Image Registration

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IMAG/e TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Overview & course schedule

Modules	Date	Topic
Registration	April 24 (Thursday)	Course introduction, geometrical transformations
	April 28 (Monday)	Point-based image registration
	May 1 (Thursday)	Intensity-based image registration
Segmentation	May 8 (Thursday)	Introduction to image segmentation
	May 15 (Thursday)	Segmentation in feature space
	May 19 (Monday)	Segmentation using graph-cuts
	May 22 (Thursday)	Statistical shape models
Deep learning for MIA	May 26 (Monday)	Convolutional neural networks
	June 2 (Monday)	Deep learning applications (registration)
	June 5 (Thursday)	Guest lecture by **Danny Ruijters** (principal scientist @ Philips, full professor @ TU/e)
	June 10 (Tuesday)	Deep learning applications (segmentation)
	June 12 (Thursday)	Unsupervised deep learning for medical image analysis

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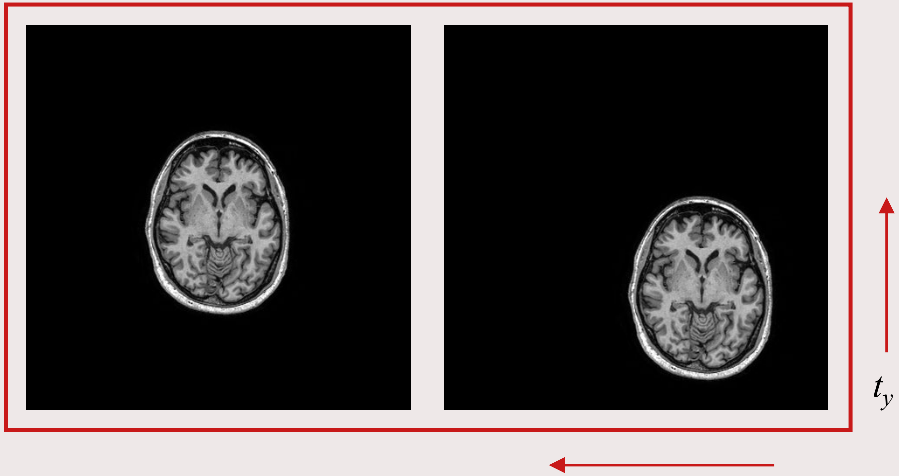
Study materials – medical image registration

- Primary: lecture slides, exercises, virtual reader
- Recommended reading – relevant sections from: Fitzpatrick, J.M., Hill, D.L. and Maurer Jr, C.R., Image registration.

Outline

- Introduction to medical image registration
- Recap of linear algebra
- Geometrical transformations



Intended learning outcome

The student can:

- name possible causes of misalignment in medical images
- name different applications of medical image registration
- classify medical image registration methods using eight different criteria
- apply the basic principles of linear algebra (i.e., matrix-vector, vector-matrix products, transpose, norms, orthogonality, determinant) to image registration tasks
- use the determinant of a transformation matrix T to predict the orientation and magnification of an object transformed with T
- compose and combine rigid and affine transformations in 2D and 3D (and rewrite them using homogeneous coordinates)
- explain the difference between affine and non-linear registrations

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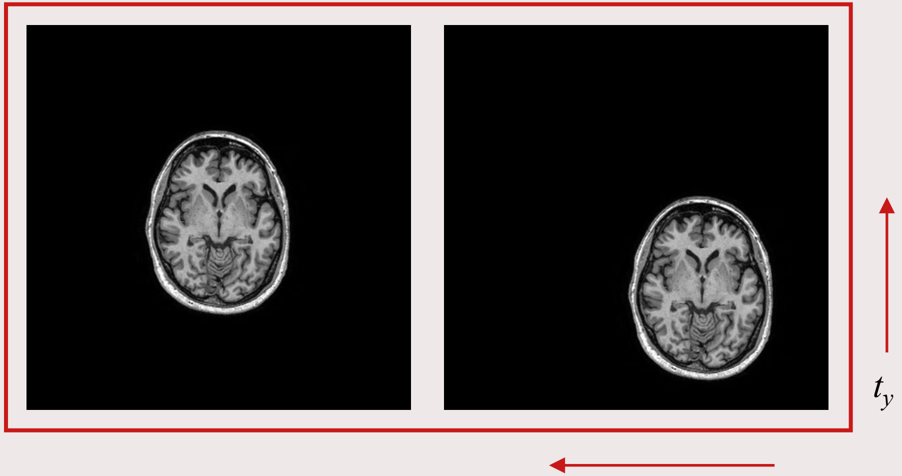
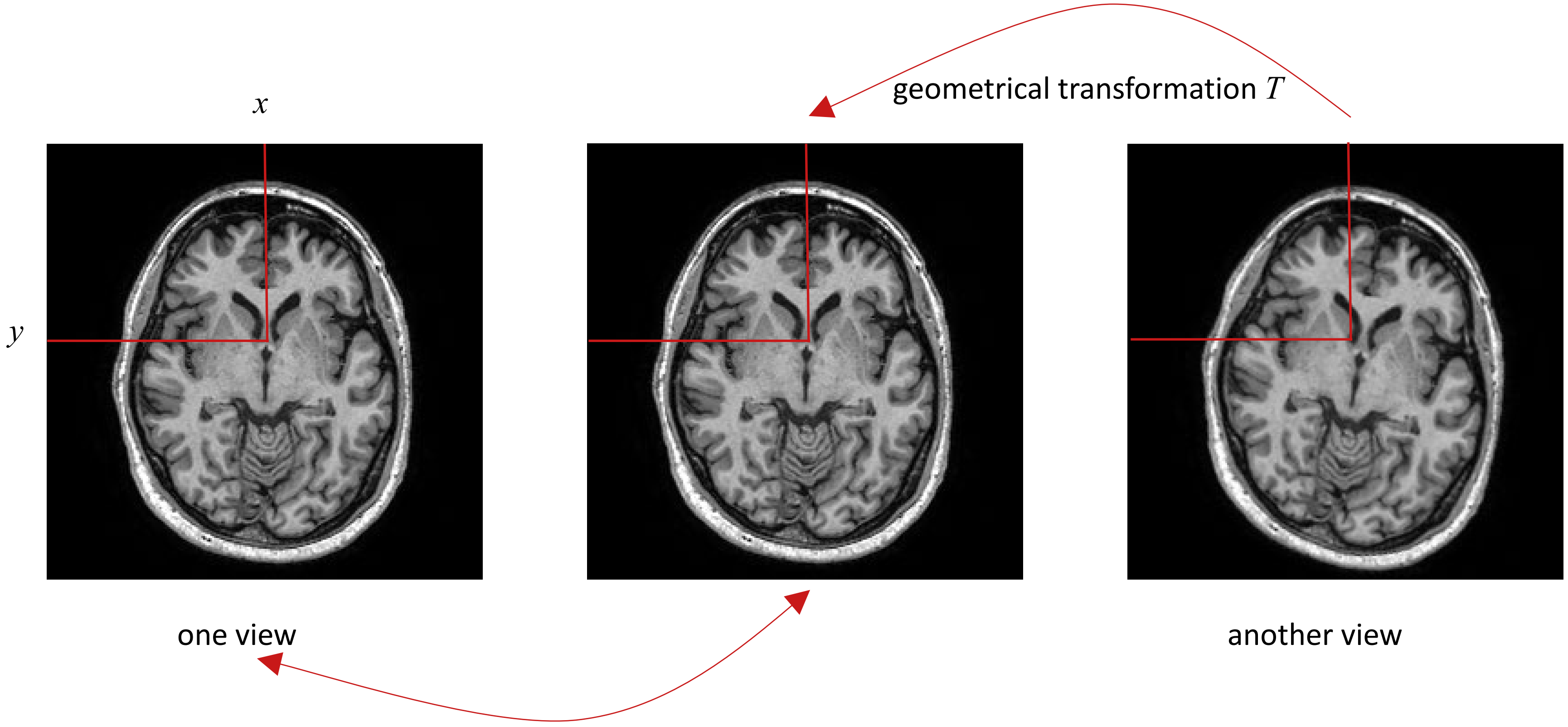


r/pics post: “I went to **Milan** to create a frame for this photo. Live frame.”

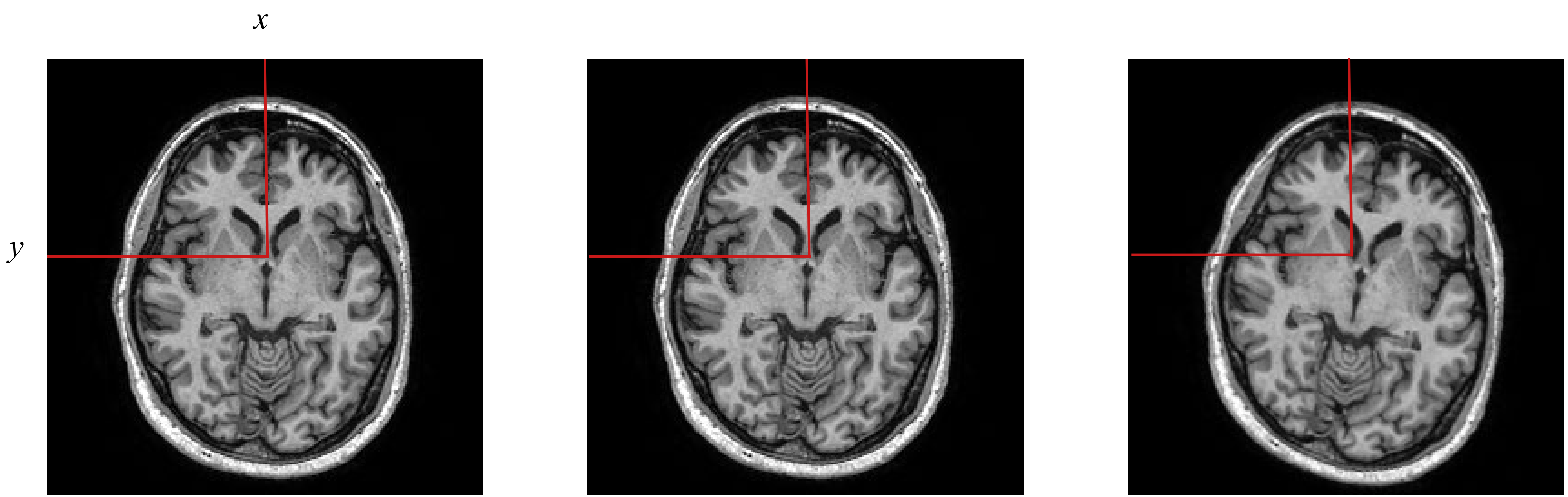
Barcelona

Image registration:

- determination of a **geometrical transformation**
- that aligns **one view** of an object
- with **another view** of that object or another object.



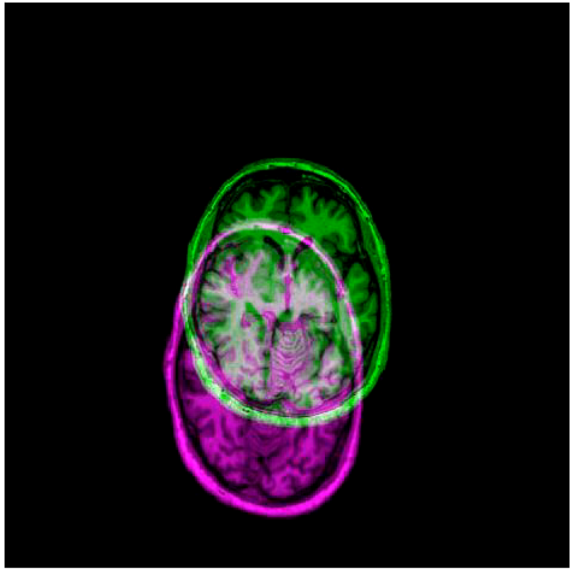
The first with the second view

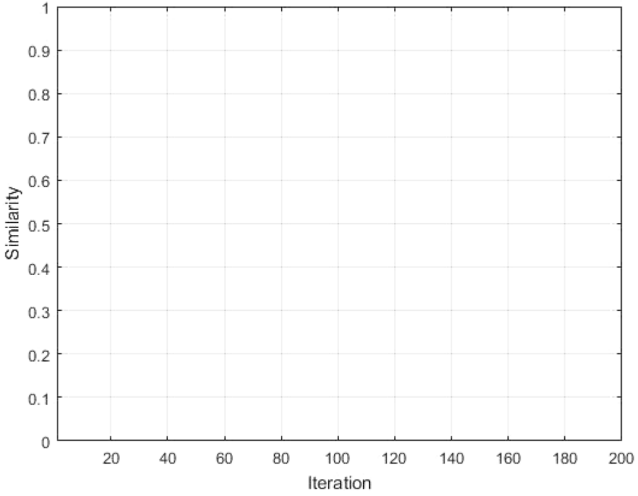


fixed image

transformed moving image

moving image





You will implement this functionality during the image registration project!

Cause of misalignment

- Imaging system
  - Distortions caused by imaging system
- Patient/subject
  - Movements of patient
  - Movements of organs due to physiology
- Operator
  - Changes in positioning of patient in scanner
  - Changes in viewing angle
  - Changes caused by interventions (e.g. surgery, chemotherapy) in between acquisition of the images
- More…

Applications of image registration

- Combining information from different sources
- Comparison: differences in (groups of) subjects
- Comparison: monitoring changes in a single subject
- Segmentation
- Motion correction
- Image-guided treatment
- Atlas, model of average anatomy

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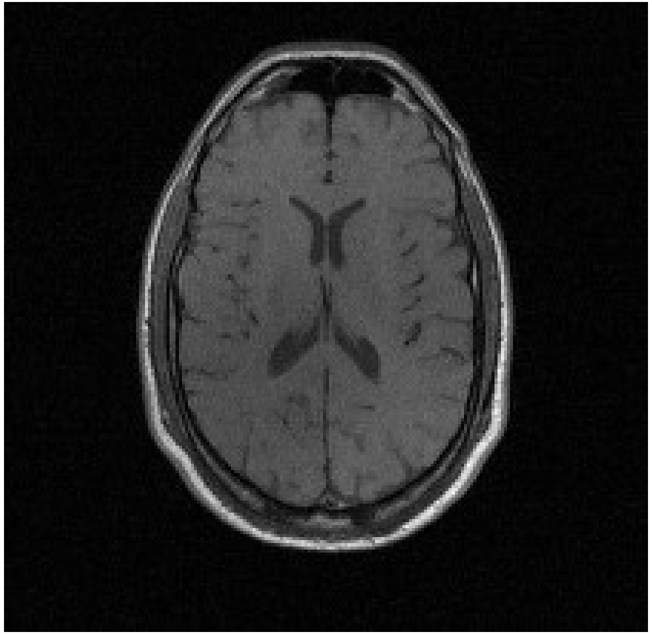
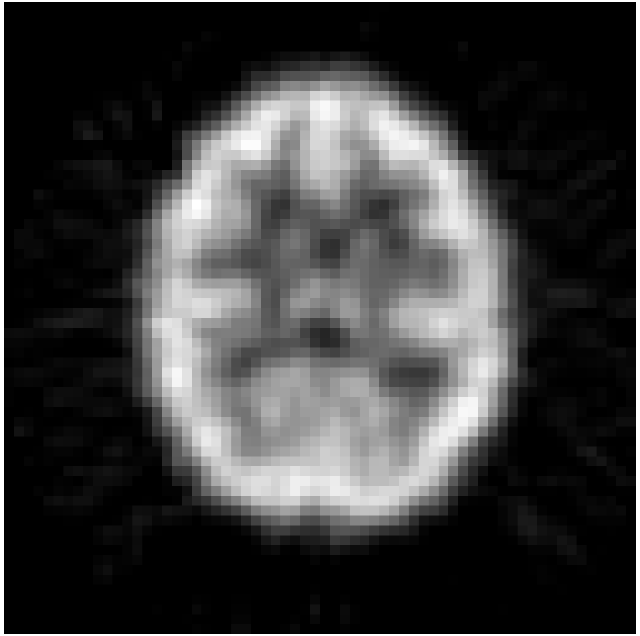


MRI, information about anatomical structures



PET, information about function

Applications of image registration

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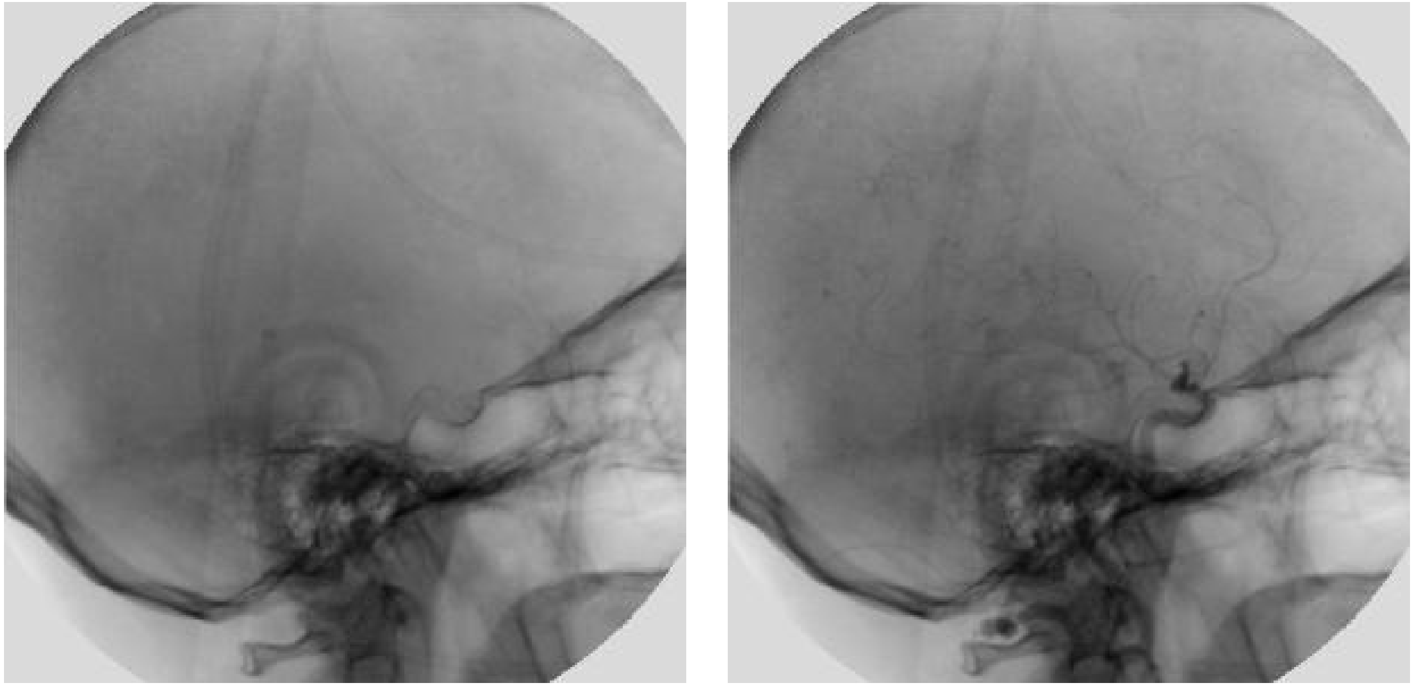
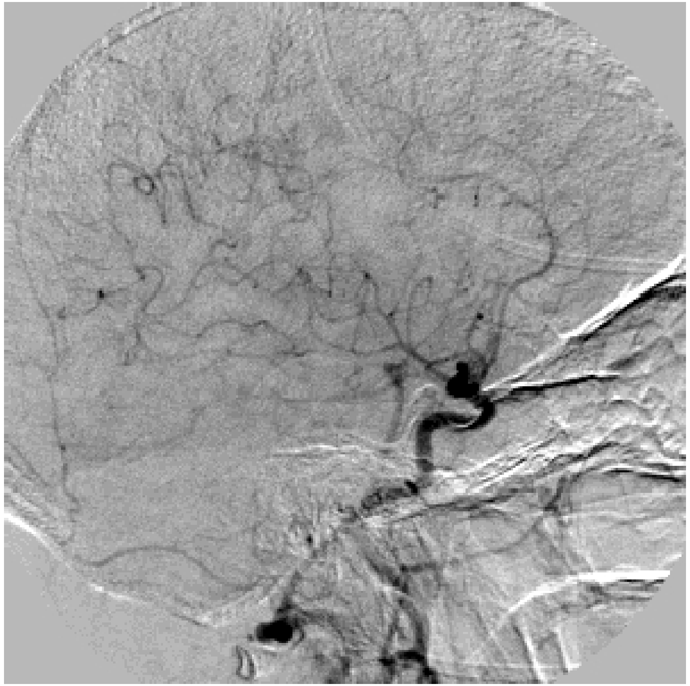
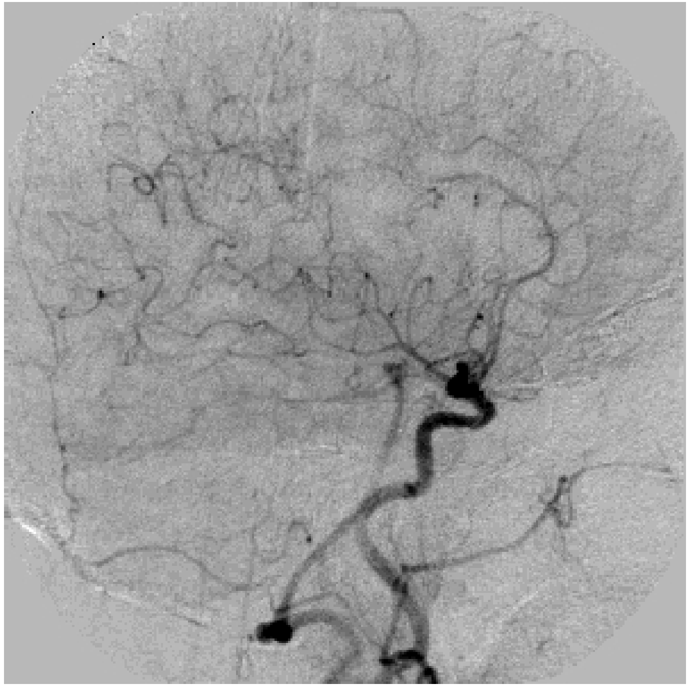


Digital subtraction angiography



Without registration



With registration

Digital subtraction angiography

Classification of image registration

- Image dimensionality: 2D, 3D, 3D + time…
- Registration basis: point sets, intensity…
- Geometrical transformations: rigid, affine, nonlinear…
- Degree of interaction: automatic, semi-automatic
- Optimization procedure: closed-form solution, iterative
- Modalities: multi-modal, intra-modal
- Subject: inter-patient, intra-patient, atlas
- Object: head, vertebra, liver…

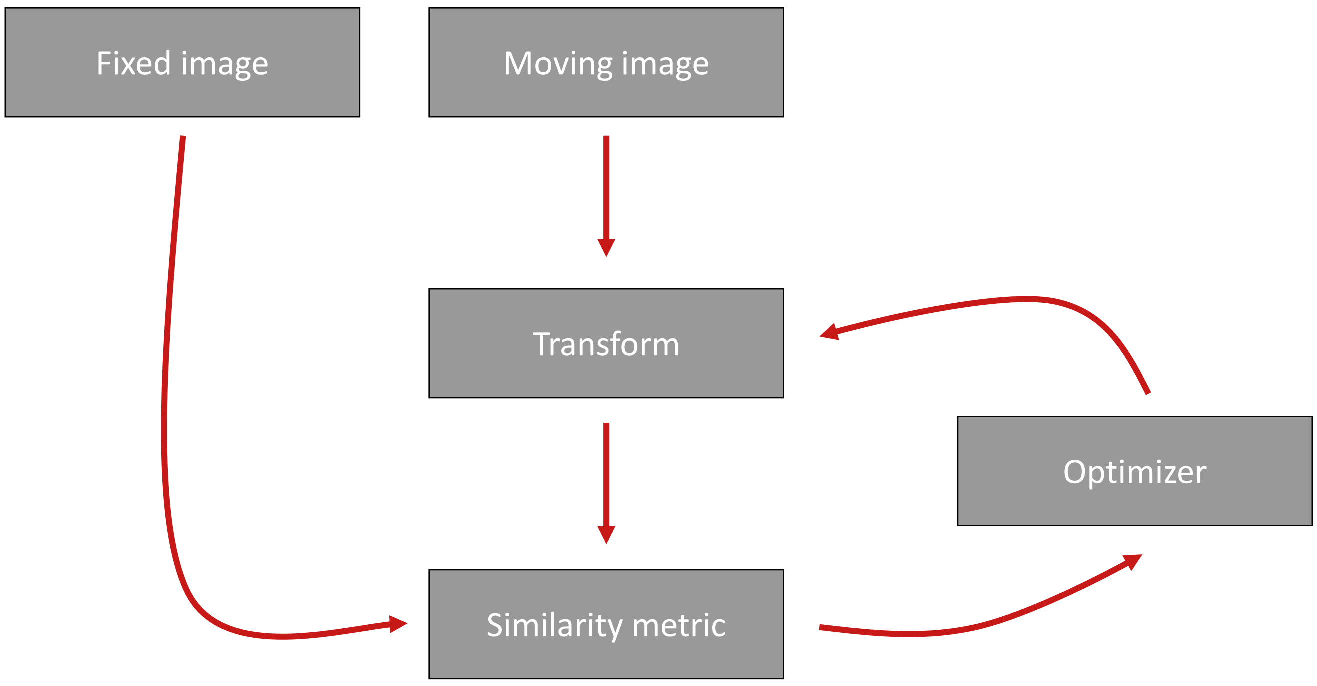
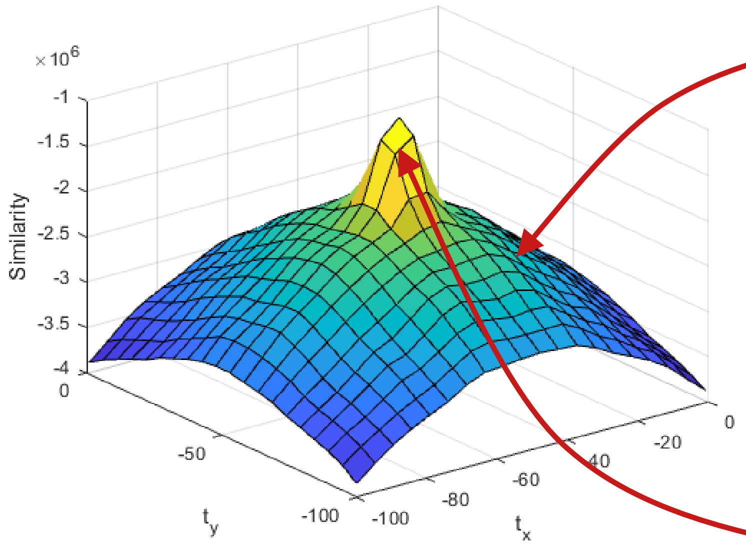
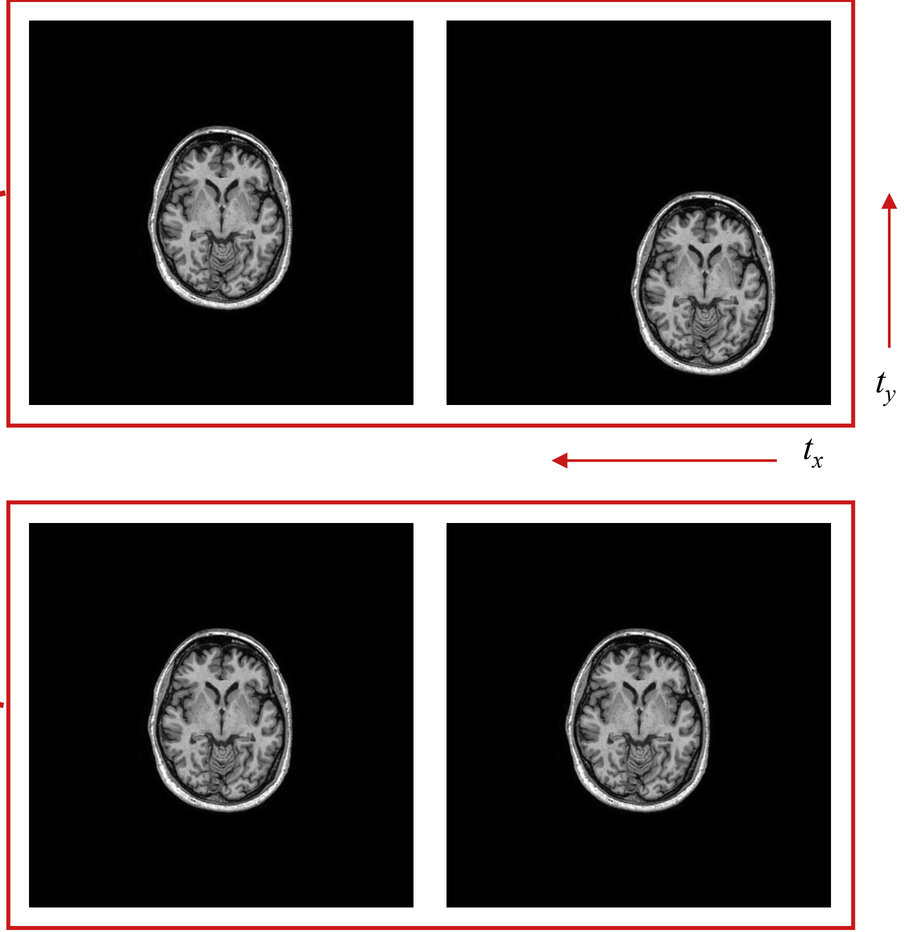


Image registration overview



Similarity as a function of translation



Outline

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Kolter, Z. Do, C., Linear Algebra Review and Reference (http://cs229.stanford.edu/section/cs229-linalg.pdf)

Topics to review:
- Matrix-vector, vector-matrix products
- Transpose
- Norms
- Orthogonality
- Determinant

Scalars

Question: Which of the following is **not** typically considered a **scalar**?

A. An integer like 7

B. A real number like 3.14

C. A grayscale pixel value

D. A 1×1 matrix

Scalars

- A scalar is a single number
- Integers, real numbers, rational numbers, etc.
- We denote it with italic font:

$$a, n, x$$

Vectors

Question: Which of the following denotes this vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$?

A. $\mathbb{R}^n$

B. $\mathbb{R}^{1\times n}$

C. $\mathbb{R}_1^n$

D. $\mathbb{R}_n$

Vectors

- A vector is a 1-D array of numbers:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.1}$$

- Can be real, binary, integer, etc.
- Example notation for type and size:

$$\mathbb{R}^n$$

Matrices

Question: What is the typical notation for a matrix with m rows and n columns?

A. $\mathbb{R}^{n \times m}$

B. $\mathbb{R}^{m \times n}$

C. $\mathbb{R}^{m}_{n}$

D. $\mathbb{R}^{n}_{m}$

Matrices

- A matrix is a 2-D array of numbers:

Row

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}. \qquad (2.2)$$

Column

- Example notation for type and shape:

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}$$

Matrix transpose

Question: If ***A*** is a 2x3 matrix, what is the dimension of ***AA****ᵀ*?

A. 2x3

B. 3x2

C. 2x2

D. 3x3

Matrix Transpose

$$(\boldsymbol{A}^\top)_{i,j} = A_{j,i}. \tag{2.3}$$

$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \boldsymbol{A}^\top = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

Figure 2.1: The transpose of the matrix can be thought of as a mirror image across the main diagonal.

$$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top. \tag{2.9}$$

Matrix (Dot) product

Question: If **u** = [1, 2, 3] and **v** = [4, -1, 0], what is the dot product of **u** and **v**?

A. -1

B. 6

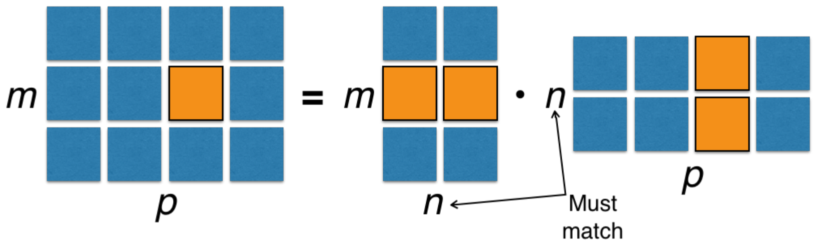
C. 4

D. 4 x 1 + (-1) x2 + 0 x3

Matrix (Dot) Product

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}. \tag{2.4}$$

$$C_{i,j} = \sum_k A_{i,k} B_{k,j}. \tag{2.5}$$



Identity matrix

Question: which of the following is the 3x3 identity matrix I_3?

A. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2.2: *Example identity matrix*: This is $\boldsymbol{I}_3$.

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{I}_n \boldsymbol{x} = \boldsymbol{x}. \tag{2.20}$$

Systems of equations

Question: A system of linear equations is consistent and has a unique solution exactly when its coefficient matrix A satisfies which condition?

A. det A = 0

B. rank A < number of unknowns

C. A is square and det A ≠ 0

D. A has more rows than columns

Systems of Equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{2.11}$$

expands to

$$\boldsymbol{A}_{1,:}\boldsymbol{x} = b_1 \tag{2.12}$$

$$\boldsymbol{A}_{2,:}\boldsymbol{x} = b_2 \tag{2.13}$$

$$\dots \tag{2.14}$$

$$\boldsymbol{A}_{m,:}\boldsymbol{x} = b_m \tag{2.15}$$

System of Linear Equation

2.0x + 4.0y +6.0z = 18

4.0x + 5.0y +6.0z = 24

3.0x + 1y - 2.0z = 4

Matrix representation

$$A = \begin{bmatrix} 2.0 & 4.0 & 6.0 \\ 4.0 & 5.0 & 6.0 \\ 3.0 & 1.0 & -2.0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 18.0 \\ 24.0 \\ 4.0 \end{bmatrix}$$

Matrix inversion

Question: If A is an invertible n × n matrix, which relation holds?

A. $A A^T = I$

B. $A^{-1} A = I$

C. $\det(A) = 0$

D. $A + A^{-1} = I$

Matrix Inversion

- Matrix inverse:

$$A^{-1}A = I_n. \tag{2.21}$$

- Solving a system using an inverse:

$$Ax = b \tag{2.22}$$

$$A^{-1}Ax = A^{-1}b \tag{2.23}$$

$$I_n x = A^{-1}b \tag{2.24}$$

- Numerically unstable, but useful for abstract analysis

Norms

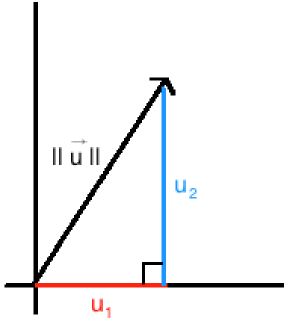
Question: Let $f: V \rightarrow \mathbb{R}$ be a candidate for a norm on a vector space V. Which of the following statements does **not** have to hold for f to qualify as a norm?

A. $f(x+y) \leq f(x)+f(y)$ for all $x,y \in V$.

B. $f(\alpha x) = |\alpha| \; f(x)$ for all scalars α and $x \in V$.

C. $f(x) \geq 0$ for all $x \in V$.

D. $f(xy) = f(x) \; f(y)$ for all $x,y \in V$.



Norms

- Functions that measure how "large" a vector is
- Similar to a distance between zero and the point represented by the vector
  - $f(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$
  - $f(\boldsymbol{x}+\boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$ (the *triangle inequality*)
  - $\forall \alpha \in \mathbb{R}, f(\alpha \boldsymbol{x}) = |\alpha| f(\boldsymbol{x})$

Special matrices and vectors

Question: Let A be a real $n{\times}n$ orthogonal matrix. Which of the following statements does NOT necessarily hold true?

A. $A^T = A^{-1}$

B. The determinant of A is either +1 or −1

C. The columns of A form an orthonormal set

D. A is a symmetric matrix, i.e., $A^T=A$

Special Matrices and Vectors

- Unit vector:

$$||\boldsymbol{x}||_2 = 1. \tag{2.36}$$

- Symmetric Matrix:

$$\boldsymbol{A} = \boldsymbol{A}^\top. \tag{2.35}$$

- Orthogonal matrix:

$$\boldsymbol{A}^\top \boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{I}. \tag{2.37}$$

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^\top$$

Determinant

Question: What is the determinant of matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

A. 0

B. 5

C. 10

D. -2

The Determinant

Symbol of Determinant

$$\det(\boldsymbol{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

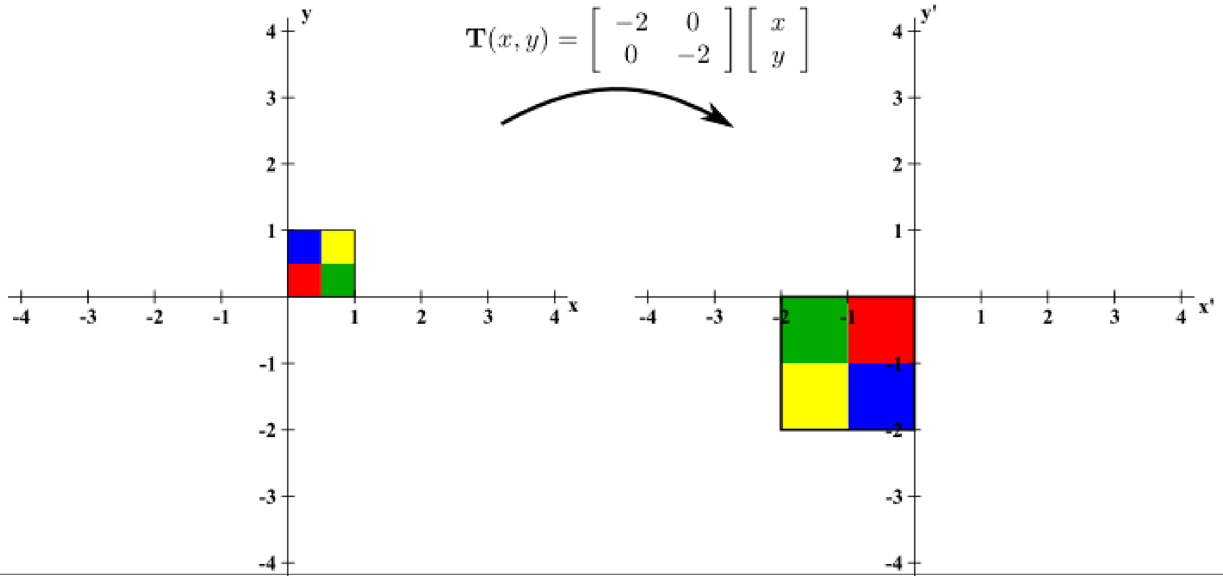
- The determinant of a square matrix maps matrices to real scalars

Question: What is the determinant of the identity matrix det (***I***) ?

Determinant of a transformation matrix **T**: the signed area of a unit square shape after transforming with **T**.

The sign reflects whether the orientation has changed.

$$\mathbf{T}(x,y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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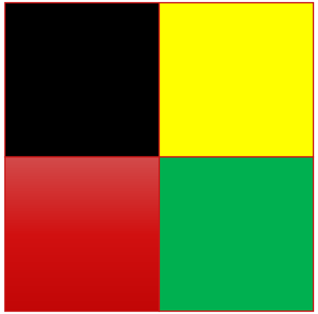
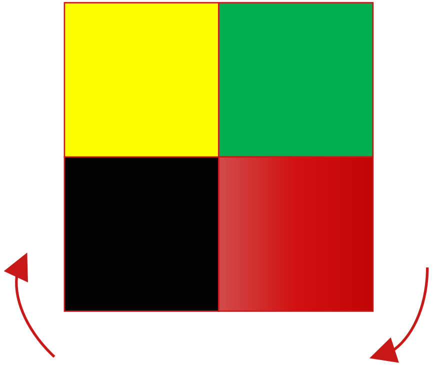


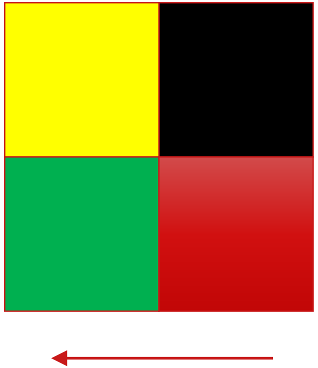
original

Positive determinant



rotation

Negative determinant

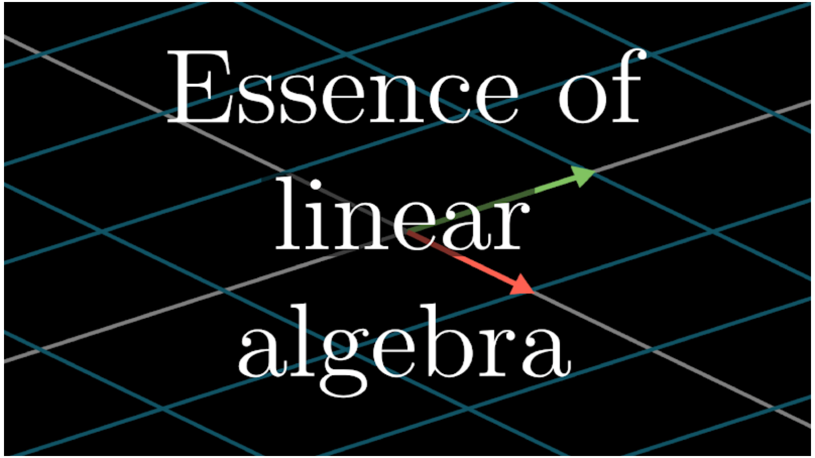


reflection

$|T|$

- $= 1$ → no magnification
- > 1 → the matrix has magnification property
- < 1 → the matrix has shrinking property
- $= 0$ → shrink any object to a dot / matrix is not invertable

Study materials



- Essence of linear algebra, 3Blue1Brown channel
- Kolter, Z. Do, C., Linear Algebra Review and Reference (http://cs229.stanford.edu/section/cs229-linalg.pdf)

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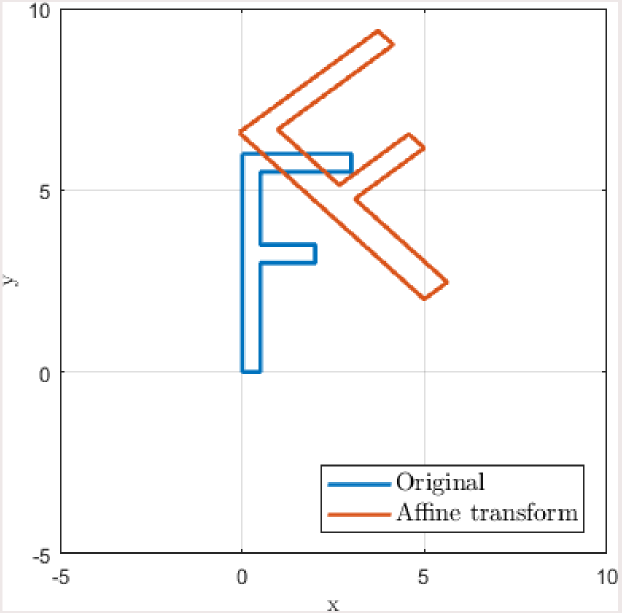
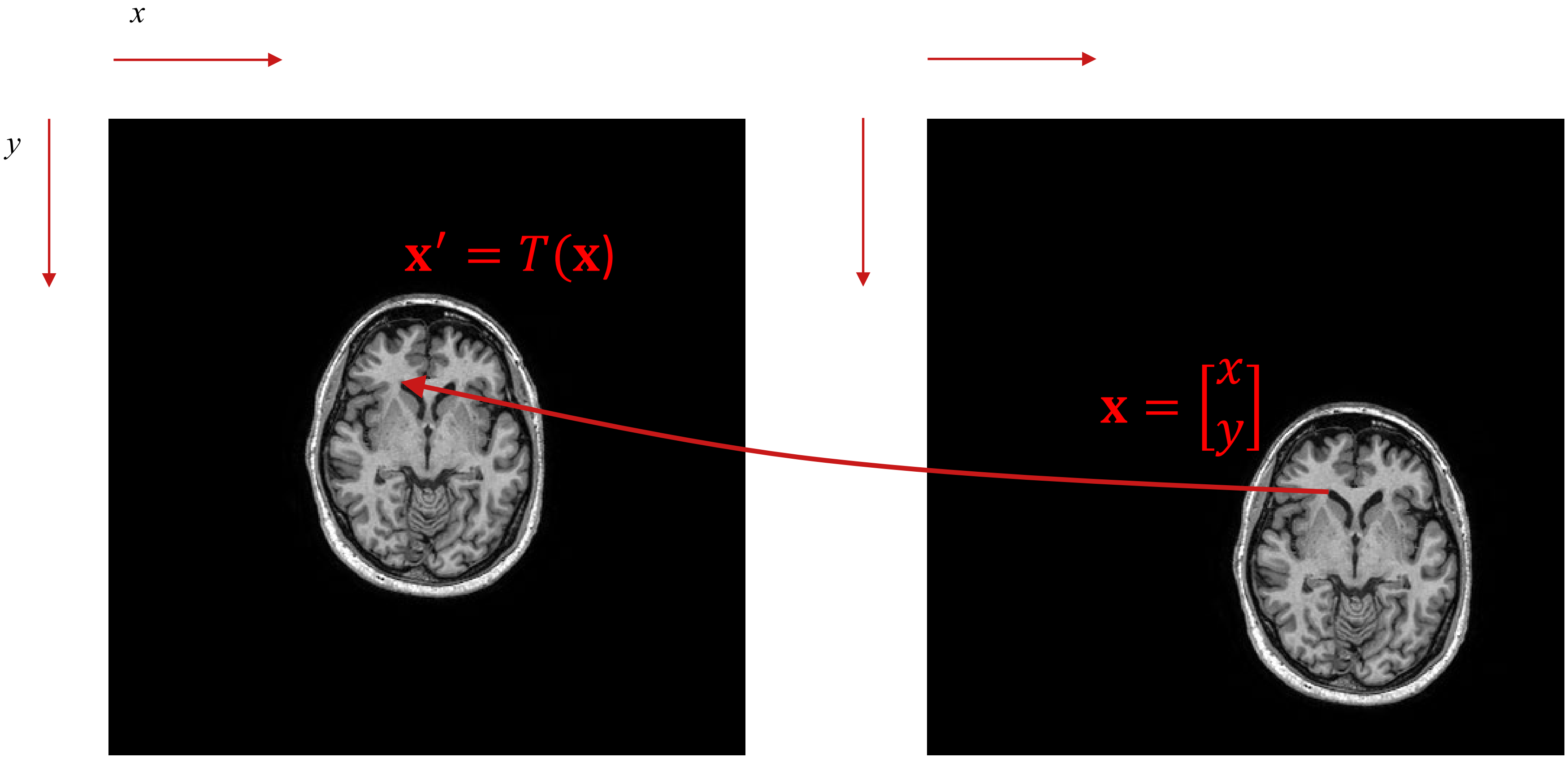



x
y
$\mathbf{x}' = T(\mathbf{x})$
$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

All examples will be for 2D geometrical shapes and images,

but they can be easily generalized to 3D.

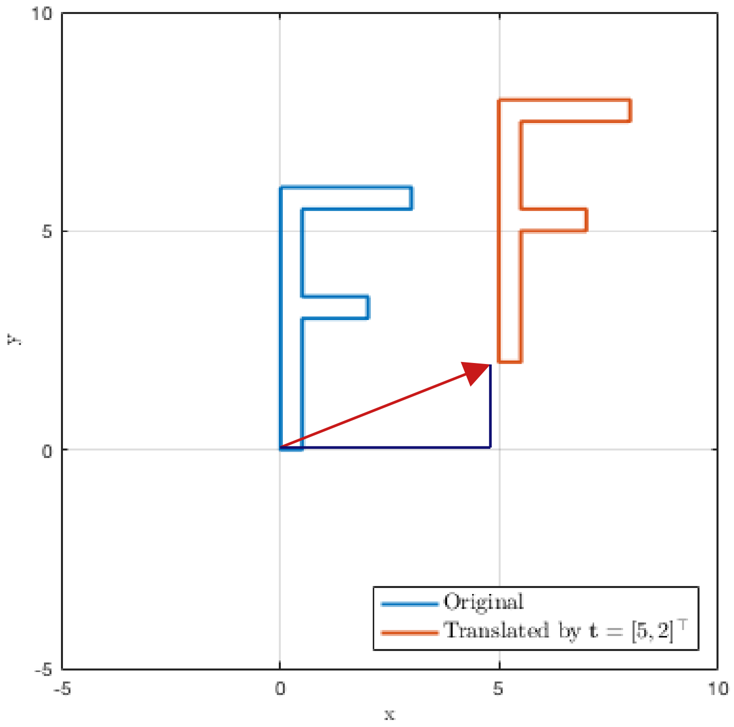
Translation

$$x' = x + t_x$$

$$y' = y + t_y$$

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$



Distance between two points in 2D:

$$D(\mathbf{x}, \mathbf{x}') = \sqrt{(x - x')^2 + (y - y')^2}$$

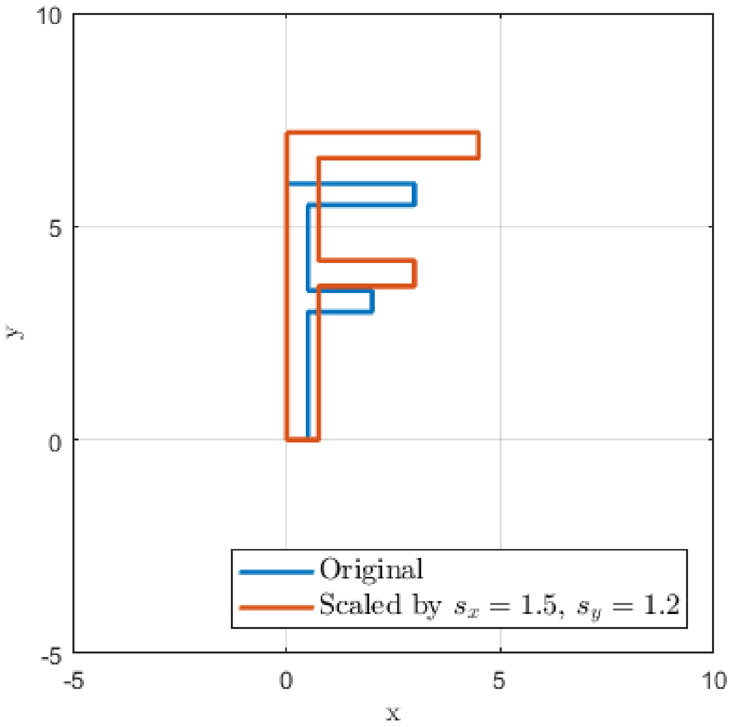
Scaling

$$x' = s_x x$$

$$y' = s_y y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

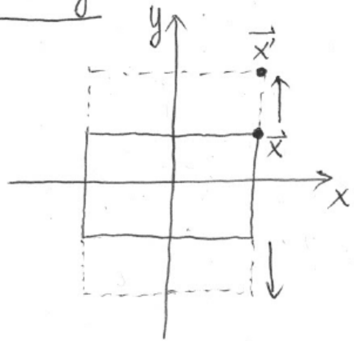
$$\mathbf{x}' = \mathbf{S}\mathbf{x} \qquad \mathbf{S} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



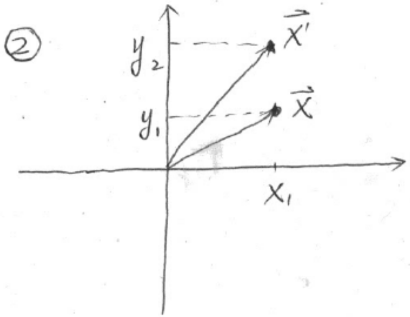
Scaling

scale ↑ 2 times in y direction

①



②



③. $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \vec{x'} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \left.\begin{matrix} x_2 = x_1 \\ y_2 = 2y_1 \end{matrix}\right\} \Rightarrow \begin{matrix} x_2 = 1 \cdot x_1 + 0 \cdot y_1 \\ y_2 = 0 \cdot x_1 + 2y_1 \end{matrix}$

④. $\vec{x'} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0 \cdot y_1 \\ 0 \cdot x_1 + 2 \cdot y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

⑤. $f(\vec{x}) = \vec{x'}$ what is $f(\cdot)$? $f(\cdot)$

Rotation

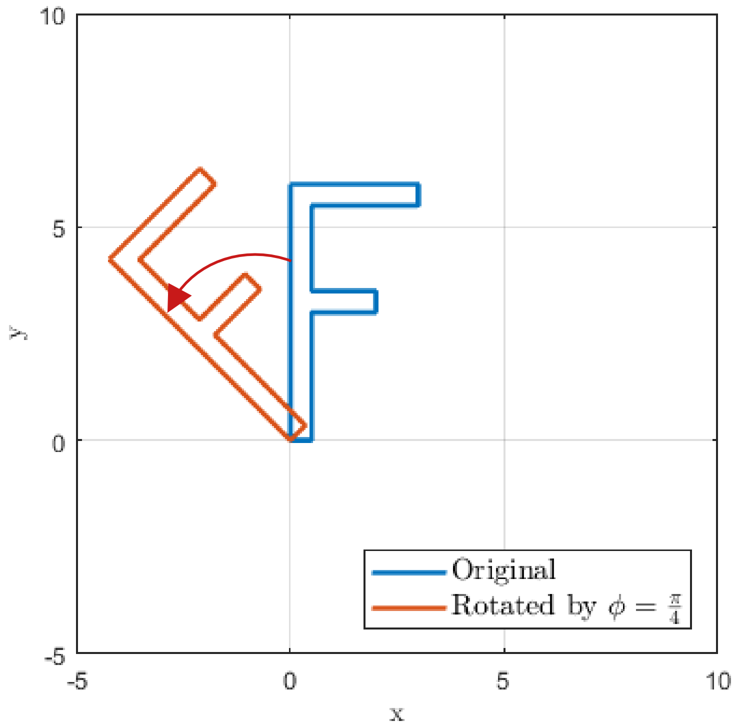
$$x' = \cos(\phi)x - \sin(\phi)y$$

$$y' = \sin(\phi)x + \cos(\phi)y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

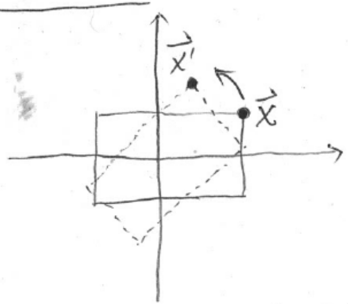
$$\mathbf{x}' = \mathbf{R}\mathbf{x}$$

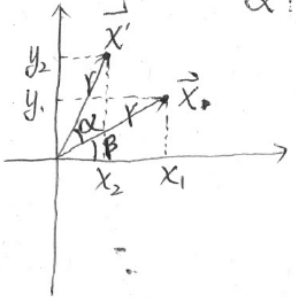
$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$



Rotation

α : angle of rotation

1. 

2. 

3. $\vec{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\quad \vec{x}' = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ $\quad x_1 = r\cos\beta \quad x_2 = r\cos(\alpha+\beta)$

$y_1 = r\sin\beta \quad y_2 = r\sin(\alpha+\beta)$

4. $\vec{x}' = f(\vec{x}) \quad f(\cdot) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$

5. $x_2 = r\cos(\alpha+\beta) = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta$

$= x_1\cos\alpha - y_1\sin\alpha$

$y_2 = r\sin(\alpha+\beta) = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta$

$= x_1\sin\alpha + y_1\cos\alpha$

6. $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha \cdot x_1 - \sin\alpha \cdot y_1 \\ \sin\alpha \cdot x_1 + \cos\alpha \cdot y_1 \end{bmatrix}$

$= \boxed{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}$

Rotation

Not every matrix can be considered a rotation matrix.

$$\mathbf{R} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

Rotation matrices:

- Are orthogonal:

$$\mathbf{R}\mathbf{R}^{-1} = \mathbf{R}\mathbf{R}^{\intercal} = \mathbf{I}$$

- Have determinant equal to 1:

$$\det(\mathbf{R}) = 1$$

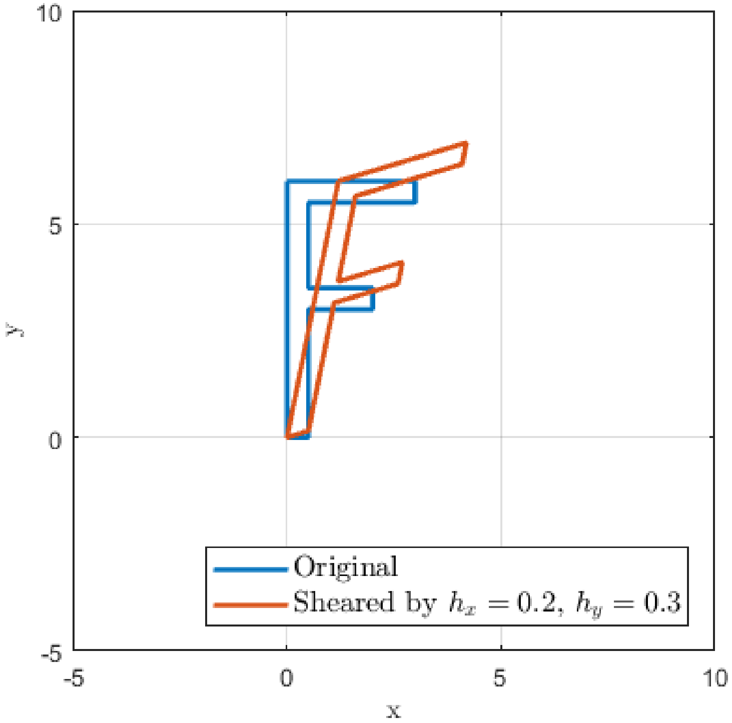
Shearing

$$x' = x + h_x y$$

$$y' = h_y x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{H}\mathbf{x} \qquad \mathbf{H} = \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix}$$



Reflection

Horizontal:

$$x' = -x$$

$$y' = y$$

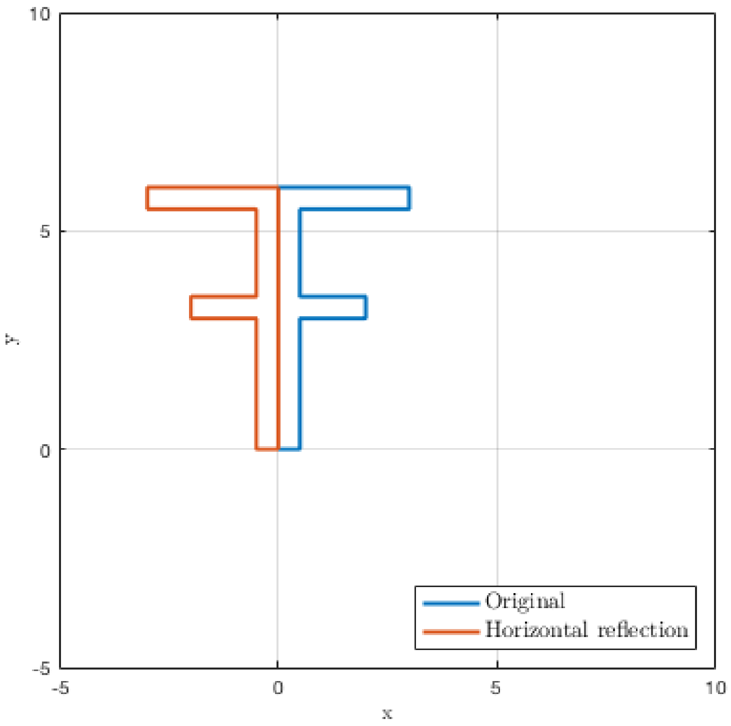
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Vertical:

$$x' = x$$

$$y' = -y$$

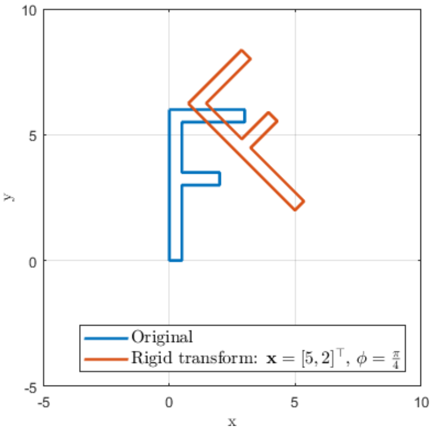
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Composition of transformations

- Rotation + translation (rigid):

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$$

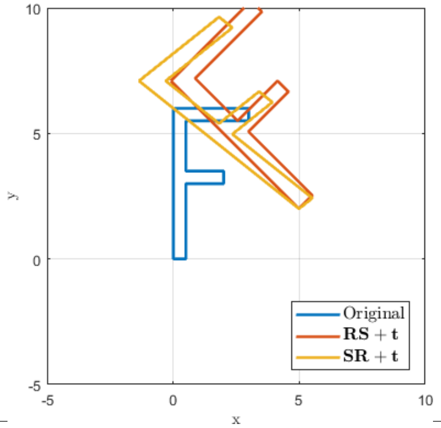


Transformations can be combined by multiplying the transformation matrices.

- Rotation, scaling + translation:

$$\mathbf{x}' = \mathbf{R}\mathbf{S}\mathbf{x} + \mathbf{t}$$

$$\mathbf{x}' = \mathbf{S}\mathbf{R}\mathbf{x} + \mathbf{t}$$



Composition of transformations

Note that matrix multiplication is not commutative:

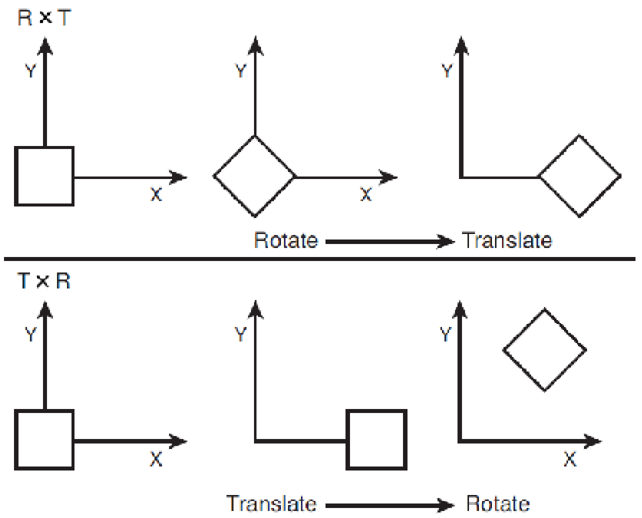
$$\mathbf{T}_1\mathbf{T}_2\mathbf{x} \neq \mathbf{T}_2\mathbf{T}_1\mathbf{x}$$

First scaling, then rotation, then translation:

$$\mathbf{x}' = \mathbf{RSx} + \mathbf{t}$$

First rotation, then scaling, then translation:

$$\mathbf{x}' = \mathbf{SRx} + \mathbf{t}$$



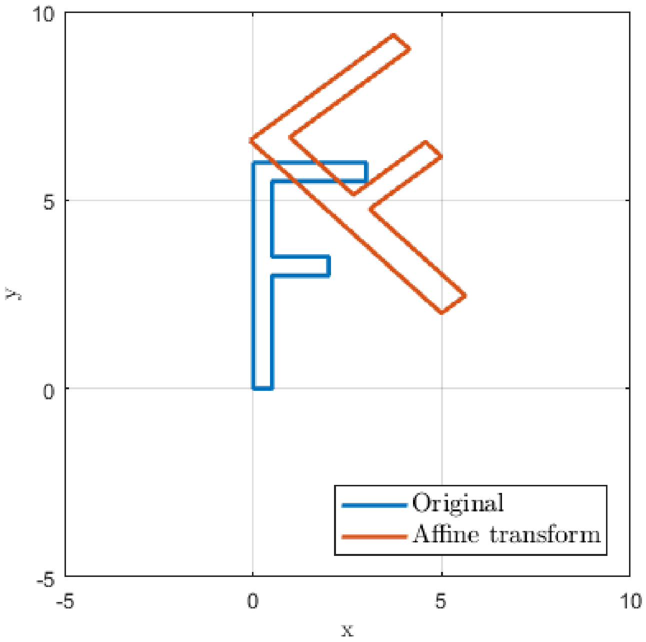
Affine transformation

→ no restriction on the transformation parameters

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$

- It can be considered as a composition of any combination of rotations, scalings, shearings, reflections, and translations.



Affine transformation

- Affine transformation has **6 parameters**: 2×2 transformation matrix and 2×1 translation vector.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

- The combination of rotation, scaling, shearing, reflection + translation has **9 parameters**: 1 rotation angle, 2 scaling parameters, 2 shearing parameters, 2 reflection parameters and 2×1 translation vector.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & r_x & r_y & t_x & t_y \end{bmatrix}$$

- However, the first 7 parameters are not **<u>independent</u>**.

- The first parameterization is more compact, the second more human-readable.

→ Affine transformation in 2D has only **6 degrees of freedom**.

Affine transformation

In medical image registration, **reflections do not usually occur,** and it can be very problematic if two images are incorrectly registered with a reflection (e.g. can cause a surgical procedure to be performed on the wrong side of the body).

Thus, reflections should be excluded from affine registration.

$$\begin{bmatrix} \phi & s_x & s_y & h_x & h_y & t_x & t_y \end{bmatrix}$$

When using the unrestricted transformation matrix, a check for reflection can be made by examining $\det(\mathbf{A})$. If a reflection has occurred $\det(\mathbf{A}) < 0$.

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{t}$$

Homogenous form

A transformation matrix and a translation vector can be combined when using homogeneous coordinates:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This largely simplifies the notation and implementation of complex transformations.

Example

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_1\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & t_{x,1} \\ 0 & 1 & t_{y,1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x,2} \\ 0 & 1 & t_{y,2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x,1} + t_{x,2} \\ 0 & 1 & t_{y,1} + t_{y,2} \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Rotation around an arbitrary point $\mathbf{x}_r = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Inverse transformation in homogenous form

Inverse transformation can be achieved by taking the inverse of the transformation matrix:

$$\mathbf{T} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}^{-1} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Affine transformation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- How many rotation angles in 3D?
- How many degrees of freedom?

Non-linear transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = ax + by + t_x$$

$$y' = cx + dy + t_y$$

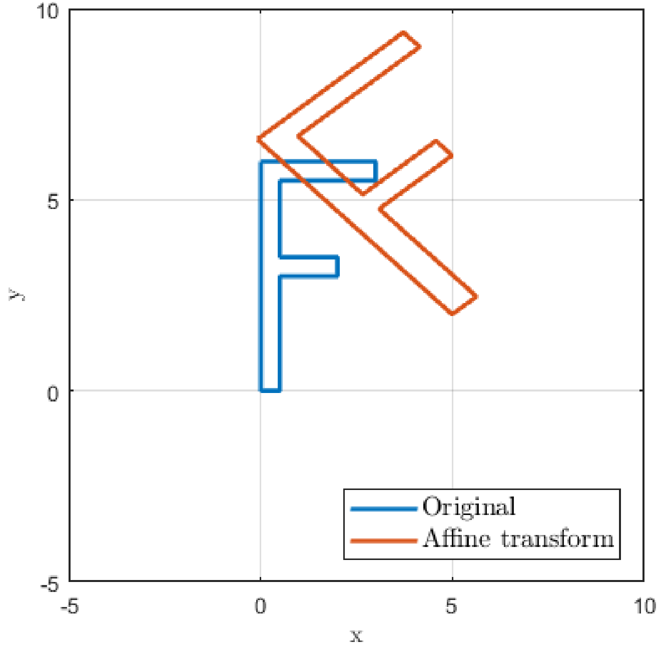
Linear polynomial

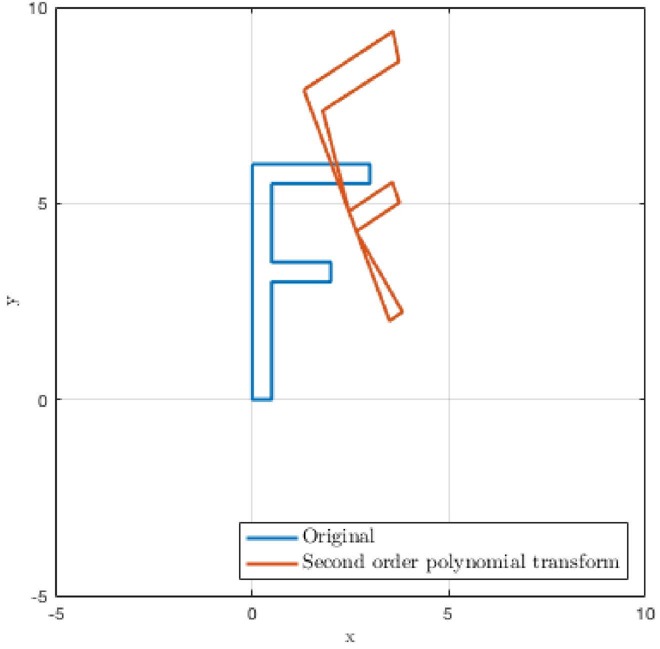
$$x' = ax + by + t_x + u_1 x^2 + u_2 y^2 + u_3 xy \ldots$$

$$y' = cx + dy + t_y + v_1 x^2 + v_2 y^2 + v_3 xy \ldots$$

Higher order polynomial

Non-linear transformations



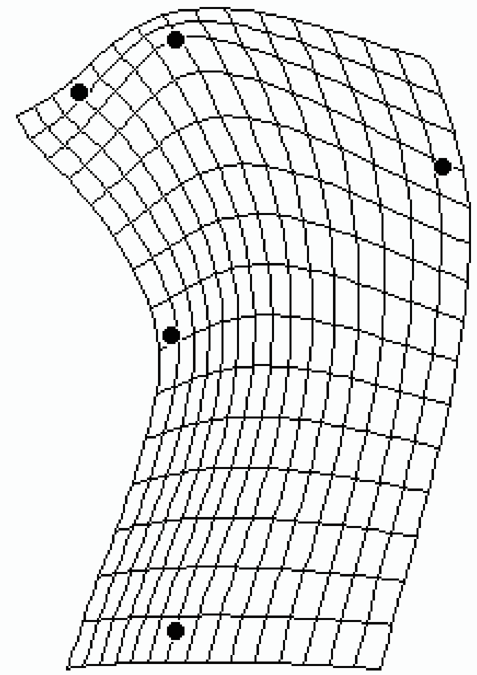


Example – thin-plate spline

$$x' = ax + by + t_x + \sum_{i=1}^{N} u_i r_i^2 \ln r_i^2$$

$$y' = cx + dy + t_y + \sum_{i=1}^{N} v_i r_i^2 \ln r_i^2$$

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2$$



Summary

The student can:

- name possible causes of misalignment in medical images
- name different applications of medical image registration
- classify medical image registration methods using eight different criteria
- apply the basic principles of linear algebra (i.e., matrix-vector, vector-matrix products, transpose, norms, orthogonality, determinant) to image registration tasks
- use the determinant of a transformation matrix T to predict the orientation and magnification of an object transformed with T
- compose and combine rigid and affine transformations in 2D and 3D (and rewrite them using homogeneous coordinates)
- explain the difference between affine and non-linear registrations

Thank you

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Next: Image transformation, point-based registration