LESSON 1
HEART OF ALGEBRA

In this lesson we will be reviewing four very basic strategies that can be used to solve a wide range of SAT math problems in all topics and all difficulty levels. Throughout this book you should practice using these four strategies whenever it is possible to do so. You should also try to solve each problem in a more straightforward way.

Start with Choice (B) or (C)

In many SAT math problems, you can get the answer simply by trying each of the answer choices until you find the one that works. Unless you have some intuition as to what the correct answer might be, then you should always start in the middle with choice (B) or (C) as your first guess (an exception will be detailed in the next strategy below). The reason for this is simple. Answers are usually given in increasing or decreasing order. So very often if choice (B) or (C) fails you can eliminate one or two of the other choices as well.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 2: HEART OF ALGEBRA

$$x - 3 = \sqrt{x + 3}$$

1. What is the solution set of the equation above?

(A) $\{1\}$
(B) $\{6\}$
(C) $\{1,6\}$
(D) There are no solutions.

See if you can answer this question by starting with choice (B) or (C).

Solution by starting with choice (B): Let's start with choice (B) and guess that the answer is $\{6\}$. We substitute 6 for x into the given equation to get

$$6 - 3 = \sqrt{6 + 3}$$
$$3 = \sqrt{9}$$
$$3 = 3$$

Since this works, we have eliminated choices (A) and (D). But we still need to check to see if 1 works to decide if the answer is (B) or (C).

We substitute 1 for x into the given equation to get

$$1 - 3 = \sqrt{1 + 3}$$
$$-2 = \sqrt{4}$$
$$-2 = 2$$

So 1 is not a solution to the given equation and we can eliminate choice (C). The answer is therefore choice (B).

Important note: Once we see that $x = 6$ is a solution to the given equation, it is **very important** that we make sure there are no answer choices remaining that also contain 6. In this case answer choice (C) also contains 6 as a solution. We therefore <u>must</u> check if 1 is a solution too. In this case it is not.

Solution by starting with choice (C): Let's start with choice (C) and guess that the answer is {1,6}. We begin by substituting 1 for x into the given equation to get the false equation $-2 = 2$ (see the previous solution for details). So 1 is not a solution to the given equation and we can eliminate choice (C). Note that we also eliminate choice (A).

Let's try choice (B) now and guess that the answer is {6}. So we substitute 6 for x into the given equation to get the true equation $3 = 3$ (see the previous solution for details).

Since this works, the answer is in fact choice (B).

Important note: Once we see that $x = 6$ is a solution to the given equation, it is **very important** that we make sure there are no answer choices remaining that also contain 6. In this case we have already eliminated choices (A) and (C), and choice (D) does not contain 6 (in fact choice (D) contains no numbers at all).

Before we go on, try to solve this problem algebraically.

Algebraic solution:

$$x - 3 = \sqrt{x+3}$$
$$(x-3)^2 = \left(\sqrt{x+3}\right)^2$$
$$(x-3)(x-3) = x + 3$$
$$x^2 - 6x + 9 = x + 3$$
$$x^2 - 7x + 6 = 0$$
$$(x-1)(x-6) = 0$$
$$x - 1 = 0 \text{ or } x - 6 = 0$$
$$x = 1 \text{ or } x = 6$$

When solving algebraic equations with square roots we sometimes generate extraneous solutions. We therefore need to check each of the *potential* solutions 1 and 6 back in the original equation. As we have already seen in the previous solutions 6 is a solution, and 1 is not a solution. So the answer is choice (B).

Notes: (1) Do not worry if you are having trouble understanding all the steps of this solution. We will be reviewing the methods used here later in the book.

(2) Squaring both sides of an equation is not necessarily "reversible." For example, when we square each side of the equation $x = 2$, we get the equation $x^2 = 4$. This new equation has two solutions: $x = 2$ and $x = -2$, whereas the original equation had just one solution: $x = 2$.

This is why we need to check for **extraneous solutions** here.

(3) Solving this problem algebraically is just silly. After finding the potential solutions 1 and 6, we still had to check if they actually worked. But if we had just glanced at the answer choices we would have already known that 1 and 6 were the only numbers we needed to check.

When NOT to Start with Choice (B) or (C)

If the word **least** appears in the problem, then start with the smallest number as your first guess. Similarly, if the word **greatest** appears in the problem, then start with the largest number as your first guess.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 2: HEART OF ALGEBRA

2. What is the greatest integer x that satisfies the inequality $2 + \frac{x}{5} < 7$?

   (A) 20
   (B) 22
   (C) 24
   (D) 25

See if you can answer this question by starting with choice (A) or (D).

Solution by plugging in answer choices: Since the word "greatest" appears in the problem, let's start with the largest answer choice, choice (D). Now $2 + \frac{25}{5} = 2 + 5 = 7$ This is just barely too big, and so the answer is choice (C).

Before we go on, try to solve this problem algebraically.

*** Algebraic solution:** Let's solve the inequality. We start by subtracting 2 from each side of the given inequality to get $\frac{x}{5} < 5$. We then multiply each side of this inequality by 5 to get $x < 25$. The greatest integer less than 25 is 24, choice (C).

Take a Guess

Sometimes the answer choices themselves cannot be substituted in for the unknown or unknowns in the problem. But that does not mean that you cannot guess your own numbers. Try to make as reasonable a guess as possible, but do not over think it. Keep trying until you zero in on the correct value.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 3: HEART OF ALGEBRA

3. Dana has pennies, nickels and dimes in her pocket. The number of dimes she has is three times the number of nickels, and the number of nickels she has is 2 more than the number of pennies. Which of the following could be the total number of coins in Dana's pocket?

   (A) 15
   (B) 16
   (C) 17
   (D) 18

See if you can answer this question by taking guesses.

*** Solution by taking a guess:** Let's take a guess and say that Dana has 3 pennies. It follows that she has $3 + 2 = 5$ nickels, and $(3)(5) = 15$ dimes. So the total number of coins is $3 + 5 + 15 = 23$. This is too many. So let's guess that Dana has 2 pennies. Then she has $2 + 2 = 4$ nickels, and she has $(3)(4) = 12$ dimes for a total of $2 + 4 + 12 = 18$ coins. Thus, the answer is choice (D).

Before we go on, try to solve this problem the way you might do it in school.

Attempt at an algebraic solution: If we let x represent the number of pennies, then the number of nickels is $x + 2$, and the number of dimes is $3(x + 2)$. Thus, the total number of coins is

$$x + (x + 2) + 3(x + 2) = x + x + 2 + 3x + 6 = 5x + 8.$$

So some possible totals are 13, 18, 23,… which we get by substituting 1, 2, 3,… for x. Substituting 2 in for x gives 18 which is answer choice (D).

Warning: Many students incorrectly interpret "three times the number of nickels" as $3x + 2$. This is not right. The number of nickels is $x + 2$, and so "three times the number of nickels" is $3(x + 2) = 3x + 6$.

Pick a Number

A problem may become much easier to understand and to solve by substituting a specific number in for a variable. Just make sure that you choose a number that satisfies the given conditions.

Here are some guidelines when picking numbers.

(1) Pick a number that is simple but not too simple. In general, you might want to avoid picking 0 or 1 (but 2 is usually a good choice).
(2) Try to avoid picking numbers that appear in the problem.
(3) When picking two or more numbers try to make them all different.
(4) Most of the time picking numbers only allows you to eliminate answer choices. So do not just choose the first answer choice that comes out to the correct answer. If multiple answers come out correct you need to pick a new number and start again. But you only have to check the answer choices that have not yet been eliminated.
(5) If there are fractions in the question a good choice might be the least common denominator (lcd) or a multiple of the lcd.
(6) In percent problems choose the number 100.
(7) Do not pick a negative number as a possible answer to a grid-in question. This is a waste of time since you cannot grid a negative number.
(8) If your first attempt does not eliminate 3 of the 4 choices, try to choose a number that's of a different "type." Here are some examples of types:
    (a) A positive integer greater than 1.
    (b) A positive fraction (or decimal) between 0 and 1.
    (c) A negative integer less than -1.
    (d) A negative fraction (or decimal) between -1 and 0.
(9) If you are picking pairs of numbers, try different combinations from (8). For example, you can try two positive integers greater than 1, two negative integers less than -1, or one positive and one negative integer, etc.

Remember that these are just guidelines and there may be rare occasions where you might break these rules. For example, sometimes it is so quick and easy to plug in 0 and/or 1 that you might do this even though only some of the answer choices get eliminated.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 3: HEART OF ALGEBRA

$$\frac{x+y}{x}=\frac{2}{9}$$

4. If the equation shown above is true, which of the following must also be true?

(A) $\frac{x}{y}=\frac{9}{11}$

(B) $\frac{x}{y}=-\frac{9}{7}$

(C) $\frac{x-y}{x}=\frac{11}{9}$

(D) $\frac{x-y}{x}=-\frac{9}{7}$

$1+\frac{y}{x}=\frac{2}{9}$

y

See if you can answer this question by picking numbers.

Solution by picking numbers: Let's choose values for x and y, say $x=9$ and $y=-7$. Notice that we chose these values to make the given equation true.

Now let's check if each answer choice is true or false.

(A) $\frac{9}{-7}=\frac{9}{11}$	False
(B) $\frac{9}{-7}=-\frac{9}{7}$	True
(C) $\frac{9-(-7)}{9}=\frac{11}{9}$ or $\frac{16}{9}=\frac{11}{9}$	False
(D) $\frac{9-(-7)}{9}=-\frac{9}{7}$ or $\frac{16}{9}=-\frac{9}{7}$	False

Since (A), (C), and (D) are each False we can eliminate them. Thus, the answer is choice (B).

Before we go on, try to solve this problem the way you might do it in school.

Algebraic solution 1: $\frac{x+y}{x} = \frac{x}{x} + \frac{y}{x} = 1 + \frac{y}{x}$. So the given equation is equivalent to $1 + \frac{y}{x} = \frac{2}{9}$. Therefore $\frac{y}{x} = \frac{2}{9} - 1 = \frac{2}{9} - \frac{9}{9} = -\frac{7}{9}$, and so $\frac{x}{y} = -\frac{9}{7}$, choice (B).

Note: Most students have no trouble at all adding two fractions with the same denominator. For example,

$$\frac{x}{x} + \frac{y}{x} = \frac{x+y}{x}$$

But these same students have trouble reversing this process.

$$\frac{x+y}{x} = \frac{x}{x} + \frac{y}{x}$$

Note that these two equations are **identical** except that the left and right hand sides have been switched. Note also that to break a fraction into two (or more) pieces, the original denominator is repeated for **each** piece.

Algebraic solution 2: We cross multiply the given equation to get

$$9(x+y) = 2x$$

We now distribute the 9 on the left to get

$$9x + 9y = 2x$$

Now we subtract $2x$ from each side of this last equation to get

$$7x + 9y = 0$$

We subtract $9y$ from each side to get $7x = -9y$.

We can get $\frac{x}{y}$ to one side by performing **cross division.** We do this just like cross multiplication, but we divide instead. Dividing each side of the equation by $7y$ will do the trick (this way we get rid of 7 on the left and y on the right).

$$\frac{x}{y} = \frac{-9}{7} = -\frac{9}{7}$$

This is choice (B).

You're doing great! Let's just practice a bit more. Try to solve each of the following problems by using one of the four strategies you just learned. Then, if possible, solve each problem another way. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 1: HEART OF ALGEBRA

5. If $3z = \frac{y-5}{2}$ and $z = 5$, what is the value of y ?

15 = 30

(A) 20
(B) 25
(C) 30
(D) 35

6. If $x > 0$ and $x^4 - 16 = 0$, what is the value of x ? ± 2

LEVEL 2: HEART OF ALGEBRA

7. If $\frac{5x}{y} = 10$, what is the value of $\frac{8y}{x}$?

$\frac{40y}{5x} = \frac{1}{10}$ $\frac{8y}{x} = 4$

(A) 4
(B) 3
(C) 2
(D) 1

LEVEL 3: HEART OF ALGEBRA

8. The cost of 5 scarves is d dollars. At this rate, what is the cost, in dollars of 45 scarves?

45. $\frac{9}{d}$ $\cancel{(45/5d)}$ $45 \frac{d}{5}$.

(A) $\frac{9d}{5}$
(B) $\frac{d}{45}$
(C) $\frac{45}{d}$
(D) $9d$

9. Bill has cows, pigs and chickens on his farm. The number of chickens he has is four times the number of pigs, and the number of pigs he has is three more than the number of cows. Which of the following could be the total number of these animals?

   (A) 28
   (B) 27
   (C) 26
   (D) 25

$p=5$ $c=20$ $c = 4p$ $p = b+3$ $b=2$

$c+p+b=x$
$4p+p+p=x+3$
$6p=x+3$

LEVEL 4: HEART OF ALGEBRA

10. For all real numbers x and y, $|x-y|$ is equivalent to which of the following?

    (A) $x+y$
    (B) $\sqrt{x-y}$
    (C) $(x-y)^2$
    (D) $\sqrt{(x-y)^2}$

11. If $k \neq \pm 1$, which of the following is equivalent to $\frac{1}{\frac{1}{k+1}+\frac{1}{k-1}}$.

    (A) $2k$
    (B) k^2-1
    (C) $\frac{k^2-1}{2k}$
    (D) $\frac{2k}{k^2-1}$

$1 \div$

let $k =$ numer

12. In the real numbers, what is the solution of the equation $4^{x+2} = 8^{2x-1}$?

    (A) $-\frac{7}{4}$
    (B) $-\frac{1}{4}$
    (C) $\frac{3}{4}$
    (D) $\frac{7}{4}$

~~$2x+4 = 2x-1$~~

~~$x+2 =$~~ $2x+4 = 6x-3$

$7 = 4x$

Answers

1. B	5. D	9. B
2. C	6. 2	10. D
3. D	7. A	11. C
4. B	8. D	12. D

Note: The full solution for question 9 has been omitted because its solution is very similar to the solution for question 3.

Full Solutions

8.

Solution by picking numbers: Let's choose a value for d, say $d = 10$. So 5 scarves cost 10 dollars, and therefore each scarf costs 2 dollars. It follows that 45 scarves cost $(45)(2) = \mathbf{90}$ dollars. **Put a nice big, dark circle around this number so that you can find it easily later.** We now substitute 10 in for d into **all** four answer choices (we use our calculator if we're allowed to).

(A) 90/5 = 18
(B) 10/45
(C) 45/10 = 4.5
(D) 9*10 = 90

Since (D) is the only choice that has become 90, we conclude that (D) is the answer.

Important note: (D) is **not** the correct answer simply because it is equal to 90. It is correct because all 3 of the other choices are **not** 90.

*** Solution using ratios:** We begin by identifying 2 key words. In this case, such a pair of key words is "scarves" and "dollars."

scarves	5	45
dollars	d	x

Notice that we wrote in the number of scarves next to the word scarves, and the cost of the scarves next to the word dollars. Also notice that the cost for 5 scarves is written under the number 5, and the (unknown) cost for 45 scarves is written under the 45. Now draw in the division symbols and equal sign, cross multiply and divide the corresponding ratio to find the unknown quantity x.

$$\frac{5}{d}=\frac{45}{x}$$
$$5x=45d$$
$$x=9d$$

So 45 scarves cost $9d$ dollars, choice (D).

10.

Solution by picking numbers: Let's choose values for x and y, let's say $x=2$ and $y=5$. Then $|x-y|=|2-5|=|-3|=\mathbf{3}$.

Put a nice big dark circle around **3** so you can find it easily later. We now substitute $x=2$ and $y=5$ into each answer choice:

(A) 7
(B) $\sqrt{-3}$
(C) $(-3)^2 = 9$
(D) $\sqrt{(-3)^2} = \sqrt{9} = 3$

Since A, B and C each came out incorrect, we can eliminate them. Therefore, the answer is choice (D).

*** Solution using the definition of absolute value:** One definition of the absolute value of x is $|x|=\sqrt{x^2}$. So $|x-y|=\sqrt{(x-y)^2}$, choice (D).

Note: Here we have simply replaced x by $x-y$ on both sides of the equation $|x|=\sqrt{x^2}$.

11.

Solution by picking a number: Let's choose a value for k, say $k=2$. Then

$$\frac{1}{\frac{1}{k+1}+\frac{1}{k-1}}=\frac{1}{\frac{1}{2+1}+\frac{1}{2-1}}=\frac{1}{\frac{1}{3}+1}=\frac{1}{\frac{1}{3}+\frac{3}{3}}=\frac{1}{\frac{4}{3}}=\mathbf{\frac{3}{4}}$$

Put a nice big, dark circle around this number so that you can find it easily later. We now substitute 2 in for k into **all** four answer choices (we use our calculator if we're allowed to).

(A) $2*2=4$
(B) $2\text{^}2-\ 1=3$
(C) $(2\text{^}2-1)/(2*2)=3/4$
(D) $(2*2)/(2\text{^}2-1)=4/3$

Since (C) is the only choice that has become $\frac{3}{4}$, we conclude that (C) is the answer.

Important note: (C) is **not** the correct answer simply because it is equal to $\frac{3}{4}$. It is correct because all 3 of the other choices are **not** $\frac{3}{4}$.

***Algebraic solution:** We multiply the numerator and denominator of the complex fraction by $(k+1)(k-1)$ to get

$$\frac{1}{\frac{1}{k+1}+\frac{1}{k-1}} \cdot \frac{(k+1)(k-1)}{(k+1)(k-1)} = \frac{(k+1)(k-1)}{(k-1)+(k+1)} = \frac{k^2-1}{2k}$$

This is choice (C).

Notes: (1) The three simple fractions within this complex fraction are $1 = \frac{1}{1}$, $\frac{1}{k+1}$, and $\frac{1}{k-1}$.

The least common denominator (LCD) of these three fractions is

$$(k+1)(k-1)$$

Note that the least common denominator is just the least common multiple (LCM) of the three denominators. In this problem the LCD is the same as the product of the denominators.

(2) To simplify a complex fraction we multiply each of the numerator and denominator of the fraction by the LCD of all the simple fractions that appear.

(3) Make sure to use the distributive property correctly here.

$$\left(\frac{1}{k+1}+\frac{1}{k-1}\right)\cdot(k+1)(k-1)$$
$$= \left(\frac{1}{k+1}\right)\cdot(k+1)(k-1) + \left(\frac{1}{k-1}\right)\cdot(k+1)(k-1)$$
$$= (k-1)+(k+1)$$

This is how we got the denominator in the second expression in the solution.

(4) Do not worry too much if you are having trouble understanding all the steps of this solution. We will be reviewing the methods used here later in the book.

12.

Solution by starting with choice (C) and using our calculator: Let's start with choice (C) and guess that $x = \frac{3}{4}$. We type in our calculator:

$$4\text{^}(3/4 + 2) \approx 45.255 \quad \text{and} \quad 8\text{^}(2 * 3/4 - 1) \approx 2.828$$

Since these two numbers are different we can eliminate choice (C).

Let's try choice (D) next:

$$4\text{^}(7/4 + 2) \approx 181.019 \quad \text{and} \quad 8\text{^}(2 * 7/4 - 1) \approx 181.019$$

Since they came out the same, the answer is choice (D).

*** Algebraic solution:** The numbers 4 and 8 have a common base of 2. In fact, $4 = 2^2$ and $8 = 2^3$. So we have $4^{x+2} = (2^2)^{x+2} = 2^{2x+4}$ and we have $8^{2x-1} = (2^3)^{2x-1} = 2^{6x-3}$. Thus, $2^{2x+4} = 2^{6x-3}$, and so $2x + 4 = 6x - 3$. We subtract $2x$ from each side of this equation to get $4 = 4x - 3$. We now add 3 to each side of this last equation to get $7 = 4x$. Finally, we divide each side of this equation by 4 to get $\frac{7}{4} = x$, choice (D).

Note: For a review of the laws of exponents see lesson 13.

LESSON 3
PASSPORT TO ADVANCED MATH

Direct Variation

The following are all equivalent ways of saying the same thing:

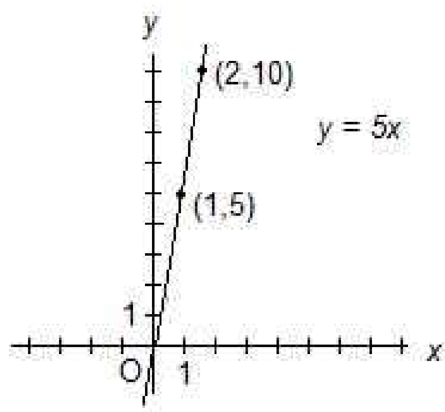
(1) y varies directly as x
(2) y is directly proportional to x
(3) $y = kx$ for some constant k
(4) $\frac{y}{x}$ is constant
(5) the graph of $y = f(x)$ is a nonvertical line through the origin.

For example, in the equation $y = 5x$, y varies directly as x. Here is a partial table of values for this equation.

x	1	2	3	4
y	5	10	15	20

Note that we can tell that this table represents a direct relationship between x and y because $\frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4}$. Here the **constant of variation** is 5.

Here is a graph of the equation.



Note that we can tell that this graph represents a direct relationship between x and y because it is a nonvertical line through the origin. The constant of variation is the slope of the line, in this case $m = 5$.

The various equivalent definitions of direct variation lead to several different ways to solve problems.

LEVEL 1: ADVANCED MATH

1. If $y = kx$ and $y = 5$ when $x = 8$, then what is y when $x = 24$? 15

Solutions

(1) We are given that $y = 5$ when $x = 8$, so that $5 = k(8)$, or $k = \frac{5}{8}$. Therefore $y = \frac{5x}{8}$. When $x = 24$, we have $y = \frac{5(24)}{8} = \mathbf{15}$.

(2) Since y varies directly as x, $\frac{y}{x}$ is a constant. So we get the following ratio: $\frac{5}{8} = \frac{y}{24}$. Cross multiplying gives $120 = 8y$, so that $y = \mathbf{15}$.

(3) The graph *of* $y = f(x)$ is a line passing through the points $(0,0)$ and $(8,5)$ The slope of this line is $\frac{5-0}{8-0} = \frac{5}{8}$. Writing the equation of the line in slope-intercept form we have $y = \frac{5}{8}x$. As in solution 1, when $x = 24$, we have $y = \frac{5(24)}{8} = \mathbf{15}$.

*** (4)** To get from $x = 8$ to $x = 24$ we multiply x by 3. So we have to also multiply y by 3. We get $3(5) = \mathbf{15}$.

.Inverse Variation

The following are all equivalent ways of saying the same thing:

(1) y varies inversely as x
(2) y is inversely proportional to x
(3) $y = \frac{k}{x}$ for some constant k
(4) xy is constant

The following is a consequence of (1), (2) (3) or (4).

(5) The graph of $y = f(x)$ is a hyperbola

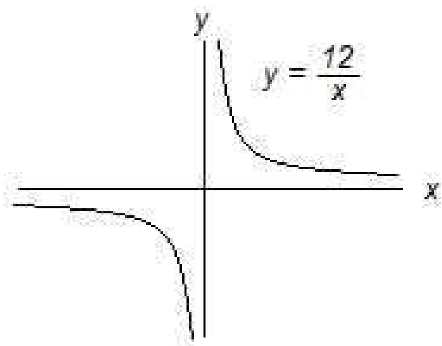
Note: (5) is not equivalent to (1), (2), (3) or (4).

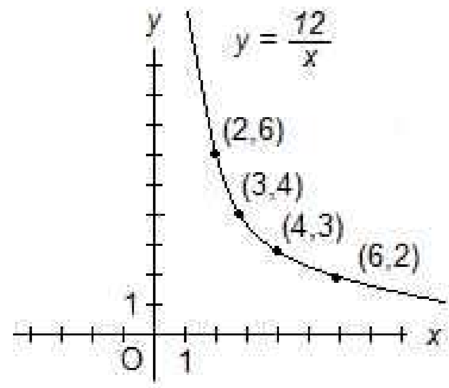
For example, in the equation $y = \frac{12}{x}$, y varies inversely as x. Here is a partial table of values for this equation.

x	1	2	3	4
y	12	6	4	3

Note that we can tell that this table represents an inverse relationship between x and y because $(1)(12) = (2)(6) = (3)(4) = (4)(3) = 12$. Here the **constant of variation** is 12.

Here is a graph of the equation. On the left you can see the full graph. On the right we have a close-up in the first quadrant.





The various equivalent definitions of inverse variation lead to several different ways to solve problems.

LEVEL 2: ADVANCED MATH

2. If $y = \frac{k}{x}$ and $y = 8$ when $x = 3$, then what is y when $x = 6$? 4

Solutions

(1) We are given that $y = 8$ when $x = 3$, so that $8 = \frac{k}{3}$, or $k = 24$. Thus, $y = \frac{24}{x}$. When $x = 6$, we have $y = \frac{24}{6} = \mathbf{4}$.

(2) Since y varies inversely as x, xy is a constant. So we get the following equation: $(3)(8) = 6y$ So $24 = 6y$, and $y = \frac{24}{6} = \mathbf{4}$.

*** (3)** $\frac{(8)(3)}{6} = \mathbf{4}$.

Functions

A function is simply a rule that for each "input" assigns a specific "output." Functions may be given by equations, tables or graphs.

Note about the notation $\boldsymbol{f(x)}$: The variable x is a placeholder. We evaluate the function f at a specific value by substituting that value in for x. For example, if $f(x) = x^3 + 2x$, then

$$f(-2) = (-2)^3 + 2(-2) = -12$$

LEVEL 4: ADVANCED MATH

x	$p(x)$	$q(x)$	$r(x)$
-2	-3	4	-3
-1	2	1	2
0	5	-1	-6
1	-7	0	-5

3. The functions p, q and r are defined for all values of x, and certain values of those functions are given in the table above. What is the value of $p(-2) + q(0) - r(1)$? $-3 + -1 + 5 = 1$

* To evaluate $p(-2)$, we look at the row corresponding to $x = -2$, and the column corresponding to $p(x)$. We see that the entry there is -3. Therefore $p(-2) = -3$. Similarly, $q(0) = -1$ and $r(1) = -5$. Finally, we have that $p(-2) + q(0) - r(1) = -3 - 1 - (-5) = -4 + 5 = \mathbf{1}$.

Try to solve each of the following problems. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 2: ADVANCED MATH

$$f(x) = 7x - 3$$
$$g(x) = x^2 - 2x + 6$$

4. The functions f and g are defined above. What is the value of $f(11) - g(4)$? 60

$$g(x) = \frac{2}{5}x + b$$

5. In the function above, b is a constant. If $g(10) = 7$, what is the value of $g(-5)$?

(A) -1
(B) 0
(C) 1
(D) 2

LEVEL 3: ADVANCED MATH

6. A function f satisfies $f(3) = 7$ and $f(7) = 1$. A function g satisfies $g(7) = 3$ and $g(1) = 4$. Find the value of $f(g(7))$. 7

7. If $g(x) = -3x - 7$, what is $g(-4x)$ equal to? f(3)

(A) $12x^2 + 28x$
(B) $12x + 7$
(C) $12x - 7$
(D) $-12x + 7$

LEVEL 4: ADVANCED MATH

8. For all real numbers x, let the function f be defined as $f(x) = 5x - 10$. Which of the following is equal to $f(3) + f(5)$?

15 20

(A) $f(4)$
(B) $f(6)$
(C) $f(7)$
(D) $f(20)$

x	$p(x)$	$q(x)$	$r(x)$
-2	-6	-2	-9
-1	-1	5	-10
0	-2	7	-3
1	5	-11	-6
2	-6	0	3

9. The functions p, q and r are defined for all values of x, and certain values of those functions are given in the table above. If the function u is defined by $u(x) = 3p(x) + q(x) - r(x)$ for all values of x, what is the value of $u(-1)$? 12

$$P(x) = \frac{20x}{98 - x}$$

10. * The function P above models the monthly profit, in thousands of dollars, for a company that sells x percent of their inventory for the month. If $90,000 is earned in profit during the month of April, what percent of April's inventory, to the nearest whole percent, has been sold?

(A) 25%
(B) 42%
(C) 56%
(D) 80%

k	-1	1	2
$f(k)$	-5	3	7

11. The table above shows some values of the linear function f. Which of the following defines f?

(A) $f(k) = k - 4$

(B) $f(k) = k - 8$

(C) $f(k) = 2k - 4$

(D) $f(k) = 4k - 1$

LEVEL 5: ADVANCED MATH

12. For all positive integers x, the function f is defined by $f(x) = (\frac{1}{b^5})^x$, where b is a constant greater than 1. Which of the following is equivalent to $f(3x)$?

(A) $\sqrt[3]{f(x)}$
(B) $(f(x))^3$
(C) $3f(x)$
(D) $\frac{1}{3}f(x)$

b^{-5}

$f(x) = b^{-5x}$

$(\quad b^{-15x}$

Answers

1. 15	5. C	9. 12
2. 4	6. 7	10. D
3. 1	7. C	11. D
4. 60	8. B	12. B

Full Solutions

6.
* $f(g(7)) = f(3) = \mathbf{7}$.

Note: $g(7)$ is given to be 3. So we replace $g(7)$ by 3 in the expression $f(g(7))$.

Do you see that we have $f(\boxed{\text{something}})$ where $\boxed{\text{something}}$ is $g(7)$? Since $g(7)$ is 3, we can replace $\boxed{\text{something}}$ by 3 to get $f(3)$. Finally, $f(3)$ is given to be 7.

7.
* $g(-4x) = -3(-4x) - 7 = 12x - 7$, choice (C).

Note: This problem can also be solved by picking a number for x. I leave the details to the reader.

8.
Solution by starting with choice (C): First note $f(3) = 5(3) - 10 = 5$ and $f(5) = 5(5) - 10 = 15$, so that $f(3) + f(5) = 5 + 15 = \mathbf{20}$.

Now, beginning with choice (C) we see that $f(7) = 5(7) - 10 = 25$.

This is a bit too big. So let's try choice (B). Then $f(6) = 5(6) - 10 = 20$. This is correct. Thus, the answer is choice (B).

Warning: Many students will compute $f(3) + f(5) = 20$ and immediately choose choice (D). Do not fall into this trap!

*** Algebraic solution:** As in the previous solution, direct computation gives $f(3) + f(5) = 20$. Setting $f(x) = 20$ yields $5x - 10 = 20$, so that $5x = 30$, and thus, $x = \frac{30}{5} = 6$. In other words, we have

$$f(6) = 20 = f(3) + f(5).$$

This is choice (B).

9.

* $u(-1) = 3p(-1) + q(-1) - r(-1) = 3(-1) + (5) - (-10)$

$$= -3 + 5 + 10 = \mathbf{12}.$$

10.

Solution by starting with choice (C): We are given that $P(x) = 90$, and being asked to approximate x. So we have $\frac{20x}{98-x} = 90$. Let's begin with choice (C) and plug in 56 for x. We have $20(56)/(98 - 56) \approx 26.67$, too small.

Let's try choice (D) next. So $20(80)/(98 - 80) \approx 88.89$. This is close, so the answer is probably choice (D). To be safe we should check the other answer choices.

(A) $20(25)/(98 - 25) \approx 6.85$
(B) $20(42)/(98 - 42) = 15$

So the answer must be choice (D).

*** Algebraic solution:**

$$\frac{20x}{98-x} = 90$$
$$20x = 90(98 - x)$$
$$20x = 8820 - 90x$$
$$110x = 8820$$
$$x = \frac{8820}{110} \approx 80.18$$

The final answer, to the nearest percent, is 80%, choice (D).

11.

*** Quick solution:** The last two columns tell us that the slope of the line is $7 - 3 = 4$. So the answer can be only choice (D).

Notes: (1) The table is telling us that the following points are on the line: $(-1, -5)$, $(1,3)$, and $(2,7)$.

(2) We can find the slope by using any two of the above points and then either using the slope formula, or plotting the points and computing $\frac{rise}{run}$. Both of these methods were covered in Lesson 2.

Since the x-coordinates of the last two points are one unit apart, we can get the slope by simply subtracting the y-coordinates: $7 - 3 = 4$.

(3) If we needed to find the y-intercept of the line, we could formally write the equation as we learned in Lesson 3, or we could use the following trick:

Since the function f is linear, "**equal jumps in $\boldsymbol{k}$ lead to equal jumps in $\boldsymbol{f(k)}$.**" But the jumps in k are **not** equal. We can make them equal however if we just slip in the number 0. The "new" table looks like this.

k	-1	0	1	2
$g(k)$	-5		3	7

Now the jumps in k are equal: x keeps increasing by 1 unit. Therefore, the jumps in $g(k)$ must be equal. It is not hard to see that the jumps in $g(k)$ need to be 4, so that the y-intercept is $g(0) = -1$.

(4) This problem could also be solved by plugging in the given points. I leave the details of this solution to the reader.

12.

Solution by picking numbers: Let's let $b = 2$. Then $f(x) = \left(\frac{1}{32}\right)^x$. Now let's plug in a value for x, say $x = 1$. Then

$$f(3x) = f(3) = \left(\frac{1}{32}\right)^3 \approx \mathbf{.00003}.$$

Put a nice, big, dark circle around this number, and now plug $x = 1$ into each answer choice.

(A) $\sqrt[3]{f(1)} = \sqrt[3]{\frac{1}{32}} \approx .315$

(B) $(f(1))^3 = \left(\frac{1}{32}\right)^3 \approx .00003$

(C) $3f(1) = 3\left(\frac{1}{32}\right) = .09375$

(D) $\frac{1}{3}(f(1)) = \frac{1}{3}\left(\frac{1}{32}\right) \approx .0104.$

We eliminate (A), (C), and (D), and therefore the answer is choice (B).

* **Algebraic solution:** $f(3x) = \left(\frac{1}{b^5}\right)^{3x} = \left(\left(\frac{1}{b^5}\right)^x\right)^3 = \left(f(x)\right)^3$, (B).

OPTIONAL MATERIAL

LEVEL 6: ADVANCED MATH

1. Suppose that z varies directly as x^2 and inversely as y^3. If $z = 9$ when $x = 3$ and $y = 2$, what is y when $z = 4.5$ and $x = 6$?

Solution

1.

* We are given that $z = \frac{kx^2}{y^3}$ for some constant k. Since $z = 9$ when $x = 3$ and $y = 2$, we have $9 = \frac{k(3)^2}{2^3} = \frac{9k}{8}$. So $k = 8$, and $z = \frac{8x^2}{y^3}$. We now substitute $z = 4.5$, $x = 6$ to get $4.5 = \frac{8(6)^2}{y^3}$. So $y^3 = \frac{8(36)}{4.5} = 64$, and therefore $y = \mathbf{4}$.

LESSON 4
STATISTICS

Change Averages to Sums

A problem involving averages often becomes much easier when we first convert the averages to sums. We can easily change an average to a sum using the following simple formula.

Sum = Average · Number

Many problems with averages involve one or more conversions to sums, followed by a subtraction.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 3: STATISTICS

1. The average of x, y, z, and w is 15 and the average of z and w is 11. What is the average of x and y?

*** Solution by changing averages to sums:** The Sum of x, y, z, and w is $15 \cdot 4 = 60$. The Sum of z and w is $11 \cdot 2 = 22$. Thus, the Sum of x and y is $60 - 22 = 38$. Finally, the Average of x and y is $\frac{38}{2} = \mathbf{19}$.

Notes: (1) We used the formula "**Sum = Average · Number"** twice here.
(2) More formally we have the following.

$$\begin{array}{rr} x + y + z + w & = 60 \\ \underline{\qquad\quad z + w} & \underline{= 22} \\ x + y \qquad\quad & = 38 \end{array}$$

Thus, $\frac{x+y}{2} = \frac{38}{2} = \mathbf{19}$.

Before we go on, try to solve this problem in two other ways.

(1) By "Picking Numbers"
(2) Algebraically (the way you would do it in school)

Solution by picking numbers: Let's let $z = w = 11$ and $x = y = 19$. Note that the average of x, y, z, and w is 15 and the average of z and w is 11. Now just observe that the average of x and y is $\mathbf{19}$.

Remarks: (1) If all numbers in a list are all equal, then the average of these numbers is that number as well.

(2) When choosing numbers to form a certain average, just "balance" these numbers around the average. In this example we chose z and w to be 11. Since 11 is 4 less than the average, we chose x and y to be 4 greater than the average.

Algebraic solution: We are given that $\frac{x+y+z+w}{4} = 15$ and $\frac{z+w}{2} = 11$. We multiply each side of the first equation by 4 and each side of the second equation by 2 to eliminate the denominators. Then we subtract the second equation from the first.

$$\begin{array}{r} x + y + z + w = 60 \\ \underline{\qquad\qquad z + w = 22} \\ x + y \qquad\quad = 38 \end{array}$$

Finally, the average of x and y is $\frac{x+y}{2} = \frac{38}{2} = \mathbf{19}$.

Important note: You should avoid this method on the actual SAT. It is too time consuming.

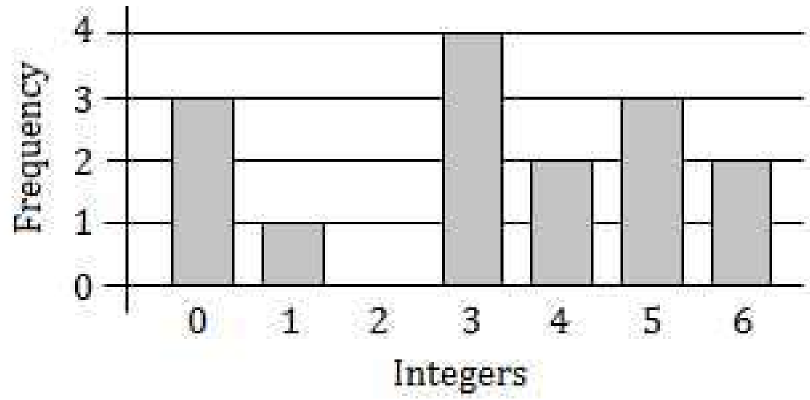
Now try to solve each of the following problems. Change averages to sums whenever possible. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 1: STATISTICS

2. The average (arithmetic mean) of four numbers is 85. If three of the numbers are 17, 58 and 83, what is the fourth number? 185

LEVEL 2: STATISTICS

3. The average (arithmetic mean) of z, 2, 16, and 21 is z. What is the value of z? 13



4. * The graph above shows the frequency distribution of a list of randomly generated integers between 0 and 6. What is the mean of the list of numbers? 3,2

LEVEL 3: STATISTICS

5. The mean length of a pop song released in the 1980's was 4 minutes and 8 seconds. The mean length of a pop song released in the 1990's was 4 minutes and 14 seconds. Which of the following must be true about the mean length of a pop song released between 1980 and 1999?

$4\frac{2}{15}$ $4\frac{7}{30}$ $4\frac{4}{30}$ $4\frac{7}{30}$

(A) The mean length must be equal to 4 minutes and 11 seconds.
(B) The mean length must be less than 4 minutes and 11 seconds.
(C) The mean length must be greater than 4 minutes and 11 seconds.
(D) The mean length must be between 4 minutes and 8 seconds and 4 minutes and 14 seconds.

LEVEL 4: STATISTICS

6. If the average (arithmetic mean) of a, b, and 23 is 12, what is the average of a and b?

$a+b+23=36$

$\frac{13}{2}$

(A) 6.5
(B) 11
(C) 13
(D) It cannot be determined from the information given.

$\frac{T+P}{n} = 35$ $\quad$ $\frac{T+(P+12)}{n} = 37$

$T+P = 35n$ $\quad$ $T+P+12 = 37n$

7. The average (arithmetic mean) age of the people in a certain group was 35 years before one of the members left the group and was replaced by someone who is 12 years older than the person who left. If the average age of the group is now 37 years, how many people are in the group? 6

$35n+12=37n$

$2n=12$

$n=6$

8. The average (arithmetic mean) of 11 numbers is j. If one of the numbers is k, what is the average of the remaining 10 numbers in terms of j and k?

(A) $\frac{k}{11}$

(B) $11j + k$

(C) $\frac{10j-k}{11}$

(D) $\frac{11j-k}{10}$

9. The average (arithmetic mean) of a, $2a$, b, and $4b$ is $2a$. What is b in terms of a?

(A) $\frac{a}{4}$

(B) $\frac{a}{2}$

(C) a

(D) $\frac{3a}{2}$

$3a+5b = 8a$

$5b = 5a$

LEVEL 5: STATISTICS

10. If $h = a + b + c + d + e + f + g$, what is the average (arithmetic mean) of a, b, c, d, e, f, g and h in terms of h?

(A) $\frac{h}{2}$

(B) $\frac{h}{3}$

(C) $\frac{h}{4}$

(D) $\frac{h}{5}$

$\frac{h}{7}$

$\frac{T}{n} = 72$ $\quad$ $\frac{T+88}{n+1} = 76$

$T - 72n = 0$ $\quad$ $T + 88 = 76n + 76$

$T - 76n = -12$

11. A group of students takes a test and the average score is 72. One more student takes the test and receives a score of 88 increasing the average score of the group to 76. How many students were in the initial group? 3

12. The average (arithmetic mean) salary of employees at an advertising firm with P employees in thousands of dollars is 53, and the average salary of employees at an advertising firm with Q employees in thousands of dollars is 95. When the salaries of both firms are combined, the average salary in thousands of dollars is 83. What is the value of $\frac{P}{Q}$? $\frac{2}{5}$

$53P + 95Q = 83$ $\quad$ $\frac{T_P}{P} = 53$ $\quad$ $P = \frac{T_P}{53}$ $\quad$ $Q = \frac{T_Q}{95}$

Answers

1. 19	5. D	9. C
2. 182	6. A	10. C
3. 13	7. 6	11. 3
4. 16/5 or 3.2	8. D	12. 2/5 or .4

Full Solutions

5.

Solution by picking numbers: Let's suppose that there are 2 pop songs from the 80's each with length 4 minutes and 8 seconds, and 4 pops songs from the 90's each with length 4 minutes and 14 seconds. Note that the given conditions are satisfied, and the combined mean is 4 minutes and 12 seconds. This eliminates choices (A) and (B).

Now let's reverse the situation, and suppose that there are 4 pop songs from the 80's each with length 4 minutes and 8 seconds, and 2 pops songs from the 90's each with length 4 minutes and 14 seconds. Note once again that the given conditions are satisfied, and the combined mean is 4 minutes and 10 seconds. This eliminates choice (C).

Since we have eliminated choices (A), (B), and (C), the answer is choice (D).

Notes: (1) The combined mean length would only be equal to 4 minutes and 11 seconds if the same exact number of pops songs were released in both the 80's and 90's.

(2) It is not true, of course, that there were only 2 pop songs released in the 80's and 4 pop songs released in the 90's. Nonetheless, we can use these simple numbers to eliminate answer choices.

(3) To actually compute the means above we use the formula

$$\text{Mean} = \frac{\text{Sum}}{\text{Number}}$$

where "Sum" is the sum of all the data, and "Number" is the amount of data. In each of the examples above there are 6 pieces of data so that the Number is 6.

(4) When computing the various means in the solution above, we need only worry about the seconds since the minutes are the same. For example, in the first paragraph, we can pretend that our data is 8, 8, 14, 14, 14, 14. The mean of this data is $\frac{8+8+14+14+14+14}{6} = \frac{72}{6} = 12$. It follows that the combined mean is 4 minutes and 12 seconds.

*** Direct solution:** Let's let a be the mean length of a pop song released in the 1980's, and b be the mean length of a pop song released in the 1990's. Since $a < b$, it follows that the combined mean m must satisfy $a < m < b$. That is, the combined mean must be between 4 minutes and 8 seconds and 4 minutes and 14 seconds, choice (D).

6.

*** Solution by changing averages to sums:** The Sum of the 3 numbers is $12 \cdot 3 = 36$. Thus $a + b + 23 = 36$, and it follows that $a + b = 13$. So the Average of a and b is $\frac{13}{2} = 6.5$, choice (A).

Solution by picking numbers: Let's let $a = 1$ and $b = 12$. We make this choice because 1 and 23 are both 11 units from 12. Then the Average of a and b is $\frac{a+b}{2} = \frac{1+12}{2} = \frac{13}{2} = 6.5$, choice (A).

7.

*** Solution by changing averages to sums:** Let n be the number of people in the group. Then originally the sum of the ages of the people in the group was $35n$. After the replacement, the new sum became $37n$. So we have

$$37n = 35n + 12$$
$$2n = 12$$
$$n = \mathbf{6}.$$

8.

*** Solution by changing averages to sums:** The Sum of the 11 numbers is $11j$. The Sum of the remaining 10 numbers (after removing k) is $11j - k$. So the Average of the remaining 10 numbers is $\frac{11j-k}{10}$, choice (D).

9.

*** Solution by changing averages to sums:** Converting the Average to a Sum we have that $a + 2a + b + 4b = (2a)(4)$. That is $3a + 5b = 8a$. Subtracting $3a$ from each side of this equation yields $5b = 5a$. Finally, we divide each side of this last equation by 5 to get $b = a$, choice (C).

10.

*** Solution by changing averages to sums:** The average of a, b, c, d, e, f, g and h is

$$\frac{a+b+c+d+e+f+g+h}{8}$$
$$=\frac{a+b+c+d+e+f+g+a+b+c+d+e+f+g}{8}$$
$$=\frac{2a+2b+2c+2d+2e+2f+2g}{8}$$
$$=\frac{2(a+b+c+d+e+f+g)}{8}$$
$$=\frac{2h}{8}$$
$$=\frac{h}{4}$$

This is choice (C).

Alternate solution by picking numbers: Let's let $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 6$, and $g = 7$. Then $h = 28$, and the average of a, b, c, d, e, f, g and h is $\frac{1+2+3+4+5+6+7+28}{8} = \frac{56}{8} = \mathbf{7}$. Put a nice big, dark circle around this number. Now plug $h = 28$ in to each answer choice.

(A) 14
(B) 9.3333…
(C) 7
(D) 5.6

Since (A), (B), and (D) are incorrect we can eliminate them. Therefore, the answer is choice (C).

11.

*** Solution by changing averages to sums:** Let n be the number of students in the initial group. Then the Sum of the scores is $72n$.

When we take into account the new student, we can find the new sum in two different ways.

(1) We can add the new score to the old sum to get $72n + 88$.

(2) We can compute the new sum directly using the formula to get $76(n + 1) = 76n + 76$.

We now set these equal to each other and solve for *n*:

$$72n + 88 = 76n + 76$$
$$12 = 4n$$
$$n = \frac{12}{4} = \mathbf{3}.$$

12.

*** Solution by changing averages to sums:** The Sum of the salaries of employees at firm P (in thousands) is $53P$.

The Sum of the salaries of employees at firm Q (in thousands) is $95Q$.

Adding these we get the Sum of the salaries of all employees (in thousands): $53P + 95Q$.

We can also get this sum directly from the problem.

$$83(P + Q) = 83P + 83Q.$$

So we have that $53P + 95Q = 83P + 83Q$.

We get P to one side of the equation by subtracting $53P$ from each side, and we get Q to the other side by subtracting $83Q$ from each side.

$$12Q = 30P$$

We can get $\frac{P}{Q}$ to one side by performing **cross division.** We do this just like cross multiplication, but we divide instead. Dividing each side of the equation by $30Q$ will do the trick (this way we get rid of Q on the left and 30 on the right).

$$\frac{P}{Q} = \frac{12}{30} = \mathbf{\frac{2}{5}}$$

So we can grid in **2/5** or **.4**.

LESSON 5
HEART OF ALGEBRA

Reminder: Before beginning this lesson remember to redo the problems from Lesson 1 that you have marked off. Do not "unmark" a question unless you get it correct.

Try a Simple Operation

Problems that ask for an expression involving more than one variable often look much harder than they are. By performing a single operation, the problem is often reduced to one that is very easy to solve. The most common operations to try are addition, subtraction, multiplication and division.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 5: HEART OF ALGEBRA

1. If $rs = 4$, $st = 7$, $rt = 63$, and $r > 0$, then $rst =$ 42

Solution by trying a simple operation: The operation to use here is multiplication.

$rs = 4$
$st = 7$
$rt = 63$
$r^2 s^2 t^2 = 1760$

$$rs = 4$$
$$st = 7$$
$$\underline{rt = 63}$$
$$(rs)(st)(rt) = 4 \cdot 7 \cdot 63$$
$$r^2 s^2 t^2 = 1764$$

Notice that we multiply all three left hand sides together, and all three right hand sides together. Now just take the square root of each side of the equation to get $rst = \mathbf{42}$.

Remark: Whenever we are trying to find an expression that involves multiplication, division, or both, **multiplying or dividing** the given equations usually does the trick.

*** Quick computation:** With a little practice, we can get the solution to this type of problem very quickly. Here, we multiply the three numbers together to get $4 \cdot 7 \cdot 63 = 1764$. We then take the square root of 1764 to get $\mathbf{42}$.

Note: If a calculator is not allowed for this problem we can get the answer quickly by rewriting $4 \cdot 7 \cdot 63$ as $2^2 \cdot 3^2 \cdot 7^2$ (this is the prime factorization – note that $4 = 2^2$ and $63 = 3^2 \cdot 7$). We can then take the square root of this product by "forgetting" the exponents and multiplying the 2, 3, and 7: $\sqrt{2^2 \cdot 3^2 \cdot 7^2} = \sqrt{2^2}\sqrt{3^2}\sqrt{7^2} = 2 \cdot 3 \cdot 7 = 42$.

Before we go on, try to solve this problem by first finding r, s and t.

Important note: You should not solve this problem this way on the actual SAT.

Solving the first equation for s gives us $s = \frac{4}{r}$. Substituting this into the second equation gives us $\left(\frac{4}{r}\right)t = 7$, or equivalently $4t = 7r$. Therefore, we have that $t = \frac{7r}{4}$. So the third equation becomes $r(\frac{7r}{4}) = 63$, or equivalently $\frac{7r^2}{4} = 63$. So $r^2 = \frac{63 \cdot 4}{7} = 36$, whence $r = 6$ (because $r > 0$). It follows that $t = \frac{7 \cdot 6}{4} = 10.5$, $s = \frac{4}{6} = \frac{2}{3}$, and $rst = 6 \cdot \left(\frac{2}{3}\right) \cdot 10.5 = \mathbf{42}$.

Systems of Linear Equations

There are many different ways to solve a system of linear equations. We will use an example to demonstrate several different methods.

LEVEL 5: HEART OF ALGEBRA

2. If $2x = 7 - 3y$ and $5y = 5 - 3x$, what is the value of x? ~~50~~ 20

*** Method 1 – elimination:** We begin by making sure that the two equations are "lined up" properly. We do this by adding $3y$ to each side of the first equation, and adding $3x$ to each side of the second equation.

$$2x + 3y = 7$$
$$3x + 5y = 5$$

We will now multiply each side of the first equation by 5, and each side of the second equation by -3.

$$5(2x+3y)=(7)(5)$$
$$-3(3x+5y)=(5)(-3)$$

Do not forget to distribute correctly on the left. Add the two equations.

$$\begin{array}{r} 10x+15y=35 \\ \underline{-9x-15y=-15} \\ x \qquad\quad = \mathbf{20} \end{array}$$

Remarks: (1) We chose to use 5 and -3 because multiplying by these numbers makes the y column "match up" so that when we add the two equations in the next step the y term vanishes. We could have also used -5 and 3.

(2) If we wanted to find y instead of x we would multiply the two equations by 3 and -2 (or -3 and 2). In general, if you are looking for only one variable, try to eliminate the one you are **not** looking for.

(3) We chose to multiply by a negative number so that we could add the equations instead of subtracting them. We could have also multiplied the first equation by 5, the second by 3, and subtracted the two equations, but a computational error is more likely to occur this way.

Method 2 – Gauss-Jordan reduction: As in method 1, we first make sure the two equations are "lined up" properly.

$$2x+3y=7$$
$$3x+5y=5$$

Begin by pushing the MATRIX button (which is 2ND x^{-1}). Scroll over to EDIT and then select [A] (or press 1). We will be inputting a 2×3 matrix, so press 2 ENTER 3 ENTER. We then begin entering the numbers 2, 3, and 7 for the first row, and 3, 5, and 5 for the second row. To do this we can simply type 2 ENTER 3 ENTER 7 ENTER 3 ENTER 5 ENTER 5 ENTER.

Note: What we have just done was create the **augmented matrix** for the system of equations. This is simply an array of numbers which contains the coefficients of the variables together with the right hand sides of the equations.

Now push the QUIT button (2ND MODE) to get a blank screen. Press MATRIX again. This time scroll over to MATH and select rref((or press B). Then press MATRIX again and select [A] (or press 1) and press ENTER.

Note: What we have just done is put the matrix into **reduced row echelon form**. In this form we can read off the solution to the original system of equations.

Warning: Be careful to use the rref(button (2 r's), and not the ref(button (which has only one r).

The display will show the following.

$$\begin{matrix}[\,[1\ 0\ 20] \\ \ [0\ 1\ -11]\,]\end{matrix}$$

The first line is interpreted as $x = 20$ and the second line as $y = -11$. In particular, $x = \mathbf{20}$.

Method 3 – substitution: We solve the second equation for y and substitute into the first equation.

$5y = 5 - 3x$ implies $y = \frac{5-3x}{5} = \frac{5}{5} - \frac{3x}{5} = 1 - \frac{3x}{5}$. So now using the first equation we have

$$2x = 7 - 3y = 7 - 3\left(1 - \frac{3x}{5}\right) = \ 7 - 3 + \frac{9x}{5} = 4 + \frac{9x}{5}.$$

Multiply each side of this equation by 5 to get rid of the denominator on the right. So we have $10x = 20 + 9x$, and therefore $x = \mathbf{20}$.

Remark: If we wanted to find y instead of x we would solve the first equation for x and substitute into the second equation.

Method 4 – graphical solution: We begin by solving each equation for y.

$$\begin{array}{ll} 2x = 7 - 3y & \quad 5y = 5 - 3x \\ 2x - 7 = -3y & \quad y = 1 - \frac{3x}{5} \\ y = -\frac{2x}{3} + \frac{7}{3} & \end{array}$$

In your graphing calculator press the Y= button, and enter the following.

Y1 = -2X/3 + 7/3
Y2 = 1 – 3X/5

Now press ZOOM 6 to graph these two lines in a standard window. It looks like the point of intersection of the two lines is off to the right. So we will need to extend the viewing window. Press the WINDOW button, and change Xmax to 50 and Ymin to -20. Then press 2nd TRACE (which is CALC) 5 (or select INTERSECT). Then press ENTER 3 times. You will see that the x-coordinate of the point of intersection of the two lines is $\mathbf{20}$.

Remark: The choices made for Xmax and Ymin were just to try to ensure that the point of intersection would appear in the viewing window. Many other windows would work just as well.

You're doing great! Let's just practice a bit more. Try to solve each of the following problems. Try a Simple Operation whenever you can. Then, if possible, solve each problem another way. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 2: HEART OF ALGEBRA

3. If $x + 5y = 25$ and $x + 9y = 11$, what is the value of $x + 7y$? 18

$\frac{85}{2}$ $-\frac{7}{2}$

$$2(x - 5) = y$$
$$\frac{y}{x} = 7$$

4. If (x, y) is the solution to the system of equations above, what is the value of xy ? 28

LEVEL 3: HEART OF ALGEBRA

5. If $x + y = 5$ and $z + w = 7$, then $xz + xw + yz + yw =$ 35

$7x + 7y$

$$5a + 2y + 3z = 23$$
$$5a + y + 2z = 15$$

6. If the equations above are true, what is the value of $y + z$? 8

$y + z = 8$

$$\begin{array}{r} 6z \\ 8 \\ 2t \\ 3 \\ +9 \\ \hline 52 \end{array} \qquad\qquad \begin{array}{r} 2z \\ 8 \\ w \\ 3 \\ +9 \\ \hline 34 \end{array}$$

$ac = \frac{7}{3}$

7. In the correctly worked addition problems above, what is the value of $4z + 2t - w$

8. If $ab = 7, bc = \frac{1}{9}, b^2 = 3$, what is the value of ac ?

LEVEL 4: HEART OF ALGEBRA

$$ax + by = 17$$
$$ax + (b + 1)y = 26$$

9. Based on the equations above, which of the following must be true?

$y = 9$

(A) $x = 13.5$
(B) $x = 18$
(C) $y = 4.5$
(D) $y = 9$

LEVEL 5: HEART OF ALGEBRA

$$k = a - b + 12$$
$$k = b - c - 17$$
$$k = c - a + 11$$

$k = b - c - 17$

10. In the system of equations above, what is the value of k ?

2

11. If $x^{15} = \frac{2}{z}$ and $x^{14} = \frac{2y}{z}$ which of the following is an expression for x in terms of y ?

(A) $3y$
(B) $2y$
(C) y
(D) $\frac{1}{y}$

$\frac{2y}{z} = \frac{2}{zx}$

$yx = 1$

$x = \frac{1}{y}$

12. If $6x = 2 + 4y$ and $7x = 3 - 3y$, what is the value of x?

9/23

Answers

1. 42	5. 35	9. D
2. 20	6. 8	10. 2
3. 18	7. 18	11. D
4. 28	8. 7/27 or .259	12. 9/23 or .391

Full Solutions

5.

Solution by trying a simple operation: The operation to use here is multiplication.

$$\begin{array}{c} x + y = 5 \\ \underline{z + w = 7} \\ (x + y)(z + w) = 35 \\ xz + xw + yz + yw = \mathbf{35} \end{array}$$

*** Quick computation:** Since $(x + y)(z + w) = xz + xw + yz + yw$, the answer is $(5)(7) = \mathbf{35}$.

Solution by picking numbers: Let's choose values for x, y, z, and w that satisfy the given conditions, say $x = 1$, $y = 4$, $z = 2$, $w = 5$. Then we have

$$xz + xw + yz + yw$$

$$= (1)(2) + (1)(5) + (4)(2) + (4)(5)$$

$$= 2 + 5 + 8 + 20 = \mathbf{35}.$$

6.

*** Solution by trying a simple operation:** The operation to use here is subtraction.

$$\begin{array}{c} 5a + 2y + 3z = 23 \\ \underline{5a + y + 2z = 15} \\ y + z = \mathbf{8} \end{array}$$

Remark: Whenever we are trying to find an expression that involves addition, subtraction, or both, **adding or subtracting** the given equations usually does the trick.

7.

Solution by trying a simple operation: Let's rewrite the equations horizontally since that is how most of us are used to seeing equations.

$$\begin{array}{c} 6z + 8 + 2t + 3 + 9 = 52 \\ 2z + 8 + w + 3 + 9 = 34 \end{array}$$

The operation to use here is subtraction. Let's go ahead and subtract term by term.

$$\begin{array}{r} 6z + 8 + 2t + 3 + 9 = 52 \\ \underline{2z + 8 + w + 3 + 9 = 34} \\ 4z + (2t - w) \qquad = \ 18 \end{array}$$

So $4z + 2t - w = \mathbf{18}$.

*** Visualizing the answer:** You can save a substantial amount of time by performing the subtraction in your head (left equation minus right equation). Note that above the lines the subtraction yields $4z + 2t - w$. This is exactly what we are looking for. Thus, we need only subtract below the lines to get the answer: $52 - 34 = \mathbf{18}$.

Solution by picking numbers: If we choose any value for z, then t and w will be determined. So, let's set z equal to 0. Then

$$\begin{array}{rr} 8 + 2t + 3 + 9 = 52 & \quad 8 + w + 3 + 9 = 34 \\ 20 + 2t = 52 & 20 + w = 34 \\ 2t = 32 & w = 14 \\ t = 16 & \end{array}$$

So $4z + 2t - w = 0 + 2(16) - 14 = 32 - 14 = \mathbf{18}$.

Remark: Any choice for z will give us the same answer. We could have chosen a value for t or w as well. But once we choose a value for one of the variables the other two are determined.

8.

*** Solution by trying a simple operation:** The operation to use here is multiplication.

$$\begin{array}{c} ab = 7 \\ \underline{bc = \frac{1}{9}} \\ (ab)(bc) = (7)(\frac{1}{9}) \\ ab^2c = \frac{7}{9} \end{array}$$

Now substitute 3 in for b^2. So we have $a(3)c = \frac{7}{9}$. Dividing each side of the equation by 3 gives us $ac = \frac{1}{3} \cdot \frac{7}{9} = \mathbf{7/27}$ or $\mathbf{.259}$.

9.

*** Solution by trying a simple operation:** First multiply out the second term on the left hand side of the second equation to get

$$ax + by + y = 26.$$

Now subtract the first equation from the second equation.

$$\begin{array}{r} ax + by + y = 26 \\ \underline{ax + by \qquad = 17} \\ y = 9 \end{array}$$

We see that the answer is choice (D).

10.

*** Solution by trying a simple operation:** Notice that when we add the three given equations, all the variables on the right hand side add to zero. So we have $3k = 12 - 17 + 11 = 6$. Therefore $k = \mathbf{2}$.

11.

*** Solution by trying a simple operation:** The operation to use here is division. We divide the left hand sides of each equation, and the right hand sides of each equation. First the left: Recall that when we divide expressions with the same base we need to subtract the exponents. So $\frac{x^{15}}{x^{14}} = x^1 = x$. Now for the right: Recall that dividing is the same as multiplying by the reciprocal. So, $\frac{2}{z} \div \frac{2y}{z} = \frac{2}{z} \cdot \frac{z}{2y} = \frac{1}{y}$. Thus, $x = \frac{1}{y}$ and the answer is choice (D).

Alternate Solution: Multiply each side of each equation by z to get

$$zx^{15} = 2 \qquad\qquad zx^{14} = 2y$$

Multiplying each side of the second equation by x yields $zx^{15} = 2xy$. So $2xy = 2$, and thus, $xy = 1$, and therefore $x = \frac{1}{y}$, choice (D).

12.

*** Solution using the elimination method:** Since we are trying to find x, we want to make y go away. So we make the two coefficients of y "match up" by multiplying by the appropriate numbers. We will multiply the first equation by 3 and the second equation by 4.

$$3(6x) = (2 + 4y)(3)$$
$$4(7x) = (3 - 3y)(4)$$

Don't forget to distribute on the right. Then add the two equations.

$$18x = 6 + 12y$$
$$\underline{28x = 12 - 12y}$$
$$46x = 18$$

Now divide each side by 46 to get $x = \mathbf{9/23}$ or $\mathbf{.391}$.

OPTIONAL MATERIAL

LEVEL 6: HEART OF ALGEBRA

1. If x and y are positive real numbers with $x^8 = \frac{z^3}{16}$ and $x^{12} = \frac{z^7}{y^4}$, what is the value of $\frac{xy}{z}$?

2. If $2x + 3y - 4z = 2$, $x - y + 5z = 6$ and $3x + 2y - z = 4$, what is the value of y ?

Solutions

1.

* $x^4 = \frac{x^{12}}{x^8} = \frac{z^7}{y^4} \div \frac{z^3}{16} = \frac{z^7}{y^4} \cdot \frac{16}{z^3} = \frac{16z^4}{y^4}$. So $x = \frac{2z}{y}$, and therefore $\frac{xy}{z} = \mathbf{2}$.

2.

*** Solution using Gauss-Jordan reduction:** Push the MATRIX button, scroll over to EDIT and then select [A] (or press 1). We will be inputting a 3×4 matrix, so press 3 ENTER 4 ENTER. Then enter the numbers 2, 3, −4 and 2 for the first row, 1, −1, 5 and 6 for the second row, and 3, 2, −1 and 4 for the third row.

Now push the QUIT button (2ND MODE) to get a blank screen. Press MATRIX again. This time scroll over to MATH and select rref((or press B). Then press MATRIX again and select [A] (or press 1) and press ENTER.

The display will show the following.

$$\begin{array}{l} [\,[1\ 0\ 0\ \ -.4] \\ \ [0\ 1\ 0\ \ \ 3.6\,] \\ \ [0\ 0\ 1\ \ \ \ \ \ 2]] \end{array}$$

The second line is interpreted as $y = \mathbf{3.6}$.

LESSON 9
HEART OF ALGEBRA

Reminder: Before beginning this lesson remember to redo the problems from Lessons 1 and 5 that you have marked off. Do not "unmark" a question unless you get it correct.

Complex Numbers

A **complex number** has the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$.

Example: The following are complex numbers:

$$2 + 3i \quad \frac{3}{2} + (-2i) = \frac{3}{2} - 2i \quad -\pi + 2.6i \quad \sqrt{-9} = 3i$$

$0 + 5i = 5i$ This is called a **pure imaginary** number.

$17 + 0i = 17$ This is called a **real number.**

$0 + 0i = 0$ This is **zero**.

Powers of $\boldsymbol{i}$: Since $i = \sqrt{-1}$, we have the following:

$i^2 = \sqrt{-1}\sqrt{-1} = -1$

$i^3 = i^2 i = -1i = -i$

$i^4 = i^2 i^2 = (-1)(-1) = 1$

$i^5 = i^4 i = 1i = i$

Notice that the pattern begins to repeat.

Starting with $i^0 = 1$, we have

$i^0 = 1$	$i^1 = i$	$i^2 = -1$	$i^3 = -i$
$i^4 = 1$	$i^5 = i$	$i^6 = -1$	$i^7 = -i$
$i^8 = 1$	$i^9 = i$	$i^{10} = -1$	$i^{11} = -i$

$\dots$

In other words, when we raise i to a nonnegative integer, there are only four possible answers:

$$1, i, -1, \text{ or } -i$$

To decide which of these values is correct, we can find the remainder upon dividing the exponent by 4.

Example: $i^{73} = i^1 = i$ because when we divide 73 be 4 we get a remainder of 1.

Notes: (1) To get the remainder upon dividing 73 by 4, you **cannot** simply divide 73 by 4 in your calculator. This computation produces the answer 18.75 which does not tell you anything about the remainder.

To find a remainder you must either do the division by hand, or use the Calculator Algorithm below.

(2) This computation can also be done quickly in your calculator, but be careful. Your calculator may sometimes "disguise" the number 0 with a tiny number in scientific notation. For example, when we type i ^ 73 ENTER into our TI-84, we get an output of −2.3E−12 + i. The expression −2.3E−12 represents a tiny number in scientific notation which is essentially 0. So this should be read as 0 + i = i.

(3) **Calculator Algorithm for computing a remainder:** Although performing division in your calculator never produces a remainder, there is a simple algorithm you can perform which mimics long division. Let's find the remainder when 73 is divided by 4 using this algorithm.

Step 1: Perform the division in your calculator: 73/4 = 18.25
Step 2: Multiply the integer part of this answer by the divisor: 18*4 = 72
Step 3: Subtract this result from the dividend to get the remainder:

73 – 72 = **1**.

Addition and subtraction: We add two complex numbers simply by adding their real parts, and then adding their imaginary parts.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

LEVEL 1: HEART OF ALGEBRA

1. For $i = \sqrt{-1}$, the sum $(2-3i)+(-5+6i)$ is

   (A) $-7+3i$
   (B) $-7+9i$
   (C) $-3-3i$
   (D) $-3+3i$

* **Solution:** $(2-3i)+(-5+6i)=(2-5)+(-3+6)i=-3+3i$, choice (D).

Multiplication: We can multiply two complex numbers by formally taking the product of two binomials and then replacing i^2 by -1.

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

LEVEL 3: HEART OF ALGEBRA

2. Which of the following complex numbers is equivalent to $(2-3i)(-5+6i)$? (Note: $i=\sqrt{-1}$)

   (A) $-7+3i$
   (B) $-7+9i$
   (C) $-3-3i$
   (D) $8+27i$

* **Solution:** $(2-3i)(-5+6i)=(-10+18)+(12+15)i=8+27i$, choice (D).

The **conjugate** of the complex number $a+bi$ is the complex number $a-bi$.

Example: The conjugate of $-5+6i$ is $-5-6i$.

Note that when we multiply conjugates together we always get a real number. In fact, we have

$$(a+bi)(a-bi)=a^2+b^2$$

Division: We can put the quotient of two complex numbers into standard form by multiplying both the numerator and denominator by the conjugate of the denominator. This is best understood with an example.

LEVEL 4: HEART OF ALGEBRA

$$\frac{1+5i}{2-3i}$$

3. If the expression above is rewritten in the form $a + bi$, where a and b are real numbers, what is the value of $b - a$?

Solution: We multiply the numerator and denominator of $\frac{1+5i}{2-3i}$ by $(2 + 3i)$ to get

$$\frac{(1+5i)}{(2-3i)} \cdot \frac{(2+3i)}{(2+3i)} = \frac{(2-15)+(3+10)i}{4+9} = \frac{-13+13i}{13} = -\frac{13}{13} + \frac{13}{13}i = -1 + i$$

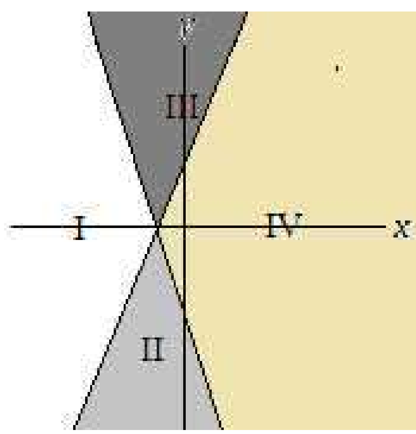
So $a = -1$, $b = 1$, and $b - a = 1 - (-1) = 1 + 1 = \mathbf{2}$.

Systems of Linear Inequalities

Let's use an example to see how to solve a system of linear inequalities.

LEVEL 5: HEART OF ALGEBRA

$$y \leq 2x + 2$$
$$y \geq -3x - 3$$



4. A system of inequalities and a graph are shown above. Which section or sections of the graph could represent all of the solutions to the system?

   (A) Section I
   (B) Section IV
   (C) Sections II and III
   (D) Sections I, II, and IV

*** Quick solution:** The line $y = 2x + 2$ has a slope of $2 > 0$, and therefore the graph is the line that moves upwards as it is drawn from left to right.

The point $(0,0)$ satisfies the inequality $y \leq 2x + 2$ since $0 \leq 2(0) + 2$, or equivalently $0 \leq 2$ is true.

It follows that the graph of $y \leq 2x + 2$ consists of sections II and IV.

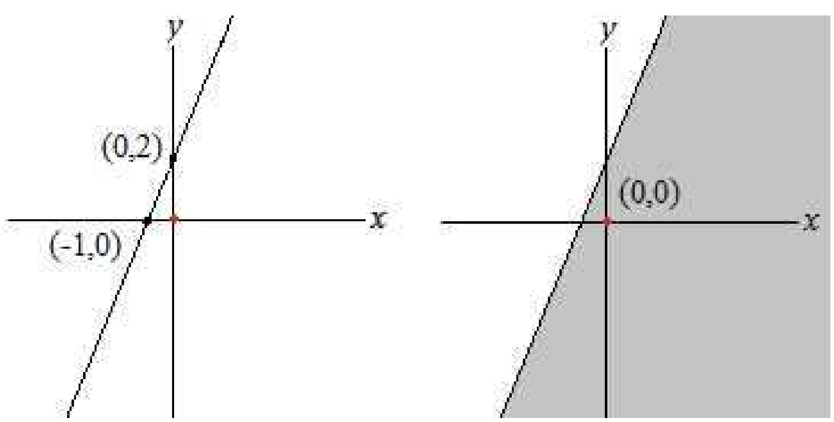
The line $y = -3x - 3$ has a slope of $-3 < 0$, and therefore the graph is a line that moves downwards as it is drawn from left to right.

$(0,0)$ satisfies the inequality $y \geq -3x - 3$ since $0 \geq -3(0) - 3$, or equivalently $0 \geq -3$ is true.

It follows that the graph of $y \geq -3x - 3$ consists of sections III and IV.

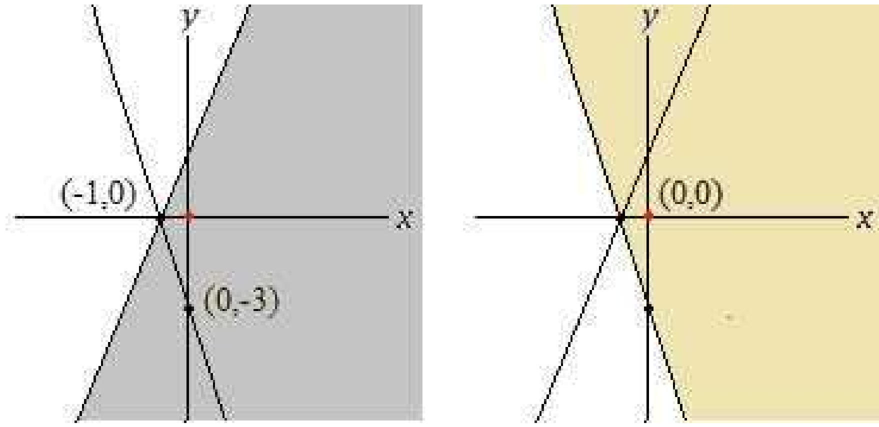
The intersection of the two solution graphs is section IV, choice (B).

Complete algebraic solution: Let's sketch each inequality, one at a time, starting with $y \leq 2x + 2$. We first sketch the line $y = 2x + 2$. There are several ways to do this. A quick way is to plot the two intercepts. We get the y-intercept by setting $x = 0$. In this case we get $y = 2 \cdot 0 + 2 = 2$. So the point $(0,2)$ is on the line. We get the x-intercept by setting $y = 0$. In this case we get $0 = 2x + 2$, so that $-2 = 2x$, and $x = -\frac{2}{2} = -1$. So the point $(-1,0)$ is on the line. This line is shown in the figure on the left below.



Now we need to figure out which direction to shade. To do this we plug any point *not on the line* into the inequality. For example, we can use $(0,0)$. Substituting this point into $y \leq 2x + 2$ gives $0 \leq 2$. Since this expression is true, we shade the region that includes $(0,0)$ as shown above in the figure on the right.

We now do the same thing for the second inequality. The intercepts of $y = -3x - 3$ are $(0, -3)$ and $(-1, 0)$. When we test $(0,0)$ we get the true statement $0 \geq -3$.



The figure on the above left shows the graph of $y = -3x - 3$ with the intercepts plotted, and the graph on the right shows the solution set of $y \geq -3x - 3$ (the shaded part).

The intersection of the two shaded regions in both figures above is the solution of the system of inequalities. This is region IV, choice (B).

Now try to solve each of the following problems. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 1: HEART OF ALGEBRA

5. If $-\frac{27}{10} < 2 - 5x < -\frac{13}{5}$, then give one possible value of $20x - 8$. -8

LEVEL 2: HEART OF ALGEBRA

6. When we subtract $2 - 3i$ from $-5 + 6i$ we get which of the following complex numbers?

   (A) $-7 + 3i$
   (B) $-7 + 9i$
   (C) $-3 - 3i$
   (D) $-3 + 3i$

LEVEL 3: HEART OF ALGEBRA

7. Michael needs a printing job completed. Photoperfect Print Shop charges a fixed fee of \$3 for the print job and 5 cents per page. Bargain Printing charges a fixed fee of \$2 for the print job and 7 cents per page. If p represents the number of pages being printed, what are all values of p for which Photoperfect Print Shop's total charge is less than Bargain Printing's total charge.

(A) $p < 20$
(B) $20 \leq p \leq 35$
(C) $35 \leq p \leq 50$
(D) $p > 50$

$300 + 5p < 200 + 7p$
$100 < 2p$
$p \; 5$

$$x + k < y$$
$$m - x > y$$

8. In the xy-plane, $(0,0)$ is a solution to the system of inequalities above. Which of the following relationships between k and m must be true?

(A) $k = -m$
(B) $k > m$
(C) $k < m$
(D) $|k| < |m|$

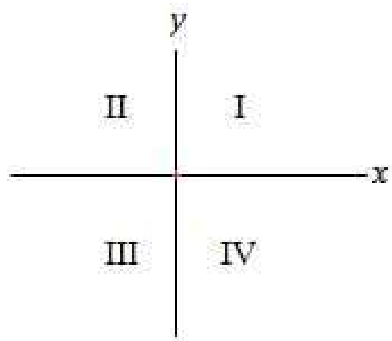
9. If $5x - 2 < 7$, which inequality represents the possible range of values of $30x - 12$?

(A) $42 < 30x - 12$
(B) $42 > 30x - 12$
(C) $32 < 30x - 12$
(D) $32 > 30x - 12$

LEVEL 4: HEART OF ALGEBRA

10. * If $i = \sqrt{-1}$, and $\frac{(3+4i)}{(-5-2i)} = a + bi$, where a and b are real numbers, then what is the value of $|b|$ to the nearest tenth?

LEVEL 5: HEART OF ALGEBRA



11. If the system of inequalities $y < 4x + 1$ and $y \geq -\frac{1}{2}x - 2$ is graphed in the xy-plane above, which quadrant contains no solutions to the system?

    (A) Quadrant I
    (B) Quadrant II
    (C) Quadrant III
    (D) There are solutions in all four quadrants.

$$y \geq -12x + 600$$
$$y \geq 3x$$

$b \geq -12a + 600$
$b \geq 3a$

12. In the xy-plane, if a point with coordinates (a, b) lies in the solution set of the system of inequalities above, what is the minimum possible value of b?

120

Answers

1. D	5. 53/5, 10.5, 10.6, 10.7	9. B
2. D	6. B	10. .5
3. 2	7. D	11. D
4. B	8. C	12.120

Full Solutions

7.

Algebraic solution: Photoperfect Print Shop's total charge for printing p pages is $.05p + 3$ dollars. Bargain Printing's total charge for printing p pages is $.07p + 2$ dollars.

We need to solve the inequality $.05p + 3 < .07p + 2$. Subtracting $.05p$ and subtracting 2 from each side of the inequality gives $1 < .02p$. Dividing by $.02$ yields $\frac{1}{.02} < p$, or $p > \frac{1}{.02} = \frac{100}{2} = 50$, choice (D).

Solution by picking a number: Let's choose a value for p, say $p = 40$ pages. Then Photoperfect Print Shop charges $3 + 40 \cdot .05 = 5$ dollars, and Bargain Printing charges $2 + 40 \cdot .07 = 4$ dollars and 80 cents. So Photoperfect charges *more than* Bargain. Therefore p CANNOT equal 40, and so we can eliminate choice (C).

Let's try $p = 100$ next. Then Photoperfect charges $3 + 100 \cdot .05 = 8$ dollars, and Bargain charges $2 + 100 \cdot .07 = 9$ dollars. So Photoperfect charges *less than* Bargain. Therefore p CAN equal 100, and so we can also eliminate choices (A) and (B).

So the answer is choice (D).

Note: We used the answer choices as a guide here to help pick numbers.

We first looked at choice (C) and chose a value for p between 35 and 50. Since that value of p did not satisfy the conclusion in the problem, we eliminate choice (C) (if it had worked, we would eliminate the other three choices).

We then looked at choice (D) next and chose a value for p greater than 50. This value of p did satisfy the conclusion in the problem allowing us to eliminate choices (A) and (B) (choice (C) is also eliminated here, but we already eliminated it with the last choice of p).

*** Solution by logical reasoning:** Since Photoperfect Print Shop's price per page is less than Bargain Printing's price per page, if we keep increasing the number of pages, then *eventually* Photoperfect's total cost will be less than Bargain Print Shop's price per page from that point on. So the answer must have the form $p > \square$, where $\square$ is some positive integer. The only answer choice of this form is choice (D).

8.

*** Solution by plugging in the point:** We replace x and y by 0 in the first equation to get $0 + k < 0$, or equivalently, $k < 0$.

We then replace x and y by 0 in the second equation to get $m - 0 > 0$, or equivalently, $m > 0$.

So we have $k < 0 < m$, so that $k < m$, choice (C).

9.

*** Solution by trying a simple operation:** We multiply each side of the given inequality by 6 to get

$$6(5x-2)<7$$
$$30x-12<42$$

This last inequality is equivalent to $42>30x-12$, choice (B).

10.

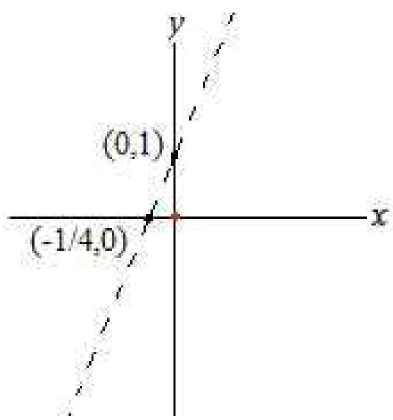
* $\frac{(3+4i)}{(-5-2i)}=\frac{(3+4i)}{(-5-2i)}\cdot\frac{(-5+2i)}{(-5+2i)}=\frac{(-15-8)+(6-20)i}{25+4}=\frac{-23-14i}{29}=-\frac{23}{29}-\frac{14}{29}i$

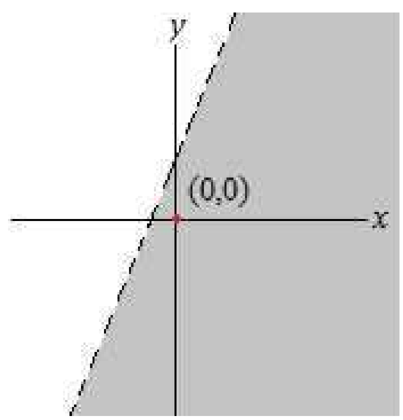
So $b=-\frac{14}{29}$.

Therefore $|b|=\frac{14}{29}\approx .4827586207$. To the nearest tenth this is $\mathbf{.5}$.

11.

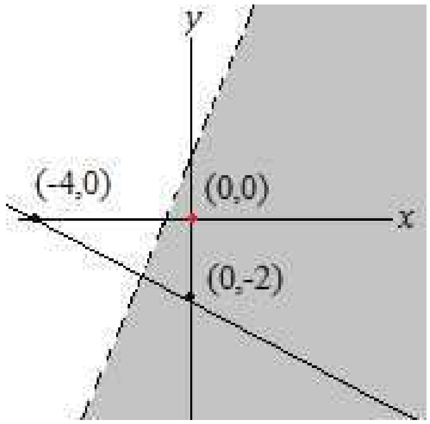
*** Complete algebraic solution:** Let's sketch each inequality, one at a time, starting with $y<4x+1$. We first sketch the line $y=4x+1$ by plotting the two intercepts. We get the y-intercept by setting $x=0$. In this case we get $y=4\cdot 0+1=1$. So the point $(0,1)$ is on the line. We get the x-intercept by setting $y=0$. In this case we get $0=4x+1$, so that $-1=4x$, and $x=-\frac{1}{4}$. So the point $(-\frac{1}{4},0)$ is on the line. This line is shown in the figure on the left below. Note that we draw a dotted line because the strict inequality $<$ tells us that points on this line are not actually solutions to the inequality $y<4x+1$.

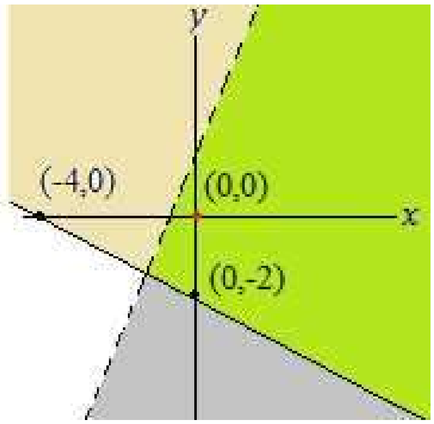




Now we need to figure out which direction to shade. To do this we plug any point *not on the line* into the inequality. For example, we can use $(0,0)$. Substituting this point into $y<4x+1$ gives $0<1$. Since this expression is true, we shade the region that includes $(0,0)$ as shown above in the figure on the right.

We now do the same thing for the second inequality. The intercepts of $y = -\frac{1}{2}x - 2$ are $(0, -2)$ and $(-4{,}0)$. When we test $(0{,}0)$ we get the true statement $0 \geq -2$.





The figure on the above left shows the graph of $y = -\frac{1}{2}x - 2$ with the intercepts plotted, and the graph on the right shows three different shadings. The rightmost shading is the solution set of the given system.

Note that there are solutions in all four quadrants, choice (D).

12.

*** Solution by solving the corresponding system of equations:** We solve the system of equations

$$y = -12x + 600$$
$$y = 3x$$

In Lesson 5 we learned several ways to solve this system. I will do it here by substitution. We have $3x = -12x + 600$, so that $15x = 600$. Therefore $x = \frac{600}{15} = 40$. It follows that $y = 3 \cdot 40 = \mathbf{120}$.

Notes: (1) It's probably not obvious to you that 120 is actually the answer to the question. But it would certainly be a good guess that if a minimum value for b exists, then it would be given by the y-coordinate of the point of intersection of the two lines.

(2) Although a minimum does not necessarily have to exist, since this is an SAT question, we are expected to give an answer. It follows that a minimum must exist, and Note (1) gives us a quick way to find it.

(3) To be certain that 120 is actually the minimum possible value of b, we should sketch the system of inequalities as was done in problems 4 and 11 from this lesson. I leave this as an exercise for the reader.