SOLUTIONS for Examples I for Time Series (S8)

1(a)(i)

$$E\{X_t\} = E\{Y_1\}\cos(ct) + E\{Y_2\}\sin(ct) = 0.$$

Also for the covariance (which for $\tau = 0$ gives the variance),

$$\begin{aligned}
E\{X_t X_{t+\tau}\} &= E\{[Y_1\cos(ct) + Y_2\sin(ct)][Y_1\cos(c[t+\tau]) + Y_2\sin(c[t+\tau])]\} \\
&= E\{Y_1^2\}\cos(ct)\cos(c[t+\tau]) + E\{Y_1Y_2\}\cos(ct)\sin(c[t+\tau]) \\
&+ E\{Y_2Y_1\}\sin(ct)\cos(c[t+\tau]) + E\{Y_2^2\}\sin(ct)\sin(c[t+\tau]) \\
&= \sigma^2\cos(ct)\cos(c[t+\tau]) + \sigma^2\sin(ct)\sin(c[t+\tau]).
\end{aligned}$$

But, since $\cos(a-b) = \cos a\cos b + \sin a \sin b$,

$$E\{X_t X_{t+\tau}\} = \sigma^2\cos(c\tau) = s_\tau.$$

Therefore the process is always stationary.

(ii) Firstly suppose that $\{X_t\}$ is strictly stationary. Then the marginal distribution of X_t is independent of $t \in \mathbb{Z}$. With $c = \pi/4$ the cases $t = 0$ and 1 give $X_0 = Y_1$ and $X_1 = (Y_1 + Y_2)/\sqrt{2}$ so that Y_1 and $(Y_1 + Y_2)/\sqrt{2}$ have the same distribution. We know that Y_1 and Y_2 are IID. From Bernstein's theorem we can conclude that Y_1 and Y_2 are Gaussian.

Now suppose that Y_1 and Y_2 are Gaussian, then $\{X_t\}$ is a Gaussian process, (all finite-dimensional marginal distributions are multivariate Gaussian). The process is (second-order) stationary by part (i), and we know that a stationary Gaussian process is strictly stationary.

(b)(i) $E\{X_t\} = E\{Y_1\}\cos(ct) = 0$. Taking $Y_2 \equiv 0$ in (a), gives

$$E\{X_t X_{t+\tau}\} = \sigma^2\cos(ct)\cos(c[t+\tau]).$$

Since t and τ are integers, the process is stationary for $c = \ell\pi, \ell \in \mathbb{Z}$ and non-stationary otherwise, i.e.,

$$s_\tau = \sigma^2\cos(\ell\pi t)\cos(\ell\pi[t+\tau]).$$

(ii) Now $\cos(\ell\pi t) = (-1)^{\ell t}$ and $\cos(\ell\pi[t+\tau]) = (-1)^{\ell(t+\tau)}$ so that

$$s_\tau = \sigma^2(-1)^{\ell t}(-1)^{\ell(t+\tau)} = \sigma^2(-1)^{\ell\tau},$$

for some choice $\ell \in \mathbb{Z}$. Hence $s_0 = \sigma^2$ and by symmetry $\rho_\tau = s_\tau/s_0 = (-1)^{|\ell\tau|},\ \tau \in \mathbb{Z}$.

(iii)

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n}\rho_{t_j-t_k}a_j a_k &= \sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{\ell(t_j-t_k)}a_j a_k \\ &= \sum_{j=1}^{n}(-1)^{\ell t_j}a_j\sum_{k=1}^{n}(-1)^{\ell t_k}a_k = [\sum_{j=1}^{n}(-1)^{\ell t_j}a_j]^2 \geq 0.\end{aligned}$$

2.

$$\begin{aligned}X_1 &= \phi X_0 + \epsilon_1 = \epsilon_1 \\ X_2 &= \phi X_1 + \epsilon_2 = \phi\epsilon_1 + \epsilon_2 \\ X_3 &= \phi X_2 + \epsilon_3 = \phi(\phi\epsilon_1 + \epsilon_2) + \epsilon_3 = \phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3.\end{aligned}$$

So $E\{X_j\} = 0$ for $j = 1, 2, 3$. Then

$$\begin{aligned}E\{X_1^2\} &= E\{\epsilon_1^2\} = \sigma_\epsilon^2 \\ E\{X_2^2\} &= E\{[\phi\epsilon_1 + \epsilon_2]^2\} = [1+\phi^2]\sigma_\epsilon^2 \\ E\{X_3^2\} &= E\{[\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]^2\} = [1+\phi^2+\phi^4]\sigma_\epsilon^2 \\ E\{X_1X_2\} &= E\{\epsilon_1[\phi\epsilon_1 + \epsilon_2]\} = \phi\sigma_\epsilon^2 \\ E\{X_1X_3\} &= E\{\epsilon_1[\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]\} = \phi^2\sigma_\epsilon^2 \\ E\{X_2X_3\} &= E\{[\phi\epsilon_1 + \epsilon_2][\phi^2\epsilon_1 + \phi\epsilon_2 + \epsilon_3]\} = \phi^3\sigma_\epsilon^2 + \phi\sigma_\epsilon^2.\end{aligned}$$

So covariance matrix is

$$\sigma_\epsilon^2\begin{bmatrix} 1 & \phi & \phi^2 \\ \phi & 1+\phi^2 & \phi(1+\phi^2) \\ \phi^2 & \phi(1+\phi^2) & 1+\phi^2+\phi^4 \end{bmatrix}.$$

This is not Toeplitz. The generated variables are not part of a stationary sequence.

This is an important example from the point of simulation. For simulation using this sort of recursive scheme (with zero boundary conditions such as $X_0 = 0$) you would have to throw away 1000's of values to be sure of removing the 'start-up transients' before keeping the generated values. Alternatively it is possible to work-out special 'stationary boundary values' so that *all* the generated sequence is stationary.

In the theory the only boundary values that can be set to zero are those at $-\infty$; strictly speaking stochastic processes run from $-\infty$ to ∞.