SOLUTIONS for Examples IV for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1] (a) Now

$$\hat{S}^{(p)}(f) = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)} \mathrm{e}^{-\mathrm{i}2\pi f\tau}.$$

So

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\,\mathrm{d}f = \sum_{\tau=-(N-1)}^{N-1} \hat{s}_\tau^{(p)} \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}f.$$

When $\tau = 0$, the integral is just 1; when $\tau \neq 0$, the integral is always zero because we have (letting $\nu = 2\pi f\tau$)

$$\begin{aligned}\int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}f &= \left\{\frac{1}{2\pi\tau}\int_{-\pi\tau}^{\pi\tau} \cos(\nu)\,\mathrm{d}\nu - \mathrm{i}\int_{-\pi\tau}^{\pi\tau} \sin(\nu)\,\mathrm{d}\nu\right\} \\ &= \frac{1}{2\pi\tau}\left\{[\sin(\pi\tau) - \sin(-\pi\tau)] + \mathrm{i}\,[\cos(\pi\tau) - \cos(-\pi\tau)]\right\} = 0.\end{aligned}$$

Hence only one term of the summation is nonzero, and we obtain

$$\int_{-1/2}^{1/2} \hat{S}^{(p)}(f)\,\mathrm{d}f = \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2,$$

as required.

(b) Here $\{X_t\}$ is a stationary process with mean value $\mu = E\{X_t\}$, and variance $s_0 = \sigma^2$. Let $\hat{s}_0 = \hat{s}_0^{(p)} = \hat{s}_0^{(u)}$. Then

$$\begin{aligned}\hat{s}_0 &= \frac{1}{N}\sum_{t=1}^{N}(X_t - \overline{X})^2 \\ &= \frac{1}{N}\sum_{t=1}^{N}([X_t - \mu] - [\overline{X} - \mu])^2 \\ &= \frac{1}{N}\sum_{t=1}^{N}\left([X_t - \mu]^2 - 2[X_t - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2\right) \\ &= \frac{1}{N}\sum_{t=1}^{N}[X_t - \mu]^2 - 2[\overline{X} - \mu][\overline{X} - \mu] + [\overline{X} - \mu]^2 \\ &= \frac{1}{N}\sum_{t=1}^{N}[X_t - \mu]^2 - [\overline{X} - \mu]^2.\end{aligned}$$

Taking the expectation of both sides and noting that $E\{\overline{X}\} = \mu$ yields

$$\begin{aligned} E\{\hat{s}_0\} &= \frac{1}{N}\sum_{t=1}^{N} E\{[X_t - \mu]^2\} - E\{[\overline{X} - \mu]^2\} \\ &= \text{var}\,\{X_t\} - \text{var}\,\{\overline{X}\} = s_0 - \text{var}\,\{\overline{X}\}, \end{aligned}$$

the desired result.

The 'unbiased' and 'biased' estimator coincide for lag $\tau = 0$; if $\overline{X}$ was replaced in the estimator by μ then both estimators would be unbiased.

[2] (a) Since $\{X_t\}$ is a white noise process, we have, for $|\tau| \neq 0$ and $t = 1, \ldots, N - |\tau|$,

$$\begin{aligned} E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} &= E\{([X_t - \mu] - [\bar{X} - \mu])([X_{t+|\tau|} - \mu] - [\bar{X} - \mu])\} \\ &= E\{(X_t - \mu)(X_{t+|\tau|} - \mu)\} \\ &\quad - \frac{1}{N}\sum_{u=1}^{N} E\{(X_t - \mu)(X_u - \mu)\} \\ &\quad - \frac{1}{N}\sum_{u=1}^{N} E\{(X_{t+|\tau|} - \mu)(X_u - \mu)\} \\ &\quad + \frac{1}{N^2}\sum_{u=1}^{N}\sum_{v=1}^{N} E\{(X_u - \mu)(X_v - \mu)\} \\ &= 0 - \frac{1}{N}\sigma^2 - \frac{1}{N}\sigma^2 + \frac{1}{N^2}N\sigma^2 = -\frac{\sigma^2}{N}. \end{aligned}$$

Hence

$$E\{\hat{s}_\tau^{(u)}\} = \frac{1}{N - |\tau|}\sum_{t=1}^{N-|\tau|} E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} = -\frac{\sigma^2}{N},$$

and

$$E\{\hat{s}_\tau^{(p)}\} = \frac{1}{N}\sum_{t=1}^{N-|\tau|} E\{(X_t - \bar{X})(X_{t+|\tau|} - \bar{X})\} = -\left(1 - \frac{|\tau|}{N}\right)\frac{\sigma^2}{N}.$$

Since for a white noise process $s_\tau = 0$ when $|\tau| \neq 0$, the magnitude of the bias for $\hat{s}_\tau^{(u)}$ is σ^2/N, while the magnitude of the bias for $\hat{s}_\tau^{(p)}$ is $(1 - |\tau|/N)\,\sigma^2/N$, which is strictly less than σ^2/N (unless $\sigma^2 = 0$, an uninteresting special case).

(b) Let

$$Q_\tau \equiv \sum_{t=1}^{N-|\tau|} (X_t - \bar{X})(X_{t+|\tau|} - \bar{X})$$

so that $\hat{s}_\tau^{(u)} = Q_\tau/(N-|\tau|)$ and $\hat{s}_\tau^{(p)} = Q_\tau/N$. Then we have

$$\operatorname{var}\{\hat{s}_\tau^{(u)}\} = \frac{\operatorname{var}\{Q_\tau\}}{(N-|\tau|)^2} \geq \frac{\operatorname{var}\{Q_\tau\}}{N^2} = \operatorname{var}\{\hat{s}_\tau^{(p)}\}.$$

Hence, recalling that mse is given by variance plus bias squared,

$$\operatorname{mse}\{\hat{s}_\tau^{(u)}\} = \frac{\operatorname{var}\{Q_\tau\}}{(N-|\tau|)^2} + \frac{\sigma^4}{N^2} > \frac{\operatorname{var}\{Q_\tau\}}{N^2} + \left(1 - \frac{|\tau|}{N}\right)^2 \frac{\sigma^4}{N^2} = \operatorname{mse}\{\hat{s}_\tau^{(p)}\}.$$

(c) The sum of the elements on the main diagonal is $\sum_{t=1}^N (X_t - \bar{X})^2 = N\hat{s}_0^{(p)}$. The sum of the elements on the τth diagonal is $N\hat{s}_\tau^{(p)}$ for $\tau = 1, \ldots, N-1$. Since $\hat{s}_\tau^{(p)} = \hat{s}_{-\tau}^{(p)}$, the sum of all the elements in the matrix is $N\sum_{\tau=-(N-1)}^{(N-1)} \hat{s}_\tau^{(p)}$. However, the sum of the uth row is $(X_u - \bar{X})\sum_{v=1}^N (X_v - \bar{X})$ which is identically zero for all u. Hence the result.

Since $\hat{s}_0^{(p)} > 0$, it follows that $\hat{s}_\tau^{(p)}$ must be negative for some value(s) of τ (a property not necessarily shared by the true autocovariance sequence).