Applied Macroeconomic Modelling

Estimating a Medium-Scale Model

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Objectives

- Understanding the core mechanism of the New Keynesian model;
- Estimating the New Keynesian model;
- Building projections on alternative future actions of macroeconomic policies.

Additional reading list

- Smets, Frank, and Rafael Wouters. "Shocks and frictions in US business cycles: A Bayesian DSGE approach." American economic review 97.3 (2007): 586-606.
- Canova, Fabio. Methods for applied macroeconomic research. Vol. 13. Princeton university press, 2007.
- Uhlig, H. F. H. V. S. "A toolkit for analyzing nonlinear dynamic stochastic models easily." (1995).

- DSGE models are now part of the toolkit of policy makers to build projections analysis based on an estimated version of Smets and Wouters (2007).
- But long way to get to this new paradigm: toy RBC model Lucas Jr (1975), medium-scale RBC model Kydland and Prescott (1982) matched on US moments, medium scale NK model Christiano et al. (2005) matched on VAR IRF, medium scale NK model Smets and Wouters (2007) through full-informations.

Plan

Households

Welfare maximizing household $j \in [0,1]$ with utility curvatures σ_C, $\sigma_H > 0$ and discount factor β:

$$\max_{\{c^H_{jt}, d_{jt}, h_{jt}\}} \sum_{\tau=0}^{\infty} \beta^\tau E_t \left\{ \frac{\varepsilon^C_t \left(c_{jt+\tau} - hC_{t-1+\tau}\right)^{1-\sigma_C}}{1-\sigma_C} - \chi \frac{h^{1+\sigma_H}_{jt+\tau}}{1+\sigma_H} \right\} \tag{1}$$

where ε^C_t an AR(1) preference shock, c^H_{jt} consumption with habits degree $h \in [0,1)$, h_{jt} hours worked and $\chi > 0$ a shift parameter. Subject to:

$$c_{jt} + i_{jt} + d_{jt} = \frac{R_{t-1}}{\pi_t} d_{jt-1} + h_{jt} w_t + k_{jt-1} z_t - T_{jt}, \tag{2}$$

$$i_{it} \varepsilon^I_t \left(1 - S\left(i_{it}/i_{it-1}\right)^2\right) = k_{it} + (1-\delta)\, k_{it-1} \tag{3}$$

where π_t inflation rate, R_{t-1} nominal interest rate of real deposits d_{jt}, w_t real wage, capital stock k_{jt-1} with return z_t and a lump-sum tax T_{jt}. Investment i_{it} subject to convex adjustment costs $S(x) = 0.5\kappa\,(x - \bar{x})^2$ from Christiano et al. (2005) and ε^I_t AR(1) shock.

Households

Imposing symmetry across households $x_{jt} = X_t$, with Lagrangian multipliers λ_t and q_t:

$$C_t : \lambda_t = (C_t - hC_{t-1})^{-\sigma_C} \tag{4}$$

$$D_t : \beta R_t E_t \{\lambda_{t+1}/\pi_{t+1}\} = \lambda_t \tag{5}$$

$$H_t : \chi h H_t^{\sigma_H} = \lambda_t^H w_t \tag{6}$$

$$I_t : q_t \varepsilon_t^I = 1 + q_t \varepsilon_t^I \frac{\kappa}{2} \left(1 + \left(3 \frac{I_t}{I_{t-1}} - 4\right) \frac{I_t}{I_{t-1}}\right) + \beta \left\{ \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \varepsilon_{t+1}^I \kappa \left(1 - \frac{I_t}{I_{t-1}}\right) \left(\frac{I_t}{I_{t-1}}\right)^2 \right\} \tag{7}$$

$$K_t : q_t = \beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \left[q_{t+1} (1 - \delta) + z_{t+1}\right] \right\} \tag{8}$$

Firms

- To introduce monopolistic competition, one needs to introduce imperfect subsitution across varieties/products.
- Final firms pack differentiated products into a consumption good sold to households.
- Intermediate producers get a rent from selling their products to final firms....
- but are subject to price rigidities.

Final Firms

- Production of final firms employs the following CES packing technology:

$$Y_t = \left(\int_0^1 y_{it}^{(\epsilon-1)/\epsilon} \mathrm{d}i \right)^{\epsilon/(\epsilon-1)} \tag{9}$$

  where $\epsilon > 1$ imperfect substitution degree between varieties i (monopolistic equilibrium for $\epsilon \to 1$, competitive equilibrium for $\epsilon \to \infty$, while for any $\epsilon \in (1, \infty)$, goods are imperfect substitutes with selling price > marginal cost)

- Perfectly competitive firms maximize profits, $P_t Y_t - p_{it} y_{it},$, under their technology (9) to obtain the optimal demand constraint for the i-th good:

$$y_{it} = \left(\frac{p_{it}}{P_t} \right)^{-\epsilon} Y_t. \tag{10}$$

  with aggregate price $P_t = (\int_0^1 p_{it}^{1-\epsilon} \mathrm{d}i)^{1/(1-\epsilon)}$.

Intermediate Firms

- Two step problem: first compute marginal cost, apply margin on marginal cost subject to nominal rigidities.
- Let's compute first marginal cost:

$$\max_{\{y_{it},h_{it},k_{it-1},e_{it},\mu_{it}\}} mc_{it}y_{it} - w_t h_{it} - z_t k_{it-1} - \tau_t e_{it} - g\left(\mu_{it}\right) y_{it}$$

  where mc_{it} marginal cost, τ_t carbon tax and $g\left(\mu_{it}\right)$ abatement cost.

- Subject to supply curve and emission curves:

$$y_{it} = \varepsilon_t^A A k_{it-1}^{\alpha} h_{it}^{1-\alpha}$$
$$e_{it} = \psi\left(1-\mu_{it}\right) y_{it}^{1-\phi}$$

  where TFP is AR ε_t^A & fixed A,capital intensity $\alpha \in [0,1]$ while carbon itensity $\psi \geq 0$, and emission elasticity ϕ.

Intermediate Firms

- The first order conditions are given by:

$$H_t : w_t = \varrho_t (1-\alpha) \frac{Y_t}{H_t}$$

$$K_{t-1} : z_t = \varrho_t \alpha \frac{Y_t}{K_{t-1}}$$

$$\mu_t : g' (\mu_t) = \tau_t \psi Y_t^{1-\phi}$$

$$Y_t : mc_t = g(\mu_t) + \tau_t (1-\phi) \psi (1-\mu_t) y_t^{-\phi} + \varrho_t$$

- One can rewrite the marginal cost as follows:

$$mc_t = g(\mu_t) + \tau_t (1-\phi) \frac{E_t}{Y_t} + \frac{1}{\varepsilon_t^A A} \left(\frac{z_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{(1-\alpha)}\right)^{1-\alpha}$$

Intermediate Firms

- In a second step, one can determine the gap between selling price p_{it} and marginal cost mc_{it} subject to nomianl rigidities through the lense of **?** menu costs:

$$\Delta_{it}^{P} = \frac{\xi}{2}\left(\frac{p_{it}}{p_{it-1}} - \bar{\pi}\right)^{2} \tag{11}$$

  where $\xi \geq 0$ is menu cost that affect the cost of adjusting prices and $\bar{\pi}$ is the long run inflation rate.

- The determination of prices reads as:

$$\max_{\{p_{it}\}} E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\lambda_{it+\tau}}{\lambda_{it}} \left[\frac{p_{it+\tau}}{P_{t+\tau}} y_{it+\tau} - \varepsilon_{t+\tau}^{P} mc_{t+\tau} y_{it+\tau} - Y_{t+\tau}\Delta_{it+\tau}^{P}\right] \tag{12}$$

$$\text{s.t. } y_{it} = (p_{it}/P_t)^{-\epsilon} Y_t$$

  where ε_t^{P} is the AR cost push shock.

Intermediate Firms

- The first order condition yields the New Keynesian Phillips Curve:

$$\pi_t\left(\pi_t-\bar{\pi}\right)=\frac{(1-\epsilon)}{\xi}+\frac{\epsilon}{\xi}\varepsilon_t^P mc_t+\beta E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t}\pi_{t+1}\left(\pi_{t+1}-\bar{\pi}\right)\right\} \tag{13}$$

- Linearization of this equation yields the one seen in the previous slides:

$$\hat{\pi}_t=\beta E_t\left\{\hat{\pi}_{t+1}\right\}+\frac{(\epsilon-1)}{\xi}\left(\hat{\varepsilon}_t^P+\widehat{mc}_t\right) \tag{14}$$

Authorities

- Monetary authority implements monetary policy through Taylor rule:

$$R_t = R_{t-1}^{\rho}\left[\bar{R}\left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right]^{1-\rho}\varepsilon_t^R \tag{15}$$

  where the smoothing coefficient $\rho \in [0,1)$, inflation stance ϕ_π, ouput gap stance ϕ_y and AR(1) shock ε_t^R. Steady states $\bar{R}$, $\bar{\pi}$ and $\bar{Y}$.
- The CO2 tax policy is given by:

$$\tau_t = \tau_0 \varepsilon_t^\tau$$

  where τ_0 is the steady state CO2 tax, ε_t^τ is a policy shock.
- Fiscal policy given by its spending side G_t

$$G_t = g^y \bar{Y} \varepsilon_t^G$$

  where G_t exogenous that is fraction of output $g^y\bar{Y}$ with AR shock ε_t^G. Implicit balance sheet: $G_t = T_t + \tau_t E_t$.

General equilibrium

- Equilibrium on the goods market:

$$Y_t = C_t + I_t + G_t + \Delta_t^P Y_t + g\left(\mu_t\right) Y_t \tag{16}$$

where $C_t = \int_0^1 C_{jt} \mathrm{d}j$, $I_t = \int_0^1 I_{jt} \mathrm{d}j$, $Y_t = \int_0^1 Y_{it} \mathrm{d}i$.

- Equilibrium on input market:

$$H_t = \int_0^1 H_{it} \mathrm{d}i = \int_0^1 H_{jt} \mathrm{d}j$$
$$K_t = \int_0^1 K_{it} \mathrm{d}i = \int_0^1 K_{jt} \mathrm{d}j$$

Plan

A general formulation

- Consider a structural model written in a state-space form:

$$\mathbb{E}_t\{f_\Theta\left(y_{t-1}, y_t, y_{t+1}, \eta_t\right)\} = 0$$

  where:
- y_t vector $N_y \times 1$ of endogenous variables;
- η_t vector $N_\eta \times 1$ of gaussian innovations, $\eta_t \sim \mathcal{N}(0, \Sigma)$.
- $f_\Theta()$ set of nonlinear equations.
- Θ vector of structural parameters.
- $\mathbb{E}_t\{\cdot\}$ the expectation scheme.

A general formulation

- How to get numerical simulations?

1. **Sequence space approximation:** impose a lifetime/horizon for fluctuations T + certainty equivalence (as we have seen in last). Convergence not guaranteed, moderately subject to curse of dimension.
2. **State-space global approximation:** construct a grid over state variables + Bellmann equation and iterate (hoping for convergence + accuracy). Computionally expensive, subject to curse of dimension, but accurate.
3. **State-space local approximation:** assume (strong assumption) that shocks are small, use taylor series expansions, and calculate a recursive solution with similar form as a VAR.

- Spoilert alert: we will use 3rd one!

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A general formulation

- Consider a function $f(x)$. A taylor expansion around a fixed point a reads as:

$$f(x) \simeq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

- Two natural questions:
  1. Which order of approximation to consider?
  2. Which value of a?

A first order taylor expansion

- Macroeconomics still dominated by linearized models. This lecture will remain on the linear version of the true nonlinear model.
- Consider a first order taylor expansion:

$$F\mathbb{E}_t\{\hat{z}_{t+1}\} + G\hat{z}_t + H\hat{z}_{t-1} + M\eta_t = 0$$

  where

  - F, G, H and M are matrices stacking first order derivarives of f_Θ
  - $\hat{z}_t = y_t - a$ is typically expressed in distance from steady state a=y. Why? linear model combined with gaussian stochastic shocks $\eta_t \sim \mathcal{N}(0, \Sigma)$ implies that $y_t - y \sim \mathcal{N}(0, \Sigma_y)$. The smallest deviations $E((y_t - a)^2)$ are obtained when 1st order expansion computed around steady state.

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A first order Taylor expansion

- One looks for a recursive solution that would locally:

$$\hat{z}_t = P\hat{z}_{t-1} + Q\eta_t$$

  where P and Q are two unknown matrices.
- We refer to slide_2.pdf for a description of the solution.

Plan

History of modern macroeconometrics

- Back in the 2,000s, dominance of time series models (e.g. VAR) to make forecasts and historical decomposition.
- DSGE models were not yet fully exploited on empirical grounds to compete with a-theoretical models.
- Estimation of DSGE models were based on weak information method: matching moments.
- Few attempts to use full-information techniques: Fair and Taylor (1983) & Ireland (2004), but subject to identification issues.

Principles of likelihood techniques

- Consider a specific Data Generating Process described by our DSGE model:

$$\hat{z}_t = P(\theta)\,\hat{z}_{t-1} + Q(\theta)\,\eta_t \text{ with } \eta_t \sim \mathcal{N}(0,\Sigma)$$
$$Y_t = H(\theta)\,\hat{z}_t + v_t, \text{ with } v_t \sim \mathcal{N}(0,R)$$

  - Where $\theta \subset \Theta$ is the subset of parameters to be estimated, the remaining being calibrated.
  - Subset of observable variables $Y_t \subset \hat{z}_t$, recognized as the set of macroeconomic time series, and map the model's equations intowhere v_t are measurement errors.

Principles of likelihood techniques

- Consider a sample Y_1, Y_2 ,..., Y_T, one can compute the a prediction error:

$$S_t(\theta) = Y_t - E_{t-1}\{Y_t(\theta)\}$$

where the prediction errors variance $\Omega_t = E(S_t S_t')$.

- Unlike VAR, in DSGE there are more endogenous variables than observable ones $\rightarrow$ use Kalman filter to update recursively by picking 'best' prediction in the Kalman filte through optimal mean squared error (MSE) estimator.

- The likelihood function is given by:

$$\log \mathcal{L}(\theta, Y_{1:T}) = -\frac{nT}{2}\log(2\pi) - \frac{1}{2}\sum_{1}^{T}\log(|\Omega_t|) - \frac{1}{2}\sum_{1}^{T} S_t{}'\Omega_t^{-1}S_t \qquad (17)$$

Principles of likelihood techniques

- So in principle, (i) compute prediction errors (i.e. sequence of shocks) $S_{\theta,t}$ that replicates a sample $Y_{1:T}$, (ii) calculate likelihood function, (iii) relative comparison of $\log \mathcal{L}\left(\theta', Y_{1:T}\right)$ against $\log \mathcal{L}\left(\theta'', Y_{1:T}\right)$ allow to determines which θ'/θ'' is the most likely to have generated the sample.

- Optimization can be implemented to determine θ_{MLE} that is the value of θ in the population of parameters that is the most likely to have generated the sample:

$$\min_{\{\theta\}} - \log \mathcal{L}\left(\theta, Y_{1:T}\right)$$

- As long as $\log \mathcal{L}\left(\theta, Y_{1:T}\right)$ is differentiable in θ, one can use gradient based methods to optimize.

Limits of likelihood techniques

- Any model is an approximate of the true data generating process.
- Does it matter? If the data-generating distribution does not belong to the model's set of probability distributions, then the model is misspecified.
- **Some illustrations:** our current medium-scale NK model does not include a financial sector or epidemic dynamics $\rightarrow$ recessions will be captured erroneously by wrong sources of fluctuations.
- Likelihood function will be higher for implausible parameter values to accomodate the wrong representation of the data given by our misspecified model.

Bayesian techniques

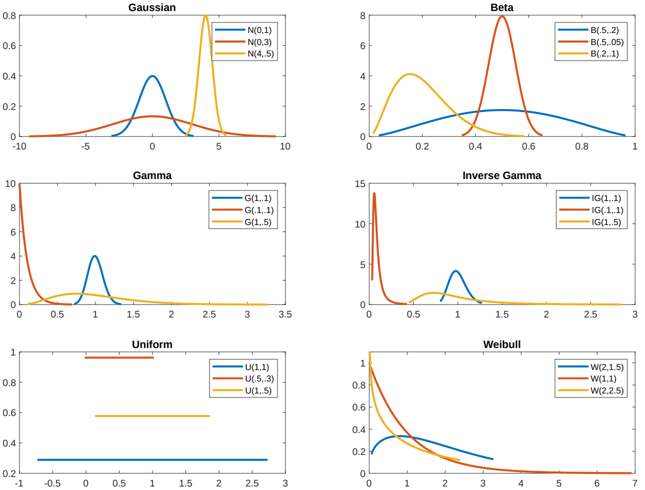
- Solution? Curb likelihood function to reasonable parameter supports.
- How? Impose prior distribution on θ.
- Recall that likelihood function is a product of probability density functions (PDF), priors are simply adding pdf $p(\theta)$ in the product of the likelihood function:
$$p(\theta|Y_{1:T}) \approx \mathcal{L}(\theta|Y_{1:T})p(\theta)$$
where $p(\theta|Y_{1:T})$ is the posterior distribution of parameters, $L(\theta|Y_{1:T})$ denotes the likelihood function [proof].
- Therefore, the Bayesian estimation is carried out by determining $\hat{\theta}_{BE}$ that simply maximizes the log posterior distribution:
$$\min_{\{\theta\}} -\log \mathcal{L}(\theta|Y_{1:T}) + \log p(\theta)$$

Setting Priors

- Dynare includes a set of priors, each prior is summarized by a shape (e.g. gaussian, beta, etc.) and a mean and standard deviation.
- How to adjust my prior? Usually have a look at already published papers, or use common sense.

Shape	Name	Support	Example(s)
normal_pdf	$\mathcal{N}\left(\mu,\sigma\right)$	$(-\infty,+\infty)$	Policy parameters, utility curvatures.
gamma_pdf	$\mathcal{G}\left(\mu,\sigma\right)$	$(0,+\infty)$	Elasticities, utility curvatures, trends.
beta_pdf	$\mathcal{B}\left(\mu,\sigma\right)$	$[0,1]$	Probabilities, AR-MA terms, shares.
inv_gamma_pdf	$\mathcal{IG}_1\left(\mu,\sigma\right)$	$(0,+\infty)$	Shocks standard deviations.
uniform_pdf	$\mathcal{U}\left(\mu,\sigma\right)$	$(-\infty,+\infty)$	Set lower/upper bounds.
weibull_pdf	$\mathcal{W}\left(\mu,\sigma\right)$	$[0,+\infty)$	Measurement error std.
inv_gamma2_pdf	$\mathcal{IG}_2\left(\mu,\sigma\right)$	$(0,+\infty)$	Standard deviations.

Setting Priors



Estimation steps:

1. Maximize $p(\theta|Y_{1:T})$ using numerical optimization techniques to find the mode of the distribution.

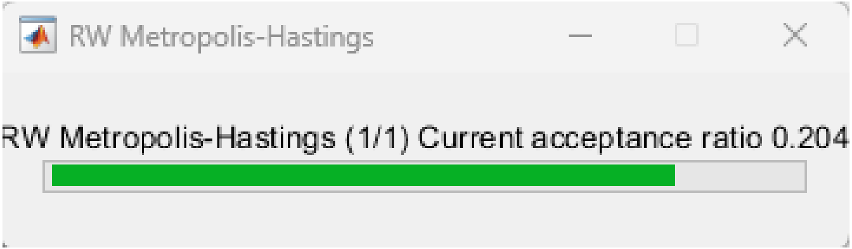
2. Use numerical integration to compute confidence intervals (parametric uncertainty) through Metropolis Hastings algorithm.

Each iteration requires: for a given candidate θ to compute prediction errors $\{S_{\theta,t}\}_1^T$, evaluate sample likelihood + posterior.

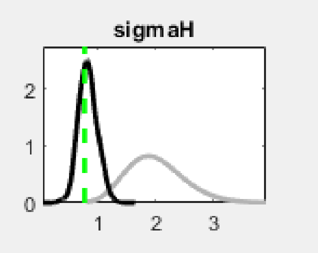
Dynare automatizes those steps, making life easier!

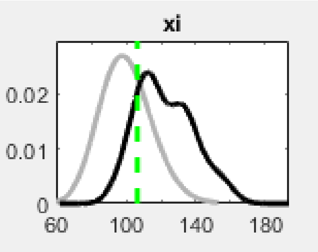
Acceptation

- Metropolis-Hastings looks to explore the parameter space. Sampling: $\theta_i^{cand} \sim \mathcal{N}(\theta_{i-1}^{cand}, c \times \Sigma_\theta)$ with parameter c that scale how large random perturbation around the mode.
- c must be set to match 20-30% of acceptance rate.
- If acceptance rate too low: chain stuck into local minimum. If too low: chain does not explore parameter space sufficiently.



Parameter Identification





- Dark line: posterior distribution $\mathcal{L}(\theta|Y_{1:T})p(\theta)$, while gray line is $p(\theta)$ for alternative parameter values θ. Green line is mean.
- Left figure: data are informative (data dominate); right figure: data are uninformative (prior dominates).
- Identification implies that prior does not dominate posterior. Here, we should calibrate xi and re-run estimation.

Plan

Out-of-Sample Forecasts

- Governments and central banks rely on DSGE model forecasts to design monetary and fiscal policies
- A forecast provides a baseline scenario, and allow to introduce some policies to evaluate how the forecast is affected

A quick recap. Suppose that we have picked $\hat{\theta}$ with highest posterior

$$\hat{z}_t = P\left(\hat{\theta}\right)\hat{z}_{t-1} + Q\left(\hat{\theta}\right)\eta_t$$

with corresponding estimated sequence of shocks $\{\eta_t\}_{t=1}^{T}$ for given $\hat{\theta}$, one wants to make some out of sample forecast over horizon $T + h$

Out-of-Sample Forecasts

- **Multi-step-ahead forecast** (iterating forward):

$$E_T[\hat{z}_{T+h}] = P\left(\hat{\theta}\right)^h \hat{z}_T$$

Since P has eigenvalues inside the unit circle (mean-reverting property), forecasts tend to revert toward steady state.

- **Confidence Intervals for Forecasts:** Since shocks are Gaussian and the system is linear, forecast errors are also Gaussian. Variance of forecast errors increases with horizon h:

$$Var\left(\hat{z}_{T+h}|z_T\right) = \sum_{j=0}^{h} P^j Q \Sigma Q' \left(P^j\right)'$$

$$E_T[\hat{z}_{T+h}] = P\left(\hat{\theta}\right)^h \hat{z}_T \pm 1.96\sqrt{Var\left(\hat{z}_{T+h}|z_T\right)}$$

Policy counterfactual

What is Counterfactual Analysis?

- Simulating "what-if" scenarios to assess how the economy would respond under alternative policy or shock conditions.
- Uses an estimated DSGE model to isolate the effects of specific shocks or policy changes.

Why Perform Counterfactuals?

- **Conditional forecasts** $\rightarrow$ What if oil prices rise by 30%?
- **Counterfactual analysis** $\rightarrow$ What if interest rates had been higher/lower?
- **Shock Decomposition** $\rightarrow$ How much of past output decline was due to demand vs. supply shocks?

Policy analysis 1: conditional forecasts

- Consider our main scenario forecast (detailed before) that is the *unconditional* forecast
- One can think about how a policy could be implemented in the future to evaluate how forecasts are sentitive to future policy changes:
  Unconditional forecast $\eta_{t+h} = 0 \rightarrow E_T[\hat{z}_{T+h}]$
  Conditional forecast $\eta_{t+h} = x \rightarrow E_T[\hat{z}^{(g)}_{T+h}]$

$$E_T[\hat{z}_{T+h}] = P\left(\hat{\theta}\right)^h \hat{z}_T + \sum_{j=0}^{h-1} P\left(\hat{\theta}\right)^j Q\left(\hat{\theta}\right) \eta_{T+h-1-j}$$

- Conditional forecasts allow policymakers to evaluate alternative scenarios (e.g., fiscal expansion).

Policy analysis 2: counterfactual simulations

- Counterfactual simulation re-evaluates past economic dynamics under an alternative policy rule, keeping the estimated historical shocks unchanged.
- This allows us to answer "what if" questions, e.g. What if the central bank had reacted more aggressively to inflation?
- The model has been estimated using real-world data, producing the sequence of structural shocks $\{\hat{\eta}_t\}_{t=1}^{T}$, with $P\left(\hat{\theta}\right)$ and $Q\left(\hat{\theta}\right)$
- Consider a policy change (e.g. $\phi_\pi \rightarrow \phi_\pi''$), compute new policy function P'' and Q''
- Resimulate feeding with $\{\hat{\eta}_t\}_{t=1}^{T}$

$$\hat{z}_t'' = P''\hat{z}_{t-1}'' + Q''\hat{\eta}_t$$

- Compare $\hat{z}_t''$ and $\hat{z}_t$ to see effects of policy changes on insample fluctuations

Policy analysis 3: shock decomposition

- What is a Shock Decomposition?
- Identify the sources of fluctuations in macroeconomic variables by decomposing them into contributions from different structural shocks.
- Since the linearized DSGE model expresses each endogenous variable as a sum of contributions from each shock, we can trace back historical fluctuations to their root causes.
- Rewriting in **shock-decomposed form**:

$$\hat{z}_t = \sum_j^N \hat{z}_t^j$$

where $\hat{z}_t^j$ represents the contribution of shock, or the path of the economy if only the $j-th$ shock is fed in policy function

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