2. Classification of 2nd-order Quasi-linear PDEs in Two Variables

Why didn't our first finite difference program work very well? There must be something wrong with (a) the mathematical problem itself, (b) our solution algorithm, or (c) our computational implementation. Usually when this happens it is because of a bug in our program. Failing that, our solution method may be at fault. But sometimes, the problem we set out to solve is not well-posed. In this lecture we will introduce some theory and illustrate some ways in which a PDE problem might be insoluble.

Most physical systems are governed by second order PDEs. In this course we discuss FDMs for solving such equations. We want our algorithms to be able to reproduce the physics and so to begin with, we must understand the physical background. We consider the equation for $u(x,\, y)$

$$au_{xx} + bu_{xy} + cu_{yy} = f \; . \tag{2.1}$$

This equation is called **quasi-linear** provided the functions a, b, c and f do not depend on u_{xx}, u_{xy} or u_{yy}. They may, however depend on x, y, u, u_x and u_y, so that (2.1) is not necessarily **linear**.

Suppose we know u, u_x and u_y along some curve Γ in (x, y)-space. From a point P on Γ we move a small vector displacement $(dx,\, dy)$ to a new point Q not on Γ. Under what circumstances can we determine uniquely the values of u, u_x and u_y at Q? We denote the change in these variables by du, $d(u_x)$ and $d(u_y)$. Then by the chain rule for partial derivatives, $du = u_x\, dx + u_y\, dy$ which is known because u_x and u_y are known along Γ. Similarly,

$$\left.\begin{aligned} d(u_x) &= u_{xx}\, dx + u_{xy}\, dy \\ d(u_y) &= u_{xy}\, dx + u_{yy}\, dy \end{aligned}\right\} \; . \tag{2.2}$$

We combine (2.1) and (2.2) in matrix form:

$$\begin{pmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix} \; . \tag{2.3}$$

a, b and c are known locally because u, u_x and u_y are known, and so the 3×3 matrix is known. Equation (2.3) will have a unique solution for u_{xx}, u_{xy} and u_{yy} unless the determinant of that matrix vanishes, that is unless

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = a(dy)^2 - b\, dxdy + c(dx)^2 = 0 \; . \tag{2.4}$$

If (2.4) holds, the equation (2.3) will have either no solution or infinitely many. This is surprising. If we can choose a direction (dx, dy) which satisfies (2.4) we have the possibility that the second derivatives u_{xx} etc. may not be uniquely defined. In other words, the solution may have discontinuities across the line PQ, with u_{xx} taking different values on each side. It is very important to know whether our solution can have this property.

Equation (2.4) is called the **Characteristic equation** of (2.1). It is a quadratic in dy/dx with solution

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \ . \tag{2.5}$$

Equation (2.1) is classified as hyperbolic, parabolic or elliptic according to whether these roots are real. Note the sign of b in the formula.

$$\text{If} \quad \left\{ \begin{array}{lll} b^2 - 4ac > 0 & \text{2 real roots, (2.1) is} & \textbf{hyperbolic} \\ b^2 - 4ac = 0 & \text{1 real root, (2.1) is} & \textbf{parabolic} \\ b^2 - 4ac < 0 & \text{0 real roots, (2.1) is} & \textbf{elliptic}. \end{array} \right\} \ . \tag{2.6}$$

For hyperbolic equations, (2.5) is an ODE for $y(x)$ which can be integrated to define two sets of curves (one for the $+$ sign, one for the $-$), called the **characteristics** of (2.1).

Example: Significance of Characteristics.

Consider the one-dimensional wave equation for $u(x,\,t)$ with constant s

$$u_{xx} = \frac{1}{s^2} u_{tt} \qquad \text{for} \quad t > 0 \qquad \text{with} \quad u(x,0) = F(x) \qquad \text{and} \quad u_t(x,0) = G(x) \ . \tag{2.7}$$

This equation has the general solution $u = f(x - st) + g(x + st)$ for any functions f and g and with the above boundary conditions we have d'Alembert's solution:

$$u(x,\,t) = \tfrac{1}{2}\big[F(x + st) + F(x - st)\big] + \frac{1}{2c}\int_{x-st}^{x+st} G(\xi)d\xi \ . \tag{2.8}$$

Using (2.5) the characteristics of (2.7) are two families of straight lines

$$\frac{dt}{dx} = \pm\frac{1}{s} \qquad \text{or} \quad x \pm st = \text{constant}. \tag{2.9}$$

From the actual solution, we see that the solution at some point P or $(x_0,\,t_0)$ with $t_0 > 0$ depends only on some of the initial data, that for which $x_0 - st_0 \leqslant x \leqslant x_0 + st_0$. Only points from which characteristics going forwards in time can reach the point P can influence the solution at P. The set of such points is called the **domain of dependence** of P. In exactly the same way not all points with $t > t_0$ can be affected by the solution at P. The collection of such points is called the **domain of influence** of P.

This behaviour is easy to understand physically, if one interprets characteristics as **curves along which information travels at a finite speed.** If something happens at P it takes a certain time before news of it reaches another point. For (2.7), the characteristics are parallel lines, but for more general hyperbolic systems they will be curved and may meet. In such cases discontinuities may form. If neighbouring characteristics touch, conflicting information arrives at the same point, leading to the creation of **shock waves** (such as sonic booms, or pressure fronts.) Such discontinuities then propagate along the characteristics.

It is clear from this example that whether or not characteristics exist is vital for the understanding and therefore the numerical modelling of a problem. They are associated with "time-like" behaviour, and have a characteristic speed associated with them, defining the rate at which information travels. In contrast elliptic problems have no "time-like" variable; x and y behave like space coordinates.

Boundary Conditions for Well-Posed Problems.

A problem involving a PDE is said to be 'well-posed' if three conditions hold:
(1) A solution exists;
(2) The solution is unique;
(3) The solution is continuous in the boundary conditions, *i.e.* small changes in the boundary conditions do not lead to large changes in the local solution. If this last condition fails to hold, the problem is non-physical and a disaster for numerical modelling.

Whether or not a problem is well-posed depends critically on whether its boundary conditions are appropriate. Typical boundary conditions are:
(a) boundary value problems (BVP); the PDE holds in some closed region and the solution is constrained all over the boundary;
(b) initial value problems (IVP); One or more constraints are given on some curve (usually $t = 0$) only partially bounding the region in which the PDE holds.

Hyperbolic Equations

From our discussion of characteristics, it is clear that hyperbolic systems should have initial value conditions. Information spreads out from the initial values at a finite rate. An example of an **ill-posed** hyperbolic BVP for $u(x, y)$ with a non-unique solution is

$$u_{xx} = u_{yy} \qquad \text{in} \quad \left\{ \begin{matrix} 0 < x < 1 \\ 0 < y < 1 \end{matrix} \right\} \qquad \text{with} \quad \left\{ \begin{matrix} u(x,0) = u(x,1) = 0 \\ u(0,y) = u(1,y) = 0 \end{matrix} \right\} . \tag{2.10}$$

This problem has the solution $u = A \sin n\pi x \sin n\pi y$ for any constant A and integer n.

Elliptic Equations

These have no characteristics; no lines along which information travels, which suggests that IVPs are inappropriate. A typical elliptic equation is **Laplace's equation**

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0 \qquad \text{in} \quad D, \tag{2.11}$$

where D is some region of (x, y)-space. It can be shown that this equation together with the boundary condition $u = f$ on the boundary ∂D gives a well-posed problem, with a smooth solution. But the problem

$$u_{xx} + u_{yy} = 1 \qquad \text{in} \quad D \qquad \text{with} \quad \widehat{\mathbf{n}} \cdot \nabla u = 0 \quad \text{on} \quad \partial D \tag{2.12}$$

can be shown to have **no solution**. Also, the IVP

$$u_{xx} + u_{yy} = 0 \qquad \text{in} \quad y > 0 \qquad \text{with} \quad u(x,0) = u_y(x,0) = 0 \tag{2.13}$$

is ill-posed, even though the solution $u(x, y) = 0$ is unique! Suppose we perturb the initial conditions so that $u(x, 0) = \varepsilon \sin nx$, and $u_y(x, 0) = 0$, where $0 < \varepsilon \ll 1$ and n is arbitrary but large. No matter how big n is, $|\sin nx| \leqslant 1$, so we are not altering the boundary condition by more than ε. The (unique) solution to this new problem is

$$u = \varepsilon \sin nx \cosh ny \simeq \varepsilon \sin nx \, \tfrac{1}{2} e^{|ny|} \qquad \text{when} \quad |ny| \gg 1 \, . \tag{2.14}$$

So a small distance away from the initial line $y = 0$, the solution is now exponentially large, whereas for the unperturbed problem it was zero. If a tiny (albeit very wiggly) perturbation to the boundary conditions can lead to a vast difference in the solution the problem is physically meaningless and impossible to model numerically. This example, due to Hadamard, shows that IVPs for elliptic equations are discontinuous in the boundary conditions.

Parabolic Equations

A typical example is the **diffusion equation** for $u(x,t)$ with constant diffusivity K:

$$u_t = K u_{xx} \qquad \text{with} \quad u(x,\,0) = f(x)\ . \tag{2.15}$$

As we know $u(x,0)$, we can calculate $u_t(x,0) = Kf''(x)$ from the equation, and so we would expect to be able to step away from $t = 0$. From (2.6), the characteristics are given by the repeated root $dt/dx = 0$ or $t =$ constant. This corresponds to an infinite speed of propagation of information. Is the solution stable? Once more we consider the boundary condition $f(x) = \varepsilon \sin nx$, so that the unique solution of (2.15) is

$$u(x,\,t) = \varepsilon \sin nx\, e^{-n^2 K t}\ . \tag{2.16}$$

When $\varepsilon = 0$ the solution is $u = 0$. When $\varepsilon > 0$ the solution decays away provided $Kt > 0$, but if $Kt < 0$ and n is large, the perturbed solution blows up once more.

Thus, parabolic equations require one initial condition and it is vital that we move "forwards in time." Physically, parabolic equations describe the smoothing out of an initial configuration towards an equilibrium. Many different initial conditions give rise to almost the same final state. This is why running the process backwards in time is an ill-posed problem. You can't un-stir a cup of tea!

Summary

Equation type	Appropriate B.C.	Method of solution
Hyperbolic	2 Initial	Step in either direction from initial line
Parabolic	1 Initial	Step in one direction only from initial line
Elliptic	1 Boundary	Must solve everywhere simultaneously

Examples of various equation types

Hyperbolic	Parabolic	Elliptic
(characteristic speed c)	(diffusivity K)	
Maxwell's equations	Heat equation	Electrostatics (Poisson)
Unsteady 1-D compressible	Unsteady incompressible N-S	Potential flow
Steady 2-D supersonic	Steady boundary layer	Steady Navier-Stokes

Clearly, an understanding of each type of equation is essential. We shall consider Finite Difference schemes appropriate to each in turn, reflecting the physics, where possible.