Homework 5 MAT452
#Problem 1a)
restart

$eqx := (3 - x) \cdot x - x \cdot y :$

$eqy := -x \cdot y + (2 - y) \cdot y :$

$sol := solve(\{eqx, eqy\}, \{x, y\})$

$$sol := \{x = 3, y = 0\}, \{x = 0, y = 0\}, \{x = 0, y = 2\} \quad (1)$$

$with(LinearAlgebra) : with(VectorCalculus) :$

$J := Jacobian([eqx, eqy], [x, y])$

$$J := \begin{bmatrix} -2\,x + 3 - y & -x \\ -y & -x - 2\,y + 2 \end{bmatrix} \quad (2)$$

$J0 := subs(sol[1], J)$

$$J0 := \begin{bmatrix} -3 & -3 \\ 0 & -1 \end{bmatrix} \quad (3)$$

$J1 := subs(sol[2], J)$

$$J1 := \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad (4)$$

$evs := Eigenvalues(J1)$

$$evs := \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad (5)$$

$Determinant(J1)$

$$6 \quad (6)$$

$Eigenvectors(J0)$

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} & 1 \\ 1 & 0 \end{bmatrix} \quad (7)$$

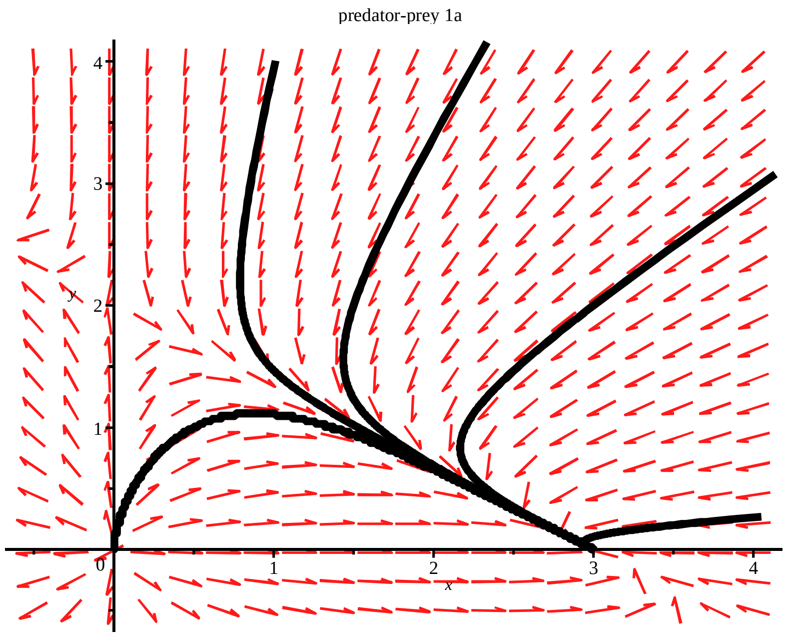
$eqxt := x'(t) = (3 - x(t)) \cdot x(t) - x(t) \cdot y(t) :$

$eqyt := y'(t) = - x(t) \cdot y(t) + (2 - y(t)) \cdot y(t) :$

$IC := [[x(0) = 0.5, y(0) = 1], [x(0) = 3.5, y(0) = 0.2], [x(0) = 3, y(0) = 2], [x(0) = .8, y(0) = 2], [x(0) = 1.5, y(0) = 2]] :$

$with(plots) : with(DEtools) :$

DEplot([eqxt, eqyt], [x(t), y(t)], t = −10 ..10, x = −0.5 ..4, y = −0.5 ..4, IC, stepsize = 0.01, title = "predator-prey 1a", linecolor = black, method = classical[rk4])



#Stable at (x=3, y=0)
#Unstable at (x=0,y=0,2)

#Problem 1b)
restart

$eqx := (3 - 2 \cdot x) \cdot x - x \cdot y :$

$eqy := -x \cdot y + (2 - y) \cdot y :$

$sol := solve(\{eqx, eqy\}, \{x, y\})$

$$sol := \{x = 0, y = 0\}, \{x = 0, y = 2\}, \left\{x = \frac{3}{2}, y = 0\right\}, \{x = 1, y = 1\} \qquad \textbf{(8)}$$

$with(LinearAlgebra) : with(VectorCalculus) :$

$J := Jacobian([eqx, eqy], [x, y])$

$$J := \begin{bmatrix} -4x + 3 - y & -x \\ -y & -x - 2y + 2 \end{bmatrix} \qquad \textbf{(9)}$$

$J0 := subs(sol[1], J)$

$$J0 := \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \qquad \textbf{(10)}$$

$J1 := subs(sol[2], J)$

$$J1 := \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \qquad \textbf{(11)}$$

$evs := Eigenvalues(J1)$

$$evs := \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \textbf{(12)}$$

$Determinant(J1)$

$$K2 \qquad \textbf{(13)}$$

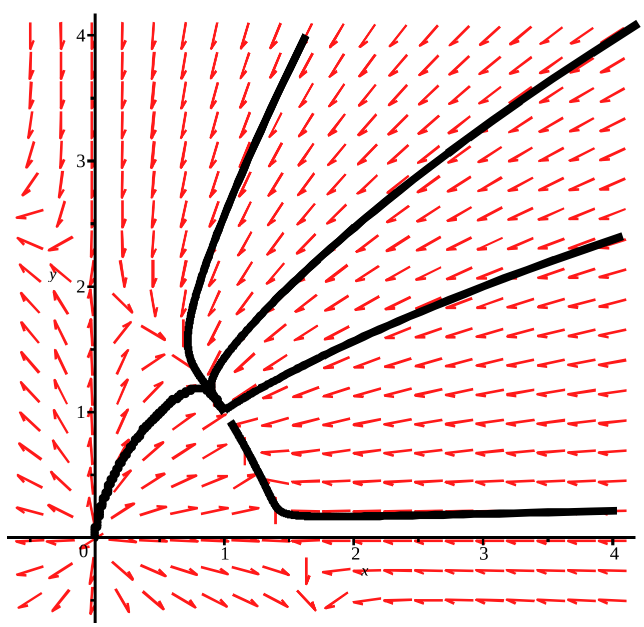
Eigenvectors(*J0*)

$$\left[\begin{array}{c} 3 \\ 2 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \qquad \textbf{(14)}$$

eqxt d $x'(t) = (3$ K $2\$\,x(t))\$\,x(t)$ K $x(t)\$y(t)$:
eqyt d $y'(t) =$ K $x(t)\$y(t)$ C $(2$ K $y(t))\$\,y(t)$:

IC d [[$x(0) = 0.5, y(0) = 1$], [$x(0) = 3.5, y(0) = 0.2$], [$x(0) = 3, y(0) = 2$], [$x(0) = .8, y(0) = 2$], [$x(0) = 1.5, y(0) = 2$]] :
with(*plots*) : *with*(*DEtools*) :
DEplot([*eqxt*, *eqyt*], [$x(t), y(t)$], $t =$ K10 ..10, $x =$ K0.5 ..4, $y =$ K0.5 ..4, *IC*, *stepsize* = 0.01, *title* = "predator-prey 1b", *linecolor* = *black*, *method* = *classical*[*rk4*])

predator-prey 1b



#Stable at (x=1,y=1)
#Unstable at (x=0,y=0),(1.33,0),(0,2)

#Problem 1c)
restart

eqx d $(3$ K $2\$\,x)\$x$ K $2\$\,x\$y$:

eqy d K$x\$y$ C $(2$ K $y)\$\,y$:
sol d *solve*({*eqx*, *eqy*}, {*x*, *y*})

$$sol \ \mathrm{d}\ \left\{x = \frac{3}{2}, y = 0\right\}, \{x = 0, y = 0\}, \{x = 0, y = 2\} \qquad \textbf{(15)}$$

with(*LinearAlgebra*) : *with*(*VectorCalculus*) :

$J := Jacobian([eqx, eqy], [x, y])$

$$J := \begin{bmatrix} -4\,x + 3 - 2\,y & -2\,x \\ -y & -x - 2\,y + 2 \end{bmatrix} \tag{16}$$

$J0 := subs(sol[1], J)$

$$J0 := \begin{bmatrix} -3 & -3 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{17}$$

$J1 := subs(sol[2], J)$

$$J1 := \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \tag{18}$$

$evs := Eigenvalues(J1)$

$$evs := \begin{bmatrix} 3 \\ 2 \end{bmatrix} \tag{19}$$

$Determinant(J1)$

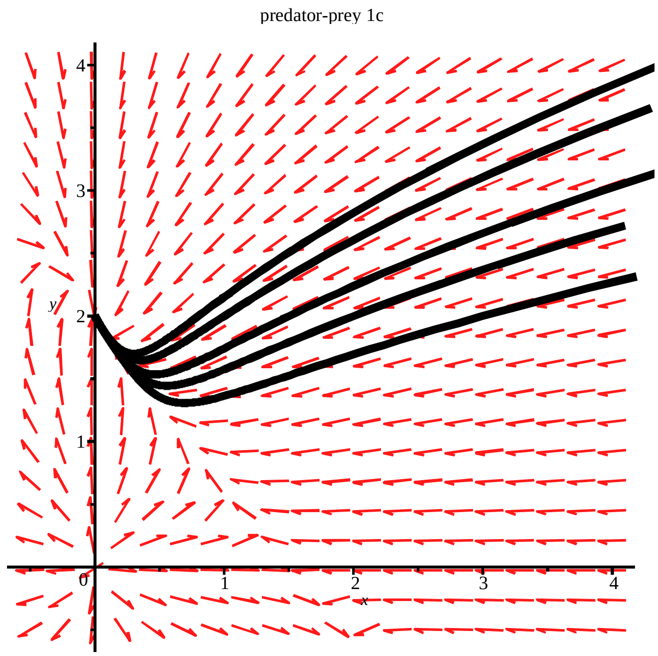
$$6 \tag{20}$$

$Eigenvectors(J0)$

$$\begin{bmatrix} \frac{1}{2} \\ -3 \end{bmatrix}, \begin{bmatrix} -\frac{6}{7} & 1 \\ 1 & 0 \end{bmatrix} \tag{21}$$

$eqxt := x'(t) = (3 - 2 \cdot x(t)) \cdot x(t) - 2 \cdot x(t) \cdot y(t) :$
$eqyt := y'(t) = - x(t) \cdot y(t) + (2 - y(t)) \cdot y(t) :$

$IC := [[x(0) = 1, y(0) = 2], [x(0) = 2, y(0) = 2], [x(0) = 3, y(0) = 2], [x(0) = .8, y(0) = 2], [x(0) = 1.5, y(0) = 2]] :$
$with(plots) : with(DEtools) :$
$DEplot([eqxt, eqyt], [x(t), y(t)], t = -10..10, x = -0.5..4, y = -0.5..4, IC, stepsize = 0.01, title = \text{"predator-prey 1c"}, linecolor = black, method = classical[rk4])$



#Stable at (x=0,y=2)
#Unstable at (x=0,1.33,y=0)

##Problem 2
restart

eqx d y^3 K $4\$x$:

eqy d y^3 K y K $3\$x$:

sol d $solve(\{eqx, eqy\}, \{x, y\})$

$$sol \text{ d } \{x=0, y=0\}, \{x=2, y=2\}, \{x = \text{K}\,2, y = \text{K}\,2\} \quad \textbf{(22)}$$

$with(LinearAlgebra) : with(VectorCalculus) :$

J d $Jacobian([eqx, eqy], [x, y])$

$$J \text{ d } \begin{bmatrix} \text{K}\,4 & 3\,y^2 \\ \text{K}\,3 & 3\,y^2\,\text{K}\,1 \end{bmatrix} \quad \textbf{(23)}$$

$J0$ d $subs(sol[1], J)$

$$J0 \text{ d } \begin{bmatrix} \text{K}\,4 & 0 \\ \text{K}\,3 & \text{K}\,1 \end{bmatrix} \quad \textbf{(24)}$$

$J1$ d $subs(sol[2], J)$

$$J1 \text{ d } \begin{bmatrix} \text{K}\,4 & 12 \\ \text{K}\,3 & 11 \end{bmatrix} \quad \textbf{(25)}$$

evs d $Eigenvalues(J1)$

$$evs \text{ d } \begin{bmatrix} 8 \\ \text{K}\,1 \end{bmatrix} \quad \textbf{(26)}$$

Determinant$(J1)$

$$-8 \tag{27}$$

Eigenvectors$(J0)$

$$\left[\begin{array}{c} -1 \\ -4 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right] \tag{28}$$

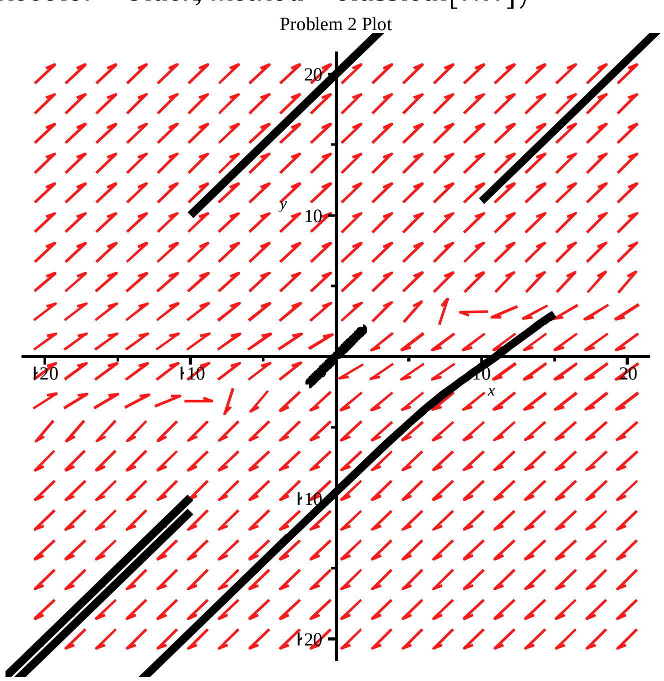
$eqxt := x'(t) = y(t)^3 - 4 \cdot x(t):$

$eqyt := y'(t) = y(t)^3 - y(t) - 3 \cdot x(t):$

$IC := [[x(0) = 1.9, y(0) = 1.9], [x(0) = -10, y(0) = -10], [x(0) = -10, y(0) = 10], [x(0) = -1.9, y(0) = -1.9], [x(0) = 15, y(0) = 3], [x(0) = 10, y(0) = 11], [x(0) = -10, y(0) = -11]]:$

$with(plots) : with(DEtools) :$

$DEplot([eqxt, eqyt], [x(t), y(t)], t = 0..20, x = -20..20, y = -20..20, IC, stepsize = 0.01, title = \text{"Problem 2 Plot"}, linecolor = black, method = classical[rk4])$



#Problem 2a)
 # Stable at (0,0)
 # Unstable at (-2,-2) and (2,2)

#Problem 2b)
#It can be seen that any initial condition on x=y stays alongthatline

#Problem 2c)
 # As each initial condition moves to t=inf it converges to x=y thus |x(t)-y(t)| would converge to 0.

#Problem 2d)

 # Appears to approach a curve that has an equation of $a \cdot x^{\frac{1}{3}}$ **for** *some a.*

##Problem 3

restart

$eqx := y:$

$eqy := x^3 - x:$

$sol := solve(\{eqx, eqy\}, \{x, y\})$

$$sol := \{x = 0, y = 0\}, \{x = 1, y = 0\}, \{x = -1, y = 0\} \tag{29}$$

$with(LinearAlgebra) : with(VectorCalculus) :$

$J := Jacobian([eqx, eqy], [x, y])$

$$J := \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix} \tag{30}$$

$J0 := subs(sol[1], J)$

$$J0 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{31}$$

$J1 := subs(sol[2], J)$

$$J1 := \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \tag{32}$$

$evs := Eigenvalues(J1)$

$$evs := \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix} \tag{33}$$

$Determinant(J1)$

$$-2 \tag{34}$$

$Eigenvectors(J0)$

$$\begin{bmatrix} \mathrm{I} \\ -\mathrm{I} \end{bmatrix}, \begin{bmatrix} -\mathrm{I} & \mathrm{I} \\ 1 & 1 \end{bmatrix} \tag{35}$$

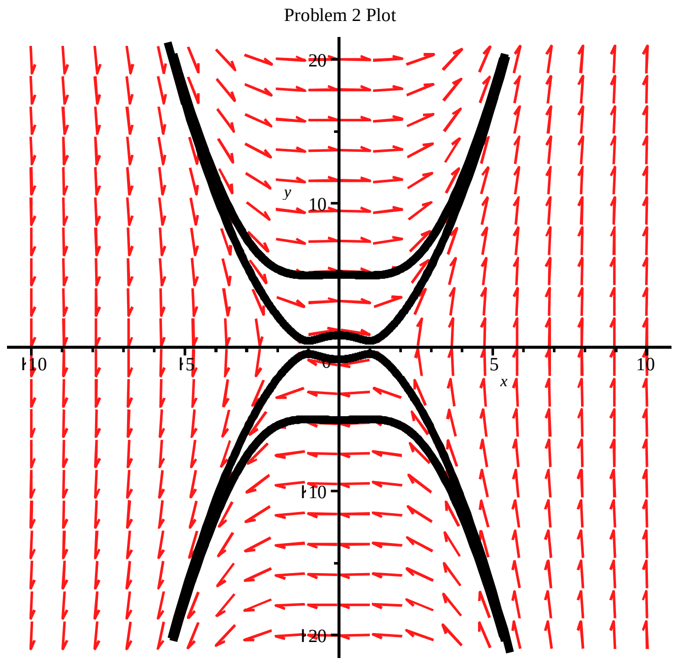
$eqxt := x'(t) = y(t) :$

$eqyt := y'(t) = x(t)^3 - x(t) :$

$IC := [[x(0) = 1.9, y(0) = 1.9], [x(0) = 1, y(0) = -5], [x(0) = -1, y(0) = 5], [x(0) = -1.9, y(0) = -1.9], [x(0) = -1, y(0) = -5], [x(0) = 1, y(0) = 5]] :$

$with(plots) : with(DEtools) :$

$DEplot([eqxt, eqyt], [x(t), y(t)], t = -20 ..20, x = -10 ..10, y = -20 ..20, IC, stepsize = 0.01, title$ = "Problem 3 Plot", $linecolor = black, method = classical[rk4])$



 # 3a) Unstable at (0,0)
 # Semistable at (-1,0) and (1,0)

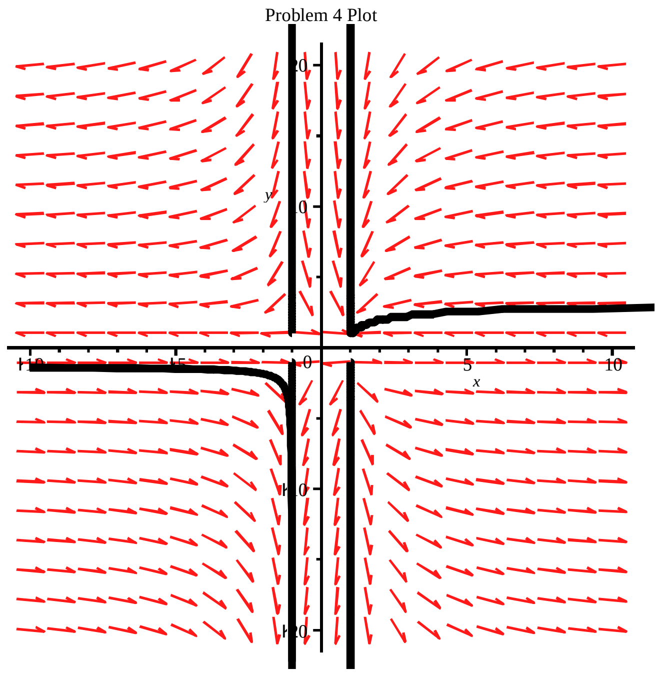
#3b) $x''\cdot x' = \left(x^3 - x x'\right) = \dfrac{d}{dt}\left(\dfrac{x^4}{4} - \dfrac{x^2}{2}\right) \Rightarrow \dfrac{x'^2}{2} + \dfrac{x^4}{4} - \dfrac{x^2}{2} = c \Rightarrow F(x, y) = 2\,x^2 + 2\,y^2 - x^4$

Problem 4
restart
$eqx := y\cdot\left(1 - x^2\right) :$
$eqy := 1 - y^2 :$
$sol := solve(\{eqx, eqy\}, \{x, y\})$

$$sol := \{x = 1, y = 1\}, \{x = 1, y = -1\}, \{x = -1, y = 1\}, \{x = -1, y = -1\} \quad \mathbf{(36)}$$

$eqxt := x'(t) = y(t)\cdot\left(1 - x(t)^2\right) :$
$eqyt := y'(t) = 1 - y(t)^2 :$
$IC := [[x(0) = 1.9, y(0) = 1.9], [x(0) = 1, y(0) = -5], [x(0) = -1, y(0) = 5], [x(0) = -1.9, y(0) = -1.9], [x(0) = -1, y(0) = -5], [x(0) = 1, y(0) = 5]] :$

$with(plots) : with(DEtools) :$
$DEplot([eqxt, eqyt], [x(t), y(t)], t = -20..20, x = -10..10, y = -20..20, IC, stepsize = 0.01, title = \text{"Problem 4 Plot"}, linecolor = black, method = classical[rk4])$



4a) This is reversible because $eqx := y\cdot(1 - x^2)$ is odd. Because $eqx(x, -y) = -eqx(x, y)$
and $eqy := 1 - y^2$ is even because $eqx(x, -y) = eqx(x, y)$.

Problem 5
restart
$eqx := y :$
$eqy := -b\cdot y - \sin(x) :$

$sol := solve(\{eqx, eqy\}, \{x, y\})$

$$sol := \{x = 0, y = 0\} \quad \mathbf{(37)}$$

$with(LinearAlgebra) : with(VectorCalculus) :$
$J := Jacobian([eqx, eqy], [x, y])$

$$J := \begin{bmatrix} 0 & 1 \\ -\cos(x) & -b \end{bmatrix} \quad \mathbf{(38)}$$

$J0 := subs(sol[1], J)$

$$J0 := \begin{bmatrix} 0 & 1 \\ -\cos(0) & -b \end{bmatrix} \quad \mathbf{(39)}$$

$J1 := subs(sol[2], J)$

$$J1 := \begin{bmatrix} 0 & 1 \\ -\cos(x) & -b \end{bmatrix} \quad \mathbf{(40)}$$

$evs := Eigenvalues(J1)$

$$evs := \begin{bmatrix} -\frac{b}{2} + \frac{\sqrt{b^2 - 4\cos(x)}}{2} \\ -\frac{b}{2} - \frac{\sqrt{b^2 - 4\cos(x)}}{2} \end{bmatrix} \tag{41}$$

Determinant(*J1*)

$$\cos(x) \tag{42}$$

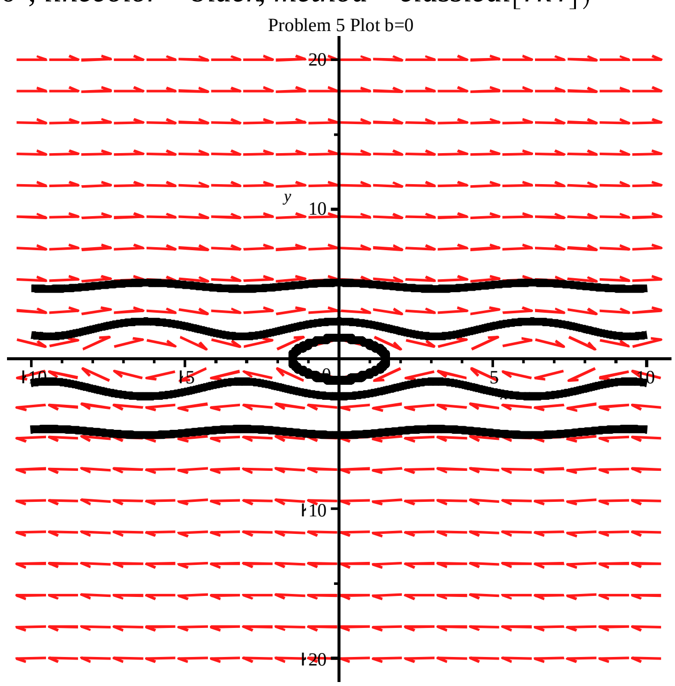
Eigenvectors(*J0*)

$$\begin{bmatrix} -\frac{b}{2} + \frac{\sqrt{b^2 - 4}}{2} \\ -\frac{b}{2} - \frac{\sqrt{b^2 - 4}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{-\frac{b}{2} + \frac{\sqrt{b^2 - 4}}{2}} & \frac{1}{-\frac{b}{2} - \frac{\sqrt{b^2 - 4}}{2}} \\ 1 & 1 \end{bmatrix} \tag{43}$$

$eqxt := x'(t) = y(t):$
$eqyt := y'(t) = -\sin(x(t)):$

$IC := [[x(0) = 1.9, y(0) = 1.9], [x(0) = 1, y(0) = -5], [x(0) = -1, y(0) = 1], [x(0) = -1.9, y(0) = -1.9], [x(0) = -1, y(0) = -5], [x(0) = 1, y(0) = 5]]:$

with(*plots*) : *with*(*DEtools*) :
DEplot([*eqxt*, *eqyt*], [$x(t)$, $y(t)$], $t = -20..20$, $x = -10..10$, $y = -20..20$, *IC*, *stepsize* = 0.01, *title* = "Problem 5 Plot b=0", *linecolor* = *black*, *method* = *classical*[*rk4*])

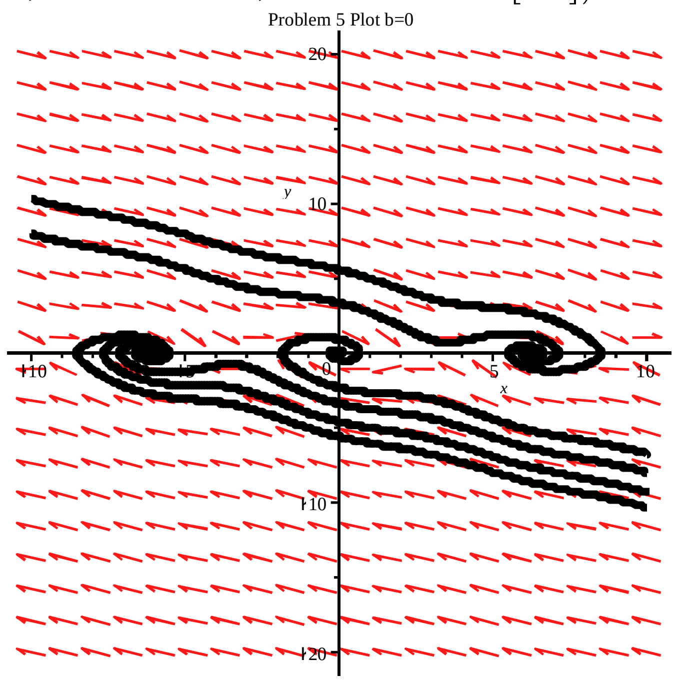


$eqxt := x'(t) = y(t):$

$eqyt := y'(t) = -.5 \cdot y(t) - \sin(x(t))$:

$with(plots) : with(DEtools) :$

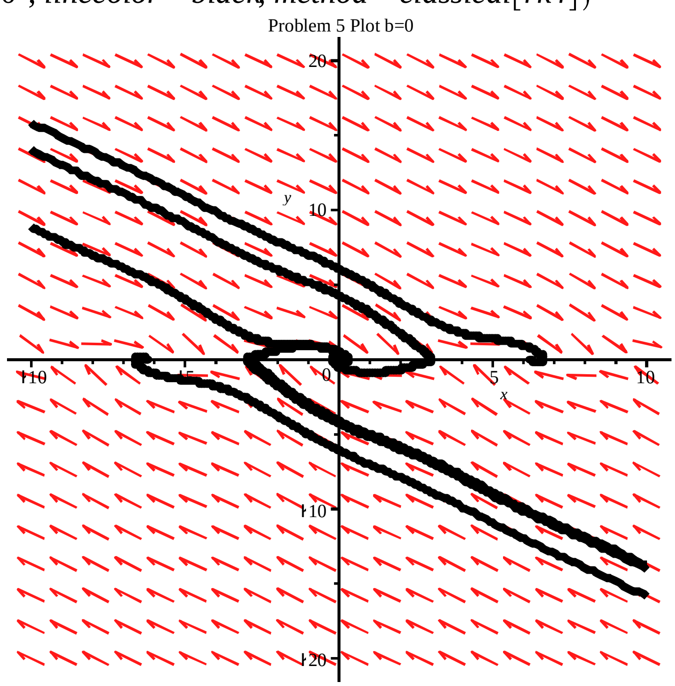
$DEplot([eqxt, eqyt], [x(t), y(t)], t = -20..20, x = -10..10, y = -20..20, IC, stepsize = 0.01, title$
$= \text{"Problem 5 Plot b=0"}, linecolor = black, method = classical[rk4])$



$eqxt := x'(t) = y(t)$:
$eqyt := y'(t) = -y(t) - \sin(x(t))$:

$with(plots) : with(DEtools) :$

$DEplot([eqxt, eqyt], [x(t), y(t)], t = -20..20, x = -10..10, y = -20..20, IC, stepsize = 0.01, title$
$= \text{"Problem 5 Plot b=0"}, linecolor = black, method = classical[rk4])$



 # 5) Equilibria at (x,y)=(-2pi,0),(0,0),(2pi,0) for all bO0
 # When b=0 all equilibria are unstable
 # When b ! 1 (0,0) is unstable and (G2pi,0) is stable
 # When bR1 all equilibria is stable