Coursework 2

Imperial College London

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M3A29 - Theory of Complex Systems

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1 Part A

The probability that a randomly, with uniform probability $P(s)$, selected site belongs to a cluster of size s existing solely of agents of type A is

$$P(s) = s \cdot (1-\rho_A)^2 \cdot (\rho_A)^s,$$

which is found in the lectures notes and slides on percolation.

2 Part B

The average size of A-clusters is:

$$\begin{aligned}
< s > &= \sum_{s=1}^{\infty} sP(s) = \sum_{s=1}^{\infty} \left[s^2 \cdot (1-\rho_A)^2 \cdot (\rho_A)^s\right] = (1-\rho_A)^2 \sum_{s=1}^{\infty} s^2 \rho_A^s, \\
&= (1-\rho_A)^2 \cdot (\rho_A + 4\rho_A^2 + 9\rho_A^3 + ...), \\
&= (1-\rho_A)^2 \cdot \rho_A \cdot (1 + 4\rho_A + 9\rho_A^2 + ...), \\
&= (1-\rho_A)^2 \cdot \rho_A \cdot (1+\rho_A) \cdot (1 + 3\rho_A + 6\rho_A^2 + ...).
\end{aligned}$$

We identify that $(1 + 3\rho_A + 6\rho_A^2 + ...)$ is the negative binomial series of $(1-\rho_A)^{-3}$, thus

$$< s > = (1-\rho_A)^2 \cdot \rho_A \cdot (1+\rho_A) \cdot \frac{1}{(1-\rho_A)^3} = \rho_A \cdot \frac{1+\rho_A}{1-\rho_A}.$$

As $\rho_A \to 1$,

$$\rho_A \cdot \frac{1+\rho_A}{1-\rho_A} \to \infty,$$

and the exponent describing the asymptotic behaviour is -1.

3 Part C

Let $P_\infty = P\{S_A = \infty\}$, and let q_A be the number of neighbours for each site. The probability that a randomly chosen site does not have an infinite amount of A-type agents is $1 - P_\infty$. From the lecture notes, equation (7.8) tells us that this is identical to $(1 - \rho_A P_\infty)^{q_A}$.

If we let $f(P_\infty) = (1 - \rho_A P_\infty)^{q_A}$, then we see that the non-zero solution for P_∞ is determined by $f'(0) = -1$. This is because $\frac{\partial}{\partial P_\infty}(1 - P_\infty) = -1$, thus

$$
\begin{aligned}
& f'(x) = -q_A \cdot \rho_A \cdot (1 - \rho_A P_\infty)^{q_A - 1} \quad (= -1), \\
\Rightarrow \quad & f'(0) = -q_A \cdot \rho_A \quad (= -1), \\
\Rightarrow \quad & \rho_A^* = \frac{1}{q_A}
\end{aligned}
$$

4 Part D

Explain the Schelling model as defined in Sec. 2.2.1:

The model, as defined in Sec. 2.2.1, is concerned with modelling the segregation that occurs when two groups with a contrasting feature occupy a fixed geographical space.

In the explanation, we can label the two groups A and B, and individuals within the groups as agent of type A and agent of type B.

The geographical space can be split up into N discrete sites, and each of these sites can be in three 'states', namely, occupied by an agent of type A, occupied by an agent of type B or be vacant.

We can then define the concept of 'closeness' between agents, and this is easily done if we model the N sites as a network. A site may be joined to a different site, and if for example, sites i and j are joined by an edge, we say that sites i and j are neighbours in the network. We can then define the concept of 'neighbourhood of site i' as the set of all neighbours of i, and this can be written as ∂i.

We may give them a property σ_i to every site i, taking three values. These values are: 1, if site i is occupied by an agent of type A, -1 if site i is occupied by an agent of type B and 0 if site i is empty.

This is a clever way of representing the state of a site, as the quantity $\sigma_i \sigma_j$ is useful in determining relationships between sites. The quantity $\sigma_i \sigma_j$ equals: 1 if i and j are occupied by agents of the same type, -1 if i and j are occupied by agents of different types, and 0 if either site is vacant.

5 Part E

Explain the mean field version in Sec. 4.1:

We first note that the mean field version described in Sec. 4.1 is an extremely simple Schelling-class model, abstracting many of the features of the more complicated model while keeping the underlying behaviour.

To achieve this simplistic model, sites in the network are grouped in pairs, such that each site has exactly one neighbour. Then, the satisfaction function can be represented by picking numbers $u, v \in [0, 1]$ and setting s_i equal to u, if i's neighbour is of the same type and equal to v otherwise. In keeping with the original Schelling model, we let $v > u$.

The initial condition in this mean field version is taking equal amounts of agents of each type, placing them randomly throughout the network, allowing for no vacancies. In each time-step, agents are allowed to move randomly by swapping places with another randomly-selected agent, according to its satisfaction function.
The transfer probabilities for this mean field model are given by

$$T_{ij}(\boldsymbol{\sigma}) = \frac{1}{N^2} \cdot (1 - s_i) \cdot s_j^{(ij)}.$$

Unpacking this equation, we observe that the factor $s_j^{(ij)}$ is the attractiveness of site j to the first agent selected, $(1 - s_i)$ introduces some inertia on the part of agent i, i.e. a measure of satisfaction with their original site and the N^{-2} factor comes from selecting two sites at random from the network, first i then j.

6 Part F

Explain how Eq.(6) follows from Eq.(5) for the simple model considered in Sec. 4.1 and 4.2:

To be able to derive Eq. (6) from Eq. (5), we first consider the interface density at time t, as defined in equation (3):

$$x(t) = \frac{\text{No. of edges between agents of opposite types}}{\text{No. of edges between agents of any type}}.$$

The function $x(t)$ is of the state $\boldsymbol{\sigma}$ of the whole system, but in the mean field, the relationship is simple enough to erase dependence on $\boldsymbol{\sigma}$.

The main feature to note is that, up to a trivial renaming of the sites, two system states of this model can differ only in the number of agents which are paired with another with a different type. Therefore, at every time-step, there are three possible outcomes to the system. The first is a swap between two agents of different types leading to them becoming homogeneous. The second is the a swap between two homogeneous agents leading to them becoming different. The last option is for no swap to occur, leading to no change to the state of the system. The first two result possibilities in a change of $\pm 4/N$ to the interface density respectively. More formally,

$$x(t+1) = \begin{cases} x(t) + 4/N & \text{with probability: p1,} \\ x(t) - 4/N & \text{with probability: p2,} \\ x(t) & \text{otherwise.} \end{cases}$$

It remains to find the probabilities p1 and p2, and obtain them in the same form as in Eq. (6).

Finding p1: If $x(t+1) = x(t) + 4/N$, then this involves swapping two homogeneous agents leading them to becoming different. There are $Nx(t)/2$ heterogeneous pairs, hence there are $(N - Nx(t))/2$ homogeneous pairs, and hence $(N - Nx(t)) \cdot (N - Nx(t))/2$ ways of choosing two of these in order. Combining this result with the transfer probabilities as defined in Sec. 4.1

$$\begin{aligned} p1 &= \frac{(N - Nx(t)) \cdot (N - Nx(t))}{2} \cdot T_{ij} = \frac{N^2 \cdot (1 - x(t))^2}{2} \cdot \frac{(1 - s_i) \cdot s_j^{(ij)}}{N^2}, \\ &= (1 - x(t))^2 \cdot \frac{(1 - s_i) \cdot s_j^{(ij)}}{2} = (1 - x(t))^2 \cdot \frac{(1 - u) \cdot v}{2}, \end{aligned}$$

where we let $v = s_i$ and $u = s_j^{(ij)}$.

If $x(t+1) = x(t) - 4/N$, then this involves swapping two heterogeneous agents leading them to becoming homogeneous. There are $Nx(t)(Nx(t)/2-1)$ ways of choosing two heterogeneous agents in order. This together with the transfer probabilities we find that

$$p2 = Nx(t)(Nx(t)/2-1)\cdot T_{ij} = Nx(t)(Nx(t)/2-1)\cdot\frac{(1-s_i)\cdot s_j^{(ij)}}{N^2},$$
$$= x(t)(x(t)-2/N)\cdot\frac{(1-s_i)\cdot s_j^{(ij)}}{2} = x(t)\left(x(t)-\frac{2}{N}\right)\cdot\frac{(1-u)\cdot v}{2},$$

where we let $u = s_i$ and $v = s_j^{(ij)}$.

The probability of $x(t+1) = x(t)$ equals $1-(p1+p2)$. This completes the derivation of Eq. (6).

7 Part G

Let $t' = t/N$ be the time re-scaled by a factor of N, then

$$\frac{t+1}{N} = t' + \frac{1}{N}.$$

So

$$\frac{dx}{dt'} = \lim_{N\to\infty}\left[\frac{x\left(t'+\frac{1}{N}\right)-x(t')}{\frac{t+1}{N}-\frac{t}{N}}\right] = \lim_{N\to\infty}\left[\frac{x\left(t'+\frac{1}{N}\right)-x(t')}{\frac{1}{N}}\right].$$

Therefore

$$\frac{dx}{dt'} = \begin{cases} 4 & \text{with probability} \quad (1-x(t))^2(1-u)v/2, \\ -4 & \text{with probability} \quad x(t)(x(t)-2/N)(1-v)u/2, \\ 0 & \text{otherwise.} \end{cases}$$

As $N\to\infty$, the time-step goes to zero, therefore the function becomes continuous and thus we are able to take the expected value, which becomes

$$\frac{dx}{dt'} = 4(1-x(t'))^2(1-u)v/2 - 4x(t')(x(t')-2/N)(1-v)u/2,$$
$$= 2(1-u)v(1-x(t'))^2 - 2(1-v)ux(t')^2 \quad \text{as} \quad N\to\infty,$$
$$= \alpha(1-x(t'))^2 - \beta x(t')^2,$$

where we let $\alpha = 2(1-u)v, \quad \beta = 2(1-v)u$.

8 Part H

In this part, I simulate the stochastic process described by Eq. (6) with parameters $u = 0.5, \quad v = 0.25$.

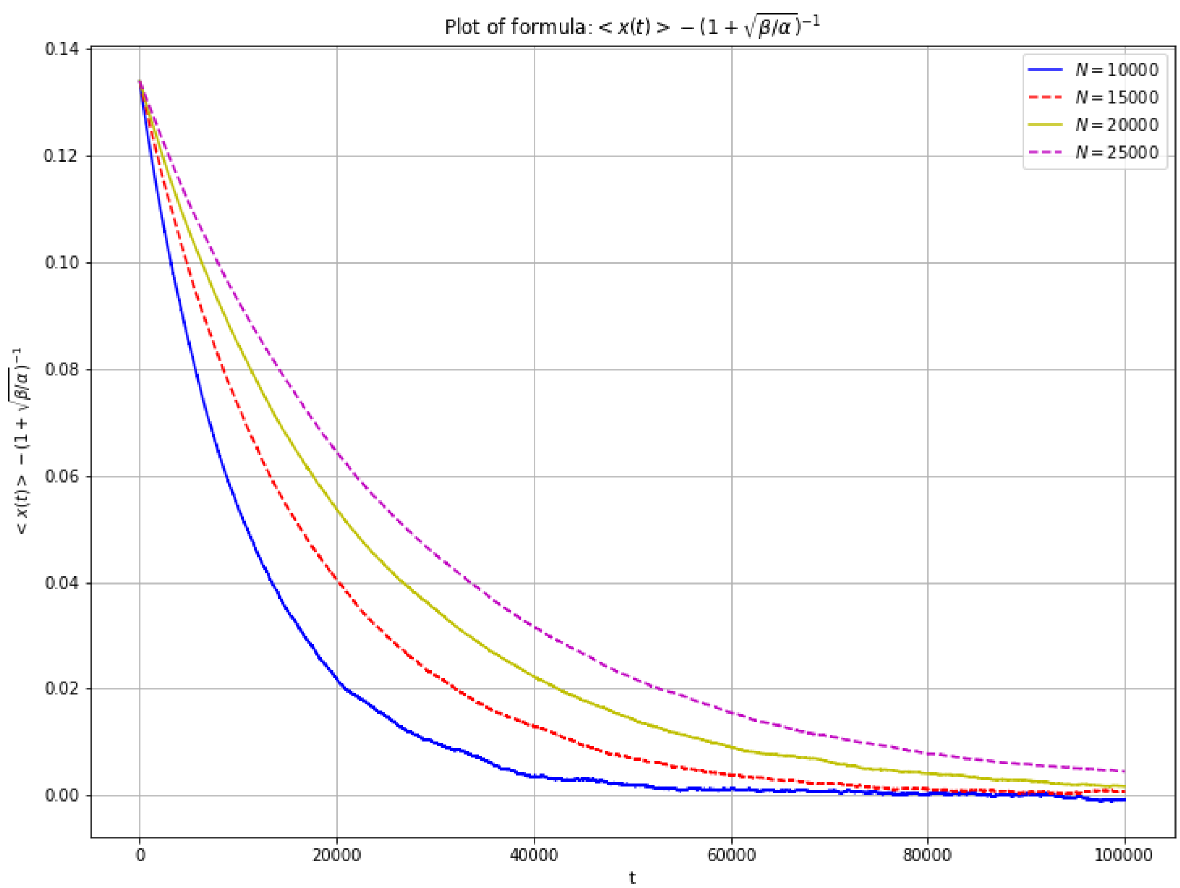
In the paper, the expression in Eq. (9) says that as $t \to \infty$,

$$x(t) \to \frac{1}{1 + \sqrt{\beta/\alpha}}.$$

This is the same as saying that the expression

$$x(t) - \frac{1}{1 + \sqrt{\beta/\alpha}} \to 0.$$

In my code (on following page), I simulate the process many times, take an average of $x(t)$ and I plot the last expression. I create a figure illustrating this last expression.



The figure clearly shows that for large t and N, the expression tends to 0, i.e. the average interface density for large values t and N is approximated by the expression in Eq. (9), as required. We can see that the larger N is, it takes longer to tend to zero.

```python
""" Code for Coursework 2 M3A29"""
import numpy as np
import matplotlib.pyplot as plt

def process_parth(params, eq, times):
    """ Function to simulate process"""
    u, v, N, t, x = params
    ave_array = [x - eq]
    x = np.ones(times)*x
    for k in range(t):
        # Outcomes as in eq.6:
        o1, o2, o3 = x + 4 / N, x - 4 / N, x
        # Probabilities as in eq.6
        p1 = (1 - x)**2 * (1 - u) * v / 2
        p2 = x * (x - 2 / N) * (1 - v) * u / 2
        p3 = 1 - p2 - p1
        # Calc. x(t + 1) for each realisation
        for i in range(times):
            x[i] = np.random.choice([o1[i], o2[i], o3[i]],
                                    p=[p1[i], p2[i], p3[i]])
        x_ave = np.mean(x - eq)  # Calculating expression
        ave_array.append(x_ave)
    return ave_array

N_array = [10000, 15000, 20000, 25000]
colours = ['b', 'r--', 'y', 'm--']
u, v, t = 0.5, 0.25, 100000
alpha, beta = 2 * (1 - u) * v, 2 * (1 - v) * u
x_0, times = 0.5, 200
times = 200  # No. of realisations
formula = 1 / (1 + np.sqrt(beta / alpha))

plt.figure(figsize=(12, 9))
for c, N in enumerate(N_array):
    params = u, v, N, t, x_0
    change_array = process_parth(params, formula, times)
    plt.plot(range(t + 1), change_array, colours[c], label=('$N = $' +
    str(N)))

plt.xlabel('t')
plt.ylabel(r'$ <x(t)> - (1 + \sqrt{ \beta / \alpha})^{-1}$')
plt.title('Plot of formula:' +
          r'$ <x(t)> - (1 + \sqrt{ \beta / \alpha})^{-1}$')
plt.grid()
plt.legend(loc='upper right')
plt.savefig('fig1.png')
plt.show()
```

9 Part I

We begin by manipulating Eq. (7) in the paper into a form where we can do separation of variables.

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha(1-x)^2 - \beta x^2, \\
&= \alpha(1 - 2x + x^2) - \beta x^2, \\
&= \alpha - 2\alpha x + (\alpha - \beta)x^2, \\
&= (\alpha - \beta)\left[x^2 - \frac{2\alpha}{\alpha - \beta}x + \frac{\alpha}{\alpha - \beta}\right], \quad \text{and completing the square:} \\
&= (\alpha - \beta)\left[\left(x - \frac{\alpha}{\alpha - \beta}\right)^2 - \frac{\alpha^2}{(\alpha - \beta)^2} + \frac{\alpha}{\alpha - \beta}\right], \\
&= (\alpha - \beta)\left[\left(x - \frac{\alpha}{\alpha - \beta}\right)^2 - \frac{\alpha\beta}{(\alpha - \beta)^2}\right].
\end{aligned}
$$

This equation can now be integrated using separation of variables:

$$
\int \frac{dx}{\frac{\alpha\beta}{(\alpha-\beta)^2} - \left(x - \frac{\alpha}{\alpha - \beta}\right)^2} = \int (\beta - \alpha) \cdot dt.
$$

This integral can be easily found by using the formula

$$
\int \frac{1}{1 - u^2} = \tanh^{-1}(u) + c.
$$

By using the formula we find that

$$
\begin{aligned}
\frac{1}{\frac{\sqrt{\alpha\beta}}{\alpha - \beta}} \tanh^{-1}\left(\frac{x - \frac{\alpha}{\alpha - \beta}}{\frac{\sqrt{\alpha\beta}}{\alpha - \beta}}\right) &= (\beta - \alpha)t + c, \\
\Rightarrow \frac{\alpha - \beta}{\sqrt{\alpha\beta}} \tanh^{-1}\left(\frac{(\alpha - \beta)x - \alpha}{\sqrt{\alpha\beta}}\right) &= (\beta - \alpha)t + c, \\
\Rightarrow \tanh^{-1}\left(\frac{(\alpha - \beta)x - \alpha}{\sqrt{\alpha\beta}}\right) &= -t\sqrt{\alpha\beta} + c_2,
\end{aligned}
$$

where $c_2 = -\frac{\sqrt{\alpha\beta}}{\alpha - \beta}c$ is a modified constant.

By using the initial condition $x(0) = 1/2$, we find that

$$\tanh^{-1}\left(-\frac{1}{2}\frac{\alpha+\beta}{\sqrt{\alpha\beta}}\right) = c_2, \qquad \text{when,}$$

Therefore

$$\begin{aligned}
\tanh^{-1}\left(\frac{(\alpha-\beta)x-\alpha}{\sqrt{\alpha\beta}}\right) &= \tanh^{-1}\left(-\frac{1}{2}\frac{\alpha+\beta}{\sqrt{\alpha\beta}}\right) - t\sqrt{\alpha\beta},\\
\Rightarrow \frac{(\alpha-\beta)x-\alpha}{\sqrt{\alpha\beta}} &= -\tanh\left[\tanh^{-1}\left(\frac{1}{2}\frac{\alpha+\beta}{\sqrt{\alpha\beta}}\right) + t\sqrt{\alpha\beta}\right],\\
&= -\frac{\frac{\alpha+\beta}{2\sqrt{\alpha\beta}} + \tanh(t\sqrt{\alpha\beta})}{1 + \frac{\alpha+\beta}{2\sqrt{\alpha\beta}}\tanh(t\sqrt{\alpha\beta})}.
\end{aligned}$$

Where we have used the addition formula for hyperbolic tangent and the fact that $\tanh(-u) = -\tanh(u)$.

Simplifying this we obtain:

$$\begin{aligned}
\Rightarrow (\beta-\alpha)x + \alpha &= \frac{\frac{\alpha+\beta}{2} + \sqrt{\alpha\beta}\tanh(t\sqrt{\alpha\beta})}{1 + \frac{\alpha+\beta}{2\sqrt{\alpha\beta}}\tanh(t\sqrt{\alpha\beta})},\\
&= \frac{\sqrt{\alpha\beta}(\alpha+\beta) + 2\alpha\beta\tanh(t\sqrt{\alpha\beta})}{2\sqrt{\alpha\beta} + (\alpha+\beta)\tanh(t\sqrt{\alpha\beta})},
\end{aligned}$$

yielding

$$(\beta-\alpha)x = \frac{\sqrt{\alpha\beta}(\alpha+\beta) + 2\alpha\beta\tanh(t\sqrt{\alpha\beta}) - \alpha\left[2\sqrt{\alpha\beta} + (\alpha+\beta)\tanh(t\sqrt{\alpha\beta})\right]}{2\sqrt{\alpha\beta} + (\alpha+\beta)\tanh(t\sqrt{\alpha\beta})}.$$

Now, taking the top of the fraction:

$$\begin{aligned}
&\sqrt{\alpha\beta}(\alpha+\beta) + 2\alpha\beta\tanh(t\sqrt{\alpha\beta}) - \alpha\left[2\sqrt{\alpha\beta} + (\alpha+\beta)\tanh(t\sqrt{\alpha\beta})\right],\\
&= \sqrt{\alpha\beta}(\alpha+\beta-2\alpha) + \alpha\tanh(t\sqrt{\alpha\beta})\left[2\beta - (\alpha+\beta)\right],\\
&= \sqrt{\alpha\beta}(\beta-\alpha) + \alpha\tanh(t\sqrt{\alpha\beta})(\beta-\alpha) = (\beta-\alpha)\left[\sqrt{\alpha\beta} + \alpha\tanh(t\sqrt{\alpha\beta})\right].
\end{aligned}$$

Substituting this back we get:

$$
\begin{aligned}
(\beta-\alpha)x &= \frac{(\beta-\alpha)\left[\sqrt{\alpha\beta}+\alpha\tanh(t\sqrt{\alpha\beta})\right]}{2\sqrt{\alpha\beta}+(\alpha+\beta)\tanh(t\sqrt{\alpha\beta})}, \\
\Rightarrow \quad x(t) &= \frac{\sqrt{\alpha\beta}+\alpha\tanh(t\sqrt{\alpha\beta})}{2\sqrt{\alpha\beta}+(\alpha+\beta)\tanh(t\sqrt{\alpha\beta})},
\end{aligned}
$$

as required, therefore we have derived Eq. (8) in the paper by explicit integration.

10 Part J

We would like to calculate t^* such that $x(t^*) = \frac{1}{2}x(0) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. We begin by setting Eq. (9) in the paper equal to 1/4 and finding t^* :

$$
\begin{aligned}
\frac{\sqrt{\alpha\beta}+\alpha\tanh(t^*\sqrt{\alpha\beta})}{(2\sqrt{\alpha\beta}+(\alpha+\beta)\tanh(t^*\sqrt{\alpha\beta}))} &= \frac{1}{4}, \\
\Rightarrow \quad 4\sqrt{\alpha\beta}+4\alpha\tanh(t^*\sqrt{\alpha\beta}) &= 2\sqrt{\alpha\beta}+(\alpha+\beta)\tanh(t^*\sqrt{\alpha\beta}), \\
\Rightarrow \quad 3\alpha\tanh(t^*\sqrt{\alpha\beta})-\beta\tanh(t^*\sqrt{\alpha\beta}) &= -2\sqrt{\alpha\beta}, \\
\Rightarrow \quad \tanh(t^*\sqrt{\alpha\beta})(3\alpha-\beta) &= -2\sqrt{\alpha\beta}, \\
\Rightarrow \quad \tanh(t^*\sqrt{\alpha\beta}) &= -\frac{2\sqrt{\alpha\beta}}{(3\alpha-\beta)}, \\
\Rightarrow \quad t^*\sqrt{\alpha\beta} &= \tanh^{-1}\left(-\frac{2\sqrt{\alpha\beta}}{(3\alpha-\beta)}\right),
\end{aligned}
$$

and by making use of the fact that $\tanh^{-1}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x})$ we find that

$$
\begin{aligned}
t^*\sqrt{\alpha\beta} &= \frac{1}{2}\ln\left(\frac{1-\dfrac{2\sqrt{\alpha\beta}}{3\alpha-\beta}}{1+\dfrac{2\sqrt{\alpha\beta}}{3\alpha-\beta}}\right), \\
\Rightarrow \quad t^*\sqrt{\alpha\beta} &= \frac{1}{2}\ln\left(\frac{3\alpha-\beta-2\sqrt{\alpha\beta}}{3\alpha-\beta+2\sqrt{\alpha\beta}}\right), \\
\Rightarrow \quad t^* &= \frac{1}{2\sqrt{\alpha\beta}}\ln\left(\frac{3\alpha-\beta-2\sqrt{\alpha\beta}}{3\alpha-\beta+2\sqrt{\alpha\beta}}\right),
\end{aligned}
$$

and therefore t^* is now found.

We now proceed to show that t^* diverges logarithmically as $\gamma \to \frac{1}{5}^-$.

Given $u = 1/2, \quad v = \gamma u = \gamma/2$, we work out that

$$\alpha = 2(1-u)v = \frac{\gamma}{2}, \quad \beta = 2(1-v)u = 1 - \frac{\gamma}{2}, \quad \sqrt{\alpha\beta} = \frac{\sqrt{\gamma(2-\gamma)}}{2}.$$

Substituting these quantities into the expression for t^* found on the previous page we find that

$$\begin{aligned} t^* &= \frac{1}{\sqrt{\gamma(2-\gamma)}} \ln\left(\frac{\frac{3\gamma}{2} - \frac{2-\gamma}{2} - \sqrt{\gamma(2-\gamma)}}{\frac{3\gamma}{2} - \frac{2-\gamma}{2} + \sqrt{\gamma(2-\gamma)}}\right), \\ &= \frac{1}{\sqrt{\gamma(2-\gamma)}} \ln\left(\frac{2\gamma - 1 - \sqrt{\gamma(2-\gamma)}}{2\gamma - 1 + \sqrt{\gamma(2-\gamma)}}\right). \end{aligned}$$

In this last equation, if we let $\gamma \to \frac{1}{5}^-$, taking each component of the equation separately, we have that

$$\begin{aligned} \frac{1}{\sqrt{\gamma(2-\gamma)}} &\to \frac{5}{3}, \\ 2\gamma - 1 - \sqrt{\gamma(2-\gamma)} &\to -1.2^-, \\ 2\gamma - 1 + \sqrt{\gamma(2-\gamma)} &\to 0^-, \\ \Rightarrow \quad \frac{2\gamma - 1 - \sqrt{\gamma(2-\gamma)}}{2\gamma - 1 + \sqrt{\gamma(2-\gamma)}} &\to +\infty \quad \text{algebraically}, \\ \Rightarrow \quad \frac{1}{\sqrt{\gamma(2-\gamma)}} \ln\left(\frac{2\gamma - 1 - \sqrt{\gamma(2-\gamma)}}{2\gamma - 1 + \sqrt{\gamma(2-\gamma)}}\right) &\to +\infty \quad \text{logarithmically}. \end{aligned}$$

Therefore, we have shown that t^* diverges logarithmically as $\gamma \to \frac{1}{5}^-$.