Systems of Autonomous Differential Equations. Equilibrium Points, Stability

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

In the case of linear systems the origin (0,0) is the equilibrium point

Linear systems with real eigenvalues:
Consider a linear system with two nonzero, real, distinct eigenvalues λ_1 and λ_2.

• If $\lambda_1 < 0 < \lambda_2$, then the origin is a **saddle point**. There are two lines in the phase portrait that correspond to straight-line solutions. Solutions along one line tend toward (0, 0) as t increases, and solutions on the other line tend away from (0, 0). All other solutions come from and go to infinity, so, in this case, the origin is unstable.
• If $\lambda_1 < \lambda_2 < 0$, then the origin is **a sink node**. All solutions tend to (0, 0) as $t \to \infty$, so the origin is asymptolically stable.
• If $0 < \lambda_1 < \lambda_2$, then the origin is **a source node**. All solutions except the equilibrium solution go to infinity as $t \to \infty$, so, the origin is unstable.

Linear systems with complex eigenvalues:
Given a linear system with complex eigenvalues $\lambda = \alpha \pm i\beta$, $\beta > 0$, the solution curves spiral around the origin in the phase plane with a period of $2\pi/\beta$. Moreover:
• If $\alpha < 0$, then the solutions spiral toward the origin. In this case the origin is called a **spiral sink**, origin is asymptolically stable.
• If $\alpha > 0$, then the solutions spiral away from the origin. In this case the origin is called a **spiral source**, the origin is unstable.
• If $\alpha = 0$, then the solutions are periodic. They return exactly to their initial conditions in the phase plane and repeat the same closed curve over and over. In this case the origin is called a **center** and it is locally asymptotically stable.

Let's consider the following linear system

$$x' = x + y$$
$$y' = x - y$$

so,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The given system is introduced by defining the two equations through the `diff` procedure.

```
> restart;
> with(DEtools): with(plots):with(linalg):
```

```
> eq1:=diff(x(t),t)=x(t)+y(t);
```

$$eq1 := \frac{d}{dt} x(t) = x(t) + y(t)$$

```
> eq2:=diff(y(t),t)=x(t)-y(t);
```

$$eq2 := \frac{d}{dt} y(t) = x(t) - y(t)$$

```
> sist:=eq1,eq2;
```

$$sist := \frac{d}{dt} x(t) = x(t) + y(t), \frac{d}{dt} y(t) = x(t) - y(t)$$

We construct the matrix of the system and we calculate the corresponding eigenvalues:

```
> A:=matrix([[1,1],[1,-1]]);
```

$$A := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

```
> eigenvals(A);
```

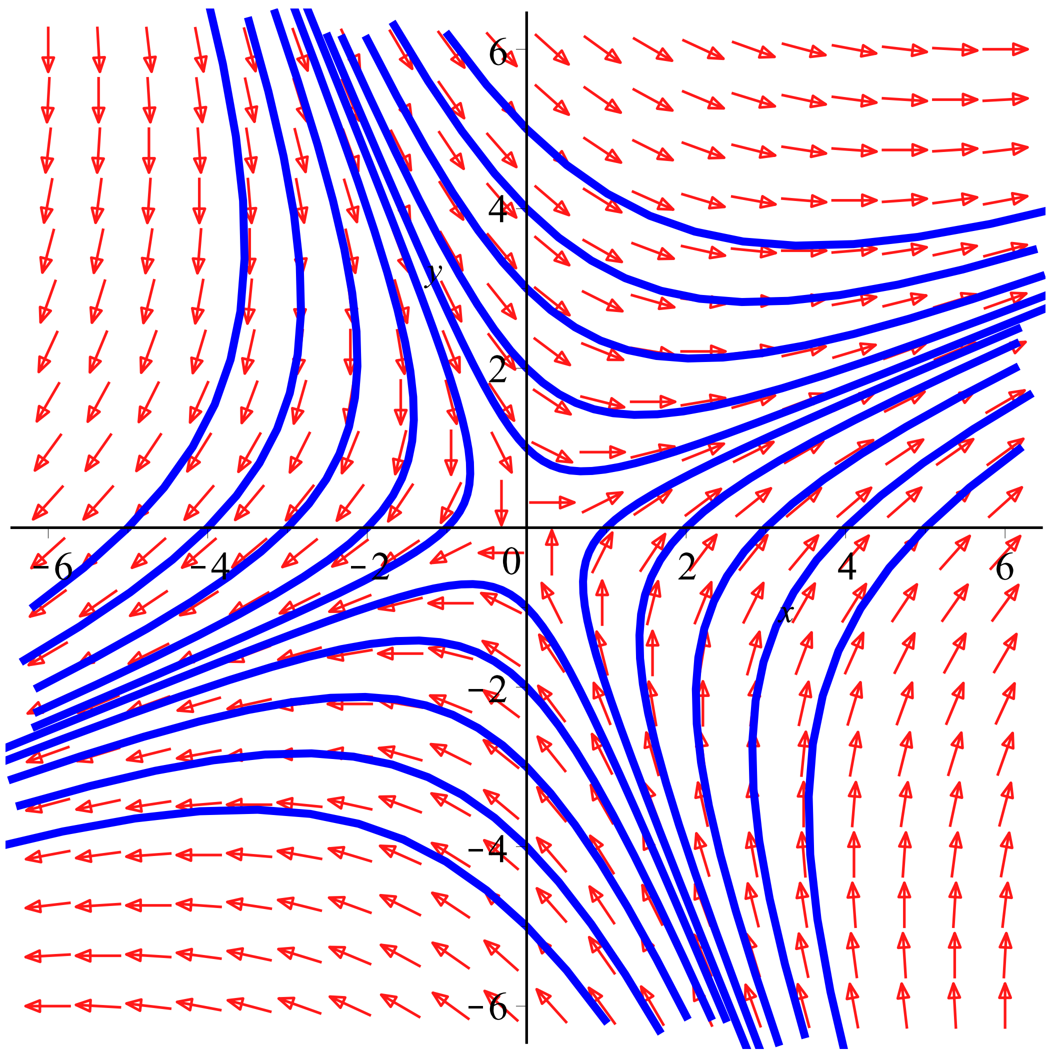
$$\sqrt{2}, -\sqrt{2}$$

The first eigenvalue is strictly positive and the second is strictly negative then the equilibrium point (0,0) is its unstable equilibrium point (a source node).
To visualize the dynamics generated by this system, the phase portrait is represented along with some orbits using the command **DEplot**

```
> in_cond:=[x(0)=0,y(0)=i]$i=1..5,[x(0)=-i,y(0)=0]$i=1..5,[x(0)=0,y(0)=-i]$i=1..5,[x(0)=i,y(0)=0]$i=1..5;
```

$in\_cond := [x(0) = 0, y(0) = 1], [x(0) = 0, y(0) = 2], [x(0) = 0, y(0) = 3], [x(0) = 0, y(0) = 4], [x(0) = 0, y(0) = 5], [x(0) = -1, y(0) = 0], [x(0) = -2, y(0) = 0], [x(0) = -3, y(0) = 0], [x(0) = -4, y(0) = 0], [x(0) = -5, y(0) = 0], [x(0) = 0, y(0) = -1], [x(0) = 0, y(0) = -2], [x(0) = 0, y(0) = -3], [x(0) = 0, y(0) = -4], [x(0) = 0, y(0) = -5], [x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = 4, y(0) = 0], [x(0) = 5, y(0) = 0]$

```
> DEplot([sist],[x(t),y(t)],t=-5..5,x=-6..6,y=-6..6,[in_cond],
  arrows=medium, linecolor=blue,stepsize=0.1);
```



Nonlinear systems. Stability of the equilibrium points

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), y(t)) \\ f_2(x(t), y(t)) \end{bmatrix}$$

The equilibrium points $X^*(x^*,y^*)$ are the real solution of the system

$$\begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The stability of the equilibrium points can be studied using the stability theorem in the first approximation for the system case:

THE STABILITY THEOREM IN THE FIRST APPROXIMATION
Let $X^*(x^*,y^*)$ be an equilibrium point for the nonlinear system.

- if Re(λ)<0 for all the eigenvalues of $J_f(X^*)$ then X^* is locally asymptotically stable

• if there exists an eigenvalue of $J_f(X^*)$ with Re(λ)>0 then X^* is unstable
where $J_f(X)$ is the jacobian of the vectorial function $f = (f_1, f_2)$

Let's consider the system

$$\begin{cases} x' = x\left(1 - \frac{x}{2} - y\right) \\ y' = y\left(x - 1 - \frac{y}{2}\right) \end{cases}$$

```
> f1:=(x,y)->x*(1-x/2-y);
```

$$f1 := (x, y) \to x\left(1 - \frac{1}{2}x - y\right)$$

```
> f2:=(x,y)->y*(x-1-y/2);
```

$$f2 := (x, y) \to y\left(x - 1 - \frac{1}{2}y\right)$$

```
> eq1:=diff(x(t),t)=f1(x(t),y(t));
```

$$eq1 := \frac{d}{dt}x(t) = x(t)\left(1 - \frac{1}{2}x(t) - y(t)\right)$$

```
> eq2:=diff(y(t),t)=f2(x(t),y(t));
```

$$eq2 := \frac{d}{dt}y(t) = y(t)\left(x(t) - 1 - \frac{1}{2}y(t)\right)$$

```
> sist2:=eq1,eq2;
```

$$sist2 := \frac{d}{dt}x(t) = x(t)\left(1 - \frac{1}{2}x(t) - y(t)\right), \frac{d}{dt}y(t) = y(t)\left(x(t) - 1 - \frac{1}{2}y(t)\right)$$

```
> EquiP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EquiP := \{x = 0, y = 0\}, \{x = 0, y = -2\}, \{x = 2, y = 0\}, \left\{x = \frac{6}{5}, y = \frac{2}{5}\right\}$$

Notice that the system has four equilibrium points.

```
> EquiP[1,1];EquiP[1,2];
```

$$x = 0$$
$$y = 0$$

First we construct the jacobian of the vectorial function $f = (f_1, f_2)$ and then we calculate the eigenvalues of $J_f(0,0)$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} 1 - x - y & -x \\ y & x - 1 - y \end{bmatrix}$$

```
> A1:=subs(EquiP[1,1],EquiP[1,2],eval(J));
```

$$A1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

```
> eigenvals(A1);
```

$$1, -1$$

So, in this case the equilibrium point (0,0) is unstable (saddle point) since the first eigenvalue is positive.

We do the same for the other equilibrium points.

```
> EquiP[2,1];EquiP[2,2];
```

$$x=0$$
$$y=-2$$

```
> A2:=subs(EquiP[2,1],EquiP[2,2],eval(J));
```

$$A2 := \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$$

```
> eigenvals(A2);
```

$$3, 1$$

In this case the equilibrium point (0, -2) is an unstable equilibrium point of the source node type.

```
> EquiP[3,1];EquiP[3,2];
```

$$x=2$$
$$y=0$$

```
> A3:=subs(EquiP[3,1],EquiP[3,2],eval(J));
```

$$A3 := \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

```
> eigenvals(A3);
```

$$-1, 1$$

Also, in this case the equilibrium point (2, 0) is an unstable equilibrium point of the saddle type.

```
> EquiP[4,1];EquiP[4,2];
```

$$x=\frac{6}{5}$$
$$y=\frac{2}{5}$$

```
> A4:=subs(EquiP[4,1],EquiP[4,2],eval(J));
```

$$A4 := \begin{bmatrix} -\frac{3}{5} & -\frac{6}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

```
> eigenvals(A4);
```

$$-\frac{2}{5}+\frac{1}{5}\,\mathrm{I}\sqrt{11},\ -\frac{2}{5}-\frac{1}{5}\,\mathrm{I}\sqrt{11}$$

In this case we have complex conjugate eigenvalues whose real part is -2/5, so the equilibrium point (6 / 5,2 / 5) is a locally asymptotically stable equilibrium point of the sink spiral type.

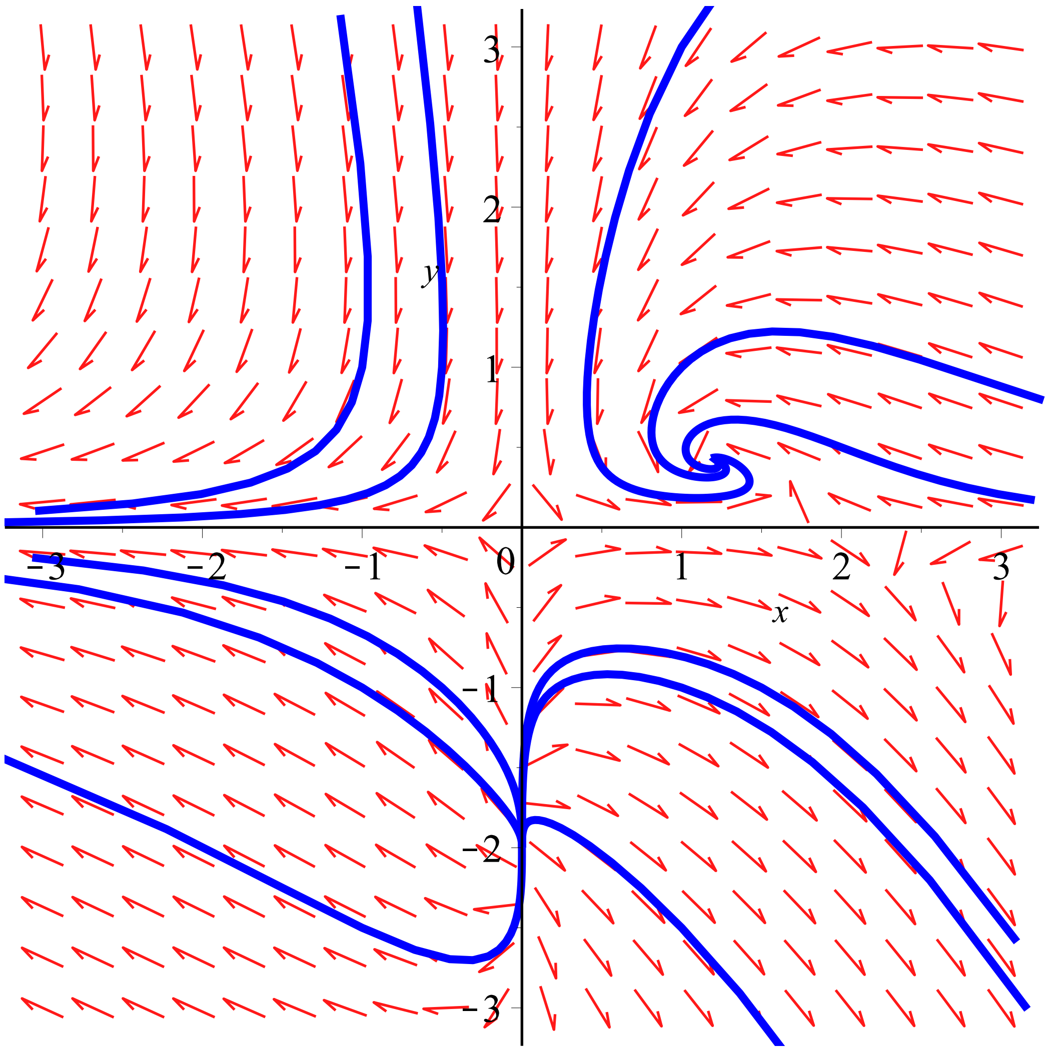
In order to represent the phase portrait, a window containing all the equilibrium points must be chosen, in this case we can choose x = -3..3, y = -3..3, and then set a list of conditions to represent some orbits.

```
> condin:=[x(0)=-1,y(0)=1],[x(0)=-0.5,y(0)=1],[x(0)=1,y(0)=1],
  [x(0)=1,y(0)=3],[x(0)=2,y(0)=0.5],[x(0)=-1,y(0)=-1],[x(0)=
```

```
-0.5,y(0)=-1],[x(0)=-1,y(0)=-2.5],[x(0)=1,y(0)=-1],[x(0)=1.5,
y(0)=-1],[x(0)=1,y(0)=-2.5];
```

$condin := [x(0) = -1, y(0) = 1], [x(0) = -0.5, y(0) = 1], [x(0) = 1, y(0) = 1], [x(0) = 1, y(0) = 3], [x(0) = 2, y(0) = 0.5], [x(0) = -1, y(0) = -1], [x(0) = -0.5, y(0) = -1], [x(0) = -1, y(0) = -2.5], [x(0) = 1, y(0) = -1], [x(0) = 1.5, y(0) = -1], [x(0) = 1, y(0) = -2.5]$

```
> DEplot([sist2],[x(t),y(t)],t=-10..10,x=-3..3,y=-3..3,
[condin],linecolor=blue,stepsize=0.1);
```



Mathematical Models for Interacting Populations

Prey-predator models

Volterra (1926) first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic.

x(t) - the prey population at the moment t

y(t) - the predator population at the moment t

Assumptions:
- the prey in the absence of any predation grows unboundedly as in the Malthus' model (unlimited resources)
- the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the number of interactions between the two populations ($x\,y$)
- in the absence of any prey for sustenance the predator's death rate results in exponential decay
- the prey's contribution to the predators' growth rate is proportional to the available prey as well as to the size of the predator population, this is proportional to the number of interactions between the two populations

$$\frac{d}{dt}\,x(t) = a\,x(t) - b\,x(t)\,y(t)$$
$$\frac{d}{dt}\,y(t) = -c\,y(t) + d\,x(t)\,y(t)$$
$$x(0) = x_0$$
$$y(0) = y_0$$

where a, b, c, d are positive.

This is known as the Lotka–Volterra model since the same equations were also derived by Lotka (1920; see also 1925) from a hypothetical chemical reaction which he said could exhibit periodic behaviour in the chemical concentrations.

```
> restart:with(DEtools):
> with(linalg):
> with(plots):
> eq1:=diff(x(t),t) =a*x(t)-b*x(t)*y(t);
```

$$eq1 := \frac{d}{dt}\,x(t) = a\,x(t) - b\,x(t)\,y(t)$$

```
> eq2:=diff(y(t),t) =-c*y(t)+d*x(t)*y(t);
```

$$eq2 := \frac{d}{dt}\,y(t) = -c\,y(t) + d\,x(t)\,y(t)$$

```
> pp_syst:=eq1,eq2;
```

$$pp\_syst := \frac{d}{dt}\,x(t) = a\,x(t) - b\,x(t)\,y(t),\ \frac{d}{dt}\,y(t) = -c\,y(t) + d\,x(t)\,y(t)$$

```
> f1:=(x,y)->a*x-b*x*y;
```

$$f1 := (x, y) \rightarrow a\,x - b\,x\,y$$

```
> f2:=(x,y)->-c*y+d*x*y;
```

$$f2 := (x, y) \rightarrow -c\,y + d\,x\,y$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EqP := \{x = 0, y = 0\},\ \left\{x = \frac{c}{d}, y = \frac{a}{b}\right\}$$

```
> EqP[1,1];
```

$$x = 0$$

```
> EqP[2,1];
```

$$x = \frac{c}{d}$$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} -b\,y + a & -x\,b \\ y\,d & d\,x - c \end{bmatrix}$$

```
> J1:=subs(EqP[1,1],EqP[1,2],eval(J));
```

$$J1 := \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

```
> eigenvals(J1);
```

$$a,\ -c$$

```
> J2:=subs(EqP[2,1],EqP[2,2],eval(J));
```

$$J2 := \begin{bmatrix} 0 & -\frac{c\,b}{d} \\ \frac{a\,d}{b} & 0 \end{bmatrix}$$

```
> assume(a>0,c>0);
> eigenvals(J2);
```

$$\mathrm{I}\sqrt{c\sim a\sim},\ -\mathrm{I}\sqrt{c\sim a\sim}$$

```
> deq:=diff(y(x),x)=f2(x,y(x))/f1(x,y(x));
```

$$deq := \frac{\mathrm{d}}{\mathrm{d}x}\,y(x) = \frac{-c\sim y(x) + d\,x\,y(x)}{a\sim x - b\,x\,y(x)}$$

```
> dsolve(deq,y(x),implicit);
```

$$-d\,x + c\sim \ln(x) - y(x)\,b + a\sim \ln(y(x)) + \_C1 = 0$$

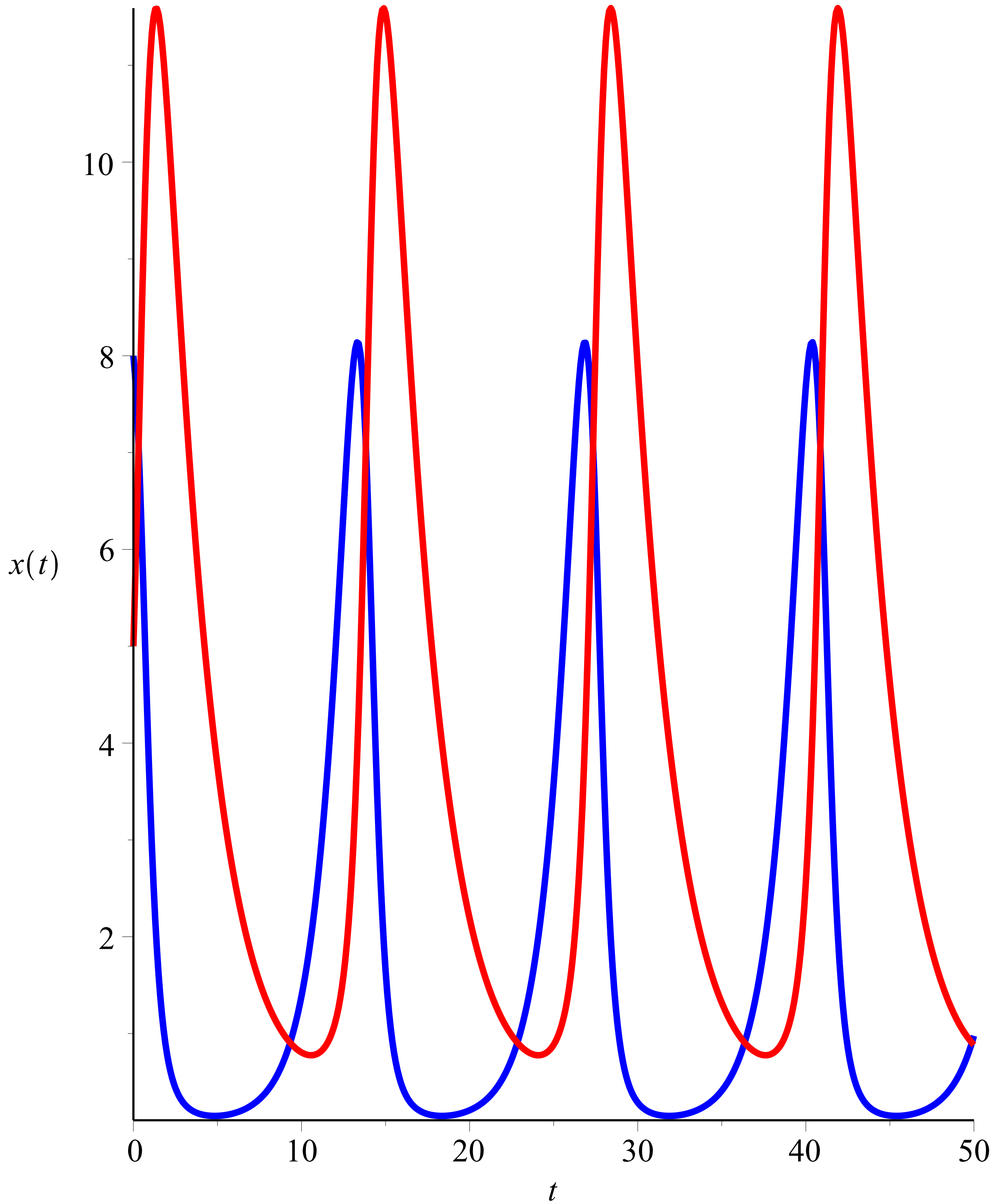
```
> a:=0.8;b:=0.2;c:=0.4;d:=0.2;x0:=8;y0:=5;
```

$$a := 0.8$$
$$b := 0.2$$
$$c := 0.4$$
$$d := 0.2$$
$$x0 := 8$$
$$y0 := 5$$

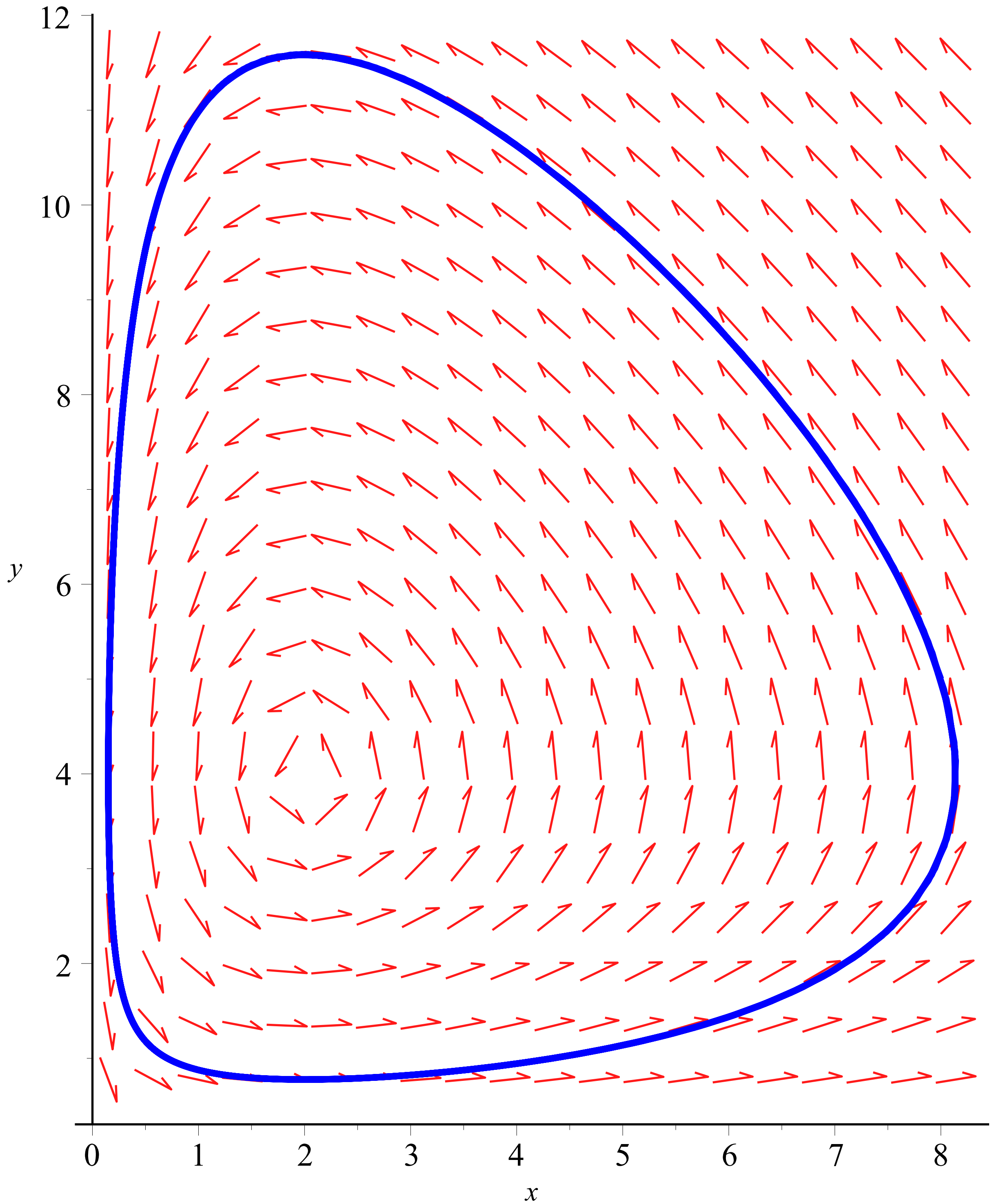
```
> EqP;
```

$$\{x = 0, y = 0\},\ \{x = 2.000000000, y = 4.000000000\}$$

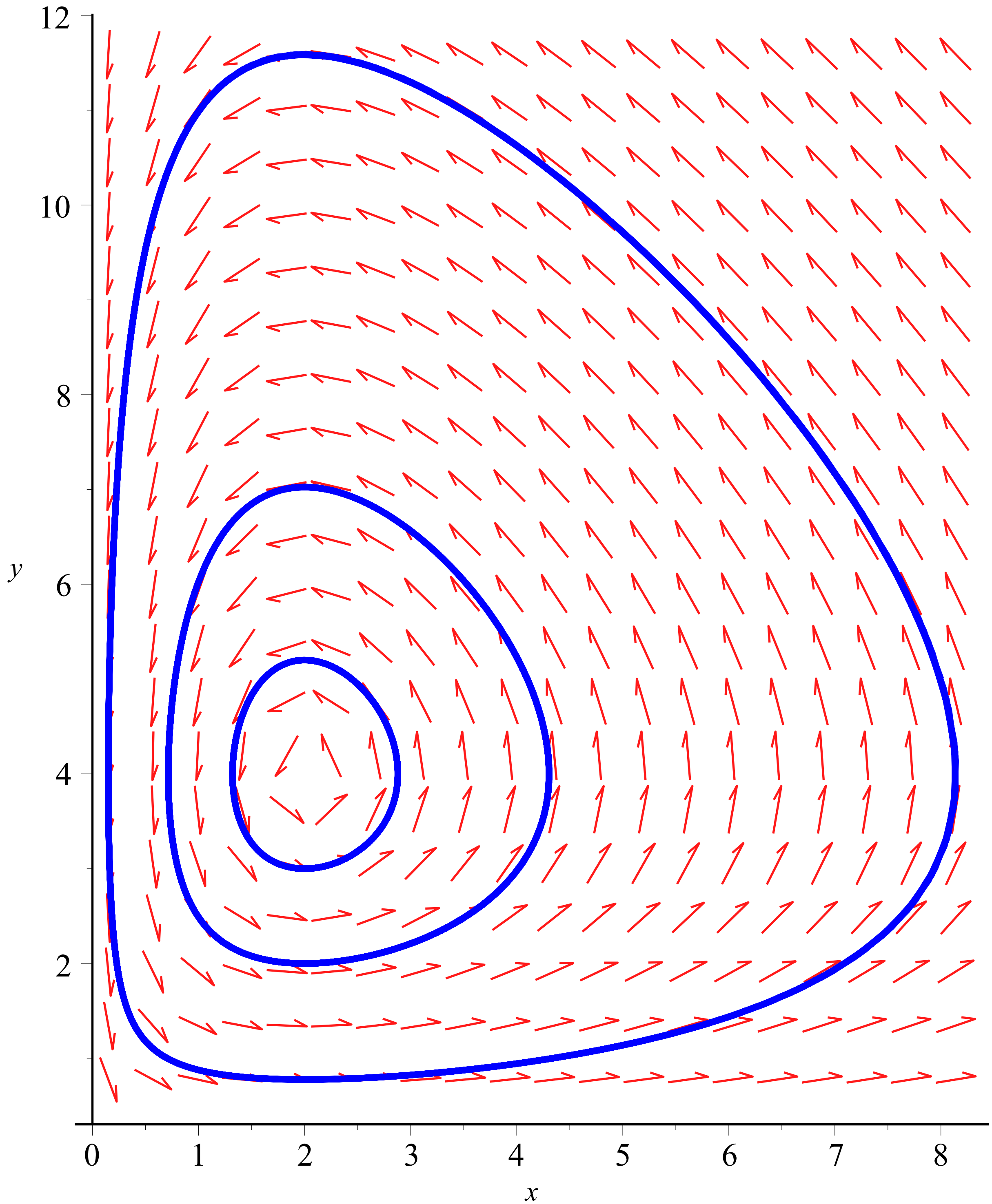
```
> gx:=DEplot([pp_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> gy:=DEplot([pp_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(gx,gy);
```




```
> DEplot([pp_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=blue);
```




```
> DEplot([pp_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0],[x(0)
  =2,y(0)=2],[x(0)=2,y(0)=3]],stepsize=0.1,linecolor=blue);
```



Since the orbits are closed then the solution are periodically with some period T. The average values of x(t) and y(t) are defined as

$$x_{average} = \frac{\int_0^T x(t)\, \mathrm{d}t}{T} \qquad y_{average} = \frac{\int_0^T x(t)\, \mathrm{d}t}{T}$$

It can be proved that average values of x(t) and y(t) are equal with the positive equilibrium point coordinates

$x_{average} = \frac{a}{b}$ $\quad$ $y_{average} = \frac{c}{d}$

In the mid 1920's the Italian biologist Umberto D'Ancona was studying the population variations of various species of fish that interact ,with each other. In the course of his research, he came across some data on per-
centages-of-total-catch of several species of fish that were cought into different Mediterranean ports in the years that spanned World War I. In particular, the data gave the percentage-of-total-catch of selachians, (sharks, skates, rays, etc.) which are not very desirable as food fish. D'Ancona was puzzled by the very large increase in the percentage of selachians during the period of the war. Obviously, he reasoned, the increase in the percentage of selachians was due to the greatly reduced level of fishing during this period. But how does the intensity of fishing affect the fish populations?

Let's include the effects of harvesting in the prey-predator model. Observe that fishing decreases the population of food fish at a rate $E\,x(t)$, and decreases the population of selachians at a rate $E\,y(t)$. The constant E reflects the intensity of fishing; i.e., the number of boats at sea and the number of nets in the water. Thus, the true state of affairs is described by the system

$$\frac{d}{dt}\,x(t) = a\,x(t) - b\,x(t)\,y(t) - E\,x(t)$$
$$\frac{d}{dt}\,y(t) = -c\,y(t) + d\,x(t)\,y(t) - E\,y(t)$$

or

$$\frac{d}{dt}\,x(t) = (a - E)\,x(t) - b\,x(t)\,y(t)$$
$$\frac{d}{dt}\,y(t) = -(c + E)\,y(t) + d\,x(t)\,y(t)$$

```
> a:='a';b:='b';c:='c';d:='d';
```

$$a := a$$
$$b := b$$
$$c := c$$
$$d := d$$

```
> f1:=(x,y)->(a-E)*x-b*x*y;
```

$$f1 := (x, y) \rightarrow (a - E)\,x - b\,x\,y$$

```
> f2:=(x,y)->-(c+E)*y+d*x*y;
```

$$f2 := (x, y) \rightarrow -(c + E)\,y + d\,x\,y$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EqP := \{x = 0, y = 0\}, \left\{x = \frac{c + E}{d}, y = -\frac{E - a}{b}\right\}$$

Hence, the average values of x(t) and y(t) are now

$$x_{average} = \frac{c+E}{d} \qquad y_{average} = \frac{a-E}{b}$$

Consequently, a moderate amount of fishing (E < a) actually increases the number of food fish, on the average, and decreases the number of selachians (predators). Conversely, a reduced level of fishing increases the number of selachians, on the average, and decreases the number of food fish. This remarkable result, which is known as Volterra's principle, explains the data of D' Ancona.

```
> eq1:=diff(x(t),t) =(a-E)*x(t)-b*x(t)*y(t);
```

$$eq1 := \frac{d}{dt} x(t) = (a-E)\,x(t) - b\,x(t)\,y(t)$$

```
> eq2:=diff(y(t),t) =-(c+E)*y(t)+d*x(t)*y(t);
```

$$eq2 := \frac{d}{dt} y(t) = -(c+E)\,y(t) + d\,x(t)\,y(t)$$

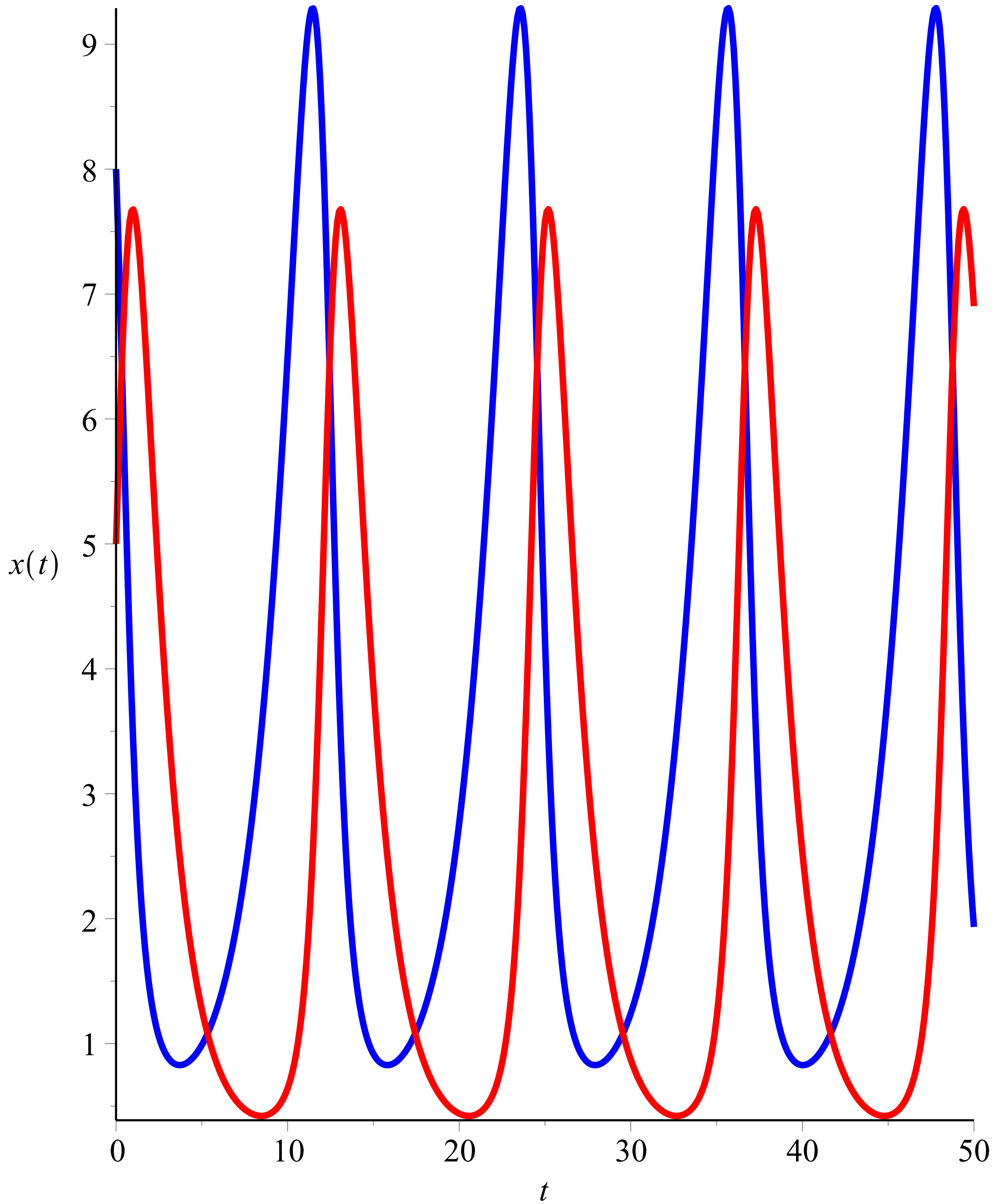
```
> a:=0.8;b:=0.2;c:=0.4;d:=0.2;x0:=8;y0:=5;E:=0.3;
```

$$a := 0.8$$
$$b := 0.2$$
$$c := 0.4$$
$$d := 0.2$$
$$x0 := 8$$
$$y0 := 5$$
$$E := 0.3$$

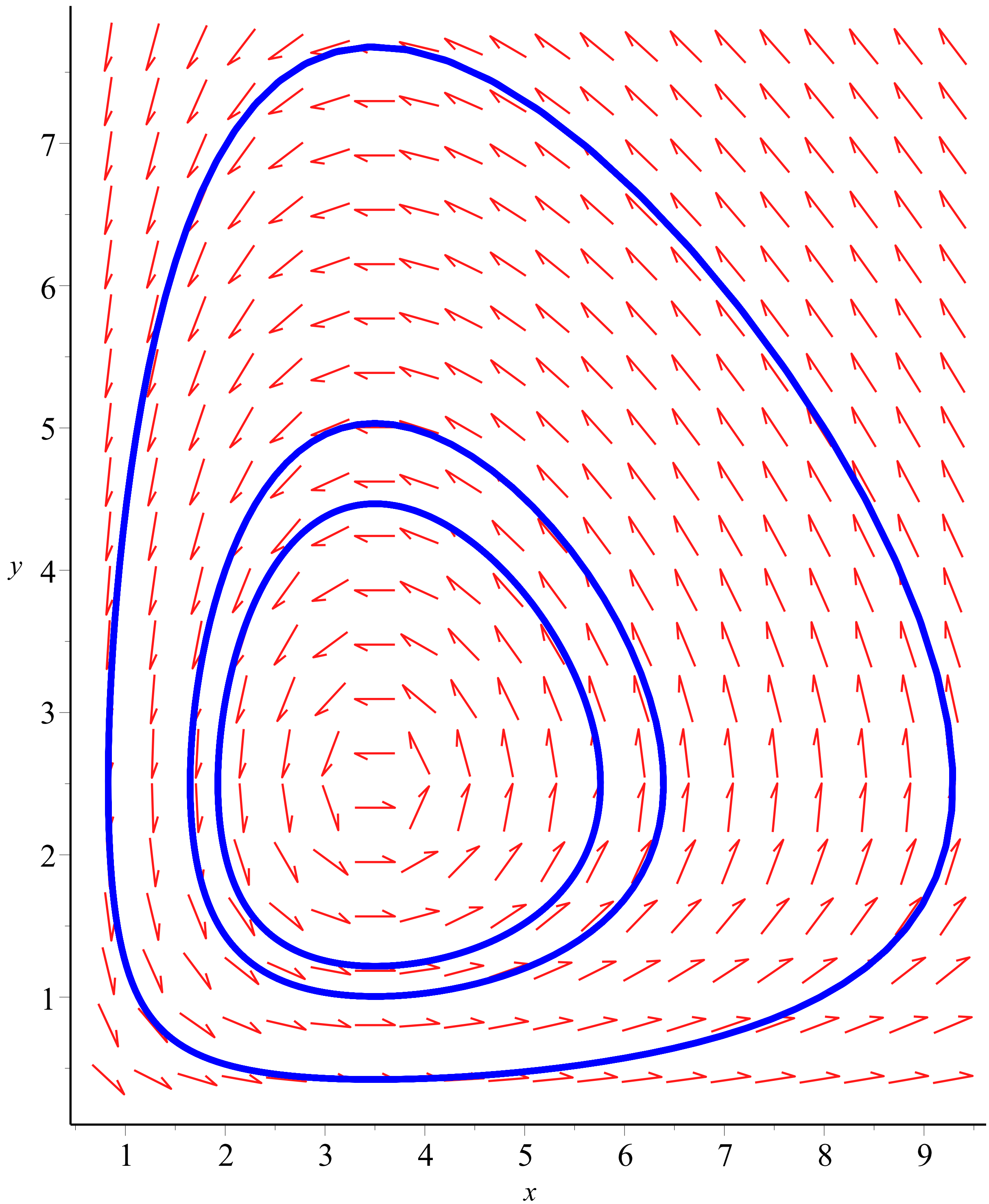
```
> h_syst:=eq1,eq2;
```

$$h\_syst := \frac{d}{dt} x(t) = 0.5\,x(t) - 0.2\,x(t)\,y(t), \frac{d}{dt} y(t) = -0.7\,y(t) + 0.2\,x(t)\,y(t)$$

```
> gx:=DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> gy:=DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(gx,gy);
```




```
> DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0],[x(0)=
  2,y(0)=2],[x(0)=2,y(0)=4]],stepsize=0.1,linecolor=blue);
```




```
> E:=0.6;
```

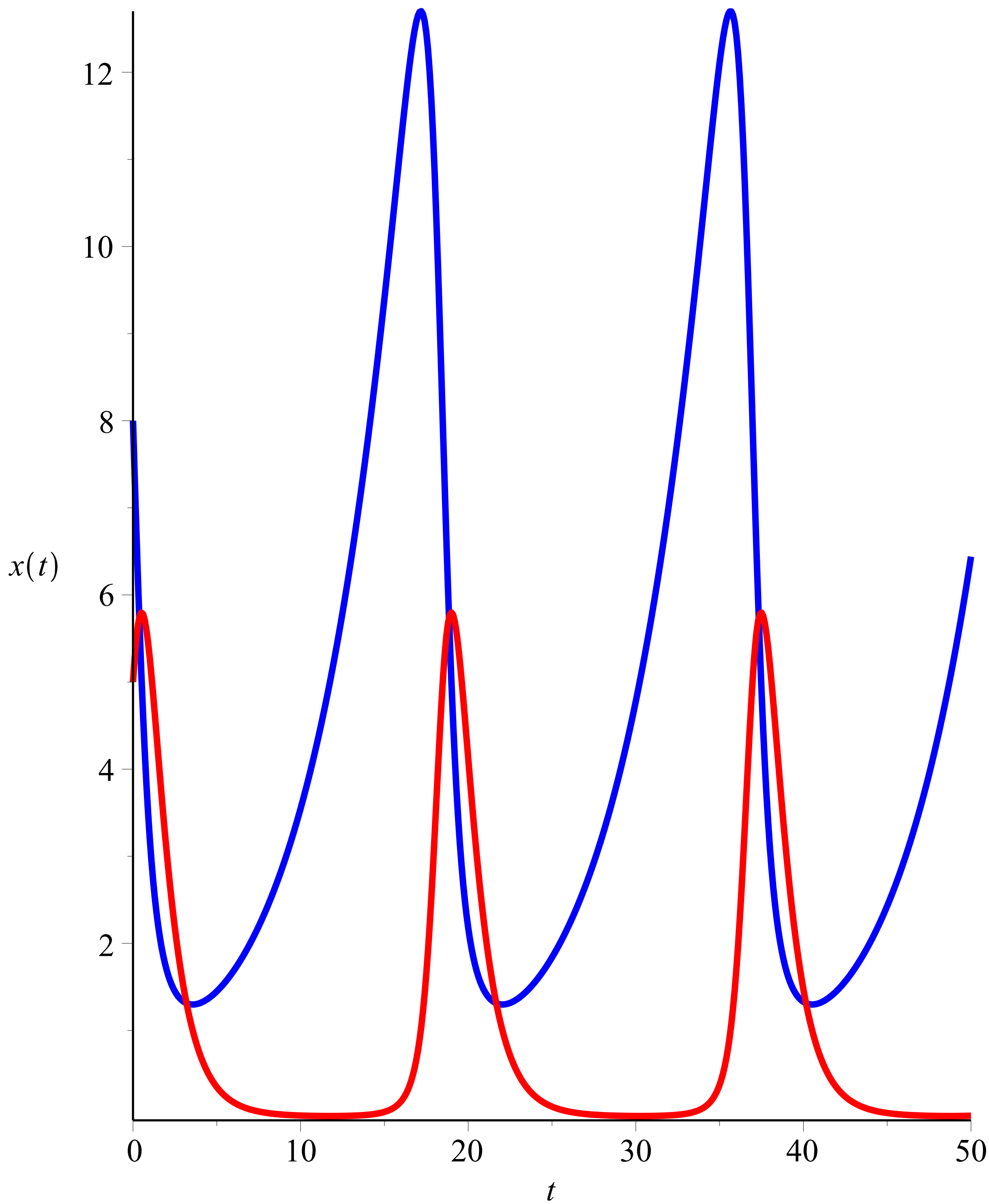
$$E := 0.6$$

```
> h_syst:=eq1,eq2;
```

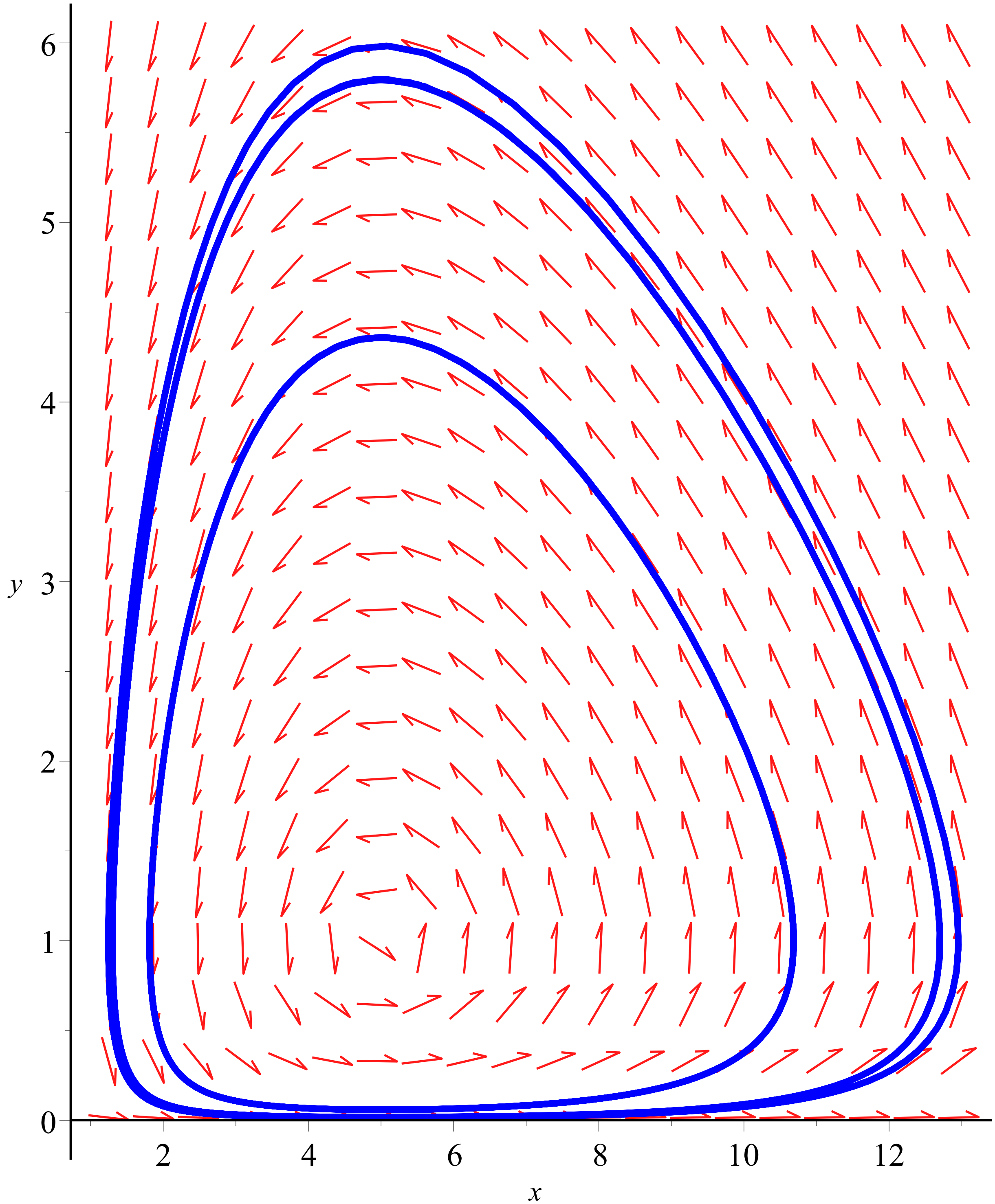
$$h\_syst := \frac{d}{dt}\, x(t) = 0.2\, x(t) - 0.2\, x(t)\, y(t),\ \frac{d}{dt}\, y(t) = -1.0\, y(t) + 0.2\, x(t)\, y(t)$$

```
> gx:=DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
```

```
> gy:=DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(gx,gy);
```




```
> DEplot([h_syst],[x(t),y(t)],t=0..50,[[x(0)=x0,y(0)=y0],[x(0)=
  2,y(0)=2],[x(0)=2,y(0)=4]],stepsize=0.1,linecolor=blue);
```



More realistic prey-predator models

Prey-predator model with intraspecies competition

Assumptions:

- the prey in the absence of any predation grows boundedly as in the logistical model (limited resources)
- the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the number of interactions between the two populations ($x\,y$)
- in the absence of any prey for sustenance the predator's death rate results in exponential decay
- the prey's contribution to the predators' growth rate is proportional to the available prey as well as to the size of the predator population, this is proportional to the number of interactions between the two populations
- in the presence of the prey there exists a competitions between the predators for the prey, this is proporitonal with the number of interaction of predators y^2

$$\frac{d}{dt}\,x(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\,x(t)\,y(t)$$
$$\frac{d}{dt}\,y(t) = -\gamma y(t) + \delta\,x(t)\,y(t) - \beta\,y(t)^2$$
$$x(0) = x_0$$
$$y(0) = y_0$$

where r, K, α, γ, δ, β are positive.

```
> restart:with(DEtools):
> with(linalg):
> with(plots):
> eq1:=diff(x(t),t)=r*x(t)*(1-x(t)/K)-alpha*x(t)*y(t);
```

$$eq1 := \frac{d}{dt}\,x(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\,x(t)\,y(t)$$

```
> eq2:=diff(y(t),t)=-gamma*y(t)+delta*x(t)*y(t)-beta*y(t)^2;
```

$$eq2 := \frac{d}{dt}\,y(t) = -\gamma y(t) + \delta\,x(t)\,y(t) - \beta\,y(t)^2$$

```
> pp_syst:=eq1,eq2;
```

$$pp\_syst := \frac{d}{dt}\,x(t) = r\,x(t)\left(1 - \frac{x(t)}{K}\right) - \alpha\,x(t)\,y(t),\ \frac{d}{dt}\,y(t) = -\gamma y(t) + \delta\,x(t)\,y(t) - \beta\,y(t)^2$$

```
> f1:=(x,y)->r*x*(1-x/K)-alpha*x*y;
```

$$f1 := (x, y) \to r\,x\left(1 - \frac{x}{K}\right) - \alpha\,x\,y$$

```
> f2:=(x,y)->-gamma*y+delta*x*y-beta*y^2;
```

$$f2 := (x, y) \to -\gamma y + \delta\,x\,y - \beta\,y^2$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EqP := \{x=0, y=0\},\ \left\{x=0, y=-\frac{\gamma}{\beta}\right\},\ \{x=K, y=0\},\ \left\{x=\frac{K\,(\alpha\gamma+\beta r)}{K\,\alpha\,\delta+\beta\,r}, y=\frac{r\,(K\,\delta-\gamma)}{K\,\alpha\,\delta+\beta\,r}\right\}$$

The equilibrium point $(0;-\frac{\gamma}{\beta})$ is not positive, so it is not realistic and the equilibrium point (

$\left(\frac{K(\beta r+\alpha\gamma)}{\delta K\alpha+\beta r}, \frac{r(-\gamma+\delta K)}{\delta K\alpha+\beta r}\right)$ is positive iff $\delta K-\gamma>0$.

$$y=0$$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} r\left(1-\frac{x}{K}\right)-\frac{r x}{K}-\alpha y & -\alpha x \\ y\delta & -2\beta y+\delta x-\gamma \end{bmatrix}$$

```
> J1:=subs(EqP[1,1],EqP[1,2],eval(J));
```

$$J1 := \begin{bmatrix} r & 0 \\ 0 & -\gamma \end{bmatrix}$$

```
> eigenvals(J1);
```

$$r, -\gamma$$

Since r > 0 this implies that (0;0) is unstable.

```
> J2:=subs(EqP[3,1],EqP[3,2],eval(J));
```

$$J2 := \begin{bmatrix} -r & -\alpha K \\ 0 & K\delta-\gamma \end{bmatrix}$$

```
> eigenvals(J2);
```

$$-r, K\delta-\gamma$$

If $\delta K-\gamma<0$ then (K;0) is local asimptotically stable.
If $\delta K-\gamma>0$ then (K;0) is unstable.

```
> J3:=subs(EqP[4,1],EqP[4,2],eval(J));
```

$$J3 := \left[\left[r\left(1-\frac{\alpha\gamma+\beta r}{K\alpha\delta+\beta r}\right)-\frac{r(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r}-\frac{\alpha r(K\delta-\gamma)}{K\alpha\delta+\beta r}, -\frac{\alpha K(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r}\right], \left[\frac{r(K\delta-\gamma)\delta}{K\alpha\delta+\beta r}, -\frac{2\beta r(K\delta-\gamma)}{K\alpha\delta+\beta r}+\frac{\delta K(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r}-\gamma\right]\right]$$

```
> J3[1,1]:=factor(expand(J3[1,1]));
```

$$J3_{1,1} := -\frac{r(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r}$$

```
> J3[2,2]:=factor(expand(J3[2,2]));
```

$$J3_{2,2} := -\frac{\beta r(K\delta-\gamma)}{K\alpha\delta+\beta r}$$

```
> eval(J3);
```

$$\begin{bmatrix} -\frac{r(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r} & -\frac{\alpha K(\alpha\gamma+\beta r)}{K\alpha\delta+\beta r} \\ \frac{r(K\delta-\gamma)\delta}{K\alpha\delta+\beta r} & -\frac{\beta r(K\delta-\gamma)}{K\alpha\delta+\beta r} \end{bmatrix}$$

```
> eigenvals(J3);
```

$$-\frac{1}{2}\frac{1}{K\alpha\delta+\beta r}\Big(\beta r\delta K+\alpha\gamma r-\beta\gamma r+\beta r^2 - \big(-4K^2\alpha^2\delta^2\gamma r-4K^2\alpha\beta\delta^2 r^2+K^2\beta^2\delta^2 r^2+4K\alpha^2\delta\gamma^2 r+2K\alpha\beta\delta\gamma r^2 - 2K\beta^2\delta\gamma r^2-2K\beta^2\delta r^3+\alpha^2\gamma^2 r^2+2\alpha\beta\gamma^2 r^2+2\alpha\beta\gamma r^3+\beta^2\gamma^2 r^2+2\beta^2\gamma r^3+\beta^2 r^4\big)^{1/2}\Big), -\frac{1}{2}\frac{1}{K\alpha\delta+\beta r}\Big(\beta r\delta K+\alpha\gamma r-\beta\gamma r+\beta r^2 + \big(-4K^2\alpha^2\delta^2\gamma r-4K^2\alpha\beta\delta^2 r^2+K^2\beta^2\delta^2 r^2+4K\alpha^2\delta\gamma^2 r+2K\alpha\beta\delta\gamma r^2 - 2K\beta^2\delta\gamma r^2-2K\beta^2\delta r^3+\alpha^2\gamma^2 r^2+2\alpha\beta\gamma^2 r^2+2\alpha\beta\gamma r^3+\beta^2\gamma^2 r^2+2\beta^2\gamma r^3+\beta^2 r^4\big)^{1/2}\Big)$$

```
> (charpoly(J3,lambda))=0;
```

$$\frac{K\alpha\delta\gamma r+K\alpha\delta\lambda^2+K\beta\delta\lambda r+K\beta\delta r^2-\alpha\gamma^2 r+\alpha\gamma\lambda r-\beta\gamma\lambda r-\beta\gamma r^2+\beta\lambda^2 r+\beta\lambda r^2}{K\alpha\delta+\beta r} = 0$$

```
> eql:=(delta*K*alpha+beta*r)*(charpoly(J3,lambda))=0;
```

$$eql := K\alpha\delta\gamma r+K\alpha\delta\lambda^2+K\beta\delta\lambda r+K\beta\delta r^2-\alpha\gamma^2 r+\alpha\gamma\lambda r-\beta\gamma\lambda r-\beta\gamma r^2+\beta\lambda^2 r + \beta\lambda r^2 = 0$$

```
> collect(eql,lambda);
```

$$(K\alpha\delta+\beta r)\lambda^2+(K\beta\delta r+\alpha\gamma r-\beta\gamma r+\beta r^2)\lambda+K\alpha\delta\gamma r+K\beta\delta r^2-\alpha\gamma^2 r-\beta\gamma r^2=0$$

```
> a1:=coeff(lhs(eql),lambda,2);
```

$$a1 := K\alpha\delta+\beta r$$

```
> a2:=factor(coeff(lhs(eql),lambda,1));
```

$$a2 := r(K\beta\delta+\alpha\gamma-\beta\gamma+\beta r)$$

```
> a3:=factor(coeff(lhs(eql),lambda,0));
```

$$a3 := r(\alpha\gamma+\beta r)(K\delta-\gamma)$$

The necessary and sufficient condition to have $\mathrm{Re}(\lambda) < 0$ is $\frac{a2}{a1} < 0$ and $\frac{a3}{a1} > 0$. In our case $a1 > 0$, thus if $a2 < 0$ and $a3 > 0$ then $\mathrm{Re}(\lambda) < 0$.

```
> a2<0;
```

$$r(K\beta\delta+\alpha\gamma-\beta\gamma+\beta r) < 0$$

```
> a3>0;
```

$$0 < r(\alpha\gamma+\beta r)(K\delta-\gamma)$$

a2 < 0 implies that $\frac{r}{\gamma}+\frac{\delta K}{\gamma}+\frac{\alpha}{\beta}<1$

a3 > 0 implies that $\delta K-\gamma>0$

Let's consider the case r = 0.3, K = 5, $\alpha = 0.2$, $\gamma = 0.6$, $\delta = 0.1$, $\beta = 0.2$, $x_0 = 3$, $y_0 = 1$

```
> f1:=(x,y,r,K,alpha)->r*x*(1-x/K)-alpha*x*y;
```

$$f1 := (x, y, r, K, \alpha) \rightarrow r\,x\left(1-\frac{x}{K}\right)-\alpha\,x\,y$$

```
> f2:=(x,y,gamma,delta,beta)->-gamma*y+delta*x*y-beta*y^2;;
```

$$f2 := (x, y, \gamma, \delta, \beta) \rightarrow -\gamma y+\delta\,x\,y-\beta\,y^2$$

```
> eq1:=diff(x(t),t)=f1(x(t),y(t),0.3,5,0.2);;
```

$$eq1 := \frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = 0.3\,x(t)\left(1-\frac{1}{5}\,x(t)\right)-0.2\,x(t)\,y(t)$$

```
> eq2:=diff(y(t),t)=f2(x(t),y(t),0.6,0.1,0.2);
```

$$eq2 := \frac{\mathrm{d}}{\mathrm{d}t}\,y(t) = -0.6\,y(t)+0.1\,x(t)\,y(t)-0.2\,y(t)^2$$

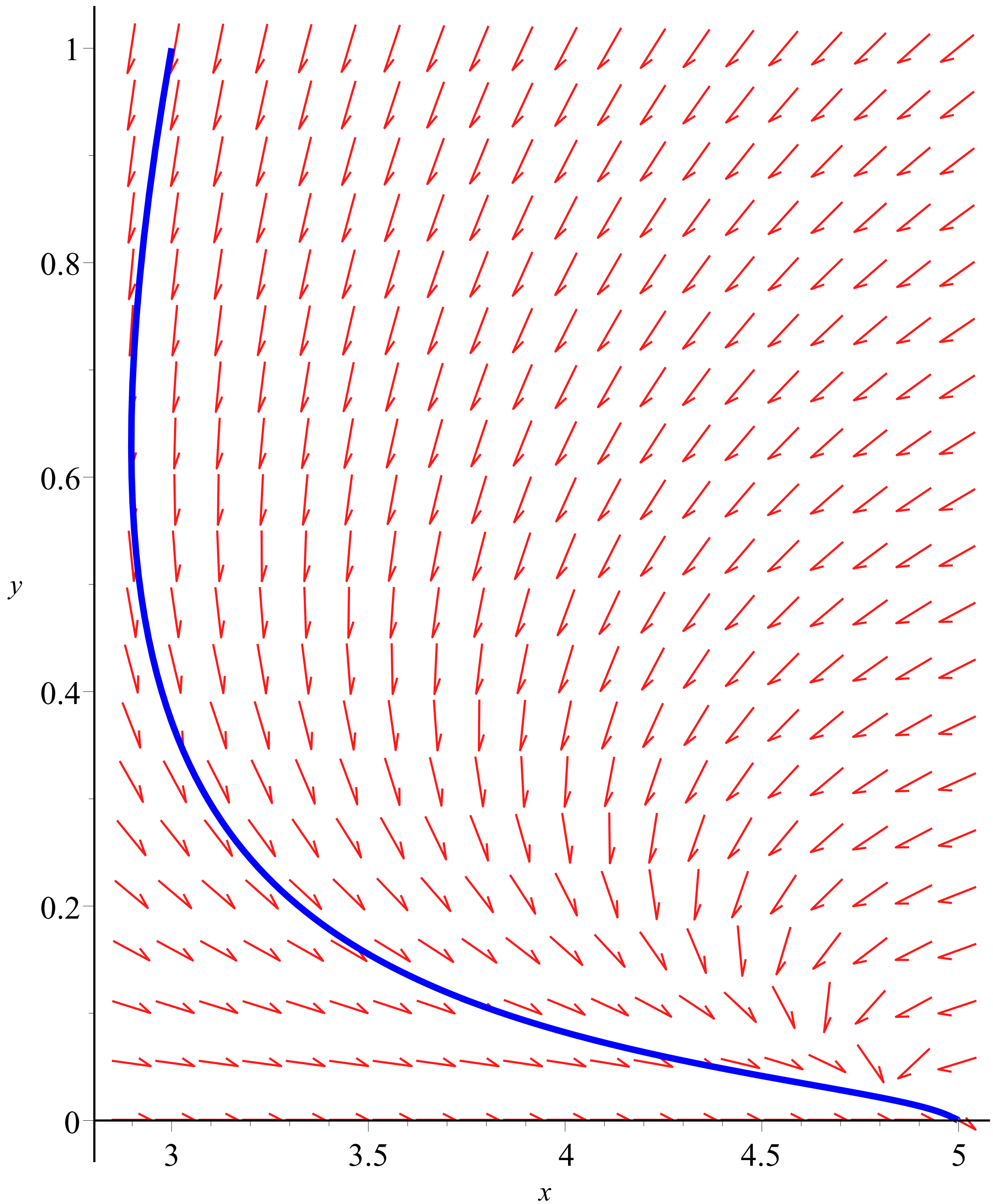
```
> syst1:=eq1,eq2;
```

$$syst1 := \frac{\mathrm{d}}{\mathrm{d}t}\,x(t) = 0.3\,x(t)\left(1-\frac{1}{5}\,x(t)\right)-0.2\,x(t)\,y(t),\ \frac{\mathrm{d}}{\mathrm{d}t}\,y(t) = -0.6\,y(t)+0.1\,x(t)\,y(t)-0.2\,y(t)^2$$

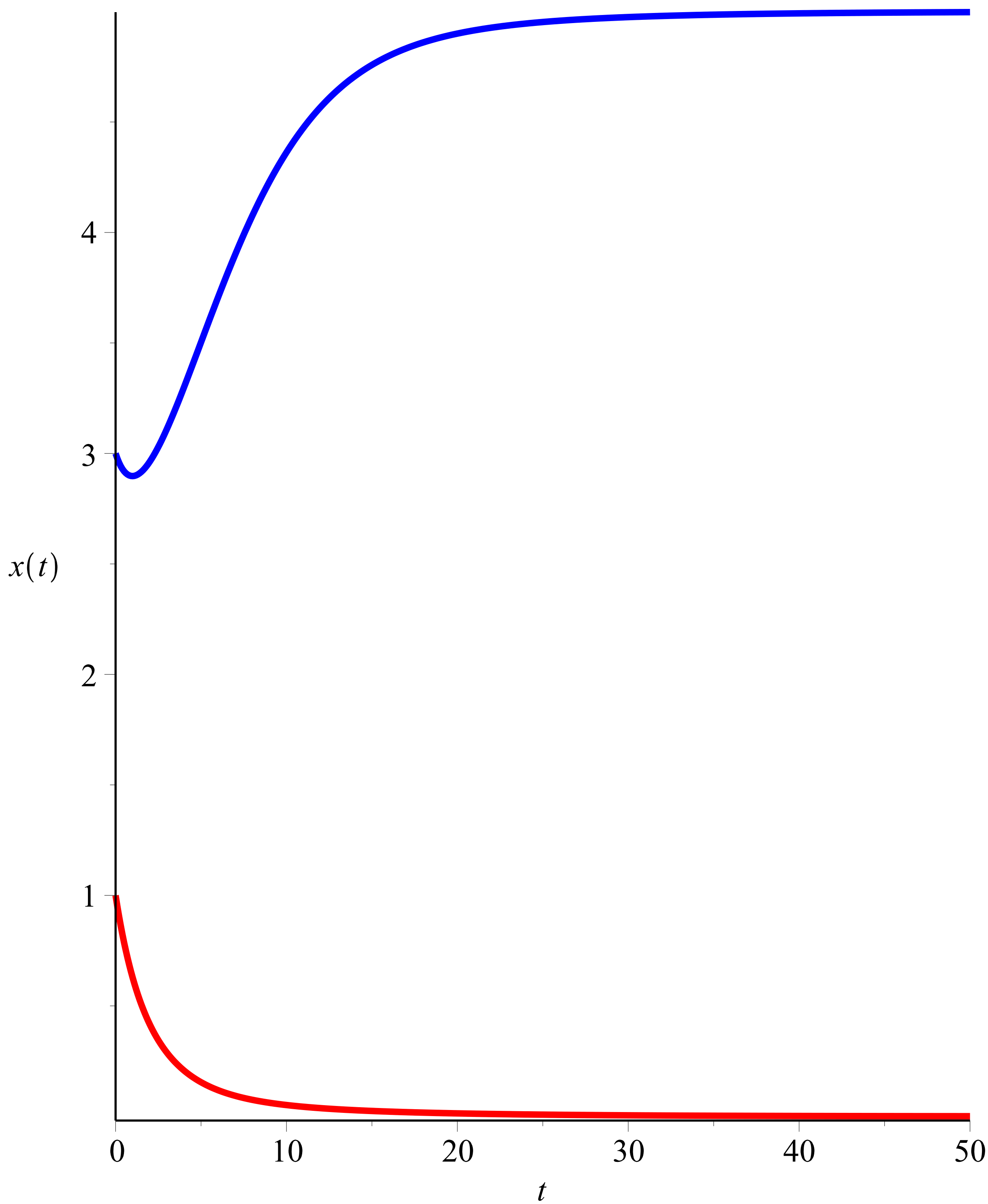
```
> EqP:=solve({f1(x,y,0.3,5,0.2)=0,f2(x,y,0.6,0.1,0.2)=0},{x,y})
  ;
```

$$EqP := \{x=0.,\ y=0.\},\ \{x=0.,\ y=-3.\},\ \{x=5.,\ y=0.\},\ \{x=5.625000000,\ y=-0.1875000000\}$$

```
> DEplot([syst1],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=blue);
```




```
> gx:=DEplot([syst1],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> gy:=DEplot([syst1],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(gx,gy);
```



Let's consider the case r = 0.3, K = 5, α = 0.2, γ= 0.3, δ = 0.2, β = 0.2, $x_0 = 3$, $y_0 = 1$

```
> eq1:=diff(x(t),t)=f1(x(t),y(t),0.3,5,0.2);
```

$$eq1 := \frac{d}{dt}\, x(t) = 0.3\, x(t) \left(1 - \frac{1}{5}\, x(t)\right) - 0.2\, x(t)\, y(t)$$

```
> eq2:=diff(y(t),t)=f2(x(t),y(t),0.3,0.2,0.2);
```

$$eq2 := \frac{d}{dt}\, y(t) = -0.3\, y(t) + 0.2\, x(t)\, y(t) - 0.2\, y(t)^2$$

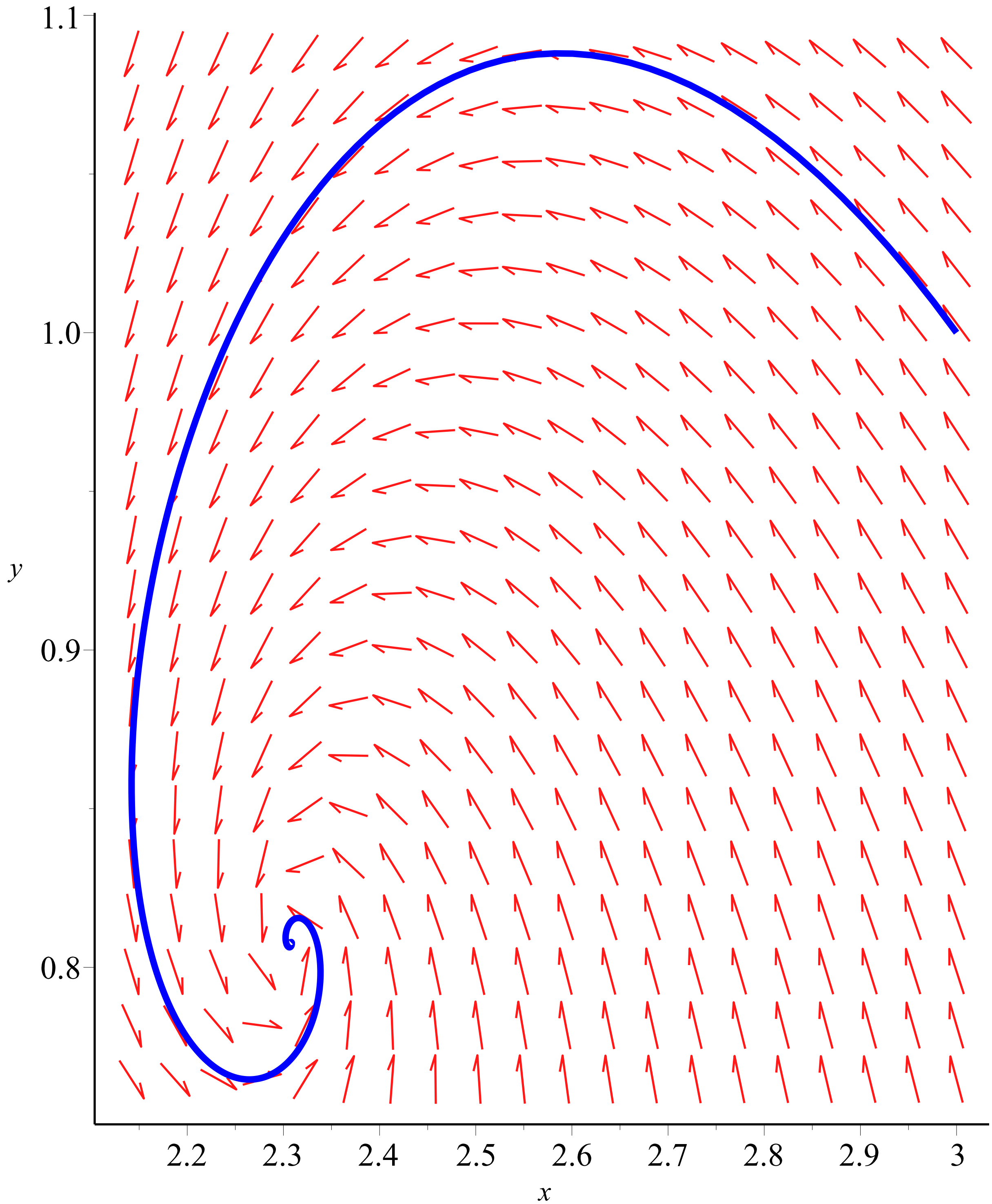
```
> syst2:=eq1,eq2;
```

$$syst2 := \frac{d}{dt}\, x(t) = 0.3\, x(t) \left(1 - \frac{1}{5}\, x(t)\right) - 0.2\, x(t)\, y(t),\ \frac{d}{dt}\, y(t) = -0.3\, y(t) + 0.2\, x(t)\, y(t) - 0.2\, y(t)^2$$

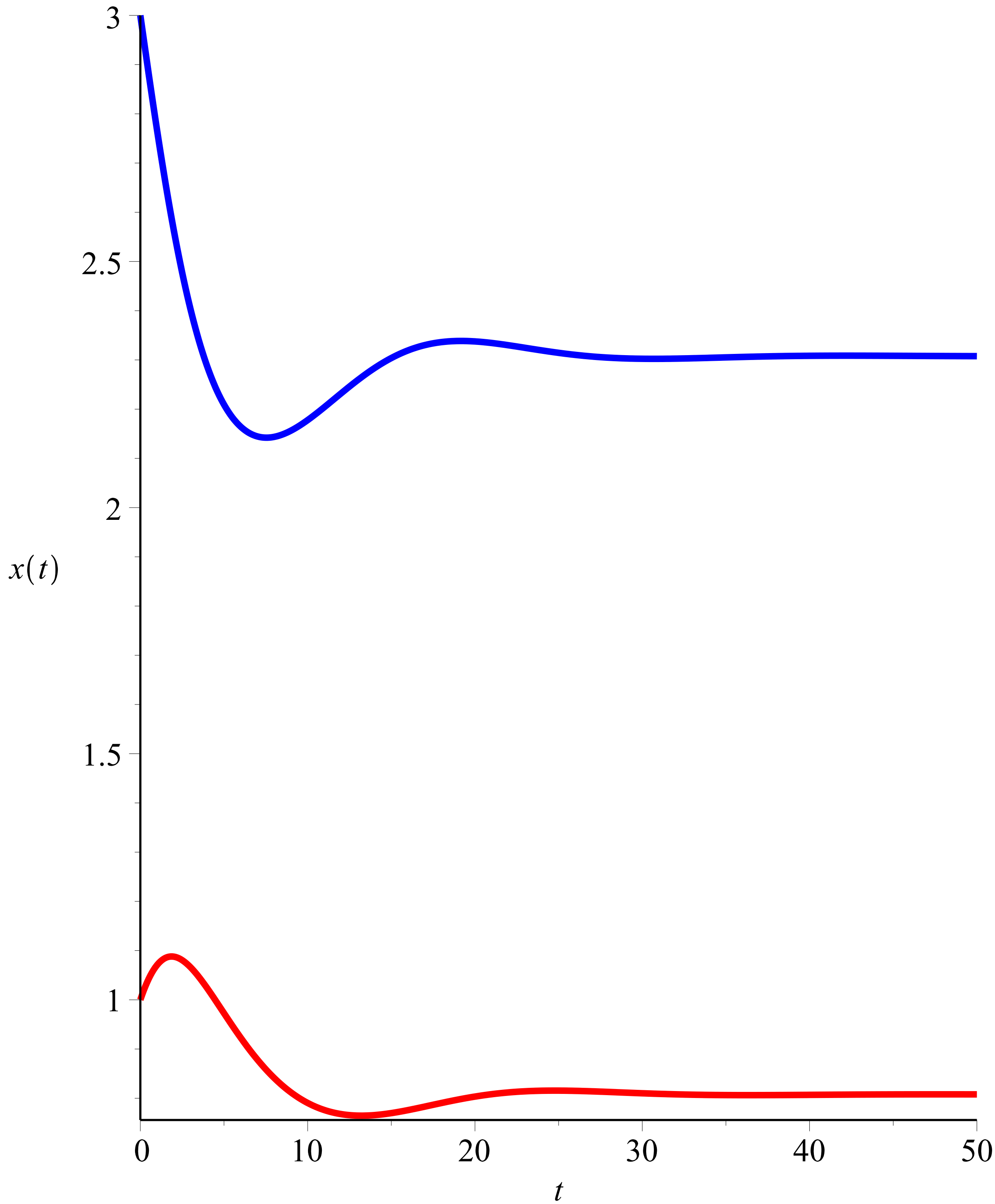
```
> EqP:=solve({f1(x,y,0.3,5,0.2)=0,f2(x,y,0.3,0.2,0.2)=0},{x,y})
  ;
```

$$EqP := \{x = 0.,\ y = 0.\},\ \{x = 0.,\ y = -1.500000000\},\ \{x = 5.,\ y = 0.\},\ \{x = 2.307692308,\ y = 0.8076923077\}$$

```
> DEplot([syst2],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=blue);
```




```
> gx:=DEplot([syst2],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> gy:=DEplot([syst2],[x(t),y(t)],t=0..50,[[x(0)=3,y(0)=1]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(gx,gy);
```



Prey-predator model with **food saturation**

In case of prey population we can assume that develops according to the logistical model in the absence of the predator and its increasing is reduced by a predation term

$$\frac{d}{dt}x(t) = r\,x(t)\left(1-\frac{x(t)}{K}\right) - x(t)\,y(t)\,R(x(t))$$

$x\,R(x)$ is the predation term. The predation term, which is the functional response of the predator to change in the prey density, generally shows some saturation effect instead of a predator response of

$b\,x$ or $\alpha\,x$ (which are linear) as in the above models. A suitable choise for R(x) is

$$R(x) = \frac{a\,x}{x+b}$$

In the case of predator equation we can take

$$\frac{d}{dt}y(t) = p\,y(t)\left(1-\frac{h\,y(t)}{x(t)}\right)$$

The term $h\,\frac{y^2}{x}$ measures the predator intraspecies competition.

Thus we obtain the following prey-predator model (Holling–Tanner model)

$$\frac{d}{dt}x(t) = r\,x(t)\left(1-\frac{x(t)}{K}\right) - \frac{a\,x(t)\,y(t)}{b+x(t)}$$

$$\frac{d}{dt}y(t) = p\,y(t)\left(1-\frac{h\,y(t)}{x(t)}\right)$$

```
> restart:with(DEtools):with(linalg):with(plots):
> eq1:=diff(x(t),t) = r*x(t)*(1-x(t)/K)-a*x(t)*y(t)/(b+x(t));
```

$$eq1 := \frac{d}{dt}x(t) = r\,x(t)\left(1-\frac{x(t)}{K}\right) - \frac{a\,x(t)\,y(t)}{b+x(t)}$$

```
> eq2:=diff(y(t),t) = p*y(t)*(1-h*y(t)/x(t));
```

$$eq2 := \frac{d}{dt}y(t) = p\,y(t)\left(1-\frac{h\,y(t)}{x(t)}\right)$$

```
> f1:=(x,y)->r*x*(1-x/K)-a*x*y/(b+x);
```

$$f1 := (x,y) \to r\,x\left(1-\frac{x}{K}\right) - \frac{a\,x\,y}{b+x}$$

```
> f2:=(x,y)->p*y*(1-h*y/x);
```

$$f2 := (x,y) \to p\,y\left(1-\frac{h\,y}{x}\right)$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$$EqP := \{x=K, y=0\}, \left\{x = RootOf\left(h\,r\,\_Z^2 + (-K\,h\,r + b\,h\,r + K\,a)\,\_Z - K\,b\,h\,r\right), y = \frac{RootOf\left(h\,r\,\_Z^2 + (-K\,h\,r + b\,h\,r + K\,a)\,\_Z - K\,b\,h\,r\right)}{h}\right\}$$

```
> eq3:=f1(h*y,y)=0;
```

$$eq3 := r\,h\,y\left(1-\frac{h\,y}{K}\right)-\frac{a\,h\,y^2}{h\,y+b}=0$$

```
> simplify(eq3);
```

$$-\frac{h\,y\,(h^2\,r\,y^2-K\,h\,r\,y+b\,h\,r\,y+K\,a\,y-K\,b\,r)}{K\,(h\,y+b)}=0$$

```
> solve(eq3,y);
```

$0,$

$$-\frac{1}{2}\,\frac{1}{h^2\,r}\Big(-K\,h\,r+b\,h\,r+K\,a-\sqrt{K^2\,h^2\,r^2+2\,K\,b\,h^2\,r^2+b^2\,h^2\,r^2-2\,K^2\,a\,h\,r+2\,K\,a\,b\,h\,r+K^2\,a^2}\Big),$$
$$-\frac{1}{2}\,\frac{1}{h^2\,r}\Big(-K\,h\,r+b\,h\,r+K\,a+\sqrt{K^2\,h^2\,r^2+2\,K\,b\,h^2\,r^2+b^2\,h^2\,r^2-2\,K^2\,a\,h\,r+2\,K\,a\,b\,h\,r+K^2\,a^2}\Big)$$

```
> a:=1;h:=4;r:=0.8;
```

$$a := 1$$
$$h := 4$$
$$r := 0.8$$

```
> a/(h*r);
```

$$0.3125000000$$

```
> K:=5;b:=1;p:=0.2;
```

$$K := 5$$
$$b := 1$$
$$p := 0.2$$

```
> syst:=eq1,eq2;
```

$$syst := \frac{d}{dt}\,x(t)=0.8\,x(t)\left(1-\frac{1}{5}\,x(t)\right)-\frac{x(t)\,y(t)}{1+x(t)},\ \frac{d}{dt}\,y(t)=0.2\,y(t)\left(1-\frac{4\,y(t)}{x(t)}\right)$$

```
> f1(x,y);f2(x,y);
```

$$0.8\,x\left(1-\frac{1}{5}\,x\right)-\frac{x\,y}{1+x}$$
$$0.2\,y\left(1-\frac{4\,y}{x}\right)$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$EqP := \{x=5.,\,y=0.\},\ \{x=3.765385341,\,y=0.9413463354\},\ \{x=-1.327885341,\,y=-0.3319713354\}$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} 0.8-0.3200000000\,x-\frac{y}{1+x}+\frac{x\,y}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{0.8\,y^2}{x^2} & 0.2-\frac{1.6\,y}{x} \end{bmatrix}$$

```
> J1:=subs(EqP[1,1],EqP[1,2],eval(J));
```

$$J1 := \begin{bmatrix} -0.800000000 & -0.8333333333 \\ 0. & 0.2 \end{bmatrix}$$

```
> eigenvals(J1);
```

$$-0.800000000000000, 0.200000000000000$$

```
> EqP[2,1],EqP[2,2];
```

$$x = 3.765385341, y = 0.9413463354$$

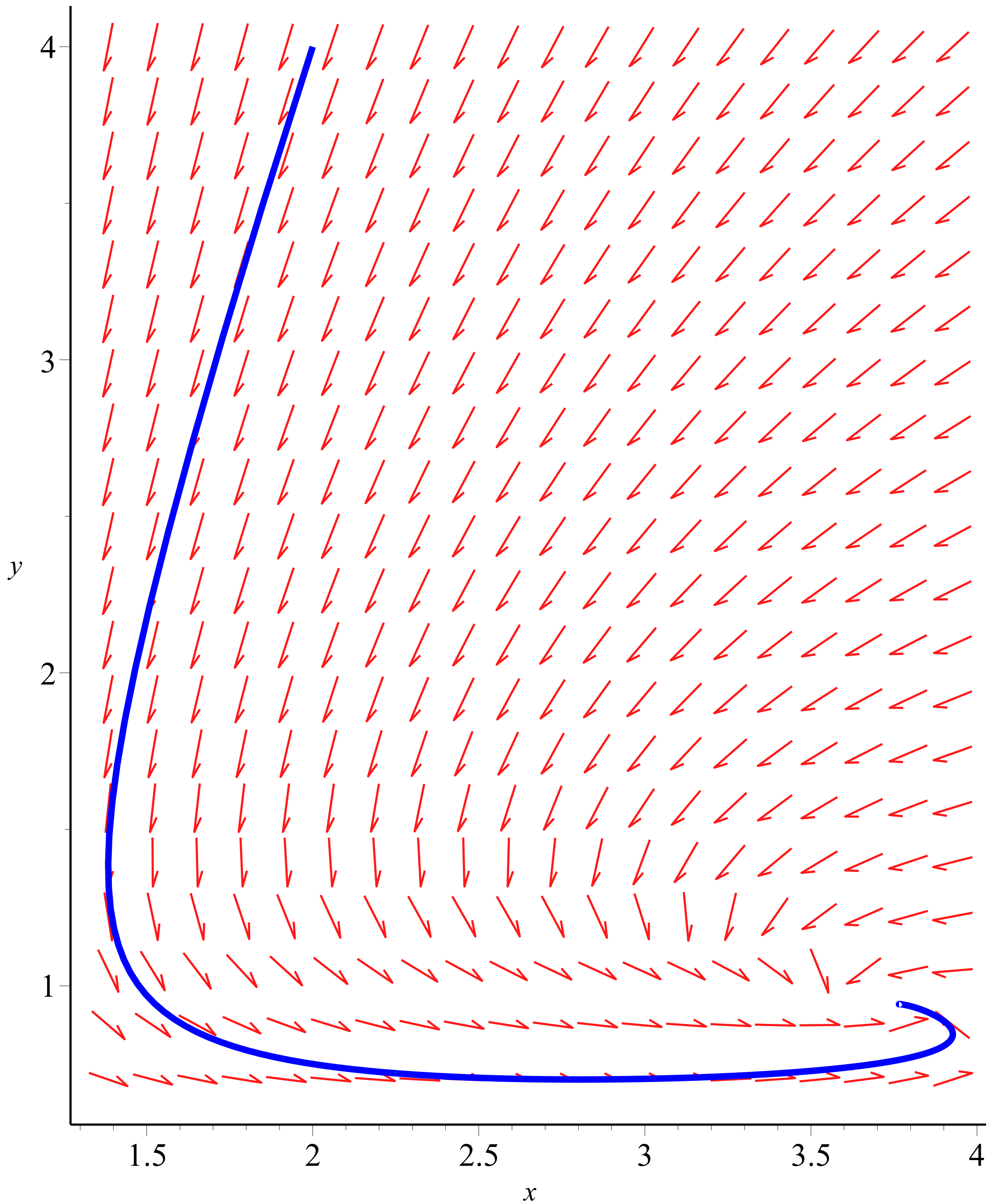
```
> J2:=subs(EqP[2,1],EqP[2,2],eval(J));
```

$$J2 := \begin{bmatrix} -0.4463760629 & -0.7901533814 \\ 0.05000000002 & -0.2000000000 \end{bmatrix}$$

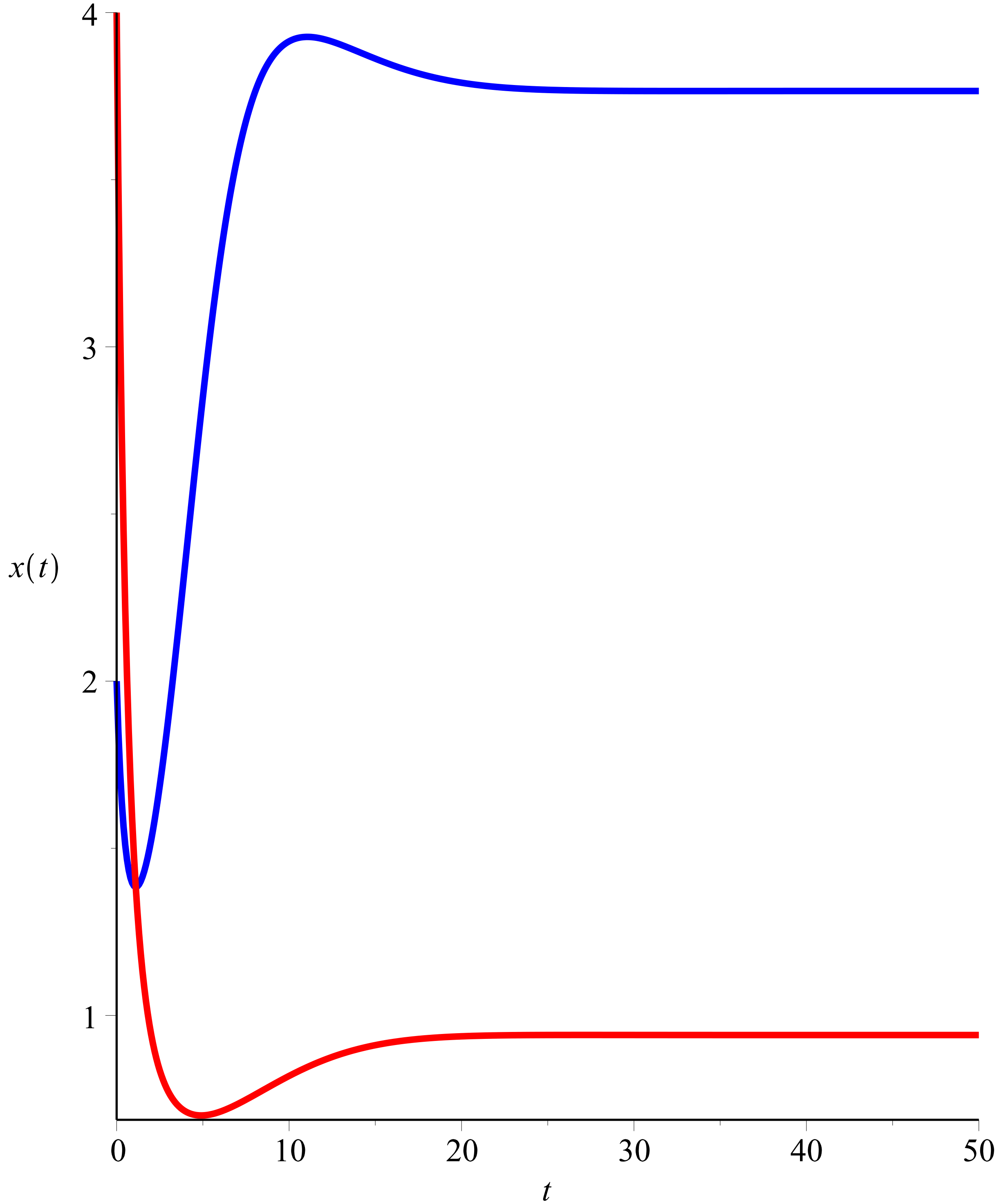
```
> eigenvals(J2);
```

$$-0.323188031450000 + 0.155988390572109\ I, -0.323188031450000 - 0.155988390572109\ I$$

```
> DEplot([syst],[x(t),y(t)],t=0..50,[[x(0)=2,y(0)=4]],stepsize=
  0.1,linecolor=blue);
```




```
> g1:=DEplot([syst],[x(t),y(t)],t=0..50,[[x(0)=2,y(0)=4]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> g2:=DEplot([syst],[x(t),y(t)],t=0..50,[[x(0)=2,y(0)=4]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(g1,g2);
```




```
> a:=1;h:=0.5;r:=1;
```

$$a := 1$$
$$h := 0.5$$
$$r := 1$$

```
> a/(h*r);
```

$$2.000000000$$

```
> K:=5;b:=1;p:=0.2;
```

$$K := 5$$
$$b := 1$$
$$p := 0.2$$

```
> syst:=eq1,eq2;
```

$$syst := \frac{d}{dt}\, x(t) = x(t)\left(1 - \frac{1}{5}\, x(t)\right) - \frac{x(t)\, y(t)}{1 + x(t)},\ \frac{d}{dt}\, y(t) = 0.2\, y(t)\left(1 - \frac{0.5\, y(t)}{x(t)}\right)$$

```
> f1(x,y);f2(x,y);
```

$$x\left(1 - \frac{1}{5}\, x\right) - \frac{x\, y}{1 + x}$$
$$0.2\, y\left(1 - \frac{0.5\, y}{x}\right)$$

```
> EqP:=solve({f1(x,y)=0,f2(x,y)=0},{x,y});
```

$EqP := \{x = 5., y = 0.\}, \{x = 0.7416573868, y = 1.483314774\}, \{x = -6.741657387, y = -13.48331477\}$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} 1 - \frac{2}{5}\, x - \frac{y}{1 + x} + \frac{x\, y}{(1 + x)^2} & -\frac{x}{1 + x} \\ \frac{0.10\, y^2}{x^2} & 0.2 - \frac{0.20\, y}{x} \end{bmatrix}$$

```
> J1:=subs(EqP[1,1],EqP[1,2],eval(J));
```

$$J1 := \begin{bmatrix} -1.000000000 & -0.8333333333 \\ 0. & 0.2 \end{bmatrix}$$

```
> eigenvals(J1);
```

$$-1., 0.200000000000000$$

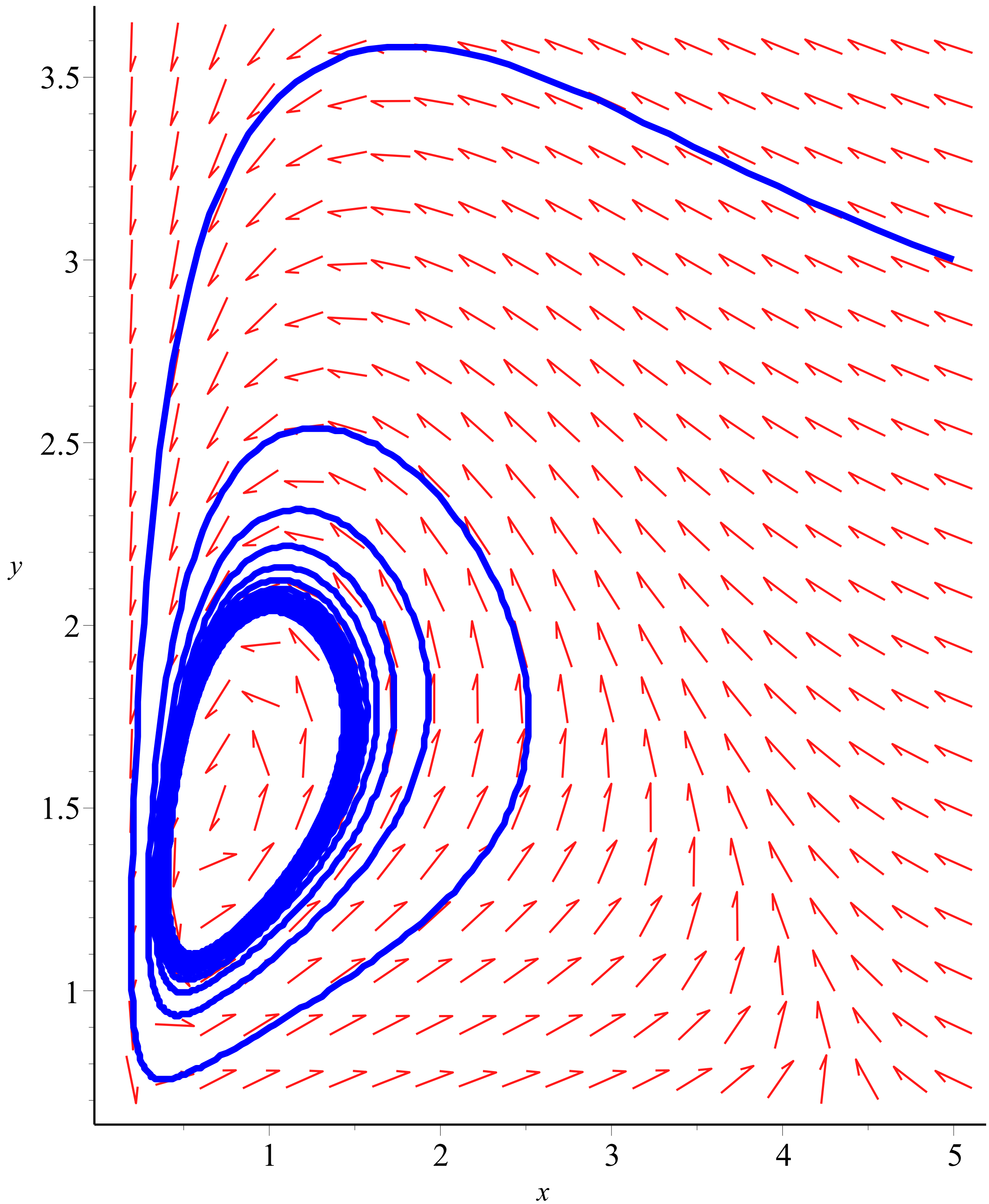
```
> J2:=subs(EqP[2,1],EqP[2,2],eval(J));
```

$$J2 := \begin{bmatrix} 0.2143381587 & -0.4258342613 \\ 0.4000000003 & -0.2000000000 \end{bmatrix}$$

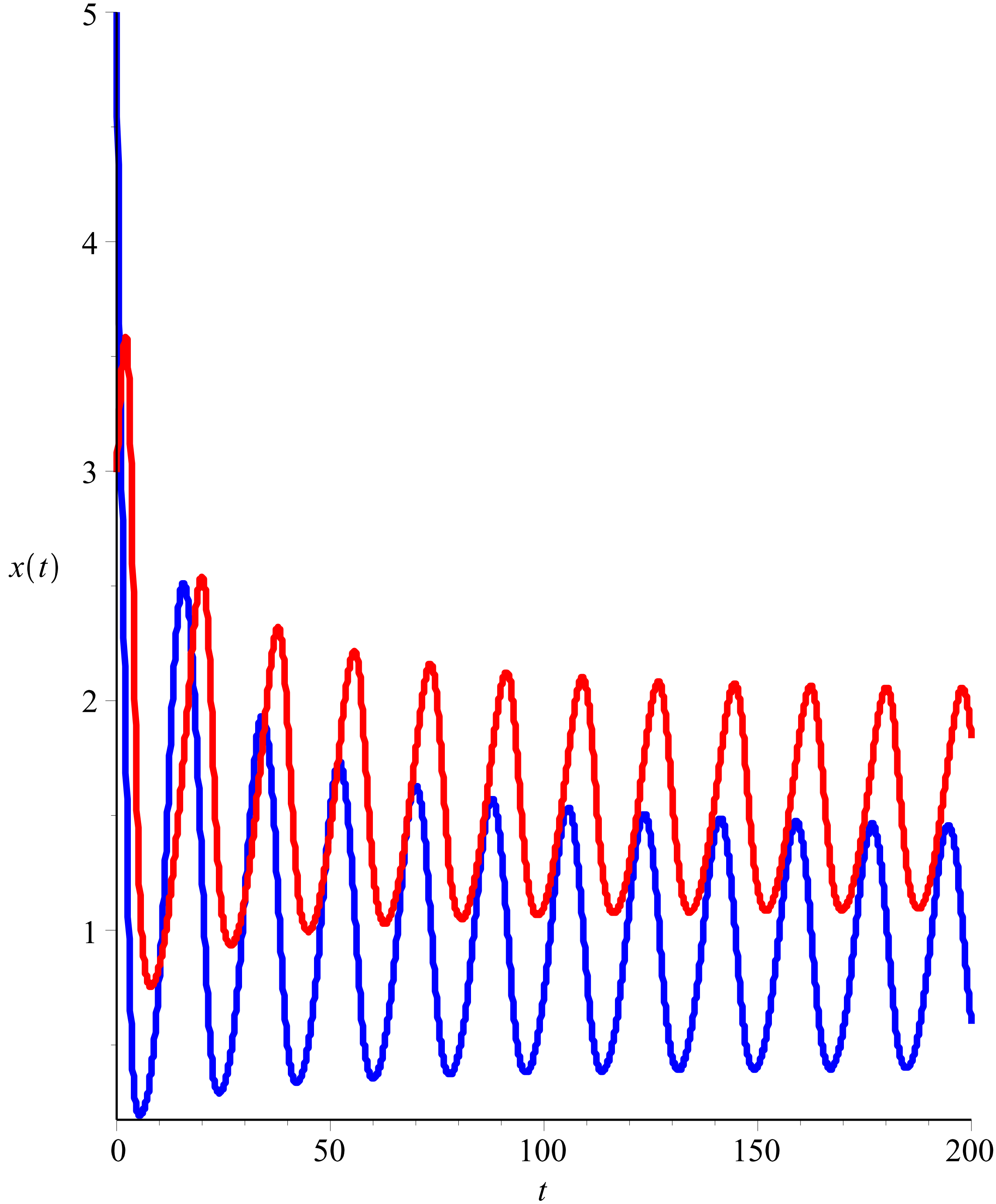
```
> eigenvals(J2);
```

$0.00716907935000000 + 0.356951925627281\, I, 0.00716907935000000 - 0.356951925627281\, I$

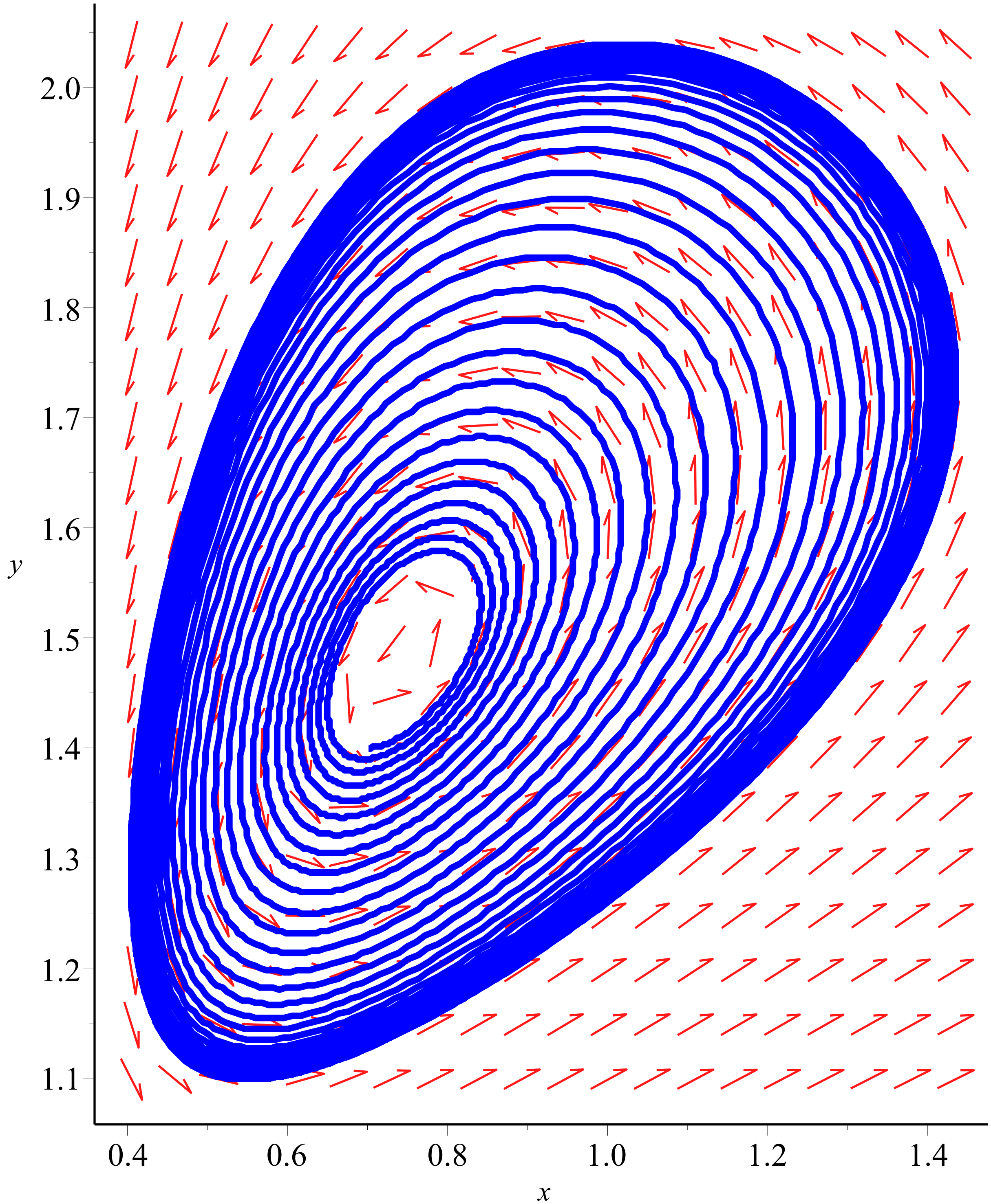
```
> DEplot([syst],[x(t),y(t)],t=0..1000,[[x(0)=5,y(0)=3]],
  stepsize=0.1,linecolor=blue);
```



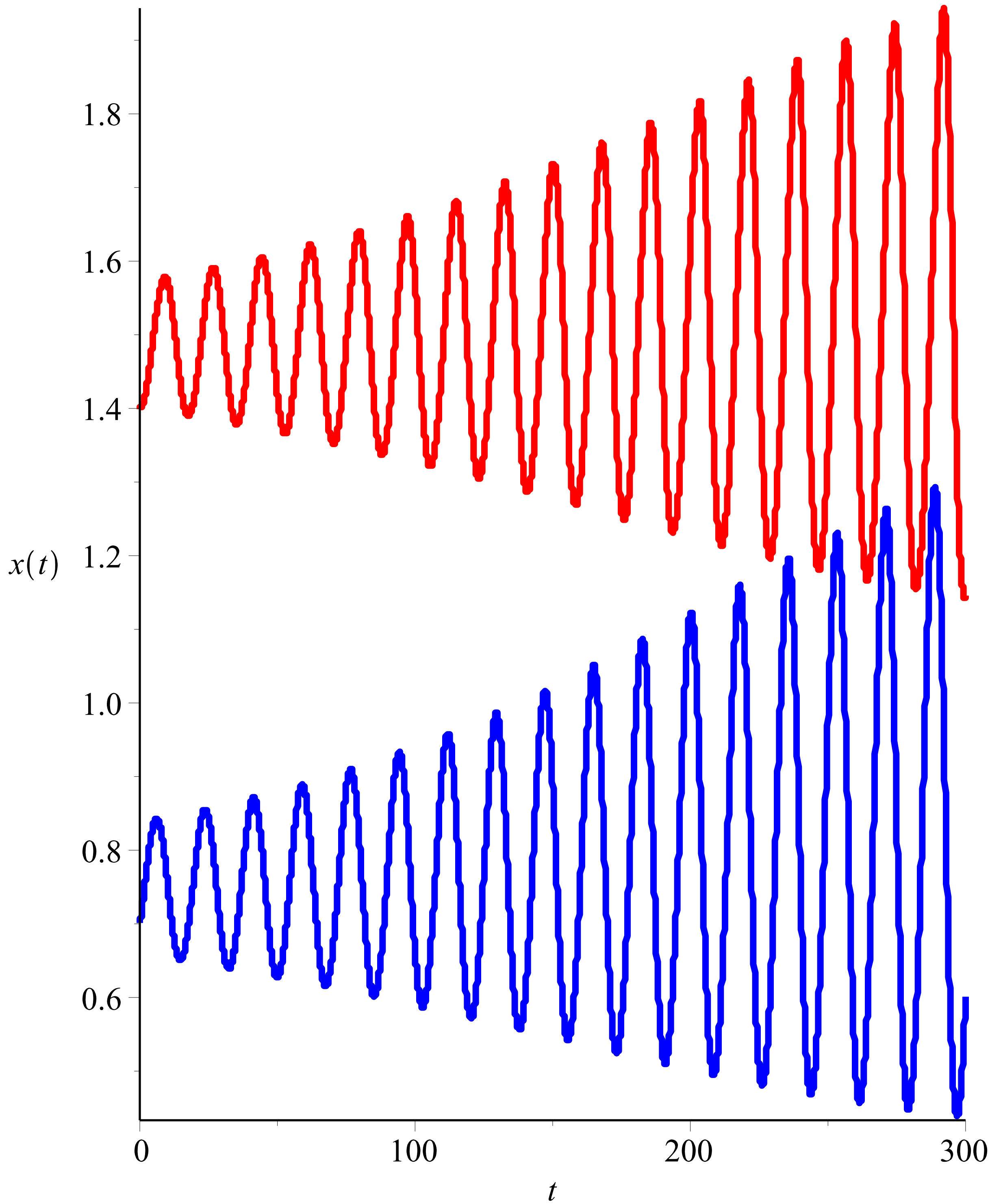

```
> g1:=DEplot([syst],[x(t),y(t)],t=0..200,[[x(0)=5,y(0)=3]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> g2:=DEplot([syst],[x(t),y(t)],t=0..200,[[x(0)=5,y(0)=3]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(g1,g2);
```




```
> DEplot([syst],[x(t),y(t)],t=0..1000,[[x(0)=0.7,y(0)=1.4]],
  stepsize=0.1,linecolor=blue);
```




```
> g1:=DEplot([syst],[x(t),y(t)],t=0..300,[[x(0)=0.7,y(0)=1.4]],
  stepsize=0.1,linecolor=blue,scene=[t,x(t)]):
> g2:=DEplot([syst],[x(t),y(t)],t=0..300,[[x(0)=0.7,y(0)=1.4]],
  stepsize=0.1,linecolor=red,scene=[t,y(t)]):
> display(g1,g2);
```



1.8
1.6
1.4
$x(t)$
1.2
1.0
0.8
0.6
0
100
200
300
t