n-pendulum

Tue Boesen

June 10, 2022

We wish to simulate a n-pendulum, with massless rods.

In order to simulate this we use the Lagrangian approach, where we have a Lagrange function, L:

$$L = T - V, \tag{1}$$

where T is the kinetic energy, and V is the potential energy.

$$V = g\sum_{i=1}^{n} m_i y_i, \tag{2}$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2, \tag{3}$$

with m being the mass of a pendulum, $g = -9.82m/s^2$ being gravity acceleration and v being the velocity.

The Lagrangian then has to obey the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{4}$$

We start by assuming all pendulums have a length of 1. The cartesian coordinates and velocities are related to the angles and angular velocities in the following way:

$$x_i = \sum_{j=1}^{i} \sin(\theta_j) \tag{5}$$

$$y_i = -\sum_{j=1}^{i} \cos(\theta_j) \tag{6}$$

$$\dot{x}_i = \sum_{j=1}^{i} \dot{\theta}_j \cos(\theta_j) \tag{7}$$

$$\dot{y}_i = \sum_{j=1}^{i} \dot{\theta}_j \sin(\theta_j) \tag{8}$$

Hence we can express the kinetic energy as:

$$
\begin{aligned}
T &= \frac{1}{2}\sum_i m_i v_i^2 && (9)\\
&= \frac{1}{2}\sum_i m_i \left(\left[\sum_{j=1}^{i}\dot{\theta}_j\cos\theta_j\right]^2 + \left[\sum_{j=1}^{i}\dot{\theta}_j\sin\theta_j\right]^2\right) && (10)\\
&= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{i}\left[\dot{\theta}_j^2\cos^2\theta_j + 2\dot{\theta}_j\cos\theta_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos\theta_k\right] + \sum_{j=1}^{i}\left[\dot{\theta}_j^2\sin^2\theta_j + 2\dot{\theta}_j\sin\theta_j\sum_{k=1}^{j-1}\dot{\theta}_k\sin\theta_k\right]\right) && (11)\\
&= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{i}\dot{\theta}_j^2 + 2\dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos\theta_j\cos\theta_k + 2\dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\sin\theta_j\sin\theta_k\right) && (12)\\
&= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{i}\dot{\theta}_j^2 + 2\dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\left[\cos\theta_j\cos\theta_k + \sin\theta_j\sin\theta_k\right]\right) && (13)\\
&= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{i}\dot{\theta}_j^2 + 2\dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos(\theta_k-\theta_j)\right) && (14)
\end{aligned}
$$

And the potential energy as:

$$
\begin{aligned}
V &= g\sum_i m_i y_i && (15)\\
&= -g\sum_i m_i\sum_{j=1}^{i}\cos\theta_j && (16)
\end{aligned}
$$

For convenience we set $m = 1$
Which gives us:

$$
\begin{aligned}
T &= \sum_i\left(\sum_{j=1}^{i}\frac{1}{2}\dot{\theta}_j^2 + \dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos(\theta_k-\theta_j)\right) && (17)\\
&= \sum_{j=1}^{n}(n-j+1)\left(\frac{1}{2}\dot{\theta}_j^2 + \dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos(\theta_k-\theta_j)\right) && (18)
\end{aligned}
$$

$$
\begin{aligned}
V &= -g\sum_i m_i\sum_{j=1}^{i}\cos\theta_j && (19)\\
&= -g\sum_i(n-i+1)\cos\theta_i && (20)
\end{aligned}
$$

Hence the Lagrangian is in angular coordinates given as:

$$
L = \sum_{j=1}^{n}(n-j+1)\left(\frac{1}{2}\dot{\theta}_j^2 + \dot{\theta}_j\sum_{k=1}^{j-1}\dot{\theta}_k\cos(\theta_k-\theta_j)\right) + g\sum_j(n-j+1)\cos\theta_j \qquad (21)
$$

We start evaluating terms needed for the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_i} &= \frac{\partial T}{\partial \dot{\theta}_i} && (22)\\
&= \frac{\partial}{\partial \dot{\theta}_i} \sum_{j=1}^{n} (n-j+1) \left(\frac{1}{2}\dot{\theta}_j^2 + \dot{\theta}_j \sum_{k=1}^{j-1} \dot{\theta}_k \cos(\theta_k - \theta_j) \right) && (23)\\
&= (n-i+1)\dot{\theta}_i + \sum_{j=1}^{n} (n-j+1) \sum_{k=1}^{j-1} \left[\delta_{ij}\dot{\theta}_k + \dot{\theta}_j \delta_{ki} \right] \cos(\theta_k - \theta_j) && (24)\\
&= (n-i+1)\dot{\theta}_i + \sum_{j=1}^{n} (n-j+1) \sum_{k=1}^{j-1} \delta_{ij}\dot{\theta}_k \cos(\theta_k - \theta_j) + \dot{\theta}_j \delta_{ki} \cos(\theta_k - \theta_j) && (25)\\
&= (n-i+1)\dot{\theta}_i + \sum_{j=1}^{n} (n-j+1)\delta_{ij} \sum_{k=1}^{j-1} \dot{\theta}_k \cos(\theta_k - \theta_j) + \sum_{j=1}^{n} (n-j+1) \sum_{k=1}^{j-1} \dot{\theta}_j \delta_{ki} \cos(\theta_k - \theta_j) && (26)\\
&= (n-i+1)\dot{\theta}_i + (n-i+1) \sum_{k=1}^{i-1} \dot{\theta}_k \cos(\theta_k - \theta_i) + \sum_{j=i+1}^{n} (n-j+1)\dot{\theta}_j \cos(\theta_i - \theta_j) && (27)\\
&= (n-i+1)\dot{\theta}_i + (n-i+1) \sum_{j=1}^{i-1} \dot{\theta}_j \cos(\theta_j - \theta_i) + \sum_{j=i+1}^{n} (n-j+1)\dot{\theta}_j \cos(\theta_i - \theta_j) && (28)\\
&= c(i)\dot{\theta}_i + c(i) \sum_{j=1}^{i-1} \dot{\theta}_j \cos(\theta_j - \theta_i) + \sum_{j=i+1}^{n} c(i,j)\dot{\theta}_j \cos(\theta_i - \theta_j) && (29)\\
&= c(i)\dot{\theta}_i + \sum_{j \neq i}^{n} c(i,j)\dot{\theta}_j \cos(\theta_j - \theta_i) && (30)
\end{aligned}
$$

where

$$c(i) = n - i + 1 \tag{31}$$

and

$$c(i,j) = n - max(i,j) + 1 \tag{32}$$

Next we calculate the time derivative of it:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} &= \frac{d}{dt} \left(c(i)\dot{\theta}_i + \sum_{j \neq i}^{n} c(i,j)\dot{\theta}_j \cos(\theta_i - \theta_j) \right) && (33)\\
&= c(i)\ddot{\theta}_i + \sum_{j \neq i}^{n} c(i,j) \left[\ddot{\theta}_j \cos(\theta_i - \theta_j) - \dot{\theta}_i\dot{\theta}_j \sin(\theta_i - \theta_j) + \dot{\theta}_j^2 \sin(\theta_i - \theta_j) \right] && (34)\\
&= \sum_{j=1}^{n} c(i,j) \left[\ddot{\theta}_j \cos(\theta_i - \theta_j) - \dot{\theta}_i\dot{\theta}_j \sin(\theta_i - \theta_j) + \dot{\theta}_j^2 \sin(\theta_i - \theta_j) \right] && (35)\\
& && (36)
\end{aligned}
$$

Next we calculate $\frac{\partial L}{\partial \theta_i}$:

$$\frac{\partial L}{\partial \theta_i} = \sum_{j=1}^{n} c(j)\dot{\theta}_j \sum_{k=1}^{j-1} \dot{\theta}_k(-\sin(\theta_k - \theta_j)\delta_{ki} + \sin(\theta_k - \theta_j)\delta_{ji} - g\sum_{j} c(j)\sin(\theta_j)\delta_{ij} \tag{37}$$

$$= -\sum_{j=1}^{n} c(j)\dot{\theta}_j \sum_{k=1}^{j-1} \dot{\theta}_k \sin(\theta_k - \theta_j)\delta_{ki} + c(i)\dot{\theta}_i \sum_{k=1}^{i-1} \dot{\theta}_k \sin(\theta_k - \theta_i) - gc(i)\sin(\theta_i) \tag{38}$$

$$= -\sum_{j=i+1}^{n} c(j)\dot{\theta}_j\dot{\theta}_i \sin(\theta_i - \theta_j) + c(i)\dot{\theta}_i \sum_{k=1}^{i-1} \dot{\theta}_k \sin(\theta_k - \theta_i) - gc(i)\sin(\theta_i) \tag{39}$$

$$= -\sum_{j=1}^{n} c(i,j)\dot{\theta}_j\dot{\theta}_i \sin(\theta_i - \theta_j) - gc(i)\sin(\theta_i) \tag{40}$$

Hence for the Euler-Lagrange equation we get:

$$\sum_{j=1}^{n} c(i,j)\left[\ddot{\theta}_j \cos(\theta_i - \theta_j) - \dot{\theta}_i\dot{\theta}_j \sin(\theta_i - \theta_j) + \dot{\theta}_j^2 \sin(\theta_i - \theta_j)\right] + \sum_{j=1}^{n} c(i,j)\dot{\theta}_j\dot{\theta}_i \sin(\theta_i - \theta_j) + gc(i)\sin(\theta_i) = 0 \tag{41}$$

Where some of the terms cancel out:

$$\sum_{j=1}^{n} c(i,j)\left[\ddot{\theta}_j \cos(\theta_i - \theta_j) + \dot{\theta}_j^2 \sin(\theta_i - \theta_j)\right] + gc(i)\sin(\theta_i) = 0 \tag{42}$$

Which we can rearange as:

$$\sum_{j=1}^{n} c(i,j)\ddot{\theta}_j \cos(\theta_i - \theta_j) = -gc(i)\sin(\theta_i) - \sum_{j=1}^{n} c(i,j)\dot{\theta}_j^2 \sin(\theta_i - \theta_j) \tag{43}$$

Which we can solve as a linear system