SOLUTIONS for Examples II for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1]

(a) We cannot assume the means are zero here. But, since $\{X_t\}$ and $\{Y_t\}$ are stationary,

$$E\{Z_t\} = E\{X_t + Y_t\} = \mu_X + \mu_Y = \mu_Z$$
$$E\{Z_{t+\tau}\} = E\{X_{t+\tau} + Y_{t+\tau}\} = \mu_X + \mu_Y = \mu_Z.$$

Then

$$\begin{aligned} s_{Z,\tau} &= E\{(Z_t - E\{Z_t\})(Z_{t+\tau} - E\{Z_{t+\tau}\})\} \\ &= E\{([X_t - \mu_X] + [Y_t - \mu_Y])([X_{t+\tau} - \mu_X] + [Y_{t+\tau} - \mu_Y])\} \\ &= E\{[X_t - \mu_X][X_{t+\tau} - \mu_X]\} + E\{[Y_t - \mu_Y][Y_{t+\tau} - \mu_Y]\} \\ &+ E\{[X_t - \mu_X][Y_{t+\tau} - \mu_Y]\} + E\{[Y_t - \mu_Y][X_{t+\tau} - \mu_X]\} \\ &= s_{X,\tau} + s_{Y,\tau}, \end{aligned}$$

as the last two expectations are zero because $\{X_t\}$ and $\{Y_t\}$ are uncorrelated.

(b) A white noise process with variance unity has an autocovariance which is unity at $\tau = 0$ and zero elsewhere. An MA(1) process with parameter $\theta_{1,1} = \psi, |\psi| < 1$, and innovations variance $\sigma_\epsilon^2 = 1$ has an autocovariance sequence of the form

$$s_\tau = \begin{cases} 1 + \psi^2, & \text{if } \tau = 0, \\ -\psi, & \text{if } |\tau| = 1, \\ 0, & \text{if } |\tau| > 1. \end{cases}$$

Hence

$$s_\tau = \begin{cases} 2 + \psi^2, & \text{if } \tau = 0, \\ -\psi, & |\tau| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

is the autocovariance sequence corresponding to the sum of uncorrelated white noise and MA(1) processes, with the parameters as above. By part (a) it follows that the stated autocovariance is a valid autocovariance sequence.

[2]

(a)

$$\begin{aligned} s_{Y,\tau} &= E\{Y_t Y_{t+\tau}\} - E\{Y_t\}E\{Y_{t+\tau}\} \\ &= E\{X_t X_{t-1} X_{t+\tau} X_{t+\tau-1}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\ &= E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} + E\{X_t X_{t+\tau}\}E\{X_{t-1} X_{t+\tau-1}\} \\ &+ E\{X_t X_{t+\tau-1}\}E\{X_{t-1} X_{t+\tau}\} - E\{X_t X_{t-1}\}E\{X_{t+\tau} X_{t+\tau-1}\} \\ &= \operatorname{cov}\{X_t, X_{t+\tau}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau-1}\} + \operatorname{cov}\{X_t, X_{t+\tau-1}\} \operatorname{cov}\{X_{t-1}, X_{t+\tau}\} \\ &= s_{X,\tau}^2 + s_{X,\tau-1} s_{X,\tau+1}. \end{aligned}$$

For an MA(1) we have

$$s_{X,\tau} = \begin{cases} \sigma_\epsilon^2(1+\theta_{1,1}^2) & \tau = 0, \\ -\sigma_\epsilon^2\theta_{1,1}, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$s_{Y,\tau} = \begin{cases} \sigma_\epsilon^4(1+3\theta_{1,1}^2+\theta_{1,1}^4) & \tau = 0, \\ \sigma_\epsilon^4\theta_{1,1}^2, & |\tau| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $s_{X,\tau} = E\{X_t X_{t+\tau}\}$ since the mean is zero. Then,

$$\begin{aligned} s_{Y,\tau} &= E\{X_t^2 X_{t+\tau}^2\} - E\{X_t^2\}E\{X_{t+\tau}^2\} \\ &= E\{X_t^2\}E\{X_{t+\tau}^2\} + 2E^2\{X_t X_{t+\tau}\} - E\{X_t^2\}E\{X_{t+\tau}^2\} \\ &= 2E^2\{X_t X_{t+\tau}\} = 2s_{X,\tau}^2. \end{aligned}$$

Then $s_{Y,0} = 2s_{X,0}^2$, so that

$$\rho_{Y,\tau} = \frac{s_{Y,\tau}}{s_{Y,0}} = \frac{2s_{X,\tau}^2}{2s_{X,0}^2} = \rho_{X,\tau}^2.$$

We take the most obvious possible model for our one case: an MA(2) process with parameters $\theta_{0,2} = -1$, (standard) and $\theta_{1,2} = 0$, and $\theta_{2,2}$. For convenience define $\theta \equiv \theta_{2,2}$.

$$s_{X,\tau} = \begin{cases} [1+\theta^2]\sigma_\epsilon^2, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ -\theta\sigma_\epsilon^2, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2, \end{cases} \quad \text{so} \quad \rho_{X,\tau} = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ -\frac{\theta}{[1+\theta^2]}, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2. \end{cases}$$

But

$$s_{Y,\tau} = \begin{cases} 200, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ 18, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2, \end{cases} \quad \text{so} \quad \rho_{X,\tau} = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } |\tau| = 1, \\ \pm\frac{3}{10}, & \text{if } |\tau| = 2, \\ 0, & \text{if } |\tau| > 2. \end{cases}$$

Let $\rho = \rho_{X,2}$, then we see that $\rho\theta^2 + \theta + \rho = 0$ which has solutions

$$\theta = \frac{-1 \pm \sqrt{[1-4\rho^2]}}{2\rho} = \frac{-1 \pm \frac{8}{10}}{\pm\frac{6}{10}} = \pm\frac{1}{3} \text{ or } \pm 3.$$

Now $s_{Y,0} = 200 = 2s_{X,0}^2 = 2[1+\theta^2]^2\sigma_\epsilon^4$. Four possible parameter combinations arise:

$$\theta = \pm 1/3 \Rightarrow \sigma_\epsilon^2 = 9; \quad \theta = \pm 3 \Rightarrow \sigma_\epsilon^2 = 1.$$