Mathematical Models

Modeling in the Frequency Domain

Electrical Networks

Mainly from Nise's Textbook Ch. 2-1

Chapter 2: Mathematical Models:
Modeling in the Frequency Domain

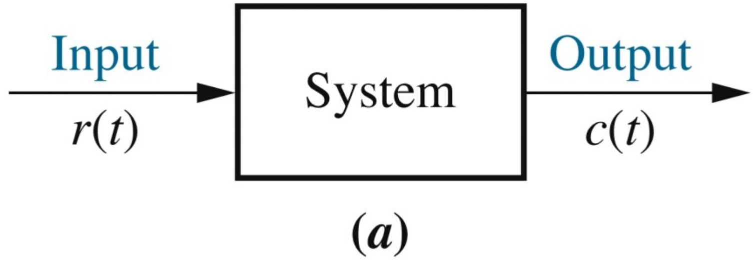
- The first step in developing a mathematical model is to apply the fundamental physical laws.
  - Kirchoff's Voltage Law - The sum of voltages around a closed path is zero.
  - Kirchoff's Current Law - The sum of currents flowing from a node is zero.
  - Newton's Laws - The sum of forces on a body is zero (considering mass times acceleration as a force).
    - The sum of moments on a body is zero.
- The model describes the relationship between the input and the output of the dynamic system.

- Applying the laws of physics generates a differential equation which can be represented in general form:

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1} c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

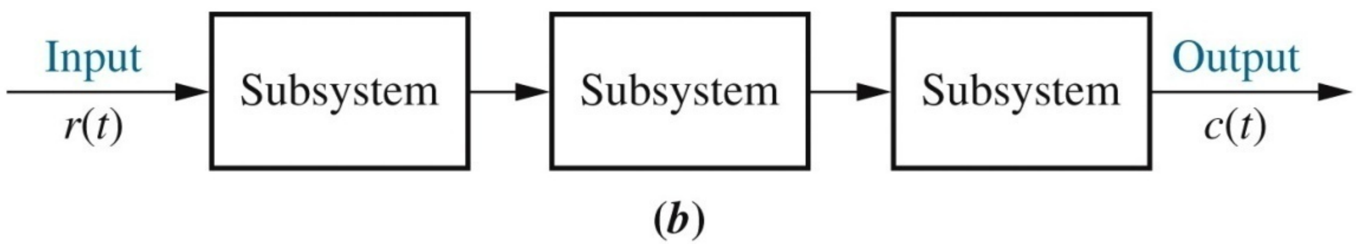
- The form and the coefficients of the differential equation are a formulation or description of the system.
- Although the differential equation relates the system to its input and output, it is not a satisfying representation from a system perspective.

- We would prefer a mathematical representation such as



in which the input, output and the system are distinct and separate parts.

- We would also like to represent cascaded interconnections, such as



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

Laplace Transformation Review

- The Laplace transform is defined as,

$$\boxed{\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt}$$

where $s = \sigma + j\omega$ is a complex variable.

- The inverse Laplace transform, which allows us to find f(t) given F(s), is

$$\boxed{\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)}$$

where

$$\begin{aligned} u(t) &= 1 \qquad t > 0 \\ &= 0 \qquad t < 0 \end{aligned}$$

- Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for t<0.

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$	Description
1.	$\delta(t)$	1	Impulse (Dirac-delta)
2.	$u(t)$	$\frac{1}{s}$	Step
3.	$tu(t)$	$\frac{1}{s^2}$	Ramp
4.	$t^n u(t)$	$\frac{n!}{s^n+1}$	Parabolic
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$	Exponential
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$	Sinusoidal
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$	

Table 2.1

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Table 2.2

Example – 1 (Laplace transform)

- Find the Laplace transform of $f(t) = Ae^{-at}u(t)$
- Since the time function does not contain an impulse function, we can replace the lower limit of the integral in the Laplace transform definition with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

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Table 2.1

Example – 2 (Inverse Laplace transform)

- Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$

  For this example we make use of the frequency shift theorem and the Laplace transform of $f(t) = tu(t)$

  If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$

  then the inverse transform of $F(s+a) = 1/(s+a)^2$

  is $e^{-at}tu(t)$.

  Hence, $f_1(t) = e^{-3t}tu(t)$

Partial-Fraction Expansion

- In order to find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a partial-fraction expansion.
- If $F_1(s) = N(s)/D(s)$ where the order of *N*(*s*) is less than the order of D(s), then a partial-fraction expansion can be made.
- If the order of *N*(*s*) is greater than or equal to the order of *D*(*s*), then *N*(*s*) must be divided by *D*(*s*) successively until the result has a remainder whose numerator is of order less than its denominator.

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \tag{2.4}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5} \tag{2.5}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right] \tag{2.6}$$

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

An example of an $F(s)$ with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)}$$

We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called residues, form the numerators.

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \qquad (2.8)$$

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Letting s approach -1 eliminates the last term and yields $K_1 = 2$. Similarly, K_2 can be found by multiplying Eq. (2.8) by $(s+2)$ and then letting s approach -2; hence, $K_2 = -2$.

$f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Example 2.3 (Laplace Transform Solution of a Differential Equation)

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2}+12\frac{dy}{dt}+32y=32u(t)$$

$$s^2Y(s)+12sY(s)+32Y(s)=\frac{32}{s}$$

Solving for the response, *Y*(*s*), yields

$$Y(s)=\frac{32}{s(s^2+12s+32)}=\frac{32}{s(s+4)(s+8)}$$

$$Y(s)=\frac{32}{s(s+4)(s+8)}=\frac{K_1}{s}+\frac{K_2}{(s+4)}+\frac{K_3}{(s+8)}$$

$$K_1=\left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0}=1$$

$$K_2=\left.\frac{32}{s(s+8)}\right|_{s\to -4}=-2$$

$$K_3=\left.\frac{32}{s(s+4)}\right|_{s\to -8}=1$$

$$Y(s)=\frac{1}{s}-\frac{2}{(s+4)}+\frac{1}{(s+8)}$$

Since each of the three component parts is represented as an F(s) in Table 2.1, *y*(*t*) is the sum of the inverse Laplace transforms of each term.

$$y(t)=(1-2e^{-4t}+e^{-8t})u(t)$$

Case 2. Roots of Denominator of *F*(*s*) are Real and Repeated

The roots of $(s+2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a *multiple root* of *multiplicity* 2.

$$F(s) = \frac{2}{(s+1)(s+2)^2} \qquad F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying Eq. (2.23) by $(s+2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3$$

Letting s approach -2, $K_2 = -2$. To find K_3 we see that if we differentiate Eq. (2.24) with respect to s,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2}K_1 + K_3$$

K_3 is isolated and can be found if we let s approach -2. Hence, $K_3 = -2$.

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Case 3. Roots of the Denom. of *F*(*s*) Are Complex or Imaginary

$$\mathscr{L}[Ae^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\mathscr{L}[Be^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2+\omega^2}$$

$$\mathscr{L}[Ae^{-at}\cos \omega t + Be^{-at}\sin \omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

K_1 is found in the usual way to be $\frac{3}{5}$. K_2 and K_3 can be found by first multiplying Eq. (2.31) by the lowest common denominator, $s(s^2+2s+5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Balancing coefficients, $\left(K_2 + \frac{3}{5}\right) = 0$ and $\left(K_3 + \frac{6}{5}\right) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

$$\boxed{f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)}$$

Now, let us see how the time function changes over time-1

$$F(s) = \frac{3}{s(s^2+2s+5)} \rightarrow \boxed{f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)}$$

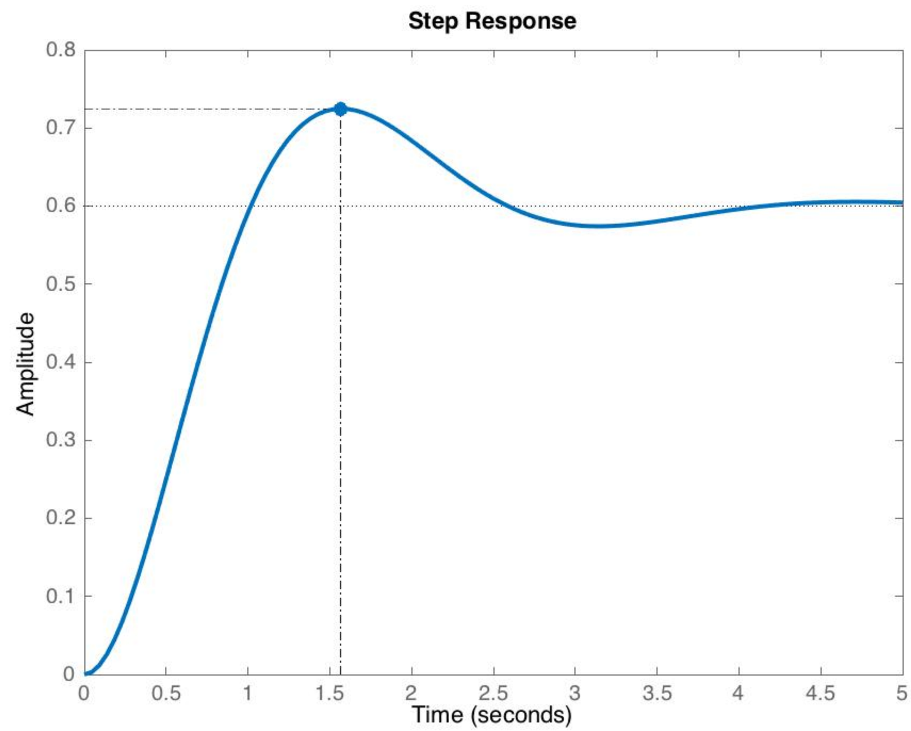
$$f(t) = 0.6 - 0.671e^{-t}\cos(2t - \phi),$$

$$\text{where } \phi = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636 \text{ rad}$$

We can plot the time response we just computed in Matlab or even in Excel as follows:

Plotting in Matlab

```
>> s=tf('s');
>> F=3/(s^2 + 2*s + 5)
>> step(F)
```



Now, let us see how the time function changes over time-2

$$F(s) = \frac{3}{s(s^2+2s+5)} \Rightarrow f(t) = 0.6 - 0.671e^{-t}\cos(2t-\phi),$$

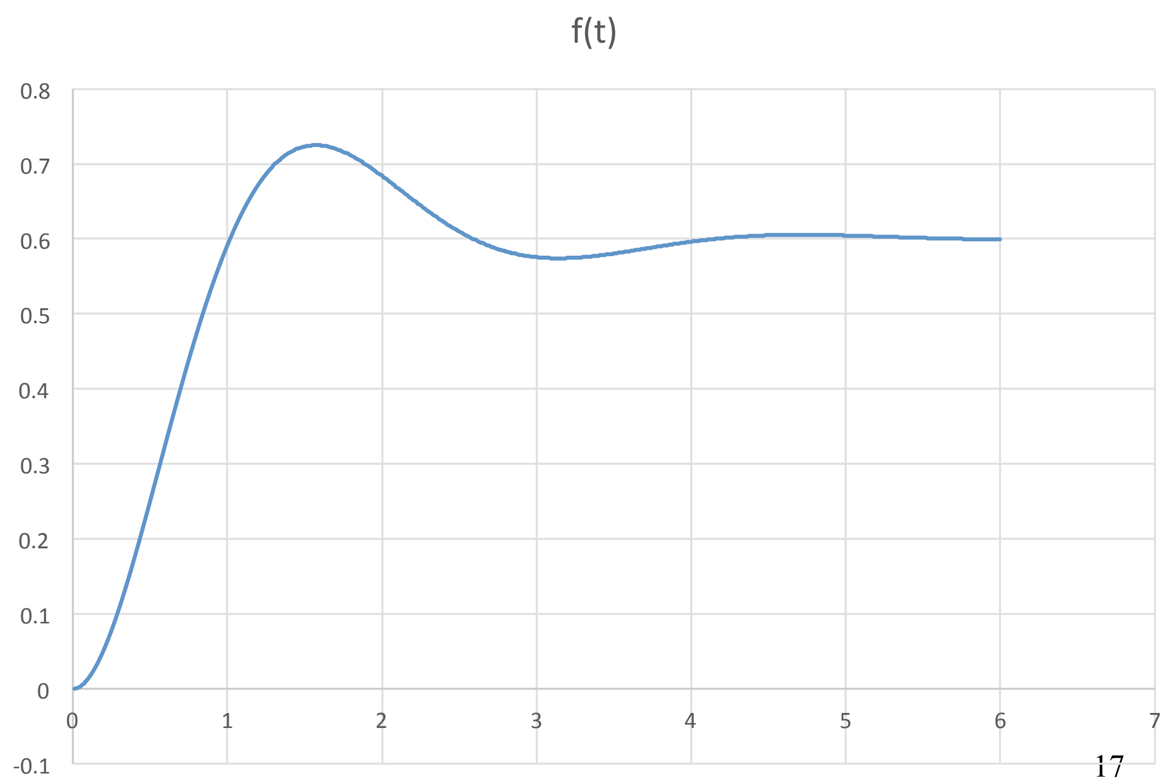
$$\text{where } \phi = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636 \text{ rad}$$

Now plotting in Excel,

=0,6-0,671*EXP(-A2)*COS(2*A2-0,4636)

Plotting in Excel

t (sec)	f(t)
0,01	-2,51816E-05
0,02	0,000418623
0,03	0,001150439
0,04	0,002164203
0,05	0,003453829
0,06	0,005013216
0,07	0,006836252
0,08	0,008916812
0,09	0,011248768
0,1	0,013825988
0,11	0,016642339
0,12	0,019691691



The Transfer Function

- We are now ready to formulate the system representation by establishing a viable definition for a function that algebraically relates a system's output to its input.
- This function will allow separation of the input, system and output into three separate and distinct parts, unlike the differential equation.
- The function will also allow us to algebraically combine mathematical representations of subsystems to yield a total system representation.

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \tag{2.50}$$

$$\begin{aligned} a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) &+ \text{initial condition terms involving } c(t) \\ = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) &+ \text{initial condition terms involving } r(t) \end{aligned} \tag{2.51}$$

The Transfer Function

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)C(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)R(s) \qquad (2.52)$$

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \qquad (2.53)$$

Notice that Eq. (2.53) separates the output, $C(s)$, the input, $R(s)$, and the system, the ratio of polynomials in s on the right. We call this ratio, $G(s)$, the transfer function and evaluate it with zero initial conditions.

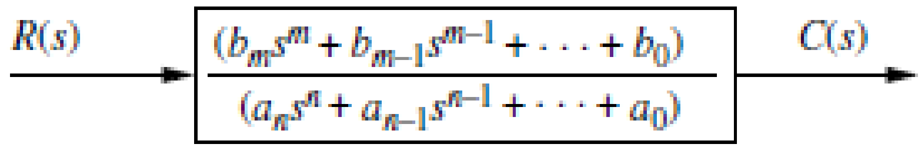


FIGURE 2.2 Block diagram of a transfer function

The transfer function can be represented as a block diagram, as shown in Figure 2.2, with the input on the left, the output on the right, and the system transfer function inside the block. Notice that the denominator of the transfer function is identical to the characteristic polynomial of the differential equation. Also, we can find the output, $C(s)$ by using

$$\boxed{C(s) = R(s)G(s)}$$

COMPUTER AIDED DESIGN

We will use MATLAB and Control System toolbox.
Included are,

- Simulink
- LTI Viewer
- SISO Design Tool
- Symbolic Math Toolbox

Example 2.4: Transfer Function for a Differential Equation

Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \tag{2.55}$$

Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \tag{2.56}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \tag{2.57}$$

Use the result of Example 2.4 to find the response, *c*(*t*) to an input, a unit step, assuming zero initial conditions.

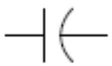
$$C(s) = R(s)G(s) = \frac{1}{s(s+2)}$$

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

$$\boxed{c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}}$$

Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical Network Transfer Functions …

- Equivalent circuits for the electric networks that we work with first consist of three passive linear components: **resistors**, **capacitors**, and **inductors**.
- We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are *Kirchhoff's laws*.
- We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero. From these relationships we can write the differential equations for the circuit.
- Then we can take the Laplace transforms of the differential equations and finally solve for the transfer function.

Example 2.6: Transfer Function—Single Loop via the Differential Equation

- Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

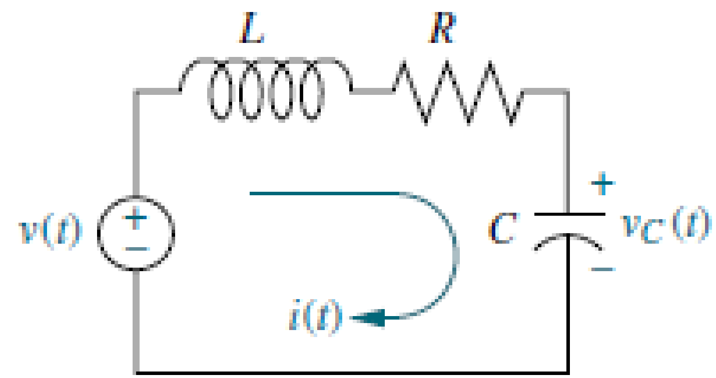


FIGURE 2.3 *RLC* network

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)$$

Changing variables from current to charge using $i(t)=\dfrac{dq(t)}{dt}$

$$L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=v(t)$$

Example 2.6: Transfer Function—Single Loop via the Differential Equation

- Substituting $q(t) = Cv_C(t)$ yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \qquad (2.64)$$

- Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \qquad (2.65)$$

- Solving for the transfer function, $\dfrac{V_C(s)}{V(s)}$ we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}}$$

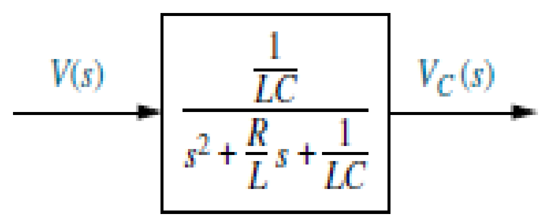


FIGURE 2.4 Block diagram of series *RLC* electrical network

Example 2.6: Transfer Function—Single Loop via the Differential Equation

- Let us now develop a technique for simplifying the solution for future problems.
  - For the capacitor, $V(s) = \frac{1}{Cs}I(s)$
  - For the resistor, $V(s) = RI(s)$
  - For the inductor, $V(s) = LsI(s)$
- Now define the following transfer function:

$$\frac{V(s)}{I(s)} = Z(s)$$

- Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current. But, unlike resistance, this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function **impedance**.

Example 2.6: Transfer Function—Single Loop via the Differential Equation

- The Laplace transform of Eq. (2.61), assuming zero initial conditions, is $\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$

$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}]$$

- <u>We summarize the steps as follows:</u>
    - Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.

Example 2.7: Transfer Function—Single Loop via Transform Methods

- Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.

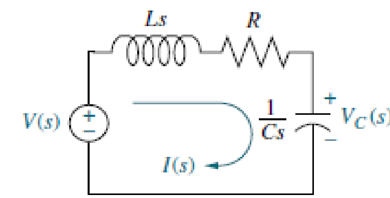


FIGURE 2.5 Laplace-transformed network

- Using Figure 2.5 and writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \qquad (2.73)$$

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \qquad (2.74)$$

- But the voltage across the capacitor, , is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s)\frac{1}{Cs} \qquad (2.75)$$

- Solving Eq. (2.75) for I(s), substituting I(s) into Eq. (2.74), and simplifying yields the same result as Eq. (2.66).

Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

- Replace passive element values with their impedances.
- Replace all sources and time variables with their Laplace transform.
- Assume a transform current and a current direction in each mesh.
- Write Kirchhoff's voltage law around each mesh.
- Solve the simultaneous equations for the output.
- Form the transfer function.

Example 2.10 Transfer Function—Multiple Loops

- Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

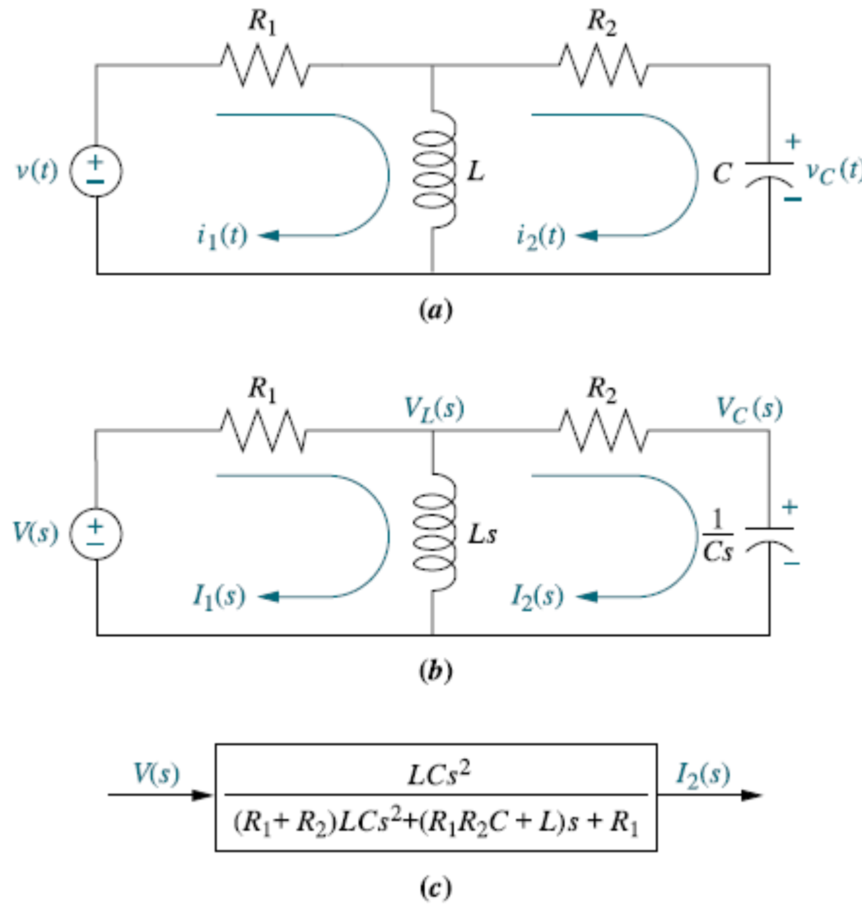


FIGURE 2.6 **a.** Two-loop electrical network; **b.** transformed two-loop electrical network; **c.** block diagram

Example 2.10 Transfer Function—Multiple Loops

- The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions.
- The result is shown in Figure 2.6(b). The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Around Mesh 2, where $I_2(s)$ flows,

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0$$

Example 2.10 Transfer Function—Multiple Loops

$$(R_1 + Ls)I_1(s) \qquad - LsI_2(s) = V(s) \tag{2.80a}$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \tag{2.80b}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta}$$

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$$

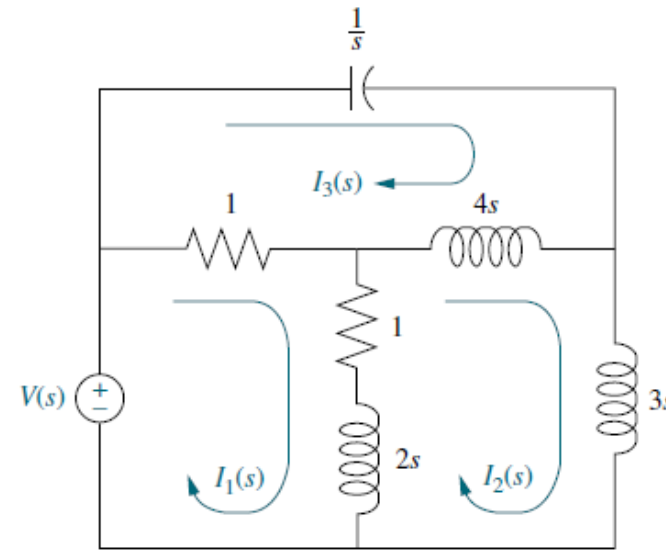
$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$$

Forming the transfer function, G(s), yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1}$$

Example 2.13 Mesh Equations via Inspection

- Write, but do not solve, the mesh equations for the network shown



$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 1}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to}\\ \text{Mesh 1 and}\\ \text{Mesh 2}\end{bmatrix} I_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to}\\ \text{Mesh 1 and}\\ \text{Mesh 3}\end{bmatrix} I_3(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 1}\end{bmatrix}$$

$$\begin{bmatrix}\text{Sum of}\\ \text{edances}\\ \text{nmon to}\\ \text{sh 1 and}\\ \text{Mesh 2}\end{bmatrix} I_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 2}\end{bmatrix} I_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to}\\ \text{Mesh 2 and}\\ \text{Mesh 3}\end{bmatrix} I_3(s) = \begin{bmatrix}\text{Sum of appied}\\ \text{voltages around}\\ \text{Mesh 2}\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to}\\ \text{Mesh 1 and}\\ \text{Mesh 3}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{common to}\\ \text{Mesh 2 and}\\ \text{Mesh 3}\end{bmatrix} I_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{around Mesh 3}\end{bmatrix} I_3(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{voltages around}\\ \text{Mesh 3}\end{bmatrix}$$

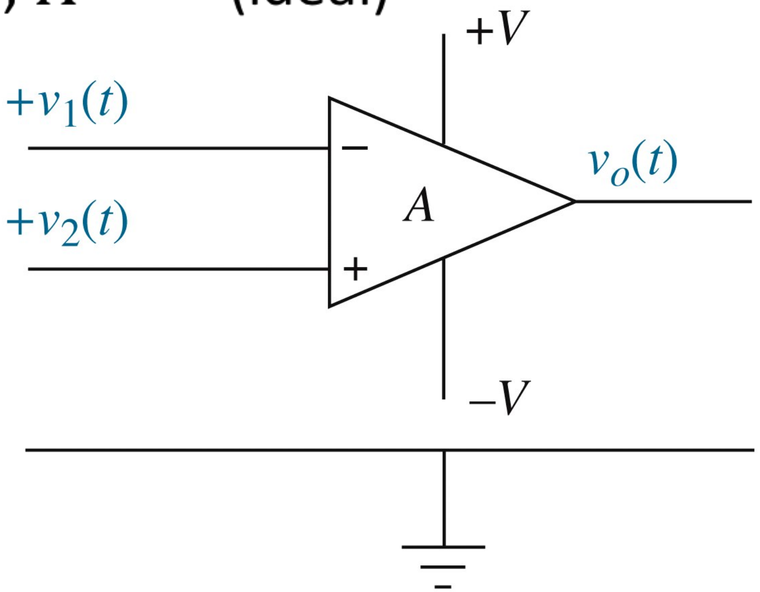
$$+(2s+2)I_1(s) - (2s+1)I_2(s) - I_3(s) = V(s)$$

$$-(2s+1)I_1(s) + (9s+1)I_2(s) - 4sI_3(s) = 0$$

$$-I_1(s) - 4sI_2(s) + \left(4s+1+\tfrac{1}{s}\right)I_3(s) = 0$$

Operational Amplifiers

- An operational amplifier is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:
  - Differential input, $v_2(t) - v_1(t)$
  - High input impedance, $Z_i = \infty$ (ideal)
  - Low output impedance, $Z_0 = 0$ (ideal)
  - High constant gain amplification, $\mathrm{A} = \infty$ (ideal)

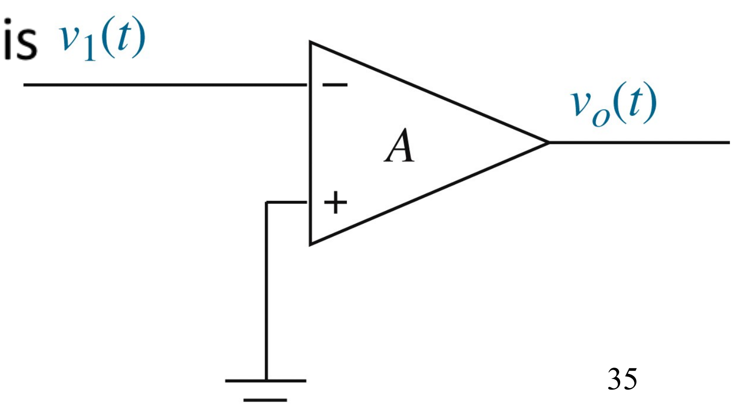


The output is given by

$$\boldsymbol{v_0(t) = A(v_2(t) - v_1(t))}$$

Inverting Operational Amplifier

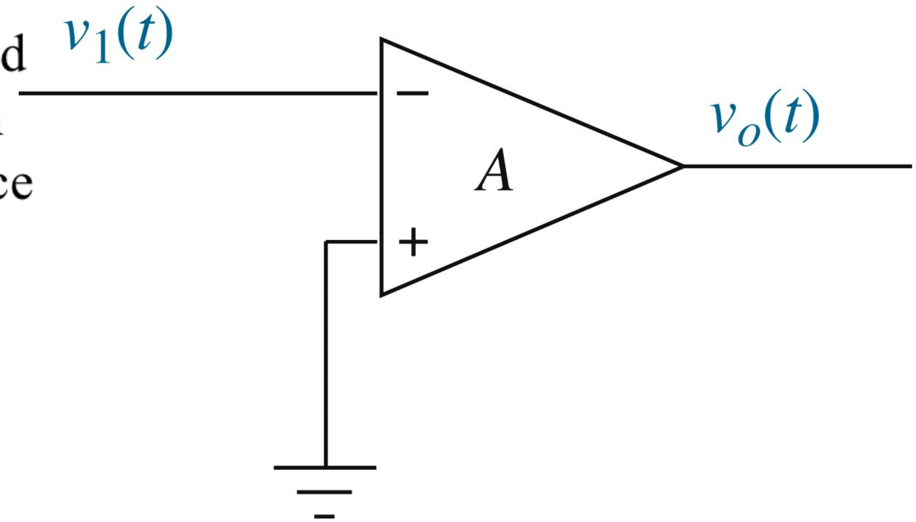
- If v2(t) is grounded, the amplifier is called an inverting operational amplifier. For the inverting operational amplifier, we have $\boldsymbol{v_0(t) = -Av_1(t)}$
- If the input impedance to the amplifier is high, then by Kirchhoff's current law, $I_a(a) = 0$ then $I_1(s) = -I_2(s)$.
- Also, since the gain A is large, v1(t)≈0.
- Thus, $\boldsymbol{I_1(s) = V_i(s)/Z_1(s)}$ and $\boldsymbol{-I_2(s) = -V_0(s)/Z_2(s)}$. Equating the two currents, $\boldsymbol{V_0(s)/Z_2(s) = -V_i(s)/Z_1(s)}$
- The transfer function of the inverting operational amplifier configured as shown in Figure 2.10(c) is



$$\boldsymbol{\frac{V_0(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

Inverting Operational Amplifier

- Since the (+) terminal is grounded and (-) terminal is used this configuration is known as **Inverting Opamp**. Hence $v_0(t) = -Av_1(t)$



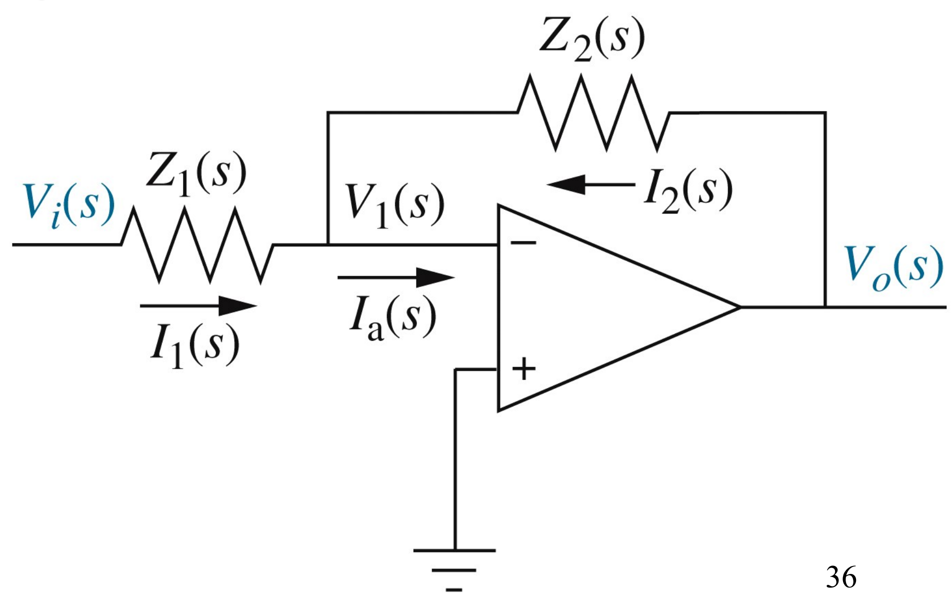
- If the input impedance is high $I_a \approx 0$ and $I_1(s) = -I_2(s)$

- Also, if A is large $v_1(t) = -\frac{v_0(t)}{\infty} = 0$

- Thus, $I_1(s) = V_i(s)/Z_1(s)$ and $I_2(s) = -V_0(s)/Z_2(s)$

- $I_1(s) = -I_2(s) \rightarrow \frac{V_i(s)}{Z_1(s)} = -\frac{V_0(s)}{Z_2(s)}$

$$\frac{\boldsymbol{V_0(s)}}{\boldsymbol{V_i(s)}} = -\frac{\boldsymbol{Z_2(s)}}{\boldsymbol{Z_1(s)}}$$



Example 2.14 Transfer Function for an Inverting Op-Amp Circuit

Example 2.14

Transfer Function—Inverting Operational Amplifier Circuit

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.11.

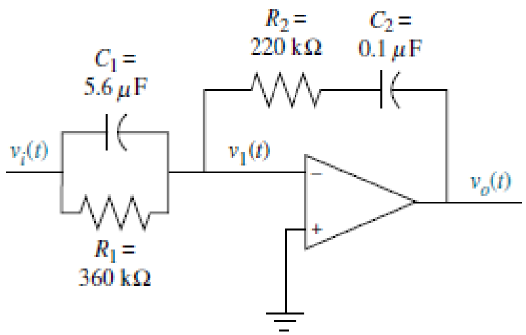


FIGURE 2.11 Inverting operational amplifier circuit for Example 2.14

SOLUTION: The transfer function of the operational amplifier circuit is given by Eq. (2.97). Since the admittances of parallel components add, $Z_1(s)$ is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1} \tag{2.98}$$

For $Z_2(s)$ the impedances add, or

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \tag{2.99}$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.97) and simplifying, we get

$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s} \tag{2.100}$$

The resulting circuit is called a PID controller and can be used to improve the performance of a control system. We explore this possibility further in Chapter 9.

Non-inverting Operational Amplifier

- Another circuit that can be analyzed for its transfer function is the non-inverting operational amplifier circuit shown in Figure 2.12.
- We now derive the transfer function. We see that

$$V_o(s) = A(V_i(s) - V_1(s)) \tag{2.101}$$

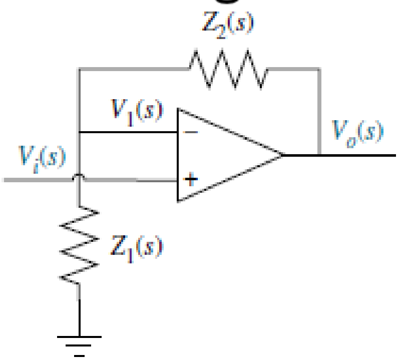


FIGURE 2.12 General noninverting operational amplifier circuit

- But, using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s) \tag{2.102}$$

- Substituting Eq. (2.102) into Eq. (2.101), rearranging, and simplifying, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))} \tag{2.103}$$

$$\boxed{\frac{V_0(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}}$$

Example 2.15 Transfer Function Noninverting Op-Amp Circuit

Example 2.15

Transfer Function—Noninverting Operational Amplifier Circuit

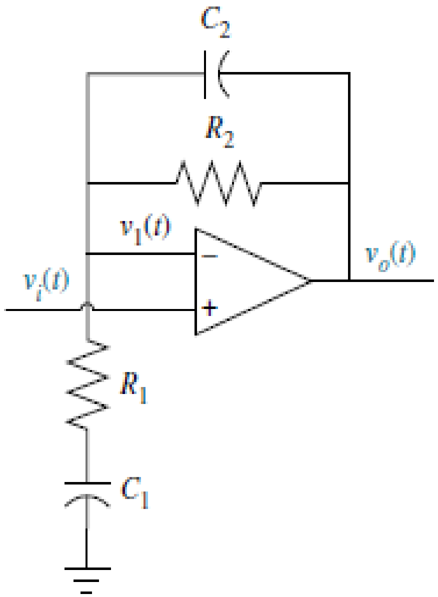


FIGURE 2.13 Noninverting operational amplifier circuit for Example 2.15

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.13.

SOLUTION: We find each of the impedance functions, $Z_1(s)$ and $Z_2(s)$, and then substitute them into Eq. (2.104). Thus,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} \tag{2.105}$$

and

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)} \tag{2.106}$$

Substituting Eqs. (2.105) and (2.106) into Eq. (2.104) yields

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1} \tag{2.107}$$

Problem: Find the transfer function of $\dfrac{V_0(s)}{V_i(s)}$

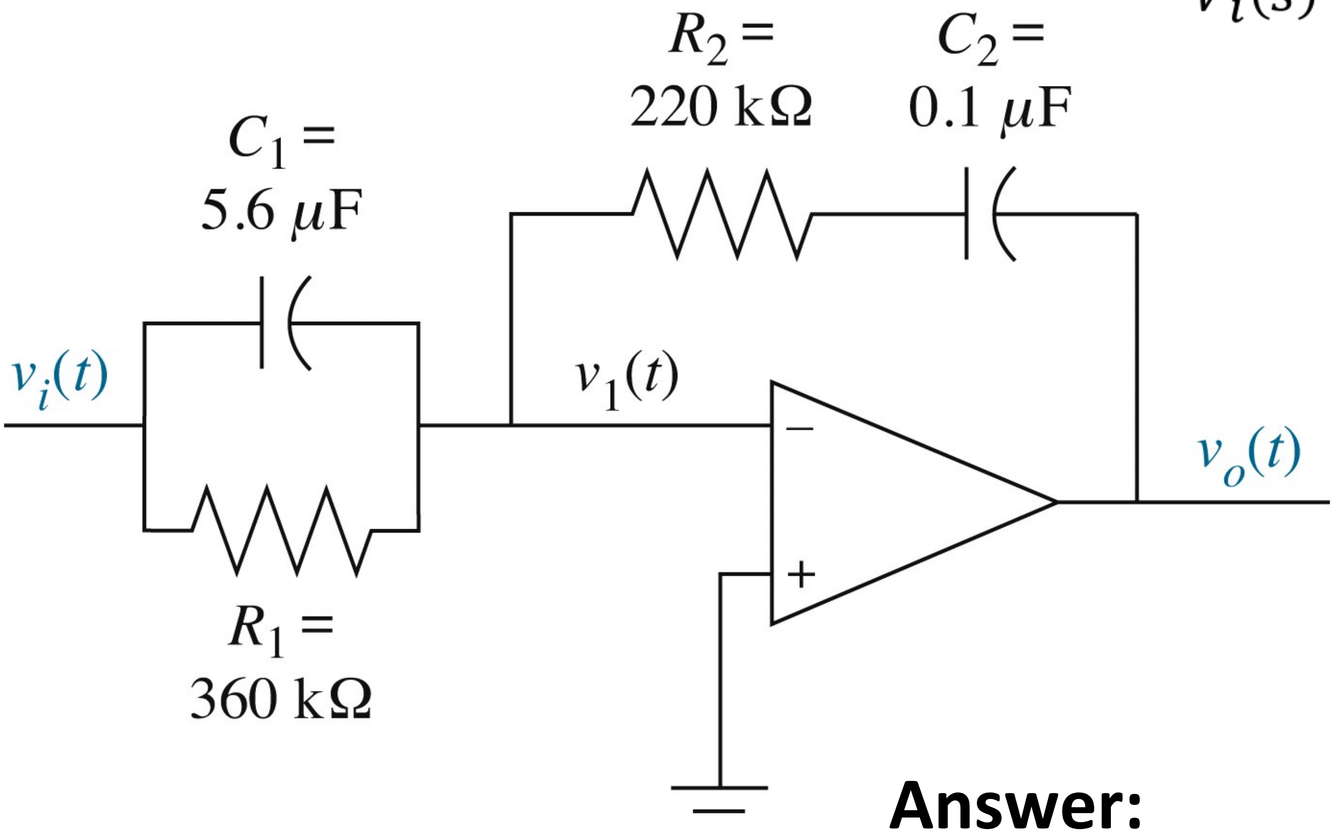


Figure 2.11

Answer:

$$\frac{V_o(s)}{V_i(s)} = -1.232\frac{s^2 + 45.95s + 22.55}{s}$$

The circuit is called a PID controller and can be used to improve the performance of a control system.

Non-inverting Operational Amplifier: The input is to the positive terminal

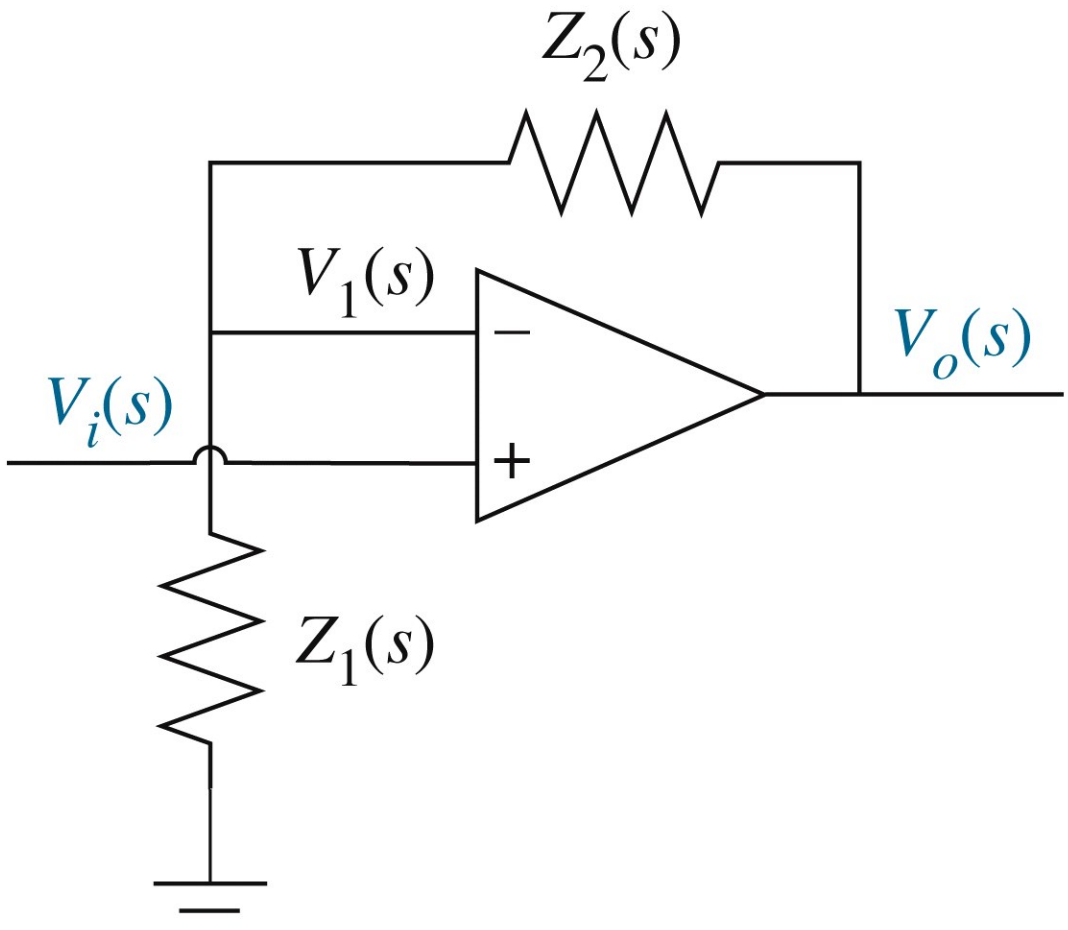


Figure 2.12

Problem: Find the transfer function of $\dfrac{V_0(s)}{V_i(s)}$

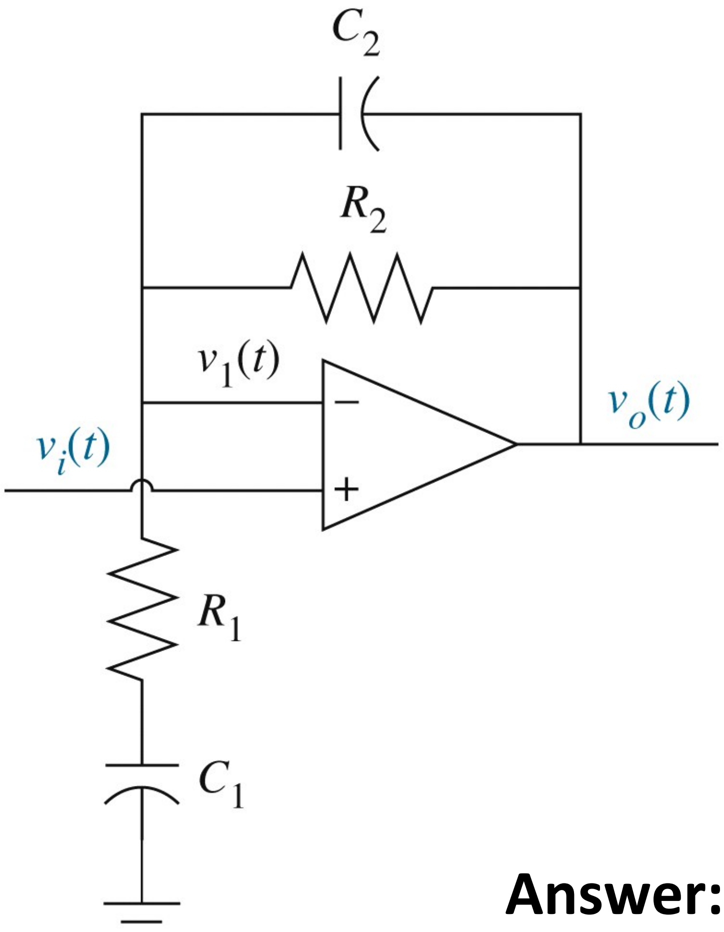


Figure 2.13

Answer:

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1}$$

PI Controller to improve the steady-state response
(we will cover this topic later in Chapter 9)

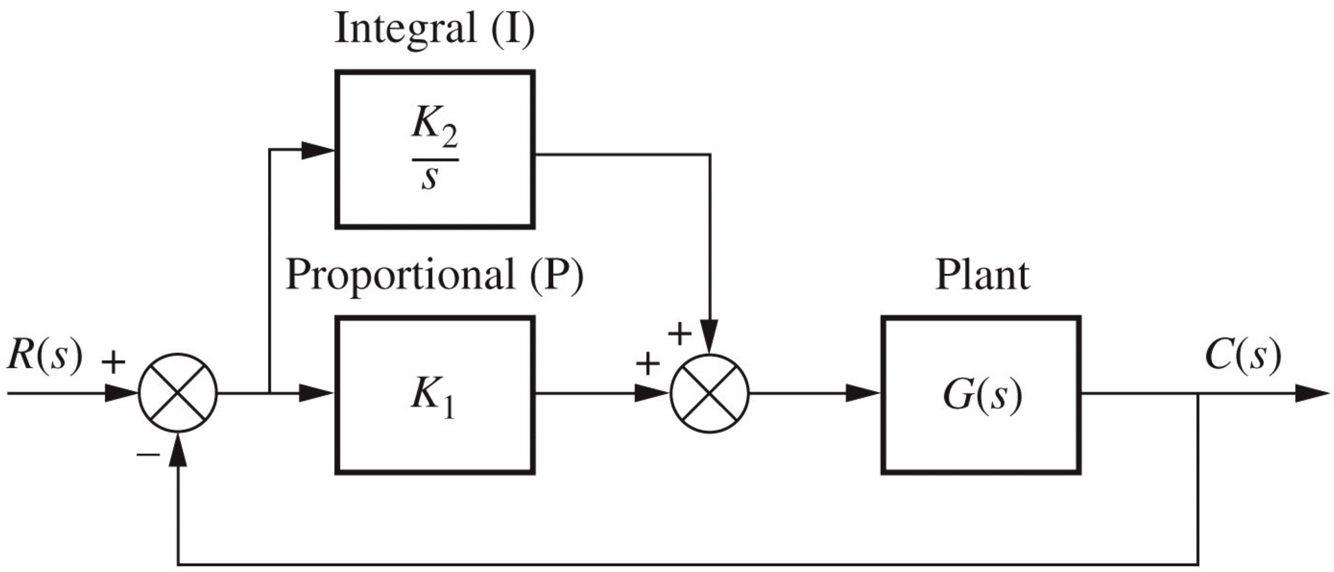


Figure 9.8

PD Controller to improve the transient response
(we will cover this topic later in Chapter 9)

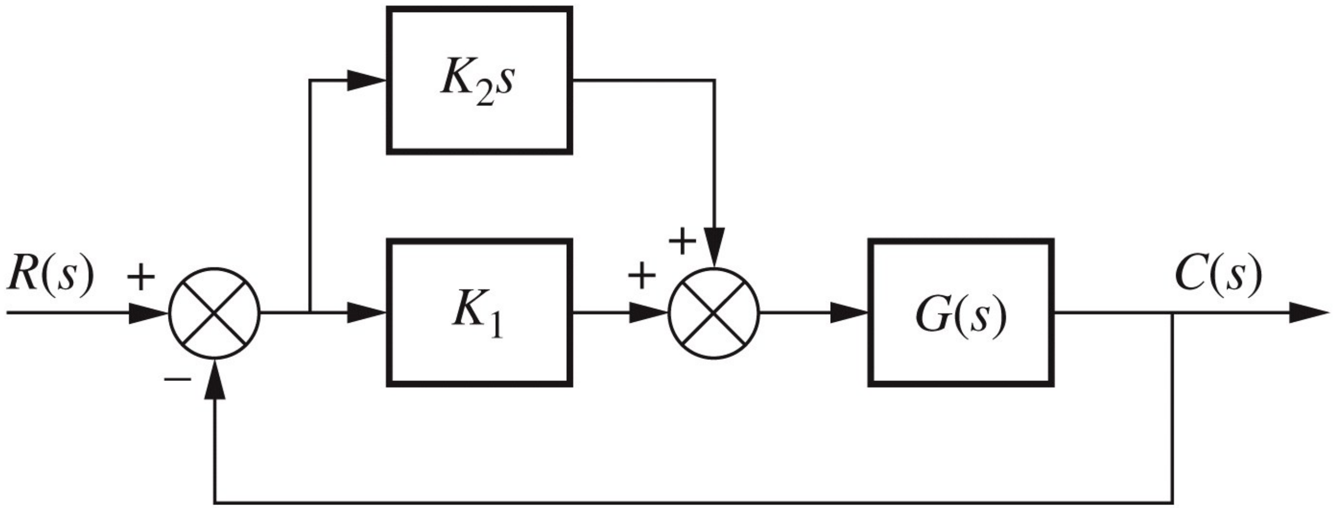


Figure 9.23

PID Controller to improve the transient and also steady-state responses (we will cover this topic later in Chapter 9)

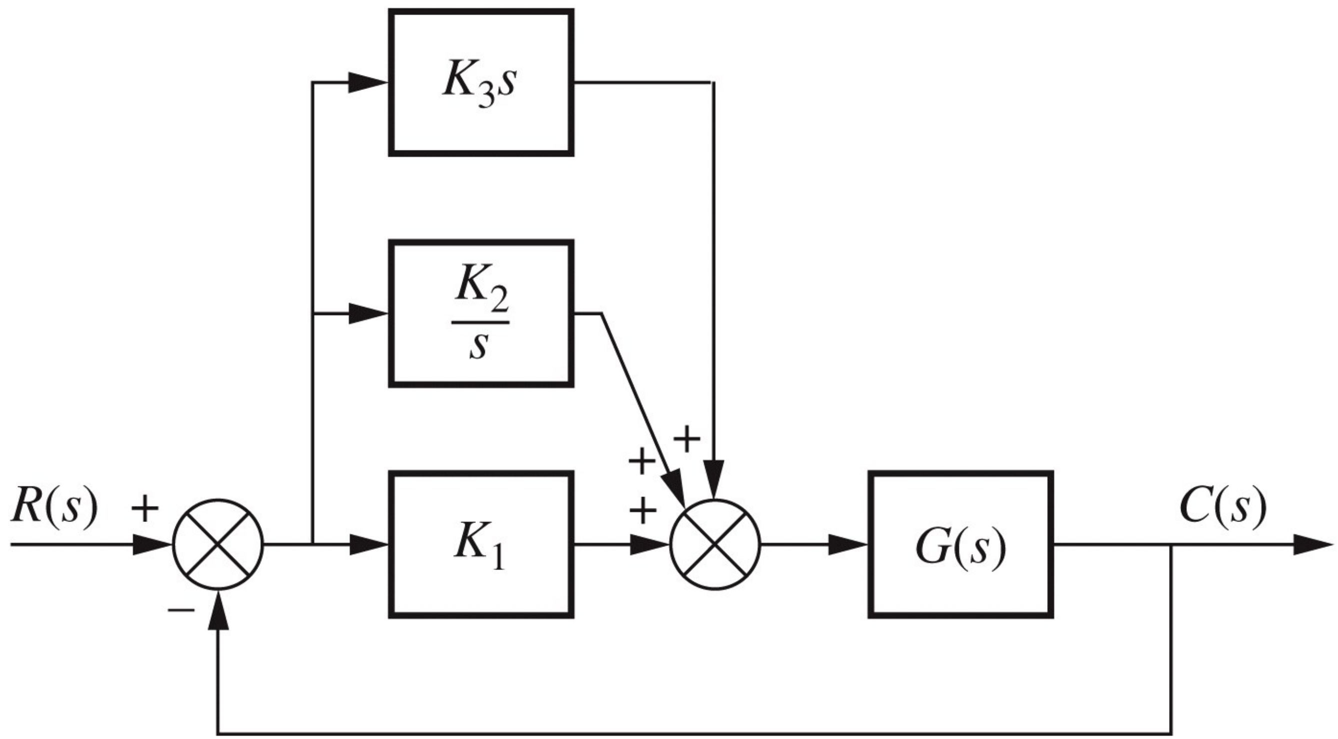


Figure 9.30

Controller realizations by means of active circuits – using an inverting OPAMP, (we will cover this topic later in Chapter 9)

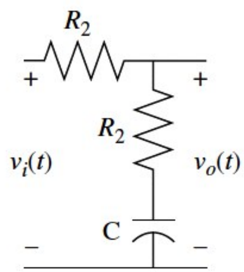
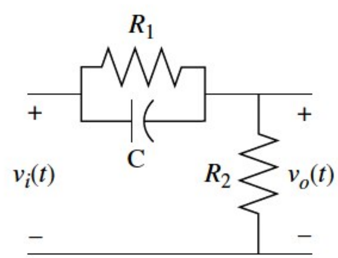
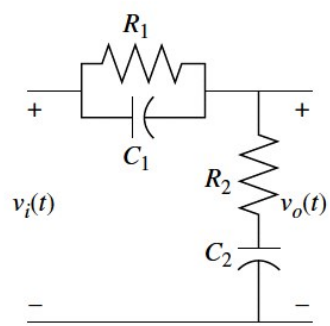
TABLE 9.10 Active realization of controllers and compensators, using an operational amplifier

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain	R_1	R_2	$-\dfrac{R_2}{R_1}$
Integration	R	C	$-\dfrac{\frac{1}{RC}}{s}$
Differentiation	C	R	$-RCs$
PI controller	R_1	R_2 C	$-\dfrac{R_2}{R_1}\dfrac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller	C R_1	R_2	$-R_2C\left(s+\dfrac{1}{R_1C}\right)$
PID controller	C_1 R_1	R_2 C_2	$-\left[\left(\dfrac{R_2}{R_1}+\dfrac{C_1}{C_2}\right)+R_2C_1s+\dfrac{\frac{1}{R_1C_2}}{s}\right]$
Lag compensation	C_1 R_1	C_2 R_2	$-\dfrac{C_1}{C_2}\dfrac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation	C_1 R_1	C_2 R_2	$-\dfrac{C_1}{C_2}\dfrac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

Table 9.10

Controller realizations by means of passive circuits, (we will cover this topic later in Chapter 9)

TABLE 9.11 Passive realization of compensators

Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1+R_2}\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$
Lead compensation		$\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}$
Lag-lead compensation		$\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$