Applied Macroeconomic Modelling

The DICE Model

Gauthier Vermandel

Objectives

- Understanding the mechanics of the climate block;
- The determination of optimal climate policy.

Additional reading list

- Barrage, Lint, and William Nordhaus. "Policies, projections, and the social cost of carbon: results from the DICE-2023 model." Proceedings of the National Academy of Sciences 121.13 (2024).
- Pindyck, Robert S. "The use and misuse of models for climate policy." Review of Environmental Economics and Policy (2017).
- Stern, Nicholas. "Stern Review: The economics of climate change." (2006).

Introduction

- DICE (Dynamic Integrated Climate-Economy model) was developed by Nordhaus (1992) and went through several (minor) updates up to the lastest 2023 snapshot;
- Simple but powerful model to provide fast estimates of the social cost of carbon and transition pathways for the world economy;
- It has become an intuitive benchmark.
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- The framework is old-fashion:
  - Solow growth model with climate externality;
  - Centralized equilibrium based on Ramsey allocations;
  - Purely deterministic setup;
- Physical capital, temperatures and carbon boxes are the main state variables that the planner uses to optimally maximize welfare.
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Outline

Plan

Carbon cycle

- Atmospheric loading of CO_2 is a system of differential equations:

$$M_t = \Phi_M M_{t-1} + \Xi_M E_t, \tag{1}$$

where $M_t = [M_t^{AT}, M_t^{UP}, M_t^{LO}]'$ comprises 3 layers of carbon: atmospheric, upper and lower ocean respectively.

- Matrices are determined by:

$$\Phi_M = \begin{bmatrix} 1-\phi_{12} & \phi_{21} & 0 \\ \phi_{12} & 1-\phi_{21}-\phi_{23} & \phi_{32} \\ 0 & \phi_{23} & 1-\phi_{32} \end{bmatrix} \text{ and } \Xi_M = \begin{bmatrix} \xi_M \\ 0 \\ 0 \end{bmatrix}$$

- Carbon emissions go into the atmosphere, before being captured by oceans through a long lasting process.

Temperatures

- Temperature anomalies $T_t = [T_t^{AT}, T_t^{OC}]$ of atmosphere and ocean respectively:

$$T_t = \Phi_T T_{t-1} + \Xi_T F_t, \tag{2}$$

- Matrices are given by:

$$\Phi_M = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \text{ and } \Xi_T = \begin{bmatrix} \xi_T \\ 0 \end{bmatrix}$$

- Cooling effects of oceans through φ_{12}, ξ_T is the heating sensitivity to radiative forcing F_t.

Radiative forcing

- Greenhouse effect via radiative forcing (W/m^2):

$$F_t = \eta \ \log\left(M_t^{AT}/M_{1750}\right)/\log(2) + F_t^{EX} \tag{3}$$

- Here, η denote the forcing (Wm-2) of equilibrium CO2 doubling ($M_t^{AT} = 2M_{1750}$).
- F_t^{EX} is an exogenous process tracking the contribution of other sources of GHG (i.e. methane) on F_t

Input structure

- World population (billion):

$$L_t = L_{t-1} \left(L_T / L_{t-1} \right)^{\ell_g}, \tag{4}$$

$\ell_g \in [0, 1]$ convergence rate, $L_T \in [0, +\infty)$ long term population.

- Law of motion of capital:

$$K_t = (1 - \delta_K) \, K_{t-1} + I_t, \tag{5}$$

$\delta_K \in [0, 1]$ is the depreciation rate.

Production of goods and emissions

- Production is Cobb-Douglas:

$$Y_t = A_t K_{t-1}^{\gamma} L_t^{1-\gamma} \tag{6}$$

where $\gamma \in [0, 1]$ is capital intensity, exogenous TFP A_t, physical capital K_{t-1}.

- Emissions (GtCO2):

$$E_t = \sigma_t \left(1 - \mu_t\right) Y_t + E_t^{\text{land}} \tag{7}$$

where σ_t decoupling rate, μ_t is abatement share, E_t^{land} is exogenous process of carbon emission from change in land use (i.e. deforestation).

Equilibrium in goods and climate

- Resource constraint:

$$Y_t \left(1 - \theta_{1,t}\mu_t^{\theta_2}\right) \Omega\left(T_t\right) = C_t + I_t \tag{8}$$

  LHS is output loss from abatement and damage, RHS is demand side.

- Damages are given by:

$$\Omega\left(T_t\right) = 1/(1 + aT_t^2) \tag{9}$$

  This expression is subject to deep uncertainty, and should be taken as a normative measure of the social loss for the society of rising temperatures. Tipping points could be introduced [Weitzman, 2012].

Exogenous trends

Total factor productivity:

$$\Delta \log A_t = g_A - \delta_a \log (A_t / A_0)$$

Decoupling rate of carbon emission to GDP:

$$\Delta \log \sigma_t = g_\sigma - \delta_\sigma \log (\sigma_t / \sigma_0)$$

Cost of abating carbon emissions:

$$\theta_{1,t} = (1 - \delta_\theta)\, \theta_{1,t-1} \sigma_t / \sigma_{t-1}$$

Non-CO2 forcing law of motion:

$$F_t^{EX} = \min\left(F_{t-1}^{EX} + \Delta_F, F_{CAP}\right)$$

Land-use law of motion:

$$E_t^{\text{land}} = (1 - \delta_e)\, E_{t-1}^{\text{land}}$$

World population dynamic:

$$L_t = L_{t-1} \left(L_T / L_{t-1}\right)^{\ell_g}$$

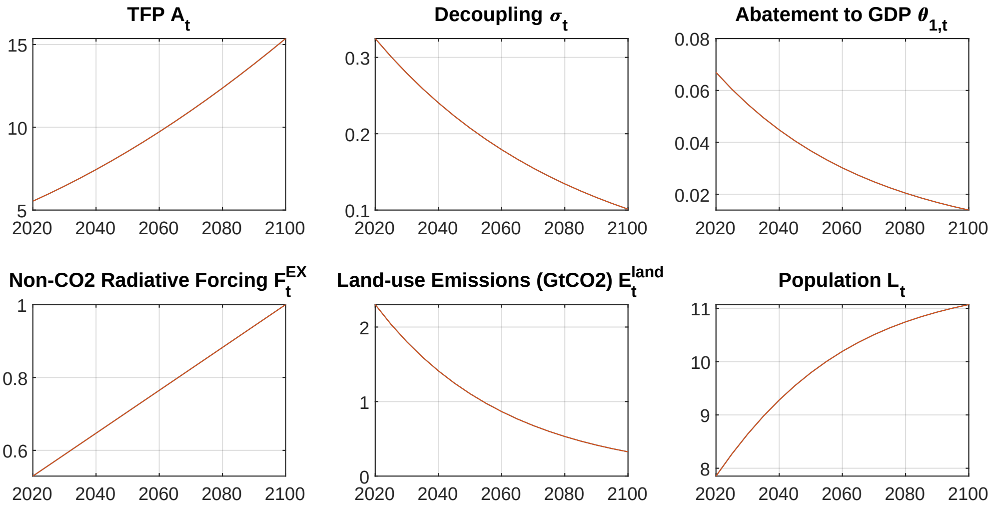

Figure. Exogenous variables

Introducing a saving rate

- How should we determine optimally C and I?

- Could use decentralized decision problem, or use old-fashion centralized equation following Ramsey allocation;

- Introduce optimal saving rate S, consumption and investment becomes:

$$C_t = (1 - S_t) \times Y_t \left(1 - \theta_{1,t}\mu_t^{\theta_2}\right) \Omega\left(T_t\right) \tag{10}$$

$$I_t = S_t \times Y_t \left(1 - \theta_{1,t}\mu_t^{\theta_2}\right) \Omega\left(T_t\right) \tag{11}$$

  $\rightarrow$implicitely assuming that income spent between C_t and I_t.

- Economists typically assume that economic decisions are taken 'optimally' by solving an optimal control problem: e.g. maximizing welfare under budget constraint. Here optimal determination of S.

Introducing a saving rate

- The planner maximizes the social welfare:

$$\max_{\{S_t,\mu_t\}} \sum_{t=0}^{\infty} \beta^t L_t \left(c_t^{1-\alpha} - 1\right) (1-\alpha)^{-1} \tag{12}$$

where $c_t = C_t/L_t$ is the per capita consumption, utility function concave.

- How to solve the problem?

1. **Numerical solver:** use solver to find sequences $\{S_t, \mu_t\}_{t=0}^{T}$ that maximizes objective (12).
2. **Ramsey solution:** maximize social welfare 12 based on N equations and $N + I$ control variables (I corresponding to the number of instruments).
3. **Dynamic programming:** use Bellman equation and value function iterations or projections.

The Ramsey (1927) problem:

$$
\begin{aligned}
&\max_{\{c_t, Y_t, K_t, T_t, M_t, \mu_t, S_t\}} \textstyle\sum_{t=0}^{\infty} \beta^t L_t \left(c_t^{1-\alpha} - 1\right) (1-\alpha)^{-1} \\
&+ \beta^t \lambda_{1,t} \left[(1 - S_t)\, Y_t \left(1 - \theta_{1,t} \mu_t^{\theta_2}\right) \Omega\left(T_t\right) - L_t c_t\right] \\
&+ \beta^t \lambda_{2,t} \left[K_t - (1 - \delta_K)\, K_{t-1} - S_t Y_t \left(1 - \theta_{1,t} \mu_t^{\theta_2}\right) \Omega\left(T_t\right)\right] \\
&+ \beta^t \lambda_{3,t} \left[Y_t - A_t K_{t-1}^{\gamma} L_t^{1-\gamma}\right] \\
&+ \beta^t \lambda_{4,t} \left[T_t - \Phi_T T_{t-1} - \Xi_T \left(\eta \ \log\left(M_t^{AT} / M_{1750}\right) / \log(2) + F_t^{EX}\right)\right] \\
&+ \beta^t \lambda_{5,t} \left[M_t - \Phi_M M_{t-1} - \Xi_M \left(\sigma_t \left(1 - \mu_t\right) Y_t + E_t^{\text{land}}\right)\right]
\end{aligned}
$$

Summary: optimal control problem with 7 control variables, 5 constraints $\rightarrow$ 12 variables/equations.
(note that climate variables include more equations + Lagrangian multipliers)

Model summary:

- 6 exogenous variables $\{A_t, \sigma_t, \theta_{1,t}, F_t^{EX}, E_t^{\text{land}}, L_t\}$ and 6 exogenous processes;
- 5 core equations (from initial model) and 7 first order conditions (from Ramsey problem);
- There are 7 core endogenous variables $\{c_t, Y_t, K_t, T_t, M_t, \mu_t, S_t\}$ and 5 Lagrangian multipliers (from Ramsey) $\{\lambda_{1,t}, ..., \lambda_{5,t}\}$;
- **In sum:** 18 equations and variables.
- **Next step:** get numerical simulations from the model.

Numeric solution

- The presence of state variables in constraints (1-5) implies that current decisions depend on future outcome; example with capital
- Stacking our equations into f:

$$\begin{bmatrix} (1-S_t)\,Y_t\left(1-\theta_{1,t}\mu_t^{\theta_2}\right)\Omega\left(T_t\right)-L_t c_t=0 \\ K_t-(1-\delta_K)\,K_{t-1}-S_tY_t\left(1-\theta_{1,t}\mu_t^{\theta_2}\right)\Omega\left(T_t\right)=0 \\ Y_t-A_tK_{t-1}^{\gamma}L_t^{1-\gamma}=0 \\ \ldots. \end{bmatrix}$$

$$\rightarrow f\left(y_{t+1},y_t,y_{t-1}\right)=0$$

where $f\left(y_{t+1},y_t,y_{t-1}\right)$ is the state-space representation of our model featuring forward and backward looking variables.

- Here, $y_t=[K_t,S_t,\mu_t,\ldots]$ is the vector of endogenous variables.

Numeric solution

A sketch of the numeric problem:

- Finite horizon problem for $t = 0, 2, ...T + 1$;
- Terminal y_{T+1} and initial y_0 conditions are given $\rightarrow$ need to numerically get $y_1, y_2, ...y_T$;
- In absence of stochastic variables $\rightarrow$ deterministic problem $\rightarrow$ perfect foresight setup where any variable in y_{t+1} corresponds to the realized variable in $t + 1$;
- Forward-looking models admit infinity of stable solutions $\rightarrow$ imposing a terminal condition is an instrument to obtain one unique solution (Boucekkine (1995));

Numeric solution

- Over the time horizon $t = 1, 2, ...T$, stacking f() over time:

$$F(Y) = \begin{bmatrix} f(y_2, y_1, y_0) \\ f(y_3, y_2, y_1) \\ \dots \\ f(y_T, y_{T-1}, y_{T-2}) \end{bmatrix}$$

with $Y = [y_t', y_{t+1}', ..., y_T']'$ and $F : \mathbb{R}^{NT} \to \mathbb{R}^{NT}$

- Y and $F(Y)$ are two vectors of size $NT \times 1$.

Numeric solution

- The goal is to numerically solve:

$$Y^* = \arg\min_{\{Y\}} |F(Y)|$$

- How? Newton–Raphson method very efficient as shown by Laffargue (1990), Boucekkine (1995) and Juillard et al. (1996). Basic idea:
  - Set an initial value $Y^{(0)}$.
  - n^{th} Newton iterations:

$$Y^{(n)} = Y^{(n-1)} - J_F\left(Y^{(n-1)}\right)^{-1} F(Y^{(n-1)})$$

    where $J_F\left(Y^{(n-1)}\right)$ is Jacobian matrix of F of dimensions $NT \times NT$.
  - Stop the iterations if $|F(Y^{(n-1)})| < \varepsilon$.

Numeric solution

- Each iteration requires to solve:

$$\begin{bmatrix} J_{1,1} & J_{1,2} & \dots & 0_N & 0_N \\ J_{2,1} & J_{2,2} & \dots & 0_N & 0_N \\ \dots & \dots & \dots & \dots & \dots \\ 0_N & 0_N & \dots & J_{T-1,T-2} & J_{T-1,T} \\ 0_N & 0_N & \dots & J_{T-1,T-1} & J_{T,T} \end{bmatrix} \begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \dots \\ \Delta y_{T-1}^{(n)} \\ \Delta y_T^{(n)} \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_{T-1} \\ f_T \end{bmatrix}$$

- The (inefficient) bruteforce way:

$$\begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \dots \\ \Delta y_{T-1}^{(n)} \\ \Delta y_T^{(n)} \end{bmatrix} = -J_F \left(\begin{bmatrix} \Delta y_1^{(n-1)} \\ \Delta y_2^{(n-1)} \\ \dots \\ \Delta y_{T-1}^{(n-1)} \\ \Delta y_T^{(n-1)} \end{bmatrix} \right)^{-1} \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_{T-1} \\ f_T \end{bmatrix}$$

Numeric solution

- Laffargue (1990) proposes using a triangular expression of $J_F\left(Y^{(n)}\right)$ to allow backward induction.

- Linear algebra yields the following: Proof

$$\begin{bmatrix} I_N & g_{1,2} & \dots & g_{1,T-1} & g_{1,T} \\ 0_N & I_N & \dots & g_{2,T-1} & g_{2,T} \\ \dots & \dots & \dots & \dots & \dots \\ 0_N & 0_N & \dots & I_N & g_{T-1,T} \\ 0_N & 0_N & \dots & 0_N & I_N \end{bmatrix} \begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \dots \\ \Delta y_{T-1}^{(n)} \\ \Delta y_T^{(n)} \end{bmatrix} = - \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_{T-1} \\ d_T \end{bmatrix} \tag{13}$$

- Principle: once matrices $g_{\tau,t}$ and d_t (for $\tau, t \in [1, T]$) are obtained, easy to get $\Delta y_t^{(n)}$ recursively by starting by last row of problem (13).

- To simulate the model, we need to calibrate initial state variables (subset of y_0) to match values observed in 2015.
- Terminal conditions computed asymptotically $t \to +\infty$ by determining y_{T+1} that satisfies:
$$f\left(y_{T+1}, y_{T+1}, y_{T+1}\right) = 0$$
- In finite horizon problem, T must be large enough to verify condition:
$$\left| f\left(y_{T+1}, y_{T}, y_{T-1}\right) - f\left(y_{T+1}, y_{T+1}, y_{T+1}\right) \right| < \varepsilon$$

- We use the (free) Dynare package embedded into MATLAB/Octave/Julia; https://www.dynare.org/
- Software developed at CEPREMAP (Centre d'Études Prospectives d'Économie Mathématique Appliquées à la Planification);
- CEPREMAP founded in 1964 to build models for government planification purpose.
- This software allows to simulate & estimate discrete time rational expectations models (perfect foresight, perturbation methods).

- $\Delta t = 5$ years, $t = \{t_0, t_1, ..., T\}$, with $t_0 = 2015$ and $T = 2600$;
- Consider two policies:

1. The mitigation policy:

$$\max_{\{c_t, Y_t, K_t, T_t, M_t, \mu_t, S_t\}} \sum_{t=0}^{\infty} \beta^t L_t \left(c_t^{1-\alpha} - 1 \right) (1-\alpha)^{-1}$$
$$s.t. \ (1) - (5)$$

2. The *laisser-faire* / Business As Usual (BAU) policy:

$$\max_{\{c_t, Y_t, K_t, T_t, M_t, S_t\}} \sum_{t=0}^{\infty} \beta^t L_t \left(c_t^{1-\alpha} - 1 \right) (1-\alpha)^{-1}$$
$$s.t. \ (1) - (5)$$
$$s.t. \ \mu_t = 0.03$$

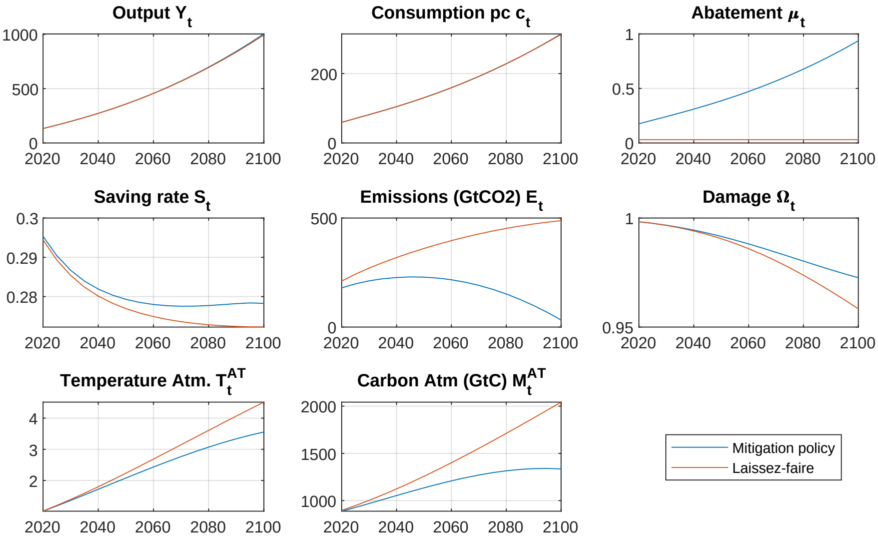

Figure. Simulations under two policy scenarios

Some striking but controversial results from DICE:

- "Optimal" to warm the planet up to $3.5°C$;
- Mitigation policy saves about 2% of climate damage;
- Net zero emission optimal by 2100, 50 years later than in Paris-Agreement;
- Main critic from DICE's policy output: those results have scientifically grounded climate inaction in policy circles...

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Social cost of carbon

- Why is it optimal to warm so much the planet according to DICE?
- An optimal carbon tax = how much the society is willing to pay to reduce emissions ($ per ton of carbon).
- This is usually referred to as the Social Cost of Carbon (SCC) in literature.
- SCC reflects the society's gains and losses from implementing the carbon tax.
- What is the model implied SCC?

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Social cost of carbon

- Get back to FOCs of the Ramsey-planner:

$$\begin{aligned} M_t^{AT} &: \lambda_{5,t}^{AT} = \beta \Phi_M \lambda_{5,t+1}^{AT} + \lambda_{4,t}^{AT} F'\left(M_t\right) \\ T_t^{AT} &: \lambda_{4,t}^{AT} = \beta \Phi_T \lambda_{4,t+1} + \lambda_{2,t}^{AT} S_t Y_t \left(1 - \theta_{1,t} \mu_t^{\theta_2}\right) \Omega'\left(T_t\right) \end{aligned}$$

$\lambda_{5,t}^{AT}$ ($\lambda_{4,t}^{AT}$) is the marginal loss from carbon (temperature) increase.

- SCC expresses the social loss into numeraire equivalents (here consumption):

$$SSC_t = -1000 \times \lambda_{5,t}^{AT} / \lambda_{1,t} \simeq -1000 \times \left(\partial W_t / \partial M_t\right) / \left(\partial W_t / \partial C_t\right)$$

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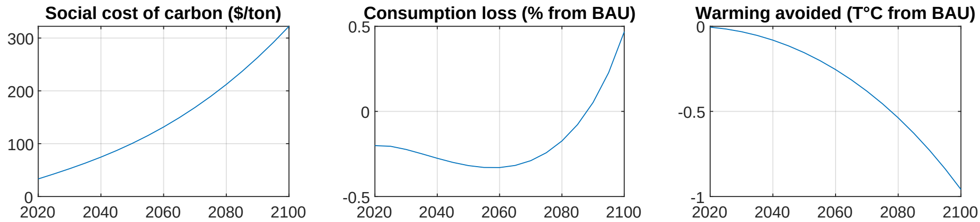
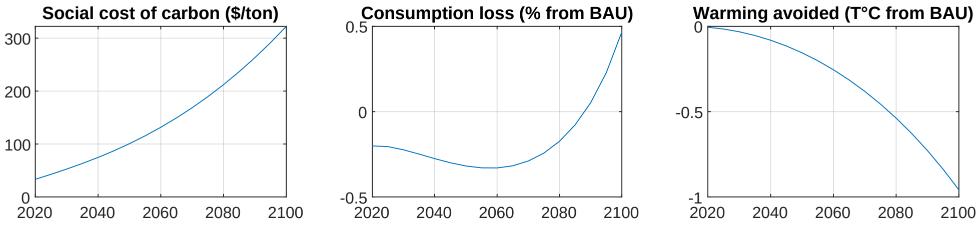


Figure. Social cost of carbon

- SCC reflect the planner's relative gains in terms of welfare from controlling M_t against its consumption losses C_t from such action;
- Optimal to cut consumption now by 0.5% in order to avoid $1°C$ of warming in 2100;
- As well off between scenarios by 2090 on current consumption grounds (but not in welfare terms).



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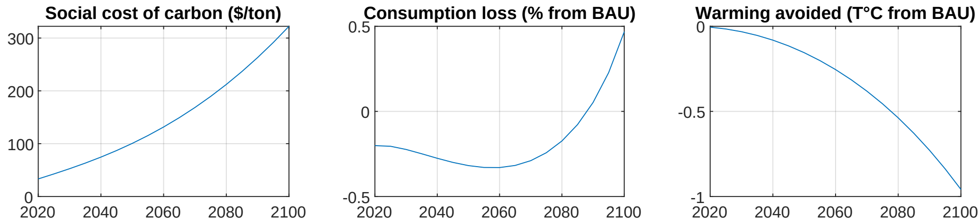

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Plan

How relevant are IAM models?

- **IAM-believiers:** they are relevant at least on normative grounds to provide discussion basis on mitigation scenarios.

- **IAM-deniers:** they are useless to guide policy scenarios → the Pindyck (2013, 2017) critique enumerates a list of flaws:

  1. "certain inputs [parameters] are arbitrary, but have huge effects on the SCC estimates the models produce";
  2. "the models' descriptions of the impact of climate change are completely ad hoc, with no theoretical or empirical foundation";
  3. "the models can tell us nothing about the most important driver of the SCC, the possibility of a catastrophic climate outcome"

- →"IAM-based analyses of climate policy create a perception of knowledge and precision, but that perception is illusory and misleading."

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The role of inputs: the discount factor

- In 2006, the main view on climate change among economists was the Nordhaus (1992)'s view: optimal to warm up to $3.5°C$ the planet.
- The Stern (2006)'s view shook this general agreement on many aspects:
  - the core argument: the price of inaction would be extraordinary and the cost of action modest ;
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$$\mathcal{W}_t = L_t U\left(c_t\right) + \beta \mathcal{W}_{t+1}$$

- β is a deep parameter that depends on several factors:

  1. psychological aspects (time-preference $\rightarrow \beta$ low),
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- Introducing a one-period financial claim:

$$\max_{\{c,b\}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_t \; [b_t (1 + r_{t-1}) - c_t - b_t]$$

$$\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} (1 + r_t) = 1 \rightarrow \ln \beta = \gamma \ln g + \ln r$$

- By observing consumption growth and real interest rate, we can pin down β;
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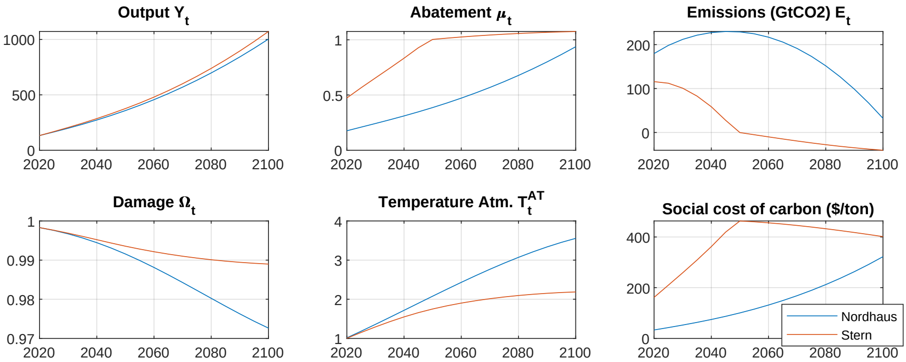

Figure. The role of the discount factor

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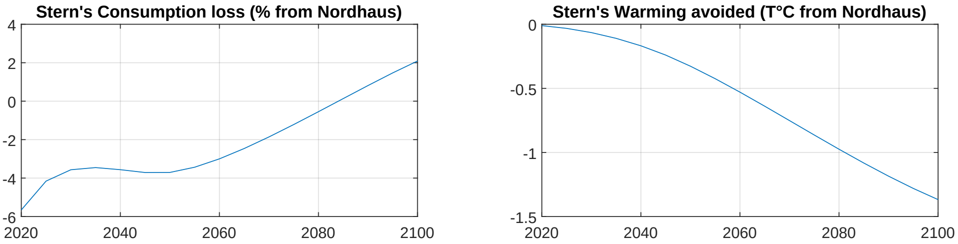

Figure. The role of the discount factor : Stern vs Nordhaus

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- The Stern (2006)'s report provides a scientific motivation for the Paris-Agreement.
- How? Consider FOC on T:

$$\begin{aligned}\lambda_{4,t}^{AT} &= \beta\Phi_T\lambda_{4,t+1} + \lambda_{2,t}^{AT} S_t Y_t \left(1-\theta_{1,t}\mu_t^{\theta_2}\right)\Omega'\left(T_t\right)\\ &= \sum_{\tau=t}^{\infty}\left(\beta\Phi_T\right)^{\tau-t} S_\tau Y_\tau \left(1-\theta_{1,\tau}\mu_\tau^{\theta_2}\right)\Omega'\left(T_\tau\right)\end{aligned}$$

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Tipping points

- Tipping point are critical threshold beyond which a system reorganizes abruptly & irreversibly.

- Natural system tipping points well identified:
  - Antarctic & Artic ice sheet disintegration: complete disintegration (at +10°C) would raise the global sea levels by 53.3 metres;
  - Amazon Rainforest destruction: a warming planet →rainforest may transform into a dry savanna landscape;
  - Sibera's Permafrost thaw: unfreezing permafrost would release methane $\nearrow$ T°C.
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- The damage function in Nordhaus (2017) augmented by Weitzman (2012) with higher order polynomial:

$$\Omega\left(T_t\right) = 1/(1 + aT_t^2 + bT_t^c)$$

where b and c are set to match potential losses as T grows: $b = 5.0703e - 06$, $c = 6.754$.

	$\Omega\left(1\right)$	$\Omega\left(3\right)$	$\Omega\left(5\right)$
$b = 0$ - Nordhaus (2017)	0.9976	0.9792	0.9443
$b > 0$ - Weitzman (2012)	0.9976	0.9712	0.7544

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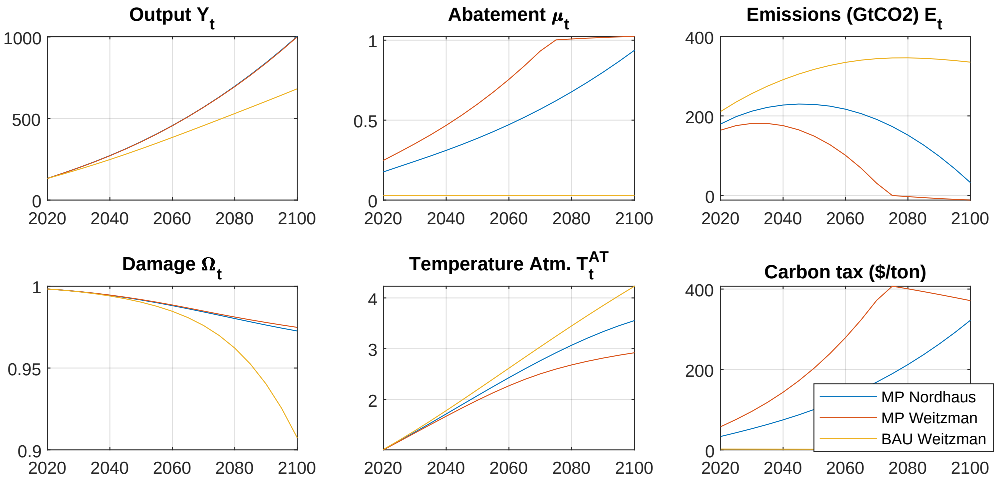

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Thank you!

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$$
\begin{aligned}
&\max_{\{c_t,Y_t,K_t\}} \textstyle\sum_{t=0}^{\infty} \beta^t L_t \left(c_t^{1-\alpha} - 1\right)(1-\alpha)^{-1} \\
&+ \beta^t \lambda_{1,t} \left[(1-S_t)\, A_t K_{t-1}^{\gamma} L_t^{-\gamma} - c_t\right] \\
&+ \beta^t \lambda_{2,t} \left[K_t - (1-\delta_K)\, K_{t-1} - S_t A_t K_{t-1}^{\gamma} L_t^{1-\gamma}\right]
\end{aligned}
$$

FOC on K:

$$
\begin{aligned}
&\gamma\beta\lambda_{1,t+1}\,(1-S_{t+1})\, A_{t+1} K_t^{\gamma-1} L_{t+1}^{-\gamma} \\
&\qquad - \beta\lambda_{2,t+1}\left[(1-\delta_K) + \gamma S_{t+1} A_{t+1} K_t^{\gamma} L_{t+1}^{1-\gamma}\right] = 0
\end{aligned}
$$

Optimal decision on K_t requires to know future $\lambda_{1,t+1}, \lambda_{2,t+1}, S_{t+1}, A_{t+1}, L_{t+1}\}$.

Go to main slides

To illustrate, consider $t = 1, 2, 3$, y_0 & y_4 given, $y_{1:3}$ unknown. We are at n-step update, $\hat{y}_t = y_t^{(n-1)}$:

$$F\left(\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix}\right) = \begin{bmatrix} f\left(\hat{y}_2, \hat{y}_1, y_0\right) \\ f\left(\hat{y}_3, \hat{y}_2, \hat{y}_1\right) \\ f\left(y_4, \hat{y}_3, \hat{y}_2\right) \end{bmatrix},$$

$$J_F\left(Y^{(n-1)}\right) = \begin{bmatrix} \frac{\partial f(\hat{y}_2,\hat{y}_1,y_0)}{\partial \hat{y}_1} & \frac{\partial f(\hat{y}_2,\hat{y}_1,y_0)}{\partial \hat{y}_2} & \frac{\partial f(\hat{y}_2,\hat{y}_1,y_0)}{\partial \hat{y}_2} \\ \frac{\partial f(\hat{y}_3,\hat{y}_2,\hat{y}_1)}{\partial \hat{y}_1} & \frac{\partial f(\hat{y}_3,\hat{y}_2,\hat{y}_1)}{\partial \hat{y}_2} & \frac{\partial f(\hat{y}_3,\hat{y}_2,\hat{y}_1)}{\partial \hat{y}_3} \\ \frac{\partial f(y_4,\hat{y}_3,\hat{y}_2)}{\partial \hat{y}_1} & \frac{\partial f(y_4,\hat{y}_3,\hat{y}_2)}{\partial \hat{y}_2} & \frac{\partial f(y_4,\hat{y}_3,\hat{y}_2)}{\partial \hat{y}_3} \end{bmatrix}$$

Percentage of zeros in $J_F\left(Y^{(n-1)}\right)$ grows in T. Note that $\frac{\partial f(\hat{y}_{t+1},\hat{y}_t,y_{t-1})}{\partial \hat{y}_\tau}$ with $\tau \in t$ is a $N \times N$ matrix.

- Recall, each Newton iteration requires to solve:

$$J_F\left(Y^{(n-1)}\right)^{-1}\Delta Y^{(n)} = -F(Y^{(n-1)})$$

$$\begin{bmatrix} J_{1,1} & J_{1,2} & 0_N \\ J_{2,1} & J_{2,2} & J_{2,3} \\ 0_N & J_{3,2} & J_{3,3} \end{bmatrix} \begin{bmatrix} \Delta y_1^{(n)} \\ \Delta y_2^{(n)} \\ \Delta y_3^{(n)} \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

  where $J_{t,\tau}$ $\frac{\partial f(\hat{y}_{t+1},\hat{y}_t,y_{t-1})}{\partial \hat{y}_\tau}$ and $f_t = f\left(\hat{y}_{t+1},\hat{y}_t,\hat{y}_{t-1}\right)$.
- Linear problem: triangular expression of J_F allows backward induction.

- 1. Solve first row to get $\Delta y_1^{(n)}$ as linear function of $\Delta y_2^{(n)}$.

$$
\begin{array}{cccccc}
I_n \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\
J_{2,1} \Delta y_1^{(n)} & +J_{2,2} \Delta y_2^{(n)} & +J_{2,3} \Delta y_3^{(n)} & = & -f_2 \\
0_N & +J_{3,2} \Delta y_2^{(n)} & J_{3,3} \Delta y_3^{(n)} & = & -f_3
\end{array}
$$

with $d_1 = J_{1,1}^{-1} f_1$ and $g_1 = J_{1,1}^{-1} J_{1,2}$.

- 2. Use first row $I_N \Delta y_1^{(n)} = -d_1 - g_1 \Delta y_2^{(n)}$ and replace to replace $\Delta y_1^{(n)}$:

$$\begin{array}{cccc} I_N \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\ 0_N & +I_N \Delta y_2^{(n)} & +g_2 \Delta y_3^{(n)} & = & -d_2 \\ 0_N & +J_{3,2} \Delta y_2^{(n)} & J_{3,3} \Delta y_3^{(n)} & = & -f_3 \end{array}$$

where $g_2 = (J_{2,2} - J_{2,1} g_1)^{-1} J_{2,3}$ and $d_2 = (J_{2,2} - J_{2,1} g_1)^{-1} (f_2 - J_{2,1} d_1)$.

- 3. Use second row $I_N \Delta y_2^{(n)} = -d_2 - g_2 \Delta y_3^{(n)}$ and replace to replace $\Delta y_2^{(n)}$:

$$\begin{array}{cccccc} I_N \Delta y_1^{(n)} & +g_1 \Delta y_2^{(n)} & +0_N & = & -d_1 \\ 0_N & +I_N \Delta y_2^{(n)} & +g_2 \Delta y_3^{(n)} & = & -d_2 \\ 0_N & +0_N & +I_N \Delta y_3^{(n)} & = & -d_3 \end{array}$$

  where $d_3 = (J_{3,3} - J_{3,2} g_2)^{-1} (f_3 - J_{3,2} d_2)$.
- Going back to stacked matrix:

$$\begin{bmatrix} I_N & g_1 & 0_0 \\ 0_N & I_N & g_2 \\ 0_N & 0_N & I_N \end{bmatrix} \Delta Y^{(n)} = - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

  $\rightarrow$ Backward induction by solving last row of $\Delta Y^{(n)}$ recursively.

- Generalization to a T horizon yields:

$$
\begin{aligned}
d_1 &= J_{1,1}^{-1} f_1 \\
d_2 &= \left(J_{2,2} - J_{2,1} g_1\right)^{-1} \left(f_2 - J_{2,1} d_1\right) \\
g_t &= \left(J_{t,t} - J_{t,t-1} g_{t-1}\right)^{-1} J_{t,t+1} \quad \text{for } t \in [2, T-1] \\
d_t &= \left(J_{t,t} - J_{t,t-1} g_{t-1}\right)^{-1} \left(f_t - J_{t,t-1} d_{t-1}\right) \quad \text{for } t \in [2, T-1] \\
d_T &= \left(J_{T,T} - J_{T,T-1} g_{T-1}\right)^{-1} \left(f_{T-1} - J_{T,T-1} d_{T-1}\right).
\end{aligned}
$$

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