LESSON 7
PASSPORT TO ADVANCED MATH

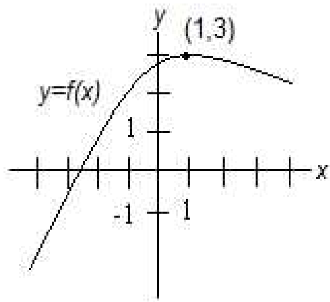
Reminder: Before beginning this lesson remember to redo the problems from Lesson 3 that you have marked off. Do not "unmark" a question unless you get it correct.

Graphs of Functions

If f is a function, then

$f(a) = b$ is equivalent to "the point (a, b) lies on the graph of f."

Example 1:



In the figure above we see that the point $(1,3)$ lies on the graph of the function f. Therefore $f(1) = 3$.

Try to answer the following question using this fact. **Do not** check the solution until you have attempted this question yourself.

LEVEL 4: ADVANCED MATH

1. In the xy-plane, the graph of the function h, with equation $h(x) = ax^2 - 16$, passes through the point $(-2,4)$. What is the value of a?

16

5

*** Solution:** Since the graph of h passes through the point $(-2,4)$, $h(-2)=4$. But by direct computation

$$h(-2)=a(-2)^2-16=4a-16.$$

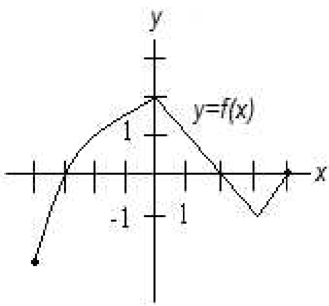
So $4a-16=4$. Therefore $4a=20$, and so $a=\mathbf{5}$.

Function Facts

Fact 1: The ***y*-intercept** of the graph of a function $y=f(x)$ is the point on the graph where $x=0$ (if it exists). There can be at most one y-intercept for the graph of a function. A y-intercept has the form $(0,b)$ for some real number b. Equivalently, $f(0)=b$.

Fact 2: An ***x*-intercept** of the graph of a function is a point on the graph where $y=0$. There can be more than one x-intercept for the graph of a function or none at all. An x-intercept has the form $(a,0)$ for some real number a. Equivalently, $f(a)=0$.

Example 2:



In the figure above we see that the graph of f has y-intercept $(0,2)$ and x-intercepts $(-3,0)$, $(2,0)$ and $(4,0)$.

The numbers -3, 2, and 4 are also called **zeros**, **roots**, or **solutions** of the function.

Fact 3: If the graph of $f(x)$ is above the x-axis, then $f(x)>0$. If the graph of f is below the x-axis, then $f(x)<0$. If the graph of f is higher than the graph of g, then $f(x)>g(x)$

Example 3: In the figure for example 2 above, observe that $f(x)<0$ for $-4\le x<-3$ and $2<x<4$. Also observe that $f(x)>0$ for $-3<x<2$.

Fact 4: As x gets very large, $\frac{1}{x}$ gets very small.

Example 4: Let $f(x)=\frac{3x^2+\frac{1}{x}}{x^2}$. Then for large x, $f(x)\approx\frac{3x^2}{x^2}=3$. So, for example $f(10^{100})\approx 3$.

Even and Odd Functions

A function f with the property that $f(-x)=f(x)$ for all x is called an **even** function. For example, $f(x)=|x|$ is an even function because

$$f(-x)=|-x|=|x|=f(x).$$

A function f with the property that $f(-x)=-f(x)$ for all x is called an **odd** function. For example, $g(x)=\frac{1}{x}$ is odd because

$$g(-x)=\frac{1}{-x}=-\frac{1}{x}=-g(x).$$

A **polynomial function** is a function for which each **term** has the form ax^n where a is a real number and n is a positive integer.

Polynomial functions with only even powers of x are even functions. Keep in mind that a constant c is the same as cx^0, and so c is an even power of x. Here are some examples of polynomial functions that are even.

$$f(x)=x^2 \qquad g(x)=4 \qquad h(x)=3x^8-2x^6+9$$

Polynomial functions with only odd powers of x are odd functions. Keep in mind that x is the same as x^1, and so x is an odd power of x. Here are some examples of polynomial functions that are odd.

$$f(x)=x^3 \qquad g(x)=x \qquad h(x)=3x^{11}-2x^5+9x$$

A quick graphical analysis of even and odd functions: The graph of an even function is **symmetrical with respect to the y-axis**. This means that the y-axis acts like a "mirror," and the graph "reflects" across this mirror.

The graph of an odd function is **symmetrical with respect to the origin**. This means that if you rotate the graph 180 degrees (or equivalently, turn it upside down) it will look the same as it did right side up.

So another way to determine if $f(-x)=f(x)$ is to graph f in your graphing calculator, and see if the y-axis acts like a mirror. Another way to determine if $f(-x)=-f(x)$ is to graph f in your graphing calculator, and see if it looks the same upside down. This technique will work for **all** functions (not just polynomials).

Try to answer the following question about even functions. **Do not** check the solution until you have attempted this question yourself.

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2. For which of the following functions is it true that $f(-x) = f(x)$ for all values of x?

   (A) $f(x) = x^2 + 5$
   (B) $f(x) = x^2 + 5x$
   (C) $f(x) = x^3 + 5x$
   (D) $f(x) = x^3 + 5$

Solution by picking numbers: Let's choose a value for x, say $x = 2$. We compute $f(-2)$ and $f(2)$ for each answer choice.

	$f(-2)$	$f(2)$
(A)	9	9
(B)	-6	14
(C)	-18	18
(D)	-3	13

Since choices (B), (C), and (D) do not match up, we can eliminate them. The answer is therefore choice (A).

Important note: (A) is **not** the correct answer simply because both computations gave the same answer. It is correct because all 3 of the other choices did **not** work. **You absolutely must check all four choices!**

*** Quick solution:** We are looking for an even function. Each answer choice is a polynomial. Therefore, the answer is the one with only even powers of x. This is choice (A) (remember that $5 = 5x^0$).

Graphical solution: Begin putting each of the four answer choices into your graphing calculator (starting with choice (B) or (C)), and choose the one that is symmetrical with respect to the y-axis. This is choice (A).

Now try to solve each of the following problems. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

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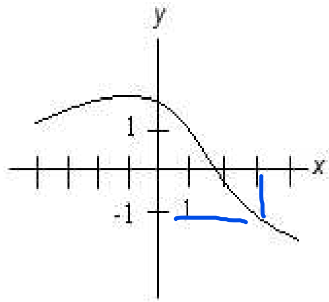
3. The function h is defined by $h(x) = 5x^2 - cx + 3$, where c is a constant. In the xy-plane, the graph of $y = h(x)$ crosses the x-axis where $x = 1$. What is the value of c ?



4. The figure above shows the graph of the function f. Which of the following is less than $f(1)$?

(A) $f(-3)$
(B) $f(-2)$
(C) $f(0)$
(D) $f(3)$

D

LEVEL 4: ADVANCED MATH

5. If $r^2s > 10^{200}$, then the value of $\frac{rs+\frac{1}{r}}{7rs}$ is closest to which of the following?

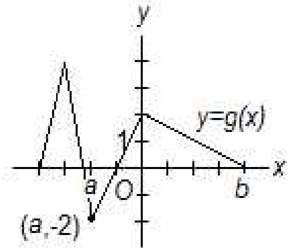
(A) 0.1
(B) 0.15
(C) 0.2
(D) 0.25

$\frac{r(rs+\frac{1}{r})}{7r^2s}$ $\quad \frac{1}{7} \cdot \frac{r^2s+1}{r^2s}$

$\frac{r^2s+1 > 10^{200}+1}{7r^2s > 7\cdot10^{200}}$ $\quad \frac{r^2s+1}{7r^2s}$

$\frac{10^{200}+10^0}{7\cdot10^{200}}$

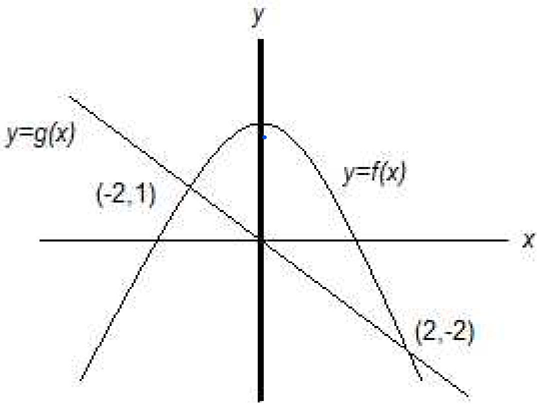
$\frac{1}{7}$ $\quad \frac{r^2s}{7r^2s}$ $\quad + \frac{1}{7} \cdot \frac{1}{r^2s}$



6. The figure above shows the graph of the function g in the xy-plane. Which of the following are true?

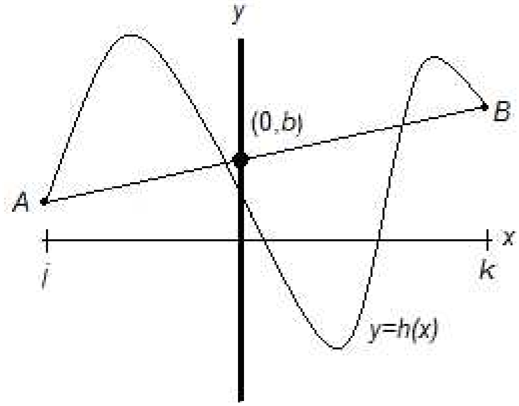
I. $g(b) = 0$
II. $g(a) + g(b) + g(0) = 0$
III. $g(a) > g(b)$

(A) None C
(B) I only
(C) I and II only
(D) I, II, and III

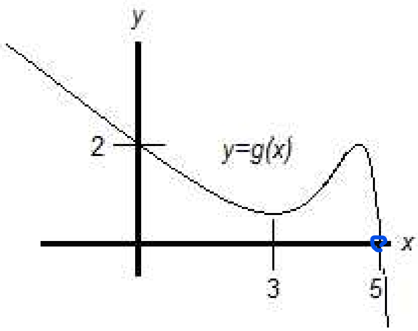


7. In the xy-plane above, the graph of the function f is a parabola, and the graph of the function g is a line. The graphs of f and g intersect at $(-2,1)$ and $(2,-2)$. For which of the following values of x is $f(x) - g(x) < 0$?

(A) -3
(B) -1
(C) 0
(D) 1



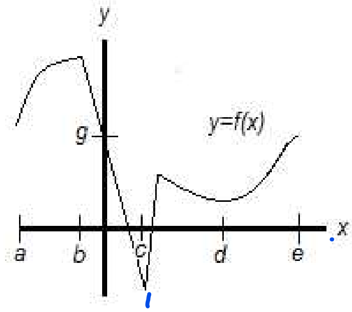
8. The figure above shows the graph of the function h and line segment $\overline{AB}$, which has a y-intercept of $(0, b)$. For how many values of x between j and k does $h(x) = b$? 3



y = 0

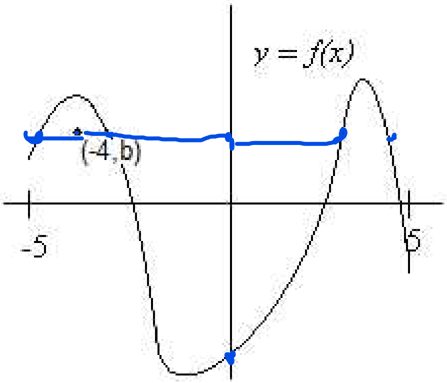
9. A portion of the graph of the function g is shown in the xy-plane above. What is the x-intercept of the graph of the function h defined by $h(x) = g(x - 1)$?

(A) (1,0)
(B) (2,0)
(C) (3,0)
(D) (6,0)



10. The figure above shows the graph of the function f on the interval $a < x < e$. Which of the following expressions represents the difference between the maximum and minimum values of $f(x)$ on this interval?

(A) $f(b - e)$
(B) $f(b - c)$
(C) $f(a) - f(e)$
(D) $f(b) - f(c)$



11. The figure above shows the graph of the function f and the point $(-4, b)$. For how many values of x between -5 and 5 does $f(x) = b$? 4

LEVEL 5: ADVANCED MATH

x	3	6	9
$f(x)$	5	a	11

a=1
b=2

8

x	6	12	24
$g(x)$	1	b	13

a=2/3
b=-3

5

12. The tables above show some values for the functions f and g. If f and g are linear functions, what is the value of $7a - 3b$? ~~72~~ 41

Answers

1. 5	5. B	9. D
2. A	6. C	10. D
3. 8	7. A	11. 4
4. D	8. 3	12. 41

Full Solutions

3.

* A graph crosses the x-axis at a point where $y = 0$. Thus, the point $(1, 0)$ is on the graph of $y = h(x)$. So,

$$0 = h(1) = 5(1)^2 - c + 3 = 5 - c + 3 = 8 - c.$$

So $8 - c = 0$, and therefore $c = \mathbf{8}$.

4.

* Let's draw a horizontal line through the point $(1, f(1))$. To do this start on the x-axis at 1 and go straight up until you hit the curve. This height is $f(1)$. Now draw a horizontal line through this point.