Examples IV for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1] Let $X_1, \ldots, X_N$ be a sample from a stationary process $\{X_t\}$ with unknown mean μ and variance σ^2. The so-called 'unbiased' and 'biased' autocovariance estimators are given, respectively, by

$$\hat{s}_\tau^{(u)} = \frac{1}{N-|\tau|} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}) \text{ and } \hat{s}_\tau^{(p)} = \frac{1}{N} \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X}).$$

(a) By writing the periodogram in terms of the biased autocovariance sequence estimator show that the integral of the periodogram is equal to the sample variance, i.e.,

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\,\mathrm{d}f = \sum_{t=1}^{N} (X_t - \overline{X})^2/N.$$

(b) Also show that

$$E\{\hat{s}_0^{(p)}\} \equiv E\{\hat{s}_0^{(u)}\} = s_0 - \mathrm{var}\{\overline{X}\},$$

and comment on this result.

[2] Let $X_1, ..., X_N$ be a sample of size N from a *white noise process* with unknown mean μ and variance σ^2.

(a) Show that, for $0 < |\tau| < N-1$,

$$E\{\hat{s}_\tau^{(u)}\} = -\frac{\sigma^2}{N} \quad \text{and} \quad E\{\hat{s}_\tau^{(p)}\} = -\left(1 - \frac{|\tau|}{N}\right)\frac{\sigma^2}{N}.$$

and hence that, for white noise, the magnitude of the bias of the 'biased' estimator $\hat{s}_\tau^{(p)}$ is less than that of the 'unbiased' estimator $\hat{s}_\tau^{(u)}$.

(b) Show that the mean square error of $\hat{s}_\tau^{(p)}$ is less than that of $\hat{s}_\tau^{(u)}$ for $0 < |\tau| < N-1$.

(c) By considering the row and diagonal sums of the $N \times N$ matrix having (u, v)th entry $(X_u - \bar{X})(X_v - \bar{X})$ for $1 \leq u, v \leq N$, show that $\sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_\tau^{(p)} = 0$.

Hence deduce that $\hat{s}_\tau^{(p)}$ must be negative for some value(s) of τ.