Lectures 8-9: Alternating Direction Implicit Methods (ADI)

The fully implicit method involves solving a large number, $(M-1)\times(N-1)$, of simultaneous equations each time-step. An important alternative is the ADI method, whose basic idea is to be implicit in only one of the x, y directions at a time, and to alternate between the two. In a complete cycle, it is only necessary to solve $(M-1)$ and then $(N-1)$ equations, which is considerably more efficient. Furthermore, the systems of equations are tri-diagonal.

During the first half time-step we are implicit in x, say, so that

$$\frac{U_{mn}^{j+1/2}-U_{mn}^{j}}{k/2}=\frac{1}{h^2}\left(\delta_x^2U_{mn}^{j+1/2}+\delta_y^2U_{mn}^{j}\right)$$

or

$$\left(1-\tfrac{1}{2}r\delta_x^2\right)U_{mn}^{j+1/2}=(1+\tfrac{1}{2}r\delta_y^2)U_{mn}^{j}\ . \tag{8.9}$$

Over the next half time-step we treat y implicitly, so that

$$\frac{U_{mn}^{j+1}-U_{mn}^{j+1/2}}{k/2}=\frac{1}{h^2}\left(\delta_x^2U_{mn}^{j+1/2}+\delta_y^2U_{mn}^{j+1}\right)$$

or

$$\left(1-\tfrac{1}{2}r\delta_y^2\right)U_{mn}^{j+1}=(1+\tfrac{1}{2}r\delta_x^2)U_{mn}^{j+1/2}\ . \tag{8.10}$$

If we want, we can eliminate the values at the half-time level by operating on (8.9) with $(1+\frac{1}{2}r\delta_x^2)$ and (8.10) with $(1-\frac{1}{2}r\delta_x^2)$ to obtain

$$\left(1-\tfrac{1}{2}r\delta_x^2\right)\left(1-\tfrac{1}{2}r\delta_y^2\right)U_{mn}^{j+1}=(1+\tfrac{1}{2}r\delta_x^2)(1+\tfrac{1}{2}r\delta_y^2)U_{mn}^{j}\ . \tag{8.11}$$

We now consider stability of the scheme to a double Fourier perturbation of the form $U_{mn}^j=\widehat{U}^j\exp(im\xi+in\eta)$. In the familiar manner, the operator δ_x^2 becomes $-4\sin^2\frac{1}{2}\xi$, while δ_y^2 becomes $-4\sin^2\frac{1}{2}\eta$. Then (8.9) and (8.10) imply

$$\left.\begin{aligned}\widehat{U}^{j+1/2}&=\frac{1-2r\sin^2\frac{1}{2}\eta}{1+2r\sin^2\frac{1}{2}\xi}\,\widehat{U}^{j}=\lambda_1\widehat{U}^{j}\\ \widehat{U}^{j+1}&=\frac{1-2r\sin^2\frac{1}{2}\xi}{1+2r\sin^2\frac{1}{2}\eta}\,\widehat{U}^{j+1/2}=\lambda_2\widehat{U}^{j+1/2}\end{aligned}\right\}\quad\text{say.}$$

Then $|\lambda_1|\leqslant 1$ for all ξ, η only if $r\leqslant 1$, which is equivalent to "$r\leqslant\frac{1}{2}$" for a time-step $\frac{1}{2}k$ rather than k. λ_2 is similar. However, over the complete time-step, $\widehat{U}^{j+1}=\lambda_1\lambda_2\widehat{U}^{j}$ and

$$|\lambda_1\lambda_2|=\left|\frac{1-2r\sin^2\frac{1}{2}\xi}{1+2r\sin^2\frac{1}{2}\xi}\right|\left|\frac{1-2r\sin^2\frac{1}{2}\eta}{1+2r\sin^2\frac{1}{2}\eta}\right|\leqslant 1 \tag{8.12}$$

for every ξ and η. Thus, although each half time-step is unstable if repeated indefinitely, the alternation of one and then the other is unconditionally stable!

ADI in Three Dimensions

The obvious generalisation to three dimensions would be

$$\left.\begin{aligned}\left(1-\tfrac{1}{3}r\delta_x^2\right)U_{lmn}^{j+1/3} &= \left(1+\tfrac{1}{3}r(\delta_y^2+\delta_z^2)\right)U_{lmn}^{j}\\ \left(1-\tfrac{1}{3}r\delta_y^2\right)U_{lmn}^{j+2/3} &= \left(1+\tfrac{1}{3}r(\delta_x^2+\delta_z^2)\right)U_{lmn}^{j+1/3}\\ \left(1-\tfrac{1}{3}r\delta_z^2\right)U_{lmn}^{j+1} &= \left(1+\tfrac{1}{3}r(\delta_x^2+\delta_y^2)\right)U_{lmn}^{j+2/3}\end{aligned}\right\} \tag{8.13}$$

However, such a scheme turns out **not** to be unconditionally stable, and so is little improvement on explicit schemes. If we consider $U_{lmn}^j = \widehat{U}^j \exp(il\xi + im\eta + in\tau)$, we get

$$\widehat{U}^{j+1/3} = \lambda_1 \widehat{U}^j \quad \textit{etc.} \qquad \text{where} \quad \lambda_1 = \frac{1-\frac{4}{3}r(\sin^2\frac{1}{2}\eta + \sin^2\frac{1}{2}\tau)}{1+\frac{4}{3}r\sin^2\frac{1}{2}\xi} = \frac{1-b-c}{1+a}\ ,$$

Then

$$|\lambda_1\lambda_2\lambda_3| = \left|\frac{(1-b-c)(1-a-b)(1-a-c)}{(1+a)(1+b)(1+c)}\right|\ . \tag{8.14}$$

Stability occurs only if a, b, c are small enough, with the worst case being $\xi = \eta = \tau = \pi$, when $a = b = c$ and the necessary condition is $|(1-2a)/(1+a)| < 1|$ giving $r \leqslant 3/2$.

A more successful generalisation is obtained by using a Crank-Nicolson idea ($\theta = \frac{1}{2}$) in the implicit direction. Consider the 2-D scheme

$$\left.\begin{aligned}U_{mn}^{*j+1} - U_{mn}^j &= \tfrac{1}{2}r\delta_x^2(U_{mn}^{*j+1} + U_{mn}^j) + r\delta_y^2 U_{mn}^j\\ U_{mn}^{j+1} - U_{mn}^j &= \tfrac{1}{2}r\delta_x^2(U_{mn}^{*j+1} + U_{mn}^j) + \tfrac{1}{2}r\delta_y^2(U_{mn}^{j+1} + U_{mn}^j)\end{aligned}\right\}\ . \tag{8.15}$$

If we eliminate U^* from (8.15) we obtain the same relation (8.11) between U^{j+1} and U^j. However, whereas the intermediate value $U^{j+1/2}$ in (8.9) is an approximation to the real solution at the $j+\frac{1}{2}$ time-level, the intermediate values in (8.15) should be regarded as a first estimate of the solution at $j+1$, which is subsequently improved upon.

In three dimensions, (8.15) generalises to the scheme with two estimates U^* and U^*_*,

$$\left.\begin{aligned}(1-\tfrac{1}{2}r\delta_x^2)U_{lmn}^{*j+1} &= \left[1+r(\tfrac{1}{2}\delta_x^2 + \delta_y^2 + \delta_z^2)\right]U_{lmn}^{j}\\ (1-\tfrac{1}{2}r\delta_y^2)U_{*lmn}^{*j+1} &= \left[1+r(\tfrac{1}{2}\delta_x^2 + \tfrac{1}{2}\delta_y^2 + \delta_z^2)\right]U_{lmn}^{j} + \tfrac{1}{2}r\delta_x^2 U_{lmn}^{*j+1}\\ (1-\tfrac{1}{2}r\delta_z^2)U_{lmn}^{j+1} &= \left[1+\tfrac{1}{2}r(\delta_x^2 + \delta_y^2 + \delta_z^2)\right]U_{lmn}^{j} + \tfrac{1}{2}r\delta_x^2 U_{lmn}^{*j+1} + \tfrac{1}{2}r\delta_y^2 U_{*lmn}^{*j+1}\end{aligned}\right\} \tag{8.16}$$

This scheme is unconditionally stable.