Finite Difference Methods; Solutions to Example Sheet 1 A. J. Mestel 1996

1. Using Taylor series,

$$u_{1/3} = u(\tfrac{1}{3}h) = u(0) + \tfrac{1}{3}hu'(0) + \tfrac{1}{2}(\tfrac{1}{3}h)^2u''(0) + O(h^3)$$
$$u_{2/3} = u(\tfrac{2}{3}h) = u(0) + \tfrac{2}{3}hu'(0) + \tfrac{1}{2}(\tfrac{2}{3}h)^2u''(0) + O(h^3)$$

Eliminating $u''(0)$ by subtracting the second equation from four times the first,

$$4u_{1/3} - u_{2/3} = 3u_0 + \tfrac{2}{3}hu'(0) + O(h^3)$$
$$\text{or} \quad \frac{12u_{1/3} - 3u_{2/3} - 9u_0}{2h} = u'(0) + O(h^2)\ .$$

2. As $p = p(\rho)$, $\nabla p = c^2\nabla\rho$, where $c^2 = dp/d\rho$. Thus $\mathbf{u}\cdot\nabla p = c^2\mathbf{u}\cdot\nabla\rho = -\rho c^2\nabla\cdot\mathbf{u}$ using the second equation (mass conservation). Taking the scalar product of the first equation (momentum conservation) with $\mathbf{u}$, we have

$$\begin{aligned} 0 &= \mathbf{u}\cdot(\rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p) = \rho(\mathbf{u}\cdot\nabla)(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}) + \mathbf{u}\cdot\nabla p \\ &= \rho(\mathbf{u}\cdot\nabla)(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}) + c^2(-\rho\nabla\cdot\mathbf{u}) \\ \text{or}\quad c^2\nabla\cdot\mathbf{u} &= \mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u})\ . \end{aligned}$$

If $\mathbf{u} = (\phi_x,\, \phi_y,\, 0)$, this equation becomes

$$\left(\phi_x\frac{\partial}{\partial x} + \phi_y\frac{\partial}{\partial y}\right)\left(\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2\right) = c^2\left(\phi_{xx} + \phi_{yy}\right)$$
$$\text{or}\quad \phi_x^2\phi_{xx} + \phi_x\phi_y\phi_{xy} + \phi_y\phi_x\phi_{xy} + \phi_y^2\phi_{yy} = c^2\left(\phi_{xx} + \phi_{yy}\right)$$
$$\text{or}\quad (\phi_x^2 - c^2)\phi_{xx} + (2\phi_x\phi_y)\phi_{xy} + (\phi_y^2 - c^2)\phi_{yy} = 0\ .$$

This is a second order quasilinear PDE for $\phi(x, y)$. To classify it, we look at

$$\begin{aligned} \text{“}b^2 - 4ac\text{”} &= 4\phi_x^2\phi_y^2 - 4(\phi_x^2 - c^2)(\phi_y^2 - c^2) \\ &= 4c^2(\phi_x^2 + \phi_y^2 - c^2) = 4c^2(|\mathbf{u}|^2 - c^2) \end{aligned}$$

Thus the equation is hyperbolic if $|\mathbf{u}|^2 > c^2$ (supersonic flow), elliptic for subsonic flow ($|\mathbf{u}|^2 < c^2$) and parabolic if $|\mathbf{u}|^2 = c^2$.

3 As the scheme is centred about $(n,\, j)$, the truncation error is second order, so that $R_n^j = O(k^2,\, h^2)$. The scheme is

$$U_n^{j+1} - U_n^{j-1} = 2r(U_{n+1}^j + U_{n-1}^j - 2U_n^j) \ .$$

We analyse the behaviour of a Fourier disturbance proportional to $\exp(in\xi)$, by seeking solutions of the form $U_n^j = f^j \exp(in\xi)$. Then

$$\begin{aligned} (f^{j+1} - f^{j-1})e^{in\xi} &= 2r(e^{i\xi} + e^{-i\xi} - 2)e^{in\xi} \\ \text{or} \quad f^{j+1} - f^{j-1} &= -(8r\sin^2 \tfrac{1}{2}\xi)f^j \end{aligned} \ .$$

This is a second order linear recurrence relation, which has the solution $f^j = (\lambda)^j$, provided λ satisfies the equation

$$\begin{aligned} &\lambda^2 + (8r\sin^2 \tfrac{1}{2}\xi)\lambda - 1 = 0 \\ &\qquad\qquad \text{or} \quad \lambda = \lambda_\pm = -4r\sin^2 \tfrac{1}{2}\xi \pm \sqrt{16r^2\sin^4 \tfrac{1}{2}\xi + 1} \ . \end{aligned}$$

The square root is bigger than one, and so $\lambda_- < -1$ for all $r > 0$, and the scheme is **unconditionally unstable**; small errors grow exponentially each timestep. Although the scheme has a small truncation error, it is unstable and hence useless inpractice.

4. We have

$$\begin{aligned} u_n^{j+1} &= \left(u + ku_t + \tfrac{1}{2}k^2u_{tt} + \tfrac{1}{6}k^3u_{ttt}\right)_n^j + O(k^4) \\ u_n^{j-1} &= \left(u - ku_t + \tfrac{1}{2}k^2u_{tt} - \tfrac{1}{6}k^3u_{ttt}\right)_n^j + O(k^4) \end{aligned}$$

and so

$$\begin{aligned} u_n^{j+1} - u_n^{j-1} &= 2k(u_t)_n^j + O(k^3) \\ u_n^{j+1} + u_n^{j-1} &= (2u + k^2u_{tt})_n^j + O(k^4) \ . \end{aligned}$$

The truncation error, R_n^j, is defined by

$$R_n^j = \frac{u_n^{j+1} - u_n^{j-1}}{2k} - \frac{1}{h^2}\left[u_{n+1}^j + u_{n-1}^j - (u_n^{j+1} + u_n^{j-1})\right] - (u_t - u_{xx})_n^j \ .$$

Using the above, we have

$$\begin{aligned} R_n^j &= \frac{(2ku_t)_n^j + O(k^3)}{2k} - \frac{1}{h^2}\left[u_{n+1}^j + u_{n-1}^j - (2u + k^2u_{tt})_n^j) + O(k^4)\right] - (u_t - u_{xx})_n^j \\ &= -\frac{1}{h^2}(u_{n+1}^j + u_{n-1}^j - 2u_n^j) + (u_{xx})_n^j + \frac{k^2}{h^2}(u_{tt})_n^j + O(k^2,\, k^4/h^2) \\ &= \frac{k^2}{h^2}u_{tt} + O(k^2,\, h^2,\, k^4/h^2) \qquad \text{as} \quad u_{xx} = \frac{1}{h^2}(u_{n+1}^j + u_{n-1}^j - 2u_n^j) + O(h^2) \ . \end{aligned}$$

Thus the scheme is **consistent** with $u_t = u_{xx}$ only if $k^2/h^2 \to 0$ as $k \to 0$ and $h \to 0$, i.e. $k \to 0$ faster than $h \to 0$. Nevertheless, this scheme is in fact unconditionally stable (we have not shown this!)

5. The scheme is two-step, and can be written in the form $A\mathbf{V}^{j+1} = B\mathbf{V}^j + C\mathbf{V}^{j-1}$ where

$$\begin{array}{c} A = (1+\theta)I \\ C = -\theta I \end{array} \qquad B = \begin{pmatrix} 1+2\theta-2r & r & 0 & \ddots & 0 \\ r & 1+2\theta-2r & r & \ddots & 0 \\ 0 & r & 1+2\theta-2r & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & r \\ 0 & 0 & \ddots & r & 1+2\theta-2r \end{pmatrix}$$

and the matrices are (say) of size $(N-1)\times(N-1)$. The stability criterion is that all $(2N-2)$ solutions of

$$det(\lambda^2 A - \lambda B - C) = 0 \qquad \text{satisfy} \quad |\lambda| \leqslant 1$$

or

$$det(B - \sigma I) = 0 \qquad \text{where} \quad \sigma = \frac{\lambda^2(1+\theta)+\theta}{\lambda} \ .$$

From lectures, the eigenvalues of B are

$$\sigma = \sigma_m = 1 + 2\theta - 2r + 2r\cos(m\pi h) \qquad \text{for} \quad m = 1 \ldots N-1, \quad h = \tfrac{1}{N} \ .$$

We note that $\sigma_m \leqslant 1 + 2\theta$. The corresponding values of λ satisfy

$$F(\lambda) = 0 \qquad \text{where} \quad F(\lambda) \equiv (1+\theta)\lambda^2 - \sigma_m\lambda + \theta \ .$$

The product of the roots is $\theta/(1+\theta) < 1$, so that if the roots are complex they have the same modulus which is less than 1, giving stability. Now $F(1) = 1 + 2\theta - \sigma_m \geqslant 0$ and $F'(1) = 2 + 2\theta - \sigma_m \geqslant 1$. From the graph of the quadratic function $F(\lambda)$ is U-shaped, there will be a root with $\lambda < -1$ if $F(-1) < 0$, with resulting instability. That is, the scheme is unstable if

$$(1+2\theta) + \sigma_m < 0 \qquad \text{or} \quad 2(1+2\theta) < 2r(1 - \cos m\pi h) \leqslant 2r \ .$$

If $F(-1) > 0$, then $F'(-1) = -2(1+\theta) - \sigma_m = -F(-1) - 1 < 0$. By considering the graph of $F(\lambda)$, we see the only possible real roots lie between -1 and +1 and so we have stability. We conclude the stability condition is

$$r\sin^2(\tfrac{1}{2}(N-1)\pi/N) \leqslant \theta + \tfrac{1}{2} \qquad \text{or approximately} \quad r \leqslant \theta + \tfrac{1}{2} \ .$$

If we set $\theta = -\frac{1}{2}$, as in question 3, we conclude the method will be unconditionally unstable, as was indeed the case.

6. The equation for $C(x,\,y)$ can be written

$$C_y = \frac{D}{u(x)} C_{xx} + \frac{Q}{u} \qquad \text{with} \quad u = u_0(d^2 - x^2) \geqslant 0 \ .$$

Since the effective diffusivity, $D/u \geqslant 0$, we must integrate in the "positive time direction" $y > 0$, that is we must solve downstream. We choose a steplength $h = 2d/10$ and a timestep k to be determined. Stability of an explicit method is guaranteed if

$$\frac{k}{h^2} \max_{grid} \left[\frac{D}{u}\right] \leqslant \tfrac{1}{2} \ .$$

Now the maximum of D/u occurs near the boundary at $x = \pm(d-h)$ as u is zero on the boundary. One gridpoint in, $u = u_0 h(2d-h)$. So we must have

$$k \leqslant \tfrac{1}{2} h^3 (2d-h) u_0 / D = 9.5 \times 10^{-4} \frac{u_0 d^4}{D} .$$

This is a poor method, as we are requiring $k = O(h^3)$ which is very small indeed.

7. Let $U^j_{lmn} = f^j \exp(il\xi + im\eta + in\tau)$. Then just as in question 3, $\delta_x^2 \to -4\sin^2\frac{1}{2}\xi$ and similarly for δ_y^2 and δ_z^2. So the equation becomes

$$f^{j+1} - f^j = -\alpha\left(\theta f^{j+1} + (1-\theta) f^j\right) \ ,$$

where

$$\alpha = 4k\left(\frac{\sin^2\frac{1}{2}\xi}{h_x^2} + \frac{\sin^2\frac{1}{2}\eta}{h_y^2} + \frac{\sin^2\frac{1}{2}\tau}{h_z^2}\right) \geqslant 0 \ .$$

So

$$f^{j+1} = f^j\left[1 - \frac{\alpha}{1+\alpha\theta}\right] = \lambda f^j \ , \qquad \text{say.}$$

For stability we need $|\lambda| \leqslant 1$. As $\alpha \geqslant 0$, we have $\lambda \leqslant 1$. Now since $1 + \alpha\theta > 0$,

$$\begin{aligned} \lambda \geqslant -1 &\iff 1 - \alpha(1-\theta) \geqslant -(1+\alpha\theta) \\ &\iff \alpha(1-2\theta) \leqslant 2 \ . \end{aligned}$$

If $\theta \geqslant \frac{1}{2}$ this is satisfied and we have unconditional stability. If $0 \leqslant \theta < \frac{1}{2}$, we need $\alpha \leqslant 2/(1-2\theta)$ for stability. The worst case is the mode with $\xi = \eta = \tau = \pi$, so we must require for stability

$$k\left[\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right] \leqslant \frac{1}{2(1-2\theta)} \ .$$