28. The method of Characteristics

Recall lecture 2, which began:

Most physical systems are governed by second order PDEs. In this course we discuss FDMs for solving such equations. We want our algorithms to be able to reproduce the physics and so to begin with, we must understand the physical background. We consider the equation for $u(x,\, y)$

$$au_{xx} + bu_{xy} + cu_{yy} = f \ . \tag{2.1}$$

This equation is called **quasi-linear** provided the functions a, b, c and f do not depend on u_{xx}, u_{xy} or u_{yy}. They may, however depend on x, y, u, u_x and u_y, so that (2.1) is not necessarily **linear**.

Suppose we know u, u_x and u_y along some curve Γ in (x, y)-space. From a point P on Γ we move a small vector displacement $(dx,\, dy)$ to a new point Q not on Γ. Under what circumstances can we determine uniquely the values of u, u_x and u_y at Q? We denote the change in these variables by du, $d(u_x)$ and $d(u_y)$. Then by the chain rule for partial derivatives, $du = u_x dx + u_y dy$ which is known because u_x and u_y are known along Γ. Similarly,

$$\left.\begin{aligned} d(u_x) &= u_{xx}dx + u_{xy}dy \\ d(u_y) &= u_{xy}dx + u_{yy}dy \end{aligned}\right\} \ . \tag{2.2}$$

We combine (2.1) and (2.2) in matrix form:

$$\begin{pmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix} \ . \tag{2.3}$$

a, b and c are known locally because u, u_x and u_y are known, and so the 3×3 matrix is known. Equation (2.3) will have a unique solution for u_{xx}, u_{xy} and u_{yy} unless the determinant of that matrix vanishes, that is unless

$$\begin{vmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = a(dy)^2 - b\, dxdy + c(dx)^2 = 0 \ . \tag{2.4}$$

If (2.4) holds, the equation (2.3) will have either no solution or infinitely many. The condition for solutions to exist, which we will use later in the course, is that

$$a\, \frac{d(u_x)}{dx} + c\, \frac{d(u_y)}{dy} = f \ . \tag{2.5}$$

We noted in lecture 2 that the solution may be discontinuous across characteristics. It is very important to know whether our solution can have this property. Equation (2.4) is called the **Characteristic equation** of (2.1). It is a quadratic in dy/dx with solution

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \ . \tag{2.6}$$

We can use equations (2.6) and (2.5) to find the solution. Equation (2.6) defines two families of characteristics, one with the +sign and one with the −sign. Starting from a given point (x, y) we can step (2.6) using some ODE-solver to find a new point point $(x + dx, y + dy$ which lies on the characteristic. Note that chars. from the same family are approximately parallel - if ever they cross, then a **shock** develops and the solution becomes singular. However the two families have different gradients and so characteristics from different families meet all the time. So of we have a set of points $\{P_n^j\}$ on some curve Γ^j at some "time", j, we can define points P_n^{j+1} to be the intersection of the +char from P_n^j and the −char from P_{n+1}^j as in the diagram, thus defining a new curve Γ^{j+1}.

Now assume u, u_x and u_y are known on Γ^j. This means that u is known after an infinitesimal step (dx, dy) by $du = u_x dx + u_y dy$, but u_x and u_y need to be found. The key is to use the condition (2.6) along the characteristics, relating how much u_x and u_y change from the known values at P_n^j and P_{n+1}^j. As we have two equations in two unknowns we can easily determine u_x and u_y at the new point P_n^{j+1} and hence we can advance the region where the solution is known to a larger region, bounded by Γ^{j+1}. This defines the method of characteristics.

An advantage of the method is that by explicitly following characteristics we are modelling the physics well. If the solution does have discontinuities, we expect to follow them accurately. A disadvantage is that it does not generalise easily to 3D, when the characteristic curves become cones. Also, we can no longer have a regular grid – the equation decides where to place the points P_n^j, and they may bunch together. One could argue, that grid points are being carried to areas where they are needed, but there is the danger of thinning of points in some regions.

This leads us on to a more fundamental question for this course – we have always tried to use a regular grid, which affords us reliable 2nd order accuracy. Should we relax this in some cases? We need points in order to resolve rapid variation. If we have a solution like $u = e^{-10x} + x^2$, we might want to have more points near $x = 0$ than near $x = 1$. To put it another way, if we use a small enough step-length to resolve the fast variation, we are being unnecessarily wasteful of computing power elsewhere. Suppose we introduce a new coordinate, $X = f(x)$. Then we can rewrite the PDE in terms of the new variable X and then use central differences in X, maintaining 2nd order accuracy. We may clutter up the PDE with derivatives of f (which may be large in some regions), but this nevertheless gives us a methodical means of redistributing points to where we think they are needed.