

1. Given a function $u(x)$ and writing $u_n = u(nh)$, show that

$$\frac{12u_{1/3} - 3u_{2/3} - 9u_0}{2h} = u'(0) + O(h^2) .$$

2. The steady equations of motion of inviscid, isentropic compressible fluid flow are

$$\left. \begin{aligned} \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= 0 \\ \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} &= 0 \end{aligned} \right\} \quad (1)$$

where \mathbf{u} is the velocity vector and the pressure, p , is a function of the density, ρ , only: $p = p(\rho)$. Defining the local sound speed c by $c^2 = \frac{dp}{d\rho}$, show that these equations imply that [skip this bit if you have no knowledge of fluid dynamics]

$$\mathbf{u} \cdot \nabla \left(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) = c^2 \nabla \cdot \mathbf{u} . \quad (2)$$

If the flow is both two-dimensional and irrotational ($\nabla \wedge \mathbf{u} = 0$), then \mathbf{u} can be written in terms of a potential, $\phi(x, y)$, so that $\mathbf{u} = (\phi_x, \phi_y, 0)$. Obtain an equation for ϕ , and determine the conditions for it to be hyperbolic, elliptic or parabolic, commenting on the physical meaning of the results.

3. The equation $u_t = u_{xx}$ may be modelled by a two-step method which is centred in both space and time, namely

$$\frac{U_n^{j+1} - U_n^{j-1}}{2k} = \frac{U_{n+1}^j - 2U_n^j + U_{n-1}^j}{h^2} . \quad (3)$$

What is the order of the truncation error of this scheme?

Use the Fourier method to investigate the stability of (3) for all positive values of $r = k/h^2$.

4. The stability behaviour of the method (3) can be improved by replacing the term $2U_n^j$ by the average $(U_n^{j+1} + U_n^{j-1})$:

$$U_n^{j+1} - U_n^{j-1} = 2r \left(U_{n+1}^j - U_n^{j+1} - U_n^{j-1} + U_{n-1}^j \right) . \quad (4)$$

Calculate the truncation error of this scheme up to and including terms of $O(k^2)$ and $O(h^2)$.

Under what circumstances is this approximation *consistent*? i.e. as $h \rightarrow 0$ and $k \rightarrow 0$, when does the truncation error of the finite difference approximation tend to zero?

5. Hard: Use the Matrix Method to examine the stability of the scheme

$$(1 + \theta) \left[\frac{U_n^{j+1} - U_n^j}{k} \right] - \theta \left[\frac{U_n^j - U_n^{j-1}}{k} \right] = \frac{U_{n+1}^j - 2U_n^j + U_{n-1}^j}{h^2} \quad (5)$$

where $0 \leq \theta \leq 1$. (Note $\theta = -\frac{1}{2}$ corresponds to (3).)

6. Pollutant is introduced at a rate $Q(x, y)$ into a river flowing with velocity $u(x)$ in the y -direction. Equilibrium between downstream advection and cross-channel diffusion is reached when the pollutant concentration $C(x, y)$ satisfies

$$uC_y = DC_{xx} + Q, \quad \text{where } D \text{ is constant.} \quad (6)$$

The river occupies $-d < x < d$, and a parabolic velocity profile $u = u_0(d^2 - x^2)$ is assumed. It is desired to solve this equation subject to boundary conditions on $x = \pm d$ and an initial condition at $y = 0$ by an explicit method. If the steplength used across the river is $1/20$ of its width, what is the maximum steplength along the river which will guarantee stability of the algorithm?

Should the direction of solution be upstream or downstream? Is the method a good one?

7. The three-dimensional diffusion equation

$$u_t = \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \quad (7)$$

is to be solved on a rectangular grid with step-lengths (h_x, h_y, h_z) using the θ -method

$$U_{lmn}^{j+1} - U_{lmn}^j = k \left[\frac{1}{h_x^2} \delta_x^2 + \frac{1}{h_y^2} \delta_y^2 + \frac{1}{h_z^2} \delta_z^2 \right] \left(\theta U_{lmn}^{j+1} + (1 - \theta) U_{lmn}^j \right), \quad (8)$$

in the usual notation. Use the Fourier method to show that this scheme is unconditionally stable for $1 \geq \theta \geq \frac{1}{2}$ while if $0 \leq \theta < \frac{1}{2}$ it is stable provided

$$k \left\{ \frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2} \right\} \leq \frac{1}{2(1 - 2\theta)}. \quad (9)$$