SOLUTIONS for Examples III for Time Series (S8)

NB: Stationarity by itself always means 'second-order' stationarity

[1] (a) The corresponding characteristic polynomial is $\Phi(z) = (1 + \frac{1}{12}z - \frac{1}{24}z^2)$ which can be factorized as $(1 - \frac{1}{6}z)(1 + \frac{1}{4}z)$ so that the roots are 6 and -4, which are both outside the unit circle, and therefore this AR(2) process is stationary.

(b) Note first that $E\{I\} = 1/2 \times 1 + 1/2 \times (-1) = 0$. We thus have $E\{X_t\} = E\{y_t I\} = y_t E\{I\} = 0$.

For the variance, note that $\text{var}\{I\} = E\{I^2\} = 1/2 \times 1^2 + 1/2 \times (-1)^2 = 1$. We thus have $\text{var}\{X_t\} = \text{var}\{y_t I\} = y_t^2 E\{I^2\} = 1$ since $y_t^2 = 1$ for all t.

For the autocovariance, we have $\text{cov}\{X_t, X_{t+\tau}\} = E\{X_t X_{t+\tau}\} = y_t y_{t+\tau} E\{I^2\} = y_t y_{t+\tau}$. Now, if either (a) $t + \tau \le 0$ and $t \le 0$ or (b) $t + \tau > 0$ and $t > 0$, then $y_t y_{t+\tau} = 1$; otherwise, $y_t y_{t+\tau} = -1$.

A requirement of stationarity is that $\text{cov}\{X_t, X_{t+\tau}\}$ be a finite number independent of t. This is not true for this stochastic process. For example, if $\tau = -5$ and $t = 0$, then $\text{cov}\{X_t, X_{t+\tau}\} = 1$; on the other hand, if $\tau = -5$ and $t = 1$, then $\text{cov}\{X_t, X_{t+\tau}\} = -1$. We conclude that X_t is not a stationary process.

[2] (a) Firstly,

$$E\{Z_t\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} C\mathrm{e}^{\mathrm{i}(2\pi f_0 t + \theta)} d\theta = C\mathrm{e}^{\mathrm{i}2\pi f_0 t} \left[\frac{\mathrm{e}^{\mathrm{i}\pi} - \mathrm{e}^{-\mathrm{i}\pi}}{\mathrm{i}2\pi} \right] = 0.$$

So,

$$\text{cov}\{Z_t, Z_{t+\tau}\} = E\{C\mathrm{e}^{-\mathrm{i}(2\pi f_0 t + \theta)} \cdot C\mathrm{e}^{\mathrm{i}(2\pi f_0 [t+\tau] + \theta)}\} = C^2 \mathrm{e}^{\mathrm{i}2\pi f_0 \tau},$$

which is finite and dependent on τ and not t. Hence, the process is stationary.

(b)

(i) Since

$$E\{Z_t\} = E\{X_t \mathrm{e}^{-\mathrm{i}2\pi f_0 t}\} = \mathrm{e}^{-\mathrm{i}2\pi f_0 t} E\{X_t\} = 0,$$

we have

$$\begin{aligned} \text{cov}\{Z_t, Z_{t+\tau}\} &= E\{Z_t^* Z_{t+\tau}\} = E\{X_t \mathrm{e}^{\mathrm{i}2\pi f_0 t} X_{t+\tau} \mathrm{e}^{-\mathrm{i}2\pi f_0 (t+\tau)}\} \\ &= \mathrm{e}^{-\mathrm{i}2\pi f_0 \tau} E\{X_t X_{t+\tau}\} = \mathrm{e}^{-\mathrm{i}2\pi f_0 \tau} s_{X,\tau}. \end{aligned}$$

So $\{Z_t\}$ is a complex-valued process with acvs $s_{Z,\tau} \equiv \mathrm{e}^{-\mathrm{i}2\pi f_0 \tau} s_{X,\tau}$. Now

$$\begin{aligned} S_Z(f) = \sum_{\tau=-\infty}^{\infty} s_{Z,\tau} \mathrm{e}^{-\mathrm{i}2\pi f \tau} &= \sum_{\tau=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi f_0 \tau} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi f \tau} \\ &= \sum_{\tau=-\infty}^{\infty} s_{X,\tau} \mathrm{e}^{-\mathrm{i}2\pi (f+f_0)\tau} = S_X(f + f_0), \end{aligned}$$

from which we see that the spectral density function of $\{Z_t\}$ is $S_Z(f) \equiv S_X(f+f_0)$.

(ii) Since $E\{Z_t\} = E\{X_t + \mathrm{i}X_{t+k}\} = 0$, we have

$$\begin{aligned}\operatorname{cov}\{Z_t, Z_{t+\tau}\} = E\{Z_t^* Z_{t+\tau}\} &= E\{(X_t - \mathrm{i}X_{t+k})(X_{t+\tau} + \mathrm{i}X_{t+\tau+k})\} \\ &= 2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}.\end{aligned}$$

So $\{Z_t\}$ is a complex-valued process with acvs $s_{Z,\tau} \equiv 2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}$. Now

$$\begin{aligned}S_Z(f) = \sum_{\tau=-\infty}^{\infty} s_{Z,\tau}\mathrm{e}^{-\mathrm{i}2\pi f\tau} &= \sum_{\tau=-\infty}^{\infty} [2s_{X,\tau} + \mathrm{i}s_{X,\tau+k} - \mathrm{i}s_{X,\tau-k}]\mathrm{e}^{-\mathrm{i}2\pi f\tau} \\ &= 2S_X(f) + \mathrm{i}\mathrm{e}^{\mathrm{i}2\pi fk}S_X(f) - \mathrm{i}\mathrm{e}^{-\mathrm{i}2\pi fk}S_X(f) \\ &= 2[1-\sin(2\pi fk)]S_X(f).\end{aligned}$$

[3] (a)

(i) The roots of the characteristic polynomial must be outside the unit circle.

(ii) For the MA(2) process the characteristic polynomial is $\Theta(z) = (1 - \frac{9}{4}z + \frac{1}{2}z^2)$ which can be factorized as $(1-2z)(1-\frac{1}{4}z)$ so that the roots are 1/2 and 4, so one is inside the unit circle, and therefore this MA(2) process is not invertible.

(b)

(i) The definition of $\{X_t\}$ implies that $\epsilon_{t-1} = \theta\epsilon_{t-2} + X_{t-1}$, $\epsilon_{t-2} = \theta\epsilon_{t-3} + X_{t-2}$ and so forth. Hence we have

$$\begin{aligned}X_t &= \epsilon_t - \theta\epsilon_{t-1} \\ &= \epsilon_t - \theta\left(\theta\epsilon_{t-2} + X_{t-1}\right) \\ &= \epsilon_t - \theta X_{t-1} - \theta^2\epsilon_{t-2} \\ &= \epsilon_t - \theta X_{t-1} - \theta^2\left(\theta\epsilon_{t-3} + X_{t-2}\right) \\ &= \epsilon_t - \theta X_{t-1} - \theta^2 X_{t-2} - \theta^3\epsilon_{t-3} \\ &\ \vdots \\ &= \epsilon_t - \sum_{j=1}^{p}\theta^j X_{t-j} - \theta^{p+1}\epsilon_{t-p-1}\end{aligned}$$

(after p such substitutions).

(ii) As $p \to \infty$, the final line above converges to the infinite autoregression

$$X_t + \sum_{j=1}^{\infty}\theta^j X_{t-j} = \epsilon_t$$

if $\lim_{p\to\infty}\theta^{p+1}\epsilon_{t-p-1} = 0$. The condition that $|\theta| < 1$ will do the trick and this is entirely consistent with 2(a)(i) since for invertibility the root of the polynomial $1 - \theta z$ must be outside the unit circle so that $|\theta| < 1$.