value problems. Typical physical examples include steady viscous flow, steady temperature distributions, equilibrium stresses in elastic structures, and steady voltage distributions. Despite the apparent diversity of the physics we shall shortly see that the governing equations for equilibrium problems are *elliptic.*†

Eigenvalue problems may be thought of as extensions of equilibrium problems wherein critical values of certain parameters are to be determined in addition to the corresponding steady-state configurations. Mathematically the problem is to find one or more constants (λ), and the corresponding functions (ϕ), such that the differential equation

$$L[\phi] = \lambda M[\phi] \tag{1-3}$$

is satisfied within D and the boundary conditions

$$B_i[\phi] = \lambda E_i[\phi] \tag{1-4}$$

hold on the boundary of D. Typical physical examples include buckling and stability of structures, resonance in electric circuits and acoustics, natural frequency problems in vibrations, and so on. The operators L and M are of elliptic type.

Propagation problems are initial value problems that have an unsteady state or transient nature. One wishes to predict the subsequent behavior of a system given the initial state. This is to be done by solving the differential equation

$$L[\phi] = f \tag{1-5}$$

within the domain D when the initial state is prescribed as

$$I_i[\phi] = h_i \tag{1-6}$$

and subject to prescribed conditions

$$B_i[\phi] = g_i \tag{1-7}$$

on the (open) boundaries. The integration domain D is open. In Fig. 1-2 we illustrate the general propagation problem. In mathematical parlance such problems are known as *initial boundary value problems.*‡ Typical physical examples include the propagation of pressure waves in a fluid, propagation of stresses and displacements in elastic systems, propagation of heat, and the development of self-excited vibrations. The physical diversity obscures the fact that the governing equations for propagation problems are *parabolic or hyperbolic.*

The distinction between equilibrium and propagation problems was well

† The original mathematical formulation of an equilibrium problem will generate an elliptic equation or system. Later mathematical approximations may change the type. A typical example is the boundary layer approximation of the equations of fluid mechanics. Those elliptic equations are approximated by the parabolic equations of the boundary layer. Yet the problem is still one of equilibrium.

‡ Sometimes only the terminology initial value problem is utilized.

Numerical methods for Partial differential Eqns 2nd-Edition Author W. F. Ames, Press NELSON 1977 Academic Press

stated by Richardson [23] when he described the first as *jury* problems and the second as *marching* problems. In equilibrium problems the entire solution is passed on by a jury requiring satisfaction of all the boundary conditions and all the internal requirements. In propagation problems the solution marches out from the initial state guided and modified in transit by the side boundary conditions.

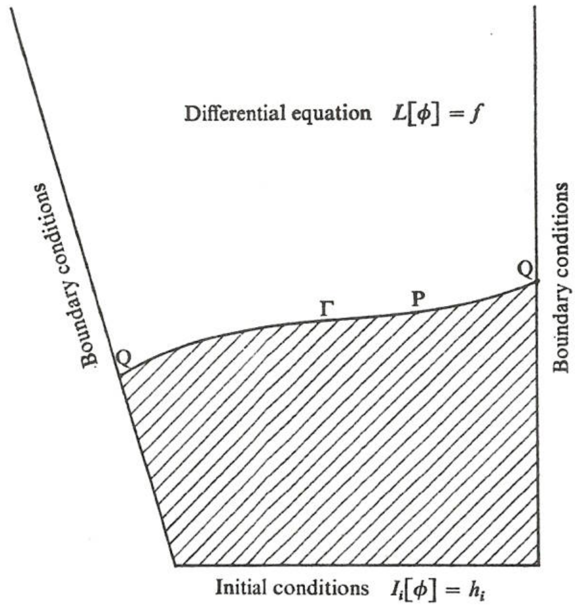

Fig. 1-2 Representation of the general propagation problem

1-2 Classification of equations

The previous physical classification emphasized the distinctive features of basically two classes of problems. These distinctions strongly suggest that the governing equations are quite different in character. From this we infer that the numerical methods for both problems must also have some basic differences. Classification of the equations is best accomplished by developing the concept of *characteristics.*

Let the coefficients $a_1, a_2, \ldots, f_1, f_2$ be functions of x, y, u, and v and consider the simultaneous first-order quasilinear system†

$$\begin{aligned} a_1u_x + b_1u_y + c_1v_x + d_1v_y &= f_1 \\ a_2u_x + b_2u_y + c_2v_x + d_2v_y &= f_2‡ \end{aligned} \tag{1-8}$$

† A quasilinear system of equations is one in which the highest order derivatives occur linearly.

‡ We shall often use the notation u_x to represent $\partial u/\partial x$.

Classify eqns based on highest Derivative term!

This set of equations is sufficiently general to represent many of the problems encountered in engineering where the mathematical model is second order.

Suppose that the solution for u and v is known from the initial state to some curve Γ.† At any boundary point P of this curve, we know the continuously differentiable values of u and v and the directional derivatives of u and v in directions *below* the curve (see Fig. 1-2).

We now seek the answer to the question: 'Is the behavior of the solution just above P uniquely determined by the information below and on the curve?' Stated alternatively: 'Are these data sufficient to determine the directional derivatives at P in directions that lie above the curve Γ?' By way of reducing this question, suppose that θ (an angle with the horizontal) specifies a direction along which σ measures distance. If u_x and u_y are known at P, then the directional derivative

$$u_\sigma|_\theta = u_x \cos\theta + u_y \sin\theta = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma} \tag{1-9}$$

is also known, so we restate the question in the simpler form: 'Under what conditions are the derivatives u_x, u_y, v_x, and v_y uniquely determined at P by values of u and v on Γ?' At P we have four relations, Eqns (1-8) and

$$\begin{aligned} du &= u_\sigma\, d\sigma = u_x\, dx + u_y\, dy \\ dv &= v_\sigma\, d\sigma = v_x\, dx + v_y\, dy \end{aligned} \tag{1-10}$$

whose matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix} \tag{1-11}$$

With u and v known at P the coefficient functions $a_1, a_2, \ldots, f_1, f_2$ are known. With the direction of Γ known, dx and dy are known; and if u and v are known along Γ, du and dv are also known. Thus, the four equations [Eqns (1-11)] for the four partial derivatives have known coefficients. A unique solution for u_x, u_y, v_x, and v_y exists if the determinant of the 4×4 matrix in Eqns (1-11) is *not zero*. If the determinant is not zero, then the directional derivatives have the same value above and below Γ.

The exceptional case, when the determinant is zero, implies that a multiplicity of solutions are possible. Thus, the system of Eqns (1-11) does not determine the partial derivatives uniquely. Consequently, discontinuities in

† We restrict this discussion to a finite domain in which discontinuities do not occur. Later developments consider the degeneration of smooth solutions into discontinuous ones. Additional information is available in Jeffrey and Taniuti [24] and Ames [25].

the partial derivatives may occur as we cross Γ. Upon equating to zero the determinant of the matrix in Eqns (1-11) we find the *characteristic equation*

$$\begin{aligned} (a_1c_2 - a_2c_1)(dy)^2 - (a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)\, dxdy \\ + (b_1d_2 - b_2d_1)(dx)^2 = 0 \end{aligned} \tag{1-12}$$

which is a quadratic equation in dy/dx. If the curve Γ (Fig. 1-2) at P has a slope such that Eqn (1-12) is satisfied, then the derivatives u_x, u_y, v_x, and v_y are not uniquely determined by the values of u and v on Γ. The directions specified by Eqn (1-12) are called *characteristic directions*; they may be real and distinct, real and identical, or not real according to whether the discriminant

$$(a_1d_2 - a_2d_1 + b_1c_2 - b_2c_1)^2 - 4(a_1c_2 - a_2c_1)(b_1d_2 - b_2d_1) \tag{1-13}$$

is positive, zero, or negative. This is also the criterion for classifying Eqns (1-8) as hyperbolic, parabolic, or elliptic. They are *hyperbolic* if Eqn (1-13) is positive—that is, has two *real* characteristic directions; *parabolic* if Eqn (1-13) is zero; and *elliptic* if there are no real characteristic directions.

Next consider the quasilinear second-order equation

$$au_{xx} + bu_{xy} + cu_{yy} = f \tag{1-14}$$

where a, b, c, and f are functions of x, y, u, u_x, and u_y. The classification of Eqn (1-14) can be examined by reduction to a system of first-order equations† or by independent treatment. Taking the latter course we ask the conditions under which a knowledge of u, u_x, and u_y on Γ (see Fig. 1-2) serve to determine u_{xx}, u_{xy},‡ and u_{yy} uniquely so that Eqn (1-14) is satisfied. If these derivatives exist we must have

$$\begin{aligned} d(u_x) &= u_{xx}\, dx + u_{xy}\, dy \\ d(u_y) &= u_{xy}\, dx + u_{yy}\, dy \end{aligned} \tag{1-15}$$

† Transformation of Eqn (1-14) into a system of first-order equations is *not unique*. This 'nonuniqueness' is easily demonstrated. Substitutions (i) $w = u_x$, $v = u_y$, and (ii) $w = u_x$, $v = u_x + u_y$ both reduce Eqn (1-14) to two first-order equations.

For (i) we find the system

$$\begin{aligned} aw_x + bw_y + cv_y &= f \\ w_y - v_x &= 0 \end{aligned}$$

and for (ii) we have

$$\begin{aligned} aw_x + (b - c)w_y + cv_y &= f \\ w_y - v_x - w_x &= 0 \end{aligned}$$

Some forms may be more convenient than others during computation. An example of this, from a paper by Swope and Ames [26], will be discussed in Chapter 4.

‡ Throughout, unless otherwise specified, we shall assume that the continuity condition, under which $u_{xy} = u_{yx}$, is satisfied.

Eqns (1-15), together with Eqn (1-14), has the matrix form

$$\begin{bmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} f \\ d(u_x) \\ d(u_y) \end{bmatrix} \tag{1-16}$$

Thus the solution for u_{xx}, u_{xy}, and u_{yy} exists, and it is unique unless the determinant of the coefficient matrix vanishes, that is

$$a(dy)^2 - b\,dy dx + c(dx)^2 = 0. \tag{1-17}$$

Accordingly, the characteristic equation for the second-order quasilinear equation is (1-17). Equation (1-14) is hyperbolic if $b^2 - 4ac > 0$, parabolic if $b^2 - 4ac = 0$, and elliptic if $b^2 - 4ac < 0$. Since a, b, and c are functions of x, y, u, u_x, and u_y, an equation may change its type from region to region.

In the hyperbolic case there are two real characteristic curves. Since the higher order derivatives are indeterminate along these curves they provide paths for the propagation of discontinuities. Indeed, shock waves and other disturbances do propagate into media along characteristics.

The characteristic directions for the linear wave equation

$$u_{xx} - \alpha^2 u_{yy} = 0 \quad (\alpha \text{ constant}) \tag{1-18}$$

are

$$(dy)^2 - \alpha^2(dx)^2 = 0$$

or

$$y \pm \alpha x = \beta. \tag{1-19}$$

These are obviously straight lines.

A more complicated example is furnished by the nozzle problem. The governing equations of steady two-dimensional irrotational isentropic flow of a gas are (see, for example, Shapiro [27]):

$$\begin{aligned} uu_x + vu_y + \rho^{-1}p_x &= 0 \\ uv_x + vv_y + \rho^{-1}p_y &= 0 \\ (\rho u)_x + (\rho v)_y &= 0 \\ v_x - u_y &= 0 \\ p\rho^{-\gamma} = \text{constant}, \qquad \frac{dp}{d\rho} &= c^2 \end{aligned} \tag{1-20}$$

where u and v are velocity components, p is pressure, ρ is density, c is the velocity of sound, and γ is the ratio of specific heats (for air $\gamma = 1.4$).

By multiplying the first of Eqns (1-20) by ρu, the second by ρv, using $dp = c^2\, d\rho$, and adding the two resulting equations we find that Eqns (1-20) are equivalent to the following pair of first-order equations for u and v,

$$\begin{aligned} (u^2 - c^2)u_x + (uv)u_y + (uv)v_x + (v^2 - c^2)v_y &= 0 \\ -u_y + v_x &= 0 \end{aligned} \tag{1-21}$$

where $5c^2 = 6c^{*2} - (u^2 + v^2)$ and the quantity c^* is a reference sound velocity chosen as the sound velocity when the flow velocity $[(u^2 + v^2)^{1/2}]$ is equal to c. This problem can be put in dimensionless form by setting

$$u' = u/c^*, \qquad v' = v/c^*, \qquad c' = c/c^*, \qquad x' = x/l, \qquad y' = y/l$$

where l is one-half the nozzle width. Inserting these values in Eqns (1-21), and dropping the primes, the dimensionless equations are

$$\begin{aligned} (u^2 - c^2)u_x + (uv)u_y + (uv)v_x + (v^2 - c^2)v_y &= 0 \\ -u_y + v_x &= 0 \end{aligned} \tag{1-22}$$

with $c^2 = 1.2 - 0.2(u^2 + v^2)$.

The characteristic directions are obtained from the particular form of Eqns (1-11)

$$\begin{bmatrix} (u^2 - c^2) & uv & uv & (v^2 - c^2) \\ 0 & -1 & 1 & 0 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ du \\ dv \end{bmatrix}$$

as

$$\left.\frac{dy}{dx}\right|_\alpha = \frac{uv + c[u^2 + v^2 - c^2]^{1/2}}{u^2 - c^2} \tag{1-23a}$$

$$\left.\frac{dy}{dx}\right|_\beta = \frac{uv - c[u^2 + v^2 - c^2]^{1/2}}{u^2 - c^2} \tag{1-23b}$$

where α and β are labels used to distinguish the two directions. When the flow is *subsonic*, $u^2 + v^2 < c^2$, the characteristics are complex, and Eqns (1-22) are therefore elliptic; when the flow is transonic, $u^2 + v^2 = c^2$ and Eqns (1-22) are parabolic; and, when $u^2 + v^2 > c^2$, the flow is supersonic and Eqns (1-22) are hyperbolic.

In Chapter 4 the characteristics will be utilized in developing a numerical method for hyperbolic systems.

PROBLEMS

1-1 In dimensionless form the *threadline* equation from Swope and Ames [26] is

$$y_{tt} + \alpha y_{xt} + \tfrac{1}{4}(\alpha^2 - 4)y_{xx} = 0 \tag{1-24}$$

where $\alpha = 2v/c$. Find the characteristics of this equation and classify it.

1-2 The one-dimensional isentropic flow of a perfect gas is governed by the equations of momentum, continuity, and gas law which are, respectively,

$$u_t + uu_x + \rho^{-1}p_x = 0 \tag{1-25a}$$

$$\rho_t + \rho u_x + u\rho_x = 0 \tag{1-25b}$$

$$p\rho^{-\gamma} = \alpha = \text{constant}, \qquad c^2 = dp/d\rho \tag{1-25c}$$