Pixelwise Gaussian Entropy

Deriving our approximation of the entropy for the observations resulting from acquiring line ℓ

We want to compute the entropy of $p(y_t \mid A^\ell, y_{<t})$

We can choose to model $p(x_t \mid y_{<t})$ as a pixelwise independent Gaussian, so that $p(x_t \mid y_{<t}) = \mathcal{N}(x_t; \bar{x}_t, S_{x_t})$ where $\bar{x}_t$ is the sample mean of the belief distribution (samples $x_t \mid y_{<t}$), and S_{x_t} is the sample pixelwise variance, i.e. a diagonal covariance with entries $\sigma^2_{x_t,i}$ for pixels i.

Given that the measurement model $y_t = U(A^\ell)x_t$ is linear (where $U(A^\ell)$ makes a matrix with 1s on the diagonal at indices contained in A^ℓ), we get that $p(y_t \mid A^\ell, y_{<t})$ is a simple transformation of $p(x_t \mid y_{<t})$:

$$p(y_t \mid A^\ell, y_{<t}) = \mathcal{N}(y_t; U(A^\ell)\bar{x}_t, U(A^\ell)S_{x_t}U(A^\ell)^\top)$$

The entropy of this distribution is then:

$$H(y_t \mid A^\ell, y_{<t}) = \frac{1}{2}\log((2\pi e)^M |U(A^\ell)S_{x_t}U(A^\ell)^\top|)$$

The matrix multiplications of S_{x_t} have the effect of selecting only the variances of the pixels revealed by A^ℓ. Given that the determinant of a diagonal matrix is just the product of its diagonal, the entropy simplifies to:

$$H(y_t \mid A^\ell, y_{<t}) = \frac{1}{2}\log((2\pi e)^M \prod_{i \in A^\ell} \sigma^2_{x_t,i})$$

Bringing the product outside of the log to become a sum, we get finally:

$$H(y_t \mid A^\ell, y_{<t}) = \sum_{i \in A^\ell} \frac{1}{2}\log((2\pi e)^{|A^\ell|}\sigma^2_{x_t,i})$$

In other words, the entropy of the measurement y_t observed by acquiring line ℓ is equal to the sum of pixelwise entropies of the pixels revealed by ℓ.

We then modify this further to make it computable, since for large $|A^\ell|$ the term $(2\pi e)^{|A^\ell|}$ causes float overflows:

$$
\begin{aligned}
H(y_t \mid A^\ell, y_{<t}) &= \sum_{i \in A^\ell} \frac{1}{2} \log\left((2\pi e)^{|A^\ell|} \sigma^2_{x_t,i}\right) \\
&= \sum_{i \in A^\ell} \frac{1}{2} \left[\log((2\pi e)^{|A^\ell|}) + \log(\sigma^2_{x_t,i})\right] \\
&= \sum_{i \in A^\ell} \frac{1}{2} \left[|A^\ell| \log((2\pi e)) + \log(\sigma^2_{x_t,i})\right]
\end{aligned}
$$

Then if $|A^\ell|$ is the same for all ℓ we can even drop the first term from the argmax.