First Order Difference Equations. Equilibrium Solutions. Stability of the Equilibrium Solutions

Difference equation is a recurrence relation between the terms of some sequence. Let (a_n) be a real numbers sequence.
First oder difference equation: $a_{n+1} = f(a_n)$
Second order difference equation: $a_{n+2} = f(a_n, a_{n+1})$
k-order difference equation: $a_{n+k} = f(a_n, a_{n+1}, \ldots, a_{n+k-1})$
Difference equations generate dynamical systems.

Long term behaviour of $a_{n+1} = r\,a_n$ for r constant

For the linear difference equation $a_{n+1} = r\,a_n$, let's consider some significant values of r.
If $r = 0$ then $a_n = 0$ (except possibly a_0) for all n>0, so there is no need for further investigation.
If $r = 1$ then $a_{n+1} = a_n = a_0$, in this case we have a constant sequence.
If $r = -1$ then (a_n) has two constant subsequences $a_{2k} = a_0$ and $a_{2k+1} = -a_0$.

What happens for some other values?

Let's cosider $a_{n+1} = r\,a_n$, for $r = 0.5$ and $a_0 = 3$

```
> a[0]:=3;r:=0.5;N:=20;
```

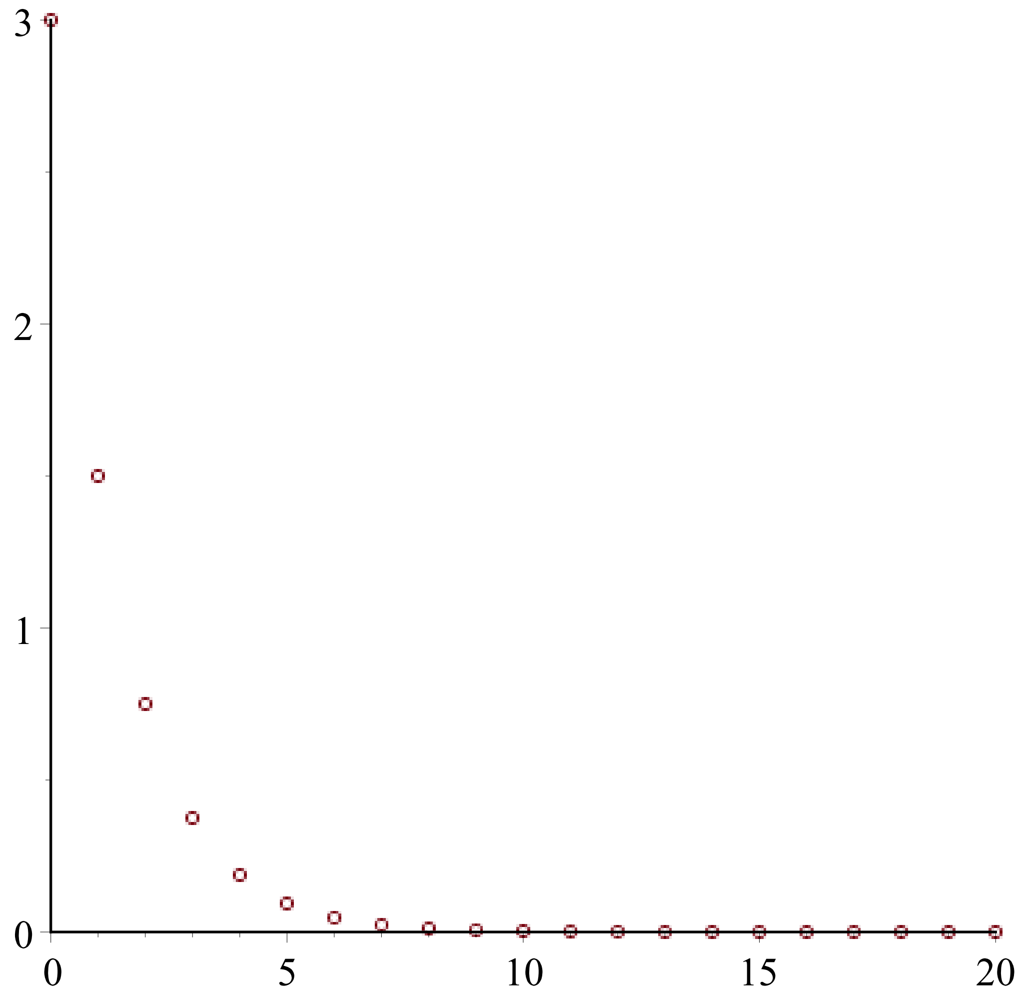
$$a_0 := 3$$
$$r := 0.5$$
$$N := 20$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]
  end do;
```

$$a_1 := 1.5$$
$$a_2 := 0.75$$
$$a_3 := 0.375$$
$$a_4 := 0.1875$$
$$a_5 := 0.09375$$
$$a_6 := 0.046875$$
$$a_7 := 0.0234375$$
$$a_8 := 0.01171875$$
$$a_9 := 0.005859375$$
$$a_{10} := 0.0029296875$$
$$a_{11} := 0.00146484375$$
$$a_{12} := 0.000732421875$$
$$a_{13} := 0.0003662109375$$
$$a_{14} := 0.0001831054688$$
$$a_{15} := 0.00009155273440$$

$$a_{16} := 0.00004577636720$$
$$a_{17} := 0.00002288818360$$
$$a_{18} := 0.00001144409180$$
$$a_{19} := 0.000005722045900$$
$$a_{20} := 0.000002861022950$$

```
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



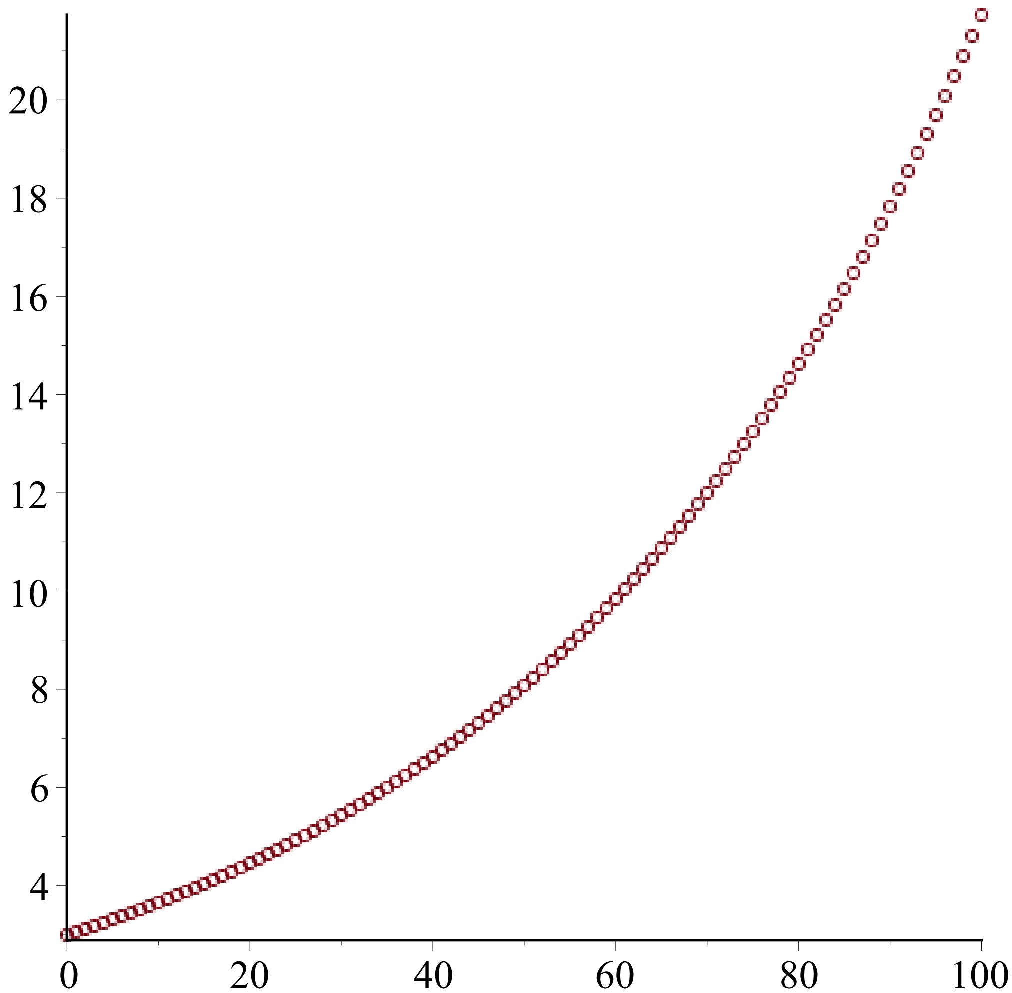
For this example we notice that the terms of the sequence are decreasing and it seems that a_n tends to 0.

Let's cosider $a_{n+1} = r\,a_n$, for $r = 1.02$ and $a_0 = 3$

```
> a[0]:=3;r:=1.02;N:=100;
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]
  end do:
```

$$a_0 := 3$$
$$r := 1.02$$
$$N := 100$$

```
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



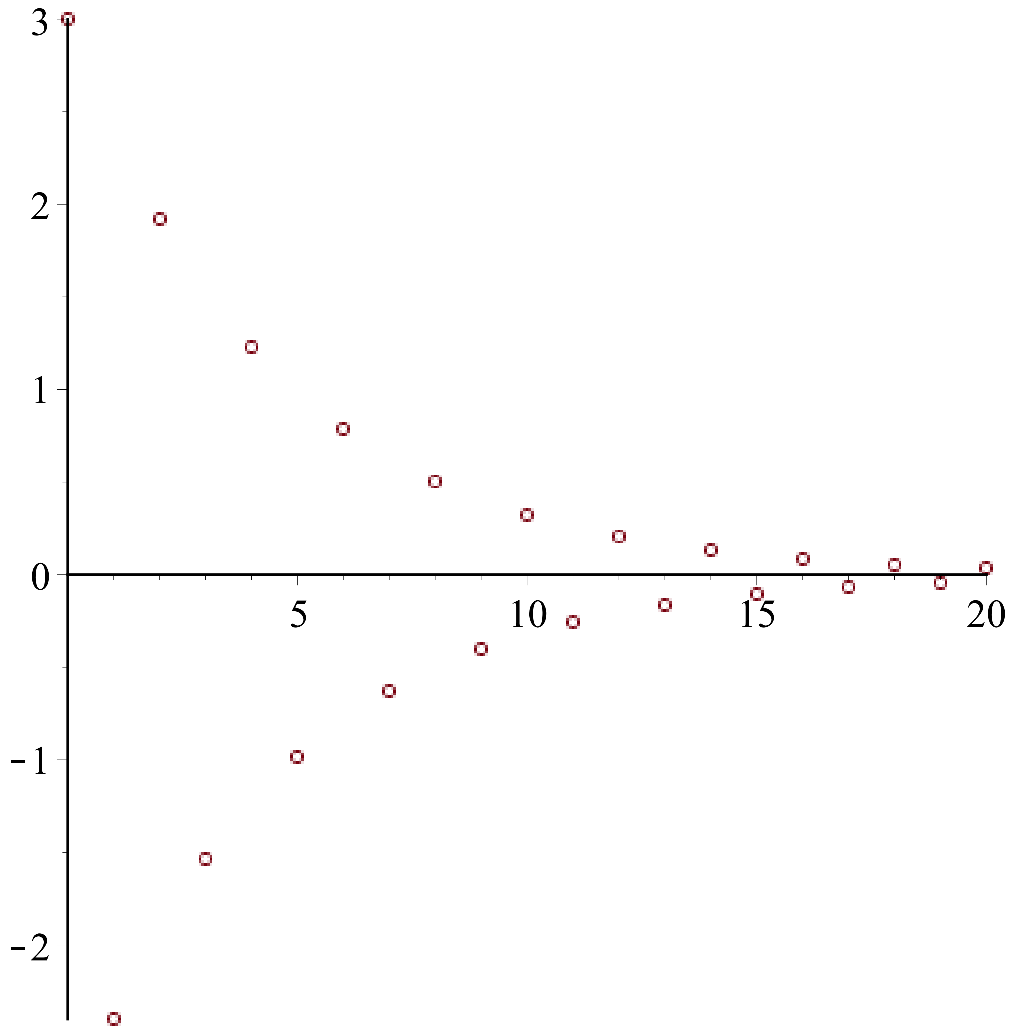
For this example we notice that the sequence is increasing and unbounded.

Let's cosider $a_{n+1} = r\, a_n$, for $r = -0.8$ and $a_0 = 3$

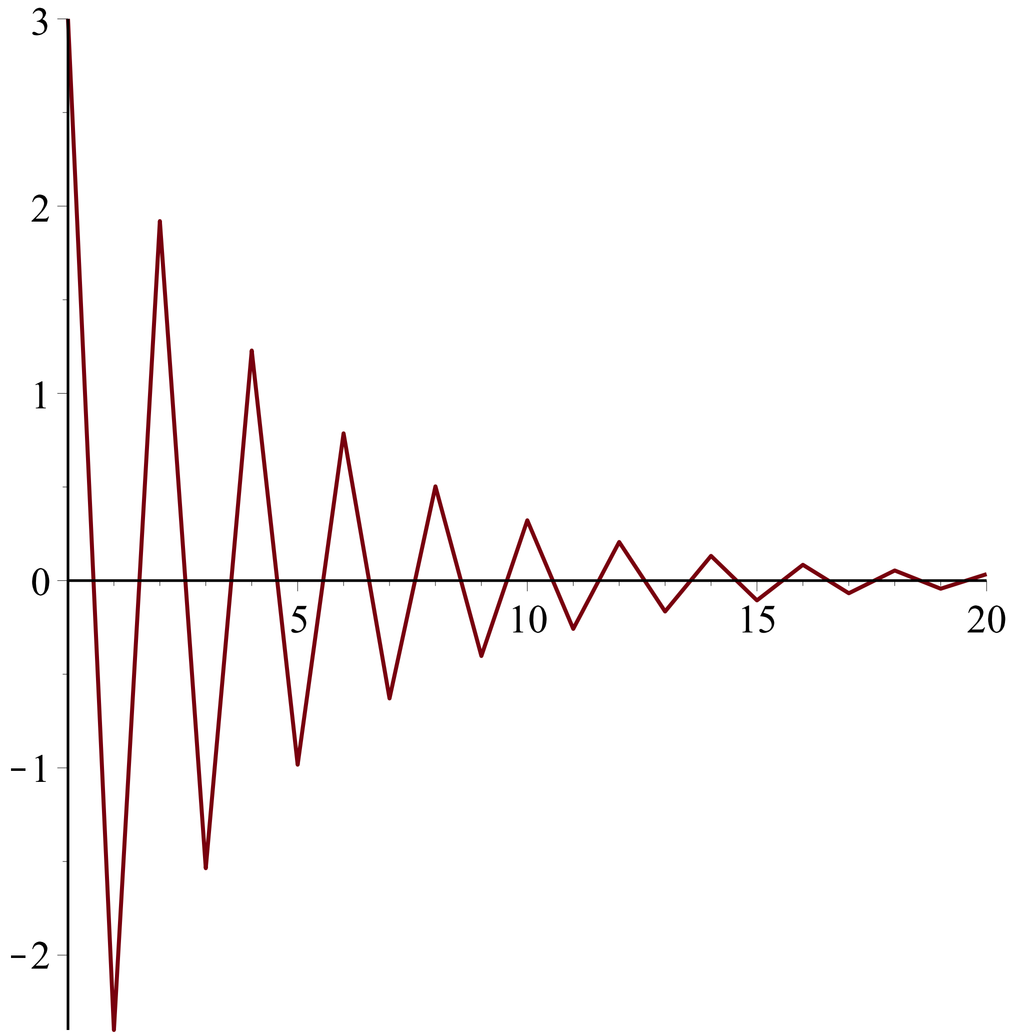
```
> a[0]:=3;r:=-0.8;N:=20;
```

$$a_0 := 3$$
$$r := -0.8$$
$$N := 20$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```




```
> plot([[n,a[n]]$n=0..N]);
```



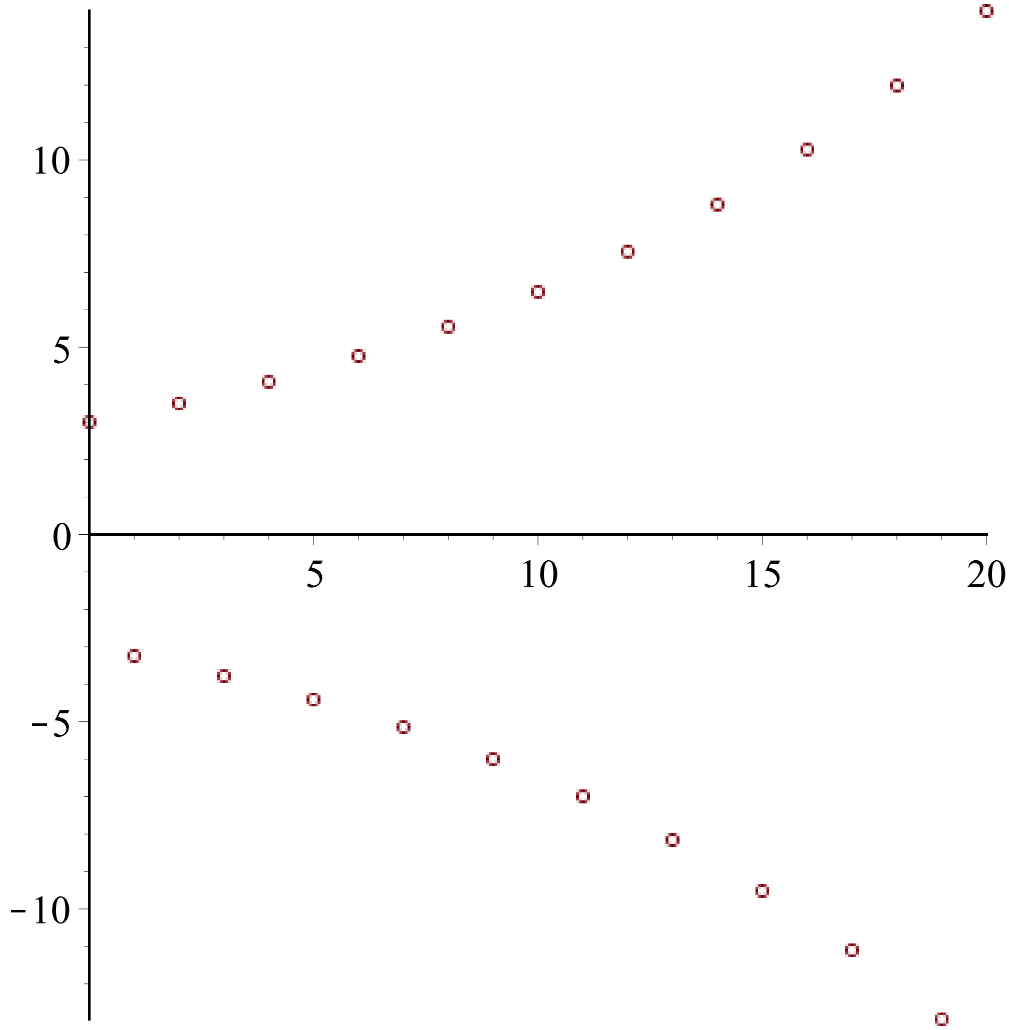
In this case the terms of the sequence oscilate and it seems that a_n tends to 0.

Let's cosider $a_{n+1} = r\, a_n$, for $r = -1.08$ and $a_0 = 3$

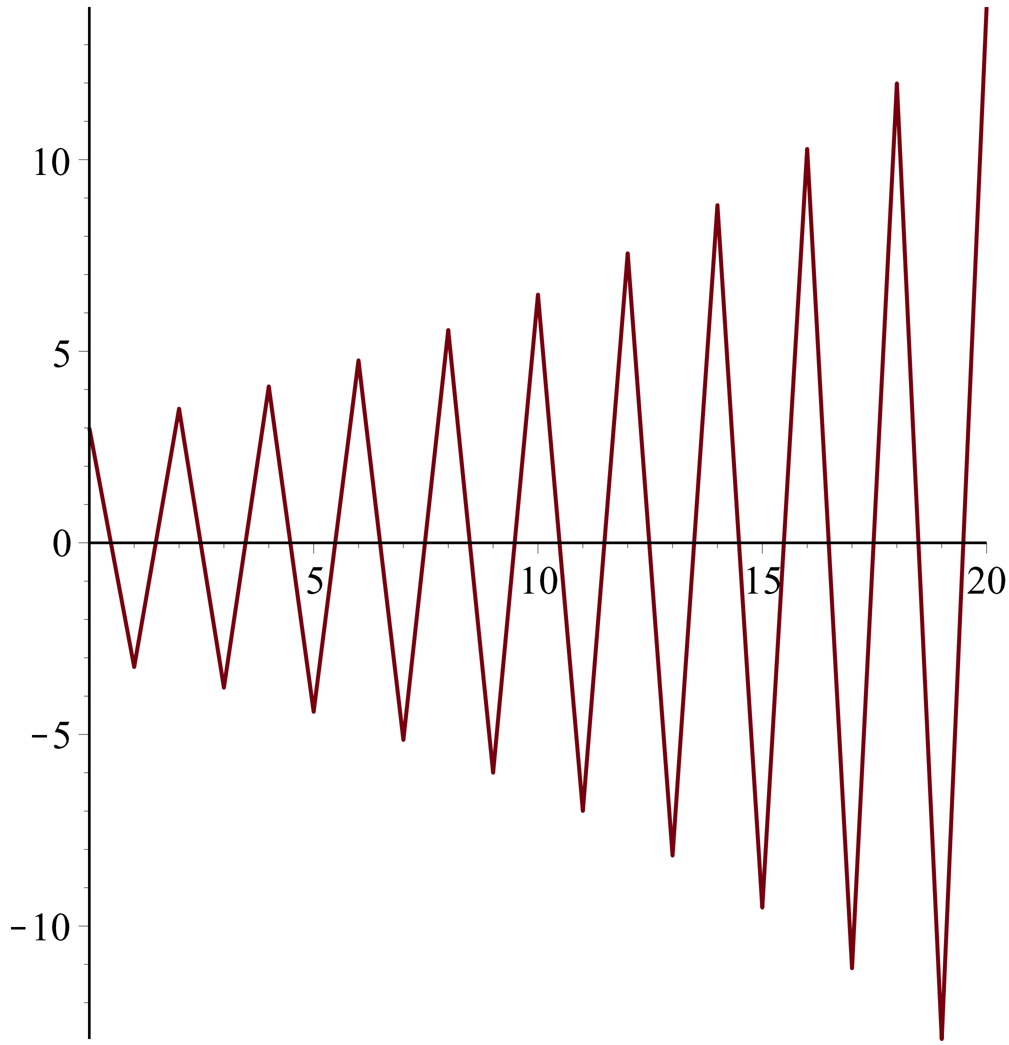
```
> a[0]:=3;r:=-1.08;N:=20;
```

$$a_0 := 3$$
$$r := -1.08$$
$$N := 20$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```




```
> plot([[n,a[n]]$n=0..N]);
```



In this case the terms of the sequence oscilate and it seems that the oscilations growth.

In fact, if we start with some initial data a_0 we have:

$a_1 = r\, a_0$

$a_2 = r\, a_1 = r^2\, a_0$

....

$a_n = r\, a_{n-1} = r^n\, a_0$

Therefore, the behaviour is given by the value of r. The expression $a_n = r^n\, a_0$ represent the solution of the difference equation. Thus, we have:

I. if $|r| < 1$ then a_n tends to 0 for all initial data a_0
II. if $r > 1$ then a_n tends to $sign(a_0) \cdot \infty$ without oscillations
III. if $r < -1$ then $|a_n|$ tends to $+\infty$ and the terms of the sequence alternate the sign.

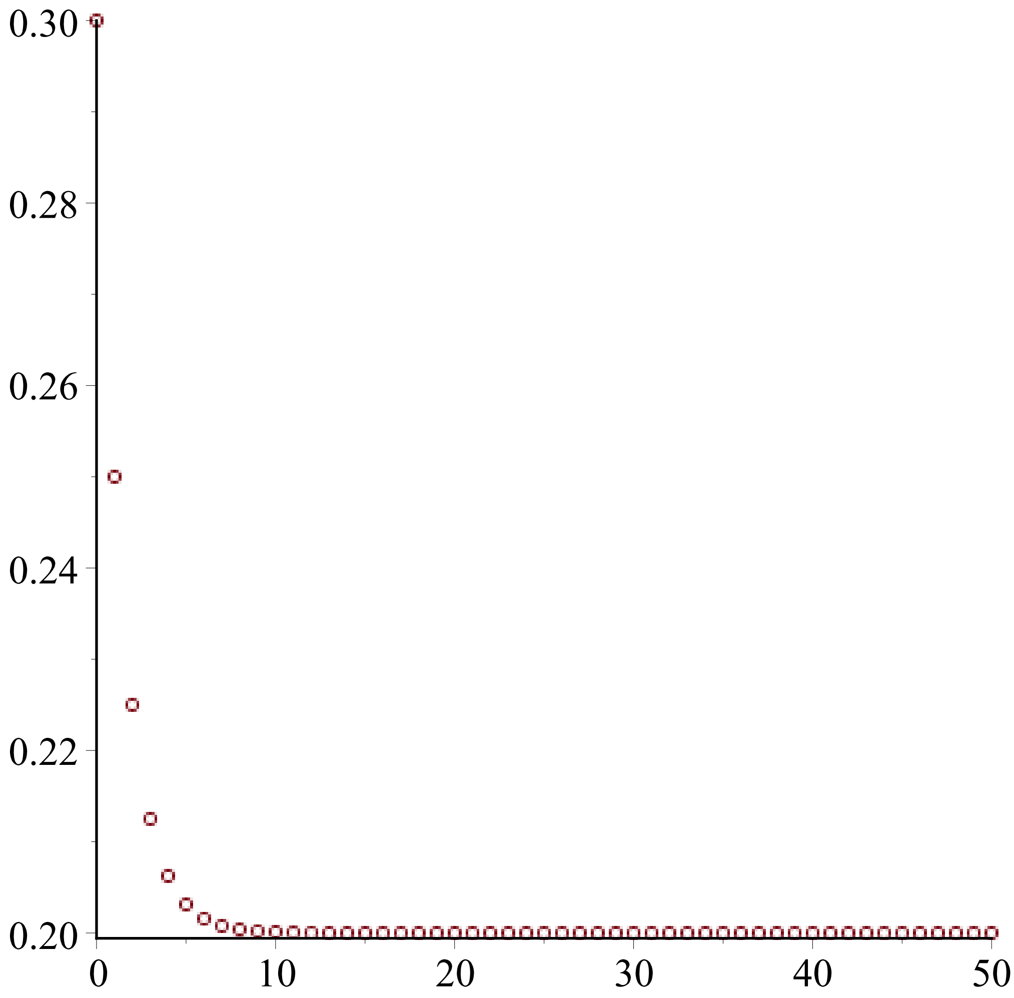
Long term behaviour of $a_{n+1} = r\, a_n + b$, where r and b are constants

Exercise: *Test the following cases:*
$a_{n+1} = 0.5\, a_n + 0.1$, for $a_0 = 0.1$, $a_0 = 0.2$ and $a_0 = 0.3$
$a_{n+1} = 1.01\, a_n - 1000$, for $a_0 = 90000$, $a_0 = 100000$ and $a_0 = 110000$

```
> a[0]:=0.3;r:=0.5;b:=0.1;N:=50;
```

$$a_0 := 0.3$$
$$r := 0.5$$
$$b := 0.1$$
$$N := 50$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]+b
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



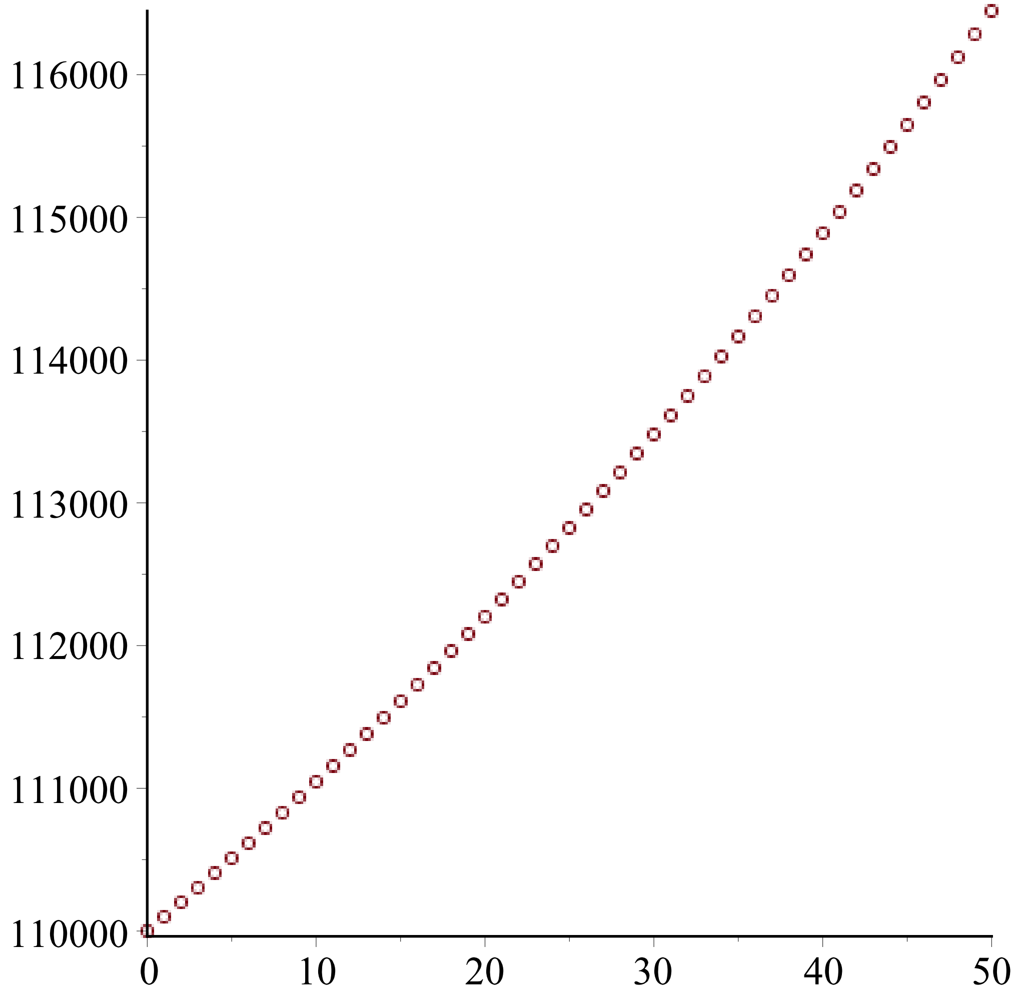

```
> a[0]:=110000;r:=1.01;b:=-1000;N:=50;
```

$$a_0 := 110000$$
$$r := 1.01$$
$$b := -1000$$
$$N := 50$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*a[i]+b
  end do:
> a[N];
```

$$1.164463180\ 10^{5}$$

```
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



Solutions of the equation $a_{n+1} = r\,a_n + b$

If $r = 0$ then $a_n = b$ for all n > 0, so we have a constant sequence (excepting a_0)

If $r = 1$ then $a_{n+1} - a_n = b$ for all n, so

$a_1 - a_0 = b$
$a_2 - a_1 = b$
....

$a_{n-1} - a_{n-2} = b$
$a_n - a_{n-1} = b$

$a_n - a_0 = n\,b$

The solution is: $a_n = n\,b + a_0$, the terms of the solution are on the line $y = b\,x + a_0$

If $r \neq 1$ and $r \neq 0$, then
$a_1 = r\,a_0 + b$
$a_2 = r\,a_1 + b = r\left(r\,a_0 + b\right) + b = r^2\,a_0 + b\,(r+1)$
$a_3 = r\,a_2 + b = r\left(r^2\,a_0 + b\,(r+1)\right)$ +b= $r^3\,a_0 + b\left(r^2 + r + 1\right)$
....
$a_n = r^n\,a_0 + b\,(r^{n-1} + ... + r + 1) = r^n\,a_0 + \dfrac{b\left(1 - r^n\right)}{1 - r} = r^n\,a_0 + \dfrac{b}{1-r} - \dfrac{b\,r^n}{1-r}$

The solution is $a_n = r^n\left(a_0 - \dfrac{b}{1-r}\right) + \dfrac{b}{1-r}$

so, we have the following situations
I. If $|r| < 1$ then $\left(a_n\right)$ tends to $a^* = \dfrac{b}{1-r}$
II. If $1 < r$ then $\left(a_n\right)$ tends to $sign\left(a_0 - \dfrac{b}{1-r}\right) \cdot \infty$
III. If $r = 1$ then $\left(a_n\right)$ tends to $sign(b) \cdot \infty$ (the graph is a line)
IV. If $r =- 1$ then $\left(a_n\right)$ has two constant subsequences $a_{2k} = a_0$ and $a_{2k+1} = -a_0 + b$.

Equilibrium solutions. Equilibrium points

A constant solution $a_n = a$, for all n, of the difference equation $a_{n+1} = f\left(a_n\right)$ is called an *equilibrium solution.*
The value a^* is called the *equilibrium point* of the difference equation.

Since the equilibrium solution is a constant sequence then the value a is a solution of the equation $a = f(a)$

For the first order linear difference equation $a_{n+1} = r\,a_n + b$ we have the equilibrium point
$a = r\,a + b$
So, $a^* = \dfrac{b}{1-r}$ if $r \neq 1$
If $r = 1$ then the equation has no equilibrium point.
If $r = 0$ and $b = 0$ then equation has an infinite number of equilibrium points (every initial value is an equilibrium point).

In the case of our examples,

$a_{n+1} = 0.5\, a_n + 0.1$ the equilibrium point is $a^* = 0.2$

$a_{n+1} = 1.01\, a_n - 1000$ the equilibrium point is $a^* = 100000$

Stability of an equilibrium point.
The equilibrium point a^* is
a) *locally stable* if for all solutions (a_n) of the difference equation with the initial condition a_0 sufficiently closed to the value a^* we have that the solutions (a_n) remains sufficiently closed to the equilibrium solution $(a_n) \equiv a^*$
b) *locally asymptotically stable* equilibrium point if for all solutions (a_n) of the difference equation with the initial condition a_0 sufficiently closed to the value a^* we have that $a_n \to a^*$ as $n \to \infty$.
If this property holds for any initial condition a_0 then we say that the equilibrium point is *globally asymptotically stable.*
c) If the equilibrium point is not locally stable then we say that is *unstable.*

In the case of the first order homogeneous linear difference equation $a_{n+1} = r\, a_n$ the solution is $a_n = r^n\, a_0$ and the equilibrium point is $a^* = 0$ if $r \neq 1$
so, we have the following situations
I. If $|r| < 1$ then the equilibrium point $a^* = 0$ is globally asymptotically stable.
II. If $1 < |r|$ then the equilibrium point $a^* = 0$ is unstable.
III. If $r = 1$ then the difference equation has an infinite number of equilibrium points and all the equilibrium points are locally stable.
IV. If $r = -1$ then the equilibrium point $a^* = 0$ is locally stable.

In the case of the first order nonhomogeneous linear difference equation $a_{n+1} = r\, a_n + b$ the solution is
$a_n = r^n \left(a_0 - \frac{b}{1-r} \right) + \frac{b}{1-r}$ and the equilibrium point is $a^* = \frac{b}{1-r}$ if $r \neq 1$
so, we have the following situations
I. If $|r| < 1$ then the equilibrium point a^* is globally asymptotically stable.
II. If $1 < |r|$ then the equilibrium point a^* is unstable.
III. If $r = 1$ then the difference equation has no equilibrium point.
IV. If $r = -1$ then then the equilibrium point $a^* = \frac{b}{2}$ is locally stable.

Nonlinear difference equation $a_{n+1} = f(a_n)$

Generally, the nonlinear difference equations can not be solved. For these equations it is important to study the behaviour of the solutions with respect to the equilibrium points, solutions of the equation $a = f(a)$

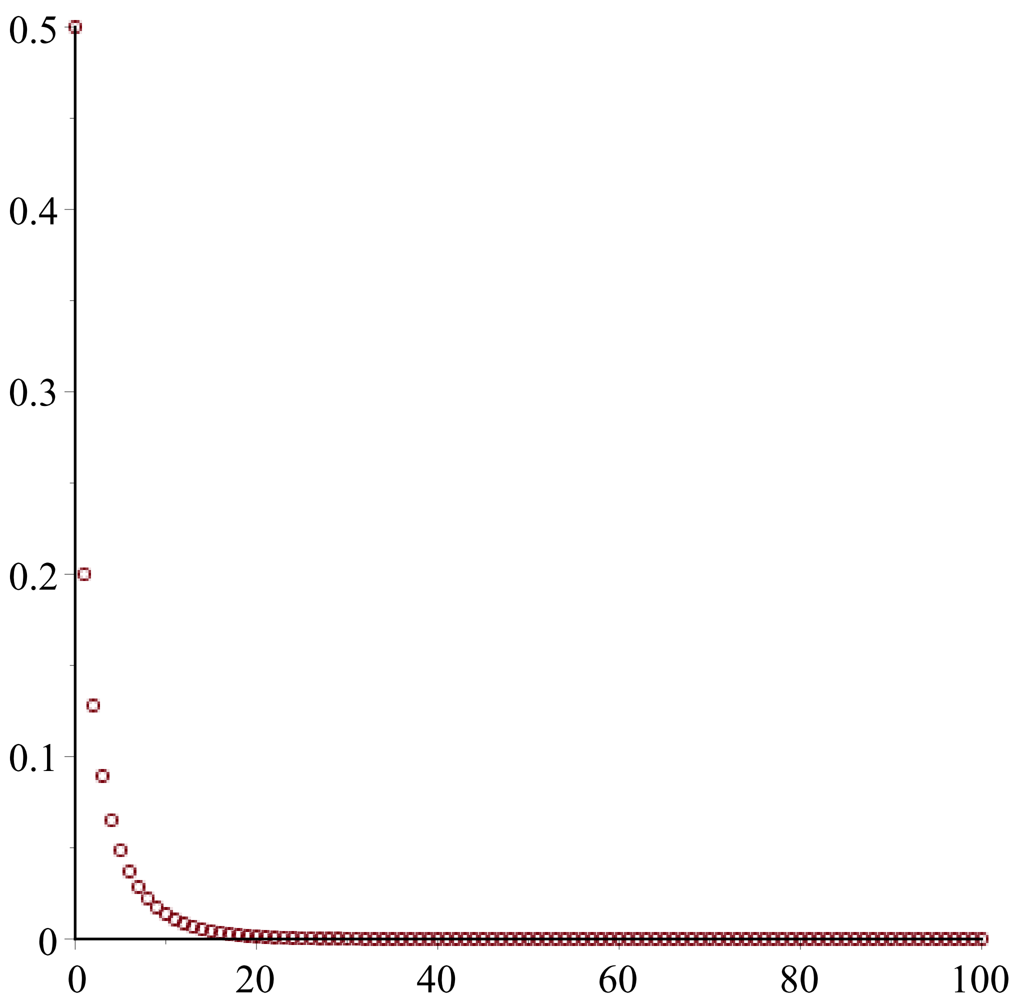
Exercise: *Plot the solutions of* $a_{n+1} = r\,(1 - a_n)\, a_n$ *for the following initial data:*
$r = 0.8$, $a_0 = 0.5$
$r = 1.5$,

$a_0 = 0.1$
$r = 2.75,\ a_0 = 0.1$
$r = 3.25,\ a_0 = 0.1$
$r = 3.525,\ a_0 = 0.1$
$r = 3.555,\ a_0 = 0.1$
$r = 3.75,\ a_0 = 0.1$

```
> a[0]:=0.5;r:=0.8;N:=100;
```

$$a_0 := 0.5$$
$$r := 0.8$$
$$N := 100$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```

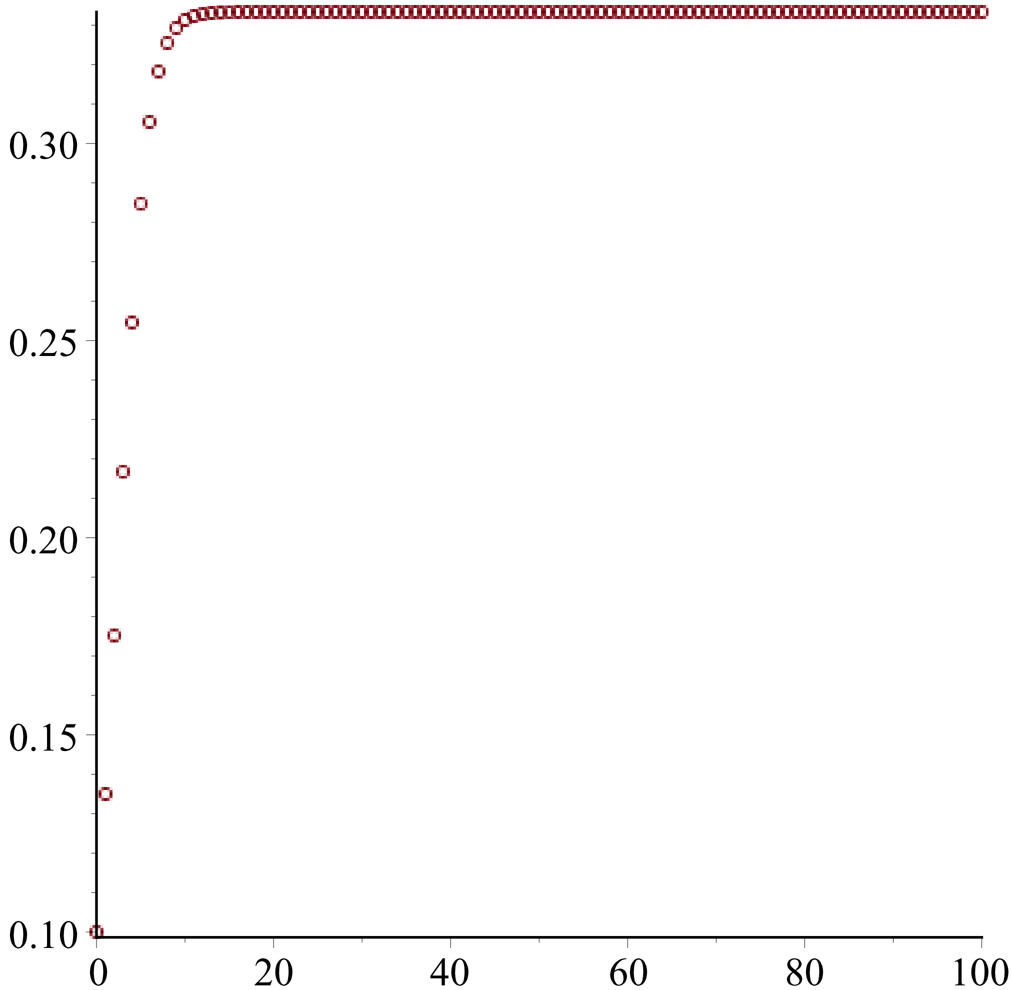



```
> a[0]:=0.1;r:=1.5;N:=100;
```

$$a_0 := 0.1$$

$$r := 1.5$$
$$N := 100$$

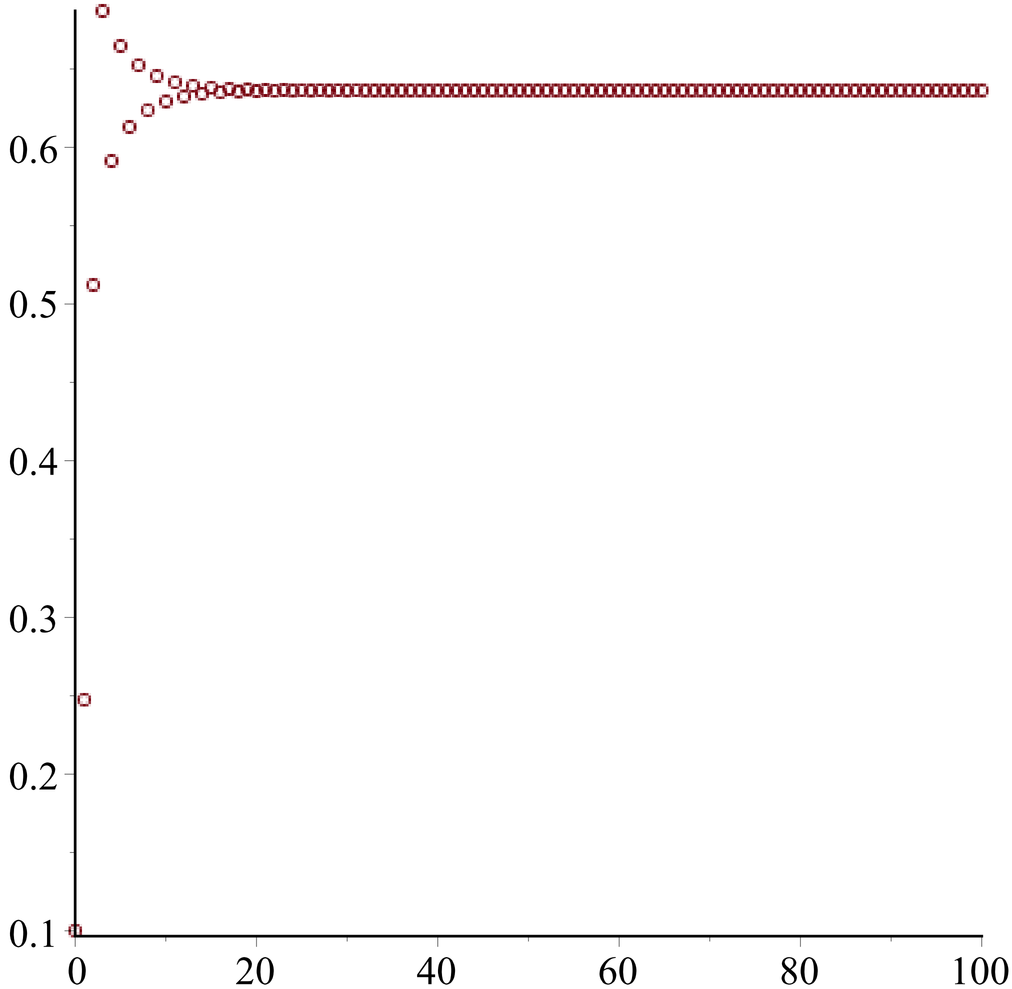
```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



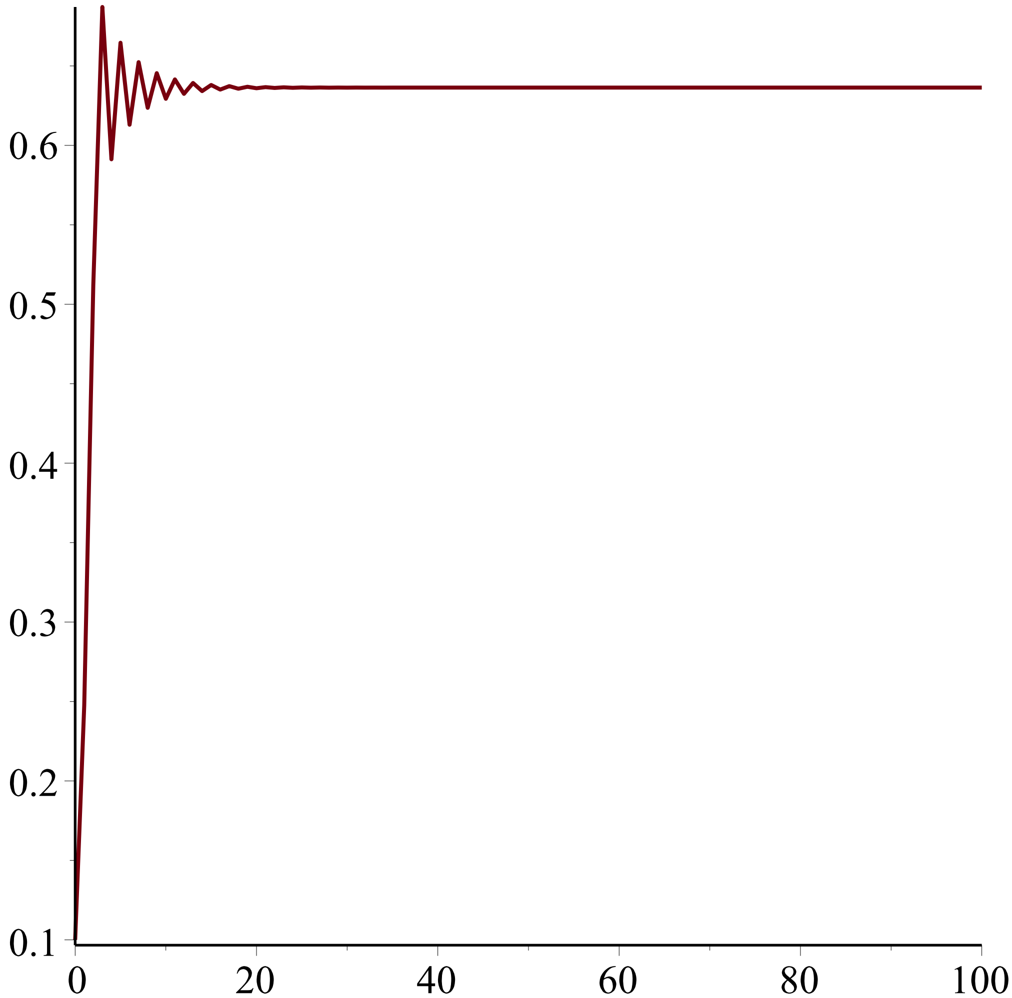

```
> r:=2.75;
```

$$r := 2.75$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



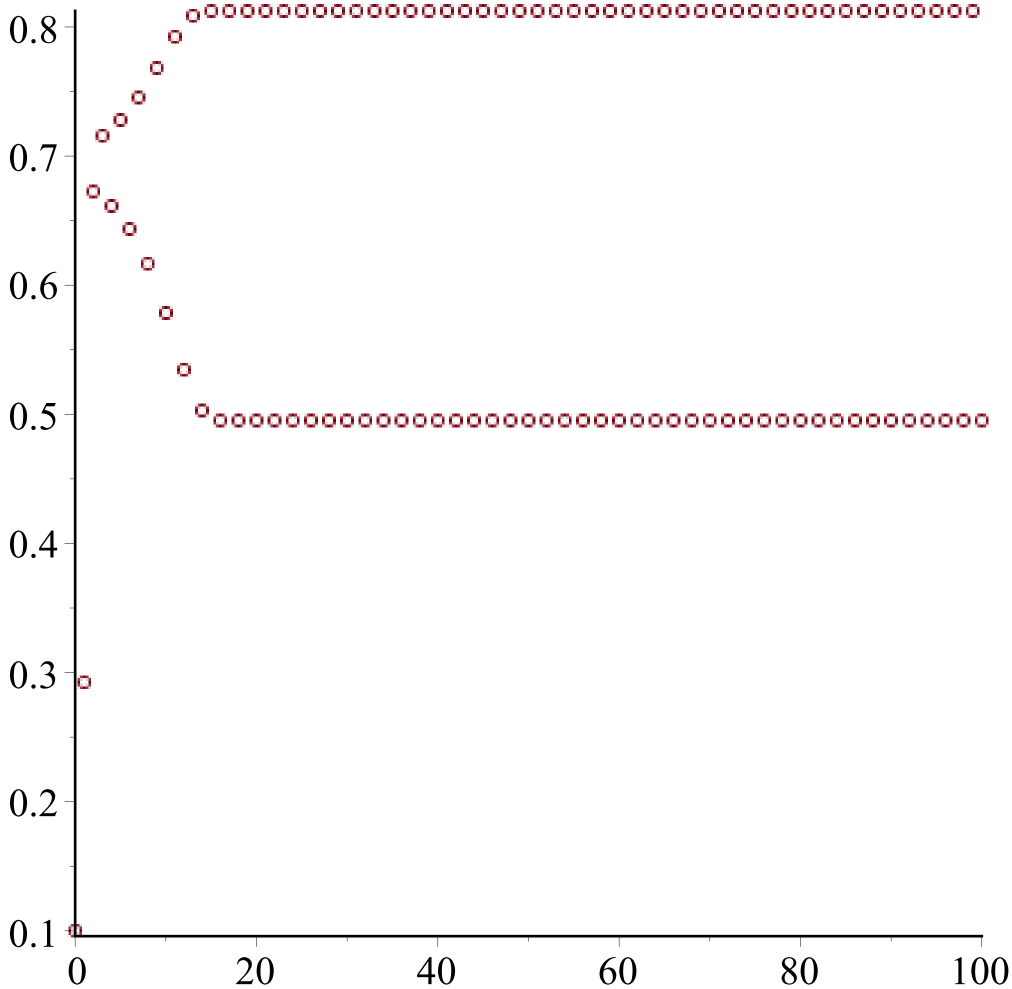

```
> plot([[n,a[n]]$n=0..N]);
```



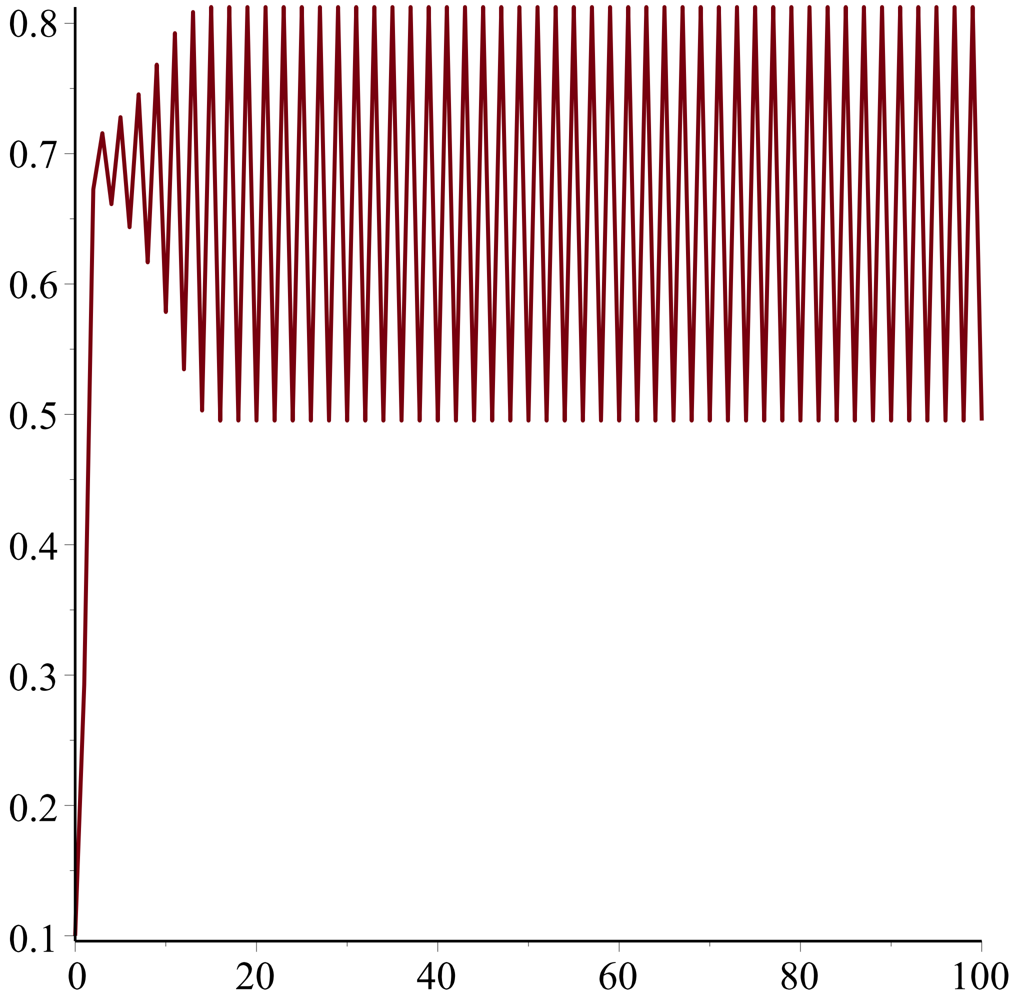

```
> r:=3.25;
```

$$r := 3.25$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



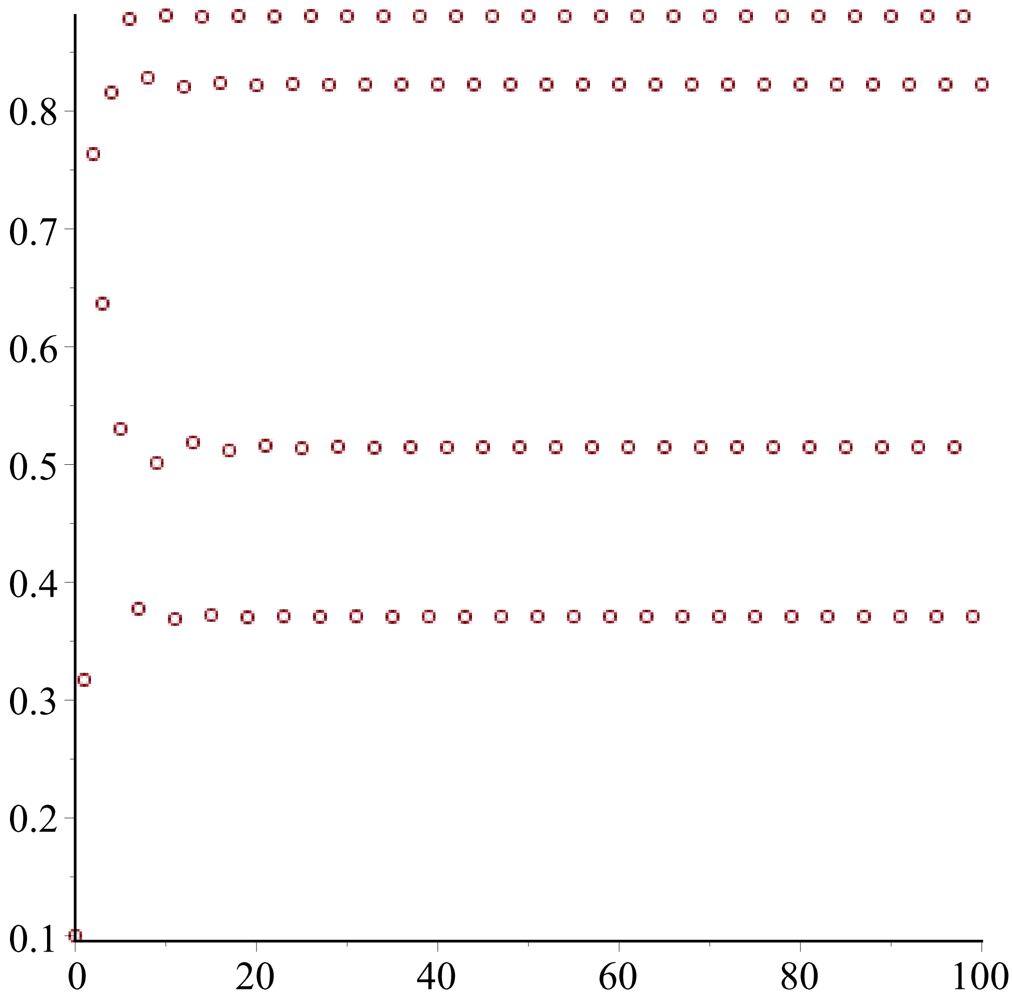

```
> plot([[n,a[n]]$n=0..N]);
```




```
> r:=3.525;
```

$$r := 3.525$$

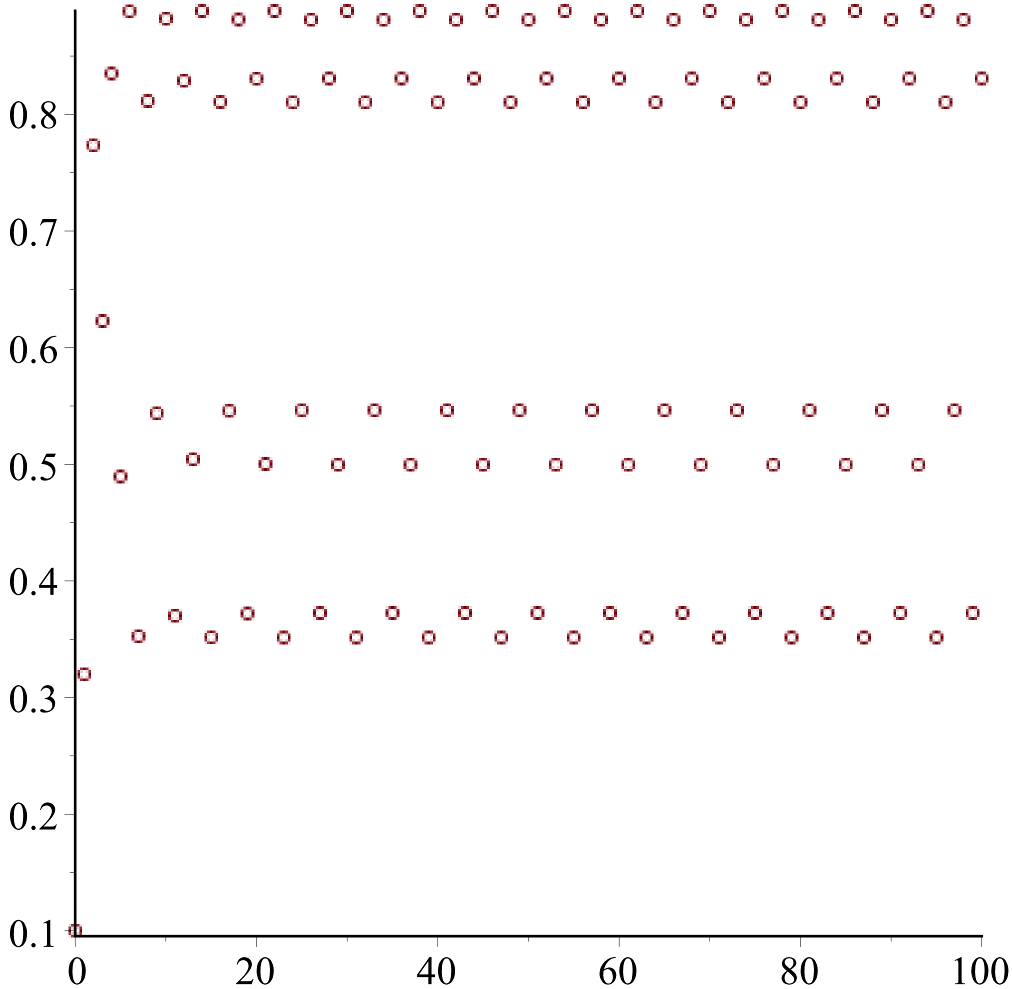
```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```




```
> r:=3.555;
```

$$r := 3.555$$

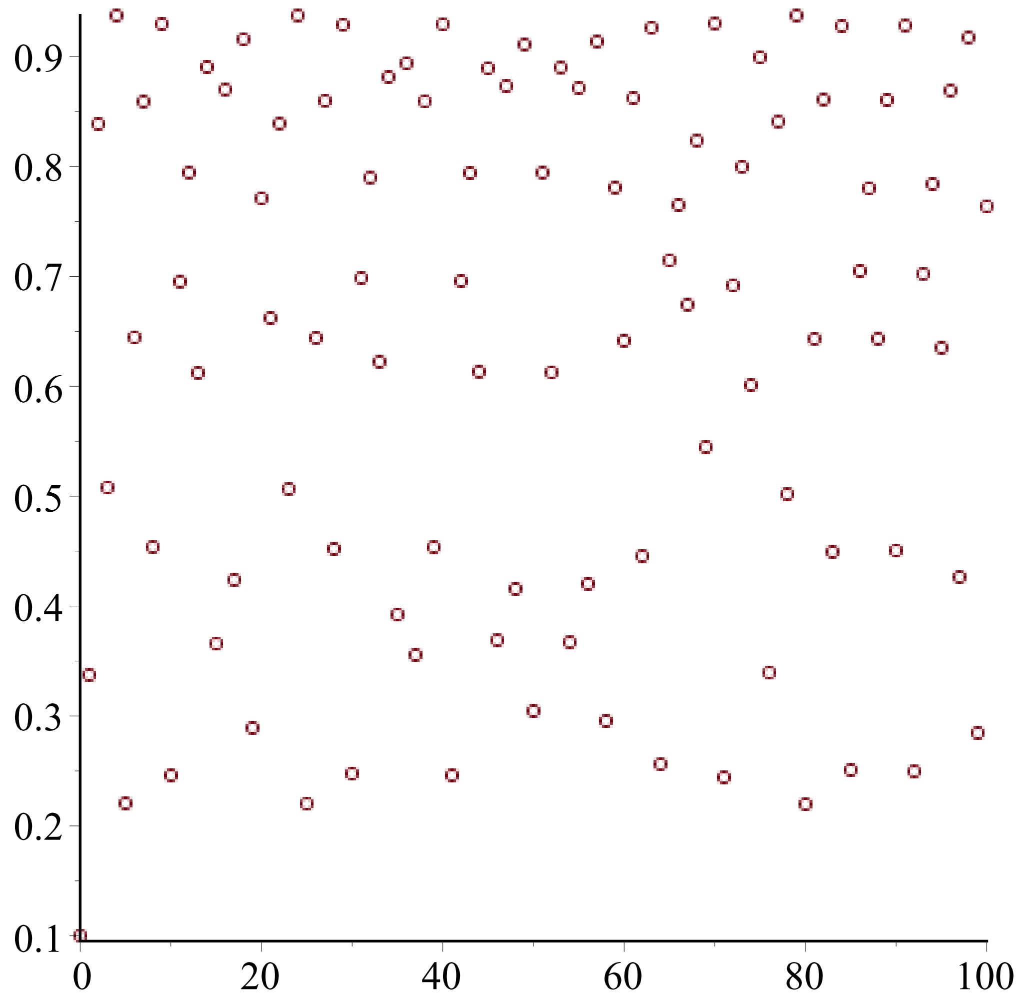
```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```




```
> r:=3.75;
```

$$r := 3.75$$

```
> for i from 0 to N-1 do
    a[i+1]:=r*(1-a[i])*a[i]
  end do:
> plot([[n,a[n]]$n=0..N],style=point,symbol=circle);
```



Let us consider the difference equation $a_{n+1} = f(a_n)$ and let a^* be an equilibrium point. From the Taylor series expansion of f in the point $x = a^*$ we have

$$f(x) = f(a^*) + f'(a^*)(x - a^*) + \frac{f''(a^*)(x - a^*)^2}{2} + \ldots$$

$$f(x) = a^* + f'(a^*)(x - a^*) + \frac{f''(a^*)(x - a)^2}{2} + \ldots$$

takeing the linear part, the difference equation $a_{n+1} = f(a_n)$ can be approximated by the difference equation

$$a_{n+1} = a^* + f'(a^*)(a_n - a^*)$$

using the notation $b_n = a_n - a^*$ we obtained so called *the linearized difference equation*

$$b_{n+1} = f'(a^*) \cdot b_n$$

Theorem (Stability in the first approximation)

Let's consider the difference equation $a_{n+1} = f(a_n)$. Suppose that a^* is an equilibrium point, i.e. $a^* = f(a^*)$, and there exists $f'(a^*)$. Then:
(i) If $|f'(a^*)| < 1$ then a^* is locally asymptotically stable;
(ii) If $|f'(a^*)| > 1$ then a^* is unstable.

Exercise: *Using the Theorem of Stability in the first approximation study the difference equation*
$a_{n+1} = r(1 - a_n)\, a_n$.

```
> restart;
> f:=x->r*(1-x)*x;
```

$$f := x \to r(1-x)\,x$$

```
> eq_p:=solve(f(x)=x,x);
```

$$eq\_p := 0, \frac{r-1}{r}$$

```
> D(f)(x);simplify(%);
```

$$-r\,x + r(1-x)$$
$$-2\,r\,x + r$$

```
> D(f)(0);
```

$$r$$

```
> D(f)((r-1)/r);simplify(%);
```

$$-r + 1 + r\left(1 - \frac{r-1}{r}\right)$$
$$-r + 2$$

$|2 - r| < 1$ is equivalent with $1 < r < 3$

Periodic Points and Periodic Cycles

Definition. Let b be in the domain of f. Then:
(i) b is called a *periodic point* of f if for some positive integer k, $f^k(b) = b$. Hence **a point is k-periodic** if it is a fixed point of f^k, that
is, if it is an equilibrium point of the difference equation
$a_{n+1} = g(a_n)$, where $g = f^k$.
The periodic orbit of b, $O(b) = \{b, f(b), f^2(b), \ldots, f^{k-1}(b)\}$, is often called a **k-cycle**.
(ii) b is called **eventually k-periodic** if for some positive integer m, $f^m(b)$ is a k-periodic point. In other words, b is eventually k-periodic if
$f^{m+k}(b) = f^m(b)$.

Theorem. (Stability of k-cycle)

Let O(b) = $\{b=a_0, f(b), f^2(b), \ldots, f^{k-1}(b)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:

(i) The k-cycle O(b) is asymptotically stable if

$$|f'(a_0) f'(a_1) \ldots f'(a_{k-1})| < 1$$

(ii) The k-cycle O(b) is unstable if

$$|f'(a_0) f'(a_1) \ldots f'(a_{k-1})| > 1$$

```
> r:='r';
```

$$r := r$$

```
> f:=x->r*(1-x)*x;
```

$$f := x \to r\,(1-x)\,x$$

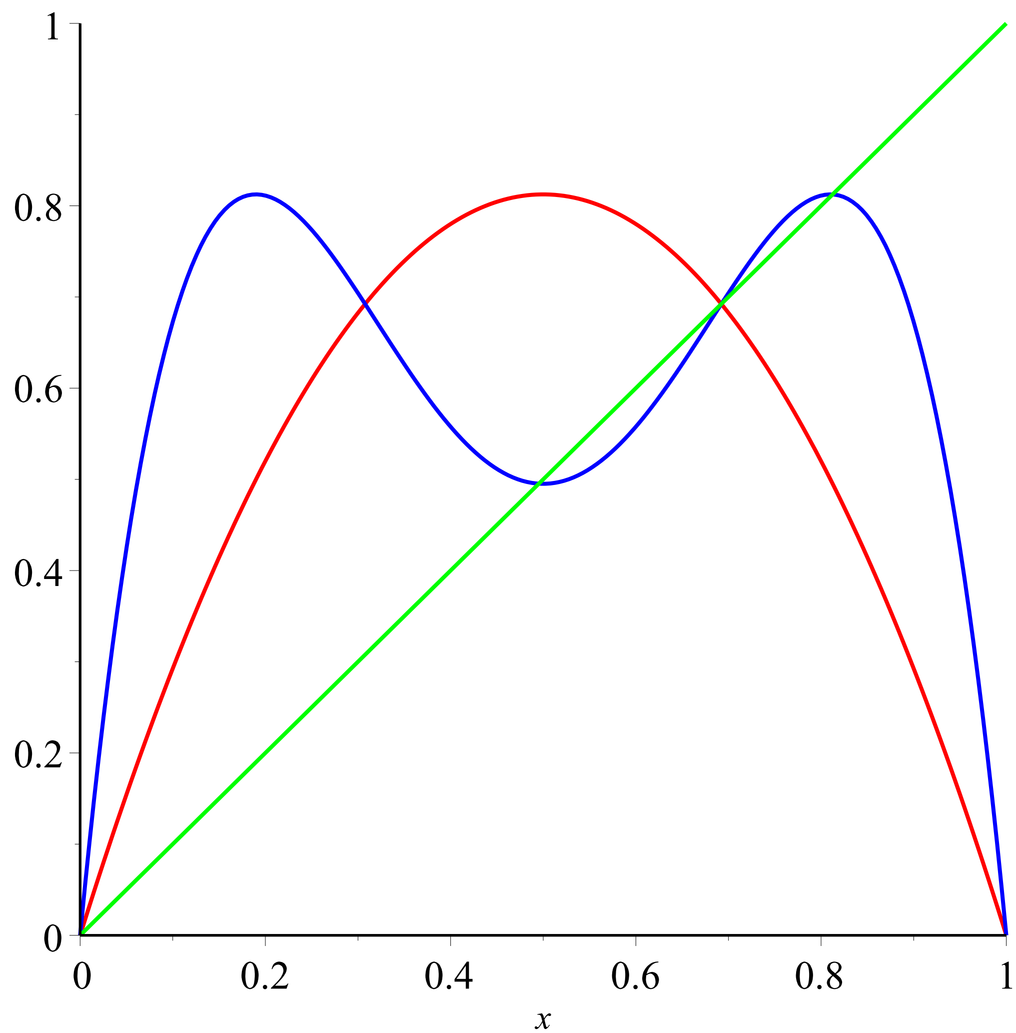
```
> g:=x->f(f(x));
```

$$g := x \to f(f(x))$$

```
> r:=3.25;
```

$$r := 3.25$$

```
> plot([f(x),g(x),x],x=0..1,color=[red,blue,green]);
```




```
> pp:=solve(g(x)=x,x);
```

$$pp := 0., 0.6923076923, 0.4952651682, 0.8124271394$$

```
> solve(f(x)=x,x);
```

$$0., 0.6923076923$$

```
> pp[1];pp[2];
```

$$0.$$
$$0.6923076923$$

pp[1] and pp[2] are equilibrium points for the difference equation $x_{n+1} = f(x_n)$, they are not a proper 2-periodic points. The 2-periodic points of this difference equation are pp[3] and pp[4].

```
> pp[3];pp[4];
```

$$0.4952651682$$
$$0.8124271394$$

```
> D(f)(x);
```

$$-6.50\,x + 3.25$$

```
> D(f)(pp[3]);D(f)(pp[4]);
```

$$0.030776406$$
$$-2.030776406$$

```
> D(f)(pp[3])*D(f)(pp[4]);
```

$$-0.06249999917$$

```
> abs(D(f)(pp[3])*D(f)(pp[4]));
```

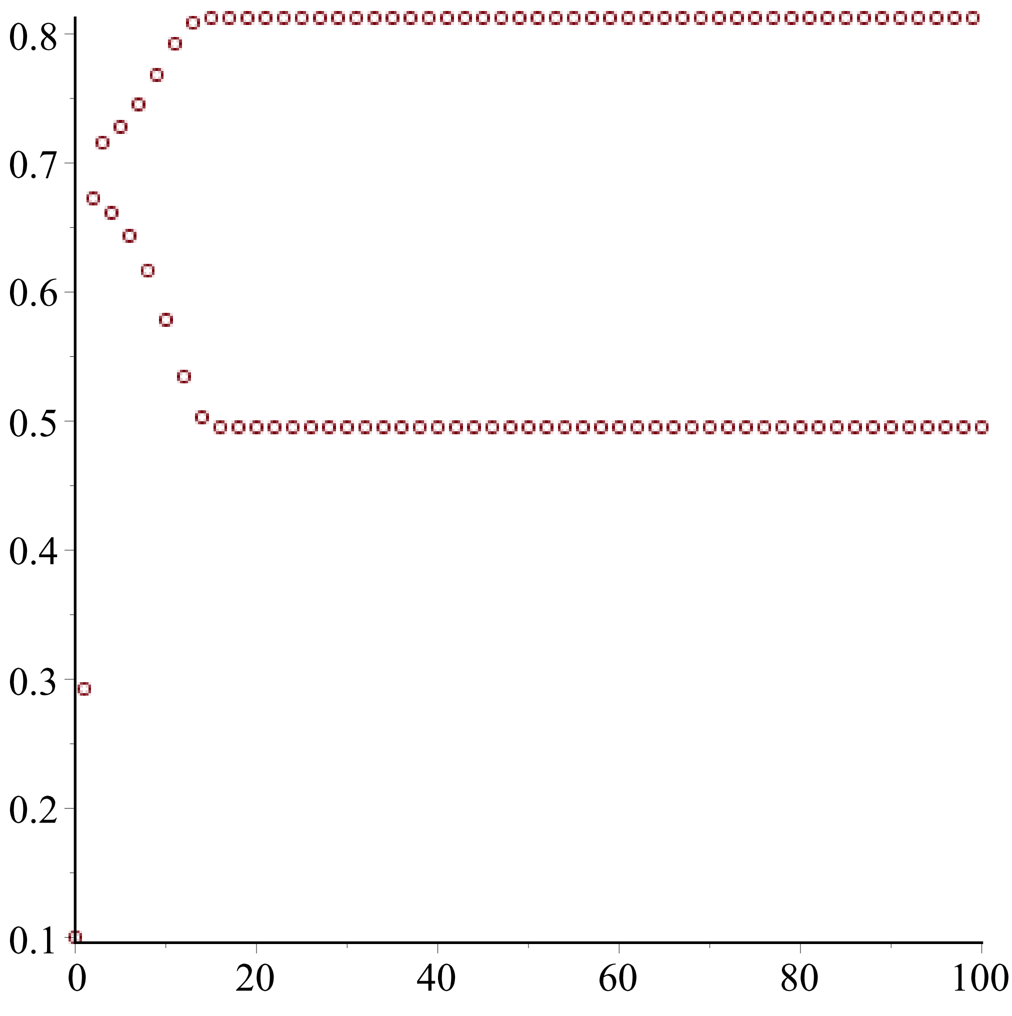
$$0.06249999917$$

So, the 2-cycle {pp[3], pp[4]} is a locally asymtotically stable 2 cycle, as you can see in the next plot

```
> a[0]:=0.1;N:=100;
```

$$a_0 := 0.1$$
$$N := 100$$

```
> for i from 0 to N-1 do
    a[i+1]:=evalf(r*(1-a[i])*a[i])
  end do:
> plot([[n,a[n]]$n=0..100],style=point,symbol=circle);
```



Let's study the 4-periodic points and check if the 4-cycle is asymptotically stable.

```
> g2:=x->(f@@4)(x);
```

$$g2 := x \to f^{(4)}(x)$$

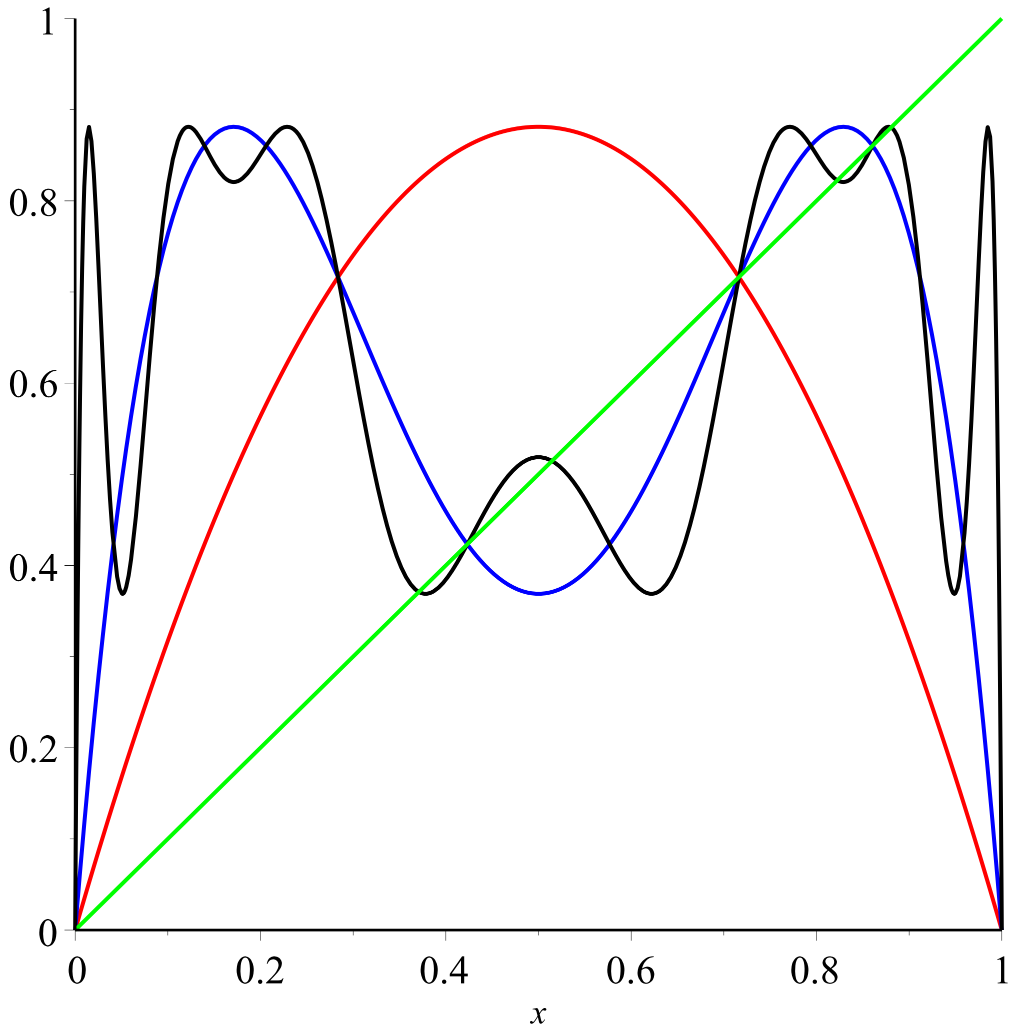
```
> g:=x->(f@@2)(x);
```

$$g := x \rightarrow f^{(2)}(x)$$

```
> r:=3.525;
```

$$r := 3.525$$

```
> plot([f(x),g(x),g2(x),x],x=0..1,color=[red,blue,black,green])
  ;
```



```
> solve(x=f(x),x);
```

$$0., 0.7163120567$$

```
> solve(x=g(x),x);
```

$$0., 0.7163120567, 0.4232189696, 0.8604689736$$

```
> p4p:=solve(g2(x)=x,x);
```

$p4p :=$ 0., 0.3709072463, 0.4232189694, 0.5146141230, 0.7163120567, 0.8225060896, 0.8604689738, 0.8804971565, 0.9860106474 + 0.006708920926 I, 0.5055172382 + 0.1724812237 I, 0.1654285294 + 0.07312465033 I, 0.04878127727 + 0.02298727936 I, 0.04878127727 − 0.02298727936 I, 0.1654285294 − 0.07312465033 I, 0.5055172382 − 0.1724812237 I, 0.9860106474 − 0.006708920926 I

Between the solutions of the equation $x = f^{(4)}(x)$ we find also the equilibrium points (the

solutions of the equation $x = f(x)$) and also 2-periodic points (the solution of the equation $x = f(f(x))$). So, 0 and 0.7163120567 are equilibrium points, 0.4232189696,0.8604689736 are 2-periodic points, 0.3709072463, 0.5146141230, 0.8225060896, 0.8804971565 are 4-periodic points.;
The next plot shows us that also in this case the 4-cycle is asymptotically stable since g2 has an asymtotically stable equilibrium point.

The 4-periodic points are:

```
> p4p[2],p4p[4],p4p[6],p4p[8];
```

0.3709072463, 0.5146141230, 0.8225060896, 0.8804971565

```
> abs(D(f)(p4p[2])*D(f)(p4p[4])*D(f)(p4p[6])*D(f)(p4p[8]));
```

0.5719004391

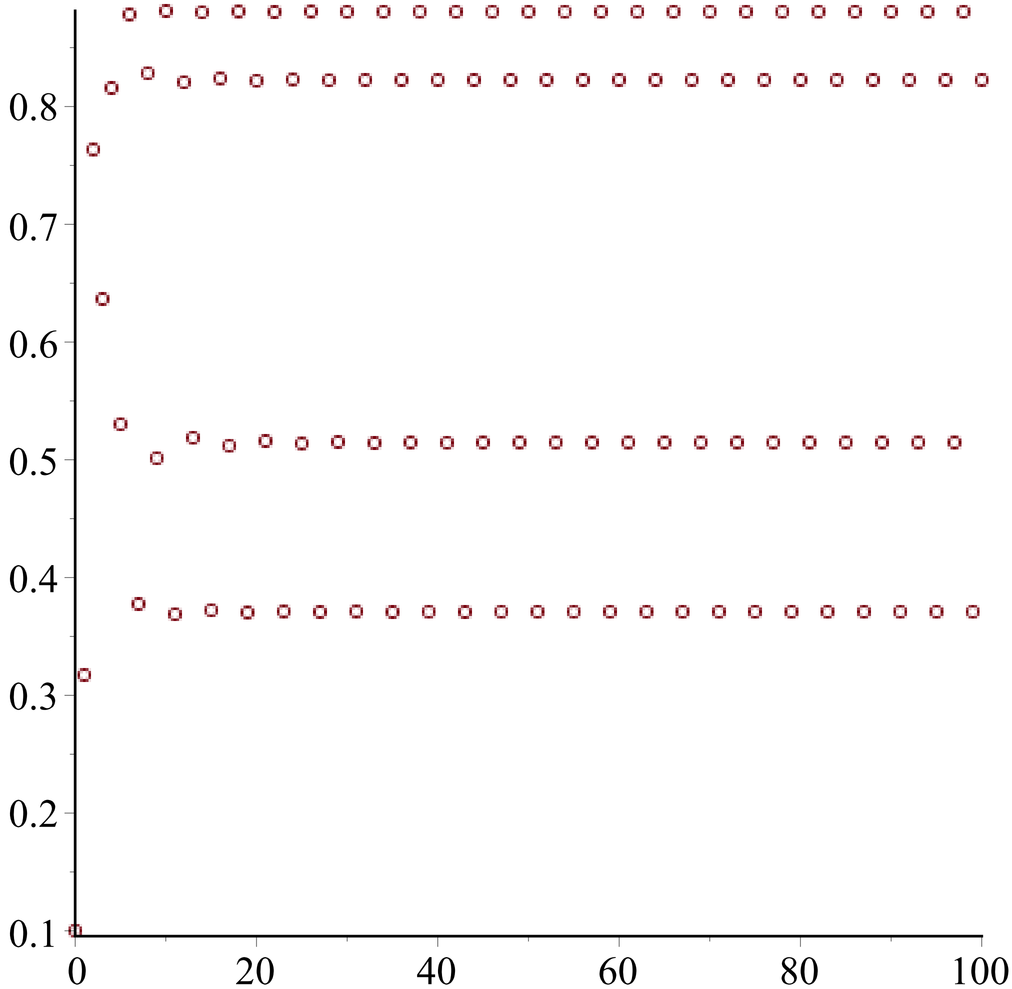
notice this value is less than 1.

```
> a[0]:=0.1;N:=100;
```

$$a_0 := 0.1$$

$$N := 100$$

```
> for i from 0 to N-1 do
    a[i+1]:=evalf(r*(1-a[i])*a[i])
  end do:
> plot([[n,a[n]]$n=0..100],style=point,symbol=circle);
```


0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0
20
40
60
80
100