SOLUTIONS for Examples V for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1] (a) Let

$$J(f) \equiv \sum_{t=1}^{N} h_t (X_t - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t}.$$

By the spectral representation theorem

$$X_t - \mu = \int_{-1/2}^{1/2} e^{i2\pi f' t} \, \mathrm{d}Z(f'),$$

where $\{Z(\cdot)\}$ is a process with orthogonal increments, and $E\{\mathrm{d}Z(f)\} = 0$. Thus

$$\begin{aligned} J(f) &= \sum_{t=1}^{N} h_t \left(\int_{-1/2}^{1/2} e^{i2\pi f' t} \, \mathrm{d}Z(f') \right) \mathrm{e}^{-\mathrm{i}2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} h_t e^{-i2\pi(f-f')t} \, \mathrm{d}Z(f') \\ &= \int_{-1/2}^{1/2} H(f - f') \, \mathrm{d}Z(f'), \end{aligned}$$

where $\{h_t\}$ and $H(\cdot)$ form a Fourier transform pair under the assumption that $\{h_t\}$ is an infinite sequence with $h_t = 0$ for $t < 1$ and $t > N$; i.e.,

$$H(f) \equiv \sum_{t=1}^{N} h_t \mathrm{e}^{-\mathrm{i}2\pi f t}.$$

Now it is given that,

$$\widehat{S}(f) \equiv |J(f)|^2 = \left| \sum_{t=1}^{N} h_t (X_t - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2.$$

Because $\{Z(\cdot)\}$ has orthogonal increments, we therefore have

$$E\{\widehat{S}(f)\} = \int_{-1/2}^{1/2} \mathcal{H}(f - f') S(f') \, \mathrm{d}f'.$$

(b) We have

$$\begin{aligned} \widetilde{S}(f) &= \left| \sum_{t=1}^{N} (h_t X_t - h_t \mu + h_t \mu - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2 \\ &= \left| \sum_{t=1}^{N} h_t (X_t - \mu) \mathrm{e}^{-\mathrm{i}2\pi f t} + \sum_{t=1}^{N} \mu (h_t - 1) \mathrm{e}^{-\mathrm{i}2\pi f t} \right|^2. \end{aligned}$$

Using the expansion $|a+b|^2 = |a|^2 + |b|^2 + ab^* + a^*b$, we obtain

$$\begin{aligned}
\widetilde{S}(f) = \widehat{S}(f) &+ \left|\sum_{t=1}^{N} \mu(h_t - 1)\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2 \\
&+ \left(\sum_{t=1}^{N} h_t(X_t - \mu)\mathrm{e}^{-\mathrm{i}2\pi ft}\right)\left(\sum_{t=1}^{N} \mu(h_t - 1)\mathrm{e}^{-\mathrm{i}2\pi ft}\right)^* \\
&+ \left(\sum_{t=1}^{N} h_t(X_t - \mu)\mathrm{e}^{-\mathrm{i}2\pi ft}\right)^*\left(\sum_{t=1}^{N} \mu(h_t - 1)\mathrm{e}^{-\mathrm{i}2\pi ft}\right).
\end{aligned}$$

If we take the expected value of both sides of the above, the last two terms on the right-hand side vanish because $E\{X_t\} = \mu$, and we have

$$E\{\widetilde{S}(f)\} = E\{\widehat{S}(f)\} + \mu^2 \left|\sum_{t=1}^{N} (h_t - 1)\mathrm{e}^{-\mathrm{i}2\pi ft}\right|^2.$$

(c) We should subtract the mean before tapering, since the mean of $\widehat{S}$ is just a smoothed version of the true spectrum, while the mean of $\widetilde{S}$ is contaminated by the μ^2 term.

[2] (a)

$$Z_t = aX_t + bY_t \quad \text{and} \quad W_t = cX_t + dY_t,$$

Then

$$\begin{aligned}
E\{Z_t W_{t+\tau}\} &= E\{[aX_t + bY_t][cX_{t+\tau} + dY_{t+\tau}]\} \\
&= E\{acX_tX_{t+\tau} + bdY_tY_{t+\tau} + bcY_tX_{t+\tau} + adX_tY_{t+\tau} \\
&= acs_{X,\tau} + bds_{Y,\tau} + bcs_{YX,\tau} + ads_{XY,\tau} \\
\Rightarrow S_{ZW}(f) &= acS_X(f) + bdS_Y(f) + bcS_{YX}(f) + adS_{XY}(f),
\end{aligned}$$

the last line following by Fourier transformation.

(b) From the form of the matrix we see that in terms of part (a)

$$a = \cos\theta; b = \sin\theta; c = -\sin\theta; d = \cos\theta.$$

With reference to part (a) put $Z_t = W_t$ so that $c = a, d = b$ and then from the result in (a):

$$\begin{aligned}
S_Z(f) &= a^2 S_X(f) + b^2 S_Y(f) + abS_{YX}(f) + abS_{XY}(f) \\
&= a^2 S_X(f) + b^2 S_Y(f) + abS^*_{XY}(f) + abS_{XY}(f) \\
&= a^2 S_X(f) + b^2 S_Y(f) + 2ab\Re\{S_{XY}(f)\} \\
&= \cos^2\theta S_X(f) + \sin^2\theta S_Y(f) + 2\cos\theta\sin\theta\Re\{S_{XY}(f)\}.
\end{aligned}$$

For $S_W(f)$ we work with c and d so that part (i) gives

$$\begin{aligned}S_W(f) &= c^2 S_X(f) + d^2 S_Y(f) + 2cd\Re\{S_{XY}(f)\}\\ &= \sin^2\theta S_X(f) + \cos^2\theta S_Y(f) - 2\cos\theta\sin\theta\Re\{S_{XY}(f)\}.\end{aligned}$$

(Alternatively these may be calculated directly as in part (a).)

(c) When $\theta = \pi/2$, $\cos\theta = 0, \sin\theta = 1$, and $a = d = 0$ and $b = 1$ and $c = -1$. Hence from part (a),

$$S_{ZW}(f) = -S_{YX}(f).$$

and from above

$$S_Z(f) = S_Y(f) \quad \text{and} \quad S_W(f) = S_X(f)$$

Then

$$\gamma^2_{ZW}(f) = \frac{|S_{ZW}(f)|^2}{S_Z(f)S_W(f)} = \frac{|-S_{YX}(f)|^2}{S_Y(f)S_X(f)} = \frac{|S_{XY}(f)|^2}{S_X(f)S_Y(f)} = \gamma^2_{XY}(f).$$

So the magnitude squared coherence is invariant to a rotation of $\pi/2$ in the coordinate system.

[3] Replacing each process by its spectral representation gives

$$E\{|Y_t - \sum_{u=-\infty}^{\infty} g_u X_{t-u}|^2\} = E\left|\int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ_Y(f) - \int_{-1/2}^{1/2} G(f)e^{i2\pi ft}\, dZ_X(f)\right|^2.$$

Expanding the modulus squared, the right-hand side can be written

$$E\left\{\left[\int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ_Y(f) - \int_{-1/2}^{1/2} G(f)e^{i2\pi ft}\, dZ_X(f)\right]\right.$$
$$\left.\left[\int_{-1/2}^{1/2} e^{-i2\pi f't}\, dZ^*_Y(f') - \int_{-1/2}^{1/2} G^*(f')e^{-i2\pi f't}\, dZ^*_X(f')\right]\right\},$$

and if we multiply out and take expectation, using that dZ_X and dZ_Y are cross-orthogonal as well as individually orthogonal, this becomes

$$\begin{aligned}&\int_{-1/2}^{1/2} [S_{YY}(f) - G(f)S^*_{XY}(f) - G^*(f)S_{XY}(f) + |G(f)|^2 S_{XX}(f)]df\\ &= \int_{-1/2}^{1/2} E\{|dZ_Y(f) - G(f)dZ_X(f)|^2\},\end{aligned}$$

giving the desired result.