Lecture 1, Calculus

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1 Sequences and numerical series of real numbers

1.1 Sequences of real numbers

Definition 1.1. *A* ***sequence*** *of real numbers (a real sequence) has the form*

$$a_0, a_1, ..., a_n, ...$$

where $a_n \in \mathbb{R}$ for all $n \geq 0$. Formally, a sequence of real numbers can be defined as a function $a : \mathbb{N} \to \mathbb{R}$, i.e. for each natural number $n \in \mathbb{N}$ we associate the real number $a_n \in \mathbb{R}$, called the ***general term*** *of the sequence. One way to denote a sequence is $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\geq 0}$, or simply (a_n).*

In some situations, a sequence must be considered as a real function defined on the set $\{n \in \mathbb{N} : n \geq n_0\}$, where $n_0 \in \mathbb{N}$ is a fixed natural number (often called *rank*), in which case the sequence will be denoted by $(a_n)_{n\geq n_0}$ (take, for example, $a_n = \frac{1}{n(n-2)}$, $n \geq 3$). Sequences most often begin with $n_0 = 0$ (for example, $a_n = n - 1$, $n \geq 0$) or $n_0 = 1$ ($a_n = \frac{\ln n}{n+1}$, $n \geq 1$). Specifically, if n is a positive integer (that is, $n_0 = 1$), then a_n is called the nth term of the sequence.

By abuse of notation, it is often convenient to write "the sequence a_n" instead of "the sequence whose general term is a_n".

Definition 1.2. *A sequence $(a_n)_{n\geq 0}$ of real numbers is called* ***bounded*** *if there exist two* ***finite*** *numbers m, M such that*

$$m \leq a_n \leq M,$$

$\forall n \geq 0$. The sequence $(a_n)_{n\geq 0}$ is called ***left-bounded*** *if there exists $m \in \mathbb{R}$, finite, such that $m \leq a_n, \forall n \geq 0$, respectively,* ***right-bounded*** *if there exists $M \in \mathbb{R}$, finite, with $a_n \leq M, \forall n \geq 0$. A sequence which is not bounded is called* ***unbounded****.*

For example, the sequence $a_n = \frac{1}{n^2}$, $n \geq 1$, is bounded because $0 < a_n \leq 1$, $\forall n \geq 1$. But the sequence $a_n = n^4, n \geq 1$, is unbounded, because it is not right-bounded, even though it is left-bounded by $m = 0$.

Definition 1.3. *A sequence $(a_n)_{n\geq 0}$ is called* ***increasing*** *(or monotonically increasing) if*

$$a_n \leq a_{n+1},$$

$\forall n \geq 0$, *respectively,* ***decreasing*** *(or monotonically decreasing) if*

$$a_n \geq a_{n+1},$$

$\forall n \geq 0$. *The sequence $(a_n)_{n\geq 0}$ is constant if $a_n = a_{n+1}$, $\forall n \geq 0$. If the above inequalities are strictly, that is, $a_n < a_{n+1}$ and $a_n > a_{n+1}$, the corresponding sequence is termed as strictly increasing, respectively, decreasing. A sequence (a_n) is called* ***monotonic*** *if it is either increasing or decreasing.*

For example, the sequence $a_n = \frac{1}{n^2}, n \geq 1$, is (strictly) decreasing because $a_n = \frac{1}{n^2} > \frac{1}{(n+1)^2} = a_{n+1}$, $\forall n \geq 1$, and the sequence $a_n = n^4, n \geq 1$, is (strictly) increasing because $a_n = n^4 < (n+1)^4 = a_{n+1}$, $\forall n \geq 1$.

Definition 1.4. *A sequence $(a_n)_{n\geq 0}$ is called* ***convergent*** *if there exists a real* ***finite*** *number $a \in \mathbb{R}$, such that $\forall \varepsilon > 0$, there is a rank $n_1 \in \mathbb{N}$ with the property that*

$$|a_n - a| < \varepsilon,$$

for all $n \in \mathbb{N}, n \geq n_1$. In this case, the number a is called ***the limit*** *of the sequence (a_n) and we denote usually by*

$$\lim_{n\to\infty} a_n = a,$$

or, simply $a_n \to a$ (specifying or not $n \to \infty$). If the limit of the sequence (a_n) does not exist or is $\pm\infty$ (infinite), the sequence (a_n) is called ***divergent****. Sometimes, if the limit exists and is $\pm\infty$, the sequence is said to be convergent to $\pm\infty$.*

Remark 1.1. *a) From this definition we learn that, if a sequence $(a_n)_{n\geq 0}$ is convergent to a, then all terms of the sequence lie in a neighborhood of a, excepting eventual a finite numbers of them, (from a_1 to a_{n_1-1}), more exactly, for $\forall \varepsilon > 0$ we have $a_n \in (a - \varepsilon, a + \varepsilon)$, for all $n \in \mathbb{N}, n \geq n_1$. The rank $n_1 \in \mathbb{N}$ depends on ε, $n_1 = n_1(\varepsilon)$.*

b) The limit of a sequence, if exists, is unique.

Remark 1.2. *When dealing with convergence of sequences, we do not use in general the definition but one or more properties described below. We list them here without proofs.*

Properties.

P1) Let $(a_n)_{n\geq 0}$ be a **monotonic** sequence (i.e. increasing or decreasing) for all $n \geq n_0$, where $n_0 \in \mathbb{N}$ is a fixed rank. If $(a_n)_{n\geq 0}$ is also **bounded** then $(a_n)_{n\geq 0}$ is convergent.

We notice from this property that, for the convergence of a_n, it suffices a_n to be monotonic from a certain rank n_0, while the first $n_0 - 1$ terms have no influence on the convergence.

P2) Any **convergent** sequence is **bounded**.

P3) The **squeezing theorem** for sequences: if there are three **sequences** such that $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $a_n \to a$, $c_n \to a$, then $b_n \to a$.

P4) Let (a_n) be a **sequence** of strictly positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l.$$

Then, if $l < 1 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$ and if $l > 1 \Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$.

P5) Let (a_n) be a **sequence** of strictly positive real numbers and assume there exists the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l.$$

Then there exists also the limit $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = l.$$

P6) **(Stolz lemma).** Let $(a_n), (b_n)$ be two **sequences** of real numbers. If (b_n) is monotonically increasing with its limit $+\infty$, and if there exists

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l,$$

then, there exists also $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=l$.

P7) If $\lim\limits_{n\to\infty}u_n=+\infty$, then $\lim\limits_{n\to\infty}\left(1+\frac{1}{u_n}\right)^{u_n}=e\approx 2.71$.

P8) (L'Hospital rule)

$$\lim_{n\to\infty}\frac{P(n)}{Q(n)}\underset{\frac{\infty}{\infty}}{\overset{\frac{0}{0}}{=}}\lim_{n\to\infty}\frac{P'(n)}{Q'(n)}.$$

Definition 1.5. *A sequence (a_n) of real numbers is called a* ***Cauchy sequence*** *(or* ***fundamental*** *sequence) if for $\forall\varepsilon>0$, there exists a rank $n_1=n_1(\varepsilon)\in\mathbb{N}$ such that*

$$|a_{n+p}-a_n|<\varepsilon,$$

$\forall n\in\mathbb{N}, n\geq n_1$ and $\forall p\in\mathbb{N}$.

Remark 1.3. *Denoting by $m=n+p$, we get an equivalent definition: the sequence (a_n) is a* ***Cauchy sequence*** *if $\forall\varepsilon>0$, there exists a rank $n_1=n_1(\varepsilon)\in\mathbb{N}$ such that, for all $m,n\in\mathbb{N}$ with $m,n\geq n_1$,*

$$|a_m-a_n|<\varepsilon.$$

Definition 1.6. *We say that $(a_{n_k})_{k\geq 1}$ is a* ***subsequence*** *of the sequence $(a_n)_{n\geq 1}$, if all terms of $(a_{n_k})_{k\geq 1}$ are extracted from the sequence $(a_n)_{n\geq 1}$.*

For example, extracting the even, respectively, the odd numbers from the sequence $a_n=n$, $n\geq 0$, we obtain two different subsequences, $b_n=2n$, $n\geq 1$, respectively, $c_n=2n-1$, $n\geq 1$.

Since any subsequence is in fact a sequence, we may denote it by (b_n), (c_n), (d_n) and so one, not necessarily by (a_{n_k}).

Remark 1.4. *a) If a* ***sequence*** *(a_n) is convergent with the limit a, then all its subsequences are convergent to the same limit a.*

b) If a ***sequence*** *(a_n) contains two different subsequences converging to two different limits, then the sequence is divergent.*

Lemma 1.1. *(Bolzano-Weierstrass). Any bounded* ***sequence*** *contains at least a convergent subsequence.*

Theorem 1.1. ***(Cauchy's general criterion for convergence of sequences).*** *The sequence (a_n) of real numbers is convergent if and only if (a_n) is a Cauchy sequence.*

Exercises

1. Show by definition that $a_n = \frac{n}{n+1}, n \geq 1$, converges to $a = 1$.

Solution. For all $\varepsilon > 0$, $|a_n - 1| < \varepsilon$ is equivalent to $\left|\frac{-1}{n+1}\right| < \varepsilon$, i.e. $\frac{1}{n+1} < \varepsilon$, which implies $n > \frac{1}{\varepsilon} - 1$. Take now $n_1 = n_1(\varepsilon) = \left[\frac{1}{\varepsilon} - 1\right] + 1 \in \mathbb{N}$, where $[x]$ denotes the integer part of the real number x *(e.g.* $[4.2] = 4$). From $[x] + 1 > x$, we get that $n > \frac{1}{\varepsilon} - 1$, for all $n \in \mathbb{N}, n \geq n_1$, which is equivalent to $|a_n - 1| < \varepsilon$, that is, the sequence (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = 1$. For example, if $\varepsilon = 0.1$, we get $n_1 = 10 \in \mathbb{N}$, which means that all terms of the sequence starting with a_{10} lie on the interval $(1-\varepsilon, 1+\varepsilon) = (0.9, 1.1)$. If $\varepsilon = 0.01$, we get $n_1 = 100$, that is, all terms starting with a_{100} lie on $(1-\varepsilon, 1+\varepsilon) = (0.99, 1.01)$. If $\varepsilon = 1$, we get $n_1 = 1$, and the terms starting with a_1 lie in the interval $(1-\varepsilon, 1+\varepsilon) = (0, 2)$. So, the smaller is ε, the smaller is the neighborhood of the limit and the more terms (but a finite number) remain outside the neighborhood.

Calculate the limit of the following sequences.

2. $a_n = \frac{n+\sqrt{n^2+1}}{n}$, $\quad b_n = \frac{n-\sqrt{n^2+1}}{n^2}$, $\quad c_n = \frac{\ln(n+1)}{n}$.

Solution. We have

$$\lim_{n\to\infty} a_n \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\left(n+\sqrt{n^2+1}\right)'}{(n)'} = \lim_{n\to\infty}\left(1 + \frac{n}{n^2+1}\right) = 1.$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\left(n-\sqrt{n^2+1}\right)\left(n+\sqrt{n^2+1}\right)}{n^2\left(n+\sqrt{n^2+1}\right)} = \lim_{n\to\infty} \frac{-1}{n^2\left(n+\sqrt{n^2+1}\right)} = 0.$$

$$\lim_{n\to\infty} c_n \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{(\ln(n+1))'}{(n)'} = \lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0.$$

3. $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$, $y_n = \frac{n^3}{3^n}$, $z_n = \frac{\sqrt[n]{n!}+n}{n}$, and

$$t_n = \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}}{n^2}, n \geq 2.$$

Solution. We have:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\sqrt{n}\,(n+1-n)}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}.$$

For y_n we evaluate

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3} < 1.$$

Now, from P4), we get $\lim\limits_{n\to\infty} a_n = 0$.

Rewriting $z_n = \sqrt[n]{\frac{n!}{n^n}} + 1$ and denoting by $u_n = \frac{n!}{n^n}$, we have

$$\frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

which, in turn, implies $\lim\limits_{n\to\infty} b_n = \frac{1}{e} + 1$. For the last sequence we denote by $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}$ and $b_n = n^2$. Since (b_n) is monotonically increasing to $+\infty$ and

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{1}{n+1}\frac{1}{2n+1} = 0,$$

by P6) $\lim\limits_{n\to\infty} t_n = 0$.

4. Use the Cauchy's general criterion for convergence to show the convergence of the sequences:

a)

$$a_n = \frac{\sin x}{2} + \frac{\sin 2x}{2^2} + \ldots + \frac{\sin nx}{2^n}.$$

Solution. Calculate $|a_{n+p} - a_n|$. It is clear that, for all n and p natural numbers, we have:

$$\begin{aligned}
|a_{n+p} - a_n| &= \left| \frac{\sin(n+1)x}{2^{n+1}} + \frac{\sin(n+2)x}{2^{n+2}} + \ldots + \frac{\sin(n+p)x}{2^{n+p}} \right| \\
&\le \frac{|\sin(n+1)x|}{2^{n+1}} + \frac{|\sin(n+2)x|}{2^{n+2}} + \ldots + \frac{|\sin(n+p)x|}{2^{n+p}} \\
&\le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^{n+p}} = \frac{1}{2^{n+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} < \frac{1}{2^n} < \frac{1}{n}.
\end{aligned}$$

But $\frac{1}{n} < \varepsilon$ for all $\varepsilon > 0$ and $n \ge n_1 = n_1(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}$, which implies that $|a_{n+p} - a_n| < \frac{1}{n} < \varepsilon$, for all $n, p \ge n_1$, which means that (a_n) is a Cauchy sequence, so convergent.

b)

$$b_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}.$$

Solution. For all non-zero $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
|b_{n+p} - b_n| &= \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2} \right| \\
&< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + ... + \frac{1}{(n+p-1)(n+p)} \\
&= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \ldots + \frac{1}{n+p-1} - \frac{1}{n+p} \\
&= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.
\end{aligned}$$

If $n_1 = n_1(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}$ then $|b_{n+p} - b_n| < \varepsilon$ for all $n, p \ge n_1$. It means that (b_n) is a Cauchy sequence, so convergent.

2 Numerical series.

Definition 2.1. *Let $(u_n)_{n\geq 1}$ be a sequence of real numbers. We call a* ***numerical series*** *associated to the sequence (u_n), the infinite sum*

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + ... + u_n + ...$$

The series $\sum_{n=1}^{\infty} u_n$ is often denoted shortly by $\sum_n u_n$, $\sum_{n\geq 1} u_n$, or simply $\sum u_n$. The sequence (s_n) with the general term

$$s_n = u_1 + u_2 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

is called ***the sequence of partial sums*** *associated to the series $\sum u_n$, and u_n is called* ***the general term of the series.***

Definition 2.2. *We say the series $\sum_n u_n$ is* ***convergent*** *if the sequence (s_n) is convergent. In this case, the limit s of the sequence (s_n),*

$$s = \lim_{n\to\infty} s_n,$$

is called ***the sum*** *of the series and one denotes by*

$$\sum_{n=0}^{\infty} u_n = s.$$

If the sequence (s_n) does not have a (unique) limit or the limit is $\pm\infty$, the series $\sum_n u_n$ is called ***divergent.***

Theorem 2.1. ***(Cauchy's general criterion for convergence of series).*** *A series of real numbers $\sum_n u_n$ is convergent if and only if the sequence of partial sums (s_n) is a Cauchy* ***sequence****, that is, for any $\varepsilon > 0$, there exists a rank $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that, for all $n \in \mathbb{N}, n \geq n_1$ and any $p \in \mathbb{N}$ we have $|s_{n+p} - s_n| < \varepsilon$, or, equivalently,*

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon.$$

Remark 2.1. *If the series* $\sum u_n$ *is convergent, then the sequence* (u_n) *is convergent to* 0, *namely* $u_n \to 0$. *If* $u_n \nrightarrow 0$, *then the series* $\sum u_n$ *is divergent.*

Remark 2.2. *If* $\sum_n u_n$ *and* $\sum_n v_n$ *are two convergent series, then the series* $\sum_n (\alpha u_n + \beta v_n)$, $\forall \alpha, \beta \in \mathbb{R}$, *is convergent.*

Definition 2.3. *We say that a series* $\sum_n u_n$ *is* ***absolutely convergent*** *if the series* $\sum_n |u_n|$ *is convergent.*

Proposition 2.1. *Any absolutely convergent series is convergent.*

Remark 2.3. *If a series* $\sum_n u_n$ *is convergent, it does not imply necessarily that it is absolutely convergent. A convergent series which is not absolutely convergent, is called* ***semi-convergent.***

Example 2.1. *We will show later that the series* $\sum_{n\geq 1} (-1)^n \frac{1}{n}$ *is semi-convergent.*

For two series $\sum_n u_n$ and $\sum_n v_n$, their **product (multiplication)** is a new series $\sum_n c_n$, with

$$c_n = u_1 v_n + u_2 v_{n-1} + ... + u_{n-1} v_2 + u_n v_1.$$

Theorem 2.2. ***(Mertens).*** *If a series* $\sum_n u_n$ *is absolutely convergent and the series* $\sum_n v_n$ *is convergent, then their product series* $\sum_n c_n$ *is convergent and we have* $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} u_n \cdot \sum_{n=1}^{\infty} v_n$.

Proposition 2.2. ***(Abel Test).*** *Assume the series of real numbers* $\sum_n u_n$ *has its sequence of partial sums* (s_n) *bounded and let* $(a_n)_{n\geq 1}$ *be a sequence of real numbers monotonically decreasing and convergent to* 0. *Then the series* $\sum_n a_n u_n$ *is convergent.*

Definition 2.4. *A series of real numbers of the form* $\sum_n (-1)^n u_n$, $u_n \geq 0$, *is called* ***an alternating series.***

Proposition 2.3. ***(Dirichlet Test).*** *Let* $\sum_n (-1)^n a_n$, $a_n > 0$, *be an alternating series. If the sequence* (a_n) *is monotonically decreasing and convergent to* 0, *then the series* $\sum_n (-1)^n a_n$ *is convergent.*

Exercises

1. Study the geometric series $\sum_{n=1}^{\infty} q^n$, with $q \neq 0$ a real number.

Solution. a) If $q = 1$, the sum $s_n = \sum_{k=1}^{n} 1 = n \to +\infty$, that is, the series is divergent.
b) If $q = -1$, the sum $s_n = -1 + 1 - 1 + 1 - ... = 0$, if n is even, respectively, $s_n = -1 + 1 - 1 + ... = -1 \neq 0$, if n is odd, which means that the sequence s_n contains two subsequences with two different limits, so that s_n is divergent. This implies the series is divergent.
c) If $q \in (-1, 1)$, the sum

$$s_n = \sum_{k=1}^{n} q^n = q + q^2 + ... + q^n = q\frac{1 - q^n}{1 - q} \to \frac{q}{1 - q},$$

because $q^n \to 0$. So the series is convergent and has the sum

$$s = \sum_{n=1}^{\infty} q^n = \lim_{n\to\infty} s_n = \frac{q}{1 - q}.$$

d) If $q > 1$, the sum

$$s_n = \sum_{k=1}^{n} q^n = q\frac{1 - q^n}{1 - q} \to +\infty,$$

because $q^n \to \infty$, so the geometric series is divergent.
e) Finally, assume $q < -1$. As $q^{2n} \to +\infty$ and $q^{2n-1} \to -\infty$, the sequence s_n is not convergent, that is the geometric series is divergent.

As a **conclusion**, the geometric series $\sum_{n=1}^{\infty} q^n$ is convergent if and only if its ration $q \in (-1, 1)$.

2. Study the convergence of the series a) $\sum_{n=1}^{\infty} \frac{1}{(n+1)n}$, b) $\sum_{n\geq 2} \frac{n-1}{n!}$, c) $\sum_{n\geq 1} \frac{n}{n^4+4}$.

Solution. a)

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{1}{(k+1)\,k} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.
\end{aligned}
$$

As $s_n \to 1$, the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)n} = 1$, so it is convergent.

b) The general term can be put in the form

$$u_n = \frac{n-1}{n!} = \frac{n}{n!} - \frac{1}{n!} = \frac{1}{(n-1)!} - \frac{1}{n!}.$$

Then

$$s_n = 1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!} = 1 - \frac{1}{n!}.$$

As $s_n \to 1$, the series is convergent to 1.

c) Taking into account that

$$n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2 = \left(n^2 + 2\right)^2 - 4n^2 = \left(n^2 + 2 - 2n\right)\left(n^2 + 2 + 2n\right),$$

the general term becomes

$$u_n = \frac{n}{(n^2 - 2n + 2)\,(n^2 + 2n + 2)} = \frac{An + B}{n^2 - 2n + 2} + \frac{Cn + D}{n^2 + 2n + 2}.$$

Identifying the coefficients, we get $u_n = \frac{1}{4}\left(\frac{1}{n^2-2n+2} - \frac{1}{n^2+2n+2}\right)$, or, equivalently

$$u_n = \frac{1}{4}\left(\frac{1}{(n-1)^2 + 1} - \frac{1}{(n+1)^2 + 1}\right).$$

Computing now the sum s_n, we get

$$
\begin{aligned}
4s_n &= 1 - \frac{1}{5} + \frac{1}{2} - \frac{1}{10} + \frac{1}{5} - \frac{1}{17} + \frac{1}{10} - ... + \frac{1}{(n-3)^2 + 1} - \frac{1}{(n-1)^2 + 1} \\
&\quad + \frac{1}{(n-2)^2 + 1} - \frac{1}{n^2 + 1} + \frac{1}{(n-1)^2 + 1} - \frac{1}{(n+1)^2 + 1} \\
&= 1 + \frac{1}{2} - \frac{1}{n^2 + 1} - \frac{1}{n^2 + 2n + 2}.
\end{aligned}
$$

As $s_n \to 3/8$, the series is convergent to $3/8$.

3. Using the Abel test show the series $\sum_{n\geq 1} \frac{\cos(nx)}{\sqrt{n}}, x \in (0, 2\pi)$, is convergent.
Solution. We compute first

$$p_n = \cos x + \cos 2x + ... + \cos(nx).$$

Denote $q_n = \sin x + \sin 2x + ... + \sin(nx)$ and $z = \cos x + i \sin x$. Using the well-known formula $z^n = (\cos x + i \sin x)^n = \cos nx + i \sin nx, n \geq 1$, one gets

$$p_n + iq_n = z + z^2 + ... + z^n = z\frac{1 - z^n}{1 - z}.$$

But

$$\begin{aligned} z\frac{1 - z^n}{1 - z} &= (\cos x + i \sin x) \frac{(1 - \cos nx) - i \sin nx}{(1 - \cos x) - i \sin x} \\ &= -\frac{1}{2} + \frac{1}{2}\frac{\cos nx - \cos(x + nx)}{1 - \cos x} + iK(n, x) \end{aligned}$$

where $K(n, x)$ is an expression depending on n and x which we do not need. Hence,

$$p_n = -\frac{1}{2} + \frac{1}{2}\frac{\cos nx - \cos(x + nx)}{1 - \cos x},$$

which, in turn, implies,

$$\begin{aligned} |p_n| &< \frac{1}{2} + \frac{1}{2}\frac{|\cos nx| + |\cos(x + nx)|}{|1 - \cos x|} \\ &< \frac{1}{2} + \frac{1}{1 - \cos x} < \infty, \end{aligned}$$

since $\cos x \neq 1$ when $x \in (0, 2\pi)$. Hence, (p_n) is bounded. As $a_n = \frac{1}{\sqrt{n}}$ is monotonically decreasing to 0, by Abel's test, the series is convergent.

4. The series $\sum_n (-1)^n \frac{1}{n}$ is convergent by the Dirichlet test because it is alternating and the sequence $a_n = \frac{1}{n}$ is monotonically decreasing and convergent to 0.