Computational Stochastic Processes
Assessed Coursework

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This is my own work unless stated otherwise.

Structure of Coursework: The coursework .zip file contains the following:

- CW_TritaTrita_01199397.pdf: (This document) Compiled LaTeX report.
- problem1.py: Code used to generate figures and output for Problem 1.
- problem2.py: Code used to generate figures and output for Problem 2.
- problem3.py: Code used to generate figures and output for Problem 3.

1 Problem 1 (Mean-Square Stability of a Numerical Integrator)

1.1 Showing X_t Mean-Square stability

The SDE in question corresponds to Geometric Brownian Motion, which can be rewritten as

$$\frac{\mathrm{d}X_t}{X_t} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}W_t.$$

If we let $Y_t = f(X_t), \quad f(x) = \ln(x), \quad f'(x) = 1/x, \quad f''(x) = -1/2x^2$ and use Itô's Formula, we can arrive at the expression

$$\mathrm{d}(\ln(X_t)) = \left(0 + \frac{1}{X_t}\mu X_t - \frac{1}{2X_t^2}\sigma^2 X_t^2\right)\mathrm{d}t + \frac{1}{X_t}\sigma X_t \,\mathrm{d}W_t = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \,\mathrm{d}W_t. \tag{1}$$

We can integrate (1) to obtain the following:

$$\begin{aligned}
\ln\left(\frac{X_t}{X_0}\right) &= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t, \\
\implies X_t &= X_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right], \\
&= \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right].
\end{aligned}$$

Now

$$\begin{aligned}
X_t\,\overline{X_t} &= \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]\overline{\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]}, \\
&= \exp\left[\left((\mu + \overline{\mu}) - \left(\frac{\sigma^2 + \overline{\sigma^2}}{2}\right)\right)t\right]\exp\left[W_t(\sigma + \overline{\sigma})\right], \\
&= \exp\left[\left(2\Re(\mu) - \left(\frac{\sigma^2 + \overline{\sigma^2}}{2}\right)\right)t\right]\exp\left[2\Re(\sigma)W_t\right].
\end{aligned}$$

Taking expectations, we can conclude that

$$\mathbb{E}[|X_t|^2] = \mathbb{E}[X_t\,\overline{X_t}] = \exp[(2\Re(\mu) + |\sigma|^2)t]$$

which implies that the mean-square stability condition holds as long as $2\Re(\mu)+|\sigma|^2 < 0$, as required.

1.2 θ-Milstein Scheme Derivation

The θ Milstein scheme, when applied to the test equation in the question, corresponds to taking $b_n = b(X_n^{\Delta t}) = \mu X_n^{\Delta t}, \sigma_n = \sigma(X_n^{\Delta t}) = \sigma X_n^{\Delta t}, \sigma_n' = \sigma'(X_n^{\Delta t}) = \sigma$. Thus

$$
\begin{aligned}
X_{n+1}^{\Delta t} &= X_n^{\Delta t} + (\theta\mu X_{n+1}^{\Delta t} + (1-\theta)\mu X_n^{\Delta t})\Delta t + \sigma X_n^{\Delta t}\Delta W_n + \frac{1}{2}\sigma^2 X_n^{\Delta t}((\Delta W_n)^2 - \Delta t),\\
\implies (1-\mu\theta\Delta t)X_{n+1}^{\Delta t} &= (1 + \mu\Delta t(1-\theta) + \sigma\Delta W_n + \sigma^2((\Delta W_n)^2 - \Delta t)/2)X_n^{\Delta t},\\
\implies X_{n+1}^{\Delta t} &= \frac{(1 + \mu\Delta t(1-\theta) + \sigma\Delta W_n + \sigma^2((\Delta W_n)^2 - \Delta t)/2}{1-\mu\theta\Delta t}X_n^{\Delta t}.
\end{aligned}
$$

Hence

$$
G(\Delta t, \Delta W_n, \mu, \sigma, \theta) = \frac{(1 + \mu\Delta t(1-\theta) + \sigma\Delta W_n + \sigma^2((\Delta W_n)^2 - \Delta t)/2}{1-\mu\theta\Delta t}. \tag{2}
$$

If we write $\Delta W_n = \xi\sqrt{\Delta t},\ \xi \sim \mathcal{N}(0,1)$, then equation (2) becomes

$$
G(\Delta t, \mu, \sigma, \theta) = \frac{(1 + \mu\Delta t(1-\theta) + \sigma\xi\sqrt{\Delta t} + \Delta t\sigma^2(\xi^2 - 1)/2}{1-\mu\theta\Delta t}. \tag{3}
$$

To obtain the expression of $Z_{n+1}^{\Delta t} = \mathbb{E}[|X_{n+1}^{\Delta t}|] = \mathbb{E}[X_{n+1}^{\Delta t}\,\overline{X_{n+1}^{\Delta t}}]$ we make use of the known moments of ξ, i.e. $\mathbb{E}[\xi] = 0, \mathbb{E}[\xi^2] = 1, \mathbb{E}[\xi^3] = 0, \mathbb{E}[\xi^4] = 3$ in the calculations to be able to ignore some of the terms in the expression

$$
\begin{aligned}
Z_{n+1}^{\Delta t} &= \mathbb{E}[G(\Delta t, \mu, \sigma, \theta)X_n^{\Delta t}\,\overline{G(\Delta t, \mu, \sigma, \theta)X_n^{\Delta t}}] = \mathbb{E}[G(\Delta t, \mu, \sigma, \theta)\,\overline{G(\Delta t, \mu, \sigma, \theta)}]\mathbb{E}[X_n^{\Delta t}\,\overline{X_n^{\Delta t}}]\\
&= R(\Delta t, \mu, \sigma, \theta)\mathbb{E}[X_n^{\Delta t}\,\overline{X_n^{\Delta t}}] = \frac{f(\Delta t, \mu, \sigma, \theta)}{g(\Delta t, \mu, \sigma, \theta)}\mathbb{E}[X_n^{\Delta t}\,\overline{X_n^{\Delta t}}] = \frac{f(\Delta t, \mu, \sigma, \theta)}{g(\Delta t, \mu, \sigma, \theta)}Z_n^{\Delta t},
\end{aligned}
$$

where

$$
\begin{aligned}
g(\Delta t, \mu, \sigma, \theta) &= (1-\mu\theta\Delta t)\,\overline{(1-\mu\theta\Delta t)},\\
&= (1-\mu\theta\Delta t)(1-\overline{\mu}\,\theta\Delta t),\\
&= 1 - 2\Re(\mu)\theta\Delta t + |\mu|^2\theta^2(\Delta t)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
f(\Delta t, \mu, \sigma, \theta) &= 1 + \mu\Delta t(1-\theta) + \overline{\mu}\,\Delta t(1-\theta) + |\mu|^2(\Delta t)^2(1-\theta)^2 + |\sigma^2|^2(\Delta t)^2/2 + |\sigma|^2\Delta t,\\
&= 1 + 2\Re(\mu)\Delta t(1-\theta) + (\Delta t)^2(|\mu|^2(1-\theta)^2 + |\sigma^2|^2/2) + |\sigma|^2\Delta t,
\end{aligned}
$$

obtained by doing some tedious algebra. Hence

$$
R(\Delta t, \mu, \sigma, \theta) = \frac{1 + 2\Re(\mu)\Delta t(1-\theta) + (\Delta t)^2(|\mu|^2(1-\theta)^2 + |\sigma^2|^2/2) + |\sigma|^2\Delta t}{1 - 2\Re(\mu)\theta\Delta t + |\mu|^2\theta^2(\Delta t)^2}
$$

1.3 Region of Mean-Square Stability for the θ-Milstein Scheme

To find the region of mean-square stability, we recall from lectures the result that a numerical scheme for solving SDEs is mean-square stable if $\mathbb{E}[|X_n^{\Delta t}|^2] \to 0$ as $n \to \infty$.

So, using the formula from the previous question, we obtain the following relation for the mean-square stability

$$\begin{aligned}\mathbb{E}[|X_n^{\Delta t}|^2] = Z_n^{\Delta t} &= R(\Delta t, \mu, \sigma, \theta) Z_{n-1}^{\Delta t}, \\ &= \prod_{j=0}^{n-1} R(\Delta t, \mu, \sigma, \theta) Z_0^{\Delta t}, \\ &= R(\Delta t, \mu, \sigma, \theta)^{n-1} \cdot 1,\end{aligned}$$

since $Z_0^{\Delta t} = \mathbb{E}[|X_0^{\Delta t}|^2] = \mathbb{E}[|1|^2] = 1$. This expression tends to 0 if $R(\Delta t, \mu, \sigma, \theta) < 1$, i.e.

$$\frac{1 + 2\Re(\mu)\Delta t(1-\theta) + (\Delta t)^2(|\mu|^2(1-\theta)^2 + |\sigma^2|^2/2) + |\sigma|^2\Delta t}{1 - 2\Re(\mu)\theta\Delta t + |\mu|^2\theta^2(\Delta t)^2} < 1.$$

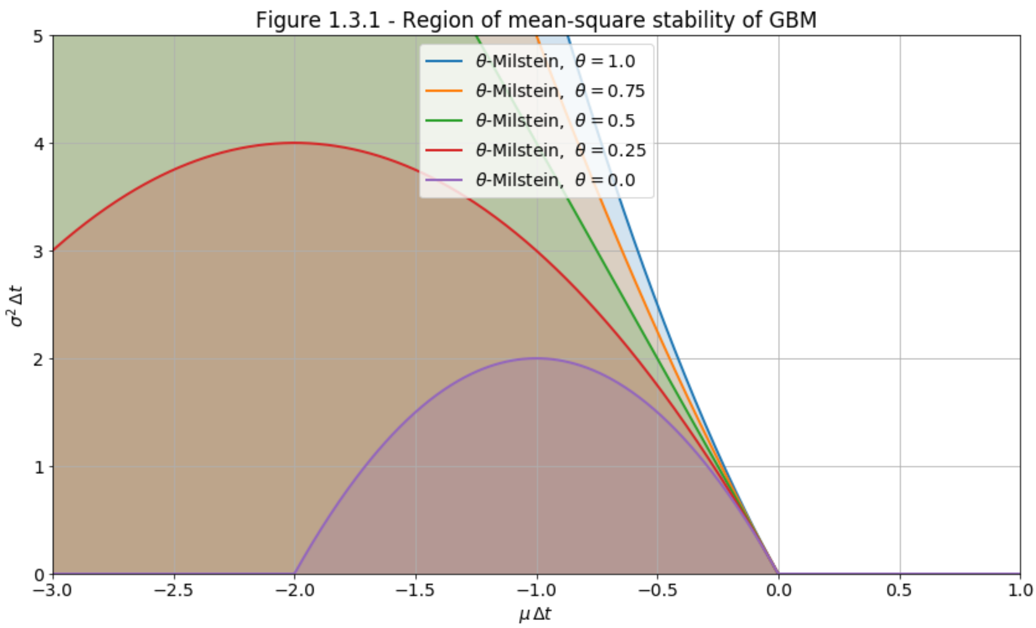
In the case of $\mu, \sigma \in \mathbb{R}$, this condition becomes

$$\frac{(1 + \mu\Delta t(1-\theta))^2 + \Delta t\sigma^2(1 + \Delta t\sigma^2/2)}{(1 - \mu\theta\Delta t)^2} < 1.$$

This condition can be simplified to give the inequality

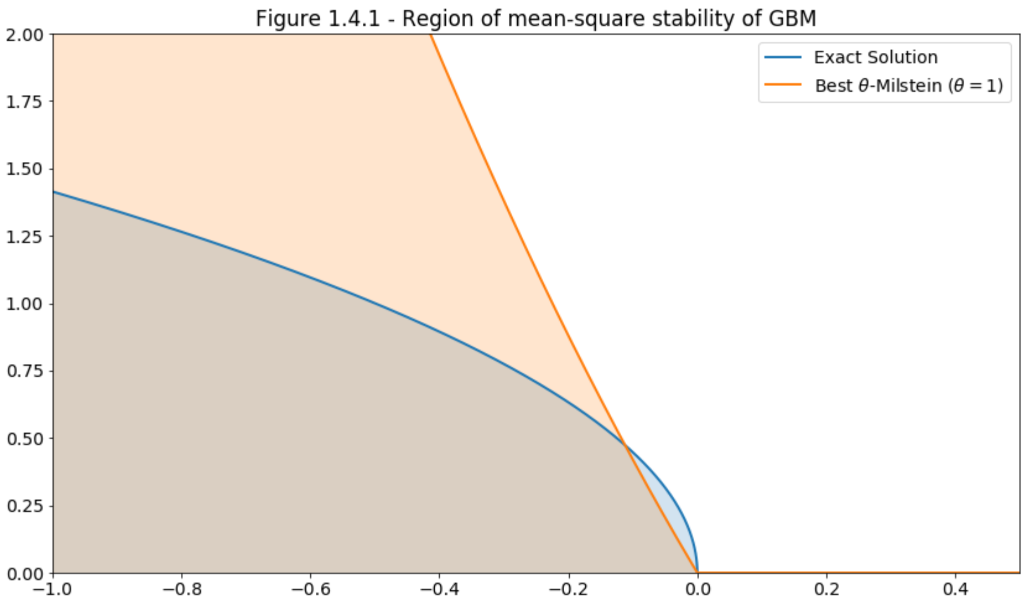
$$(\Delta t\sigma^2)^2 + 2\Delta t\sigma^2 + 2(2\Delta t\mu + (\Delta t\mu)^2(1-2\theta)) < 0 \tag{4}$$

which is a quadratic inequality in $\Delta t\sigma^2$. The stability region for $\theta = 0, 0.25, 0.5, 0.75, 1$ is:



1.4 A-stability for Numerical Scheme

There are no values of θ for which the θ-Milstein scheme is A-stable. This can be seen in the figure below.



As we can see in figure 1.3.1, the θ-Milstein scheme with largest region of stability is the $\theta = 1$ scheme. Since the exact solution has a region of parameters which is not covered by the 1-Milstein scheme we can conclude that there are no values of θ for which the θ-Milstein scheme is A-stable.

1.5 Implementation of Scheme

To show the required condition, we need to substitute the values $\theta = 1/4, \mu = -1, \sigma = 1$ into equation (4) to imply that the numerical scheme is numerically stable if and only if

$$\begin{aligned} &(\Delta t)^2 + 2\Delta t + 2(2\Delta t(-1) + (\Delta t(-1))^2(1 - 2\cdot(0.25))) < 0, \\ \iff &\Delta t^2 - \Delta t < 0, \quad \iff \Delta t(\Delta t - 1) < 0, \end{aligned}$$

which can be satisfied iff $\Delta t < \Delta t* := 1$ as required.

We now implement the method for these parameters and generate 10^5 replicas of the numerical solution over 100 time steps.

The implementation can be found in the file `problem1.py` and the output is:

```
Estimate for dt=2: 51902449.77023178
Estimate for dt=0.5: 6.79775768477993e-70
```

As we can see, in the first case, the estimate is very large, suggesting that the numerical method is not stable, whereas for the second case, the estimate is essentially 0, satisfying the stability condition which is that the expectation should tend to zero as n tends to infinity.

2 Problem 2 (Inference for SDEs)

2.1 Analytical Solution

We wish to solve the Ornstein-Uhlenbeck Equation:

$$\mathrm{d}X_t = -\theta(X_t - \mu)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad X_0 = (1+x) \tag{5}$$

where $\mu = -1,\ \theta = 1,\ \sigma = \sqrt{2}, x \sim U(0,1)$.

We can introduce the first change of variables $Y_t = X_t - \mu$, which renders the following equation

$$\mathrm{d}Y_t = -\theta Y_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{6}$$

Using a further change of variables $Z_t = f(t, Y_t),\ f(x) = e^{\theta t}x$, hence

$$\frac{\partial f}{\partial t} = \theta e^{\theta t}x, \quad \frac{\partial f}{\partial x} = e^{\theta t}, \quad \frac{\partial^2 f}{\partial t^2} = 0,$$

so using Itô's Formula we obtain:

$$\mathrm{d}(f(Y_t)) = \sigma e^{\theta t}\,\mathrm{d}W_t, \tag{7}$$

and we can integrate to obtain

$$\begin{aligned}
Z_t &= Z_0 + \sigma\int_0^t e^{\theta s}\,\mathrm{d}W_s,\\
\implies e^{\theta t}Y_t &= e^{\theta\cdot 0}Y_0 + \sigma\int_0^t e^{\theta s}\,\mathrm{d}W_s,\\
\implies Y_t &= e^{-\theta t}Y_0 + \sigma\int_n^t e^{\theta(s-t)}\,\mathrm{d}W_s,\\
\implies (X_t - \mu) &= e^{-\theta t}(X_0 - \mu) + \sigma\int_0^t e^{\theta(s-t)}\,\mathrm{d}W_s,\\
\implies X_t &= \mu + e^{-\theta t}(X_0 - \mu) + \sigma\int_0^t e^{\theta(s-t)}\,\mathrm{d}W_s.
\end{aligned}$$

Upon substituting our values we can obtain the solution

$$X_t = -1 + e^{-t}(2+x) + \sqrt{2}\int_0^t e^{s-t}\,\mathrm{d}W_s. \tag{8}$$

To calculate $\mathbb{E}[X_T^2]$, we first compute the following:

$$X_T^2 = 1 - 2e^{-T}(x+2) - 2\sqrt{2}e^{-T}(x+2)I_T - 2\sqrt{2}I_T + e^{-2T}(x+2)^2 + 2I_T^2,$$

where $I_T = \int_0^T e^{\theta(s-T)}\,\mathrm{d}W_s = \int_0^T e^{s-T}\,\mathrm{d}W_s$.

Taking expectations, and using the fact that $\mathbb{E}[x] = 1/2, \mathbb{E}[x^2] = 1/3\ \mathbb{E}[I_T] = 0,\ \mathbb{E}[I_T x] = \mathbb{E}[x I_T] = 0$, since I_T, x are independent, we obtain:

$$\mathbb{E}[X_T^2] = 1 - 5e^{-T} + \frac{19}{3}e^{-2T} + 2\mathbb{E}[I_T^2].$$

We can compute $\mathbb{E}[I_T^2]$ using Itô's isometry:

$$\begin{aligned}
\mathbb{E}[I_T^2] = \mathbb{E}\left(\int_0^T e^{s-T}\,\mathrm{d}W_s\right)^2 &= \int_0^T \mathbb{E}\left[\left(e^{s-T}\right)^2\right]\mathrm{d}s,\\
= \int_0^T e^{2(s-T)}\,\mathrm{d}s &= \left[\frac{1}{2}e^{2(s-T)}\right]_0^T = \frac{1}{2}(1 - e^{-2T}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}[X_T^2] &= 1 - 5e^{-T} + \frac{19}{3}e^{-2T} + (1 - e^{-2T}),\\
\implies \mathbb{E}[X_1^2] &= 2 - 5e^{-1} + \frac{16}{3}e^{-2} \approx 0.882.
\end{aligned}$$

2.2 Itô Integral being Normally Distributed

To show the desired result, we begin by showing that I follows a normal distribution.

Firstly, we note that we can write the integral I_N as

$$I_N = \int_0^T f_N(s)\,\mathrm{d}W_s = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in P_n} f_N(t_{i-1})(W_{t_i} - W_{t_{i-1}})$$

where P_n denotes the set of partitions of the interval $[0, T]$. We know that the Wiener process W_t is normally distributed, and hence, using the second fact given, $(W_{t_i} - W_{t_{i-1}})$ is normally distributed, hence the whole sum $\sum_{[t_{i-1},t_i]\in P_n} f_N(t_{i-1})(W_{t_i} - W_{t_{i-1}})$ is normally distributed. Using the third given fact about the convergence of a sequence of random variables, we conclude that I_N is a random variable that is normally distributed.

Using the first and third given facts, we can show that

$$\begin{aligned}
I = \lim_{N\to\infty} \int_0^T f_N(s)\,\mathrm{d}W_s &= \int_0^T \lim_{N\to\infty} f_N(s)\,\mathrm{d}W_s,\\
&= \int_0^T f(s)\,\mathrm{d}W_s,
\end{aligned}$$

and hence I has been shown to be normally distributed.

It remains to find the expectation and variance of I.

To find the expectation, we note that

$$
\begin{aligned}
\mathbb{E}\left[I\right] &= \lim_{N\to\infty} \mathbb{E}\left[I_N\right], \\
&= \mathbb{E}\left[\lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in P_n} f_N(t_{i-1})(W_{t_i} - W_{t_{i-1}})\right], \\
&= \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in P_n} f_N(t_{i-1})\mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})\right], \\
&= 0,
\end{aligned}
$$

since $\mathbb{E}\left[W_t\right] = 0$. For the variance, we can use Itô's isometry to obtain

$$
\mathrm{Var}[I] = \mathrm{Var}\left[\int_0^T f(s)\,\mathrm{d}W_s\right] = \mathbb{E}\left[\left(\int_0^T f(s)\,\mathrm{d}W_s\right)^2\right] = \int_0^T |f(s)|^2\,\mathrm{d}W_s.
$$

and hence the result has been shown.

2.3 Iterative Numerical Scheme for Ornstein-Uhlenbeck SDE

It can be shown that the solution to the process at time t given the solution at time τ, we can use the following equation

$$
\begin{aligned}
X_t &= \mu + e^{-\theta(t-\tau)}(X_\tau - \mu) + \sigma \int_\tau^t e^{\theta(s-t)}\,\mathrm{d}W_s, \\
&= -1 + e^{-(t-\tau)}(X_\tau + 1) + \sqrt{2}\int_\tau^t e^{s-t}\,\mathrm{d}W_s
\end{aligned}
$$

We can discretise this as the following

$$
X_{n+1}^{\Delta t} = -1 + e^{-(t_{n+1}-t_n)}(X_n^{\Delta t} + 1) + \sqrt{2}\int_{t_n}^{t_{n+1}} e^{s-t_{n+1}}\,\mathrm{d}W_s
$$

and using part 2, the stochastic integral is normally distributed, hence

$$
\int_{t_n}^{t_{n+1}} e^{s-t_{n+1}}\,\mathrm{d}W_s = \xi\sqrt{\int_{t_n}^{t_{n+1}} e^{2(s-t_{n+1})}\,\mathrm{d}s} = \xi\sqrt{\frac{1}{2}(1-\exp(2(t_n - t_{n+1})))}
$$

and since $\Delta t = t_{n+1} - t_n$, our iterative scheme is

$$
X_{n+1}^{\Delta t} = -1 + e^{-\Delta t}(X_n^{\Delta t} + 1) + \xi\sqrt{1-e^{-2\Delta t}}, \quad \xi \sim N(0,1) \tag{9}
$$

In general, the scheme is

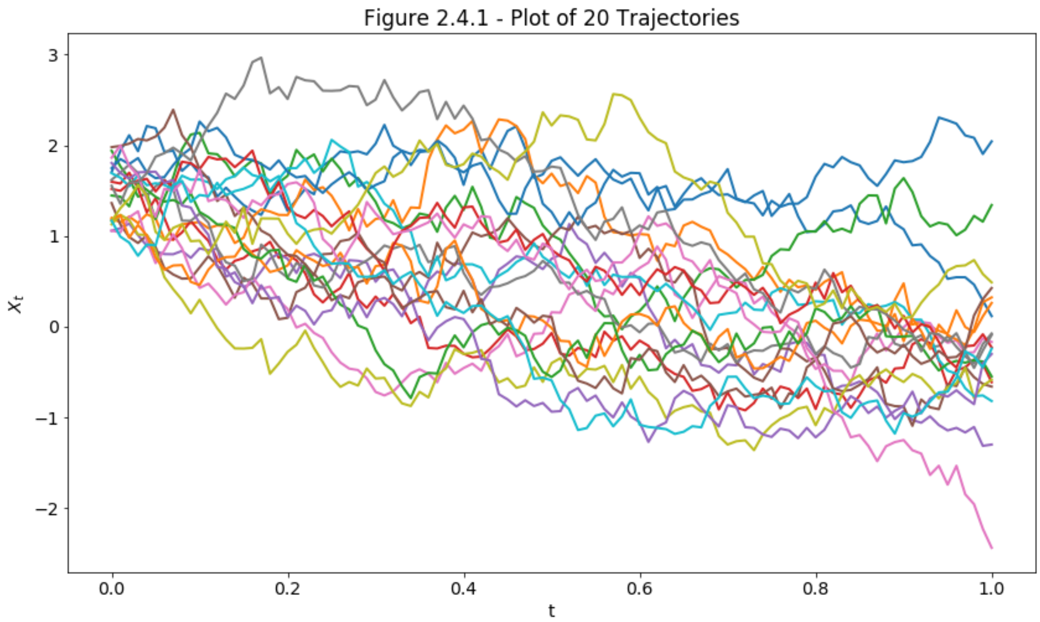
$$
X_{n+1}^{\Delta t} = \mu + e^{-\theta\Delta t}(X_n^{\Delta t} - \mu) + \sigma\xi\sqrt{\frac{1}{2\theta}(1-e^{-2\theta\Delta t})}, \quad \xi \sim N(0,1) \tag{10}
$$

2.4 Implementation of Scheme

The scheme is implemented in the file `problem2.py`. The following output contains the 99% confidence intervals for $\mathbb{E}[X_T]$ and $\mathbb{E}[X_T^2]$

```
The 99% CI for E[X_t^2]: [0.8627457588585971, 0.9272205299405393]
```

The 20 trajectories of the numerical solution are below.



Comments:

Since the calculate theoretical value of the expectation being ≈ 0.882 lies inside our confidence interval, we can be confident that the implementation is correct.

2.5 PDF of $\hat{X}$

Since $t_0 = 0, X_0 = 1 + x$ and $x \sim U(0,1)$, then $X_0 \sim U(1,2)$.

If we assume we are given X_{n-1}, then $(X_n|X_{n-1}) \sim N(\mu + e^{-\theta\Delta t}(X_{n-1} - \mu), \sigma^2(1 - e^{-2\theta\Delta t})/2\theta)$.

If we expand the iterative scheme, we can obtain the formula

$$
\begin{aligned}
X_n^{\Delta t} &= \mu + e^{-\theta\Delta t}(X_{n-1}^{\Delta t} - \mu) + \xi_n\sigma\sqrt{\frac{1}{2\theta}(1 - e^{-2\theta\Delta t})}, \\
&= \mu + e^{-2\theta\Delta t}(X_{n-2}^{\Delta t} - \mu) + \sigma(\xi_n + \xi_{n-1}e^{-\theta\Delta t})\sqrt{\frac{1}{2\theta}(1 - e^{-2\theta\Delta t})}, \\
&= \dots, \\
&= \mu + e^{-n\theta\Delta t}(X_0 - \mu) + \sum_{k=1}^{n}\xi_k e^{-\theta(n-k)\Delta t}\sigma\sqrt{\frac{1}{2\theta}(1 - e^{-2\theta\Delta t})}, \\
&= xe^{-n\theta\Delta t} + \mu + (1-\mu)e^{-n\theta\Delta t} + \sum_{k=1}^{n}\xi_k e^{-\theta(n-k)\Delta t}\sigma\sqrt{\frac{1}{2\theta}(1 - e^{-2\theta\Delta t})}
\end{aligned}
$$

where each of the $\xi_i \sim N(0,1)$.

Since $x \sim U(0,1)$, then we can say that $X_n^{\Delta t}$ is the sum of two independent random variables

$$\alpha \sim U(0, e^{-n\theta\Delta t}) := U(a,b), \tag{11}$$

and

$$\beta \sim N\left(\mu + (1-\mu)e^{-n\theta\Delta t}, \sum_{k=1}^{n}\frac{\sigma^2\exp\left(-2\theta(n-k)\Delta t\right)}{2\theta}(1 - e^{-2\theta\Delta t})\right) := N(c, d^2) \tag{12}$$

where f_1, f_2 are the PDFs of α, β respectively. We can find the PDF by using the convolution of two independent random variables

$$
\begin{aligned}
f_{\hat{X}}(x) &= \int_{-\infty}^{\infty} f_1(x)f_2(x-u)\,\mathrm{d}u, \\
&= \frac{1}{b-a}\int_a^b \frac{1}{d^2\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-u-c}{d}\right)^2\right]\mathrm{d}u,
\end{aligned}
$$

as required. Alternatively, if treat X_0 as a constant, since it does not affect our drift, we can do estimations with the PDF (the density of the law w.r.t. the Lebesgue measure) of $\hat{X} = (X_0, X_1, ..., X_N)$ as given by

$$f_{\hat{X}}^N(x_0, ..., x_N; \theta, \sigma, \mu) = \left|\frac{1}{\sqrt{2\pi\beta^2}}\right|^N \exp\left(-\frac{1}{2\beta^2}\sum_{k=0}^{N-1}|x_{k+1} - \mu - e^{-\theta\Delta t}(x_k - \mu)|^2\right)$$

where

$$\beta^2 = \sigma\sqrt{\frac{1}{2\theta}(1 - e^{-2\theta\Delta t})}$$

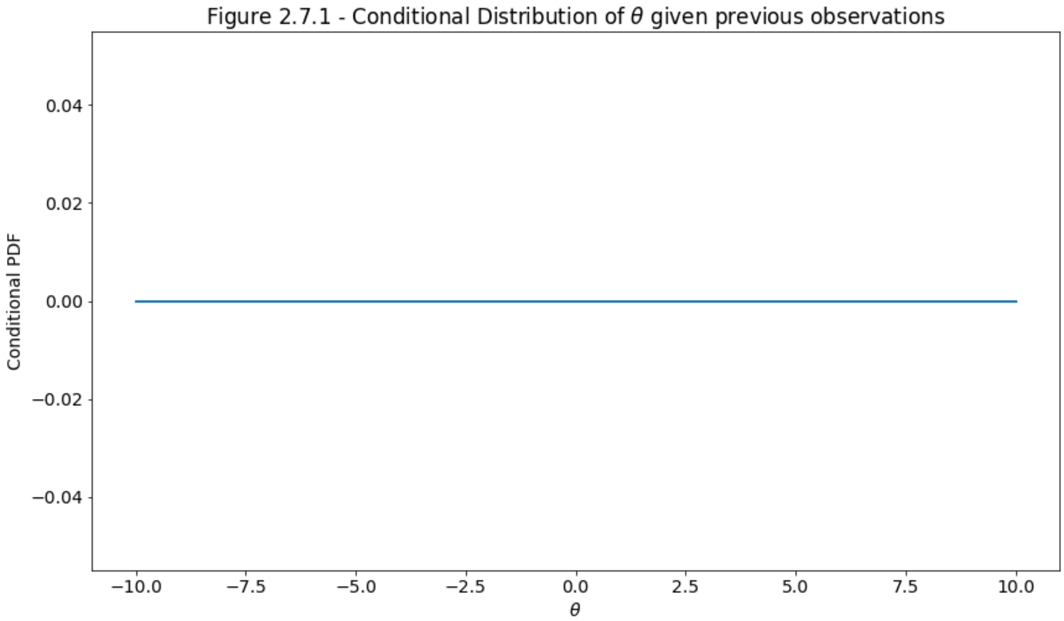
2.6 MLE for θ based on this sequence

We generate 10^7 observations of the process (4) with parameters $\mu = -1, \theta = 1, \sigma = \sqrt{2}$. We wish to estimate the MLE of θ based on this data. Since we are assuming that μ, σ are known, we shall minimise our likelihood function which in this case it is given by the PDF of $\hat{X}$ evaluated at the data given with respect to θ.

After running our code, the output is: `Theta MLE = 1.001073 - Real Theta = 1`. This is very close to the real value of theta, indicating that our procedure works well.

2.7 Bayesian Calculations

The code and plot for this section can be found in the file `problem2.py`. Unfortunately, I wasn't able to get the conditional distribution working and as such, I got the following plot.



The theory for this section can be found hand-written below.

P2.Q7

We are given that $\theta \sim N(2,1)$. From the lectures / notebooks, we know that the joint PDF

$$f_{\theta,\hat{x}}(\theta, x_0, \dots, x_N) = f_\theta(\theta)\, f_{\hat{x}|\theta}(\hat{x}|\theta)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-2)^2\right) \cdot \left|\frac{1}{\sqrt{2\pi\beta^2}}\right|^N \exp\left(\frac{-1}{2\beta^2} \sum_{k=0}^{N} \left| x_{k+1} - \mu - e^{-\theta\Delta t} \times (x_k - \mu)\right|^2\right)$$

The conditional probability distribution

$$f_{\theta|\hat{x}}(\theta|x_0, \dots, x_N) = \frac{f_{\theta,\hat{x}}(m, \hat{x})}{\int_{\mathbb{R}} f_{\theta,\hat{x}}(\theta, \hat{x})\, d\theta}$$

Hence, we only need to calculate $f_{\theta,\hat{x}}$ to retrieve $f_{\theta|\hat{x}}$ up to a constant factor.

3 Problem 3 (Numerical Method for a Stratonovich SDE)

3.1 Chain rule for Stratonovich SDE

We begin by exploring the sum

$$
\begin{aligned}
\sum_{j=0}^{N-1}(W_{t_j^N}+W_{t_{j+1}^N})(W_{t_{j+1}^N}-W_{t_j^N}) &= \sum_{j=0}^{N-1}(W^2_{t_{j+1}^N}-W^2_{t_j^N}),\\
&= W^2_{t_N^N}-W^2_{t_0^N}
\end{aligned}
$$

Now using the definition of the integral, we conclude that, in the mean-square sense,

$$
\begin{aligned}
\int_0^T W_s \circ \mathrm{d}W_s &= \lim_{N\to\infty}\sum_{j=0}^{N-1}(W_{t_j^N}+W_{t_{j+1}^N})(W_{t_{j+1}^N}-W_{t_j^N})/2,\\
&= \lim_{N\to\infty}(W^2_{t_N^N}-W^2_{t_0^N})/2,\\
&= (W_t^2-W_0^2)/2 = \frac{W_t^2}{2},
\end{aligned}
$$

which is what we want, since $W_0 = 0$, as required.

Using the assumptions given, with $Y_t := h(X_t) = X_t^{m+1} \implies h'(X_t) = (m+1)X_t^m$

$$
\begin{aligned}
Y_t - Y_0 &= \int_0^t h'(X_s)b(s,\omega)\,\mathrm{d}s + \int_0^t h'(X_s)\sigma(s,\omega)\circ \mathrm{d}W_s,\\
\implies X_t^{m+1} - X_0^{m+1} &= \int_0^t (m+1)X_t^m \circ \mathrm{d}W_s,\\
\implies \frac{X_t^{m+1}}{(m+1)} &= \frac{X_0^{m+1}}{(m+1)} + \int_0^t X_t^m \circ \mathrm{d}W_s,
\end{aligned}
$$

hence if we take $X_t = W_t$, we arrive at the solution, since $W_0 = 0$

$$
\int_0^t W_s^m \circ \mathrm{d}W_s = \frac{W_t^{m+1}}{m+1}, \qquad m \in \mathbb{N}_{>0}.
$$

3.2 Solution to Geometric Brownian Motion Stratonovich SDE

We can compute the solution by using the chain rule (7) in the same way as we use it in classical calculus. Therefore, if we find a suitable anti-derivative function, we will be able to solve the SDE using rules analogous to classical calculus.

We proceed as follows

$$\begin{aligned}
\mathrm{d}X_t &= \mu X_t \,\mathrm{d}t + \sigma X_t \circ \mathrm{d}W_t, \\
\implies \frac{\mathrm{d}X_t}{X_t} &= \mu \,\mathrm{d}t + \sigma \circ \mathrm{d}W_t, \\
\implies \int_0^t \frac{\mathrm{d}X_s}{X_s} &= \int_0^t \mu \,\mathrm{d}s + \int_0^t \sigma \circ \mathrm{d}W_s, \\
\implies \ln(X_t) - \ln(X_0) &= \mu t + \sigma W_t, \quad \text{using previous part}, \\
\implies \ln(X_t) &= \mu t + \sigma W_t, \quad \text{since } X_0 = 1, \\
\implies X_t &= \exp\left(\mu t + \sigma W_t\right),
\end{aligned}$$

which is the solution to the Stratonovich SDE at hand.

3.3 Showing 2 identities

The proof of the first identity can be seen in the following hand-written picture:

P3. Q3

Proving $(t-s)\,J_{(1)}^{s,t} = J_{(0,1)}^{s,t} + J_{(1,0)}^{s,t}$.

$$J_{(1)}^{s,t} = \int_s^t 1 \circ dV^1_{u_1} = \int_s^t 1 \circ dW_{u_1} = W_t - W_s$$

Hence $(t-s)\,J_{(1)}^{s,t} = (t-s)(W_t - W_s)$

$$J_{(0,1)}^{s,t} = \int_s^t \int_s^{u_1} 1 \circ dV^0_{u_2} \circ dV^1_{u_1} = \int_s^t \int_s^{u_1} 1 \, du_2 \circ dW_{u_1}$$

$$= \int_s^t (u_1 - s) \circ dW_{u_1} = \underbrace{\int_s^t u_1 \circ dW_{u_1}}_{①} - s(W_t - W_s)$$

$$J_{(1,0)}^{s,t} = \int_s^t \int_s^{u_1} 1 \circ dV^1_{u_2} \circ dV^0_{u_1} = \int_s^t \int_s^{u_1} 1 \circ dW_{u_2} \, du_1$$

$$= \int_s^t (W_{u_1} - W_s) \, du_1 = \underbrace{\int_s^t W_{u_1} \, du_1}_{②} - W_s(t-s)$$

$$① + ② = \int_s^t W_{u_1} \, du_1 + \int_s^t u_1 \circ dW_{u_1}$$

$= t W_t - s W_s$, by using (7) with mutatis mutandis.

$$\text{Thus, } J_{(0,1)}^{s,t} + J_{(1,0)}^{s,t} = tW_t - sW_s - s(W_t - W_s) - W_s(t-s)$$
$$= tW_t - tW_s - sW_t + sW_s$$
$$= (t-s)(W_t - W_s)$$

Therefore $J_{(1)}^{s,t} = J_{(0,1)}^{s,t} + J_{(1,0)}^{s,t}$, as required.

The proof for the second identity goes as follows:

P3. Q3. P.2 <u>Second Identity</u>

We begin by writing $J^{s,t}_{(1,1)} = \int_s^t \int_s^{u_1} 1 \circ dV^1_{u_2} \circ dV^1_{u_1}$

$$= \int_s^t \int_s^{u_1} 1 \circ dW_{u_2} \circ dW_{u_1} = \int_s^t (W_{u_1} - W_s) \circ dW_{u_1}$$

$$= \frac{W_t^2}{2} - \frac{W_s^2}{2} - W_s(W_t - W_s) = \frac{(W_t - W_s)^2}{2}$$

Hence $(s-t) J^{s,t}_{(1,1)} = (s-t) \frac{(W_t - W_s)^2}{2}$

Now $J^{s,t}_{(1,1,0)} = \int_s^t \int_s^{u_1} \int_s^{u_2} 1 \circ dV^1_{u_3} \circ dV^1_{u_2} \circ dV^0_{u_1}$

$$= \int_s^t \int_s^{u_1} (W_{u_2} - W_s) \circ dW_{u_2} \, du_1$$

$$= \int_s^t \left[\frac{W_{u_1}^2}{2} - \frac{W_s^2}{2} - W_s (W_{u_1} - W_s) \right] du_1$$

$$= \underbrace{\int_s^t \frac{W_{u_1}^2}{2} \, du_1}_{1} - \underbrace{W_s \int_s^t W_{u_1} \, du_1}_{2} + \underbrace{\frac{W_s^2}{2}(t-s)}_{7}$$

Similarly $J^{s,t}_{(1,0,1)} = \int_s^t \int_s^{u_1} \int_s^{u_2} 1 \circ dV^1_{u_3} \circ dV^0_{u_2} \circ dV^1_{u_1}$

$$= \int_s^t \int_s^{u_1} (W_{u_2} - W_s) \, du_2 \circ dW_{u_1}$$

$$= \ldots = \underbrace{\int_s^t \int_s^{u_1} W_{u_2} \, du_2 \circ dW_{u_1}}_{3} - \underbrace{W_s \int_s^t u_1 \circ dW_{u_1}}_{4} + \underbrace{s W_s (W_t - W_s)}_{8}$$

Now, $J^{s,t}_{(0,1,1)} = \int_s^t \int_s^{u_1} \int_s^{u_2} 1 \circ d V^0_{u_3} \circ d V^1_{u_2} \circ d V^1_{u_1}$

$$= \int_s^t \int_s^{u_1} (u_2 - s) \circ dW_{u_2} \circ dW_{u_1}$$

$$= \dots = \underbrace{\int_s^t \int_s^{u_1} u_2 \circ dW_{u_2} \circ dW_{u_1}}_{(5)} - \left\{ \underbrace{\frac{sW_t^2}{2} + sW_sW_t - s\frac{W_s^2}{2}}_{(9)} \right\}$$

Now, (2) + (4) $= W_s \left[\int_s^t W_{u_1} du_1 + \int_s^t u_1 \circ dW_{u_1} \right]$

$$= W_s \left(tW_t - sW_s \right) = \underbrace{tW_sW_t - sW_s^2}_{(10)}$$

And $(3)+(5) = \int_s^t \int_s^{u_1} W_{u_2} du_2 \circ dW_{u_1} + \int_s^t \int_s^{u_1} u_2 \circ dW_{u_2} \circ dW_{u_1}$

$$= \int_s^t \left[\int_s^{u_1} W_{u_2} du_2 + \int_s^{u_1} u_2 \circ dW_{u_2} \right] \circ dW_{u_1}$$

$$= \int_s^t \left(u_1 W_{u_1} - sW_s \right) \circ dW_{u_1} = \dots$$

$$= \underbrace{\int_s^t u_1 W_{u_1} \circ dW_{u_1}}_{(6)} - \underbrace{sW_sW_t + sW_s^2}_{(11)}$$

And (1) + (6) $= \int_s^t \frac{W_{u_1}^2}{2} du_1 + \int_s^t u_1 W_{u_1} \circ dW_{u_1}$

$$= \underbrace{\frac{tW_t^2}{2} - \frac{sW_s^2}{2}}_{(12)}$$

<u>Summing all the terms:</u>

(7) + (8) − (9) − (10) − (11) + (12)

$$= \frac{W_s^2}{2}(t-s) + sW_s(W_t - W_s) - \frac{sW_t^2}{2} + sW_sW_t - \frac{sW_s^2}{2}$$

$$- tW_sW_t + sW_s^2 + \frac{tW_t^2}{2} - \frac{sW_s^2}{2}$$

$$= \dots = (s-t)\frac{(W_s - W_t)^2}{2}, \text{ as required}$$

//

3.4 Taylor Methods for Stratonovich SDEs

$\boxed{\text{P3.Q4.}}$

The SDE $dX_t = b(X_t)\,dt + \sigma(X_t)\circ dW_t,\ X_t = x_0$ can be written in the form:

$$X_t - X_s = \int_s^t b(X_{u_1})\,du_1 + \int_s^t \sigma(X_{u_1})\circ dW_{u_1}$$

Using the chain rule for Stratonovich SDEs (7), we can obtain

$$f(X_t) = f(X_s) + \int_s^t b(X_{u_1})\,f'(X_{u_1})\,du_1 + \int_s^t \sigma(X_{u_1})\,f'(X_{u_1})\circ dW_{u_1}$$

(+)
$$= f(X_s) + \int_s^t \mathcal{L}_0 f(X_{u_1})\,du_1 + \int_s^t \mathcal{L}_1 f(X_{u_1})\circ dW_{u_1}.$$

If we let $f(x) \equiv x$, we recover our original equation and $\mathcal{L}_0 f = b$, $\mathcal{L}_1 f = \sigma$.

We now apply (+) to the function $f = b$ and $f = \sigma$ to obtain:

$$X_t - X_s = \underbrace{\int_s^t \Big[\, b(X_s)}_{J_{(0)}^{s,t} f_{(0)}(x_s)} + \int_s^{u_1} \mathcal{L}_0 b(X_{u_2})\,du_2 + \dots + \int_s^{u_1} \mathcal{L}_1 b(X_{u_2})\circ dW_{u_2} \Big]\,du_1$$

$$+ \underbrace{\int_s^t \Big[\, \sigma(X_s)}_{J_{(1)}^{s,t} f_{(1)}(x_s)} + \int_s^{u_1} \mathcal{L}_0 \sigma(X_{u_2})\,du_2 + \int_s^{u_2} \mathcal{L}_1 \sigma(X_{u_2})\circ dW_{u_2} \Big]\circ dW_{u_1}$$

(Upon applying the formula over and over)

$$= J_{(0)}^{s,t} f_{(0)}(X_s) + J_{(1)}^{s,t} f_{(1)}(X_s)$$

$$+ J_{(0,0)}^{s,t} f_{(0,0)}(X_s) + J_{(0,1)}^{s,t} f_{(0,1)}(X_s) + J_{(1,0)}^{s,t} f_{(1,0)}(X_s)$$

$$+ J_{(1,1)}^{s,t} f_{(1,1)}(X_s) + \dots \quad \text{(higher multiplicity integrals)}$$

We can obtain the higher multiplicity integrals by applying (+) again.

3.5 Working Out the Strong Order 2 Numerical Scheme

P3.Q5

The set $A_2 = \{ \alpha \in \{0,1\}^n : n > 0 \text{ and } \sum_i (2-\alpha_i) \leq 4 \}$
$= \{ (0), (1), (0,0), (0,1), (1,0), (1,1), (1,1,0), (0,1,1), (1,0,1), (1,1,1), (1,1,1,1) \}$

So we need to work out the terms with these indices.

$$f_{(0)}(x) = \mathcal{L}_0 V(x) = \mathcal{L}_0 x = b(x) = \mu x$$
$$f_{(1)}(x) = \mathcal{L}_1 V(x) = \mathcal{L}_1 x = \sigma(x) = \sigma x$$

$$f_{(0,0)}(x) = \mathcal{L}_0 \mathcal{L}_0 V(x) = \mathcal{L}_0 \mathcal{L}_0 x = \mathcal{L}_0 \mu x = \mu^2 x$$
$$f_{(1,1)}(x) = \mathcal{L}_1 \mathcal{L}_1 V(x) = \mathcal{L}_1 \mathcal{L}_1 x = \mathcal{L}_1 \sigma x = \sigma^2 x$$
$$f_{(0,1)}(x) = f_{(1,0)}(x) = \mu \sigma x$$

$$f_{(1,1,1)}(x) = \sigma^3 x, \quad f_{(0,1,1)} = f_{(1,1,0)} = f_{(1,0,1)} = \mu \sigma^2 x.$$

$$f_{(1,1,1,1)}(x) = \sigma^4 x.$$

Now for the integrals:

$$J_{(0)}^{t_n,t_{n+1}} f_{(0)}(x) = \mu x \int_{t_n}^{t_{n+1}} du_1 = \mu x \,\Delta t$$

$$J_{(1)}^{t_n,t_{n+1}} f_{(1)}(x) = \sigma x \int_{t_n}^{t_{n+1}} 1 \circ dW_{u_1} = \sigma x \,\Delta W.$$

$$J_{(0,0)}^{t_n,t_{n+1}} f_{(0,0)}(x) = \mu^2 x \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} du_2 \, du_1 = \mu^2 x \,\frac{\Delta t^2}{2}$$

$$J_{(1,1)}^{t_n,t_{n+1}} f_{(1,1)}(x) = \sigma^2 x \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} 1 \circ dW_{u_2} \circ dW_{u_1} = \sigma^2 x \,\frac{\Delta W^2}{2}.$$

A pattern is beginning to emerge.

$$J_{(1,1,1)}^{t_n,t_{n+1}} f_{(1,1,1)}(x) = \sigma^3 x \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} \int_{t_n}^{u_2} 1 \circ dW_{u_3} \circ dW_{u_2} \circ dW_{u_1}$$
$$= \sigma^3 x \,\frac{\Delta W^3}{3!} = \sigma^3 x \,\frac{\Delta W^3}{6}$$

$$J^{t_n,t_{n+1}}_{(1,1,1,1)} f_{(1,1,1,1)}(x) = \dots = \sigma^4 x \, \frac{\Delta W^4}{4!}$$

$$J^{t_n,t_{n+1}}_{(0,1)} f_{(0,1)}(x) + J_{(1,0)} f_{(1,0)}(x)$$

$$= \underset{\substack{\parallel \\ f_{(1,0)}(x)}}{f_{(0,1)}(x)} \left[J^{t_n,t_{n+1}}_{(0,1)} + J^{t_n,t_{n+1}}_{(1,0)} \right] = f_{(0,1)}(x) \left[\Delta t \, J^{t_n,t_{n+1}}_{(1)} \right]$$

$$= \mu \sigma \Delta t \, \Delta W$$

$$\left(J^{t_n,t_{n+1}}_{(1,0,1)} + J^{t_n,t_{n+1}}_{(0,1,1)} + J^{t_n,t_{n+1}}_{(1,1,0)} \right) f_{(0,1,1)}(x)$$

$$= \left(\Delta t \, J^{t_n,t_{n+1}}_{(1,1)} \right) \mu \sigma^2 x$$

$$= \Delta t \, \frac{\Delta W^2}{2!} \mu \sigma^2 x$$

Hence, we have all the terms that we need, and the numerical scheme is

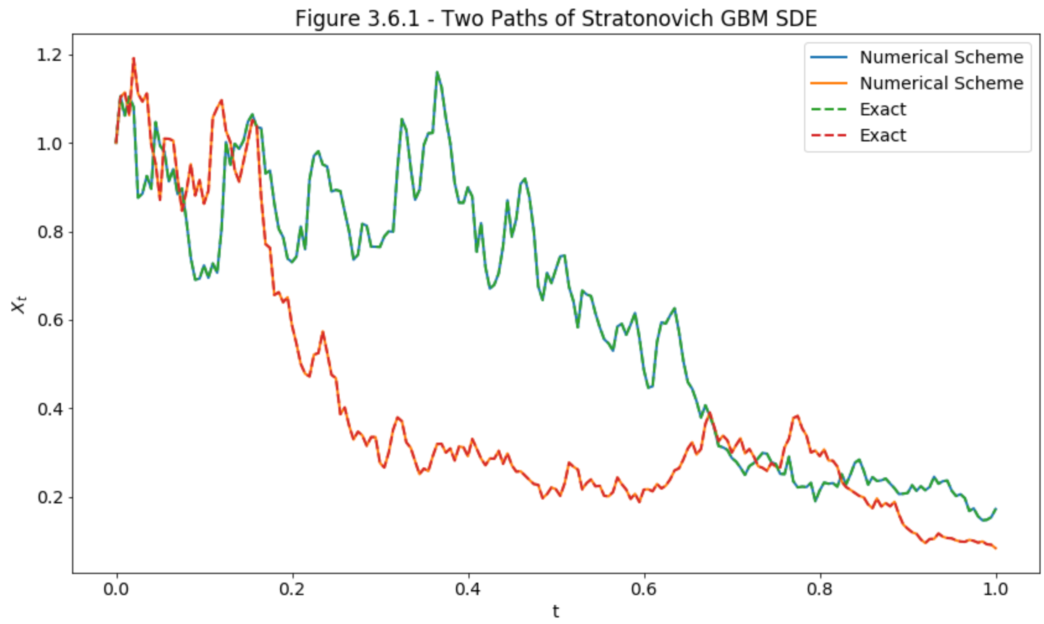
$$X^{\Delta t}_{n+1} = X^{\Delta t}_{n} \Big[1 + \mu \Delta t + \sigma \Delta W + \frac{\mu^2}{2} (\Delta t)^2 + \frac{\sigma^2}{2} (\Delta W)^2 + \mu \sigma \Delta t \Delta W + \frac{\mu \sigma^2 \Delta t (\Delta W)^2}{2} + \frac{\sigma^3}{6} (\Delta W)^3 + \frac{\sigma^4}{24} (\Delta W)^4 \Big].$$

The numerical Scheme is:

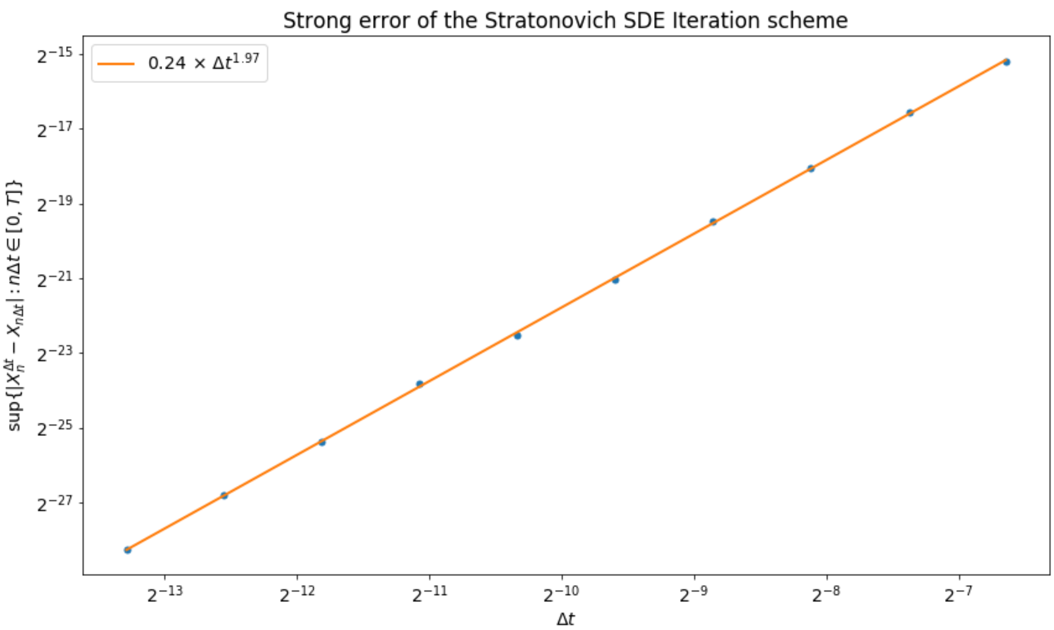
$$X^{\Delta t}_{n+1} = X^{\Delta t}_{n}(1 + \mu\Delta t + \sigma\Delta W + \frac{\mu^2}{2}(\Delta t)^2 + \frac{\sigma^2}{2}(\Delta W)^2 + \mu\sigma\Delta t\Delta W$$
$$+ \frac{\mu}{2}\sigma^2\Delta t(\Delta W)^2 + \frac{\sigma^3}{6}(\Delta W)^3 + \frac{\sigma^4}{24}(\Delta W)^4)$$

3.6 Implementation and Verification of Scheme

The implementation for this scheme can be found in the `problem3.py` file. We begin by veryfing that the scheme computes the desired solution by comparing 2 paths below.



To verify the order of strong convergence, we have done some numerical experiments found in the file `problem3.py` and plotted the strong order error in the following figure:



From this graph, we can see that the exponent is close to 2, verifying strong convergence order 2.