1 Show that the 3-dimensional ADI method given below is an unconditionally stable approximation to the equation $u_t = \nabla^2 u$.

$$\left.\begin{aligned}
(1-\tfrac{1}{2}r\delta_x^2)U^{*j+1}_{lmn} &= \left[1+r(\tfrac{1}{2}\delta_x^2+\quad\delta_y^2+\delta_z^2)\right]U^{\,j}_{lmn}\\
(1-\tfrac{1}{2}r\delta_y^2)U^{*j+1}_{*lmn} &= \left[1+r(\tfrac{1}{2}\delta_x^2+\tfrac{1}{2}\delta_y^2+\delta_z^2)\right]U^{\,j}_{lmn}+\tfrac{1}{2}r\delta_x^2U^{*j+1}_{lmn}\\
(1-\tfrac{1}{2}r\delta_z^2)U^{\,j+1}_{lmn} &= \left[1+\tfrac{1}{2}r(\delta_x^2+\quad\delta_y^2+\delta_z^2)\right]U^{\,j}_{lmn}+\tfrac{1}{2}r\delta_x^2U^{*j+1}_{lmn}+\tfrac{1}{2}r\delta_y^2U^{*j+1}_{*lmn}
\end{aligned}\right\}\qquad (1)$$

2 Show that the implicit θ-method for the hyperbolic equation $u_{tt} = u_{xx}$ can be written

$$\delta_t^2 U_n^j = s^2\delta_x^2(1+\theta\delta_t^2)U_n^j\ , \qquad (2)$$

where $s = k/h$. Show that the truncation error of this scheme is $O(h^4 + k^4)$ for the particular value $\theta = (s^2-1)/12s^2$. Would this be a good value to choose in practice?

3 The 'Box scheme' for solving the advection equation $u_t + Vu_x = 0$ (V constant) is similar to the implicit Crank-Nicolson scheme, only it treats t and x derivatives similarly:

$$\frac{1}{2}\left[\frac{(U_{n+1}^{j+1}-U_{n+1}^{j})}{k}+\frac{(U_{n}^{j+1}-U_{n}^{j})}{k}\right]+\frac{V}{2}\left[\frac{(U_{n+1}^{j+1}-U_{n}^{j+1})}{h}+\frac{(U_{n+1}^{j}-U_{n}^{j})}{h}\right]=0\ . \qquad (3)$$

Calculate the amplification factor λ for an x-dependence like $e^{in\xi}$ and deduce that the scheme is conservative. Obtain the phase shift in the form

$$\mathrm{Arg}(\lambda) = -2\tan^{-1}\left[q\tan(\xi/2)\right], \quad \text{where} \quad q = \frac{kV}{h}\ . \qquad (4)$$

Observe that the low harmonics (ξ small) are modelled well by the finite difference approximation. Show also that when the Courant number $q = 1$, the phase factor is exactly correct for all ξ. Deduce that the approximation is exact in this case.

4 Examine the stability of the two-step Lax-Wendroff method for the conservation equation $\underline{u}_t + \left[\underline{f}(\underline{u})\right]_x = 0$, namely

$$\left.\begin{aligned}
\underline{U}^{\,j+1/2}_{n+1/2} &= \tfrac{1}{2}\left[\underline{U}^{\,j}_{n}+\underline{U}^{\,j}_{n+1}\right]-\tfrac{1}{2}s\left[\underline{f}^{\,j}_{n+1}-\underline{f}^{\,j}_{n}\right]\\
\underline{U}^{\,j+1}_{n} &= \underline{U}^{\,j}_{n}-s\left[\underline{f}^{\,j+1/2}_{n+1/2}-\underline{f}^{\,j+1/2}_{n-1/2}\right]
\end{aligned}\right\}\qquad (5)$$

for the case when $\underline{f}(\underline{u}) = \mathbf{A}\underline{u}$, where $\mathbf{A}$ is a constant matrix. Writing $\underline{U}_n^j = (\lambda)^j e^{in\xi}\underline{V}$, show that

$$\left[2s^2\sin^2(\tfrac{1}{2}\xi)\mathbf{A}^2 + is\sin(\xi)\mathbf{A} + (\lambda-1)\mathbf{I}\right]\underline{V} = 0\ . \qquad (6)$$

Assuming the eigenvectors of $\mathbf{A}$ are complete, the only non-zero solutions to (6) occur when $\underline{V}$ is an eigenvector of $\mathbf{A}$, so that

$$2s^2\sin^2(\tfrac{1}{2}\xi)\mu^2 + is\sin(\xi)\mu + (\lambda-1) = 0\ , \qquad (7)$$

where μ is an eigenvalue of $\mathbf{A}$. Assuming all values of μ are real, show that the stability condition is $s \max \mid \mu \mid \leq 1$. If, conversely, μ takes complex values (in conjugate pairs, of course), show that the low harmonics ($\xi \to 0$) grow exponentially for any value of s. Interpret this behaviour in terms of the existence of characteristics and the underlying nature of the partial differential equation.

5 Show that for a square grid with steplength h, the Laplacian may be approximated by

$$\frac{1}{2h^2}\left[u_{m+1,n+1} + u_{m+1,n-1} + u_{m-1,n-1} + u_{m-1,n+1} - 4u_{m,n}\right] = (\nabla^2 u)_{m,n} + O(h^2)\ . \quad (8)$$

Examine the truncation errors of (8) and the usual 5-point formula ($\nabla^2 \approx (\delta_x^2 + \delta_y^2)/h^2$) and hence find a linear combination of the two which enables an $O(h^4)$ approximation to $\nabla^2 u = f$ to be made.

6 Consider the solution of the Poisson equation with Dirichlet boundary conditions,

$$u_{xx} + u_{yy} = f \qquad \text{in D}, \qquad u = g \qquad \text{on a curved boundary C.} \quad (9)$$

Using a square mesh, we need to approximate the Laplacian near the boundary. Suppose that the curved boundary intersects the grid at the points A and B whose coordinates are $(x,\ y) = (M + \theta,\ N)h$, and $(x,\ y) = (M,\ N + \phi)h$ respectively, where $0 < \theta \leq 1$, $0 < \phi \leq 1$, as in the figure. Calculate approximations to u_{xx} and u_{yy} at the mesh point $(Mh,\ Nh)$ and hence show that a reasonable approximation to (9) at that point is

$$\frac{2g(A)}{\theta(1+\theta)} + \frac{2g(B)}{\phi(1+\phi)} + \frac{2U_{M-1,N}}{1+\theta} + \frac{2U_{M,N-1}}{1+\phi} - 2\left(\frac{1}{\theta} + \frac{1}{\phi}\right) U_{M,N} = h^2 f_{M,N}\ , \quad (10)$$

where $g(A)$ denotes g evaluated at the point A. What is the order of the truncation error of (10)?