Lecture 23. The Navier-Stokes equations and Convection in 2D

An important application of many of the techniques of this course is to the equations of Fluid Dynamics. The motion of an incompressible Newtonian fluid is defined by its velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ which satisfy the equations

$$\nabla \cdot \mathbf{u} = 0, \tag{23.1}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}, \tag{23.2}$$

where $\mathbf{F}$ is an external force such as gravity. These equations are tremendously important with very wide application. Here ν is a constant parameter, the inverse of the Reynolds number if the variables have been made non-dimensional. It measures the relative strength of diffusion and advection. The character of the equations is very different if ν is large or small, because ν multiplies the highest derivative in the equation.

In this course we shall only consider two-dimensional flows, of the form

$$\mathbf{u} = (u(x, y, t),\, v(x, y, t), 0) \qquad p = p(x, y, t). \tag{23.3}$$

Flows of this form can be represented by a single scalar function, $\psi(x, y, t)$. The incompressibility condition (23.1) can be satisfied by writing

$$\mathbf{u} = \nabla \times (0, 0, \psi) \qquad u = \psi_y, \quad v = -\psi_x.$$

Then the vorticity, $\nabla \times \mathbf{u} = (0, 0, \omega)$ where

$$\omega = -\nabla^2 \psi. \tag{23.4}$$

Now taking the curl of (23.2), we can eliminate the pressure field. The resulting vorticity equation only has a z-component, which is

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + G, \tag{23.5}$$

where $G = \widehat{\mathbf{z}} \cdot \nabla \times \mathbf{F}$. We could if we wish substitute for ω to obtain a single equation in ψ.

What boundary conditions shall we impose? The normal thing to would be to specify the velocity on the (solid walls), so that on a stationary boundary we would have $\psi = \psi_x = \psi_y = 0$.

We than have to solve the nonlinear advection diffusion equation (23.5) along with the elliptic equation (23.4). Our operator splitting ideas are very useful here, especially when we extend out multigrid ideas so that they can be used for the diffusion equation

$$u_t = \nu(u_{xx} + u_{yy}). \tag{23.6}$$

But we will attenpt a more complicated problem:

Rayleigh-Bénard Convection

An interesting fluid dynamical is known as **Convection**.

Convection occurs because fluids expand when heated. Their density decreases and they become buoyant, in accordance with Archimedes' principle. If T denotes the excess temperature The body force $\mathbf{F} = -\beta T \mathbf{g}$ where $\mathbf{g}$ is the gravitational acceleration and β is the coefficient of expansion. Gravity can then drive fluid motion, which then advects heat around. Suitably non-dimensionalised, the governing equations for the temperature T and velocity $\mathbf{u}$ are

$$\left.\begin{aligned} T_t + \mathbf{u}\cdot\nabla T &= \nabla^2 T \\ P^{-1}(\omega_t + \mathbf{u}\cdot\nabla\omega) &= RT_x + \nabla^2\omega \\ \omega &= -\nabla^{"}\psi \end{aligned}\right\} \tag{23.7}$$

There are two parameters in the problem: The Prandtl number, $P = \nu/\kappa$ is the ratio of the fluid kinematic viscosity to the thermal diffusivity. The Rayleigh number R is a measure of the thermal driving force, as defined last lecture. In deriving (23.7), it is assumed that the density decreases linearly with temperature, and that the fluid speed is much less than the speed of sound, permitting the Boussinesq approximation. Gravity acts in the y-direction.

The problem is known as Rayleigh-Bénard convection. There is a purely conductive solution, with $\mathbf{u} = 0$ and $T = T(y)$. What we expect to happen is that for $R \leqslant R_c$ the only solution is the conductive solution, but for $R > R_c$ this solution is unstable, and a flow (*convection*) develops. We want to investigate the critical value of R, and the convection patterns formed for different values of P, R.

Our solution plan is to use operator splitting. If T, ψ and ω are known at time t, then we:

(a) advect T
(b) diffuse T
(c) advect ω
(d) diffuse ω
(e) find new ψ and hence new velocity
(f) derive new boundary condition on ω from new ψ
(g) repeat,