)$. This equation describes the growth of a mosquito population. Mosquitoes lay eggs, some of which hatch as soon as conditions are favorable, while others remain dormant for a year or two. In this model, it is assumed that eggs are dormant for one year at most.

   (a) Find the equilibrium points;

   (b) Study the stability of the equilibrium points;

   (c) Make numerical simulations.

3. Flour Beetles model

$$x_{n+1} = \alpha x_n + \beta x_{n-2} \cdot e^{-c_1 x_{n-2} - c_2 x_n}$$

   where $\alpha, \beta > 0$.

   (a) Find the equilibrium points;

   (b) Study the stability of the equilibrium points;

   (c) Make numerical simulations.

Hint: In the case of Exercises 2 and 3 use the **Schur-Cohn Criterion**

(a) for the equation the equation

$$q^2 = p_1 q + p_2$$

we have:

$|q_{1,2}| < 1$ if and only if

$$|p1| < 1 - p_2 < 2.$$

(b) for the equation the equation

$$q^3 = p_1 q^2 + p_2 q + p_3$$

we have:

$|q_{1,2,3}| < 1$ if and only if

$$|p_3 + p_1| < 1 - p_2 \text{ and } |p_1 p_3 + p_2| < 1 - p_3^2.$$

4. Study the stability of the equilibrium point (0,0) for the following systems. For the cases where stability occurs, verify this property numerically.

(a) $\begin{cases} x_{n+1} &= x_n + 4y_n \\ y_{n+1} &= \dfrac{1}{4}x_n + y_n \end{cases}$

(b) $\begin{cases} x_{n+1} &= x_n - \dfrac{1}{4}y_n \\ y_{n+1} &= \dfrac{1}{4}x_n + \dfrac{1}{2}y_n \end{cases}$

5. Find and study the stability of the equilibrium points for the following systems. For locally asymptotically stable equilibrium points, determine some initial conditions for which the corresponding solution converges to that equilibrium point.

(a) $\begin{cases} x_{n+1} &= x_n + \dfrac{1}{6}x_n(1 - x_n - y_n) \\ y_{n+1} &= y_n(1 + x_n - y_n) \end{cases}$

(b) $\begin{cases} x_{n+1} &= \dfrac{1}{3}x_n(5 - x_n - y_n) \\ y_{n+1} &= \dfrac{1}{3}y_n(3y_n - x_n) \end{cases}$

(c) $\begin{cases} x_{n+1} &= \dfrac{1}{3}x_n(1 + x_n + y_n) \\ y_{n+1} &= \dfrac{1}{2}y_n(1 - x_n + y_n) \end{cases}$