Examples II for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

A general hint for some parts of the two questions: the autocovariance sequence of an MA(q) process is given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q}\theta_{j+|\tau|,q}, & \text{if } |\tau| \le q, \\ 0, & \text{if } |\tau| > q. \end{cases}$$

[1](a) If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary sequences — i.e., X_s and Y_t are uncorrelated for every s and t — show that $\{Z_t\}$ defined by $Z_t = X_t + Y_t$, is stationary with autocovariance sequence given by $s_{Z,\tau} = s_{X,\tau} + s_{Y,\tau}$, where $\{s_{X,\tau}\}$ is the autocovariance sequence for $\{X_t\}$ and $\{s_{Y,\tau}\}$ is the autocovariance sequence for $\{Y_t\}$.

(b) Determine, with full justification, whether

$$s_\tau = \begin{cases} 2+\psi^2, & \tau = 0; \\ -\psi, & |\tau| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

(where ψ is a constant with $|\psi| < 1$), is a valid autocovariance sequence.

[2] Let $\{X_t\}$ be a Gaussian (normal) stationary process with a mean of zero and autocorrelation sequence $\rho_{X,\tau}$.

You will need to use the following version of the Isserlis Theorem: If X_j, X_k, X_l, X_m are any four real-valued Gaussian random variables with zero mean then

$$E\{X_jX_kX_lX_m\} = E\{X_jX_k\}E\{X_lX_m\} + E\{X_jX_l\}E\{X_kX_m\} + E\{X_jX_m\}E\{X_kX_l\}.$$

(a) Define $Y_t = X_tX_{t-1}$. Find the autocovariance sequence $s_{Y,\tau}$ of $\{Y_t\}$ in terms of the autocovariance sequence $s_{X,\tau}$ of $\{X_t\}$.

If $\{X_t\}$ is an MA(1) process, give the form of $s_{Y,\tau}$ in terms of $\theta_{1,1}$ and σ_ϵ^2.

(b) Show that the autocorrelation sequence for $Y_t = X_t^2$ is given by $\rho_{Y,\tau} = \rho_{X,\tau}^2$.

Find *one* model for $\{X_t\}$, such that $Y_t = X_t^2$ has autocovariance sequence $\{s_{Y,\tau}\}$ given by

$$s_{Y,\tau} = \begin{cases} 200, & \tau = 0; \\ 0, & |\tau| = 1; \\ 18, & |\tau| = 2; \\ 0, & |\tau| > 2, \end{cases}$$

giving all parameter value combinations that satisfy the stated form of $\{s_{Y,\tau}\}$.