Examples I for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1]

(a) Let Y_1 and Y_2 be independent and identically distributed random variables, each with mean zero and variance σ^2, and let c be a constant.

(i) Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct) + Y_2 \sin(ct),$$

and hence show that the process is stationary.

(ii) For the case $c = \pi/4$ show that $\{X_t\}$ is *strictly* stationary if and only if Y_1 and Y_2 each have the Gaussian (normal) distribution.

[*You will need to use Bernstein's theorem which states that if U and V are IID random variables, and $(U + V)/\sqrt{2}$ has the same distribution as U and V, then U and V are Gaussian (normal). Also recall that* $\cos(\pi/4) = 1/\sqrt{2}$.]

(b) Let Y_1 be a random variable with mean zero and variance σ^2, and let c be a constant.

(i) Find the mean and covariance of $\{X_t\}$ defined by

$$X_t = Y_1 \cos(ct),$$

and hence determine when the process is stationary.

(ii) Show that when the process in (b)(i) is stationary that its autocorrelation sequence must be of the form

$$\rho_\tau = (-1)^{|\ell\tau|}, \qquad \tau \in \mathbb{Z},$$

for some $\ell \in \mathbb{Z}$.

(iii) The sequence $\{\rho_\tau\}$ is positive semidefinite, if, for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n$ contained in $\mathbb{Z}$, and for any set of nonzero real numbers $a_1, a_2, \ldots, a_n$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{t_j - t_k} a_j a_k \geq 0.$$

Show that this inequality condition is satisfied by $\rho_\tau = (-1)^{|\ell\tau|}$ for $\ell, \tau \in \mathbb{Z}$.

[2] Let $\{X_t\}$ be defined by

$$X_t = \phi X_{t-1} + \epsilon_t, \qquad t = 1, 2, 3, \ldots$$

where $X_0 = 0$. Find the 3×3 covariance matrix of X_1, X_2 and X_3. Here $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2.