Lecture 17. The Lax-Wendroff method and conservation equations

One explicit method which attempts to get the second order terms correct is due to Lax and Wendroff. We know that

$$\frac{u_n^{j+1} - u_n^j}{k} = \left[u_t + \tfrac{1}{2} k u_{tt}\right]_n^j + O(k^2) \ .$$

So if $u_t = -c u_x$ with c constant, then $u_{tt} = -c u_{xt} = c^2 u_{xx}$, and we may write

$$\frac{u_n^{j+1} - u_n^j}{k} = -c \frac{\Delta u_n^j}{2h} + \frac{kc^2}{2h^2} \delta^2 u_n^j + O(k^2,\, h^2)$$

This leads to the centred scheme (16.3F) on the last sheet

$$U_n^{j+1} = U_n^j - \tfrac{1}{2} q \Delta U_n^j + \tfrac{1}{2} q^2 \delta^2 U_n^j \tag{17.1}$$

Note that this formula requires c to be constant. However, a similar result can be derived for the **conservation** equation $u_t + [f(u,\, x)]_x = 0$. In general, an equation of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \{\mathbf{b}\} = Q$$

can be interpreted as a conservation law for a physical quantity of density ρ, for which Q is a source and $\mathbf{b}$ is its *flux*. Over any fixed volume V bounded by the surface ∂V, normal $\widehat{\mathbf{n}}$,

$$\frac{d}{dt} \int_V \rho \, dV = \int_V Q \, dV - \int_{\partial V} \mathbf{b} \cdot \widehat{\mathbf{n}} \, dS \ ,$$

so that the rate of increase of the physical quantity in V is equal to the total rate of generation within V less the amount that flows out of V across the boundary.

Lax-Wendroff schemes for simultaneous conservation equations.

More generally, let $\mathbf{u}$ and $\mathbf{f}[\mathbf{u}]$ be p-vectors satisfying

$$\mathbf{u}_t + (\mathbf{f}[\mathbf{u}])_x = 0 \tag{17.2}$$

Equation (17.2) represents the conservation of p physical quantities. Ideally, we would like our FDM to conserve them also. For example, we could write

$$\mathbf{U}_n^{j+1} - \mathbf{U}_n^j = -\tfrac{1}{2} s \Delta \mathbf{F}_n^j \qquad \text{where} \quad \mathbf{F}_n^j = \mathbf{f}\left[\mathbf{U}_n^j\right] \qquad \text{and} \quad s = k/h \ . \tag{17.3}$$

This scheme is **conservative** because if we sum over all values of n the right-hand-side all cancels (apart possibly from boundary terms if we are on a finite region of x), so that $\sum_n \mathbf{U}_n^{j+1} = \sum_n \mathbf{U}_n^j$. However, we have seen that (17.3) may be unstable. Alternatively we might rewrite (17.2) in the form

$$(u_l)_t + \sum_{m=1}^{p} A_{lm} (u_m)_x = 0 \qquad \text{for} \quad l = 1 \ldots p \qquad \text{where} \quad A_{lm} = \frac{\partial (f_l)}{\partial (u_m)} \ , \tag{17.4}$$

and u_m is the m-th component of $\mathbf{u}$. If $A[\mathbf{u}]$ is the matrix whose (l,m)th element is A_{lm}, we could then approximate

$$\mathbf{U}_n^{j+1} - \mathbf{U}_n^j = -\tfrac{1}{2} sA\left(\Delta \mathbf{U}_n^j\right) \tag{17.5}$$

but this would **not** be conservative unless A is constant. The Lax-Wendroff idea applied to (17.2) gives

$$\mathbf{u}_{tt} = -(\mathbf{f}[\mathbf{u}])_{xt} = -(A\mathbf{u}_t)_x = \big(A(\mathbf{f})_x\big)_x$$

so that

$$\begin{aligned}(\mathbf{u}_{tt})_n^j &= \frac{1}{h}\left[A_{n+1/2}^j\,(\mathbf{f}_x)_{n+1/2}^j - A_{n-1/2}^j\,(\mathbf{f}_x)_{n-1/2}^j\right] + O(h^2) \\ &= \frac{1}{h^2}\left[A_{n+1/2}^j\left(\mathbf{f}_{n+1}^j - \mathbf{f}_n^j\right) - A_{n-1/2}^j\left(\mathbf{f}_n^j - \mathbf{f}_{n-1}^j\right)\right] + O(h^2)\end{aligned} \quad . \tag{17.6}$$

In the above, to the same order of accuracy we can approximate

$$A_{n+1/2}^j \equiv A[\mathbf{u}_{n+1/2}^j] = A\left[\tfrac{1}{2}(\mathbf{u}_{n+1}^j + \mathbf{u}_n^j)\right] + O(h^2)\ .$$

Using the estimate (17.6) for $\mathbf{u}_{tt}$, we obtain

$$\mathbf{U}_n^{j+1} = \mathbf{U}_n^j - \tfrac{1}{2}s\Delta\mathbf{F}_n^j + \tfrac{1}{2}s^2\left[A_{n+1/2}^j\left(\mathbf{F}_{n+1}^j - \mathbf{F}_n^j\right) - A_{n-1/2}^j\left(\mathbf{F}_n^j - \mathbf{F}_{n-1}^j\right)\right]\ . \tag{17.7}$$

However while (17.7) is a second order scheme, it is still non-conservative if A depends on $\mathbf{u}$, and moreover it is necessary to calculate $A_{n+1/2}^j$. A superior two-step version of the Lax-Wendroff scheme is due to Richtmyer. The system

$$\left.\begin{aligned}\mathbf{U}_{n+1/2}^{j+1/2} &= \tfrac{1}{2}\left(\mathbf{U}_n^j + \mathbf{U}_{n+1}^j\right) - \tfrac{1}{2}s\left(\mathbf{F}_{n+1}^j - \mathbf{F}_n^j\right) \\ \mathbf{U}_n^{j+1} &= \mathbf{U}_n^j - s\left(\mathbf{F}_{n+1/2}^{j+1/2} - \mathbf{F}_{n-1/2}^{j+1/2}\right)\end{aligned}\right\} \tag{17.8}$$

has truncation error $O(k^2+h^2)$, is **conservative** and stable if the Courant condition holds. When A is constant, so that $\mathbf{F} = A\mathbf{U}$, it reduces to (17.7). For the one-dimensional case $p = 1$, with $f = cu$, we obtain the two-step scheme (16.3D) with a different step-size.

The two-step Lax-Wendroff scheme (17.8) may be extended to two space dimensions without difficulty. Consider the equation for $\mathbf{u}(x,\, y,\, t)$

$$\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = 0 \qquad \text{and take} \quad h_x = h_y = h\ .$$

We define $\mathbf{U}$ at the half time-levels for $(p,\, q) = (m,\, n+1/2)$ or $(m+1/2,\, n)$ by

$$\mathbf{U}_{pq}^{j+1/2} = \tfrac{1}{4}\left[\mathbf{U}_{p+,q}^j + \mathbf{U}_{p-,q}^j + \mathbf{U}_{p,q+}^j + \mathbf{U}_{p,q-}^j\right] - \tfrac{1}{2}s\left[\mathbf{F}_{p+,q}^j - \mathbf{F}_{p-,q}^j + \mathbf{G}_{p,q+}^j - \mathbf{G}_{p,q-}^j\right]\ ,$$

where $p\pm$ denotes $p \pm 1/2$ and similarly $q\pm$. At the integer time levels, for $(p,\, q) = (m,\, n)$ or $(m+1/2,\, n+1/2)$,

$$\mathbf{U}_{pq}^{j+1} = \mathbf{U}_{pq}^j - s\left[\mathbf{F}_{p+,q}^{j+1/2} - \mathbf{F}_{p-,q}^{j+1/2} + \mathbf{G}_{p,q+}^{j+1/2} - \mathbf{G}_{p,q-}^{j+1/2}\right]\ . \tag{17.9}$$

The resulting scheme is second order and conservative.