CMPE 352
Signal Processing & Algorithms

Spring 2019

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Review Questions (1)

- What is the information provided by the <u>signal spectrum</u>?

  The spectrum of the signal $x(t)$ shows the set of frequencies contained in the signal. The spectrum extends over a certain *bandwidth*.

  The magnitude spectrum $|X(j\omega)|$ indicates the strength (amplitude) of the sinusoidal component at frequency ω, for all ω.
  The phase spectrum $\measuredangle X(j\omega)$ indicates the phase of the sinusoidal component at frequency ω, for all ω.

- What is the information provided by the <u>system frequency response</u>?

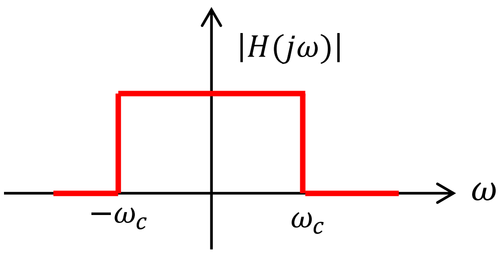
  The magnitude of the system frequency response $|H(j\omega)|$ indicates by how much the system will amplify/attenuate the sinusoidal component of the input signal at frequency ω, for all ω.

  The phase of the system frequency response $\measuredangle H(j\omega)$ indicates by how much the system will shift in phase the sinusoidal component of the input signal at frequency ω, for all ω.

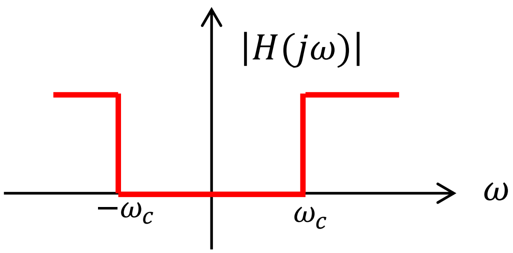
Review Questions (2)

- What are the three basic filter types?

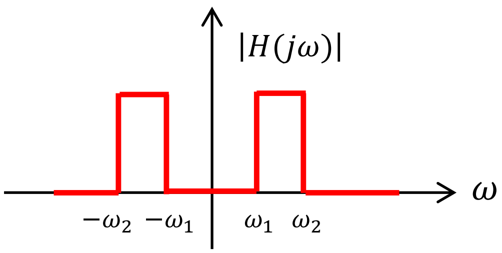
Low-pass



High-pass



Band-pass



- What is ω_c called?

The <u>cutoff</u> frequency

Review Questions (3)

- If the system frequency response is $H(j\omega)$ and the input signal spectrum is $X(j\omega)$, what is the output signal spectrum?

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

- If $H(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)}$ and $X(j\omega) = |X(j\omega)|e^{j\measuredangle X(j\omega)}$ what is the magnitude spectrum of the output signal $Y(j\omega)$?

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

- If $H(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)}$ and $X(j\omega) = |X(j\omega)|e^{j\measuredangle X(j\omega)}$ what is the phase spectrum of the output signal $Y(j\omega)$?

$$\measuredangle Y(j\omega) = \measuredangle H(j\omega) + \measuredangle X(j\omega)$$

Review Questions (4)

- What is the statement of the Sampling Theorem?

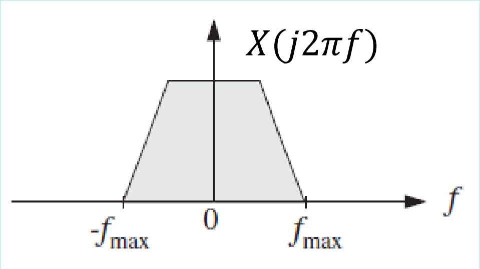
  The sampling theorem states that for accurate representation of a signal $x(t)$ by its time samples $x(nT_s)$, two conditions must be met:

1. The signal $x(t)$ must be bandlimited, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency f_{max} and no frequencies beyond that
2. The sampling rate f_s must be chosen to be at least twice the maximum frequency f_{max} contained in the signal, that is,

$$f_s \geq 2f_{max}$$

or, in terms of the sampling time interval:

$$T_s \leq \frac{1}{2f_{max}}$$



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Typical Sampling Frequencies

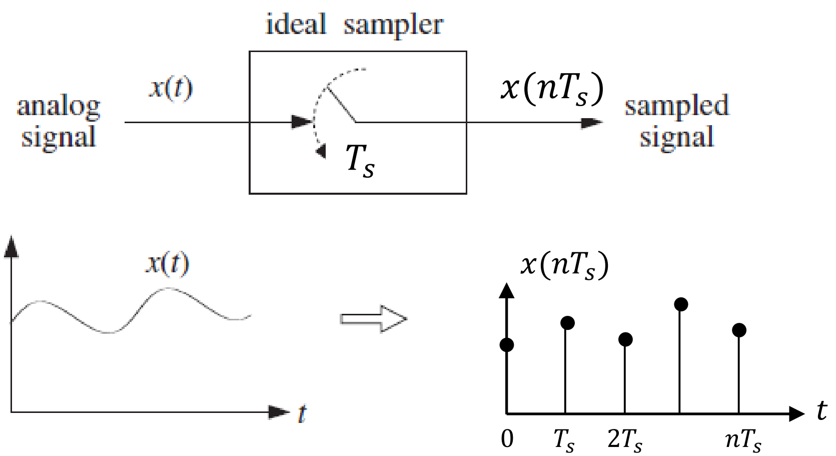
- The values of f_{max} and proper selection of f_s depend on the application
- Typical sampling frequencies for some common applications:

application	$f_{\max}$	f_s
geophysical	500 Hz	1 kHz
biomedical	1 kHz	2 kHz
mechanical	2 kHz	4 kHz
speech	4 kHz	8 kHz
audio	20 kHz	40 kHz
video	4 MHz	8 MHz

- Note: The frequency range $[-f_s/2, +f_s/2]$ plays an important role and is called the Nyquist interval

Sampling Process

- During the sampling process, the analog signal $x(t)$ is periodically measured every T_s (s). Thus, time is discretized in units of the sampling interval T_s: $nT_s,\ n = 0,1,2,3 \ldots.$



- For system design purposes, two questions must be answered:

1. How fast should one choose the sampling frequency $f_s = 1/T_s$, or: how small should one choose the sampling interval (sampling period) T_s ?

2. What is the effect of sampling on the frequency spectrum?

- Note: to answer these questions we will make use of what we learned from Fourier (Series or Transform): ”any” signal is a linear combination of sinusoidal signals

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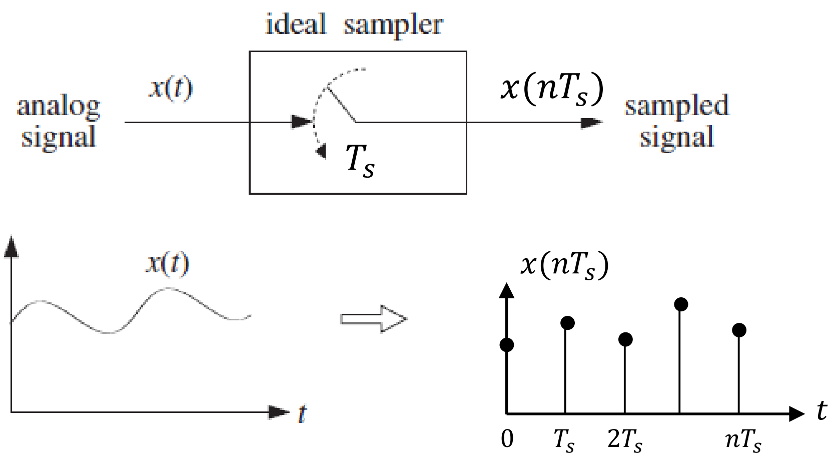

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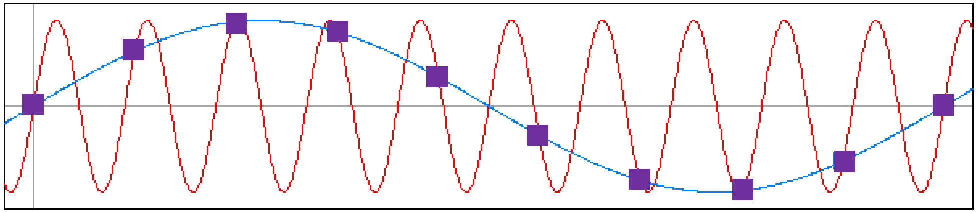
Sampling of Sinusoidal Signals (1)

- Consider the following case of two sinusoids of different frequencies (the red and blue sinusoids)



- What do you notice at the sampling points (purple square-markers)?

  The red and blue sinusoids have identical samples!

- What does this imply?

  In the digital domain, you cannot tell from which sinusoid the samples were taken! Hence seen from the digital domain, the frequencies of the red and blue sinusoids are "equivalent".

- **Question**: In general, for a given sampling frequency f_s, which sinusoids (that is, sinusoids of what frequencies) lead to identical samples?

Sampling of Sinusoidal Signals (2)

- Consider the sinusoid: $x(t) = \sin(2\pi f_1 t)$
  and its sampled version $x(nT_s) = \sin(2\pi f_1 nT_s)$ [obtained by setting $t = nT_s$].

- Define also the following family of sinusoids, for $m = \pm 1, \pm 2, \ldots$,

$$x_m(t) = \sin[2\pi(f_1 + mf_s)t]$$

  and their sampled versions

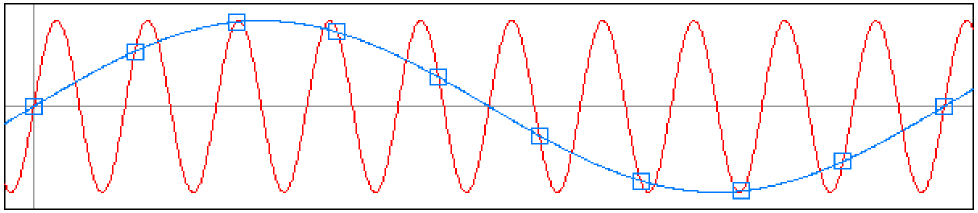
$$x_m(nT_s) = \sin[2\pi(f_1 + mf_s)nT_s].$$

- We have that

$$x_m(nT_s) = \sin[2\pi f_1 nT_s + 2\pi mnf_s T_s] = \sin[2\pi f_1 nT_s + 2\pi mn] = \sin(2\pi f_1 nT_s)$$

- We find that, although all signals $x_m(t)$ are of different frequencies, <u>their sampled values $x_m(nT_s)$ are the identical and equal to</u> $x(nT_s)$

<u>When sampled at rate f_s, the digital samples of sinusoids at frequencies f_1 and $f_1 \pm mf_s$ are all identical</u>



Spectrum Replication -- Aliasing

- Therefore, the set of frequencies,

$$\dots, f_1 - mf_s, \dots, f_1 - 2f_s\ , f_1 - f_s, f_1, f_1 + f_s, f_1 + 2f_s, \dots, f_1 + mf_s, \dots.$$

are "equivalent" to each other: the effect of sampling was to replace the original frequency f_1 with this <u>replicated set of frequencies</u>.

- This is called <u>spectrum replication</u>.

- We say that the frequencies $f_1 \pm mf_s$ ($m = 0{,}1{,}2$ ...) are <u>alias* frequencies.</u>

- <u>Alias frequency</u>: each of a set of signal frequencies which, when sampled at a given uniform rate, would give the same set of sampled values (and thus might be incorrectly substituted for one another when reconstructing the original signal!).

** used to indicate that a named person is also known or more familiar under another specified name (example: Eric Blair, alias George Orwell)*

Example

A sinusoidal signal has frequency 1 kHz and a second sinusoidal signal has frequency 3 kHz. Give the value of a sampling frequency that would cause the sampled versions of the two sinusoidal signals to be the alias of each other.

If the first signal is sampled at 2 kHz, the basic frequency and alias frequencies are (in kHz):

..., -7, -5, -3, -1, 1, 3, 5, 7, ..

If the second signal is sampled at 2 kHz, the basic frequency and alias frequencies are (in kHz):

..., -7, -5, -3, -1, 1, 3, 5, 7, ..

Hence if sampled at $f_s = 2$ kHz, the two digital signals are indistinguishable.

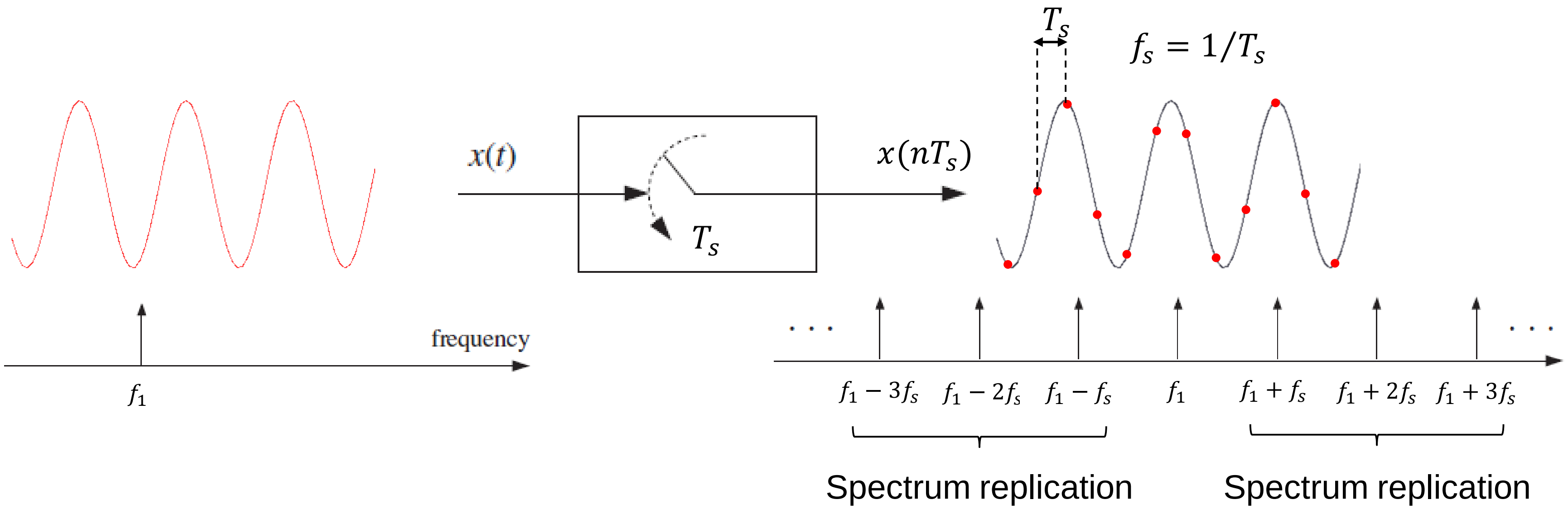
Example

Stroboscopic effect

https://www.youtube.com/watch?v=d23IGrmkph4

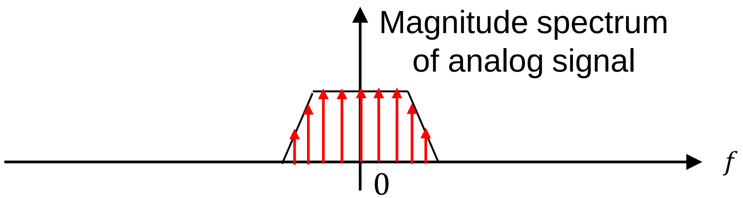
Spectrum Replication

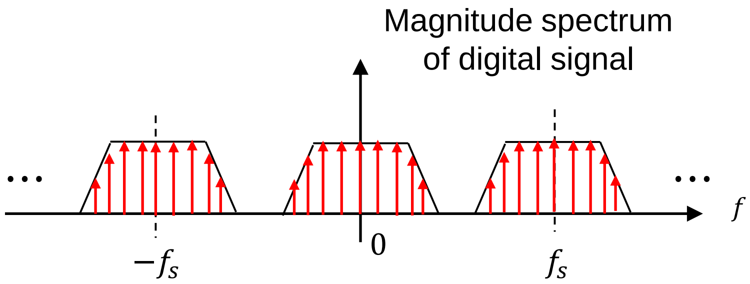
- Hence, in the case of a single sinusoid of frequency f_1 that is sampled at frequency f_s, the spectrum of the sampled signal can be represented as



- What happens for a general (non sinusoidal) signal ?

Spectrum Replication – General Case



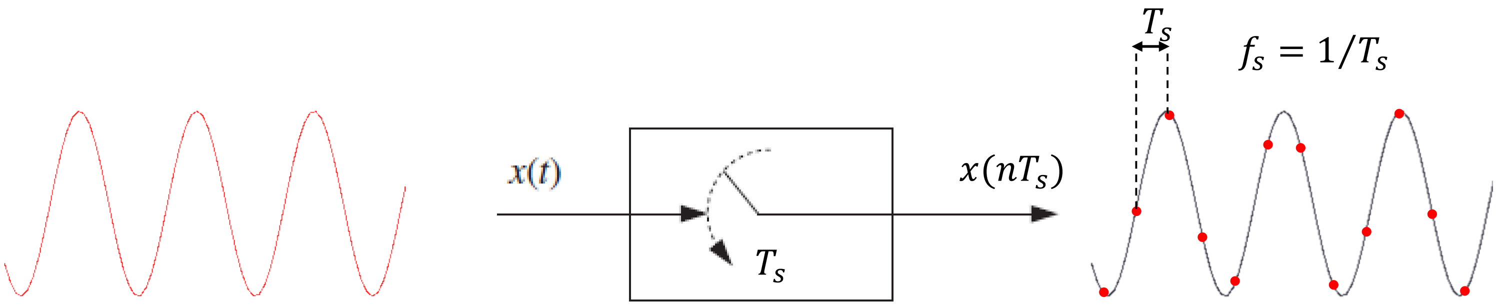


The discrete-time spectrum is obtained by repeating the continuous-time spectrum with a repetition period of f_s

(Note: also, a multiplying factor appears in the spectrum of the discrete-time signal)

Spectrum Replication

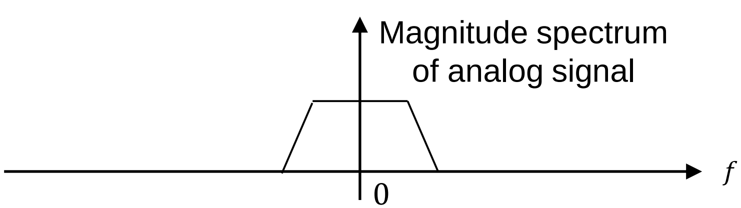
- Hence, in the case of a single sinusoid of frequency f_1 that is sampled at frequency f_s, the spectrum of the sampled signal can be represented as

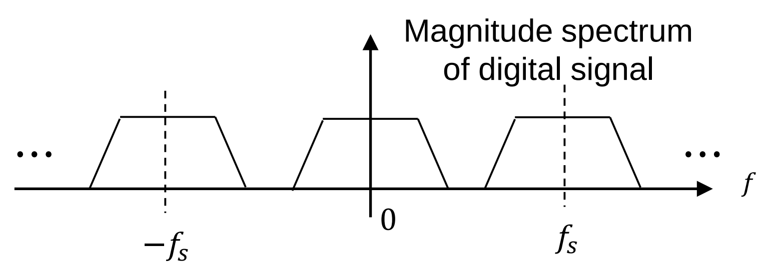


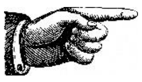
f_1

$\underbrace{f_1 - 3f_s \quad f_1 - 2f_s \quad f_1 - f_s}_{\text{Spectrum replication}} \quad f_1 \quad \underbrace{f_1 + f_s \quad f_1 + 2f_s \quad f_1 + 3f_s}_{\text{Spectrum replication}}$

- For a general signal that is sampled at frequency f_s, spectrum replication in the frequency domain occurs in a similar way

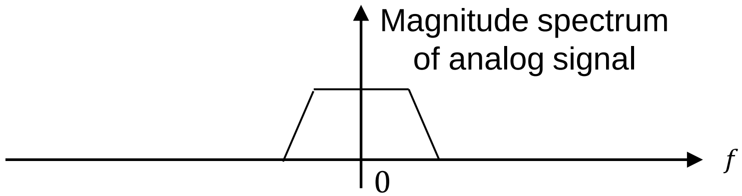


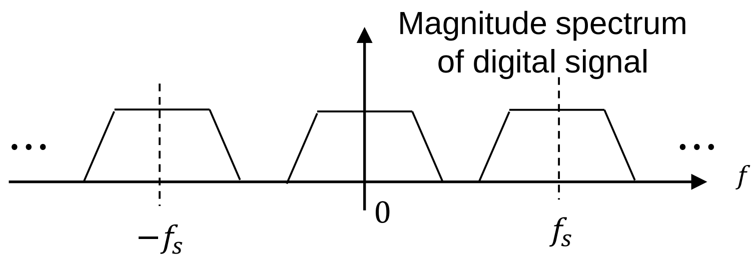




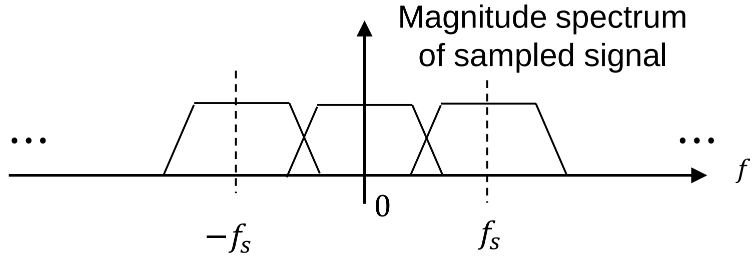
Sampling results in the periodic repetition of the signal spectrum

What is the effect of the choice for f_s?





What happens if we make f_s too small?

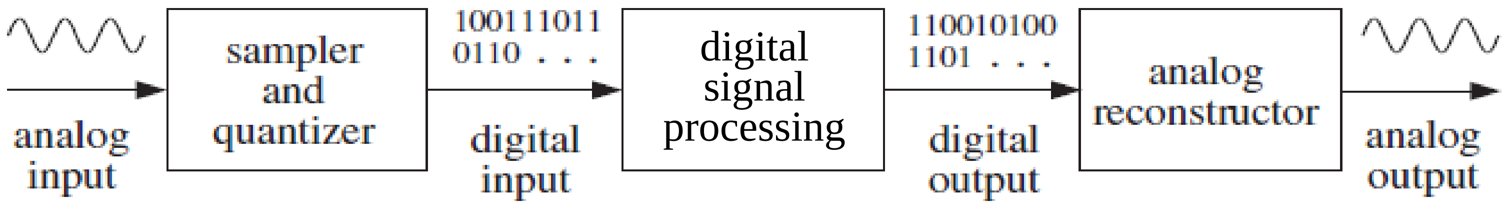


Hence if f_s is <u>too small</u> ($f_s < 2f_{max}$) then the <u>digital spectra overlap</u> – this is probably not a desirable result! Hence the sampling theorem must be respected:

$$f_s \geq 2f_{max}$$

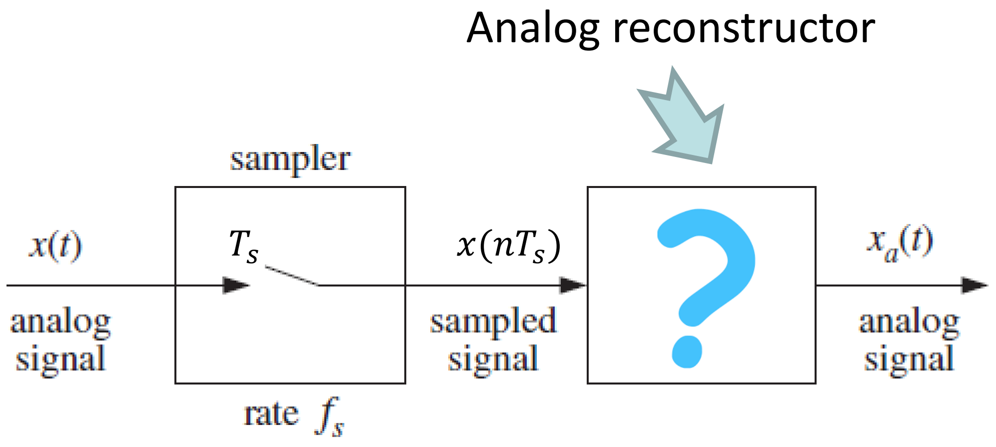
Analog Reconstruction (1)

- Recall our general diagram:

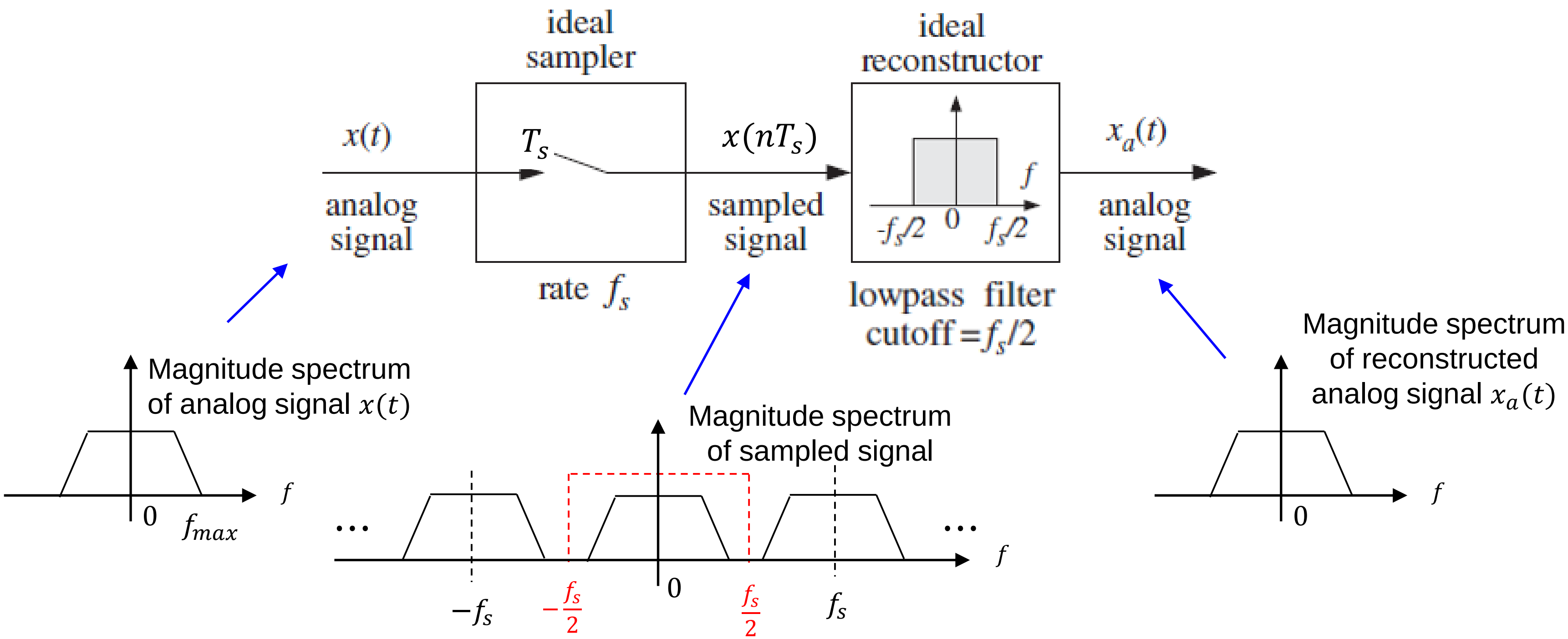


- Suppose we have sampled the signal $x(t)$ at sampling frequency f_s and then we want to reconvert it to analog form

- For this, we need a device called ″analog reconstructor″

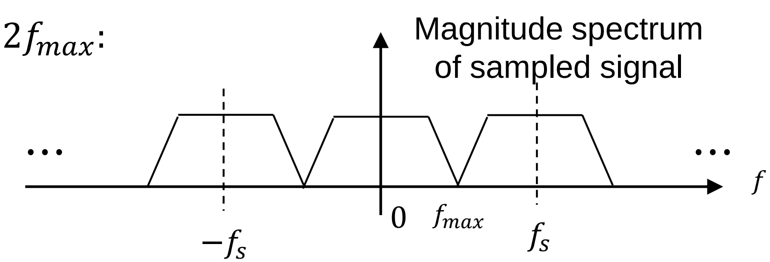


Analog Reconstruction (2)



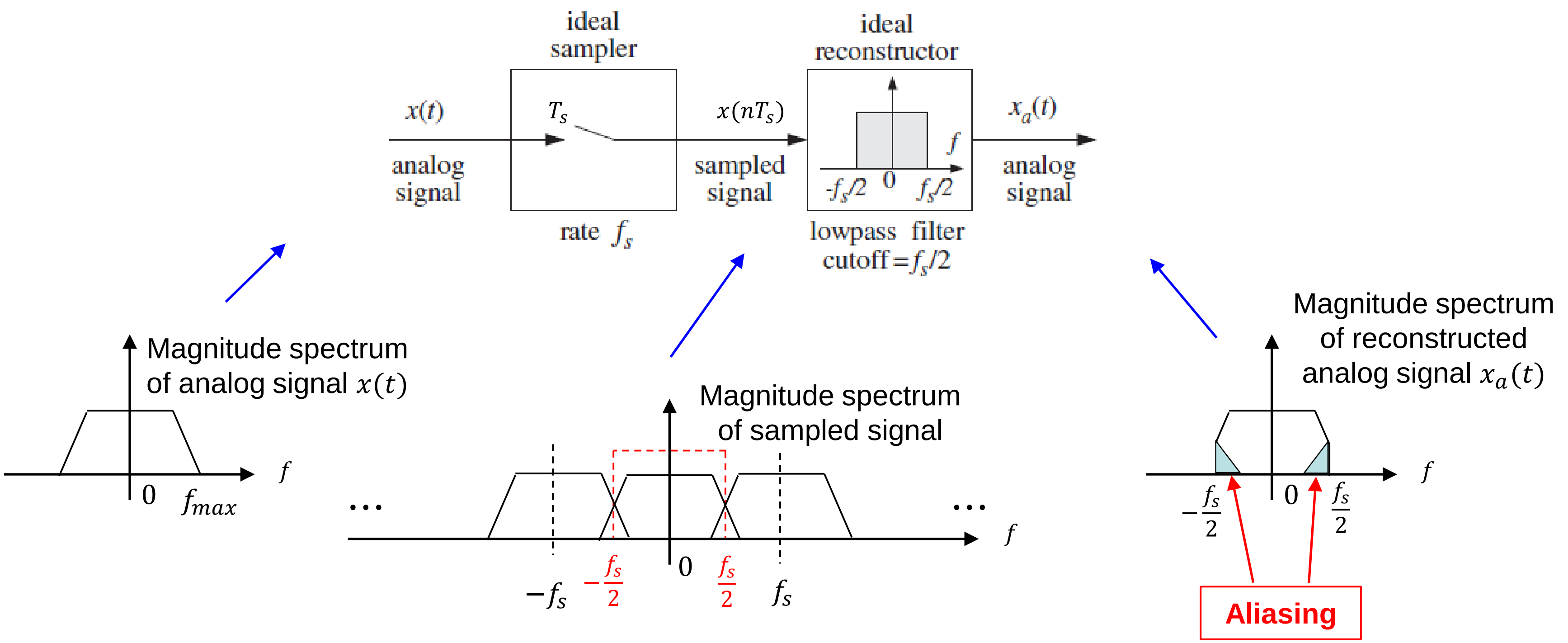
Note that in the above figure $f_s > 2f_{max}$.

Note also that if $f_s = 2f_{max}$:



Analog Reconstruction (3)

What happens if the sampling theorem is not respected: $f_s < 2f_{max}$?



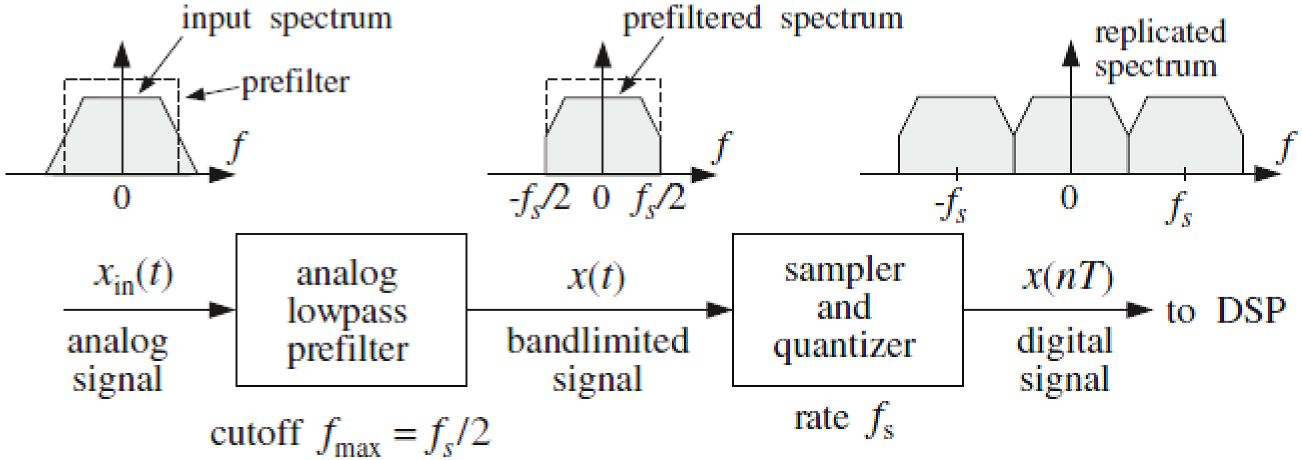
Hence in order to avoid aliasing, the sampling theorem must be respected:

$$f_s \geq 2f_{max}$$

Analog Reconstruction and Aliasing (1)

Therefore, the practical implications of the sampling theorem are quite important. Since most signals are not bandlimited, they must be made so by lowpass filtering before sampling.

In order to sample a signal at a desired rate f_s and satisfy the conditions of the sampling theorem, the signal must be prefiltered by a lowpass analog filter, known as an antialiasing prefilter. The cutoff frequency of the prefilter, f_{max}, must be taken to be at most equal to the Nyquist frequency $f_s/2$, that is, $f_{max} \leq f_s/2$.
The output of the analog prefilter will then be bandlimited to maximum frequency f_{max} and may be sampled properly at the desired rate f_s.

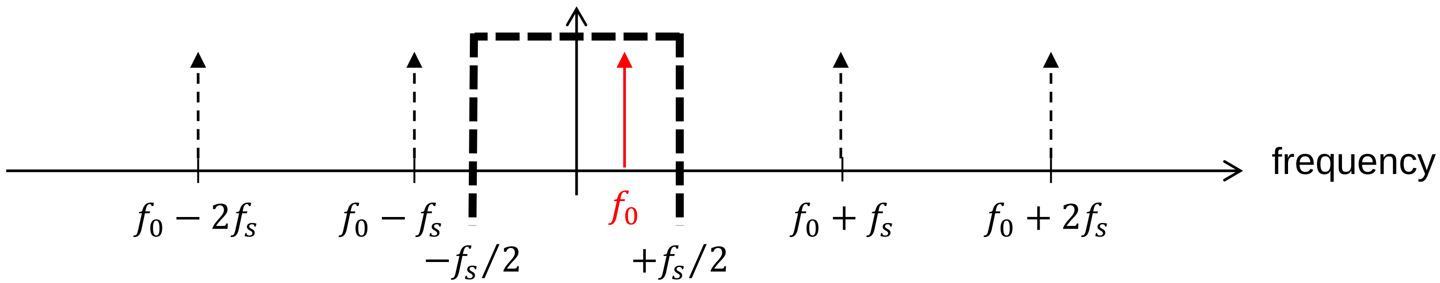


Analog Reconstruction and Aliasing (2)

In summary, potential aliasing effects that can arise at the reconstruction phase of DSP operations can be avoided if one makes sure that all frequency components of the signal to be sampled satisfy the sampling theorem condition, $|f| \leq f_s/2$ that is, all frequency components lie within the Nyquist interval. This is ensured by employing a lowpass antialiasing prefilter, which removes all frequencies beyond the Nyquist frequency $f_s/2$.

Example 1

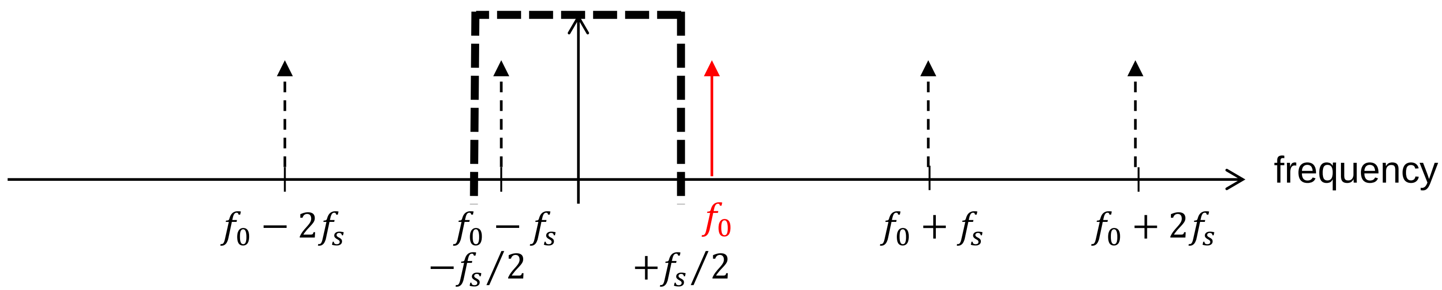
An analog sinusoidal signal with frequency f_0 is sampled at frequency f_s. The frequencies before and after sampling are shown below. When the digital signal is reconverted to analog form, will it have the same frequency as the original analog signal?



Yes, because $\frac{f_s}{2} > f_0$

Example 2

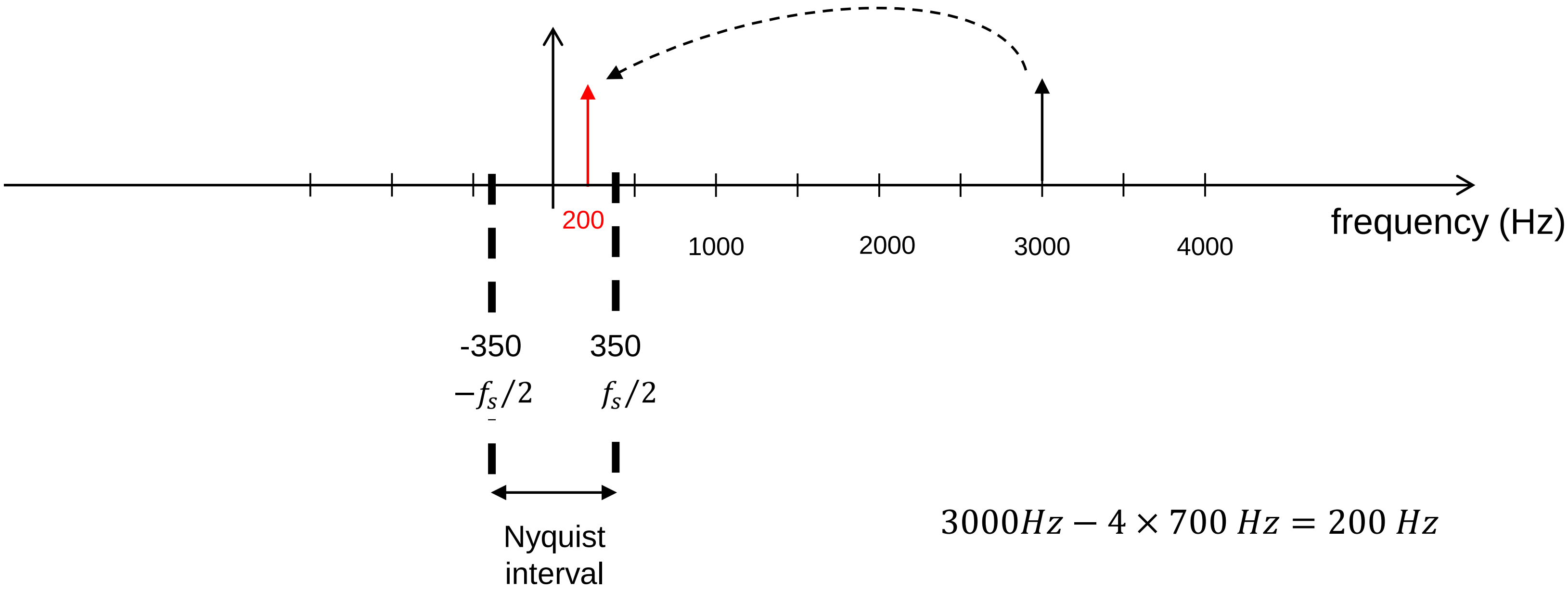
An analog sinusoidal signal with frequency f_0 is sampled at frequency f_s (see below). When the digital signal is reconverted to analog form, will it have the same frequency as the original analog signal?



No, because $\frac{f_s}{2} < f_0$

Example 3

What is the alias frequency (in the Nyquist interval) of frequency 3000 Hz if the sampling frequency is 700 Hz?



$$3000Hz - 4 \times 700\ Hz = 200\ Hz$$