▼ Mathematical models given by sytems of difference equations

▼ Model for a Population Divided into Two Age Classes

In the development of a population, the contribution to demographic growth is given by the adult population. To obtain a more realistic model that takes this aspect into account, we consider the population divided into two age classes:

x_n - the size of the young population n the season n

y_n - the size of the adult population in the season n

x_0, y_0 - the sizes of the two populations at the initial moment

Naturally, we can assume that the size of the young population in the next season is proportional to the size of the adult population in the previous season, that is.

$$x_{n+1} = b\, y_n$$

b - the fertility rate of the adult population.

From one season to the next, a certain fraction c of the young population moves into the adult class, and a certain fraction s of the adult population survives. This means that the adult population in the next season is given by $c \cdot x_n$, the young population that becomes adult, and $s \cdot y_n$, that is:

$$y_{n+1} = c\, x_n + s\, y_n$$

If $c = 1$, then all members of the young population become adults at the end of the season, while if $c < 1$, then only part of them become adults, and the rest die.
Thus, we obtain a homogeneous linear system of the form:

$$\begin{cases} {}^{-}x_{n+1} = b\, y_n \\ y_{n+1} = c\, x_n + s\, y_n \end{cases}$$

where $0 < b,\, 0 < c, s \leq 1$

We denote by

$$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & b \\ c & s \end{pmatrix}$$

then the system can be written as:
$$X_{n+1} = A \cdot X_n$$

$$X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The solution of the system, starting from the initial vector X_0 is of the form:

$$X_n = A^n \cdot X_0$$

The equilibrium point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable if

$\left|\lambda_{1,2}\right| < 1$ where $\lambda_{1,2}$ are the eigenvalues of matrix A

$$det\left(\lambda\, I_2 - A\right) = 0 \Rightarrow \lambda^2 - s\,\lambda - b\,c = 0$$

otherwise, it is unstable if at least one eigenvalue satisfies $\left|\lambda\right| > 1$

```
> with(linalg):
> A:=matrix([[0,b],[c,s]]);
```
$$A := \begin{bmatrix} 0 & b \\ c & s \end{bmatrix}$$

```
> charpoly(A,lambda);
```
$$-b\,c + \lambda^2 - \lambda\,s$$

```
> eigenvals(A);
```
$$\frac{1}{2}\,s + \frac{1}{2}\,\sqrt{4\,b\,c + s^2},\ \frac{1}{2}\,s - \frac{1}{2}\,\sqrt{4\,b\,c + s^2}$$

For the equation the equation
$$\lambda^2 = p_1\,\lambda + p_2$$
where $p_1 = s$ and $p_2 = b\,c$
We have:
$\quad \left|\lambda_{1,2}\right| < 1$ if and only if $\left|p_1\right| < 1 - p_2 < 2$, so

$|s| = s < 1 - b\cdot c < 2 \Rightarrow s + b\,c < 1$

```
> s:=1/3;b:=2;c:=1/4;s+b*c;
```
$$s := \frac{1}{3}$$
$$b := 2$$
$$c := \frac{1}{4}$$
$$\frac{5}{6}$$

```
> A:=matrix([[0,b],[c,s]]);
```
$$A := \begin{bmatrix} 0 & 2 \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

```
> charpoly(A,lambda);
```

$$\lambda^2 - \frac{1}{3}\lambda - \frac{1}{2}$$

```
> eigenvals(A);evalf(%);
```

$$\frac{1}{6} + \frac{1}{6}\sqrt{19}, \frac{1}{6} - \frac{1}{6}\sqrt{19}$$

$$0.8931498242, -0.5598164908$$

```
> f1:=(x,y)->b*y;
```

$$f1 := (x, y) \rightarrow b\, y$$

```
> f2:=(x,y)->c*x+s*y;
```

$$f2 := (x, y) \rightarrow c\, x + s\, y$$

```
> x[0]:=100;y[0]:=20;N:=100;
```
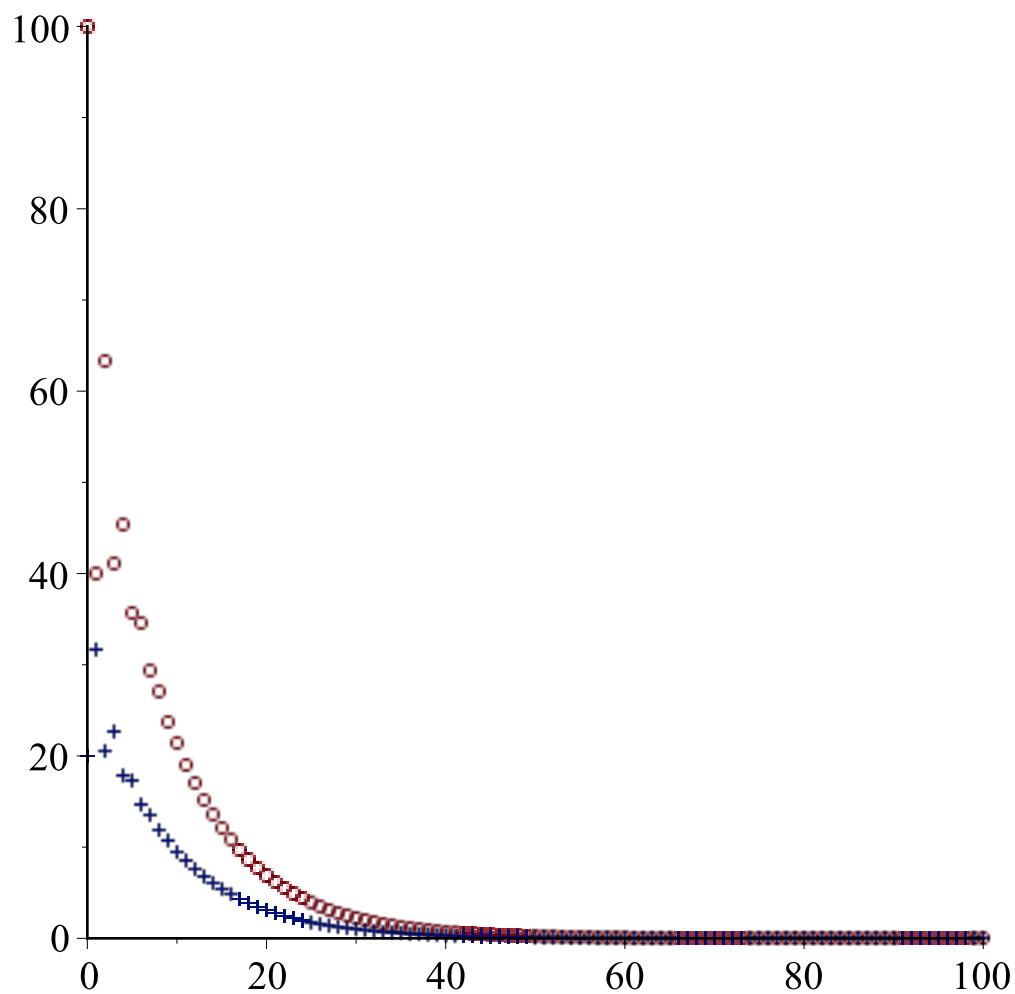
$$x_0 := 100$$

$$y_0 := 20$$

$$N := 100$$

```
> for i from 0 to N-1 do
    x[i+1]:=evalf(f1(x[i],y[i]));
    y[i+1]:=evalf(f2(x[i],y[i]));
  end do:
> plot([[[n,x[n]]$n=0..N],[[n,y[n]]$n=0..N]],style=point,
  symbol=[circle,cross]);
```

```
> s:=1/3;b:=2;c:=1/2;s+b*c;
```

$$s := \frac{1}{3}$$

$$b := 2$$

$$c := \frac{1}{2}$$

$$\frac{4}{3}$$

```
> A:=matrix([[0,b],[c,s]]);
```

$$A := \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

```
> charpoly(A,lambda);
```

$$\lambda^2 - \frac{1}{3}\lambda - 1$$

```
> eigenvals(A);evalf(%);
```

$$\frac{1}{6} + \frac{1}{6}\sqrt{37}, \ \frac{1}{6} - \frac{1}{6}\sqrt{37}$$

1.180460422, −0.8471270883

```
> f1:=(x,y)->b*y;
```

$$f1 := (x, y) \rightarrow b\,y$$

```
> f2:=(x,y)->c*x+s*y;
```

$$f2 := (x, y) \rightarrow c\,x + s\,y$$

```
> x[0]:=100;y[0]:=20;N:=30;
```
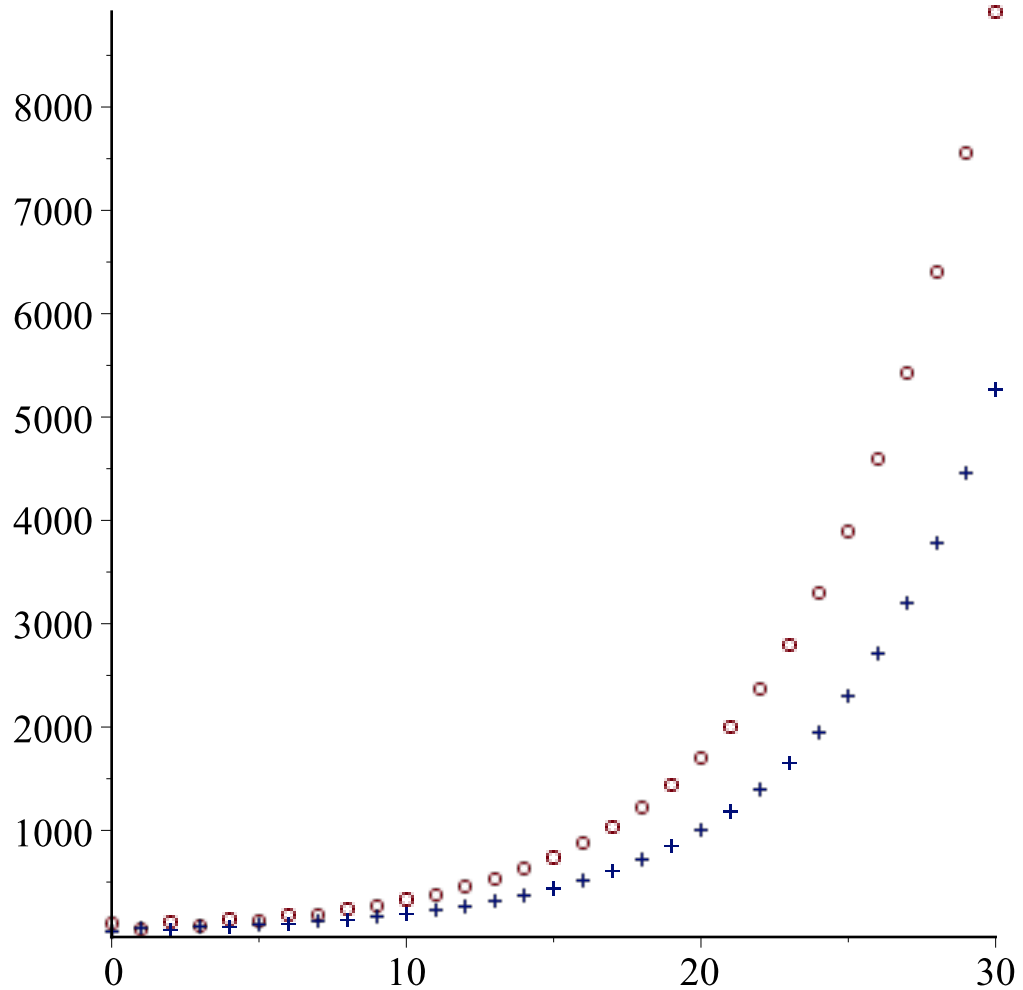
$$x_0 := 100$$
$$y_0 := 20$$
$$N := 30$$

```
> for i from 0 to N-1 do
    x[i+1]:=evalf(f1(x[i],y[i]));
    y[i+1]:=evalf(f2(x[i],y[i]));
  end do:
> plot([[[n,x[n]]$n=0..N],[[n,y[n]]$n=0..N]],style=point,
  symbol=[circle,cross]);
```



```
> s:=0;b:=4;c:=1/4;s+b*c;
```

$$s := 0$$
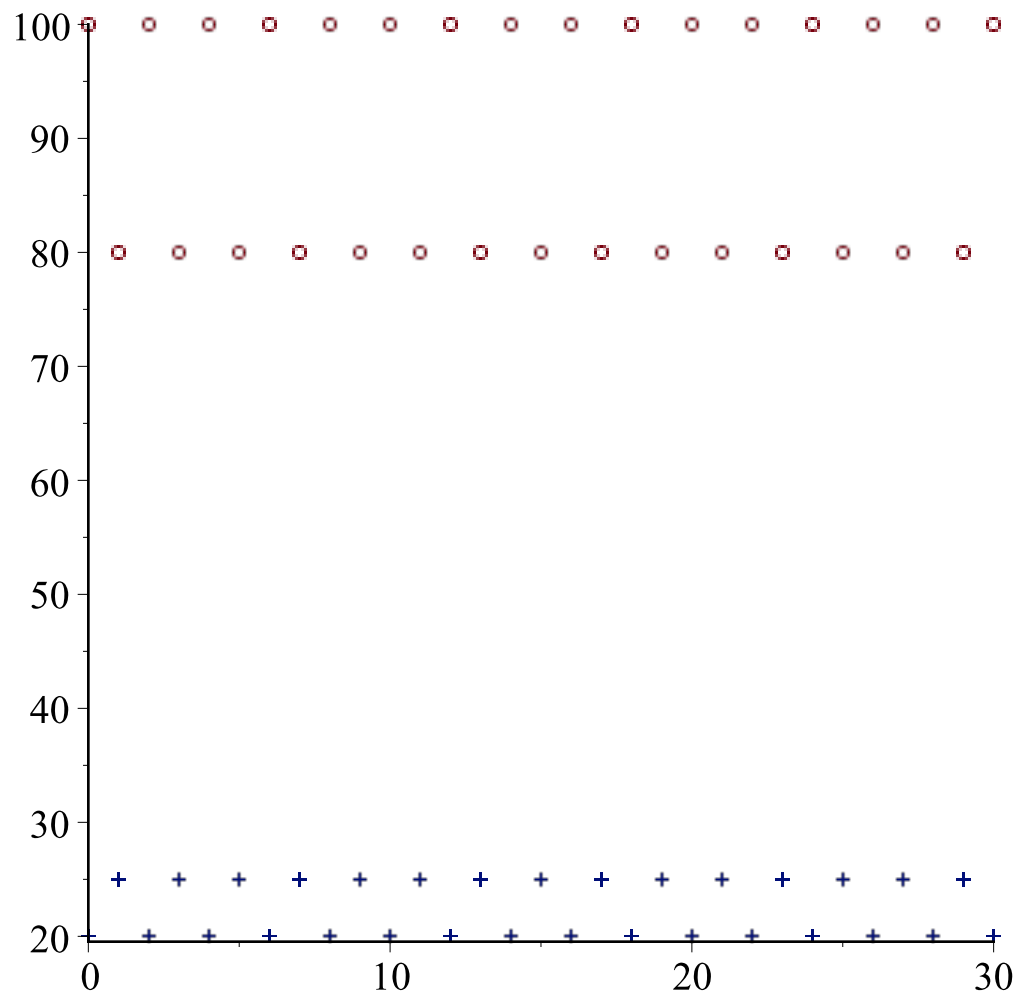$$b := 4$$

$$c := \frac{1}{4}$$
$$1$$

```
> A:=matrix([[0,b],[c,s]]);
```

$$A := \begin{bmatrix} 0 & 4 \\ \dfrac{1}{4} & 0 \end{bmatrix}$$

```
> charpoly(A,lambda);
```

$$\lambda^2 - 1$$

```
> eigenvals(A);
```

$$1, -1$$

```
> f1:=(x,y)->b*y;
```

$$f1 := (x, y) \rightarrow b\,y$$

```
> f2:=(x,y)->c*x+s*y;
```

$$f2 := (x, y) \rightarrow c\,x + s\,y$$

```
> x[0]:=100;y[0]:=20;N:=30;
```

$$x_0 := 100$$
$$y_0 := 20$$
$$N := 30$$

```
> for i from 0 to N-1 do
    x[i+1]:=evalf(f1(x[i],y[i]));
    y[i+1]:=evalf(f2(x[i],y[i]));
  end do:
> plot([[[n,x[n]]$n=0..N],[[n,y[n]]$n=0..N]],style=point,
  symbol=[circle,cross]);
```

A Nonlinear Model for a Population Divided into Two Age Classes

A more realistic model is obtained if we take into account the phenomenon of competition among members of the adult population for resources. This implies that the development of the adult population is reduced by a factor proportional to the total number of possible interactions among members of this population, that is, by $d\,y_n^2$ where d is called the competition factor. Under these conditions, we obtain a nonlinear model:

$$\begin{cases} x_{n+1} = b\,y_n \\ y_{n+1} = c\,x_n + s\,y_n - d\,y_n^2 \end{cases}$$

```
> restart;with(linalg):
> f1:=(x,y)->b*y;
```
$$f1 := (x, y) \rightarrow b\,y$$
```
> f2:=(x,y)->c*x+s*y-d*y^2;
```

$$f2 := (x, y) \rightarrow c\,x + s\,y - d\,y^2$$

```
> eqp:=solve({f1(x,y)=x,f2(x,y)=y},{x,y});
```
$$eqp := \{x = 0, y = 0\},\ \left\{x = \frac{b\,(b\,c + s - 1)}{d},\ y = \frac{b\,c + s - 1}{d}\right\}$$

The second equilibrium point is positive if $b\,c + s - 1 > 0$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```
$$J := \begin{bmatrix} 0 & b \\ c & -2\,d\,y + s \end{bmatrix}$$

```
> A1:=subs(x=0,y=0,evalm(J));
```
$$A1 := \begin{bmatrix} 0 & b \\ c & s \end{bmatrix}$$

```
> eigenvals(A1);
```
$$\frac{1}{2}\,s + \frac{1}{2}\,\sqrt{4\,b\,c + s^2},\ \frac{1}{2}\,s - \frac{1}{2}\,\sqrt{4\,b\,c + s^2}$$

```
> charpoly(A1,lambda);
```
$$-b\,c + \lambda^2 - \lambda\,s$$

We have the same characteristic equation as in the linear case
$$\lambda^2 = p_1\,\lambda + p_2$$
where $p_1 = s$ and $p_2 = b\,c$

So:
$$\left|\lambda_{1,2}\right| < 1 \text{ if and only if } \left|p_1\right| < 1 - p_2 < 2, \text{ so}$$

$$|s| = s < 1 - b \cdot c < 2 \Rightarrow s + b\,c < 1$$
(0,0) is locally asymptotically stable if $s + b\,c < 1$

```
> eqp[2,1];eqp[2,2];
```
$$x = \frac{b\,(b\,c + s - 1)}{d}$$
$$y = \frac{b\,c + s - 1}{d}$$

```
> A2:=subs(eqp[2,1],eqp[2,2],evalm(J));
```
$$A2 := \begin{bmatrix} 0 & b \\ c & -2\,b\,c - s + 2 \end{bmatrix}$$

```
> eigenvals(A2);
```
$$-b\,c - \frac{1}{2}\,s + 1 + \frac{1}{2}\,\sqrt{4\,b^2\,c^2 + 4\,b\,c\,s - 4\,b\,c + s^2 - 4\,s + 4},\ -b\,c - \frac{1}{2}\,s + 1$$
$$-\frac{1}{2}\,\sqrt{4\,b^2\,c^2 + 4\,b\,c\,s - 4\,b\,c + s^2 - 4\,s + 4}$$

```
> pol:=charpoly(A2,lambda);
```
$$pol := 2\,b\,c\,\lambda - b\,c + \lambda^2 + \lambda\,s - 2\,\lambda$$

```
> ech:=collect(pol,lambda)=0
```

$$ech := \lambda^2 + (2\,b\,c + s - 2)\,\lambda - b\,c = 0$$

```
> coeff(pol,lambda,2);
```
$$1$$

```
> coeff(pol,lambda,1);
```
$$2\,b\,c + s - 2$$

```
> coeff(pol,lambda,0);
```
$$-b\,c$$

```
> p1:=-coeff(pol,lambda,1);
```
$$p1 := -2\,b\,c - s + 2$$

```
> p2:=-coeff(pol,lambda,0);
```
$$p2 := b\,c$$

We have the characteristic equation
$$\lambda^2 = p_1\,\lambda + p_2$$
where $p_1 = -2\,b\,c - s + 2$ and $p_2 = b\,c$

So:
$$\left|\lambda_{1,2}\right| < 1 \text{ if and only if } |p_1| < 1 - p_2 < 2,\text{ so}$$

$|-2\,b\,c - s + 2| < 1 - b\cdot c < 2 \Rightarrow$
$b\cdot c - 1 < 2\,b\,c + s - 2 < 1 - b\cdot c \Rightarrow$
If
$1 < bc + s$ and $3\,b\,c + s < 3$
then
$\left(\dfrac{b\,(b\,c + s - 1)}{d}, \dfrac{b\,c + s - 1}{d}\right)$ is locally asymptotically stable.

```
> s:=1/2;b:=1;c:=1/4;d:=1/100;s+b*c;
```
$$s := \frac{1}{2}$$
$$b := 1$$
$$c := \frac{1}{4}$$
$$d := \frac{1}{100}$$
$$\frac{3}{4}$$

```
> eqp:=solve({f1(x,y)=x,f2(x,y)=y},{x,y});
```
$$eqp := \{x = 0, y = 0\},\ \{x = -25, y = -25\}$$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```
$$J := \begin{bmatrix} 0 & 1 \\ \dfrac{1}{4} & \dfrac{1}{2} - \dfrac{1}{50}\,y \end{bmatrix}$$

```
> A1:=subs(x=0,y=0,evalm(J));
```

$$A1 := \begin{bmatrix} 0 & 1 \\ \dfrac{1}{4} & \dfrac{1}{2} \end{bmatrix}$$

```
> eigenvals(A1);evalf(%);
```

$$\frac{1}{4}\sqrt{5} + \frac{1}{4}, \ \frac{1}{4} - \frac{1}{4}\sqrt{5}$$

$$0.8090169942, \ -0.3090169942$$

```
> x[0]:=5;y[0]:=10;N:=100;
```
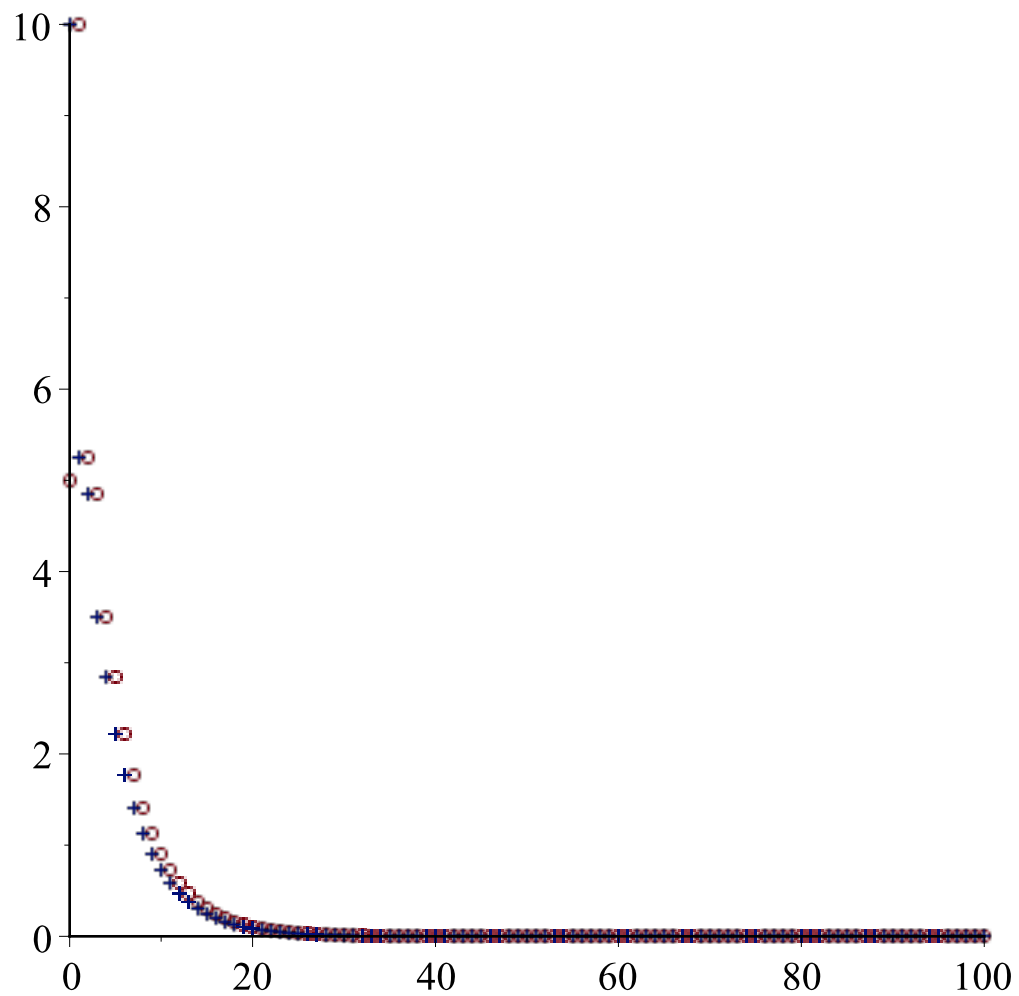
$$x_0 := 5$$

$$y_0 := 10$$

$$N := 100$$

```
> for i from 0 to N-1 do
    x[i+1]:=evalf(f1(x[i],y[i]));
    y[i+1]:=evalf(f2(x[i],y[i]));
  end do:
> plot([[[n,x[n]]$n=0..N],[[n,y[n]]$n=0..N]],style=point,
  symbol=[circle,cross]);
```

```
> s:=0.8;b:=2;c:=1/4;d:=1/100;s+b*c;
```

$$s := 0.8$$
$$b := 2$$
$$c := \frac{1}{4}$$
$$d := \frac{1}{100}$$
$$1.300000000$$

```
> eqp:=solve({f1(x,y)=x,f2(x,y)=y},{x,y});
```

$$eqp := \{x = 0., y = 0.\}, \{x = 60., y = 30.\}$$

```
> J:=jacobian([f1(x,y),f2(x,y)],[x,y]);
```

$$J := \begin{bmatrix} 0 & 2 \\ \dfrac{1}{4} & 0.8 - \dfrac{1}{50}y \end{bmatrix}$$

```
> A1:=subs(x=0,y=0,evalm(J));
```

$$A1 := \begin{bmatrix} 0 & 2 \\ \dfrac{1}{4} & 0.8 \end{bmatrix}$$

```
> eigenvals(A1);
```
$$-0.412403840463596, \ 1.21240384046360$$

```
> A2:=subs(eqp[2,1],eqp[2,2],evalm(J));
```

$$A2 := \begin{bmatrix} 0 & 2 \\ \dfrac{1}{4} & 0.2000000000 \end{bmatrix}$$

```
> eigenvals(A2);
```
$$-0.614142842854285, \ 0.814142842854285$$

```
> x[0]:=3;y[0]:=10;N:=100;
```
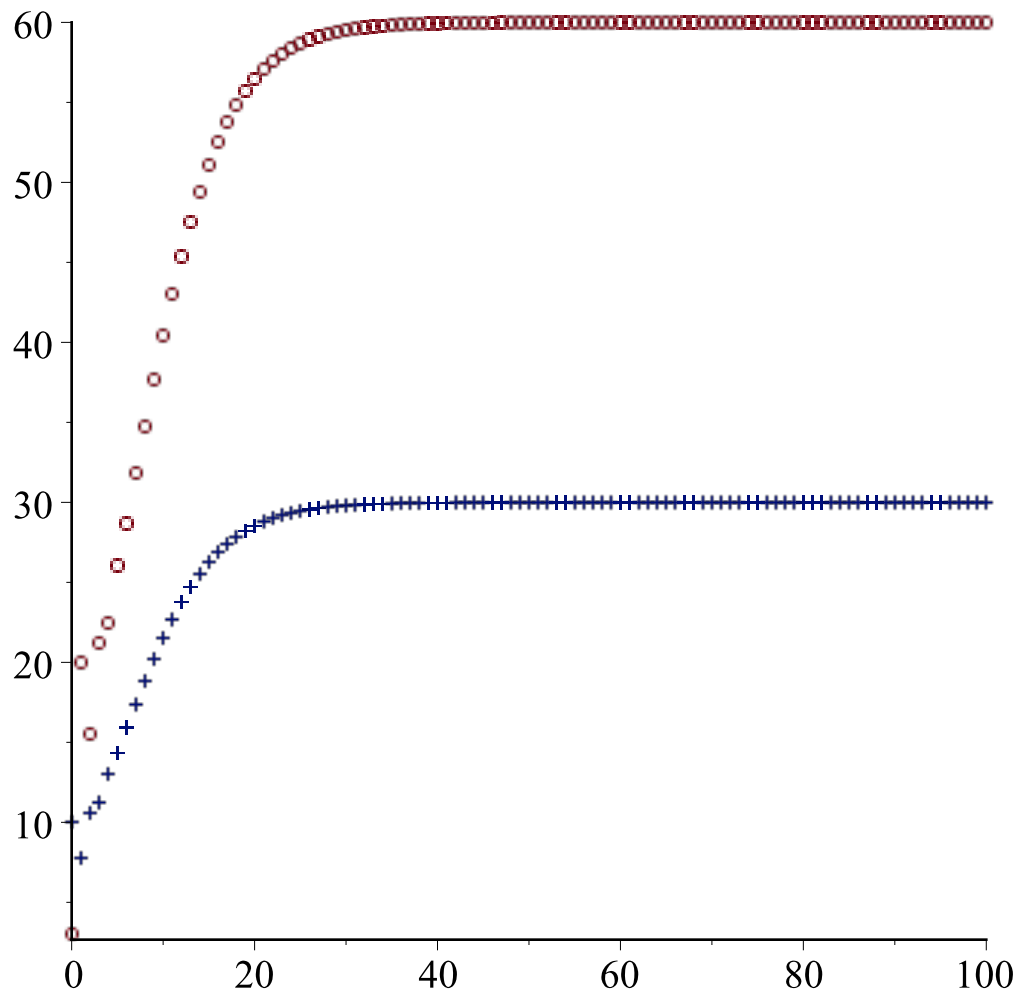$$x_0 := 3$$
$$y_0 := 10$$
$$N := 100$$

```
> for i from 0 to N-1 do
    x[i+1]:=evalf(f1(x[i],y[i]));
    y[i+1]:=evalf(f2(x[i],y[i]));
  end do:
> plot([[[n,x[n]]$n=0..N],[[n,y[n]]$n=0..N]],style=point,
  symbol=[circle,cross]);
```

▼ Leslie-Type Models

Leslie's model for population growth was developed in 1940 and was based on the premise that adult members of a population contribute differently to demographic growth depending on age. This characteristic can be modeled by dividing the population into several age classes, each having a characteristic fertility rate.

Let m be the number of age classes into which the population is divided, and denote by x_n^j the size of the population in class j during season n, where $j=1,2,\ldots,m$.

Each age class j has a fertility rate b_j, $j=1,2,\ldots,m$.

At the end of a season, a certain fraction c_j of class j moves into class $j+1$, for $j=1,2,\ldots,m-1$. The factor c_j represents the survival rate or can be viewed as the probability that a member of class j moves into class $j+1$. For a realistic model, we assume that $b_j \geq 0$, $c_j \in (0,1]$.

The first class represents the young population, and its size in the next season is given by the demographic contribution of the other classes:

$$x_{n+1}^1 = b_1 \cdot x_n^1 + b_2 \cdot x_n^2 + \ldots + b_m \cdot x_n^m$$

The size of the population from the class $j+1$ in the next season is given by the size of the population from the class j that survives:

$$x_{n+1}^{j+1} = c_j \cdot x_n^j, \quad j=1, \ldots, m\text{-}1$$

In vectorial form the system can be written as:

$$
\begin{pmatrix} x_{n+1}^1 \\ x_{n+1}^2 \\ x_{n+1}^3 \\ \ldots \\ x_{n+1}^m \end{pmatrix}
=
\begin{pmatrix}
b_1 & b_2 & b_3 & \ldots & b_m \\
c_1 & 0 & 0 & \ldots & 0 \\
0 & c_2 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & c_{m-1} & 0
\end{pmatrix}
\cdot
\begin{pmatrix} x_n^1 \\ x_n^2 \\ x_n^3 \\ \ldots \\ x_n^m \end{pmatrix}
$$

or

$$X_{n+1} = L \cdot X_n$$

where:

$$
X_n = \begin{pmatrix} x_n^1 \\ x_n^2 \\ x_n^3 \\ \ldots \\ x_n^m \end{pmatrix}
\quad \text{and} \quad
L = \begin{pmatrix}
b_1 & b_2 & b_3 & \ldots & b_m \\
c_1 & 0 & 0 & \ldots & 0 \\
0 & c_2 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & c_{m-1} & 0
\end{pmatrix}
$$

L - Leslie matrix

The model solution is given by:

$$X_n = L^n \cdot X_0$$

Theorem
Let L be a Leslie matrix such that:
(i) $b_j \geq 0$, for $j=1,2,\ldots,m$;
(ii) at least two consecutive values b_j are strictly positive;
(iii) $c_j \in (0,1]$, for $j=1,2,\ldots,m\text{-}1$.
Then:
(a) The matrix L has an unique strictly positive eigenvalue λ_1;
(b) λ_1 is a simple eigenvalue;

(c) The eigenvector corresponding to λ_1 has all components positive;

(d) Any other eigenvalue of L, $\lambda_i \neq \lambda_1$, $i \neq 1$, satisfies $|\lambda_i| < \lambda_1$.

The eigenvalue λ_1 is also called the *strictly dominant eigenvalue.*

In the long term, the distribution of the population across age classes stabilizes, and this distribution is proportional to the eigenvector Y_1 corresponding to the stricly domonant eigenvalue . If the vector Y_1 is normalized, meaning the sum of its components equals 1, it can provide information about the percentage distribution of the population across age classes.

```
> restart;
> with(linalg):
> L:=matrix([[0,3,1],[0.3,0,0],[0,0.5,0]]);
```

$$L := \begin{bmatrix} 0 & 3 & 1 \\ 0.3 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

```
> X[0]:=vector([1000,2000,3000]);
```

$$X_0 := \begin{bmatrix} 1000 & 2000 & 3000 \end{bmatrix}$$

```
> S:=X[0][1]+X[0][2]+X[0][3];
```

$$S := 6000$$

```
> Xp[0]:=vector([X[0][1]/S,X[0][2]/S,X[0][3]/S]);
```

$$Xp_0 := \begin{bmatrix} \dfrac{1}{6} & \dfrac{1}{3} & \dfrac{1}{2} \end{bmatrix}$$

```
> evalm(L&*X[0]);
```

$$\begin{bmatrix} 9000 & 300.0 & 1000.0 \end{bmatrix}$$

```
> N:=50;
```

$$N := 50$$

```
> for i from 0 to N-1 do
    X[i+1]:=evalm(L&*X[i]);
    S:=X[i+1][1]+X[i+1][2]+X[i+1][3];
    Xp[i+1]:=vector([X[i+1][1]/S,X[i+1][2]/S,X[i+1][3]/S]);
  end do:
> plot([[[n,Xp[n][1]]$n=0..N],[[n,Xp[n][2]]$n=0..N],[[n,Xp[n]
  [3]]$n=0..N]],style=point,symbol=[circle,cross,diamond]);
```
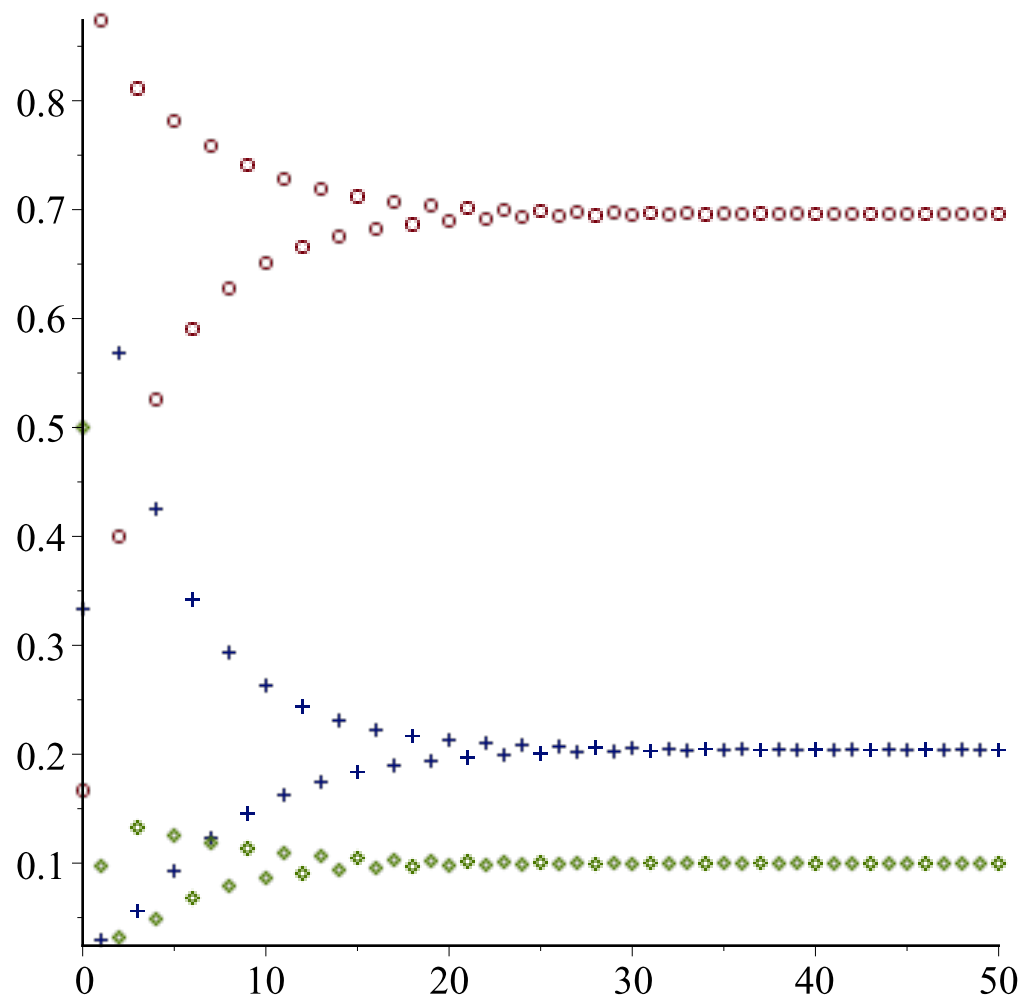
```
> Xp[N][1];Xp[N][2];Xp[N][3]
```
$$0.6960812662$$
$$0.2041616000$$
$$0.09975713358$$
```
> eigenvals(L);
```
$$1.02304502025578, -0.850689378936424, -0.172355641319359$$
```
> charpoly(L, lambda);
```
$$\lambda^3 - 0.9\,\lambda - 0.15$$
```
> Y:=eigenvects(L);
```
$$Y := \left[1.023045023, 1, \left\{\left[\begin{array}{ccc} 2.929402072 & 0.8590243857 & 0.4198370403 \end{array}\right]\right\}\right], \left[-0.8506893789,\right.$$
$$1, \left\{\left[\begin{array}{ccc} -2.649056944 & 0.9342036033 & -0.5490862037 \end{array}\right]\right\}\right], \left[-0.1723556413, 1,\right.$$
$$\left\{\left[\begin{array}{ccc} 0.4240673100 & -0.7381260702 & 2.141287795 \end{array}\right]\right\}\right]$$
```
> YY:=Y[1,3,1];
```
$$YY := \left[\begin{array}{ccc} 2.929402072 & 0.8590243857 & 0.4198370403 \end{array}\right]$$
```
> S:=YY[1]+YY[2]+YY[3];
```
$$S := 4.208263498$$

```
> Yp:=vector([YY[1]/S,YY[2]/S,YY[3]/S]);
```

$$Yp := \begin{bmatrix} 0.6961070934 & 0.2041279939 & 0.09976491265 \end{bmatrix}$$

..