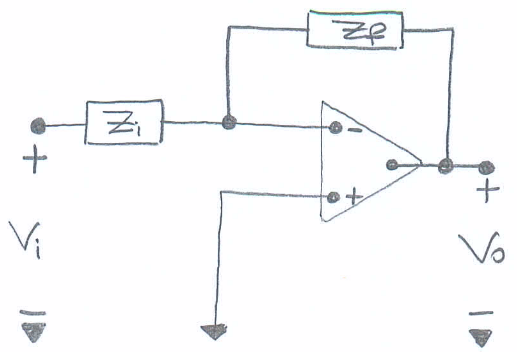
Selected Problems IV

Problem 1) Find the transfer function V_o/V_i for the circuit shown as



where Z_f is the equivalent impedance of the feedback circuit, and Z_i is the equivalent impedance of the input circuit, and op amp is ideal.

Solution. We have

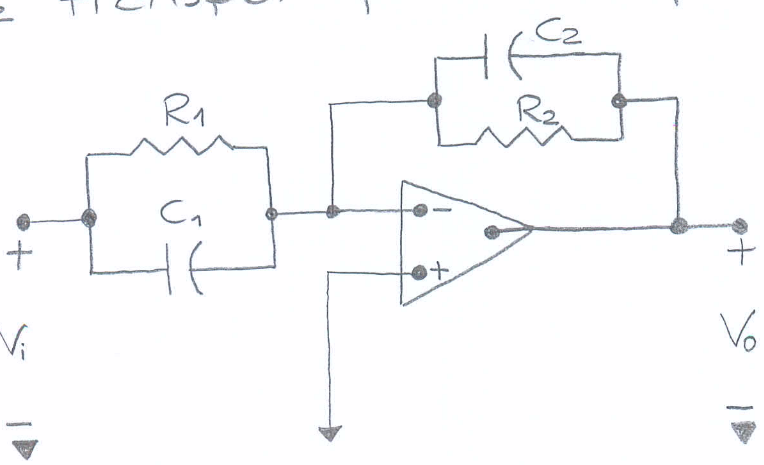
$$V_n = V_p = 0 \quad , \quad i_n = i_p = 0$$

KCL at inverting input of the op amp:

$$\frac{0 - V_i}{Z_i} + \frac{0 - V_o}{Z_f} = 0$$

$$\Rightarrow \quad -\frac{V_i}{Z_i} - \frac{V_o}{Z_f} = 0 \quad \Rightarrow \quad \frac{V_o}{V_i} = -\frac{Z_f}{Z_i}$$

Problem 2) Use the results of Problem 1 to find the transfer function of the following circuit:



a. What is the gain of the circuit as $\omega \to 0$?
b. What is the gain of the circuit as $\omega \to \infty$?
c. Do your answers to (b) and (c) make sense in terms of known circuit behavior?

Solution. We calculate

$$Z_i : \quad R_1 // \frac{1}{sC_1} \quad \Rightarrow \quad Z_i = \frac{R_1 \cdot (1/sC_1)}{R_1 + (1/sC_1)} = \frac{R_1}{sR_1C_1+1}$$

$$Z_f : \quad R_2 // \frac{1}{sC_2} \quad \Rightarrow \quad Z_f = \frac{R_2 \cdot (1/sC_2)}{R_2 + (1/sC_2)} = \frac{R_2}{sR_2C_2+1}$$

Hence;

$$\frac{V_o}{V_i} = -\frac{R_2 / sR_2C_2+1}{R_1 / sR_1C_1+1} = -\frac{R_2(sR_1C_1+1)}{R_1(sR_2C_2+1)}$$

$$= -\frac{sR_1R_2C_1 + R_2}{sR_1R_2C_2 + R_1}$$

$$= -\frac{\cancel{R_1R_2}C_1\left(s + \frac{1}{R_1C_1}\right)}{\cancel{R_1R_2}C_2\left(s + \frac{1}{R_2C_2}\right)}$$

$$= -\frac{C_1}{C_2}\,\frac{s + (1/R_1C_1)}{s + (1/R_2C_2)} \triangleq H(s)$$

a. We let $s = j\omega$ in $H(s)$ to get

$$H(j\omega) = -\frac{C_1}{C_2}\,\frac{j\omega + (1/R_1C_1)}{j\omega + (1/R_2C_2)}$$

$$H(j\cdot 0) = -\frac{C_1}{C_2}\,\frac{1/R_1C_1}{1/R_2C_2}$$

$$= -\frac{\cancel{C_1}}{\cancel{C_2}}\,\frac{R_2\cancel{C_2}}{R_1\cancel{C_1}} = -\frac{R_2}{R_1}$$

b. $$\lim_{\omega\to\infty} H(j\omega) = \lim_{\omega\to\infty} -\frac{C_1}{C_2}\,\frac{\cancel{j\omega}\left(1+\frac{1}{j\omega R_1 C_1}\right)}{\cancel{j\omega}\left(1+\frac{1}{j\omega R_2 C_2}\right)}$$

$$= -\frac{C_1}{C_2}$$

c. As $\omega \to 0$, the capacitors behave as an open-circuit

↳ the circuit becomes a resistive inverting amplifier with a gain of $-\frac{R_2}{R_1}$

As $\omega \to \infty$, the capacitors behave as a short-circuit

↳ we obtain an indeterminate situation, i.e. $v_n \to v_i$ BUT $v_n = 0$ (ideal op amp)

Moreover;

- the gain of the ideal op amp is infinite

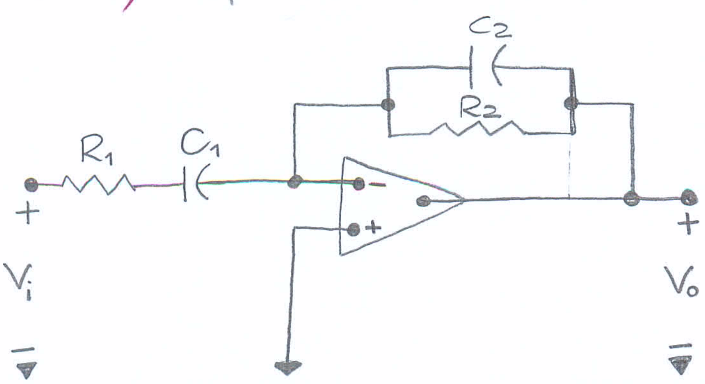
↳ $v_o \to \infty \cdot 0$ INDETERMINATE case

However;

- for finite values of ω, $H(j\omega) \to -\frac{C_2}{C_1}$

↳ the circuit approaches a purely capacitive inverting amplifier

Problem 3) Repeat Problem 2 using the circuit shown as



Solution. We have

$$Z_i = R_1 + \frac{1}{sC_1}, \qquad Z_f = \frac{R_2 \cdot (1/sC_2)}{R_2 + (1/sC_2)} = \frac{R_2}{sR_2C_2+1}$$

then

$$\frac{V_o}{V_i} \triangleq H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2/(sR_2C_2+1)}{(sR_1C_1+1)/sC_1}$$

$$= -\frac{sR_2C_1}{(sR_1C_1+1)(sR_2C_2+1)}$$

a. We let $s = j\omega$ to obtain

$$H(j\omega) = -\frac{j\omega R_2 C_1}{(j\omega R_1 C_1+1)(j\omega R_2 C_2+1)}$$

$$H(j0) = -\frac{0.R_2C_1}{(0.R_1C_1+1)(0.R_2C_2+1)} = 0$$

b.

$$\lim_{\omega\to\infty} H(j\omega) = \lim_{\omega\to\infty} -\frac{\cancel{j\omega R_2 C_2}}{j\omega R_1 C_1\left(1+\frac{1}{j\omega R_1C_1}\right)\cancel{j\omega R_2 C_2}\left(1+\frac{1}{j\omega R_2C_2}\right)}$$

$$= 0$$

c. As $\omega \to 0$, the capacitor C_1 becomes open-circuit and disconnects V_i and V_n

↳ thus, $v_o = v_n = 0$

As $\omega \to \infty$, the capacitor C_2 becomes short-circuit and connects output with the inverting input of the op amp ↳ thus, $v_o = v_n = 0$

PS 4.4

Problem 4) Consider the circuit in Problem 1.

a. Design a low-pass filter with a passband gain of 15 dB and a cutoff frequency of 10 kHz. Assume a 5nF capacitor is available.

b. Draw the circuit diagram and label all components.

Solution. We choose

$$Z_i \equiv R_1 \,, \quad Z_f : R_2 // \frac{1}{sC}$$

$$\Rightarrow \quad H(s) = -\frac{Z_f}{Z_i}$$

$$= -\frac{\dfrac{R_1 \cdot (1/sC)}{R_1 + \frac{1}{sC}}}{R_2}$$

$$= -\frac{R_1}{R_2} \frac{1}{sR_1C + 1}$$

$$= -\frac{R_1}{R_2} \frac{1/R_1C}{s + (1/R_1C)} \triangleq -K \frac{\omega_c}{s + \omega_c}$$

where

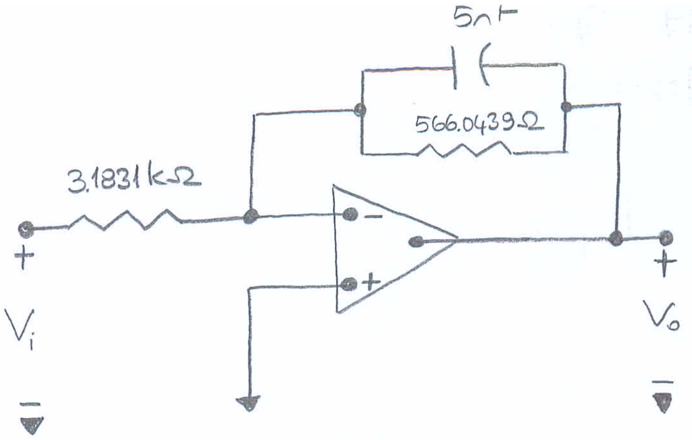
$$\omega_c = \frac{1}{R_1 C} \,, \quad K = \frac{R_1}{R_2}$$

$$2\pi \cdot 10 \cdot 10^3 = \frac{1}{R_1 \cdot 5 \cdot 10^{-9}} \Rightarrow R_1 = \frac{10000}{\pi} \cong 3.1831 \text{ k}\Omega$$

$$20 \log_{10} |H(j0)| = 15 \Rightarrow \overset{4}{\cancel{20}} \log_{10} K = \overset{3}{\cancel{15}}$$

$$\Rightarrow K = 10^{3/4} = 5.6234 \Rightarrow R_2 = \frac{3.1831 \cdot 10^3}{5.6234} = 566.0439 \,\Omega$$

b.



Problem 5) Design an op amp-based high pass filter with a cutoff frequency of 300 Hz and a passband gain of 5 using a 100nF capacitor.

Solution. We let Z_i, Z_f in the circuit of Problem 1 to be chosen as

$$Z_i = R_1 + \frac{1}{sC}, \qquad Z_f = R_2$$

then we get

$$H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2}{R_1 + \frac{1}{sC}}$$

$$= -\frac{sR_2C}{sR_1C+1}$$

$$= -\frac{R_2}{R_1}\,\frac{s}{s+(1/R_1C)}$$

$$\triangleq -K\frac{s}{s+\omega_c}$$

where

$$K = \frac{R_2}{R_1}, \qquad \omega_c = \frac{1}{R_1C}$$

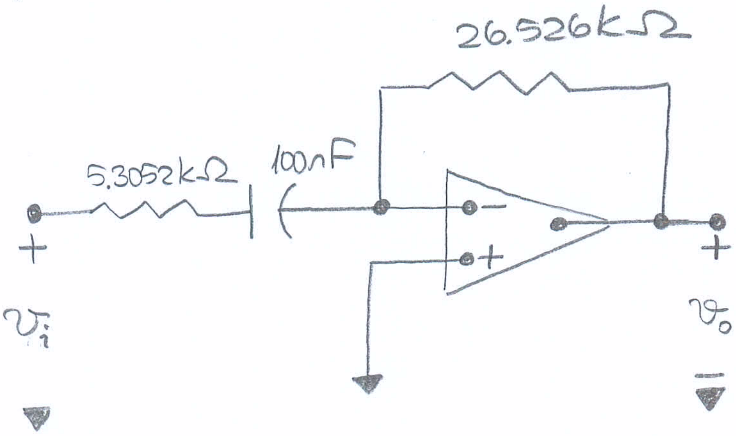
$$R_1 = \frac{1}{2\pi\cdot(300)\cdot 100\cdot 10^{-9}} = 5.3052\ \mathrm{k\Omega}$$

and

$$R_2 = 5 \cdot 5.3052 \cdot 10^3 = 26.526\, k\Omega$$

Hence ;

- we shall draw the op amp-based highpass filter circuit as



Problem 6) The input to the high pass filter designed in Problem 5 is 150 cosωt mV.

a. Suppose the power supplies are ±Vcc. What is the smallest value of Vcc that will cause the op amp to operate in its linear region?

b. Find the output voltage when $\omega = \omega_c$.

Solution. It follows from Problem 5 that we have

$$\omega_c = 2\pi \cdot 300 = 600\pi = 1885 \text{ rad/s}$$

$$K = 5$$

$$\Rightarrow \quad H(s) = -5 \frac{s}{s+1885} = \frac{V_o}{V_i} \quad , \quad V_i = 150 \angle 0^\circ \text{ mV}$$

$$\Rightarrow \quad H(j\omega) = -5 \frac{j\omega}{j\omega + 1885}$$

and

$$H_{max} = \lim_{\omega \to \infty} |H(j\omega)| = 5$$

Hence;

$$V_{cc} \geq H_{max} \cdot (V_i)_{max} = 5 \cdot 150 = 750 \text{ mV}$$

b.

$$H(j\omega_c) = -5 \frac{j \cdot 1885}{j1885 + 1885} = 5 \frac{1885 \angle -90^\circ}{1885\sqrt{2} \angle 45^\circ}$$

$$= \frac{5}{\sqrt{2}} \angle -135^\circ$$

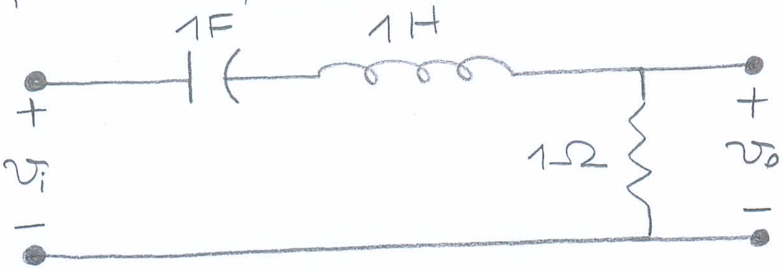
$$V_o = H(j1885) \cdot V_i$$

$$= \frac{5}{\sqrt{2}} \angle -135^\circ \cdot 0{,}15 \angle 0^\circ$$

$$= 0.5303 \angle -135^\circ$$

$$\Rightarrow \quad v_o(t) = 0{,}5303 \cos(\omega t - 135^\circ) \text{ V}$$

Problem 7) The voltage transfer function of the prototype bandpass filter shown as



$$\frac{V_o}{V_i} = H(s) = \frac{\left(\frac{1}{Q}\right) s}{s^2 + \left(\frac{1}{Q}\right)s + 1}$$

Show that if the circuit is scaled in both magnitude and frequency, the scaled transfer function is

$$H'(s) = \frac{\left(\frac{1}{Q}\right)\left(\frac{s}{k_f}\right)}{\left(\frac{s}{k_f}\right)^2 + \left(\frac{1}{Q}\right)\left(\frac{s}{k_f}\right) + 1}$$

Solution. Let k_m and k_f denote magnitude and frequency scaling factor, respectively

$$R' \to k_m R$$

$$L' \to \frac{k_m}{k_f} L$$

$$C' \to \frac{1}{k_m k_f} C$$

$$H'(s) = \frac{k_m R}{\dfrac{1}{s\dfrac{C}{k_m k_f}} + s\dfrac{k_m}{k_f}L + k_m R}$$

$$= \frac{\cancel{k_m} R}{\dfrac{\cancel{k_m} k_f}{sC} + s\dfrac{\cancel{k_m}}{k_f} L + \cancel{k_m} R}$$

$$= \frac{R}{\dfrac{k_f}{sC} + \dfrac{s}{k_f} L + R}$$

$$= \frac{R.(sC/k_f)}{1 + \left(\dfrac{s}{k_f}\right)^2 LC + \left(\dfrac{s}{k_f}\right) RC}$$

$$= \frac{\left(\dfrac{R}{L}\right)\left(\dfrac{s}{k_f}\right)}{\left(\dfrac{s}{k_f}\right)^2 + \dfrac{R}{L}\left(\dfrac{s}{k_f}\right) + \dfrac{1}{LC}} \overset{\Delta}{=} \frac{\beta\left(\dfrac{s}{k_f}\right)}{\left(\dfrac{s}{k_f}\right)^2 + \beta\left(\dfrac{s}{k_f}\right) + \omega_0^2}$$

Note that;

- for a prototype bandpass filter, we have

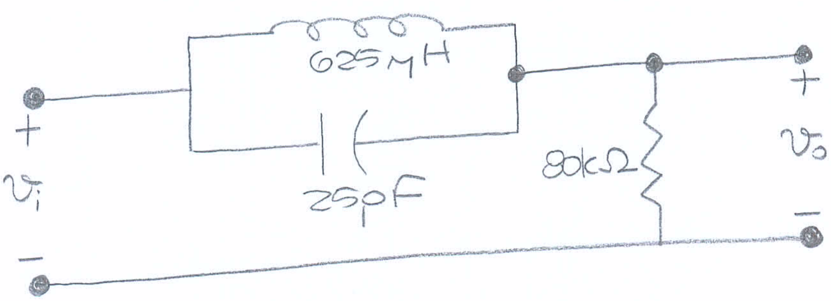
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \cdot 1}} = 1 \text{ rad/sec}$$

$$Q = \frac{\omega_0}{\beta} \Rightarrow \beta = \frac{1}{Q}$$

Hence;

$$H'(s) = \frac{\left(\frac{1}{Q}\right)\left(\frac{s}{k_f}\right)}{\left(\frac{s}{k_f}\right)^2 + \left(\frac{1}{Q}\right)\left(\frac{s}{k_f}\right) + 1}$$

Problem 8) Consider the following bandreject filter.



Scale this bandreject filter to get a center frequency of 500 krad/s using a 50μH inductor. Determine the values of the resistor, the capacitor and bandwidth of the scaled filter.

Solution. We first calculate the center frequency of the unscaled filter:

$$\omega_0 = \frac{1}{\sqrt{625 \cdot 10^{-6} \cdot 25 \cdot 10^{-12}}} = \frac{1}{25 \cdot 5 \cdot 10^{-9}} = \frac{1000}{125} \cdot 10^{6} = 8 \text{ Mrad/s}$$

$$\Rightarrow k_f = \frac{500 \cdot 10^3}{8 \cdot 10^6} = 62.5 \cdot 10^{-3}$$

and

$$50\mu H = \frac{k_m}{k_f} \cdot 625\mu H$$

$$\Rightarrow \quad 5\cancel{0} = \frac{k_m}{62\cancel{5}.10^{-3}} \cdot \frac{1\cancel{0}}{\cancel{625}} \Rightarrow \quad k_m = 5.10^{-3}$$

Hence;

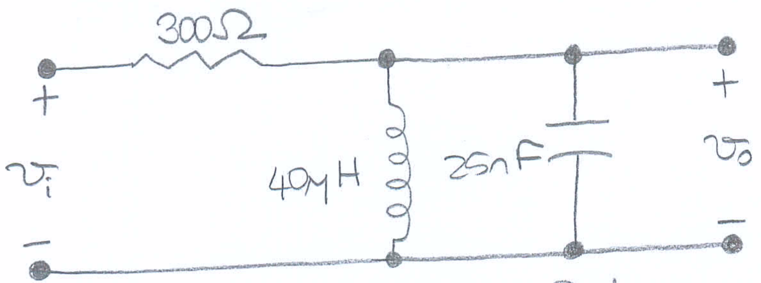
$$C' = \frac{C}{k_m k_f} = \frac{\overset{5}{\cancel{25}}.10^{-\cancel{12}\,6}}{\cancel{5}.\cancel{10^{-3}} \cdot 62.5.\cancel{10^{-3}}} = 8.10^{-8} = 80\,nF$$

and

$$R' = k_m R = 5.10^{-3} \cdot 80.10^{3} = 400\Omega$$

$$\beta' = \frac{1}{R'C'} = \frac{1}{400 \cdot 80.10^{-3}} = \frac{10^6}{32} = 31250 \text{ rad/sec}$$

Problem 9) Consider the following bandpass filter shown as



Scale this bandpass filter so that the center frequency is 250 kHz and the quality factor is 7.5 using a 10nF capacitor. Determine the value of the resistor, the inductor, and the two cutoff frequencies of the scaled filter.

Solution. We first calculate the center frequency of the unscaled filter:

$$\omega_0 = \frac{1}{\sqrt{40\cdot 10^{-6}\cdot 25\cdot 10^{-9}}} = \frac{1}{\sqrt{10^{-12}}} = 10^6 \text{ rad/sec}$$

$$\Rightarrow k_f = \frac{2\pi\cdot 250\cdot 10^{\cancel{3}}}{10^{\cancel{6}3}} = 0.5\pi = 1.5708$$

- for the magnitude scaling factor, we have

$$\cancel{10}^{2}\cdot\cancel{10^{-9}} = \frac{5\ \cancel{25\cdot 10^{-9}}}{1.5708\cdot k_m} \Rightarrow k_m = 1.5915$$

Hence ;

$$R' = 1.5915\cdot 300 = 477.4648\,\Omega$$

$$L' = \frac{1.5915}{1.5708}\cdot 40\ mH = 40.5271\ mH$$

$$\beta' = \frac{1}{R'\cdot C'} = \frac{1}{477.4648\cdot 10\cdot 10^{-9}} = 210k \text{ rad/sec}$$

$$\omega'_{c1} = -\frac{\beta'}{2} + \sqrt{\left(\frac{\beta'}{2}\right)^2 + (\omega'_0)^2}$$

$$= -\frac{210k}{2} + \sqrt{\left(\frac{210k}{2}\right)^2 + (2\pi\cdot 250k)^2}$$

$$= 1469.3 \text{ rad/sec}$$

$$\omega'_{c2} = \frac{\beta'}{2} + \sqrt{\left(\frac{\beta'}{2}\right)^2 + (\omega'_0)^2}$$

$$= \frac{210k}{2} + \sqrt{\left(\frac{210k}{2}\right)^2 + (2\pi\cdot 250k)^2}$$

$$= 1679.3 \text{ rad/sec}$$