1

Exponents & Radicals

Here are the laws of exponents you should know:

Law	Example
$x^1 = x$	$3^1 = 3$
$x^0 = 1$	$3^0 = 1$
$x^m \cdot x^n = x^{m+n}$	$3^4 \cdot 3^5 = 3^9$
$\dfrac{x^m}{x^n} = x^{m-n}$	$\dfrac{3^7}{3^3} = 3^4$
$(x^m)^n = x^{mn}$	$(3^2)^4 = 3^8$
$(xy)^m = x^m y^m$	$(2 \cdot 3)^3 = 2^3 \cdot 3^3$
$\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$	$\left(\dfrac{2}{3}\right)^3 = \dfrac{2^3}{3^3}$
$x^{-m} = \dfrac{1}{x^m}$	$3^{-4} = \dfrac{1}{3^4}$

Many students don't know the difference between

$$(-3)^2 \text{ and } -3^2$$

Order of operations (PEMDAS) dictates that parentheses take precedence. So,

$$(-3)^2 = (-3) \cdot (-3) = 9$$

Without parentheses, exponents take precedence:

$$-3^2 = -3 \cdot 3 = -9$$

The negative is not applied until the exponent operation is carried through. Make sure you understand this so you don't make this common mistake. Sometimes, the result turns out to be the same, as in:

$$(-2)^3 \text{ and } -2^3$$

Make sure you see why they yield the same result.

EXERCISE 1: Evaluate WITHOUT a calculator. Answers for this chapter start on page 251.

1. $(-1)^4$ 1
2. $(-1)^5$ -1
3. $(-1)^{10}$ 1
4. $(-1)^{15}$ -1
5. $(-1)^8$ 1
6. -1^8 -1
7. $-(-1)^8$ -1
8. $(-3)^3$ -3
9. -3^3 -9
10. $-(-3)^3$ 3
11. $-(-6)^2$ -6
12. $-(-4)^3$ 4
13. $2^3 \times 3^2 \times (-1)^5$ -64
14. $(-1)^4 \times 3^3 \times 2^2$ 36
15. $(-2)^3 \times (-3)^4$ $-5\ 48$
16. 3^0 1
17. 6^{-1} $\frac{1}{6}$
18. 4^{-1} $\frac{1}{4}$
19. 5^0 1
20. 3^2 9
21. 3^{-2} $\frac{1}{9}$
22. 5^3 125
23. 5^{-3} $\frac{1}{125}$
24. 7^2 49
25. 7^{-2} $\frac{1}{49}$
26. 10^3 1000
27. 10^{-3} $\frac{1}{1000}$

EXERCISE 2: Simplify so that your answer contains only positive exponents. Do NOT use a calculator. The first two have been done for you. Answers for this chapter start on page 251.

1. $3x^2 \cdot 2x^3 = 6x^5$
2. $2k^{-4} \cdot 4k^2 = \dfrac{8}{k^2}$
3. $5x^4 \cdot 3x^{-2}$ $15x^2$
4. $7m^3 \cdot -3m^{-3}$ -21
5. $(2x^2)^{-3}$ $(2x)^{-6}$
6. $-3a^2b^{-3} \cdot 3a^{-5}b^8$ $-9a^{-3} \cdot b^5$
7. $\dfrac{3n^7}{6n^3}$ $\dfrac{n^4}{2}$
8. $(a^2b^3)^2$ (a^4a^6)
9. $\left(\dfrac{xy^4}{x^3y^2}\right)$ $\dfrac{x}{x^3} \cdot \dfrac{y^4}{y^2}$ $\dfrac{1}{x^2} \cdot y^2$
10. $-(-x)^3$
11. $(x^2y^{-1})^3$ $x^6 \cdot \dfrac{1}{y^3}$
12. $\dfrac{6u^4}{8u^2}$ $\dfrac{6}{8}u^2$
13. $2uv^2 \cdot -4u^2v$ $2uv(v-2u)$
14. $\dfrac{x^2}{x^{-3}}$ x^5
15. $\dfrac{3x^4}{(x^{-2})^2}$ $3x^8$
16. $\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$ x
17. $x^2 \cdot x^3 \cdot x^4$ x^9
18. $(x^2)^{-3} \cdot 2x^3$ $\dfrac{1}{x^6} \cdot 2x^3 = \dfrac{2}{x^3}$
19. $(2m)^2 \cdot (3m^3)^2$ $2m^2$ $(6m^4)^2$
20. $(a^{-1} \cdot a^{-2})^2$ $(a)^{-6}$
21. $(b^{-2})^{-3} \cdot (b^3)^2$ 1
22. $\dfrac{(m^2n)^3}{(mn^2)^2}$ $\dfrac{m^6n^3}{m^2n^4} = \dfrac{m^4}{n^{-1}}$
23. $\dfrac{1}{x^{-2}}$ x^2
24. $\dfrac{mn}{m^2n^3}$ $\dfrac{1}{mn^2}$
25. $\dfrac{k^{-2}}{k^{-3}}$ k
26. $\left(\dfrac{m^2}{n^3}\right)^3$ $\dfrac{m^6}{n^9}$
27. $\left(\dfrac{x^2y^3z^4}{x^{-3}y^{-4}z^{-5}}\right)$ $x^5y^7z^9$

EXAMPLE 1: If $3^{x+2} = y$, then what is the value of 3^x in terms of y?

A) $y+9$ B) $y-9$ C) $\dfrac{y}{3}$ D) $\dfrac{y}{9}$

Let's avoid the trouble of finding what x is. Here we notice that the 2 in the exponent is the only difference between the given equation and what we want. So using our laws of exponents, let's extract the 2 out:

$$3^{x+2} = 3^x \cdot 3^2 = y$$

$$3^x = \frac{y}{9}$$

The answer is $\boxed{(D)}$.

EXAMPLE 2: If $3^{a+1} = 3^{-a+7}$, what is the value of a? $a+1 = -a+7$

Here we see that the bases are the same. The exponents must therefore be equal. $2a = 6$ $a = 3$

$$a+1 = -a+7$$

$$2a = 6$$

$$a = \boxed{3}$$

EXAMPLE 3: If $2a - b = 4$, what is the value of $\frac{4^a}{2^b}$?

Realize that 4 is just 2^2.

$$\frac{4^a}{2^b} = \frac{(2^2)^a}{2^b} = \frac{2^{2a}}{2^b} = 2^{2a-b} = 2^4 = \boxed{16}$$

Square roots are just fractional exponents:

$$x^{\frac{1}{2}} = \sqrt{x}$$

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

But what about $x^{\frac{2}{3}}$? The 2 on top means to square x. The 3 on the bottom means to cube root it:

$$\sqrt[3]{x^2}$$

We can see this more clearly if we break it down:

$$x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$$

The order in which we do the squaring and the cube-rooting doesn't matter.

$$x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2$$

The end result just looks prettier with the cube root on the outside. That way, we don't need the parentheses.

EXAMPLE 4: Which of the following is equal to $\sqrt[4]{x^5}$?

A) x B) $x^5 - x^4$ C) $x^{\frac{5}{4}}$ D) $x^{\frac{4}{5}}$

The fourth root equates to a fractional exponent of $\frac{1}{4}$, so

$$\sqrt[4]{x^5} = x^{\frac{5}{4}}$$

Answer $\boxed{\text{(C)}}$.

The SAT will also test you on simplifying square roots (also called "surds"). To simplify a square root, factor the number inside the square root and take out any pairs:

$$\sqrt{48} = \sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3} = \sqrt{\boxed{2\cdot 2}\cdot\boxed{2\cdot 2}\cdot 3} = 2\cdot 2\sqrt{3} = 4\sqrt{3}$$

In the example above, we take a 2 out for the first $\boxed{2\cdot 2}$. Then we take another 2 out for the second pair $\boxed{2\cdot 2}$. Finally, we multiply the two 2's outside the square root to get 4. Of course, a quicker route would have looked like this:

$$\sqrt{48} = \sqrt{\boxed{4\cdot 4}\cdot 3} = 4\sqrt{3}$$

Here's one more example:

$$\sqrt{72} = \sqrt{\boxed{2\cdot 2}\cdot\boxed{3\cdot 3}\cdot 2} = 2\cdot 3\sqrt{2} = 6\sqrt{2}$$

To go backwards, take the number outside and put it back under the square root as a pair:

$$6\sqrt{2} = \sqrt{6\cdot 6\cdot 2} = \sqrt{72}$$

EXAMPLE 5: If $4\sqrt{3} = \sqrt{3x}$, what is the value of x?

A) 4 B) 12 C) 16 D) 48

Solution 1: Moving the 4 back inside, we get

$$4\sqrt{3} = \sqrt{3\cdot 4\cdot 4} = \sqrt{48}$$

Now equating the stuff inside the square roots,

$$\sqrt{48} = \sqrt{3x}$$

$$48 = 3x$$

$$16 = x$$

Answer $\boxed{(C)}$.

Solution 2: Square both sides:

$$(4\sqrt{3})^2 = (\sqrt{3x})^2$$
$$16\cdot 3 = 3x$$
$$16 = x$$

EXERCISE 3: Simplify the radicals or solve for x. Do NOT use a calculator. Answers for this chapter start on page 251.

1. $\sqrt{12}$ $2\sqrt{3}$

2. $\sqrt{96}$ $4\sqrt{6}$

3. $\sqrt{45}$ $3\sqrt{5}$

4. $\sqrt{18}$ $3\sqrt{2}$

5. $2\sqrt{27}$ $6\sqrt{3}$

6. $3\sqrt{75}$ $15\sqrt{3}$

7. $\sqrt{32}$ $4\sqrt{2}$

8. $\sqrt{200}$ $10\sqrt{2}$

9. $\sqrt{8}$ $2\sqrt{2}$

10. $\sqrt{128}$ $8\sqrt{2}$

11. $5\sqrt{2} = \sqrt{x}$ $x = 50$

12. $3\sqrt{x} = \sqrt{45}$ $x = 5$

13. $2\sqrt{2} = \sqrt{4x}$ $x = 2$

14. $4\sqrt{6} = 2\sqrt{3x}$ $\sqrt{4} \cdot \sqrt{3}$ $\sqrt{12x}$ $x = 8$

15. $3\sqrt{14} = \sqrt{6x}$ $\sqrt{126}$ $x = 21$

16. $4\sqrt{3x} = 2\sqrt{6}$ $\sqrt{16 \cdot 3x}$ $\sqrt{24}$ $\frac{1}{2}$

17. $3\sqrt{8} = x\sqrt{2}$ $\sqrt{72} = 6\sqrt{2}$ $6\sqrt{2}$

18. $x\sqrt{x} = \sqrt{216}$ $6\sqrt{6}$

CHAPTER EXERCISE: Answers for this chapter start on page 251.

A calculator should NOT be used on the following questions.

1

If $a^{-\frac{1}{2}} = 3$, what is the value of a?

$\log \frac{1}{\sqrt{a}} = 3$

A) -9

B) $\frac{1}{9}$

C) $\frac{1}{3}$

D) 9

2

Let $n = 1^2 + 1^4 + 1^6 + 1^8 + \ldots + 1^{50}$

What is the value of n?

$\frac{50-2}{2} + 1$

A) 10

B) 20

C) 25

D) 30

3

If $4^{2n+3} = 8^{n+5}$, what is the value of n?

$4n+6 = 3n+15$

$n = 9$

A) 6

B) 7

C) 8

D) 9

4

If $\frac{2^x}{2^y} = 2^3$, then x must equal

$2^{x-y} = 3$

A) $y+3$

B) $y-3$

C) $3-y$

D) $3y$

5

If $3^x = 10$, what is the value of 3^{x-3}?

A) $\frac{10}{3}$

B) $\frac{10}{9}$

C) $\frac{10}{27}$

D) $\frac{27}{10}$

6

If $x^2y^3 = 10$ and $x^3y^2 = 8$, what is the value of x^5y^5?

$x^5y^5 = 80$

A) 18

B) 20

C) 40

D) 80

7

If a and b are positive even integers, which of the following is greatest?

$a, b > 0$

A) $(-2a)^b$

B) $(-2a)^{2b}$

C) $(2a)^b$

D) $2a^{2b}$

8

Which of the following is equivalent to $x^{\frac{2a}{b}}$, for all values of x?

A) $\sqrt[b]{ax^2}$

B) $\sqrt[b]{x^{2a}}$

C) $\sqrt[b]{x^{a+2}}$

D) $\sqrt[2a]{x^b}$

$(x^{2a})^{\frac{1}{b}}$

$\sqrt[b]{2a}$

9

If $x^2 = y^3$, for what value of z does $x^{3z} = y^9$?

A) -1
B) 0
C) 1
D) 2

$\frac{x^{3z}}{x^2} = \frac{y^9}{y^3}$

$3z - 4 = 2$

$z = 2$

$x^{3z-2} = y^6$

$x^{3z-4} = y^3$

10

If $2^{x+3} - 2^x = k(2^x)$, what is the value of k?

A) 3
B) 5
C) 7
D) 8

$2^x(2^3 - 1) = k(2^x)$

11

If $\sqrt{x\sqrt{x}} = x^a$, then what is the value of a?

A) $\frac{1}{2}$
B) $\frac{3}{4}$
C) 1
D) $\frac{4}{3}$

$x^{\frac{1}{2}}$

12

$$2\sqrt{x+2} = 3\sqrt{2}$$

If $x > 0$ in the equation above, what is the value of x?

A) 2.5
B) 3
C) 3.5
D) 4

13

If $x^{ac} \cdot x^{bc} = x^{30}$, $x > 1$, and $a + b = 5$, what is the value of c?

A) 3
B) 5
C) 6
D) 10

$x^{c(a+b)}$

A calculator is allowed on the following questions.

14

If $n^3 = x$ and $n^4 = 20x$, where $n > 0$, what is the value of x?

$x = \frac{n^4}{20}$

8000

$n^3 = \frac{n^4}{20}$

$20n^3 = n^4$

$20 = n$

15

If $x^8y^7 = 333$ and $x^7y^6 = 3$, what is the value of xy?

$\frac{x^8y^7}{x^7y^6} = 111$

$xy = 111$

4

Proportion

Imagine we have a triangle. We know that the area of a triangle is $A = \frac{1}{2}bh$.

Now let's say we triple the height. What happens to the area?

Well, if we triple the height, the new height is $3h$. The new area is then

$$A_{new} = \frac{1}{2}b(3h) = 3\left(\frac{1}{2}bh\right) = 3A_{old}$$

See what happened? The terms were rearranged so that we could clearly see the new area is three times the old area. We put the "3" out in front of the old formula.

This technique is extremely important because it saves us time on tough proportion problems. We could've made up numbers for the base and the height and calculated everything out, and while that's certainly a strategy you should have in your toolbox, it would've taken much longer and left us more open to silly mistakes.

Let's do a few more complicated examples.

EXAMPLE 1: The radius of a circle is increased by 25%. By what percent does the area of the circle increase?

56%

$\pi r^2 = \pi(1.25r)^2$

$(1{,}25)^2 \cdot r^2 \cdot \pi$

$1{,}5625 \cdot S_{old}$

Let the original area be A_{old}. If the original radius is r, then the new radius is $1.25r$.

$$A_{new} = \pi(1.25r)^2 = (1.25)^2(\pi r^2) = 1.5625(\pi r^2) = 1.5625A_{old}$$

We can see that the area increases by $\boxed{56.25\%}$.

The idea is to get a number in front of the old formula. In the previous example, that number turned out to be 1.5625. Also note that the $1.25r$ was wrapped in parentheses so that the whole thing gets squared. It would've been incorrect to have $A_{new} = \pi(1.25)r^2$ because we wouldn't be squaring the new radius.

EXAMPLE 2: The length of a rectangle is increased by 20%. The width is decreased by 20%. Which of the following accurately describes the change in the area of the rectangle?

A) Increases by 10% B) Increases by 10% C) Decreases by 4% D) Stays the same

r × l = 1,2 , 0,80 .r.s
= 96% .r.s

Originally, $A = lw$. Now,

$$A_{new} = (1.20l)(0.80w) = 0.96lw = 0.96A_{old}$$

The area has decreased by 4%. Answer $\boxed{(C)}$. Most students think the answer is (D). It's not.

EXAMPLE 3:

$$F = \frac{9q_1q_2}{r^2}$$

The force of attraction between two particles can be determined by the formula above, in which F is the force between them, r is the distance between them, and q_1 and q_2 are the charges of the two particles. If the distance between two charged particles is doubled, the resulting force of attraction is what fraction of the original force?

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{1}{8}$ D) $\frac{1}{16}$

$\frac{9q_1q_2}{(2r)^2}$

$$F_{new} = \frac{9q_1q_2}{(2r)^2} = \left(\frac{1}{2}\right)^2\left(\frac{9q_1q_2}{r^2}\right) = \frac{1}{4}\left(\frac{9q_1q_2}{r^2}\right) = \frac{1}{4}F_{old}$$

Answer $\boxed{(B)}$. Notice how we do not let constants like the "9" in the formula affect the result. In getting a number out front, students often make the mistake of mixing that number up with numbers that were originally in the formula.

EXAMPLE 4: The volume of a cube is tripled. The length of each side must have been increased by approximately what percent?

A) 3% B) 12% C) 33% D) 44%

$V = (x s)^3$ 3
$= x^3 \cdot s^3$

Now we have to solve backwards. Keep in mind that the volume of a cube is $V = s^3$ where s is the length of each side. Even though this problem is a little different, we can still apply the same process as before: increase each side by some factor and rearrange the terms to extract a number. Only this time, we have to use x.

$$V_{new} = (xs)^3$$
$$V_{new} = x^3s^3 = x^3V_{old}$$

Notice how we were still able to extract something out in front, x^3. That x^3 must be equal to 3 if the new volume is to be triple the old volume:

$$x^3 = 3$$
$$x = \sqrt[3]{3} \approx 1.44$$

Each side must have been increased by approximately 44%. Answer $\boxed{(D)}$.

CHAPTER EXERCISE: Answers for this chapter start on page 258.

A calculator is allowed on the following questions.

1

$$P = \frac{V^2}{R}$$

$p = \frac{\left(\frac{V}{2}\right)^2}{R} = \frac{\frac{1}{4}V^2}{R}$

Electric power P is related to the voltage V and resistance R by the formula above. If the voltage were halved, how would the electric power be affected?

A) The electric power would be 4 times greater.
B) The electric power would be 2 times greater.
C) The electric power would be halved.
D) The electric power would be a quarter of what it was.

2

Julie has a square fence that encloses her garden. She decides to expand her garden by making each side of the fence 10 percent longer. After this expansion, the area of Julie's garden will have increased by what percent?

$1.1a \cdot 1.1a$

A) 20%
B) 21%
C) 22%
D) 25%

3

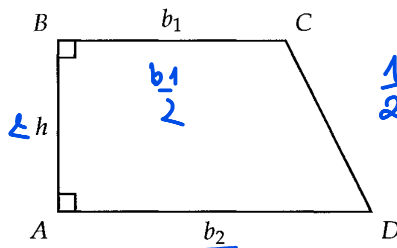
A right circular cone has a base radius of r and a height of h. If the radius is decreased by 20 percent and the height is increased by 10 percent, which of the following is the resulting percent change in the volume of the cone?

A) 10% decrease
B) 12% decrease
C) 18.4% decrease
D) 29.6% decrease

$\frac{1}{3}\pi \cdot r^2 \cdot h$

$\frac{1}{3}\pi \cdot (0.8r)^2 \cdot 1.1h$

4



$\frac{1}{2}\left(\frac{b_1}{2}+\frac{b_2}{2}\right) \cdot 2h$

The area of the trapezoid above can be found using the formula $\frac{1}{2}(b_1 + b_2)h$. If lengths BC and AD are halved and the height is doubled, how would the area of the trapezoid change?

A) The area would be increased by 50 percent.
B) The area would stay the same.
C) The area would be decreased by 25 percent.
D) The area would be decreased by 50 percent.

5

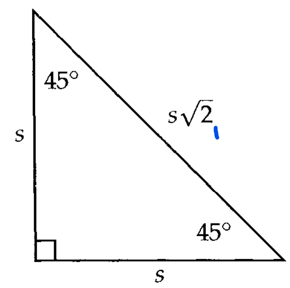
Calvin has a sphere that is four times bigger than the one Kevin has in terms of volume. The radius of Calvin's sphere is how many times greater in length than the radius of Kevin's sphere (rounded to the nearest hundredth)?

A) 1.44
B) 1.59
C) 1.67
D) 2.00

$V = \frac{4}{3}\pi r^3$

$V_{calvin} = \frac{4}{3}\pi (xr)^3$

6



In the triangle above, the lengths of the sides relate to one another as shown. If a new triangle is created by decreasing s such that the area of the new triangle is 64 percent of the original area, s must have been decreased by what percent?

A) 8%
B) 20%
C) 25%
D) 30%

$S_{old} = \frac{1}{2} s \cdot s$

$\frac{1}{2}(xs)^2$

Questions 7-8 refer to the following information.

$$L = 4\pi d^2 b$$

The total amount of energy emitted by a star each second is called its luminosity L, which is related to d, its distance (meters) away from Earth, and b, its brightness measured in watts per square meter, by the formula above.

7

If one star is three times as far away from Earth as another, and twice as bright, its luminosity is how many times greater than that of the other star?

A) 8
B) 9
C) 16
D) 18

$L = 4 \cdot \pi \cdot (3d)^2 \cdot 2b$

$9 \cdot 2$

8

Astronomers see two equally bright stars, Star A and Star B, in the night sky, but the luminosity of Star A is one-ninth the luminosity of Star B. The distance of Star A from Earth is what fraction of the distance of Star B from Earth?

A) $\frac{1}{27}$
B) $\frac{1}{9}$
C) $\frac{1}{3}$
D) $\frac{2}{3}$

$A = \frac{1}{9} B$

$4\pi d^2 b = \frac{1}{9} 4\pi d^2 b$

$\frac{d_A^2}{d_B^2} = \frac{1}{9}$

5

Rates

I've found rate problems to be pretty polarizing—some students just "get" them intuitively, others get completely lost. Most of the rate problems on the SAT will be pretty straightforward, but for the ones that aren't, I highly recommend using conversion factors to setup the solution (if you've gone through chemistry, you should know what I'm talking about). Conversion factors are a fool-proof way to approach a lot of these problems, but they can be slow-going for stronger problem solvers. I'll be covering both the straightforward, intuitive approaches and the conversion factor approach throughout the examples in this chapter.

EXAMPLE 1: A bicycle manufacturer can produce 20 bicycles per hour. How many hours would it take the manufacturer to produce 320 bicycles?

16 1 h 20 bicycles
320

Easy enough. We divide the total by the rate to get $320 \div 20 = \boxed{16}$ hours.

EXAMPLE 2: A rocket has 360 gallons of fuel left after 2 hours of flight, and only 100 gallons after 6 hours of flight. It burns n gallons of fuel for every hour of flight, where n is a constant. What is the value of n?

4 260
4 n 65

Here, we are figuring out the rate. In $6 - 2 = 4$ hours of flight, the rocket burned $360 - 100 = 260$ gallons of fuel. Therefore, the rocket burns $\dfrac{260}{4} = \boxed{65}$ gallons of fuel every hour.

EXAMPLE 3: A box at the supermarket can hold 6 oranges each. Each orange costs 20 cents. Given that the supermarket has a budget of $500 to stock oranges, how many boxes will the supermarket be able to completely fill?

1 6 8
416 n 2500

If each orange is 20 cents, then a dollar would be enough for 5 oranges. Five hundred dollars would be enough for $500 \times 5 = 2500$ oranges, which would fill $2500 \div 6 = 416.67$ boxes. Given that the question asks for full boxes, the answer is $\boxed{416}$.

The examples above were quite straightforward and didn't really call for writing out full conversion factors, but what if we wanted to use conversion factors for Example 3? What would've the solution looked like?

$$500 \cancel{\text{dollars}} \times \frac{100 \cancel{\text{cents}}}{1 \cancel{\text{dollar}}} \times \frac{1 \cancel{\text{orange}}}{20 \cancel{\text{cents}}} \times \frac{1 \text{ box}}{6 \cancel{\text{oranges}}} = 416.67 \text{ boxes}$$

The rest of the examples in this chapter are done with conversion factors to teach you how they're used, even though there may be more "casual" solutions.

EXAMPLE 4: A car can travel 1 mile in 1 minute and 15 seconds. At this rate, how many miles can the car travel in 1 hour?

In most rate problems, you'll start with what the question is asking for. We need to convert that 1 hour to a distance that the car travels. The car's rate is 1 mile every 75 seconds.

$$1 \cancel{\text{hour}} \times \frac{60 \cancel{\text{minutes}}}{1 \cancel{\text{hour}}} \times \frac{60 \cancel{\text{seconds}}}{1 \cancel{\text{minute}}} \times \frac{1 \text{ mile}}{75 \cancel{\text{seconds}}} = \frac{60 \times 60 \text{ miles}}{75} = \boxed{48} \text{ miles}$$

The units should cancel as you go along. If the units are canceling, chances are we're doing things right. Notice that the "miles" unit at the end is the unit we wanted to end up with. This is another sign that we've done things right.

EXAMPLE 5: Tom drives 30 miles at an average rate of 50 miles per hour. If Leona drives at an average rate of 40 miles per hour, how many more minutes will it take her to travel the same distance?

We have to figure out how long it takes Tom to drive 30 miles:

$$30 \text{ miles} \times \frac{1 \text{ hour}}{50 \text{ miles}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 36 \text{ minutes}$$

Leona will take

$$30 \text{ miles} \times \frac{1 \text{ hour}}{40 \text{ miles}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} = 45 \text{ minutes}$$

So,

$$45 - 36 = \boxed{9} \text{ minutes}$$

EXAMPLE 6: To prepare for class, Mr. Chu has to print a number of booklets with p pages per booklet. If every 5 pages cost c cents to print and he spent a total of d dollars, how many booklets did Mr. Chu print in terms of $p, c,$ and d?

A) $\dfrac{cp}{500d}$ B) $\dfrac{100d}{cp}$ C) $\dfrac{500d}{cp}$ D) $\dfrac{5d}{cp}$

$$d \cancel{\text{dollars}} \times \frac{100 \cancel{\text{cents}}}{1 \cancel{\text{dollar}}} \times \frac{5 \cancel{\text{pages}}}{c \cancel{\text{cents}}} \times \frac{1 \text{ booklet}}{p \cancel{\text{pages}}} = \frac{500d}{cp} \text{ booklets}$$

The answer is $\boxed{\text{(C)}}$.

CHAPTER EXERCISE: Answers for this chapter start on page 260.

A calculator should NOT be used on the following questions.

1

A carpenter lays x bricks per hour for y hours and then lays $\frac{x}{2}$ bricks for $2y$ more hours. In terms of x and y, how many bricks did he lay in total?

xy + xy

A) $2xy$

B) $\frac{5}{2}xy$

C) $5xy$

D) $\frac{3}{2}x + 3y$

2

Tim's diet plan calls for 60 grams of protein per day. If Tim were to meet this requirement by only eating a certain protein bar that contains 30 grams of protein, how many protein bars would he have to buy to last a week?

14

2 bar / day

3

An electronics company sells computer monitors and releases a new model every year. With each new model, the company increases the screen size by a constant amount. In 2005, the screen size was 15.5 inches. In 2011, the screen size was 18.5 inches. Which of the following best describes how the screen size changed between 2005 and 2011?

A) The company increases the screen size by 0.5 inch every year.

B) The company increases the screen size by 1 inches every year.

C) The company increases the screen size by 2 inches every year.

D) The company increases the screen size by 3 inches every year.

4

As a submarine descends into the deep ocean, the pressure it must withstand increases. At an altitude of -700 meters, the pressure is 50 atm (atmospheres), and at an altitude of -900 meters, the pressure is 70 atm. For every 10 meters the submarine descends, the pressure it faces increases by n, where n is a constant. What is the value of n?

A) 0.1

B) 1

C) 2

D) 10

200 20

10 n

5

An empty pool can be filled in 5 hours if water is pumped in at 300 gallons an hour. How many hours would it take to fill the pool if water is pumped in at 500 gallons an hour?

8 hours

n 500

5 360

500

5 300

6

If a apples cost d dollars, which of the following expressions gives the cost of 20 apples, in dollars?

A) $\frac{20a}{d}$

B) $\frac{20d}{a}$

C) $\frac{a}{20d}$

D) $\frac{20}{ad}$

a d

20

2

7

At a school, there are a grade levels with b students in each grade. If the school buys n stickers to be distributed equally among the students, which of the following gives the number of stickers each student receives?

A) $\frac{ab}{n}$

B) $\frac{an}{b}$

C) $\frac{bn}{a}$

D) $\frac{n}{ab}$

8

During a race on a circular race track, a racecar burns fuel at a constant rate. After lap 4, the racecar has 22 gallons left in its tank. After lap 7, the racecar has 18 gallons left in its tank. Assuming the racecar does not refuel, after which lap will the racecar have 6 gallons left in its tank?

A) Lap 13

B) Lap 15

C) Lap 16

D) Lap 19

1 n

3 4

9

By 1:00 PM, a total of 40 boxes had been unloaded from a delivery truck. By 3:30PM, a total of 65 boxes had been unloaded from the same truck. If boxes are unloaded from the truck at a constant rate, what is the total number of boxes that will have been unloaded from the truck by 7:00PM?

70

$2\frac{1}{2}$ 25

7

10

Amy buys d dollars worth of groceries each week and spends a fourth of those dollars on fruit. In terms of d, how many weeks will it take Amy to spend a total of $100 just on fruit?

A) $\frac{400}{d}$

B) $\frac{25}{d}$

C) $\frac{d}{25}$

D) $\frac{d}{400}$

d 1

$\frac{1}{4}$ $\frac{1}{4d}$

11

An internet service provider charges a one time setup fee of $100 and $50 each month for service. If c customers join at the same time and are on the service for m months, which of the following expressions represents the total amount, in dollars, the provider has charged these customers?

A) $100c + 50m$
B) $100c + 50cm$
C) $150cm$
D) $100m + 50cm$

12

A manufacturing plant increases the temperature of a chemical compound by d degrees Celsius every m minutes. If the compound has an initial temperature of t degrees Celsius, which of the following expressions gives its temperature after x minutes, in degrees Celsius?

A) $\dfrac{mx + t}{d}$
B) $\dfrac{md + t}{x}$
C) $t + \dfrac{d}{mx}$
D) $t + \dfrac{dx}{m}$

m d
1 $\frac{d}{m} \cdot x$

13

At a shop for tourists, the price of one souvenir is a dollars. Each additional souvenir purchased after the first is discounted by 40 percent. If James buys n souvenirs, where $n > 1$, which of the following represents the total cost of the souvenirs?

A) $a + (n - 1)(0.4a)$
B) $a + (n - 1)(0.6a)$
C) $a + n(0.6a)$
D) $0.6an$

14

A cupcake store employs bakers to make boxes of cupcakes. Each box contains x cupcakes and each baker is expected to produce y cupcakes each day. Which of the following expressions gives the number of boxes needed for all the cupcakes produced by $3x$ bakers working for 4 days?

A) $12x^2y$
B) $\dfrac{3y}{4}$
C) $\dfrac{12x^2}{y}$
D) $12y$

10x cupcakes
1 x
12xy 12xy

15

At a math team competition, there are m schools with n students from each school. The host school wants to order enough pizza such that there are 2 slices for each student. If there are 8 slices in one pizza, which of the following gives the number of pizzas the host school must order?

A) $\dfrac{mn}{8}$
B) $\dfrac{mn}{4}$
C) $\dfrac{m + 2n}{8}$
D) $2mn$

n·m
slices stu
2 1
2nm nm

16

$$P\left(1+\frac{r}{100}\right)^5$$

The expression above gives the population of leopards after five years during which an initial population of P leopards grew by r percent each year. Which of the following expressions gives the percent increase in the leopard population over these five years?

A) $\left(1+\frac{r}{100}\right)^5$

B) $\dfrac{\left(1+\frac{r}{100}\right)^5-1}{\left(1+\frac{r}{100}\right)^5}\times 100$

C) $\left[\left(1+\frac{r}{100}\right)^5-1\right]\times 100$

D) $\left(1+\frac{r}{100}\right)^5\times 100$

A calculator is allowed on the following questions.

17

Henry drives 150 miles at 30 miles per hour and then another 200 miles at 50 miles per hour. What was his average speed, in miles per hour, for the entire journey, to the nearest hundredth?

A) 38.89
B) 40.00
C) 42.33
D) 43.58

50m, 20m/s
1
$\frac{2}{5}$ m/s

18

A rolling ball covers a distance of 2400 feet in 4 minutes. What is the ball's average speed, in inches per second? (12 inches = 1 foot)

120 2400 4

200 4
1

19

Idina can type 90 words in 2.5 minutes. How many words can she type in 12 minutes? 432

90 2.5
12

20

A painter can cover a circular region with a radius of 3 feet with paint in 2 minutes. At this rate, how many minutes will it take the painter to cover a circular region with a radius of 6 feet with paint?

4 3ε 2 m
6 x

21

A "slow" clock falls behind at the same rate every hour. It is set to the correct time at 4:00 AM. When the clock shows 5:00 AM the same day, the correct time is 5:08 AM. When the clock shows 10:30 AM that day, what is the correct time?

A) 11:02 AM
B) 11:18 AM
C) 11:22 AM
D) 12:18 PM

1 1:08

22

A salesman at a tea company makes a $15 commission on every $100 worth of products that he sells. If a jar of tea leaves is $20, how many jars would he have to sell to make $180 in commission?

23

A train covers 32 kilometers in 14.5 minutes. If it continues to travel at the same rate, which of the following is closest to the distance it will travel in 2 hours?

A) 54 kilometers
B) 265 kilometers
C) 364 kilometers
D) 928 kilometers

24

One liter is equivalent to approximately 33.8 ounces. Mark has plastic cups that can each hold 12 ounces of liquid. At most, how many of these plastic cups could a two liter bottle of soda fill?

A) 5
B) 6
C) 7
D) 8

25

Brett currently spends $160 each month on gas. His current car is able to travel 30 miles per gallon of gas. He decides to switch his current car for a new car that is able to travel 40 miles per gallon of gas. Assuming the price of gas stays the same, how much will he spend on gas each month with the new car?

A) $100
B) $120
C) $130
D) $140

26

An 8 inch by 10 inch piece of cardboard costs $2.00. If the cost of a piece of cardboard is proportional to its area, what is the cost of a piece of cardboard that is 16 inches by 20 inches?

A) $4.00
B) $8.00
C) $12.00
D) $16.00

27

Margaret can buy 4 jars of honey for 9 dollars, and she can sell 3 jars of honey for 15 dollars. How many jars of honey would she have to buy and then sell to make a total profit of 132 dollars?

5 jar

48

$4 \to 9$

$1 \to 2{,}25$ jar

$2{,}25x$

$5x - 2{,}25x = 132$

28

In one hour, Jason can install at least 6 windows but no more than 8 windows. Which of the following could be a possible amount of time, in hours, that Jason takes to install 100 windows in a home?

A) 12
B) 16
C) 17
D) 18

29

$$1 \text{ fluid ounce} = 29.6 \text{ milliliters}$$
$$1 \text{ cup} = 16 \text{ fluid ounces}$$

A chemistry teacher is planning to run a class experiment in which each student must measure out 100 milliliters of vinegar in a graduated cylinder. The class is limited to using 6 cups of vinegar. Given the information above, what is the maximum number of students who will be able to participate in this experiment? 28

3, 37

30

Yoona runs at a steady rate of 1 yard per second. Jessica runs 4 times as fast. If Jessica gives Yoona a head start of 30 yards in a race, how many yards must Jessica run to catch up to Yoona?

40

6

Expressions

Algebraic expressions are just combinations of numbers and variables. Both $x^2 + y$ and $\dfrac{3m - k}{2}$ are examples of expressions. In this chapter, we'll cover some fundamental techniques that will allow you to deal with questions involving expressions quickly and effectively.

1. Combining Like Terms

When combining like terms, the most important mistake to avoid is putting terms together that look like they can go together but can't. For example, you cannot combine $b^2 + b$ to make b^3, nor can you combine $a + ab$ to make $2ab$. To add or subtract, the variables have to completely match.

EXAMPLE 1:

$$2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2)$$

Which of the following is equivalent to the expression above?

A) $-6a^2b^2$ B) $3a^2 - 11a^2b^2 - 18b^2$ C) $3a^2 - 11a^2b^2 + 2b^2$ D) $5a^2 + 2a^2b^2 + 2b^2$

$4a^2 - 6a^2b^2 - 8b^2$

$a^2 - 5a^2b^2 + 10b^2$

$3a^2 - 11a^2b^2 + 2b^2$

$$\begin{aligned}2(2a^2 - 3a^2b^2 - 4b^2) - (a^2 + 5a^2b^2 - 10b^2) &= 4a^2 - 6a^2b^2 - 8b^2 - a^2 - 5a^2b^2 + 10b^2\\ &= 3a^2 - 11a^2b^2 + 2b^2\end{aligned}$$

Answer $\boxed{\text{(C)}}$.

2. Expansion and Factoring

EXAMPLE 2:

$$2(x-4)(2x+3)$$

$(2x-8)(2x+3)$

Which of the following is equivalent to the expression above?

A) $4x^2-10x-24$ B) $4x^2+10x-24$ C) $4x^2+10x+24$ D) $8x^2-20x-24$

Some people like to expand using a method called FOIL (first, outer, inner, last). If you haven't heard of it, that's totally fine. After all, it's the same thing as distributing each term. First, we distribute the "2."

$$2(x-4)(2x+3)=(2x-8)(2x+3)$$

Notice that it applies to just one of the two factors. Either one is fine, but NOT both.

$$\begin{aligned}(2x-8)(2x+3)&=4x^2+6x-16x-24\\&=4x^2-10x-24\end{aligned}$$

Answer $\boxed{(A)}$.

Now when it comes to factoring and expansion, there are several key formulas you should know:

- $(a+b)^2=a^2+2ab+b^2$
- $(a-b)^2=a^2-2ab+b^2$
- $a^2-b^2=(a+b)(a-b)$

Memorize these forwards and backwards. They show up very often.

EXAMPLE 3: Which of the following is equivalent to $4x^4-9y^2$?

A) $(2x^2+9y)(2x^2-y)$ B) $(4x^2+3y)(x^2-3y)$ C) $(x^2+3y)(4x^2-3y)$ D) $(2x^2+3y)(2x^2-3y)$

Part of what makes for a top SAT score is pattern recognition. Once you've done enough practice, you should be able to recognize the question above as a difference of two squares, a variation of the a^2-b^2 formula. The SAT will rarely test you on those formulas in a straightforward way. Be on the lookout for variations that match the pattern. With more practice, you'll get better and better at noticing them.

Using the formula $a^2-b^2=(a+b)(a-b)$, we can see that $a=2x^2$ and $b=3y$. Therefore,

$$4x^4-9y^2=(2x^2+3y)(2x^2-3y)$$

Answer $\boxed{(D)}$.

EXAMPLE 4:

$$16x^4 - 8x^2y^2 + y^4$$

Which of the following is equivalent to the expression shown above?

A) $(4x^2 + y^2)^2$ B) $(2x - y)^4$ C) $(2x + y)^2(2x - y)^2$ D) $(4x + y)^2(x - y)^2$

Using the formula $(a - b)^2 = a^2 - 2ab + b^2$ (in reverse), we can see that $a = 4x^2$ and $b = y^2$. Therefore,

$$16x^4 - 8x^2y^2 + y^4 = (4x^2 - y^2)^2$$

This is not in the answer choices. We have to take it one step further and apply the $a^2 - b^2$ formula to the expression inside the parentheses.

$$(4x^2 - y^2)^2 = [(2x + y)(2x - y)]^2 = (2x + y)^2(2x - y)^2$$

Answer $\boxed{(C)}$.

3. Combining Fractions

When you're adding simple fractions,

$$\frac{1}{3} + \frac{1}{4}$$

the first step is to find the least common multiple of the denominators. We do this so that we can get a common denominator. In a lot of cases, it's just the product of the denominators, as it is here, $3 \times 4 = 12$.

$$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \cdot \frac{4}{4} + \frac{1}{4} \cdot \frac{3}{3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

Now when we're adding fractions with expressions in the denominator, the idea is the same.

EXAMPLE 5:

$$\frac{1}{x+2} + \frac{2}{x-2}$$

Which of the following is equivalent to the expression above?

A) $\frac{3x-2}{(x+2)(x-2)}$ B) $\frac{3x+2}{(x+2)(x-2)}$ C) $\frac{3}{(x+2)(x-2)}$ D) $\frac{2}{(x+2)(x-2)}$

The common denominator is just the product of the two denominators: $(x + 2)(x - 2)$. So now we multiply the top and bottom of each fraction by the factor they don't have:

$$\frac{1}{x+2} + \frac{2}{x-2} = \frac{1}{x+2} \cdot \frac{x-2}{x-2} + \frac{2}{x-2} \cdot \frac{x+2}{x+2} = \frac{x-2}{(x+2)(x-2)} + \frac{2(x+2)}{(x+2)(x-2)} = \frac{(x-2) + 2(x+2)}{(x+2)(x-2)}$$

$$= \frac{3x+2}{(x+2)(x-2)}$$

Answer $\boxed{(B)}$.

4. Flipping (Dividing) Fractions

What's the difference between $\dfrac{\frac{1}{2}}{3}$ and $\dfrac{1}{\frac{2}{3}}$?

The difference is where the longer fraction line is. The first is $\frac{1}{2}$ divided by 3. The second is 1 divided by $\frac{2}{3}$. They're not the same.

$$\frac{\frac{1}{2}}{3} = \frac{1}{2} \div 3 = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\frac{1}{\frac{2}{3}} = 1 \div \frac{2}{3} = 1 \times \frac{3}{2} = \frac{3}{2}$$

The shortcut is to flip the fraction that is in the denominator. So,

$$\frac{a}{\frac{b}{c}} = \frac{ac}{b}$$

If the fraction is in the numerator, then the following occurs:

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

EXAMPLE 6: If $x > 1$, which of the following is equivalent to $\dfrac{x}{\frac{1}{x-1} + \frac{1}{x+1}}$? $\dfrac{x}{2x}$

A) $\dfrac{2x^2}{(x-1)(x+1)}$ B) $\dfrac{2}{(x-1)(x+1)}$ C) $\dfrac{x(x-1)(x+1)}{2}$ D) $\dfrac{(x-1)(x+1)}{2}$ $(x-1)(x+1)$

First, combine the two fractions on the bottom with the common denominator $(x-1)(x+1)$.

$$\frac{1}{x-1} + \frac{1}{x+1} = \frac{x+1}{(x-1)(x+1)} + \frac{x-1}{(x-1)(x+1)} = \frac{2x}{(x-1)(x+1)}$$

Next, substitute this back in and flip it.

$$\frac{x}{\frac{2x}{(x-1)(x+1)}} = \frac{x(x-1)(x+1)}{2x} = \frac{(x-1)(x+1)}{2}$$

Answer $\boxed{(D)}$.

5. Splitting fractions

EXAMPLE 7: Which of the following is equivalent to $\frac{30+c}{6}$?

A) $\frac{5+c}{6}$ B) $\frac{10+c}{2}$ C) $5+c$ D) $5+\frac{c}{6}$

S

We can split the fraction into two:

$$\frac{30+c}{6} = \frac{30}{6} + \frac{c}{6} = 5 + \frac{c}{6}$$

The answer is $\boxed{D}$. This is just the reverse of adding fractions.

Note that while you can split up the numerators of fractions, you cannot do so with denominators. So,

$$\frac{3}{x+y} \neq \frac{3}{x} + \frac{3}{y}$$

In fact, you cannot break up a fraction like $\frac{3}{x+y}$ any further.

CHAPTER EXERCISE: Answers for this chapter start on page 263.

A calculator should NOT be used on the following questions.

1

Which of the following is equivalent to $6x^2y + 6xy^2$?

A) $6xy(x+y)$
B) $12xy(x+y)$
C) $6x^2y^2(y+x)$
D) $12x^3y^3$

2

If $a > 0$, then $\frac{1}{a} + \frac{3}{4}$ is equivalent to which of the following?

A) $\frac{3+4a}{4a}$
B) $\frac{4+3a}{4a}$
C) $\frac{7}{4a}$
D) $\frac{4}{a+4}$

3

Which of the following is equivalent to $(x^2+y)(y+z)$?

A) $x^2z + y^2 + yz$
B) $x^2y + x^2z + y^2 + yz$
C) $x^2y + y^2 + x^2z$
D) $x^2 + x^2z + y^2 + yz$

4

Which of the following is equivalent to $\frac{4+8x}{12x}$?

A) $\frac{1+8x}{3x}$
B) $\frac{4+2x}{3x}$
C) $\frac{1+2x}{3x}$
D) 1

$\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$

5

Which of the following is equivalent to $3x^4 - 3$?

A) $3(x^2+1)^2$
B) $3(x^2-1)^2$
C) $3(x^3-1)(x+1)$
D) $3(x^2+1)(x+1)(x-1)$

$3(x^4-1)$

6

$$(x+1)^2 + 2(x+1)(y+1) + (y+1)^2$$

Which of the following is equivalent to the expression shown above?

A) $(x+y+1)^2$
B) $(x+y+2)^2$
C) $(x+y)^2 + 2$
D) $(x+y)^2 - x - y$

7

Which of the following is equivalent to $\frac{xy - x^2}{xy - y^2}$?

A) $-\frac{y}{x}$
B) $\frac{y}{x}$
C) $\frac{x}{y}$
D) $-\frac{x}{y}$

$\frac{x(y-x)}{xy-y^2}$

$\frac{x}{y} \cdot \frac{(y-x)}{(x-y)}$

$\frac{x}{y} \cdot \frac{-(y-x)}{(x-y)}$

$-\frac{x}{y} \cdot \frac{\cancel{-y+x}}{\cancel{x-y}}$

47

8

If $x > 1$, which of the following is equivalent to $\dfrac{1}{\dfrac{x-1}{2}+\dfrac{x+5}{3}}$?

$3x-3+2x+16$

$\dfrac{1}{\dfrac{5x+7}{6}}$ $\quad \dfrac{6}{5x}$

A) $\dfrac{5x+7}{6}$

B) $\dfrac{6}{2x+4}$

C) $\dfrac{6}{5x+7}$

D) $\dfrac{1}{30x+42}$

9

$$\frac{2+\frac{1}{x}}{2-\frac{1}{x}}$$

$\dfrac{\dfrac{2x+1}{x}}{\dfrac{2x-1}{x}}$

The expression above is equivalent to which of the following?

$\dfrac{2x+1}{2}$

A) $\dfrac{2x-1}{2x+1}$

B) $\dfrac{2x+1}{2x-1}$

C) $\dfrac{4x^2-1}{x^2}$

D) -1

A calculator is allowed on the following questions.

10

$$3x^3+8x^2-4x$$
$$7x^2-11x-7$$

Which of the following is the sum of the two polynomials above?

A) $3x^3+x^2-15x-7$

B) $3x^3+15x^2-15x-7$

C) $10x^5-7x-7$

D) $15x^4+3x^3-15x^2-7$

11

$$(5a+3\sqrt{a})-(2a+5\sqrt{a})$$

Which of the following is equivalent to the expression above?

$3a$

A) $-2a\sqrt{a}$

B) $a\sqrt{a}$

C) $3a-2\sqrt{a}$

D) $3a+8\sqrt{a}$

12

If $y \neq 0$, what is the value of $\dfrac{9(2y)^2+2(6y)^2}{8(3y)^2}$?

$\dfrac{3}{2}$

$\dfrac{9.4y^2+2.36y^2}{8.9y^2}$

$= \dfrac{36y^2+72y^2}{72y^2}$

7

Manipulating & Solving Equations

On the SAT, there is a huge emphasis on equations. To get these types of questions right, you must learn how to isolate the variables and expressions you want. First, we'll cover several useful techniques in dealing with equations that you may already be familiar with.

1. Don't forget to combine like terms

You should be ruthless in finding like terms and combining them. Doing so will simplify things and allow you to figure out the next step.

EXAMPLE 1: If $2(a+b+2c+3d+1) = 3a+2b+4c+6d$, find the value of a.

The same four variables are on both sides of the equation, a, b, c and d. That should tell you to distribute on the left side first and then **combine like terms**. Sounds simple but you won't believe how many students forget to do this, especially in the middle of a more complex problem.

$$2(a+b+2c+3d+1) = 3a+2b+4c+6d$$

The $b, c,$ and d variables cancel quite nicely.

$$2a+\cancel{2b}+\cancel{4c}+\cancel{6d}+2 = 3a+\cancel{2b}+\cancel{4c}+\cancel{6d}$$

$$\boxed{2} = a$$

2. Square and square root correctly

When squaring equations to remove a square root, the most important thing to remember is that you're not squaring individual elements—you're squaring the entire side.

EXAMPLE 2:

ab = a2-2ab + b2

$$\sqrt{ab} = a - b$$

If $a > 0$ and $b > 0$, the equation above is equivalent to which of the following?

A) $ab = a^2 - b^2$ B) $ab = a^2 + b^2$ C) $2ab = a^2 - b^2$ D) $3ab = a^2 + b^2$

The square root in the problem should scream to you that the equation should be squared. Most students know the square root should be eliminated, but here's the common mistake they make:

$$ab = a^2 - b^2$$

They square each individual element. However, this is WRONG. When modifying equations, you must apply any given operation to the entire SIDE, like so:

$$(\sqrt{ab})^2 = (a - b)^2$$

If it helps, wrap each side in parentheses before applying the operation. By the way, the same holds true for all other operations, including multiplication and division. When you multiply or divide both sides of an equation, what you're actually doing is wrapping each side in parentheses, but because of the distributive property, it just so happens that multiplying or dividing each individual element gets you the same result. For example, if we had the equation

$$x + 2 = y$$

and we wanted to multiply both sides by 3, what we're actually doing is

$$3(x + 2) = 3(y)$$

which turns out to be the same as

$$3x + 6 = 3y$$

Anyway, back to the problem

$$(\sqrt{ab})^2 = (a - b)^2$$
$$ab = a^2 - 2ab + b^2$$
$$3ab = a^2 + b^2$$

The answer is $\boxed{D}$.

Another common mistake is squaring each side before the square root is isolated on one side. For example,

$$\sqrt{x - 2} + 3 = x$$

Don't square each side until you've moved the "3" on the left to the right:

$$\sqrt{x - 2} = x - 3$$

And now we can square both sides and go from there.

Now, when it comes to taking the square root of an equation, most students forget the plus or minus (±).

Always remember that an equation such as $x^2 = 25$ has two solutions:

$$\sqrt{x^2} = \sqrt{25}$$
$$x = \pm 5$$

However, this only applies when you're taking the square root to **solve an equation**. By definition, square roots always refer to the positive root. So, $\sqrt{9} = 3$, NOT ± 3. And $\sqrt{x} = -3$ is not possible (except when working with non-real numbers, which we'll look at in a future chapter). The plus or minus is only necessary when the square root is used as a tool to solve an equation. That way, we get all the possible solutions to the equation.

EXAMPLE 3: If $(x+3)^2 = 121$, what is the sum of the two possible values of x?

$$(x+3)^2 = 121$$
$$\sqrt{(x+3)^2} = \pm\sqrt{121}$$
$$x + 3 = \pm 11$$
$$x = -3 \pm 11$$

So x could be either 8 or -14. The sum of those two possibilities is $\boxed{-6}$.

3. Cross-multiply when fractions are set equal to each other

Whenever a fraction is equal to another fraction,

$$\frac{a}{b} = \frac{c}{d}$$

you can cross-multiply: $ad = bc$.

EXAMPLE 4: If $\frac{4}{5}x = \frac{10}{3}$, what is the value of x?

$$\frac{4}{5}x = \frac{10}{3}$$
$$12x = 50$$
$$x = \boxed{\frac{25}{6}}$$

EXAMPLE 5: If $\frac{5}{x-2} - \frac{3}{x+2} = 0$, what is the value of x?

$$\frac{5}{x-2} - \frac{3}{x+2} = 0$$
$$\frac{5}{x-2} = \frac{3}{x+2}$$
$$5(x+2) = 3(x-2)$$
$$5x + 10 = 3x - 6$$
$$2x = -16$$
$$x = \boxed{-8}$$

4. Factoring should be in your toolbox

Some equations have variables that are tougher to isolate. For a lot of these equations, you will have to do some shifting around to factor out the variable you want.

EXAMPLE 6:

$$b = \frac{a}{3a+c}$$

Which of the following expresses a in terms of b and c?

A) $\frac{bc}{1-3b}$ B) $\frac{bc}{3b+1}$ C) $\frac{1-3b}{bc}$ D) $\frac{3b+1}{bc}$

$b(3a+c) = a$
$3ab + \cancel{a} bc = a$
$bc = a - 3ab$
$bc = a(1-3b)$
$\frac{bc}{1-3b}$

$$b = \frac{a}{3a+c}$$
$$b(3a+c) = a$$
$$3ab + bc = a$$
$$bc = a - 3ab$$
$$bc = a(1-3b)$$
$$\frac{bc}{1-3b} = a$$

See what we did? We expanded everything out and put every term containing a on the right side. Then we were able to factor out a and isolate it. The answer is $\boxed{(A)}$.

EXAMPLE 7:

$$x^4 + 3x^3 + x + 3 = 0$$

What is one possible real value of x for which the equation above is true?

$$x^4 + 3x^3 + x + 3 = 0$$
$$x^3(x+3) + (x+3) = 0$$
$$(x+3)(x^3+1) = 0$$
$$x = -3 \text{ or } -1$$

Once we factored out x^3 from the first two terms, further factoring was possible with the $(x+3)$ term. How would you know to do this? Experience.

5. Treat complicated expressions as one unit

EXAMPLE 8:

$$x^3 + x^2 + x = \frac{x\sqrt{x - \frac{1}{x}}}{m\left(x + \frac{1}{x}\right)}$$

Which of the following gives m in terms of x?

(A) $m = \dfrac{(x^4 + x^3 + x^2)\sqrt{x - \frac{1}{x}}}{\left(x + \frac{1}{x}\right)}$

(B) $m = \dfrac{\sqrt{x - \frac{1}{x}}}{(x^4 + x^3 + x^2)\left(x + \frac{1}{x}\right)}$

(C) $m = \dfrac{(x^3 + x^2 + x)\left(x + \frac{1}{x}\right)}{x\sqrt{x - \frac{1}{x}}}$

(D) $m = \dfrac{x\sqrt{x - \frac{1}{x}}}{(x^3 + x^2 + x)\left(x + \frac{1}{x}\right)}$

Don't let the big and complicated expressions freak you out. Treat these complicated expressions as one unit or variable, like so:

$$A = \frac{B}{mC}$$

Multiply both sides by m.

$$mA = \frac{B}{C}$$

Divide both sides by A.

$$m = \frac{B}{AC}$$

Finally, plug the original expressions back in.

$$m = \frac{x\sqrt{x - \frac{1}{x}}}{(x^3 + x^2 + x)\left(x + \frac{1}{x}\right)}$$

Answer $\boxed{(D)}$.

EXAMPLE 9:

$$(x+1)^2 + 5(x+1) - 24 = 0$$

If $x > 0$, for what real value of x is the equation above true?

Treat $(x+1)$ as one unit and call it A.

$$\begin{aligned}(x+1)^2 + 5(x+1) - 24 &= 0\\ A^2 + 5A - 24 &= 0\\ (A+8)(A-3) &= 0\\ (x+1+8)(x+1-3) &= 0\\ (x+9)(x-2) &= 0\\ x &= -9 \text{ or } 2\end{aligned}$$

Because the question stipulates that $x > 0$, the answer is $\boxed{2}$.

6. Be comfortable solving for expressions, rather than any one variable

EXAMPLE 10: If $3x + 9y = 9$, what is the value of $x + 3y$?

Get in the habit of looking for what you want **before** you solve for anything specific. Is there any way to get the answer without solving for x and y?

Yes! Dividing both sides of the given equation by 3 gives $x + 3y = \boxed{3}$.

EXAMPLE 11: If $\frac{x}{y} = 3$, what is the value of $\frac{y}{2x}$?

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) $\frac{3}{2}$

Here, we have no choice but to solve for the expression. We're given x over y but we want y over x. We can flip the given equation to get

$$\frac{y}{x} = \frac{1}{3}$$

Then we can divide both sides by 2 to obtain the $\frac{y}{2x}$ we're looking for.

$$\frac{y}{2x} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

The answer is $\boxed{(A)}$.

7. Guess and check when you're out of options

When all else fails and you don't have any answer choices or a calculator to work from, it never hurts to guess and check small numbers.

EXAMPLE 12:

$$x^2(x^3 - 4) = 4^x$$

If x is an integer, what is one possible solution to the equation above?

$x^5 - 4$

If you have to do a question this complicated without a calculator or any answer choices, you know it has to be solvable through basic guess and check. There's simply no other way.

It would be silly to show you every step of guess and check on this page. Just remember to start with numbers like 0, 1, 2, and -1. In this case, the answer is $\boxed{2}$.

EXERCISE 1: Isolate the variable in **bold**. Answers for this chapter start on page 264.

1. $A = \pi \boldsymbol{r}^2$ $\sqrt{\frac{A}{\pi}}$
2. $C = 2\pi \boldsymbol{r}$ $\frac{C}{2\pi}$
3. $A = \frac{1}{2}\boldsymbol{b}h$ $\frac{2A}{h}$
4. $V = lw\boldsymbol{h}$ $\frac{V}{lh}$
5. $V = \pi r^2 \boldsymbol{h}$ $\frac{V}{\pi r^2}$
6. $V = \pi \boldsymbol{r}^2 h$ $\sqrt{\frac{V}{\pi h}}$
7. $c^2 = a^2 + \boldsymbol{b}^2$ $\sqrt{c^2 - a^2}$
8. $V = \boldsymbol{s}^3$ $\sqrt[3]{V} = s$
9. $S = 2\pi r\boldsymbol{h} + 2\pi r^2$ $S - 2\pi rh - 2\pi r^2 =$
10. $\frac{\boldsymbol{a}}{b} = \frac{c}{d}$ $a = \frac{cb}{d}$
11. $\frac{a}{b} = \frac{c}{\boldsymbol{d}}$ $\frac{cb}{a} = d$
12. $y = \boldsymbol{m}x + b$ $\frac{y-b}{x}$
13. $m = \frac{\boldsymbol{y_2} - y_1}{x_2 - x_1}$ $m(x_2 - x_1) + y_1$
14. $m = \frac{y_2 - y_1}{x_2 - \boldsymbol{x_1}}$
15. $v^2 = u^2 + 2\boldsymbol{a}s$ $\frac{v^2 - u^2}{2s}$
16. $\frac{a}{b} = \frac{x}{\boldsymbol{y}^2}$ $\sqrt{\frac{bx}{a}}$
17. $t = 2\pi\sqrt{\frac{L}{\boldsymbol{g}}}$ $\frac{t^2}{4\pi^2}$ $\frac{L4\pi^2}{t^2} = g$
18. $A = \pi r\sqrt{\boldsymbol{p} + q}$
19. If $X = \frac{X+1}{Y+Z}$, find X in terms of Y and Z.
20. If $x(y+2) = y$, find y in terms of x.
21. If $\frac{a}{b} = \frac{a+1}{2c}$, find a in terms of b and c.
22. If $t = \frac{2}{3}ax$, find ax in terms of t.
23. If $3x + 6y = 7z$, find $x + 2y$ in terms of z.
24. If $x + 5 = 2b$, find $2x + 10$ in terms of b.
25. If $\frac{a-1}{2t} = a$, find $4t$ in terms of a.
26. If $\frac{p-h}{p+h} = \frac{2}{3}$, find $\frac{p}{h}$.
27. If $\frac{1+2r}{1-t} = \frac{1}{2}$, find t in terms of r.
28. If $x^y = z$, then find x^{2y} in terms of z.
29. If $\frac{4^{x+1}}{x^3 - x^2} = p(x^5 - x^4)$, what is p in terms of x?
30. If $2^x\left(x^3 - \frac{1}{x}\right) = m(x^2+1) - \frac{1}{x^2}$, what is m in terms of x?
31. If $\frac{\sqrt{x}+1}{5x^2 - 3} - x^3 = \frac{1}{nx}$, what is n in terms of x?
32. If $a(b^2+2) + c = 5(c+1)^3$, what is a in terms of b and c?
33. If $k(x^2+4) + ky = \frac{7x^2+3}{2}$, what is k in terms of x and y?
34. If $ax + 3a + x + 3 = b$, what is x in terms of a and b?

CHAPTER EXERCISE: Answers for this chapter start on page 264.

A calculator should NOT be used on the following questions.

1

If $a + b = -2$, then $(a + b)^3 =$

A) 4

B) 0

C) -4

D) -8

2

For what value of n is $(n - 4)^2 = (n + 4)^2$?

$n^2 - 8n + 16 = n^2 + 8n + 16$

0

3

If $\frac{1}{a} \times \frac{b}{c} = 1$, what is the value of $b - ac$?

A) -3

B) 0

C) 2

D) It cannot be determined from the information given.

4

If $3x - 8 = -23$, what is the value of $6x - 7$?

A) -5

B) -21

C) -30

D) -37

5

If $\frac{4}{9} = \frac{8}{3}m$, what is the value of m?

A) $\frac{1}{6}$

B) $\frac{2}{3}$

C) $\frac{5}{6}$

D) 6

6

If $3x + 1 = -8$, what is the value of $(x + 2)^3$?

A) -1

B) 1

C) 8

D) 125

7

If $\frac{4}{k+2} = \frac{x}{3}$, where $k \neq -2$, what is k in terms of x?

A) $\frac{12 - 2x}{x}$

B) $\frac{12 + 2x}{x}$

C) $\frac{x}{12 + 2x}$

D) $12x - 2$

$\frac{k+2}{4} = \frac{3}{x}$

$\frac{12}{x} - 2$

8

If $(x - 3)^2 = 36$ and $x < 0$, what is the value of x^2?

$x^2 - 6x + 9 = 36$

9

9

$$f = p\left(\frac{(1+i)^n - 1}{i}\right)$$

The formula above gives the future value f of an annuity based on the monthly payment p, the interest rate i, and the number of months n. Which of the following gives p in terms of f, i, and n?

A) $\dfrac{fi}{(1+i)^n - 1}$

B) $\dfrac{(1+i)^n - 1}{fi}$

C) $\dfrac{f - i}{(1+i)^n - 1}$

D) $fi + 1 - (1+i)^n$

10

If $\dfrac{m}{2n} = 2$, what is the value of $\dfrac{n}{2m}$?

A) $\dfrac{1}{8}$

B) $\dfrac{1}{4}$

C) $\dfrac{1}{2}$

D) 1

$\frac{2n}{m} = \frac{1}{28}$

11

If $x < 0$ and $x^2 - 12 = 4$, what is the value of x?

A) -16

B) -8

C) -4

D) -2

12

If $x^2 + 7 = 21$, then what is the value of $x^2 + 3$? 17

14

13

$$x^2(x^4 - 9) = 8x^4$$

If $x > 0$, for what real value of x is the equation above true?

3

14

If $\dfrac{2\sqrt{x+4}}{3} = 6$ and $x > 0$, what is the value of x?

77

$x+4$

15

$$20 - \sqrt{x} = \frac{2}{3}\sqrt{x} + 10$$

If $x > 0$, for what value of x is the equation above true?

.36

A calculator is allowed on the following questions.

16

If $8 + 5x$ is twice $x - 5$, what is the value of x?

A) -6
B) -3
C) $-\frac{7}{3}$
D) -2

17

If $\frac{x}{6} = \frac{x + 12}{42}$, what is the value of $\frac{6}{x}$?

A) $\frac{1}{3}$
B) 2
C) 3
D) 6

18

c 23

$$d = a\left(\frac{c+1}{24}\right)$$

Doctors use Cowling's rule, shown above, to determine the right dosage d, in milligrams, of medication for a child based on the adult dosage a, in milligrams, and the child's age c, in years. Ben is a patient who is in need of a certain medication. If a doctor uses Cowling's rule to prescribe Ben a dosage that is half the adult dosage, what is Ben's age, in years?

A) 7
B) 9
C) 11
D) 13

19

If $3(x - 2y) - 3z = 0$, which of the following expresses x in terms of y and z?

A) $\frac{2y + 3z}{3}$
B) $2y + z$
C) $y + 2z$
D) $6y + 3z$

$-6y$ $-3z$

20

$$xy^2 + x - y^2 - 1 = 0$$

If the equation above is true for all real values of y, what must the value of x be?

$xy^2 - y^2 + x - 1 = 0$

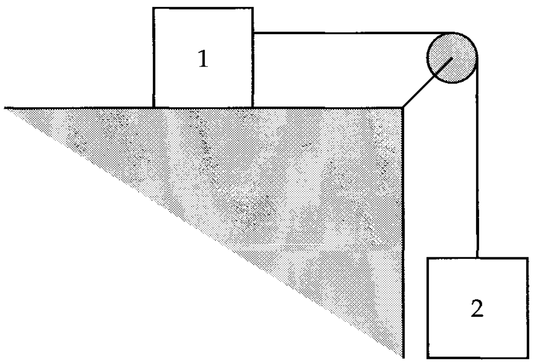
$y^2(x - 1) + (x - 1) = 0$

$(x - 1)(y^2 + 1)$

$a = 1$

$1 = \frac{1}{2}\left(\frac{c+1}{24}\right)$

Questions 21-22 refer to the following information.



In the figure above, two objects are connected by a string which is threaded through a pulley. Using its weight, object 2 moves object 1 along a flat surface. The acceleration a of the two objects can be determined by the following formula

$$a = \frac{m_2 g - \mu m_1 g}{m_1 + m_2}$$

where m_1 and m_2 are the masses of object 1 and object 2, respectively, in kilograms, g is the acceleration due to Earth's gravity measured in $\frac{\text{m}}{\text{sec}^2}$, and μ is a constant known as the coefficient of friction.

21

Which of the following expresses μ in terms of the other variables?

A) $\mu = \dfrac{a(m_1 + m_2)}{m_1 m_2 g^2}$

B) $\mu = \dfrac{a(m_1 + m_2)}{m_2 g - m_1 g}$

C) $\mu = \dfrac{m_2 g - a(m_1 + m_2)}{m_1 g}$

D) $\mu = \dfrac{a(m_1 + m_2) - m_2 g}{m_1 g}$

22

If the masses of both object 1 and object 2 were doubled, how would the acceleration of the two objects be affected?

A) The acceleration would stay the same.

B) The acceleration would be halved.

C) The acceleration would be doubled.

D) The acceleration would be quadrupled (multipled by a factor of 4).

$$a = \frac{2m_2 g - \mu 2m_1 g}{2m_1 + 2m_2}$$

$$= \frac{2(m_2 g - \mu m_1 g)}{2(m_1 + m_2)}$$

Questions 23-24 refer to the following information.

$$V = P(1-r)^t$$

The value V of a car depreciates over t years according to the formula above, where P is the original price and r is the annual rate of depreciation.

23

Which of the following expresses r in terms of V, P, and t?

A) $r = 1 - \sqrt[t]{\frac{V}{P}}$

B) $r = 1 + \sqrt[t]{\frac{V}{P}}$

C) $r = \sqrt[t]{\frac{V}{P}} - 1$

D) $r = 1 - \frac{\sqrt[t]{V}}{P}$

$\frac{V}{P} = (1-r)^t$

$\sqrt[t]{\frac{V}{P}} = (1-r)$

$-\sqrt{\frac{V}{P}} + 1$

24

If a car depreciates to a value equal to half its original price after 5 years, then which of the following is closest to the car's annual rate of depreciation?

A) 0.13

B) 0.15

C) 0.16

D) 0.2

$r = 1 - \sqrt[t]{\frac{1}{2}}$

$r = 1 - \sqrt[5]{\frac{1}{2}}$

8

More Equation Solving Strategies

In this chapter, we'll touch on two equation solving strategies that are necessary for some of the tougher questions.

1. Matching coefficients

EXAMPLE 1: If $(x+a)^2 = x^2 + 8x + b$, what is the value of b?

It's hard to see anything meaningful right away on both sides of the equation. So let's expand the left side first and see if that takes us anywhere.

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$$

So now we have

$$x^2 + 2ax + a^2 = x^2 + 8x + b$$

We can match up the coefficients.

$$x^2 + \underline{2a}x + \underline{a^2} = x^2 + \underline{8}x + \underline{b}$$

So,

$$2a = 8$$

$$a^2 = b$$

Solving the equations, $a = 4$ and $b = \boxed{16}$.

2. Clearing denominators

When you solve an equation like $\frac{1}{2}x + \frac{1}{3}x = 10$, a likely first step is to get rid of the fractions, which are harder to work with. How do we do that? By multiplying both sides by 6. But where did that 6 come from? 2 times 3. So this is what you're actually doing when you multiply by 6:

$$\frac{1}{2}x \cdot (2 \cdot 3) + \frac{1}{3}x \cdot (2 \cdot 3) = 10 \cdot (2 \cdot 3)$$

$$\frac{1}{\cancel{2}}x \cdot (\cancel{2} \cdot 3) + \frac{1}{\cancel{3}}x \cdot (2 \cdot \cancel{3}) = 10 \cdot (2 \cdot 3)$$

$$3x + 2x = 60$$

We got rid of the fractions by clearing the denominators. Here's the takeaway: we can do the same thing even when there are variables in the denominators.

EXAMPLE 2:

$$\frac{3}{x} + \frac{5}{x+2} = 2$$

If x is a solution to the equation above and $x > 0$, what is the value of x?

In the same way we multiplied by $2 \cdot 3$ before, we can multiply by $x(x+2)$ here.

$$\frac{3}{x} \cdot x(x+2) + \frac{5}{x+2} \cdot x(x+2) = 2 \cdot x(x+2)$$

$$\frac{3}{\cancel{x}} \cdot \cancel{x}(x+2) + \frac{5}{\cancel{x+2}} \cdot x\cancel{(x+2)} = 2x(x+2)$$

$$3(x+2) + 5x = 2x^2 + 4x$$

$$3x + 6 + 5x = 2x^2 + 4x$$

$$0 = 2x^2 - 4x - 6$$

$$0 = x^2 - 2x - 3$$

$$0 = (x-3)(x+1)$$

$x = 3$ or $x = -1$ but because $x > 0$, $x = \boxed{3}$.

Here's one final example that showcases both of the strategies in this chapter.

EXAMPLE 3:

$$\frac{3x}{x+1} + \frac{5}{ax+2} = \frac{-6x^2 + 11x + 5}{(x+1)(ax+2)}$$

In the equation above, $x \neq -\frac{2}{a}$ and a is a constant. What is the value of a?

A) -6 B) -2 C) 2 D) 6

Let's clear the denominators by multiplying both sides by $(x+1)(ax+2)$:

$$\frac{3x}{x+1} \cdot (x+1)(ax+2) + \frac{5}{ax+2} \cdot (x+1)(ax+2) = \frac{-6x^2 + 11x + 5}{(x+1)(ax+2)} \cdot (x+1)(ax+2)$$

$$\frac{3x}{\cancel{x+1}} \cdot \cancel{(x+1)}(ax+2) + \frac{5}{\cancel{ax+2}} \cdot (x+1)\cancel{(ax+2)} = \frac{-6x^2 + 11x + 5}{\cancel{(x+1)(ax+2)}} \cdot \cancel{(x+1)(ax+2)}$$

$$3x(ax+2) + 5(x+1) = -6x^2 + 11x + 5$$

$$3ax^2 + 6x + 5x + 5 = -6x^2 + 11x + 5$$

Comparing the coefficients of the x^2 term on either side, $3a = -6$. Therefore, $a = -2$. Answer $\boxed{(B)}$.

CHAPTER EXERCISE: Answers for this chapter start on page 269.

A calculator should NOT be used on the following questions.

1

If $(2x+3)(ax-5) = 12x^2 + bx - 15$ for all values of x, what is the value of b?

A) 6
B) 8
C) 10
D) 12

6

$2ax^2 - 10x + 3ax - 15$

$2ax^2 - x(3a - 10)$

2

If $(x+3y)^2 = x^2 + 9y^2 + 42$, what is the value of x^2y^2?

49

$x^2 + 6xy + 9y^2 = x^2 + 9y^2 + 42$

$6xy = 9y$

$9y^2 = 42$

3

If $\frac{ab+a}{b} = \frac{a}{b} + 5$ for all values of b, what is the value of a?

5

$\frac{ab+a}{b} = \frac{a+5b}{b}$

$ab + a = a + 5b$

$ab - 5b = -6$

$b(a-5)$

4

If $\frac{1}{x} - \frac{1}{x-4} = 1$, what is the value of x?

2

$(x-4) - x = x(x-4)$

$-4 = x^2 - 4x$

5

If $n < 0$ and $4x^2 + mx + 9 = (2x+n)^2$, what is the value of $m + n$?

A) -15
B) -9
C) -3
D) 12

$4x^2 + mx + 9 = 4x^2 + 4xn + n^2$

$m = 4, -3$

$n = -3$

6

If $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$, what is x in terms of p and y?

A) $p - y$
B) $\frac{py}{p+y}$
C) $\frac{py}{p-y}$
D) $\frac{py}{y-p}$

$\frac{1}{x} = \frac{1}{p} - \frac{1}{y}$

$\frac{1}{x} = \frac{y-p}{py}$

$x = \frac{py}{y-p}$

7

$$(2x+a)(3x+b) = 6x^2 + cx + 7$$

In the equation above, a and b are integers. If the equation is true for all values of x, what are the two possible values of c?

A) 8 and 12
B) 14 and 21
C) 15 and 18
D) 17 and 23

$6x^2 + 2bx + 3ax + ab$

$6x^2 + x(2b+3a) + ab$

$2b + 3a = c$

$ab = 7$

$b = \frac{7}{a}$

$2 \cdot \frac{7}{a} + 3a = c$

$-4 = x^2 - 4$

0

8

$$\frac{12x^2+mx+23}{2x+1} = 6x - 18 + \frac{41}{2x+1}$$

In the equation above, m is a constant and $x \neq -\frac{1}{2}$. What is the value of m?

A) -42
B) -36
C) -30
D) 42

$\frac{12x^2+mx+23}{2x+1}$

9

$$(x^3+kx^2-3)(x-2) = x^4+7x^3-18x^2-3x+6$$

In the equation above, k is a constant. If the equation is true for all values of x, what is the value of k?

A) -9
B) 5
C) 7
D) 9

$x^4 - 2x^3 + kx^3 - 3x + 6$

10

$$\frac{3}{n-1} + \frac{2n}{n+1} = 3$$

If $n > 0$, for what value of n is the equation above true?

$3(n+1) + 2n(n-1) = 3$

11

What is one possible solution to the equation $\frac{12}{x+2} - \frac{2}{x-2} = 1$?

4

12

$$\frac{4}{x-1} + \frac{2}{x+1} = \frac{35}{x^2-1}$$

If $x > 1$, what is the solution to the equation above?

$\frac{4(x+1)}{(x-1)(x+1)} + \frac{2(x-1)}{(x+1)(x-1)}$

$\frac{4x+4+2x-2}{(x-1)(x+1)} = \frac{35}{x^2-1}$

$\frac{6x+2}{(x-1)(x+1)} = \frac{35}{(x+1)(x-1)}$

$6x+2=35$
$6x = 33$
$x = \frac{33}{6}$

11

Word Problems

For many students, solving word problems is a frustrating experience. They require you to translate the question before you can even do the math. The examples and the exercises in this chapter will show you how to handle the full range of word problems that are tested. You will develop an instinct for translating words into math, setting the right variables, and finally solving for the answer. Experience is the best guide.

EXAMPLE 1: The sum of three consecutive integers is 72. What is the largest of these three integers?

The most important technique in solving word problems is to let a variable be one of the things you don't know. In this problem, we don't know any of the three integers, so we let the smallest one be x. It doesn't matter which number we set as x, as long as we're consistent throughout the problem.

So if x is the smallest, then our consecutive integers are

$$x, x+1, x+2$$

Because they sum to 72, we can make an equation:

$$\begin{aligned} x+(x+1)+(x+2) &= 72 \\ 3x+3 &= 72 \\ 3x &= 69 \\ x &= 23 \end{aligned}$$

And because x is the smallest, our three consecutive integers must be

$$23, 24, 25$$

The largest one is $\boxed{25}$.

What if we had let x be the largest integer? Our three integers would've been

$$x-2, x-1, x$$

And our solution would've looked like this:

$$\begin{aligned}(x-2)+(x-1)+x &= 72\\ 3x-3 &= 72\\ 3x &= 75\\ x &= 25\end{aligned}$$

And because x was set to be the largest of the three integers this time, we're already at the answer!

On SAT word problems, think about which unknown you want to set as a variable. Often times, that unknown will be what the question is asking for. Other times, it will be an unknown you specifically choose to make the problem easier to set up and solve. And sometimes, as was the case in Example 1, it doesn't matter which unknown you pick; you'll end up with the same answer with the same effort.

EXAMPLE 2: One number is 3 times another number. If they sum to 44, what is the larger of the two numbers?

In this problem, we want to set x to be the smaller of the two numbers. That way, the two numbers can be expressed as

$$x \text{ and } 3x$$

If we let x be the larger of the two, we would have to work with

$$x \text{ and } \frac{x}{3}$$

and fractions are yucky.

Setting up our equation,

$$\begin{aligned}x+3x &= 44\\ 4x &= 44\\ x &= 11\end{aligned}$$

Be careful—we're not done yet! The question asks for the larger of the two, so we have to multiply x by 3 to get $\boxed{33}$.

EXAMPLE 3: What is a number such that the square of the number is equal to 2.7% of its reciprocal?

Let the number we're looking for be x.

$$x^2 = .027 \times \frac{1}{x}$$

Multiply both sides by x to isolate it.

$$x^3 = .027$$

Cube root both sides.

$$x = \boxed{.3}$$

EXAMPLE 4: Albert is 7 years older than Henry. In 5 years, Albert will be twice as old as Henry. How old is Albert now?

Let x be Albert's age now. We could've assigned x to be Henry's age, but as we mentioned earlier, assigning the variable to be what the question is asking for is typically the faster route. Now at this point, some of you might be thinking of assigning another variable to Henry's age. While that would certainly work, it would only add more steps to the solution. Try to stick to one variable unless the question clearly calls for more.

If Albert is x years old now, then Henry must be $x - 7$ years old.
Five years from now, Albert will be $x + 5$ and Henry will be $x - 2$ years old.

$$x + 5 = 2(x - 2)$$
$$x + 5 = 2x - 4$$
$$x = \boxed{9}$$

EXAMPLE 5: Jake can run 60 yards per minute. Amy can run 120 yards per minute for the first 10 minutes but then slows down to 20 yards per minute thereafter. If they start running at the same time, after how many minutes t will both Jake and Amy have run the same distance, assuming $t > 10$?

The problem already gives us a variable t to work with. We want to equate Jake's distance run with Amy's.

Jake's distance: $60t$
Amy's distance: $120(10) + 20(t - 10)$

$$60t = 120(10) + 20(t - 10)$$
$$60t = 1,200 + 20t - 200$$
$$40t = 1,000$$
$$t = 25$$

After $\boxed{25}$ minutes, they will have run the same distance.

EXAMPLE 6: At a pharmaceutical company, research equipment must be shared among the scientists. There is one microscope for every 4 scientists, one centrifuge for every 3 scientists, and one freezer for every 2 scientists. If there is a total of 52 pieces of research equipment at this company, how many scientists are there?

Let x be the number of scientists. Then the number of microscopes is $\frac{x}{4}$, the number of centrifuges is $\frac{x}{3}$, and the number of freezers is $\frac{x}{2}$.

$$\frac{x}{4} + \frac{x}{3} + \frac{x}{2} = 52$$

Multiply both sides by 12 to get rid of the fractions,

$$3x + 4x + 6x = 52 \cdot 12$$
$$13x = 624$$
$$x = \boxed{48}$$

EXAMPLE 7: Mark and Kevin own $\frac{1}{4}$ and $\frac{1}{3}$ of the books on a shelf, respectively. Lori owns the rest of the books. If Kevin owns 9 more books than Mark, how many books does Lori own?

Let x be the total number of books. Mark then has $\frac{1}{4}x$ books and Kevin has $\frac{1}{3}x$ books. Kevin owns 9 more than Mark, so

$$\frac{1}{3}x - \frac{1}{4}x = 9$$

Multiplying both sides by 12,

$$\begin{aligned} 4x - 3x &= 108 \\ x &= 108 \end{aligned}$$

The total number of books is 108. Mark owns $\frac{1}{4} \times 108 = 27$ books and Kevin owns $\frac{1}{3} \times 108 = 36$ books. Lori must then own $108 - 27 - 36 = \boxed{45}$ books.

EXAMPLE 8: A group of friends wants to split the cost of renting a cabin equally. If each friend pays \$130, they will have \$10 too much. If each friend pays \$120, they will have \$50 too little. How much does it cost to rent the cabin?

We have two unknowns in this problem. We'll let the number of people in the group be n and the cost of renting a cabin be c. From the information given, we can come up with two equations (make sure you see the reasoning behind them):

$$\begin{aligned} 130n - 10 &= c \\ 120n + 50 &= c \end{aligned}$$

In the first equation, $130n$ represents the total amount the group pays, but because that's 10 dollars too much, we need to subtract 10 to arrive at the cost of rent, c. In the second equation, $120n$ represents the total amount the group pays, but this time it's 50 dollars too little, so we need to add 50 to arrive at c. Substituting c from the first equation into the second, we get

$$\begin{aligned} 120n + 50 &= 130n - 10 \\ -10n &= -60 \\ n &= 6 \end{aligned}$$

So there are 6 friends in the group. And

$$c = 130n - 10 = 130 \cdot 6 - 10 = 770$$

The cost of renting the cabin is $\boxed{770}$.

EXAMPLE 9: Of the 200 jellybeans in a jar, 70% are green and the rest are red. How many green jellybeans must be removed so that 60% of the remaining jellybeans are green?

The answer is NOT 20. You can't just take 10% of the green jellybeans away because as you do that, the total number of jellybeans also goes down. We first find that there are $\frac{7}{10} \times 200 = 140$ green jellybeans. We need to remove x of them so that 60% of what's left is green:

$$\frac{\text{green jellybeans left}}{\text{total jellybeans left}} = 60\%$$

$$\frac{140 - x}{200 - x} = \frac{6}{10}$$

Cross multiplying,

$$\begin{aligned} 10(140 - x) &= 6(200 - x) \\ 1,400 - 10x &= 1,200 - 6x \\ 200 &= 4x \\ x &= 50 \end{aligned}$$

$\boxed{50}$ green jellybeans need to be removed.

EXAMPLE 10: Altogether, David and Robert have 120 baseball cards. David gives Robert one third of his cards and then 10 more cards. Robert now has five times as many cards as David. How many cards did Robert have originally?

Solution 1: This question is really tough and tricky. When David gives Robert some of his cards, David loses at the same time Robert gains. We could set a variable for David and another variable for Robert, but that solution is a little messier (see Solution 2).

Instead, let's work backwards. If x is the number of cards David ends up with, then Robert ends up with $5x$ cards. Because there are 120 cards altogether,

$$\begin{aligned} x + 5x &= 120 \\ 6x &= 120 \\ x &= 20 \end{aligned}$$

So David has 20 cards and Robert has 100 cards at the end. Let's rollback another transaction. David had given Robert 10 cards. So before that happened, David must have had $20 + 10 = 30$ cards. Rollback another transaction and we see that David had given a third of his cards away to get down to the 30 that we just calculated. Well if he had given away a third, then the 30 he had left must have represented two-thirds of the cards he had at the start. Let d be the number of cards David had at the start.

$$\begin{aligned} \frac{2}{3}d &= 30 \\ 2d &= 90 \\ d &= 45 \end{aligned}$$

So David had 45 cards at the start, which means Robert must have had $120 - 45 = \boxed{75}$ cards at the start.

Solution 2: Let x be the number of cards David starts with and y be the number of cards Robert starts with. Here are the equations I would set up:

$$x + y = 120$$
$$y + \frac{1}{3}x + 10 = 5(x - \frac{1}{3}x - 10)$$

Multiplying the second equation by 3,

$$x + y = 120$$
$$3y + x + 30 = 15x - 5x - 150$$

Shifting things over,

$$x + y = 120$$
$$9x - 3y = 180$$

At this point, we can use substitution or elimination. I'm going to use substitution. From the first equation, $x = 120 - y$. Plugging this into the second equation,

$$\begin{aligned} 9(120 - y) - 3y &= 180 \\ 1080 - 12y &= 180 \\ -12y &= -900 \\ y &= \boxed{75} \end{aligned}$$

CHAPTER EXERCISE: Answers for this chapter start on page 277.

A calculator should NOT be used on the following questions.

1

Which of the following represents the square of the sum of x and y, decreased by the product of x and y?

A) $x^2 + y^2 - xy$
B) $x^2y^2 - xy$
C) $(x+y)^2 - (x+y)$
D) $(x+y)^2 - xy$

2

On a 100 cm ruler, lines are drawn at 10, X, and 98 cm. The distance between the lines at X and 98 cm is three times the distance between the lines at X and 10 cm. What is the value of X? 32

$|x - 98| = 3 \cdot |x - 10|$

3

If 5 is added to the square root of x, the result is 9. What is the value of $x + 2$? 18

$\sqrt{x} + 5 = 9$

2

4

A grocery store sells tomatoes in boxes of 4 or 10. If Melanie buys x boxes of 4 and y boxes of 10, where $x \geq 1$ and $y \geq 1$, for a total of 60 tomatoes, what is one possible value of x? 5

$4x + 10y = 60$

5 4

5

A retail store has monthly fixed costs of \$3,000 and monthly salary costs of \$2,500 for each employee. If the store hires x employees for an entire year, which of the following equations represents the store's total cost c, in dollars, for the year?

A) $c = 3{,}000 + 2{,}500x$
B) $c = 12(3{,}000 + 2{,}500x)$
C) $c = 12(3{,}000) + 2{,}500x$
D) $c = 3{,}000 + 12(2{,}500x)$

6

Susie buys 2 pieces of salmon, each weighing x pounds, and 1 piece of trout, weighing y pounds, where x and y are integers. The salmon cost \$3.50 per pound and the trout cost \$5 per pound. If the total cost of the fish was \$77, which of the following could be the value of y?

A) 4
B) 5
C) 6
D) 7

$3.5 \cdot 2x + 5y = 77$

$\frac{3.5 \times 2x - 77}{5} = y$

A calculator is allowed on the following questions.

7

If 75% of 68 is the same as 85% of n, what is the value of n? 66

8

The Pirates won exactly 4 of their first 15 games. They then played N remaining games and won all of them. If they won exactly half of all the games they played, what is the value of N?

7

9

Alice and Julie start with the same number of pens. After Alice gives 16 of her pens to Julie, Julie then has two times as many pens as Alice does. How many pens did Alice have at the start?

48

$x - 16 = x + 16$

$x - 16 = 2 \cdot (x + 16)$

10

At a Hong Kong learning center, $\frac{1}{4}$ of the students take debate, $\frac{1}{6}$ of the students take writing, and $\frac{1}{8}$ of the students take science. The rest take math. If 33 students take math, what is the total number of students at the learning center?

A) 60
B) 66
C) 72
D) 78

11

Ian has 20 football cards, and Jason has 44 baseball cards. They agree to trade such that Jason gives Ian 2 baseball cards for every card Ian gives to Jason. After how many such trades will Ian and Jason each have an equal number of cards?

$20 - x = 44 - 2y$

$44 - 2 \cdot 10$

24

A) 9
B) 10
C) 11
D) 12

12

If 3 is subtracted from 3 times the number x, the result is 21. What is the result when 8 is added to half of x?

$3x - 3 = 21$

A) 1
B) 5
C) 8
D) 12

13

At a store, the price of a tie is k dollars less than three times the price of a shirt. If a shirt costs \$40 and a tie costs \$30, what is the value of k?

30

$43s - t = k$

$3

14

A bakery gave out coupons to celebrate its grand opening. Each coupon was worth either $1, $3, or $5. Twice as many $1 coupons were given out as $3 coupons, and 3 times as many $3 coupons were given out as $5 coupons. The total value of all the coupons given out was $360. How many $3 coupons were given out?

A) 40
B) 45
C) 48
D) 54

$\frac{x}{2} = x$

x

15

Alex, Bob, and Carl all collect seashells. Bob has half as many seashells as Carl. Alex has three times as many seashells as Bob. If Alex and Bob together have 60 seashells, how many seashells does Carl have?

A) 15
B) 20
C) 30
D) 40

$B = \frac{C}{2}$

$A = 3B$

$A + B$

$A + 3B = 60 - 3B$

$\frac{C}{2} + 3B = 6C$

$\frac{a}{2} + 3 \cdot \frac{C}{2} - - = 6C$

10

A line with a slope of $\frac{2}{3}$ passes through the points $(1,4)$ and $(x,10)$. What is the value of x?

A) 4
B) 6
C) 8
D) 10

$b = \frac{10}{3}$

11

If $f(x)$ is a linear function such that $f(2) \leq f(3)$, $f(4) \geq f(5)$, and $f(6) = 10$, which of the following must be true?

A) $f(3) < f(0) < f(4)$
B) $f(0) = 0$
C) $f(0) > 10$
D) $f(0) = 10$

$f(0)$ $f(2)$ $f(3)$ $f(4)$ $f(5)$

12

$$y = \frac{a}{b}x + c$$

$$y = \frac{d}{e}x + c$$

$-\frac{1}{\frac{a}{b}} = \frac{-b}{a}$

The equations of two perpendicular lines in the xy-plane are shown above, where $a, b, c, d,$ and e are constants. If $0 < \frac{a}{b} < 1$, which of the following must be true?

A) $\frac{d}{e} < -1$

B) $-1 < \frac{d}{e} < 0$

C) $0 < \frac{d}{e} < 1$

D) $\frac{d}{e} > 1$

It's important that you don't get tricked into choosing a rate that looks right but ultimately doesn't fit the context set by the variables x and y (in this case, T and V). Answer (A) is wrong because we're not dealing with the number of homes sold; we're dealing with the value of a home. Answer (D) is wrong because the numbers in the equation aren't on a per-square-foot basis. Always be aware of the variables you're working with.

EXAMPLE 2: The maximum height of a plant h, in inches, can be determined by the equation $h = \frac{4x+6}{5}$, where x is the amount of fertilizer, in grams, used to grow the plant.

PART 1: According to the equation, one more gram of fertilizer would increase the maximum height of a plant by how many inches?

PART 2: To raise the maximum height of a plant by exactly one inch, how many more grams of fertilizer should be used in growing the plant?

Part 1 Solution: This question is essentially asking for the change in h for every 1 unit increase in x. This is the slope. From the equation, we can see that the slope is $\frac{4}{5}$, or $\boxed{0.8}$. To make this even clearer, we can put the equation into $y = mx + b$ form by splitting up the fraction: $h = \frac{4}{5}x + \frac{6}{5}$. Note that when we're dealing with changes in x and y, the y-intercept b is irrelevant because it's a constant that's always there.

Part 2 Solution: Because this question is asking for the change in x for every 1 unit increase in h, the reverse of Part 1, we need to rearrange the equation so that we have x in terms of h.

$$h = \frac{4x+6}{5}$$

$$5h = 4x + 6$$

$$5h - 6 = 4x$$

$$x = \frac{5}{4}h - \frac{3}{2}$$

Now we can see that x increases by $\frac{5}{4}$, or $\boxed{1.25}$, when h increases by 1. The answer is just the slope of our new equation. A shortcut for this type of question is to take the reciprocal of the slope of the original equation. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

EXAMPLE 3:

$$T = 65 - 6m$$

A can of soda is put into a freezer. The temperature T of the soda, in degrees Fahrenheit, can be found by using the equation above, where m is the number of minutes the can has been in the freezer. What is the decrease in the temperature of the soda, in degrees Fahrenheit, for every 5 minutes the can is left in the freezer?

The slope of -6 represents the change in the temperature for every 1 minute the can is left in the freezer. So for every 5 minutes, the temperature of the soda decreases by $5 \times 6 = \boxed{30}$ degrees Fahrenheit.

CHAPTER EXERCISE: Answers for this chapter start on page 281.

A calculator should NOT be used on the following questions.

1

The water level h, in feet, in a large aquarium can be modeled by $h = 100 - 3d$, where d is the number of days that have passed since the aquarium was last refilled. Based on the model, how does the water level change each day?

A) Decreases by 3 feet

B) Increases by 3 feet

C) Decreases by 100 feet

D) Increases by 100 feet

2

The number of loaves of bread b remaining in a bakery each day can be estimated by the equation $b = 200 - 18h$, where h is the number of hours that have passed since the store's opening. What is the meaning of the value 18 in this equation?

A) The bakery sells all its loaves of bread in 18 hours.

B) The bakery sells 18 loaves of bread each hour.

C) The bakery sells a total of 18 loaves of bread each day.

D) There are 18 loaves of bread left in the bakery at the end of each day.

3

A membership website offers video tutorials on programming. The number of members, m, subscribed to the site can be estimated by the equation $m = 500 + 200n$, where n is the number of videos available on the site. Based on the equation, which of the following statements is true?

A) For every one additional video, the site gains 500 new members.

B) The site initially made 200 videos available to members.

C) The site was able to get 500 members without any available videos.

D) The site gains 500 new members for every 200 additional videos available on the site.

4

$$s = 10 - 2h$$

A recipe suggests sweetening honey tea with sugar. The equation above can be used to determine the amount of sugar s, in teaspoons, that should be added to a tea beverage with h teaspoons of honey. What is the meaning of the 2 in the equation?

A) For every teaspoon of honey in the beverage, two more teaspoons of sugar should be added.

B) For every teaspoon of honey in the beverage, two fewer teaspoons of sugar should be added.

C) For every two teaspoons of honey in the beverage, one more teaspoon of sugar should be added.

D) For every two teaspoons of honey in the beverage, one fewer teaspoon of sugar should be added.

5

The monthly salary of a salesperson at a used car dealership is determined by the expression $1,000 + 2,000xc$, where x is the salesperson's commission rate and c is the number of cars sold by the salesperson. Which of the following statements is the best interpretation of the number 2,000 in the context of this problem?

A) The average price of a used car at the dealership

B) The base monthly salary of a salesperson at the dealership

C) The average monthly commission earned by each salesperson at the dealership

D) The average number of cars sold by the dealership each month

6

$$p = 2,000s + 15,000$$

A state government uses the equation above to estimate the average population p for a town with s schools. Which of the following best describes the meaning of the number 2,000 in the equation?

A) The average number of students at each school in a town

B) The average number of schools in each town

C) The estimated increase in a town's population for each additional school

D) The estimated population of a town without any schools

7

$$h = 100 - 4t$$

The equation above can be used to model the number of hours h until a gallon of milk held at a temperature of t, in degrees Celsius, goes sour. Based on the model, which of the following is the best interpretation of the number 4 in the equation?

A) An increase of $1°C$ will make a gallon of milk go sour 4 hours faster.

B) An increase of $1°C$ will make 4 gallons of milk go sour 1 hour faster.

C) An increase of $4°C$ will make a gallon of milk go sour 1 hour faster.

D) An increase of $4°C$ will make a gallon of milk go sour 4 hours faster.

8

An antique lamp was sold at an auction. The price p of the lamp, in dollars, during the auction can be modeled by the equation $p = 900 - 10t$, where t is the number of seconds left in the auction. According to the model, what is the meaning of the 900 in the equation?

A) The starting auction price of the lamp

B) The final auction price of the lamp

C) The increase in the price of the lamp per second

D) The time it took to auction off the lamp, in seconds

9

$$y = 1.30x - 1.50$$

A bank teller uses the equation above to exchange U.S. dollars into euros, where y is the euro amount and x is the U.S. dollar amount. Which of the following is the best interpretation of the 1.50 in the equation?

A) The bank charges 1.50 euros to do the currency exchange.
B) The bank charges 1.50 U.S. dollars to do the currency exchange.
C) One U.S. dollar is worth 1.50 euros.
D) One euro is worth 1.50 U.S. dollars.

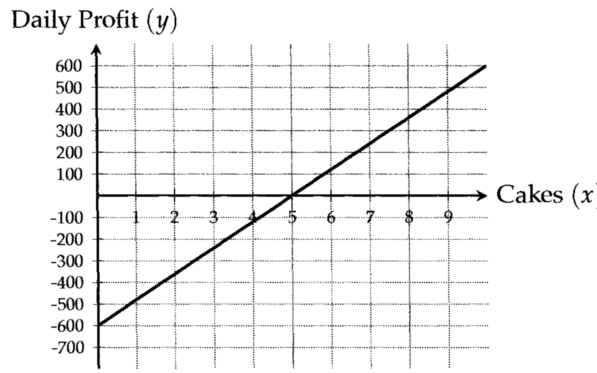
A calculator is allowed on the following questions.

10

$$t = \frac{2x + 9}{5}$$

The equation above models the time t, in seconds, it takes to load a web page with x images. Based on the model, by how many seconds does each image increase the load time of a web page?

Questions 11-13 refer to the following information.



The relationship between the daily profit y, in dollars, of a bakery and the number of cakes sold by the bakery is graphed in the xy-plane above.

11

What does the slope of the line represent?

A) The price of each cake
B) The profit generated from each cake sold
C) The daily profit generated from all the cakes that were sold
D) The number of cakes that need to be sold to make a daily profit of 100 dollars

12

Which of the following is the best interpretation of the y-intercept in the context of this problem?

A) The price of each cake
B) The cost of making each cake
C) The daily costs of running a bakery
D) The daily cost of making the cakes that weren't able to be sold

13

What does it mean that $(5, 0)$ is a solution to the equation of the line?

A) The bakery needs to sell 5 cakes per day to cover its daily expenses.

B) Each cake must be sold for at least 5 dollars to cover the cost of making it.

C) It costs 5 dollars to make each cake.

D) Each day, the bakery gives the first 5 cakes away for free.

14

$$T = 56 + 5h$$

To warm up his room, Patrick turns on the heater. The temperature T of his room, in degrees Fahrenheit, can be modeled by the equation above, where h is the number of hours since the heater started running. Based on the model, what is the temperature increase, in degrees Fahrenheit, for every 30 minutes the heater is turned on?

15

$$2y - x = 14$$

Alice owns a pet frog but would like to add turtles to the same tank. The local veterinarian uses the equation above to determine the total amount of water y, in gallons, that should be held in the tank for x turtles to thrive alongside Alice's frog. Based on the equation, which of the following must be true?

I. One additional gallon of water can support two more turtles.
II. One additional turtle requires two more gallons of water.
III. One more turtle requires an additional half a gallon of water.

A) II only

B) III only

C) I and II only

D) I and III only

16

$$C = 1.5 + 2.5x$$

A local post office uses the equation above to determine the cost C, in dollars, of mailing a shipment weighing x pounds. An increase of 10 dollars in the mailing cost is equivalent to an increase of how many pounds in the weight of the shipment?

A) 2

B) 2.5

C) 4

D) 5

14 Functions

A function is a machine that takes an input, transforms it, and spits out an output. In math, functions are denoted by $f(x)$, with x being the input. So for the function

$$f(x) = x^2 + 1$$

every input is squared and then added to one to get the output. It's important to understand that x is a completely arbitrary label—it's just a placeholder for the input. In fact, I can put in whatever I want as the input, including values with x in them:

$$\begin{aligned}
f(2x) &= (2x)^2 + 1 \\
f(a) &= a^2 + 1 \\
f(b+1) &= (b+1)^2 + 1 \\
f(\star) &= (\star)^2 + 1 \\
f(\text{Panda}) &= (\text{Panda})^2 + 1
\end{aligned}$$

Notice the careful use of parentheses. In the first equation, for example, $(2x)^2$ is not the same as $2x^2$. Wrap the input in parentheses and you'll never go wrong.

EXAMPLE 1: If $f(x) = (x+1)^x$, then what is the value of $f(0) + f(1) + f(2) + f(3)$?

Just plug in the inputs.

$$\begin{aligned}
f(0) + f(1) + f(2) + f(3) &= (0+1)^0 + (1+1)^1 + (2+1)^2 + (3+1)^3 \\
&= 1^0 + 2^1 + 3^2 + 4^3 \\
&= 1 + 2 + 9 + 64 \\
&= \boxed{76}
\end{aligned}$$

EXAMPLE 2:

$$f(x) = \frac{4}{x^2 - 10x + 25}$$

For what value of x is the function f above undefined?

Because we can't divide by 0, **a function is undefined when the denominator is zero.** Setting the denominator to zero,

$$x^2 - 10x + 25 = 0$$
$$(x-5)^2 = 0$$
$$x = \boxed{5}$$

EXAMPLE 3: If $f(x-1) = 6x$ and $g(x) = x+3$, what is the value of $f(g(2))$?

Whenever you see composite functions (functions of other functions), start from the inside and work your way out. First,

$$g(2) = 2 + 3 = 5$$

Now we have to figure out the value of $f(5)$.

Well, we can plug in $x = 6$ into $f(x-1) = 6x$ to get $f(5) = 6(6) = \boxed{36}$.

EXAMPLE 4: Functions f and g are defined by $f(x) = x + 1$ and $g(x) = \frac{x}{2}$. If $f(g(f(k))) = 10$, what is the value of k?

Again, we start from the inside and work our way out:

$$f(k) = k + 1$$
$$g(k+1) = \frac{k+1}{2}$$
$$f\left(\frac{k+1}{2}\right) = \frac{k+1}{2} + 1$$

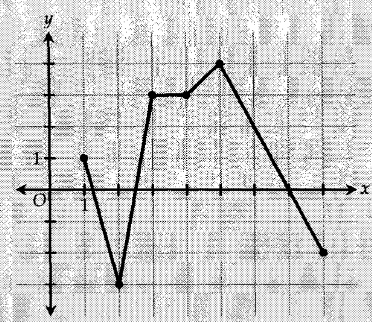
Finally,

$$\frac{k+1}{2} + 1 = 10$$
$$\frac{k+1}{2} = 9$$
$$k + 1 = 18$$
$$k = \boxed{17}$$

As we've mentioned, a function takes an input and returns an output. Well, these input and output pairs allow us to graph any function as a set of points in the xy-plane, with the input as x and the output as y. In fact, $y = x^2 + 1$ is the same as $f(x) = x^2 + 1$. Both $f(x)$ and y are the same thing—they're used to denote the output. The only reason we use y is that it's consistent with the y-axis being the y-axis.

Anytime $f(x)$ is used in a graphing question, think of it as the y. So if a question states that $f(x) > 0$, all y values are positive and the graph is always above the x-axis. It's extremely important that you learn to think of points on a graph as the inputs and outputs of a function.

EXAMPLE 5:



The graph of $f(x)$ is shown in the xy-plane above. For what value of x is $f(x)$ at its maximum?

Again, when it comes to graphs, interpret $f(x)$ as the y. This question is asking for the point on the graph with the highest y-value. That point is $(5,4)$. The x-value there is $\boxed{5}$.

EXAMPLE 6: If the function with equation $y = ax^2 + 3$ crosses the point $(1,2)$, what is the value of a?

Remember—a point is just an input and an output, an x and a y. Because $(1,2)$ is a point on the graph of the function, we can plug in 1 for x and 2 for y.

$$2 = a(1)^2 + 3$$
$$2 = a + 3$$
$$a = \boxed{-1}$$

EXAMPLE 7: If the function $y = x^2 + 2x - 4$ contains the point $(m, 2m)$ and $m > 0$, what is the value of m?

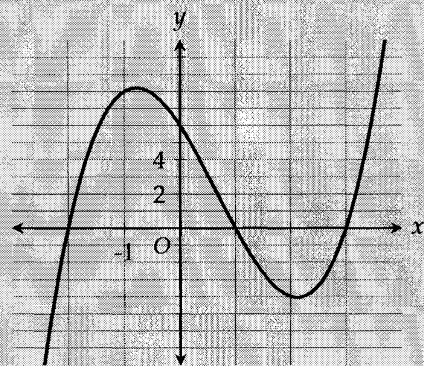
It's important not to get intimidated by all the variables. The question gives us a point on the graph, so let's plug it in.

$$y = x^2 + 2x - 4$$
$$2m = m^2 + 2m - 4$$
$$0 = m^2 - 4$$

From here we can see that $m = \pm 2$. The question states that $m > 0$, so $\boxed{m = 2}$.

The **zeros, roots, and x-intercepts** of a function are all just different terms for the same thing—the values of x that make $f(x) = 0$. Graphically, they refer to the values of x where the function crosses the x-axis.

EXAMPLE 8:



The graph of $f(x) = x^3 - 2x^2 - 5x + 6$ is shown in the xy-plane above.

PART 1: How many distinct zeros does the function f have?

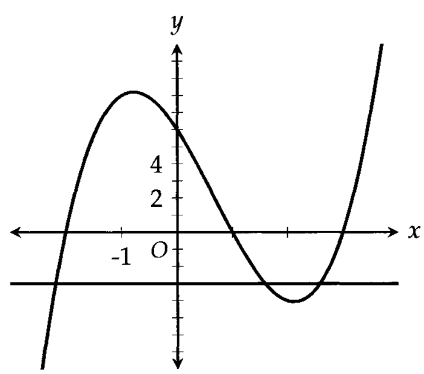
PART 2: If k is a constant such that $f(x) = k$ has 1 solution, which of the following could be the value of k?

A) -3 B) 1 C) 5 D) 9

Part 1 Solution: The graph crosses the x-axis three times, so f has $\boxed{3}$ distinct zeros. From the graph, we can see that these zeros are $-2, 1$, and 3.

Part 2 Solution: This question is quite involved, so don't panic if you feel lost during the explanation. Read all the way through and then go back to the bits that were confusing. I promise you'll be able to make sense of everything.

To truly understand this question, first realize that a constant is just a function. No matter the input, we always get the same output. In this question, we can write it as $y = k$ or $g(x) = k$. So let's say $k = -3$. What does $y = -3$ look like? A horizontal line at -3!



Now when a question asks for the **solutions** to $f(x) = k$, it's merely referring to the intersection points of $f(x)$ and the horizontal line $y = k$. In general, if a question sets two functions equal to each other, $f(x) = g(x)$, and asks you about the solutions, it's referring to the intersection points. After all, it's only at the intersection points that the value of y is the same for both functions. In this particular case, $g(x)$ just happens to be a constant function, $g(x) = k$.

The number of solutions is equivalent to the number of intersection points. So if $k = -3$ as shown above, there must be 3 solutions to $f(x) = -3$, as represented by the 3 intersection points. The solutions themselves are the x-values of those points. We can estimate them to be $-2.2, 1.6$, and 2.6.

Getting back to the original problem, we have to choose a k such that there is only one solution. Now we're thinking backwards. Instead of being given the constant, we have to choose it. Where might we place a horizontal line so that there's only one intersection point? Certainly not at -3 because we just showed how that would result in 3 solutions.

Well, looking back at the graph, we could place one just above 8 or just below -4. Horizontal lines at these values would intersect with $f(x)$ just once. Looking at the answer choices, 9 is the only one that meets our condition. Answer $\boxed{(D)}$.

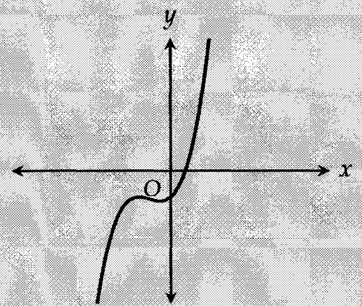
Let's take a moment to revisit part 1. In part 1, we found the number of intersection points between $f(x)$ and the x-axis. But realize that the x-axis is just the horizontal line $y = 0$. In counting the number of intersection points between $f(x)$ and the horizontal line $y = 0$, what you were really doing is finding the number of solutions to $f(x) = 0$.

If you didn't grasp everything in this example the first time through, it's ok. Take your time and go through it again, making sure you fully understand each of the concepts. The SAT will throw quite a few questions at you related to the zeros of functions as well as the solutions to $f(x) = g(x)$.

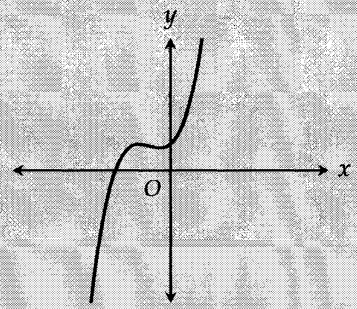
Hopefully by now, you're starting to see constants as horizontal lines. So for instance, if $f(x) > 5$, that means the entire graph of f is above the horizontal line $y = 5$. Thinking of constants in this way will help you on a lot of SAT graph questions.

EXAMPLE 9: Which of the following could be the graph of $y = x^3 + 2x^2 + x + 1$?

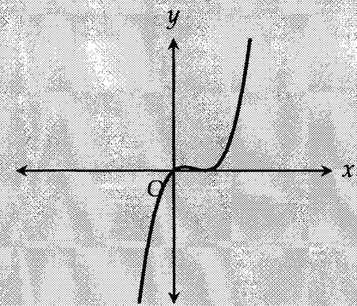
(A)



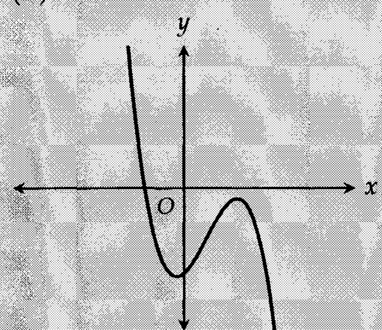
(B)



(C)



(D)



Although the given function looks complicated and you might be tempted to graph it on your calculator, this is the easiest question ever! All you have to do is find a point that's certain to be on the graph and eliminate the graphs that don't have that point. So what's an easy point to find and test?

Plug in $x = 0$ to get $y = 1$. Now which graphs contain the coordinate $(0, 1)$? Only graph $\boxed{(B)}$.

By the way, numbers like 0 and 1 are particularly good for finding "easy" points to use for this strategy.

CHAPTER EXERCISE: Answers for this chapter start on page 283.

A calculator should NOT be used on the following questions.

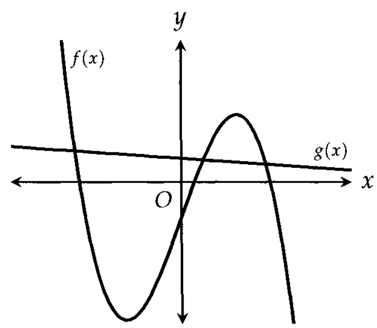
1

x	y
0	20
1	21
3	29

The table above displays several points on the graph of the function f in the xy-plane. Which of the following could be $f(x)$?

A) $f(x) = 20x$
B) $f(x) = x + 20$
C) $f(x) = x - 20$
D) $f(x) = x^2 + 20$

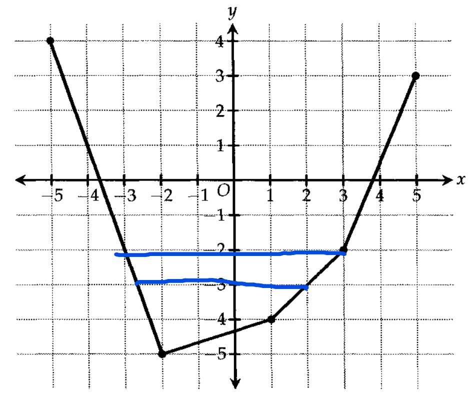
2



In the portion of the xy-plane shown above, for how many values of x does $f(x) = g(x)$?

A) None
B) One
C) Two
D) Three

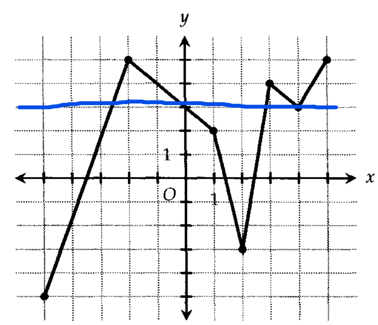
3



The graph of the function f is shown in the xy-plane above. If $f(a) = f(3)$, which of the following could be the value of a?

A) -4
B) -3
C) -2
D) 1

4



The function f is graphed in the xy-plane above. For how many values of x does $f(x) = 3$?

A) Two
B) Three
C) Four
D) Five

5

For which of the following functions is it true that $f(-3) = f(3)$?

A) $f(x) = \frac{2}{x}$

B) $f(x) = \frac{x^3}{3}$

C) $f(x) = 3x^2 + 1$

D) $f(x) = x + 2$

6

The function f is defined by $f(x) = 3x + 2$ and the function g is defined by $g(x) = f(2x) - 1$. What is the value of $g(10)$? 61

7

If $f(x) = \frac{16 + x^2}{2x}$ for all $x \neq 0$, what is the value of $f(-4)$?

A) -8

B) -4

C) 4

D) 8

8

x	0	1	2
$f(x)$	-2	3	18

Several values of the function f are given in the table above. If $f(x) = ax^2 + b$ where a and b are constants, what is the value of $f(3)$?

A) 23

B) 39

C) 43

D) 56

9

If $f(x) = x^2$, for which of the following values of c is $f(c) < c$?

A) $\frac{1}{2}$

B) 1

C) $\frac{3}{2}$

D) 2

10

If the graph of the function f has x-intercepts at -3 and 2, and a y-intercept at 12, which of the following could define f?

A) $f(x) = (x + 3)^2(x - 2)$

B) $f(x) = (x + 3)(x - 2)^2$

C) $f(x) = (x - 3)^2(x + 2)$

D) $f(x) = (x - 3)(x + 2)^2$

11

$$f(x) = x^2 + 1$$
$$g(x) = x^2 - 1$$

The functions f and g are defined above. What is the value of $f(g(2))$?

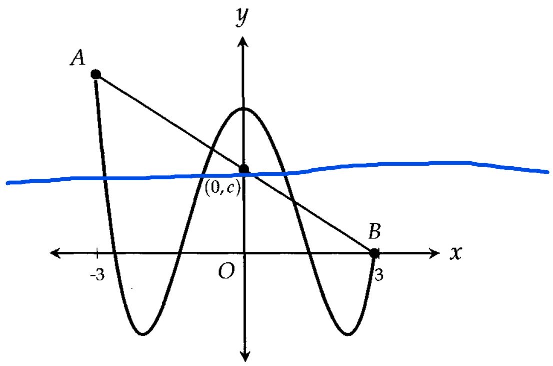
A) 3

B) 5

C) 10

D) 17

12



The graph of the function f and line segment $\overline{AB}$ are shown in the xy-plane above. For how many values of x between -3 and 3 does $f(x) = c$?

3

13

x	$f(x)$
-4	3
-2	5
0	2
2	16
3	4
4	8

The table above gives some values for the function f. If $g(x) = 2f(x)$, what is the value of k if $g(k) = 8$?

A) 2
B) 3
C) 4
D) 8

14

$$f(x) = \sqrt{x-2}$$

The function f is defined above for all $x \geq 2$. Which of the following is equal to $f(18) - f(11)$?

A) $f(3)$
B) $f(5)$
C) $f(6)$
D) $f(7)$

A calculator is allowed on the following questions.

15

$$y = \frac{x+1}{x-1}$$

Which of the following points in the xy-plane is NOT on the graph of y?

A) $(-2, \frac{1}{3})$
B) $(-1, 0)$
C) $(0, -1)$
D) $(1, 2)$

16

Let the function g be defined by $g(x) = \sqrt{3x}$. If $g(a) = 6$, what is the value of a?

A) 3
B) 6
C) 9
D) 12

Questions 17-18 refer to the following information.

x	$f(x)$	$g(x)$
-2	3	4
-1	5	2
0	-2	-3
1	3	5
2	6	7
3	7	1

The functions f and g are defined for the six values of x shown in the table above.

17

What is the value of $f(g(-1))$?

A) -2
B) 3
C) 5
D) 6

18

If $g(c) = 5$, what is the value of $f(c)$?

A) -2
B) 3
C) 5
D) 6

19

If $f(x) = -3x + 5$ and $\frac{1}{2}f(a) = 10$, what is the value of a?

A) -8
B) -5
C) 5
D) 8

20

x	$f(x)$
0	-4
1	-8
2	3
3	6
4	7
5	2
6	4
7	5

Several values of the function f are given in the table above. If the function g is defined by $g(x) = f(2x-1)$, what is the value of $g(3)$?

A) 2
B) 6
C) 5
D) 7

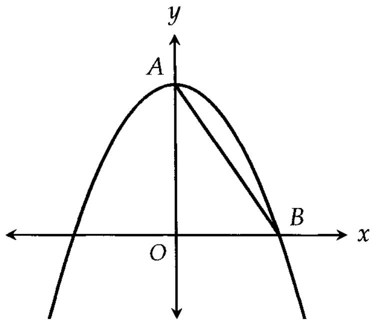
21

$$f(x) = 4x - 3$$
$$g(x) = 3x + 5$$

The functions f and g are defined above. Which of the following is equal to $f(8)$?

A) $g(1)$
B) $g(3)$
C) $g(5)$
D) $g(8)$

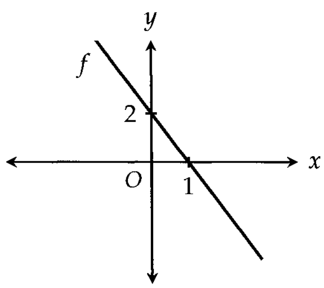
22



The graph of the function $y = 9 - x^2$ is shown in the xy-plane above. What is the length of $\overline{AB}$?

A) $3\sqrt{2}$
B) $3\sqrt{10}$
C) 9
D) $9\sqrt{10}$

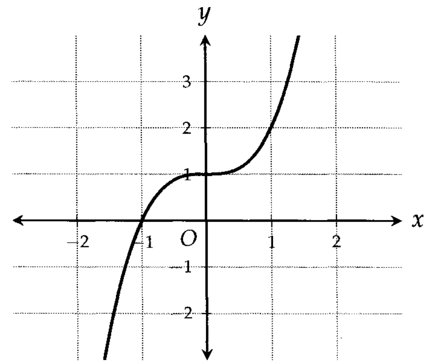
23



The function f is graphed in the xy-plane above. If the function g is defined by $g(x) = f(x) + 4$, what is the x-intercept of $g(x)$?

A) -3
B) -1
C) 3
D) 4

24



The function $f(x) = x^3 + 1$ is graphed in the xy-plane above. If the function g is defined by $g(x) = x + k$, where k is a constant, and $f(x) = g(x)$ has 3 solutions, which of the following could be the value of k?

A) -1
B) 0
C) 1
D) 2

25

In the xy-plane, the function $y = ax + 12$, where a is a constant, passes through the point $(-a, a)$. If $a > 0$, what is the value of a?

3

$-a \cdot a + 12 = a$

$-a^2 + 12 = a$

$-a^2 - a + 12 = 0$