4. Explicit Method for the General Non-linear Parabolic Problem in 1-D

Last time we proved that the simple explicit method for $u_t = u_{xx}$ would work provided $r < 1/2$ where $r = k/h^2$. Experiments with the program AdvDiff.m, show that if $r > 1/2$, the method breaks down seriously as small errors get amplified out of proportion. We can understand that by solving the scheme exactly, using separation of variables. Suppose at time $t = 0$ there is a small error proportional to $\varepsilon e^{in\xi}$ and we neglect all subsequent truncation errors R_n^j. Then from (3.6), the error z_n^j obeys the equation

$$z_n^{j+1} = r z_{n+1}^j + (1-2r) z_n^j + r z_{n-1}^j, \qquad z_n^0 = \varepsilon e^{in\xi}. \tag{4.1}$$

We can find **separable** solutions to this equation with $z_n^j = \varepsilon(\lambda)^j \exp(in\xi)$ provided

$$\lambda = re^{i\xi} + (1-2r) + re^{-i\xi} = 1 - 2r(1-\cos\xi) = 1 - 4r\sin^2 \tfrac{1}{2}\xi. \tag{4.2}$$

If this error is not to grow, then we require $|\lambda| \leqslant 1$. Now clearly $\lambda < 1$, but we need to ensure that $\lambda > -1$. The worst case is when $\xi = \pi$, and then we must require $1 - 4r \geqslant -1$ or $r \leqslant 1/2$. If this constraint is violated, we expect the errors to grow. Note that the worst case $\xi = \pi$ corresponds to a perturbation $\exp(in\xi)$ which alternates between $+1$ and -1 on neighbouring gridpoints. This is quite a common instability of finite difference methods. This is an example of the Fourier or Von Neumann stability method.

The program also considered the effect of an advecting velocity V, solving $u_t + Vu_x = au_{xx}$. It was found that even if $r < 1/2$, the method could be unstable. Let us try to generalise the Maximum Principle Analysis from last lecture to more general parabolic PDEs.

Consider the problem defined for an arbitrary function Φ,

$$u_t = \Phi(x,\ t,\ u,\ u_x,\ u_{xx}) \qquad \text{in} \quad 0 < x < 1 = Nh, \qquad 0 < t < T = Jk. \tag{4.3}$$

For physical sense, we will assume that the effective diffusivity is positive and bounded, so that

$$A \geqslant \frac{\partial \Phi}{\partial u_{xx}} \geqslant a > 0$$

for some constants A and a. A simple, centred, explicit method replaces

$$\left.\begin{array}{lll} u_x & \text{by} & \dfrac{1}{2h}\triangle U_n^j \equiv \dfrac{U_{n+1}^j - U_{n-1}^j}{2h} \\[2ex] u_{xx} & \text{by} & \dfrac{1}{h^2}\delta^2 U_n^j \equiv \dfrac{U_{n+1}^j + U_{n-1}^j - 2U_n^j}{h^2} \end{array}\right\} + O(h^2) \tag{4.4}$$

so that

$$U_n^{j+1} = U_n^j + k\Phi\left[nh,\ jk,\ U_n^j,\ \frac{\triangle U_n^j}{2h},\ \frac{\delta^2 U_n^j}{h^2}\right] \tag{4.5}$$

As before, we define the local error $z_n^j = u_n^j - U_n^j$. Now u obeys the same equation as U with a truncation error $R_n^j = O(k,\ h^2)$ added in. Furthermore,

$$\begin{aligned} &\Phi\left[nh,\ jk,\ u_n^j,\ \frac{\triangle u_n^j}{2h},\ \frac{\delta^2 u_n^j}{h^2}\right] - \Phi\left[nh,\ jk,\ U_n^j,\ \frac{\triangle U_n^j}{2h},\ \frac{\delta^2 U_n^j}{h^2}\right] \\ &\qquad = \frac{\partial \Phi}{\partial u} z_n^j + \frac{\partial \Phi}{\partial u_x}\frac{\triangle z_n^j}{2h} + \frac{\partial \Phi}{\partial u_{xx}}\frac{\delta^2 z_n^j}{h^2} + O((z)^2)\ . \end{aligned} \tag{4.6}$$

Subtracting (4.5) from the equation involving u and using the above, gives

$$z_n^{j+1} = r\left[\frac{\partial\Phi}{\partial u_{xx}} - \frac{h}{2}\frac{\partial\Phi}{\partial u_x}\right] z_{n-1}^j + r\left[\frac{\partial\Phi}{\partial u_{xx}} + \frac{h}{2}\frac{\partial\Phi}{\partial u_x}\right] z_{n+1}^j + \left[1 + k\frac{\partial\Phi}{\partial u} - 2r\frac{\partial\Phi}{\partial u_{xx}}\right] z_n^j + kR_n^j \tag{4.7}$$

Now the Maximum Principle argument requires the three coefficients in square brackets to be positive (see the arguments of §3, equation (3.4) etc.). Suppose therefore that

$$A \geqslant \frac{\partial\Phi}{\partial u_{xx}} \geqslant a > 0, \qquad \left|\frac{\partial\Phi}{\partial u_x}\right| \leqslant b\ , \qquad C \geqslant \frac{\partial\Phi}{\partial u} \geqslant c\ , \tag{4.8}$$

where $b > 0$, c and C are constants. Then

$$\frac{\partial\Phi}{\partial u_{xx}} \pm \tfrac{1}{2}h\frac{\partial\Phi}{\partial u_x} \geqslant a - \tfrac{1}{2}hb \qquad \text{and} \quad 1 + k\frac{\partial\Phi}{\partial u} - 2r\frac{\partial\Phi}{\partial u_{xx}} \geqslant 1 + kc - 2rA. \tag{4.9}$$

For the MPA, we therefore require

$$a - \tfrac{1}{2}hb \geqslant 0 \qquad \text{and} \quad 1 + kc - 2rA \geqslant 0. \tag{4.10}$$

If (4.10) holds then we can show in the notation of §3 that

$$||z^j|| \leqslant \frac{e^{Cjk} - 1}{C} D(k + h^2) \qquad \text{for} \quad 1 \leqslant j \leqslant J \tag{4.11}$$

By choosing k and h small enough for fixed $T = Jk$ we can ensure that the errors are as small as we choose. (Note that the exponential growth in the error term is due to the PDE itself possessing exponentially growing solutions. Compare the equation $u_t = cu$.) We conclude that the explicit method will work for the nonlinear equation (4.3) provided (4.10) holds. These conditions are **sufficient** but may be overcautious.

As an example, consider Burgers' equation

$$u_t + uu_x = \nu u_{xx} \qquad \text{so that} \qquad \Phi = \nu u_{xx} - uu_x\ , \tag{4.12}$$

where ν is constant. Then

$$A = a = \nu, \qquad b = \max_{x,t}\left[|u|\right], \qquad c = \min_{x,t}\left[-u_x\right],\ C = \max_{x,t}\left[-u_x\right]\ . \tag{4.13}$$

So the explicit method for this equation is stable if

$$\begin{aligned} \nu - \tfrac{1}{2}h\max\big[|u|\big] \geqslant 0 \qquad &\text{or} \qquad |u| \leqslant \frac{2\nu}{h} \\ \text{and} \quad 1 + k\min\big[-u_x\big] - 2r\nu \geqslant 0 \qquad &\text{or} \qquad -u_x \geqslant \frac{2r\nu - 1}{k}\ . \end{aligned} \tag{4.14}$$

The first of these conditions is a restriction on the size of the spatial steplength h for large values of the advective velocity u. In the absence of diffusion ($\nu = 0$) the centred scheme given by (4.5) is always unstable (see later). The second condition is a generalisation of the '$r < 1/2$' relation with which we are familiar. We note that if $2r\nu > 1$, no value of k will guarantee a stable scheme, and as $k \to 0$ the stability condition is violated.