3. The Explicit Method for the 1-D Diffusion Equation

Let us consider the problem for $u(x,\, t)$:

$$\left.\begin{aligned} u_t = u_{xx} \qquad &\text{in} \quad 0 < x < 1,\ t > 0 \\ \text{with} \quad u(0,\, t) = 0\ , \quad u(1,\, t) = 0\ , \quad &u(x,\, 0) = f(x) \end{aligned}\right\} . \tag{3.1}$$

We define a regular, rectangular grid $(x_n,\, t^j)$ for $0 \leqslant n \leqslant N$, $0 \leqslant j \leqslant J$ of lengths $(h,\, k)$, so that $Nh = 1$, $x_n = nh$, $t^j = jk$. We shall seek an approximation U_n^j to the exact solution evaluated on the grid points, $u_n^j \equiv u(nh,\, jk)$. The boundary conditions require $u_0^j = 0$, $u_N^j = 0$ and $u_n^0 = f_n \equiv f(x_n)$. Recalling the results from §1, we have

$$\frac{u_n^{j+1} - u_n^j}{k} = \frac{\partial u_n^j}{\partial t} + \tfrac{1}{2} k \frac{\partial^2 u_n^j}{\partial t^2} + O(k^2)\ ,$$

and

$$\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} = \frac{\partial^2 u_n^j}{\partial x^2} + \tfrac{1}{12} h^2 \frac{\partial^4 u_n^j}{\partial x^4} + O(h^4)\ .$$

Using the above in the equation $u_t = u_{xx}$, we have

$$\frac{u_n^{j+1} - u_n^j}{k} = \frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + R_n^j\ , \tag{3.2}$$

where the **Truncation Error,** R_n^j is

$$R_n^j = \tfrac{1}{12} h^2 \frac{\partial^4 u_n^j}{\partial x^4} - \tfrac{1}{2} k \frac{\partial^2 u_n^j}{\partial t^2} + O(k^2, h^4)\ . \tag{3.3}$$

The simplest explicit method neglects terms of $O(k,\, h^2)$. If we write

$$r = k/h^2\ ,$$

neglect R_n^j and replace u by U in (3.2), we obtain the scheme

$$U_n^{j+1} = rU_{n+1}^j + (1 - 2r)U_n^j + rU_{n-1}^j\ , \tag{3.4}$$

with the boundary conditions

$$U_0^j = 0\ , \qquad U_N^j = 0 \qquad \text{and} \quad U_n^0 = f_n \equiv f(x_n)\ .$$

With a little thought, we see that these boundary conditions and repeated use of (3.4) enable us to calculate U_n^j everywhere. We must now consider how accurate the approximation is. We shall perform a simple "**Maximum Principle Analysis**" to show that under some conditions U_n^j can be made as close as we choose to the real solution u_n^j, for all n and j.

Truncation Error and Solution Error

In general, suppose a PDE $Lu = f$, where L is some differential operator, is approximated on a suitable grid $(nh,\, jk)$, at which points u and f take the values u_n^j and f_n^j, by the FDM $MU_n^j = f_n^j$, where M is a difference operator. The **solution error**, z_n^j and the **truncation error**, R_n^j, are defined by

$$z_n^j \equiv u_n^j - U_n^j \ , \qquad \text{and} \quad R_n^j \equiv (Lu)_n^j - Mu_n^j \ . \tag{3.5}$$

Returning to our specific equation, subtracting (3.4) from (3.2), and using (3.5), we obtain

$$z_n^{j+1} = rz_{n+1}^j + (1-2r)z_n^j + rz_{n-1}^j + kR_n^j \qquad \text{and} \quad z_0^j = z_N^j = z_n^0 = 0 \ . \tag{3.6}$$

Now

$$|z_n^{j+1}| \leqslant |r|\ |z_{n+1}^j| + |(1-2r)|\ |z_n^j| + |r|\ |z_{n-1}^j| + |kR_n^j|.$$

Since $R_n^j = O(k, h^2)$, over the finite interval $0 < t < T \equiv Jk$ we can find a positive constant A such that $|R_n^j| \leqslant A(|k| + h^2)$. We shall also define the norm

$$||z^j|| \equiv \max_{n=0\ldots N} |z_n^j| \ ,$$

which is the maximum error over all the points at a fixed time-level j. Then $|z_n^j| \leqslant ||z^j||$ for all n, and so we have

$$|z_n^{j+1}| \leqslant \Big(|r| + |1-2r| + |r|\Big)||z^j|| + A(k^2 + |k|h^2)$$

As this is true for all values of n, it is true for that value which maximises its LHS, and so

$$||z^{j+1}|| \leqslant \Big(|r| + |1-2r| + |r|\Big)||z^j|| + A(k^2 + |k|h^2)$$

We now **assume** that $0 < r \leqslant \frac{1}{2}$, so that the quantities inside modulus signs are positive. Then the maximum possible error at the time-level $(j+1)$ is related to the maximum at time j by

$$||z^{j+1}|| \leqslant ||z^j|| + A(k^2 + |k|h^2)$$

Now the initial error, $||z^0||$, is zero because we know the solution exactly at $t = 0$. Applying the above repeatedly therefore implies

$$||z^1|| \leqslant |k|A(|k| + h^2) \ , \qquad ||z^2|| \leqslant 2|k|A(|k| + h^2) \qquad \text{and} \quad ||z^j|| \leqslant j|k|A(|k| + h^2) \ .$$

We have therefore shown that provided $0 < r \leqslant \frac{1}{2}$, the maximum possible error at time $t^j \equiv jk$ is $O(k,\, h^2)t^j$, which can be made as small as we choose by choosing small enough steplengths, k and h. Note that $r < 0$ would correspond to $k < 0$, which involves stepping **backwards in time**, which we saw in §2 leads to an ill-posed problem for this **Parabolic** equation. We have proved nothing yet about the case $r > \frac{1}{2}$, but we will find that for such values of r the FDM (3.2) is numerically unstable.