Chapter II - Part 1

FINITE DIMENSIONAL VECTOR SPACES

1 Definitions and Properties

Vector Spaces. Linearly Independent Systems of Vectors

Let $(\mathbb{K}, +, \cdot)$ be a commutative field.
Definition 1.1. *The set* $V \neq \emptyset$ *together with the operations:*

$$+ : \mathrm{V} \times \mathrm{V} \to \mathrm{V},$$

called vector addition, and

$$\cdot : \mathbb{K} \times \mathrm{V} \to \mathrm{V},$$

called scalar multiplication is a **vector space** *if the following properties hold:*
(VS1) $(V, +)$ *is an abelian group;*
(VS2) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$, *for any* $\alpha, \beta \in \mathbb{K}$ *and* $x \in V$; *(associativity of scalar multiplication)*
(VS3) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$, *for any* $\alpha \in \mathbb{K}$ *and* $x, y \in V$; *(distributive laws for scalar)*
(VS4) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$, *for any* $\alpha, \beta \in \mathbb{K}$ *and* $x \in V$; *(multiplication over addition)*
(VS5) $1 \cdot x = x$, *for any* $x \in V, 1 \in \mathbb{K}$ *(the identity property for scalar multiplication).*

The elements of the vector space V are called *vectors* and the elements of the field K are called *scalars.*

Example 1.1.
- $V = \{0\}$ is a vector space, called *the trivial vector space.* This means that no smaller vector space is possible.

- Let $\mathbb{K}^n = \{(x_1, x_2, ...x_n),\ x_i \in \mathbb{K}, i = \overline{1,n}\}$. Then $(\mathbb{K}^n, +, \dot{})$ is a vector space over $\mathbb{K}$, where

$$+ : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n,\ \ \cdot : \mathbb{K} \times \mathbb{K}^{\mathrm{n}} \to \mathbb{K}^{\mathrm{n}},$$

$$(a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) \overset{def}{=} (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$$

  and

$$\alpha(a_1, a_2, ..., a_n) \overset{def}{=} (\alpha a_1, \alpha a_2, ..., \alpha a_n).$$

  If $\mathbb{K} = \mathbb{R}$ then we obtain *the real vector space* $\mathbb{R}^n$; if $\mathbb{K} = \mathbb{C}$ then we obtain *the complex vector space* $\mathbb{C}^n$.

- The set $M_{m,n}(\mathbb{C})$ of all matrices $m \times n$ with the usual operations of matrix addition and scalar multiplication is a vector space.

- Let $\mathbb{R}_n[X]$ be the set of polynomials of degree $\leq n$, with real coefficients. We define addition of polynomials by adding corresponding coefficients and the scalar multiplication by the multiplication of each coefficient with the scalar. Then $\mathbb{R}_n[X]$ is a vector space under these operations.

- Let V be the set of all real-functions defined on $\mathbb{R}$. We define addition of functions as:

$$(f + g)(x) = f(x) + g(x)$$

  for every $x \in \mathbb{R}$, and the scalar multiplication as:

$$(\alpha f)(x) = \alpha f(x),$$

  for every $x \in \mathbb{R}$. Then V is a vector space under these operations.

Properties of vector spaces Let V be a vector space. Then, for any vectors u, v from V and any scalars α, β from $\mathbb{K}$, the following properties hold:

i) $\alpha \cdot v = 0 \Rightarrow \alpha = 0 \vee v = 0$;

ii) $\alpha \cdot v = \beta \cdot v \Rightarrow \alpha = \beta$, for any $v \neq 0$;

iii) $\alpha \cdot v = \alpha \cdot u \Rightarrow v = u$, for any $\alpha \neq 0$.

Counterexample: Let V be the set of all real functions f defined on $[0, 100]$ such that $f(10) = 5$. Then V is *not* a vector space because:

$$(f + g)(10) = f(10) + g(10) = 5 + 5 = 10 \neq 5,$$

so $f + g \notin V$. Therefore, V is not closed under addition and cannot be a vector space.

Definition 1.2. *Let V be a vector space and $S = \{v_1, v_2, ..., v_n\}$ be a finite set of vectors from V. We define* **a linear combination of the vectors from** *S to be the sum*

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_n \cdot v_n,$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are scalars in $\mathbb{K}$.

Definition 1.3. *Let $S = \{v_1, v_2, ..., v_n\}$ be a finite non-empty subset of V. S is* **linearly dependent** *iff there exists the scalars $\alpha_1, \alpha_2, ..., \alpha_n$ not all zero such that $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_n \cdot v_n = 0$ (iff the zero vector can be expressed as a nontrivial linar combination of the vectors of S).*

S is **linearly independent** *iff it is not linearly dependent.*

Remark 1.1. 1. The empty set $\emptyset$ is linearly independent.
2. Any subset of V containing the zero vector is linearly dependent.

Definition 1.4. *The system $S \subset V$ is called* **maximal linearly independent** *iff the system S is linearly independent and $S \cup \{v\}$ is linearly dependent, for any $v \in V$.*

Proposition 1.1. *Let $S = \{v_1, v_2, ..., v_n\} \subset V$ be a set of vectors. Then the following assertions hold:*
i) If S is linearly independent then any subset of S is linearly independent.
ii) If there exists a linearly dependent subset of S, then S is linearly dependent.

Proposition 1.2. *The set $S = \{v_1, v_2, ..., v_n\} \subset V$ is linearly dependent iff at least one vector of S can be expressed as a a linear combination of the other vectors of S.*

Definition 1.5. *We say that the set $S = \{v_1, v_2, ..., v_n\} \subset V$* **spans** *the vector space V if:*

$$\forall v \in \mathrm{V} \exists\, \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K} \text{ s.t. } v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_n \cdot v_n,$$

(any vector of V can be written as a linear combination of the vectors of S).

Proposition 1.3. *Let $S = \{v_1, v_2, ..., v_n\} \subset V$ be a set of vectors. Then the following assertions are equivalent:*
i) S is a maximal linear independent set.
ii) S is a linear independent set and spans V.
iii) Any vector of V can be uniquely expressed as a linear combination of the vector of S.

Basis for Vector Spaces

Definition 1.6. *Let $B = \{v_1, v_2, ..., v_n\} \subset V$ be a set of vectors. Then B is a* **basis** *for V if B is an ordered set, linearly independent and it spans V.*

Example 1.2.

1. The set $B = \{1\}$ is a basis for the vector space $\mathbb{K}$.
2. The set $B_s = \{v_1 = (1, 0, ..., 0), v_2 = (0, 1, ..., 0), ..., v_n = (0, 0, ..., 1)\}$ is a basis for the vector space $\mathbb{R}^n$, called *the standard basis of* $\mathbb{R}^n$.
3. The set $B = \{E_{11}, E_{12}, ..., E_{mn}\}$, where:

$$E_{11} = \begin{pmatrix} 1 & 0 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}, ...,$$

$$E_{mn} = \begin{pmatrix} 0 & 0 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 \end{pmatrix},$$

   is a basis for the vector space $M_{m,n}(\mathbb{R})$ called *the standard basis of* $M_{m,n}(\mathbb{R})$.

4. The set

$$B = \left\{p_0 = X^n, p_1 = X^{n-1}, ..., p_{n-1} = X, p_n = 1\right\}$$

   is a basis for the vector space $\mathbb{R}_n[X]$ called *the standard basis of* $\mathbb{R}_n[X]$.

Proposition 1.4. *Let $B = \{v_1, v_2, ..., v_n\} \subset V$ be a basis for V. Then any vector of V can be* **uniquely** *written as a linear combination of the vectors of the bases B :*

$$\forall v \in V \ \exists! \ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R} \text{ s.t. } v = \alpha_1 \cdot v_1 + ... + \alpha_n \cdot v_n.$$

The scalars $\alpha_1, \alpha_2, ..., \alpha_n$ are called the **coordinates of the vector v related to the basis B** :

$$[v]_B = (\alpha_1, \alpha_2, ..., \alpha_n).$$

We say that v **is expressed in $B-$coordinates.**

Proposition 1.5. *Let $S, T \subset V$ such that S spans V and T is linearly independent. Then card $T \leq$ card S.*

Proposition 1.6. *Let B_1, B_2 be basis in the vector space V. Then card $B_1 =$ card B_2.*

It follows that if V has one basis containing n vectors, then every basis in V has the same number of vectors.

Definition 1.7. *If the vector space V has a basis B containing a finite number of elements, then V is called* **finite dimensional.** *In this case,* **the dimension of** V *is the number of elements in any basis for V. In particular, card $B = \dim V$.*

If V has no finite basis, then V is **infinite dimensional.**

Example 1.3.

1. $\dim \mathbb{R}^n = n$.
2. $\dim M_{m,n}(\mathbb{R}) = mn$.
3. $\dim \mathbb{R}_n[X] = n + 1$.
4. If $V = \{0\}$ then $\dim V = 0$.

Remark 1.2. In all that follows, we will study only finite vector spaces. Some results we will present here does not hold for infinite vector spaces.

Remark 1.3. To create a basis for a vector space V of dimension n consists in finding n linearly independent vectors in V (a maximal linearly independent system).

Proposition 1.7. *Let V be a finite vector space.*
1. If $S \subset V$ that spans V then $\dim V \leq$ card S. Moreover, $\dim V =$ card S iff S is a basis for V.
2. If $S \subset V$ is linearly independent then $\dim S \leq$ card V. Moreover, $\dim V =$ card S iff S is a basis for V.
3. If $\dim S >$ card V then S is linearly dependent.