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Generalized maximum-likelihood generalized extreme-value quantile estimators for hydrologic data

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Abstract. The three-parameter generalized extreme-value (GEV) distribution has found wide application for describing annual floods, rainfall, wind speeds, wave heights, snow depths, and other maxima. Previous studies show that small-sample maximum-likelihood estimators (MLE) of parameters are unstable and recommend L moment estimators. More recent research shows that method of moments quantile estimators have for $-0.25 < \kappa < 0.30$ smaller root-mean-square error than L moments and MLEs. Examination of the behavior of MLEs in small samples demonstrates that absurd values of the GEV-shape parameter κ can be generated. Use of a Bayesian prior distribution to restrict κ values to a statistically/physically reasonable range in a generalized maximum likelihood (GML) analysis eliminates this problem. In our examples the GML estimator did substantially better than moment and L moment quantile estimators for $-0.4 \le \kappa \le 0$.

1. Introduction

The generalized extreme-value (GEV) distribution, introduced by *Jenkinson* [1955], has found many applications in hydrology. It was recommended for at-site flood frequency analysis in the United Kingdom [*Natural Environment Research Council* (*NERC*), 1975], for rainfall frequency in the United States by *Willeke et al*. [1995], and for sea waves [*de Haan and de Ronde*, 1998]. For regional frequency analysis the GEV distribution has received special attention since the introduction of the index-flood procedure based on probability-weighted moments (PWM) by *Wallis* [1980] and *Greis and Wood* [1981]. Many studies in regional frequency have used the GEV distribution [*Hosking et al*., 1985b; *Wallis and Wood*, 1985; *Lettenmaier et al*., 1987; *Hosking and Wallis*, 1988; *Chowdhury et al*., 1991; *Stedinger and Lu*, 1995; *Madsen et al*., 1997b]. In practice, it has been used to model a wide variety of natural extremes, including floods, rainfall, wind speeds, wave height, and other maxima. The physical origin of these maxima suggests that their distributions may be one of the extreme value (EV) types spanned by the GEV distribution (EV types I, II, and III). Mathematically, the GEV distribution is very attractive because its inverse has a closed form, and parameters are easily estimated by moments and L moments [*Hosking et al*., 1985a; *Hosking*, 1990].

In general, the length of a gauged record restricts the precision of at-site estimators. Thus it is important to use efficient estimation methods for quantiles and parameters, as well as to use additional information such as regional or/and historical data [*Jin and Stedinger*, 1989]. *Hosking et al*. [1985a] showed that the probability weighted moments (PWM) or equivalent L moments (LM) estimators for the GEV distribution are better than maximum-likelihood estimators (MLE) in terms of bias and variance for sample sizes varying from 15 to 100. More recently, *Madsen et al*. [1997a] showed that the method of moments (MOM) quantile estimators have smaller RMSE (root-mean-square error) for $-0.25 < \kappa < 0.30$ than LM and MLE when estimating the 100-year event for sample sizes of 10–50. MLEs are preferable only when $\kappa > 0.3$ and the sample sizes are modest ($n \ge 50$).

This paper examines the behavior of MLEs in small samples and demonstrates that, indeed, absurd values of the GEV shape parameter κ can be generated in small samples. This explains the unstable behavior of small-sample maximum likelihood (ML) quantile estimators reported by *Hosking et al*. [1985a]. In contrast to MLEs the MOM restricts $\hat{\kappa}$ to values greater than $-1/3$, whereas LM estimators restrict $\hat{\kappa}$ to values greater than -1 (S. G. Coles and M. J. Dixon, Likelihood-based inference for extreme value models, submitted to *Extremes*, 1998).

The three parameters ξ, α, and κ of the GEV distribution are related to its range (for $\kappa < 0$, $\xi + \alpha/\kappa \le x$; for $\kappa > 0$, $x \le \xi + \alpha/\kappa$). Since the range of the distribution depends on unknown parameters, the regularity conditions for maximum-likelihood estimation are not necessarily satisfied. For $\kappa \le 0.50$ the desirable asymptotic properties of efficiency and normality of MLEs hold, but the same is not true for $\kappa > 0.5$ [*Smith*, 1985; *Cheng and Iles*, 1987; *Davison and Smith*, 1990]. Indeed, if $\kappa > 1$, the density $\rightarrow +\infty$ as $\xi + \alpha/\kappa \rightarrow x_{(N)}$ (the largest observation). As a result, MLEs do not exist. In such nonregular cases ($\kappa > 0.5$) the ML method can perform satisfactorily if the likelihood is appropriately modified, for instance, (1) by grouping the observations and working with the product of probability increments instead of the product of densities or (2) by using the largest observation as the bound and then employing ML to estimate the two remaining parameters with the remaining observations [*Smith*, 1985; *Cheng and Iles*, 1987]. Nevertheless, these modifications of the likelihood function do not address the small-sample properties of ML estimation. Our

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interest in flood frequency analysis is for $\kappa \leq 0$ and certainly $\kappa < 0.5$, for which the asymptotic properties of MLEs do hold.

Here a Bayesian framework is employed to restrict estimated κ values to a statistically/physically reasonable range for annual flood series. That range for κ is described by a prior distribution function which assigns weights to different values of κ within the allowed range. This paper demonstrates that such a prior distribution function, combined with at-site sample data, produces more accurate T-year event estimators over the range of κ values of interest than do moment and L moment estimators with small and moderate sample sizes. The new estimator is called the generalized maximum likelihood (GML) estimator.

2. Literature Review

The computational aspects of maximum-likelihood estimation of the GEV distribution parameters have been developed by *Jenkinson* [1969], *Otten and Van Montfort* [1980], *Prescott and Walden* [1980, 1983], and *Hosking* [1985]. The *NERC* [1975] uses the method of scoring which employs the expected value of the Fisher information matrix for computing MLEs of the GEV distribution. Instead of the method of scoring, *Prescott and Walden* [1983] recommend use of the Newton-Raphson method to identify the MLEs for the GEV distribution. When compared with the method of scoring, the method of Newton-Raphson improves the speed of convergence and identifies more accurate values for the variances of the MLEs [*Hosking*, 1985]. *Hosking* [1985] presents an algorithm based on the Newton-Raphson method with some modifications to improve the speed and rate of convergence. *MacLeod* [1989] made small modifications to Hosking's algorithm in order to prevent some run time errors and to improve the inversion of the Hessian matrix.

Hosking et al. [1985a] show that small-sample MLE parameter estimators are very unstable and recommend probability-weighted moment (PWM) estimators, which are equivalent to L moment estimators [*Hosking*, 1990]. Both are linear combinations of order statistics. L moments are analogous to ordinary moments, and they can be used to summarize theoretical probability distributions and sample characteristics [*Hosking and Wallis*, 1997].

An advantage of maximum-likelihood estimators is that they can employ censored information without difficulty. To take advantage of the efficiency of L moments in small samples for the GEV distribution with regional flood data and historical/paleoflood information, *Jin and Stedinger* [1989] used a combination of both ML and L moment estimators.

When the shape parameter κ for the at-site GEV distribution is not known, it is difficult to identify reliable flood quantile estimators with short and moderate sample sizes. *Lu and Stedinger* [1992] show that when the shape parameter is misrepresented, a two-parameter quantile estimator (Gumbel/L moments) can have smaller MSE than the corresponding three-parameter quantile estimator, depending on the true shape parameter and the sample size. Similar work was reported by *Rosbjerg et al*. [1992] for partial duration series.

More recently, *Christopeit* [1994] has shown that the method of moments provides reasonable GEV parameter estimates for the distribution of earthquake magnitudes. *Madsen et al*. [1997a, b] address the relative precision of GEV flood-quantile estimators obtained with both partial duration and annual maximum series using MLE, MOM, and L moment estimators.

3. GEV Distribution

The generalized extreme-value distribution (GEV) incorporates Gumbel's type I ($\kappa = 0$), Frechet's type II ($\kappa < 0$), and the Weibull or type III ($\kappa > 0$) distributions. The GEV distribution has cumulative distribution function

$$\begin{aligned} F(x) &= \exp\left\{-\left[1-\kappa\frac{(x-\xi)}{\alpha}\right]^{1/\kappa}\right\} \qquad \kappa \neq 0 \\ &= \exp\left\{-\exp\left[-\frac{(x-\xi)}{\alpha}\right]\right\} \qquad \kappa = 0, \end{aligned} \tag{1}$$

where $\xi + \alpha/\kappa \leq x < \infty$ for $\kappa < 0$, $-\infty < x < +\infty$ for $\kappa = 0$, and $-\infty < x \leq \xi + \alpha/\kappa$ for $\kappa > 0$. Here ξ, α, and κ are the location, scale, and shape parameters, respectively.

Quantiles of the GEV distribution are given in terms of the parameters and the cumulative probability p by

$$\begin{aligned} x_p &= \xi + \frac{\alpha}{\kappa}\left[1-(-\ln(p))^{\kappa}\right] \qquad \kappa \neq 0 \\ &= \xi - \alpha \ln(-\ln(p)) \qquad \kappa = 0. \end{aligned} \tag{2}$$

3.1. L Moments

The L moment estimators for the GEV distribution [*Hosking et al*., 1985a] are

$$\begin{gathered} \hat{\xi} = \hat{\lambda}_1 - \frac{\hat{\alpha}}{\hat{\kappa}}\{1-\Gamma(1+\hat{\kappa})\} \qquad \hat{\alpha} = \frac{\hat{\lambda}_2\hat{\kappa}}{(1-2^{-\hat{\kappa}})\Gamma(1+\hat{\kappa})} \\ \hat{\kappa} = 7.8590c + 2.9554c^2, \\ c = 2/(3+\hat{\tau}_3) - \log(2)/\log(3). \end{gathered} \tag{3}$$

Here the final $\hat{\kappa}$ function is a very good approximation for $\hat{\kappa}$ in the range $(-0.5, 0.5)$ [*Hosking et al*., 1985a]. The L moment estimators $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\tau}_3 = \hat{\lambda}_3/\hat{\lambda}_2$ (L skewness) were obtained by using an unbiased estimator of the first three PWMs defined as

$$\beta_r = \xi + \frac{\alpha}{\kappa}\left[1-(r+1)^{-\kappa}\Gamma(1+\kappa)\right]/(r+1). \tag{4}$$

The unbiased estimator of β_r is [*Landwehr et al*., 1979; *Hosking and Wallis*, 1995]

$$b_r = \sum_{i=1}^{n}\left[\frac{(i-1)(i-2)(i-3)\ldots(i-r)}{n(n-1)(n-2)\ldots(n-r)}x_{(i)}\right] \tag{5}$$

$$r = 0, 1, 2, \ldots,$$

where the $x_{(i)}$ are the ordered observations from a sample of size $n\{x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}\}$ and where $\lambda_1 = \beta_0$, $\lambda_2 = 2\beta_1 - \beta_0$, and $\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$ [*Hosking*, 1990; see also *Wang*, 1996].

3.2. Moment Estimators

The moment estimators of the parameters of the GEV distribution are given by

$$\hat{\xi} = \hat{\mu} - \frac{\hat{\alpha}}{\hat{\kappa}}\{1-\Gamma(1+\hat{\kappa})\}$$

$$\hat{\alpha} = \frac{\hat{\sigma}|\hat{\kappa}|}{\{\Gamma(1+2\hat{\kappa}) - [\Gamma(1+\hat{\kappa})]^2\}^{1/2}} \tag{6}$$

$$\hat{\gamma} = \text{sign}(\hat{\kappa}) \cdot \frac{-\Gamma(1+3\hat{\kappa}) + 3\Gamma(1+\hat{\kappa})\Gamma(1+2\hat{\kappa}) - 2[\Gamma(1+\hat{\kappa})]^3}{\{\Gamma(1+2\hat{\kappa}) - [\Gamma(1+\hat{\kappa})]^2\}^{3/2}},$$

where sign($\hat{\kappa}$) is plus or minus 1 depending on the sign of $\hat{\kappa}$, $\Gamma(\,)$ is the gamma function, and $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\gamma}$ are the sample mean, standard deviation, and skewness, respectively [*Stedinger et al.*, 1993; *Madsen et al.*, 1997a]. There is no explicit solution of the skewness equation for κ, which requires an iterative solution. For $\hat{\kappa} > -1/3$ the first three moments of the GEV distribution are finite; The rth moment exists only if $\hat{\kappa} > -1/r$.

3.3. Maximum Likelihood Estimators

If the set $\{x_i\}$ are independent and identically distributed from a GEV distribution, then the log-likelihood function for a sample of n observations $\{x_1, x_2, \ldots, x_n\}$ is

$$\ln[L(\theta|x)] = -n\ln(\alpha) + \sum_{i=1}^{n}\left[\left(\frac{1}{\kappa} - 1\right)\ln(y_i) - (y_i)^{1/\kappa}\right], \tag{7}$$

where $\theta = (\xi, \alpha, \kappa)$ and $y_i = [1 - (\kappa/\alpha)(x - \xi)]$ [*Hosking et al.*, 1985a]. The MLE of ξ, α, and κ can be identified by solving the following system of equations, which correspond to setting to zero the first derivatives of $\ln[L(\theta|x)]$ with respect to each parameter [*Hosking*, 1985]. Thus

$$\frac{1}{\alpha}\sum_{i=1}^{S}\left[\frac{1-\kappa-(y_i)^{1/\kappa}}{y_i}\right] = 0$$

$$-\frac{S}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{S}\left[\frac{1-\kappa-(y_i)^{1/\kappa}}{y_i}\left(\frac{x_i-\xi}{\alpha}\right)\right] = 0 \tag{8}$$

$$-\frac{1}{\kappa^2}\sum_{i=1}^{S}\left\{\ln(y_i)[1-\kappa-(y_i)^{1/\kappa}] + \frac{1-\kappa-(y_i)^{1/\kappa}}{y_i}\,\kappa\left(\frac{x_i-\xi}{\alpha}\right)\right\} = 0.$$

The Newton-Raphson method was used to solve the likelihood equations above following *Hosking* [1985] and *MacLeod* [1989].

4. Maximum Likelihood in Small Samples

Hosking et al. [1985a] points out that the justification of the ML method is based on large-sample theory and that little work had been done to evaluate the GEV-MLE performance with small and moderate samples. Results reported by *Hosking et al.* [1985a] indicate that quantiles obtained by L moments are biased but preferable to ML estimators because MLEs are more variable. Reexamination of the behavior of MLEs in small samples demonstrates that absurd values of the GEV shape parameter κ can be generated. To illustrate the problem with estimation of κ in small samples, consider the following 15 numbers generated in the course of this study: −0.3955, −0.3948, −0.3913, −0.3161, −0.1657, 0.3129, 0.3386, 0.5979, 1.4713, 1.8779, 1.9742, 2.0540, 2.6206, 4.9880, and 10.3371 from a GEV distribution with $\xi = 0$, $\alpha = 1$, and $\kappa = -0.20$.

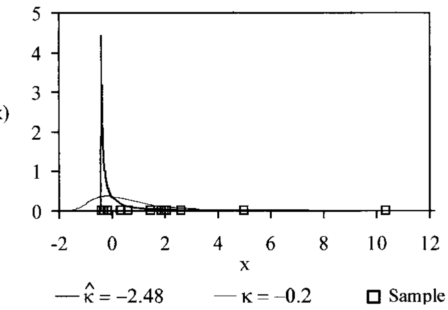

Figure 1. True ($\kappa = -0.20$) and "estimated" ($\hat{\kappa} = -2.48$) distribution for a small sample (sample values indicated by squares).

The maximum-likelihood estimators for this sample are

$$\hat{\xi} = -0.20, \qquad \hat{\alpha} = 0.53, \qquad \hat{\kappa} = -2.48.$$

The true value for the 0.999 quantile ($\xi = 0$, $\alpha = 1$, and $\kappa = -0.20$) is 14.9, while the estimated value ($\hat{\xi} = -0.20$, $\hat{\alpha} = 0.53$, and $\hat{\kappa} = -2.48$) for the same quantile is of the order of 6×10^6. This value corresponds to a contribution of about 10^9 to the MSE (mean-square error) of the 99.9 percentile estimator. Figure 1 compares the true distribution and "estimated" distribution for this small sample.

5. Generalized Maximum-Likelihood Estimators

5.1. A Prior Distribution for κ

Suppose that the true parameter κ of the GEV distribution with probability density function (pdf) $f(x|\xi, \alpha, \kappa)$ is a random variable whose range is $[\kappa_L, \kappa_U]$ with prior density $\pi(\kappa)$. Hydrologic experience indicates that $-0.30 \le \kappa \le 0$ is the most likely range for κ. Several studies show that floods seem to have heavier tails ($\kappa < 0$) than a Gumbel distribution ($\kappa = 0$) [*Farquharson et al.*, 1992; *Madsen and Rosbjerg*, 1997]. In addition, examination of GEV pdfs reveals that reasonable shapes are obtained for κ between −0.3 and perhaps +0.3. Figure 2 shows the pdf of the GEV distribution for different values of κ. In Figure 2, with $\kappa = -0.1$ and −0.3, both pdfs are positively skewed with a long upper tail, as one would expect for flood peaks and similar strictly positive phenomena such as annual rainfall maxima. In addition, the GEV distribution only has finite variance when $\kappa > -0.5$ and has finite skew when

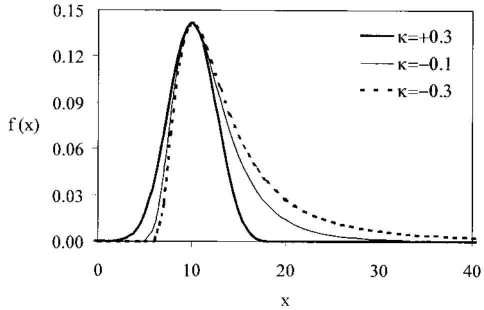

Figure 2. Probability density function of the generalized extreme value (GEV) distribution for $\kappa = -0.3$, −0.1, and +0.3 (where $\xi = 10$ and $\alpha = 2.6$).

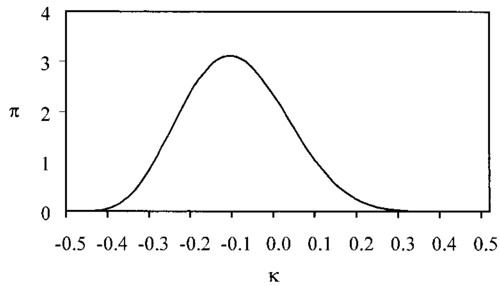

Figure 3. Geophysical prior for the GEV shape parameter κ. Beta distribution with mean $= -0.10$ and variance $= 0.122^2$ is shown.

$\kappa > -1/3$. For $\kappa < 0$ the EV type II has a Pareto tail such that $F(x) = 1 - [1 + (|\kappa|/\alpha)(x - \xi)]^{-|1/\kappa|}$ as $x \to \infty$, and for $\kappa > 0$ the EV type III has polynomial contact such that $F(x) = 1 - [(\kappa/\alpha)(\xi + \alpha/\kappa - x)]^{1/\kappa}$ as $x \to \xi + \alpha/\kappa$. Overall, a reasonable prior distribution for the shape parameter κ for annual maximum floods would look like the pdf in Figure 3.

The prior employed here is the beta distribution, $\pi(\kappa) = (0.5 + \kappa)^{p-1}(0.5 - \kappa)^{q-1}/B(p, q)$, between $[-0.5, +0.5]$, with $p = 6$ and $q = 9$, where $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q)$. It has $E[\kappa] = -0.10$ and $\mathrm{Var}[\kappa] = (0.122)^2$. Figure 3 shows the prior for κ, referred to here as our geophysical prior. This prior is relatively flat for κ values between -0.30 and 0.15, which is the range of interest. If regional information from a number of sites can be pooled to develop a more informative prior distribution for κ (as done by *Madsen and Rosbjerg* [1997]), improvements in extreme quantile estimators (exceedance probability $\leq 1\%$) should result.

5.2. Properties of Generalized Maximum-Likelihood Estimators

Once the prior $\pi(\kappa)$ is chosen, the joint density (or the generalized-likelihood function) is computed as $\mathrm{GL}(\xi, \alpha, \kappa|x) = L(\xi, \alpha, \kappa|x)\pi(\kappa)$, which shows the relationship between the generalized-likelihood (GL) function and the likelihood function. Thus $\ln[\mathrm{GL}(\xi, \alpha, \kappa|x)]$ equals the expression in (7) plus $\ln[\pi(\kappa)]$. One could be more rigorous and include as well a joint prior for ξ and α, but we have not done so. See *Kuczera* [1999] for a full Bayesian procedure in flood frequency analysis. Such a full Bayesian analysis is appropriate when one truly has regional information on the location and scale of the distribution, which we do not have in our general situation.

The generalized maximum likelihood estimator (GMLE) of ξ, α, and κ can be identified by maximizing the generalized log-likelihood function, which corresponds to the mode of the Bayesian posterior distribution of the parameters [*Berger*, 1985, p. 133]. Again, the Newton-Raphson method was used to compute the generalized-likelihood estimators.

With GMLEs, as the sample size increases, the information contained in the likelihood should dominate the information provided by the prior distribution [*Robert*, 1994, p. 138], so asymptotically the two have the same properties. In general, GMLEs will have the desired asymptotic optimal properties if both the likelihood and the prior satisfy a few regularity conditions, and they may even have these properties in other cases [*Lehmann and Casella*, 1996]. With the geophysical prior our GMLE will inherit the desirable asymptotic optimal properties of the GEV-MLEs for $-0.5 < \kappa < 0.5$.

6. Results

Monte Carlo simulations were performed for sample sizes 15–100, location parameter $\xi = 0$, scale parameter $\alpha = 1$, and shape parameters in the range $-0.4 \leq \kappa \leq +0.4$. For each sample size, 10,000 replicates were generated, and the bias, variance, and RMSE (root-mean-square error) of the 0.001–0.999 quantiles were computed for four estimation procedures: ML, LM, MOM, and GML.

Problems in the Newton-Raphson algorithm for ML estimation are well documented in the literature [*Hosking*, 1985; *Madsen et al.*, 1997a]. Nonconvergence of the Newton-Raphson algorithm for MLEs, as well as the generation of very extreme quantiles in small samples, is also experienced here. Generation of very extreme quantiles was associated with extremely negative values of κ, which resulted in unstable estimates of the RMSE for the ML estimation. Use of the GML method eliminated these problems.

For different values of κ the RMSE of quantile estimators for the GEV distribution are compared in Figures 4 and 5 and Table 1 for sample sizes of 25 and 100. For negative values of κ, GML estimation does better than the ML, LM, and MOM estimators in terms of RMSE. The RMSE decreases as the sample size increases for all methods, but the performance of the GML is still best.

Table 1 shows the bias and RMSE of the 0.001–0.999 quantile estimators for LM, MOM, and GML estimation when $\kappa = -0.10$, the mean of the geophysical prior. The GML method always performs as well as the LM and MOM methods and has

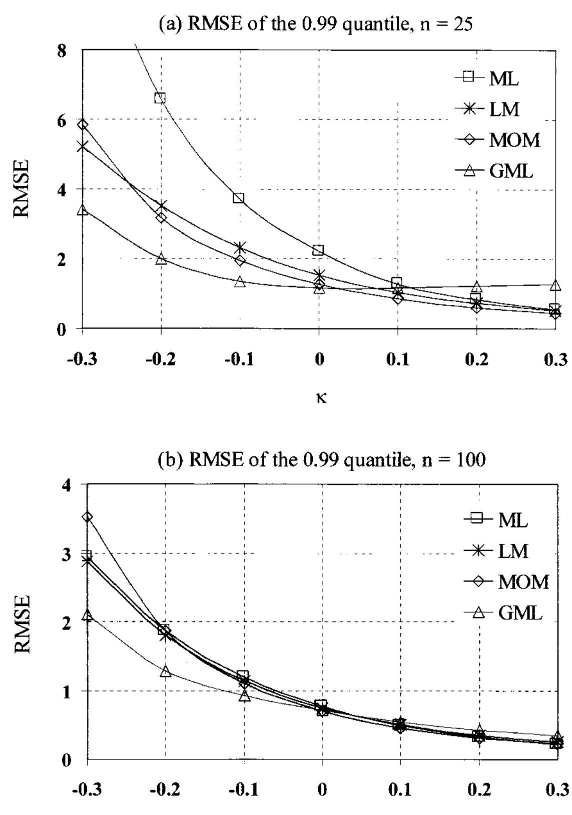

Figure 4. Root-mean-square error (RMSE) of the 0.99 quantile estimators (ML, maximum likelihood; LM, *L* moments; MOM, method of moments; and GML, generalized maximum likelihood). (a) Sample size 25 and (b) sample size 100.

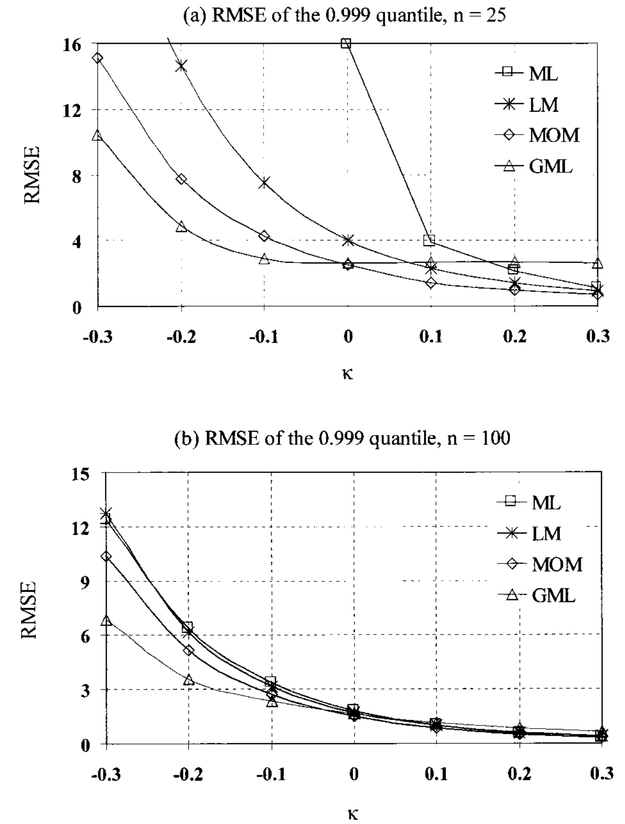

Figure 5. RMSE of the 0.999 quantile estimators (ML, maximum likelihood; LM, L moments; MOM, method of moments; and GML, generalized maximum likelihood). (a) Sample size 25 and (b) sample size 100.

the smallest RMSE for $p \geq 0.98$ and $p \leq 0.01$. Moreover, across the range of quantiles considered, the GML quantile estimators are relatively unbiased. MOM estimators, which have the second smallest RMSE for flood quantile estimators ($p \geq 0.98$), exhibit a large downward bias. In this case the GML quantile estimators have the smallest bias and the smallest RMSE. (See also Table 2 which includes results for $\kappa = -0.2$ and 0.0.)

Table 2 reports for $\kappa = -0.20$ and 0.0, the relative bias, standard deviation, and RMSE of upper quantile estimators. In order to compare the results for different quantiles, the results are presented for the ratio $\hat{x}_p/x_p$, where $\hat{x}_p$ is the estimator of the pth quantile [*Hosking et al.*, 1985a]. (For each κ, ξ was chosen so that $Pr[X \leq 0] = 1\%$ with $\alpha = 1$; *Stedinger and Lu* [1995] show this is a realistic choice.) For $\kappa \leq 0$ the GMLE are less variable than both the LM and MOM estimators but are more biased than the LM estimators. When κ is positive, the MOM method has the smallest bias and smallest standard deviation.

For fixed $\kappa = 0$, MLEs perform as well or better than L moments and the method of moments [*Landwehr et al.*, 1979]. Therefore ML was used here to estimate the two-parameter GEV/ML estimator with $\kappa = -0.10$ and to compare it with the three-parameter GEV/GML estimators for the 100-year event. Figure 6 shows the combinations of κ and n (sample size) for which the MSE of the two-parameter GEV/ML estimator with $\kappa = -0.10$ is equal to 0.5, 1.0, 1.5, and 2.0 times the MSE of the three-parameter GEV/GML estimator for the 100-year quantile. Similar figures are given by *Lu and Stedinger* [1992] and *Rosbjerg et al.* [1992]. For the 100-year event the region (κ, N) for which the two-parameter GEV/ML estimator with $\kappa = -0.10$ would be preferable to the three-parameter GEV/GML estimator is fairly narrow, indicating that the three-parameter

Table 1. Bias and RMSE of ML, LM, MOM, and GML Quantile Estimators of GEV Quantiles for $\kappa = -0.10$

		Bias				RMSE			
P	x_p	ML	LM	MOM	GML	ML	LM	MOM	GML
				n = 25					
0.001	−1.76	0.01	−0.09	−0.39	0.09	0.47	0.46	0.57	0.31
0.01	−1.42	0.03	−0.05	−0.26	0.08	0.32	0.33	0.41	0.26
0.1	−0.80	0.04	0.00	−0.08	0.05	0.20	0.20	0.25	0.20
0.2	−0.46	0.04	0.01	−0.01	0.03	0.20	0.20	0.22	0.20
0.5	0.37	0.01	0.01	0.08	−0.01	0.27	0.27	0.28	0.25
0.8	1.62	−0.05	−0.02	0.08	−0.06	0.42	0.42	0.47	0.42
0.9	2.52	−0.06	−0.05	0.00	−0.09	0.65	0.62	0.68	0.57
0.98	4.77	0.14	−0.03	−0.42	−0.14	2.19	1.58	1.46	1.06
0.99	5.84	0.42	0.05	−0.68	−0.14	3.71	2.32	1.95	1.35
0.998	8.61	2.10	0.57	−1.52	−0.07	13.57	5.33	3.45	2.31
0.999	9.95	3.67	1.04	−1.98	0.01	25.14	7.51	4.27	2.89
				n = 100					
0.001	−1.76	0.01	−0.02	−0.20	0.02	0.19	0.21	0.30	0.17
0.01	−1.42	0.01	−0.01	−0.13	0.02	0.14	0.15	0.22	0.13
0.1	−0.80	0.01	0.00	−0.05	0.01	0.10	0.10	0.12	0.10
0.2	−0.46	0.01	0.00	−0.01	0.01	0.10	0.10	0.11	0.10
0.5	0.37	0.00	0.00	0.04	0.00	0.13	0.13	0.14	0.13
0.8	1.62	−0.01	0.00	0.05	−0.01	0.21	0.21	0.23	0.21
0.9	2.52	−0.02	−0.01	0.02	−0.02	0.32	0.31	0.34	0.30
0.98	4.77	0.00	0.00	−0.17	−0.01	0.83	0.80	0.79	0.67
0.99	5.84	0.04	0.01	−0.30	0.01	1.20	1.15	1.10	0.93
0.998	8.61	0.22	0.14	−0.72	0.11	2.52	2.39	2.13	1.81
0.999	9.95	0.37	0.25	−0.95	0.19	3.35	3.16	2.73	2.34

True parameters are $\varepsilon = 0$, $\alpha = 1$, and $\kappa = -0.1$. Sample sizes N are 25 and 100. Results are obtained from 10,000 simulations. Here x_p is the true pth quantile. Definitions are as follows: P, probability, RMSE, root-mean-square error; GEV, generalized extreme value; ML, maximum likelihood; LM, L moments; MOM, moments; and GML, generalized maximum likelihood.

Table 2. Relative Bias, Standard Deviation, and RMSE of Quantile Estimators of GEV Quantiles

		$x_{0.90} = 4.16$			$x_{0.99} = 8.87$			$x_{0.999} = 16.22$		
N	Method	Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
				$\xi = 1.32$, $\alpha = 1$, and $\kappa = -0.20$.						
15	LM	−0.03	0.23	0.23	0.00	0.50	0.50	0.17	1.17	1.18
	MOM	0.01	0.30	0.30	−0.17	0.37	0.41	−0.32	0.40	0.52
	GML	−0.07	0.20	0.22	−0.15	0.22	0.27	−0.23	0.25	0.34
25	LM	−0.02	0.18	0.19	0.00	0.40	0.40	0.11	0.90	0.91
	MOM	0.02	0.25	0.25	−0.13	0.33	0.36	−0.28	0.40	0.48
	GML	−0.05	0.16	0.17	−0.13	0.19	0.23	−0.19	0.23	0.30
50	LM	−0.01	0.14	0.14	0.00	0.29	0.29	0.06	0.56	0.56
	MOM	0.03	0.17	0.17	−0.09	0.26	0.27	−0.20	0.32	0.38
	GML	−0.03	0.12	0.12	−0.09	0.16	0.18	−0.13	0.22	0.26
100	LM	−0.01	0.10	0.10	0.00	0.20	0.20	0.03	0.38	0.38
	MOM	0.03	0.12	0.13	−0.07	0.20	0.22	−0.16	0.28	0.32
	GML	−0.02	0.08	0.08	−0.05	0.14	0.15	−0.08	0.20	0.22

		$x_{0.90} = 3.78$			$x_{0.99} = 6.13$			$x_{0.999} = 8.44$		
N	Method	Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
				$\xi = 1.53$, $\alpha = 1$, and $\kappa = 0.0$						
15	LM	−0.01	0.17	0.17	0.02	0.33	0.33	0.14	0.68	0.69
	MOM	−0.02	0.17	0.17	−0.08	0.24	0.25	−0.12	0.33	0.35
	GML	−0.01	0.16	0.16	0.08	0.20	0.22	0.21	0.28	0.35
25	LM	−0.01	0.13	0.13	0.01	0.25	0.25	0.07	0.47	0.47
	MOM	−0.01	0.13	0.13	−0.06	0.20	0.20	−0.09	0.29	0.30
	GML	0.00	0.13	0.13	0.08	0.17	0.19	0.18	0.26	0.32
50	LM	−0.01	0.10	0.10	0.01	0.17	0.17	0.04	0.30	0.31
	MOM	−0.01	0.10	0.10	−0.03	0.15	0.15	−0.05	0.23	0.23
	GML	0.01	0.09	0.09	0.05	0.14	0.15	0.12	0.23	0.26
100	LM	0.00	0.07	0.07	0.00	0.12	0.12	0.02	0.20	0.21
	MOM	0.00	0.07	0.07	−0.02	0.11	0.11	−0.03	0.17	0.18
	GML	0.01	0.07	0.07	0.03	0.11	0.12	0.07	0.18	0.19

GML quantile estimator is very well behaved even for small sample sizes. *Lu and Stedinger* [1992] show that the two-parameter Gumbel/L moment estimator performs better than the three-parameter GEV/L moment estimator over a much broader region. For $n = 100$ the two-parameter GEV/ML estimator with $\kappa = -0.10$ would be preferable to the three-parameter GEV/GML estimator for κ in the range [−0.05, −0.15] (Figure 6). If regional information from other sites can be pooled to develop a more informative prior, additional improvements in the performance of the three-parameter GEV/GML estimator would be expected.

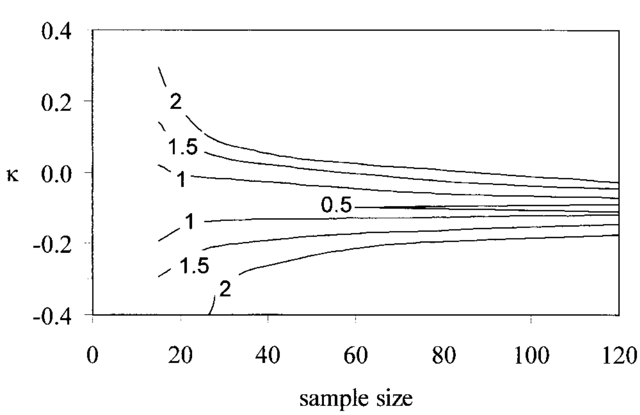

Figure 6. The combinations of sample size and shape parameter κ for which the two-parameter GEV/ML estimator with $\kappa = -0.10$ has a mean-square error (MSE) that is 0.5, 1.0, 1.5 and 2.0 times the MSE of the three-parameter GEV/GML estimator for the 100-year event.

Table 3 shows how more precise priors improve the precision of quantile estimators for samples from GEV distributions when $\kappa = -0.10$ and when values of κ are random values

Table 3. RMSE of 0.99 and 0.999 GML Quantile Estimators Where the Standard Deviation of the Prior Distribution Varies From 0 to 0.150 for $n = 50$

	RMSE of Quantile Estimator			
Standard Deviation of $\pi(\kappa)$	True $\kappa = -0.10$		True $\kappa \sim \pi(\kappa)$	
	$\hat{x}_{0.99}$	$\hat{x}_{0.999}$	$\hat{x}_{0.99}$	$\hat{x}_{0.999}$
0.150	1.280	3.246	1.696	5.185
0.100	1.023	2.259	1.338	3.540
0.050	0.822	1.440	0.987	2.144
0.025	0.790	1.295	0.847	1.559
0.000*	0.836	1.367	0.836	1.367

*Here value of κ is always −0.10, and the RMSE reflects only uncertainty in $\hat{\xi}$ and $\hat{\alpha}$.

drawn from a prior with the indicated standard deviation. The RMSE of the quantile estimators are larger when the true κ for each sample is drawn from the prior distribution for κ. However, in both cases the quantile estimators become more precise as the variance of the prior for κ decreases.

7. Conclusions

This paper compares the performance of GML, ML, MOM, and LM quantile estimators for the GEV distribution using Monte Carlo simulation. MLEs for the GEV distribution can yield unreasonable κ estimates in small samples and poor performance for quantile estimators.

Use of Bayesian prior distributions to restrict estimated κ values to a statistically/physically reasonable range in a generalized maximum-likelihood (GMLE) analysis eliminated this problem. The GML quantile estimator employes a beta prior distribution with $p = 6$ and $q = 9$ in the range $[-0.5, +0.5]$ for which $E[\kappa] = -0.10$ and $\mathrm{Var}[\kappa] = 0.015$. The GML does better than MOM and LM at estimating quantiles for $\kappa \leq 0$, which includes the general range of interest. If κ is thought to be positive in a region, then a more appropriate prior should be adopted.

When regional information from a number of sites can be pooled to develop a more informative prior distribution for κ (as done by *Madsen and Rosbjerg* [1997]), then substantial improvements in extreme quantile estimators (exceedance probability $\leq 1\%$) should result.

Comparisons of two-parameter GEV/ML with $\kappa = -0.10$ and three-parameter GEV/GML quantile estimators were performed for realistic record lengths. Two-parameter GEV/ML estimator with $\kappa = -0.10$ is better than the three-parameter GEV/GML quantile estimator in a narrow region of κ-n space.

An advantage of maximum-likelihood estimators is that they can efficiently employ historical information [*Jin and Stedinger*, 1989]. Cases with censoring or historical information are not as easily or as well addressed by LM and MOM. In addition, they need not have the restricted range of LM and moment estimators when very thick tailed distributions are appropriate. L moments are still useful to compute unbiased estimators of κ for use in regionalization, provided κ is not too negative, to describe characteristics of data and as a basis of goodness-of-fit tests [*Hosking and Wallis*, 1997].

In summary, for flood-like distributions which correspond in general to negative values of κ (heavy-tailed distributions), the results of this study indicate that the GEV/GML quantile estimators should be preferred for at-site analysis to L moment and method of moment estimators. If regional information about the shape parameter is available, it can be incorporated through use of a more informative prior distribution for κ.

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