Examples V for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1] Suppose that $X_1, \ldots, X_N$ is a sample of a second-order stationary process $\{X_t\}$ with mean μ. Let $\{h_t\}$ be a data taper, standardized so that $\sum_{t=1}^N h_t^2 = 1$. Define the direct spectral estimator where the mean is subtracted *before* the time series is tapered as

$$\widehat{S}(f) = \left| \sum_{t=1}^{N} h_t (X_t - \mu) e^{-i2\pi ft} \right|^2 ,$$

and the direct spectral estimator where the mean is subtracted *after* the time series is tapered as

$$\widetilde{S}(f) = \left| \sum_{t=1}^{N} (h_t X_t - \mu) e^{-i2\pi ft} \right|^2 .$$

(a) Show that the mean of the direct spectral estimator $\widehat{S}(f)$ is given by

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{H}(f - f') S(f') df' ,$$

where $\mathcal{H}(f) = \left| \sum_{t=1}^N h_t e^{-i2\pi ft} \right|^2$.

(b) Use the complex-number expansion $|a+b|^2 = |a|^2 + |b|^2 + ab^* + a^*b$ to show that

$$E\{\widetilde{S}(f)\} = E\{\widehat{S}(f)\} + \mu^2 \left| \sum_{t=1}^{N} (h_t - 1) e^{-i2\pi ft} \right|^2 .$$

(c) Is it thus better to subtract the mean before or after the time series is tapered? Give your reasoning.

[2] (a) Let $\{X_t\}$ and $\{Y_t\}$ be two real-valued and zero mean jointly stationary stochastic processes, and let

$$Z_t = aX_t + bY_t \quad \text{and} \quad W_t = cX_t + dY_t,$$

where a, b, c and d are real-valued constants. Show that

$$S_{ZW}(f) = ac\, S_X(f) + bd\, S_Y(f) + bc\, S_{YX}(f) + ad\, S_{XY}(f),$$

where $S_{ZW}(f)$ is the cross-spectrum of $\{Z_t\}$ and $\{W_t\}$, $S_{XY}(f)$ is the cross-spectrum of $\{X_t\}$ and $\{Y_t\}$, and $S_X(f)$ and $S_Y(f)$ are the power spectra of $\{X_t\}$ and $\{Y_t\}$, respectively.

(b) Consider rotating the coordinate system through an angle θ so that the time series referred to the new axes are given by

$$\begin{bmatrix} Z_t \\ W_t \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix}.$$

Show that

$$\begin{aligned} S_Z(f) &= \cos^2\theta \; S_X(f) + \sin^2\theta \; S_Y(f) + 2\sin\theta\cos\theta \; \Re\{S_{XY}(f)\} \\ S_W(f) &= \sin^2\theta \; S_X(f) + \cos^2\theta \; S_Y(f) - 2\sin\theta\cos\theta \; \Re\{S_{XY}(f)\}, \end{aligned}$$

where $S_Z(f)$ and $S_W(f)$ are the power spectra of $\{Z_t\}$ and $\{W_t\}$, respectively, and $\Re\{\cdot\}$ denotes the real-part function.

(c) Hence show that the magnitude squared coherence $\gamma^2_{ZW}(f)$ satisfies

$$\gamma^2_{ZW}(f) = \gamma^2_{XY}(f)$$

when $\theta = \pi/2$, and interpret this result.

[3] Suppose the stationary process $\{Y_t\}$ is the result of filtering the zero mean stationary process $\{X_t\}$ by the LTI filter $\{g_u\}$ plus an added zero mean stationary noise process $\{\eta_t\}$, i.e., $Y_t = \sum_{u=-\infty}^{\infty} g_u X_{t-u} + \eta_t$, where all processes and $\{g_u\}$ are real-valued.

Using the spectral representations of the processes show that

$$E\left\{ \left| Y_t - \sum_{u=-\infty}^{\infty} g_u X_{t-u} \right|^2 \right\} = \int_{-1/2}^{1/2} E\{|\mathrm{d}Z_Y(f) - G(f)\mathrm{d}Z_X(f)|^2\},$$

(where $G(f)$ is the Fourier transform of $\{g_u\}$) explaining each step carefully.