STABILITY

(Nise's CSE Textbook Ch.6 and more)

Definition of Stability

- Stability is the most important system specification. If a system is unstable, transient response and steady-state errors are moot points.

- An unstable system cannot be designed for a specific transient response or steady-state error requirement.

- There are many definitions for stability, depending upon the kind of system or the point of view. In this section, we limit ourselves to linear, time-invariant systems.

Definition of Stability, *cnts*.

The total response of a system is the sum of the forced and natural responses. Using these concepts, we present the following definitions of stability, instability, and marginal stability:

- **A linear, time-invariant system is <u>stable</u> if the natural response approaches zero as time approaches infinity.**

- **A linear, time-invariant system is <u>unstable</u> if the natural response grows without bound as time approaches infinity.**

- **A linear, time-invariant system is <u>marginally stable</u> if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.**

Definition of Stability, *cnts*.

The alternate definition of stability, one that regards the total response and implies the first definition based upon the natural response, is this:

- **A system is stable if every bounded input yields a bounded output. We call this statement the <u>bounded-input, bounded-output (BIBO)</u> definition of stability.**

The alternate definition of instability, one that regards the total response, is this:

- **A system is unstable if any bounded input yields an unbounded output.**

The Stability of a Closed-Loop System

If the closed-loop system poles are in the left half of the plane and hence have a negative real part, the system is stable.

- **Stable systems have closed-loop transfer functions with poles only in the left half-plane.**

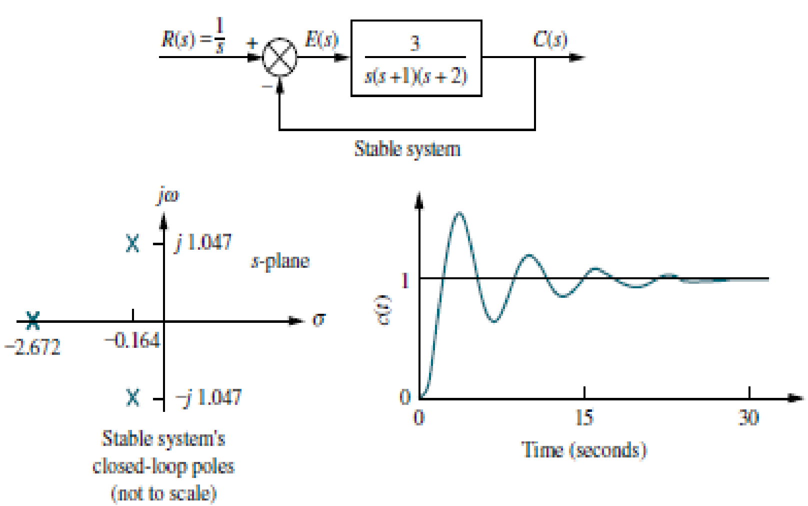
- **Unstable systems have closed-loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.**

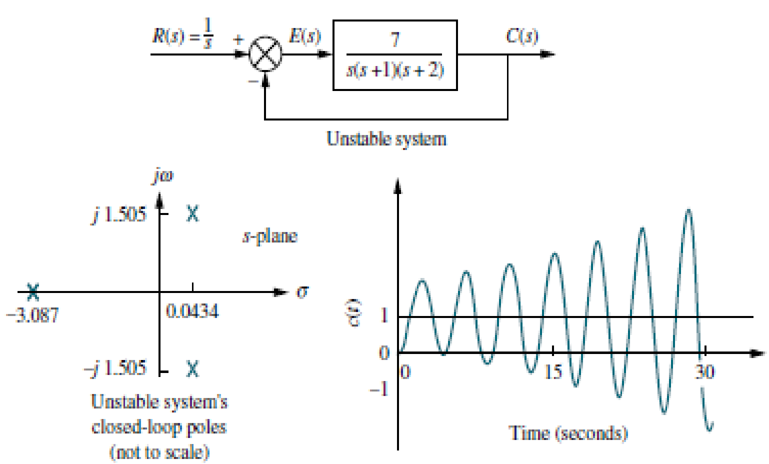
- **Marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.**

The Stability of a Closed-Loop System, *cnts*…

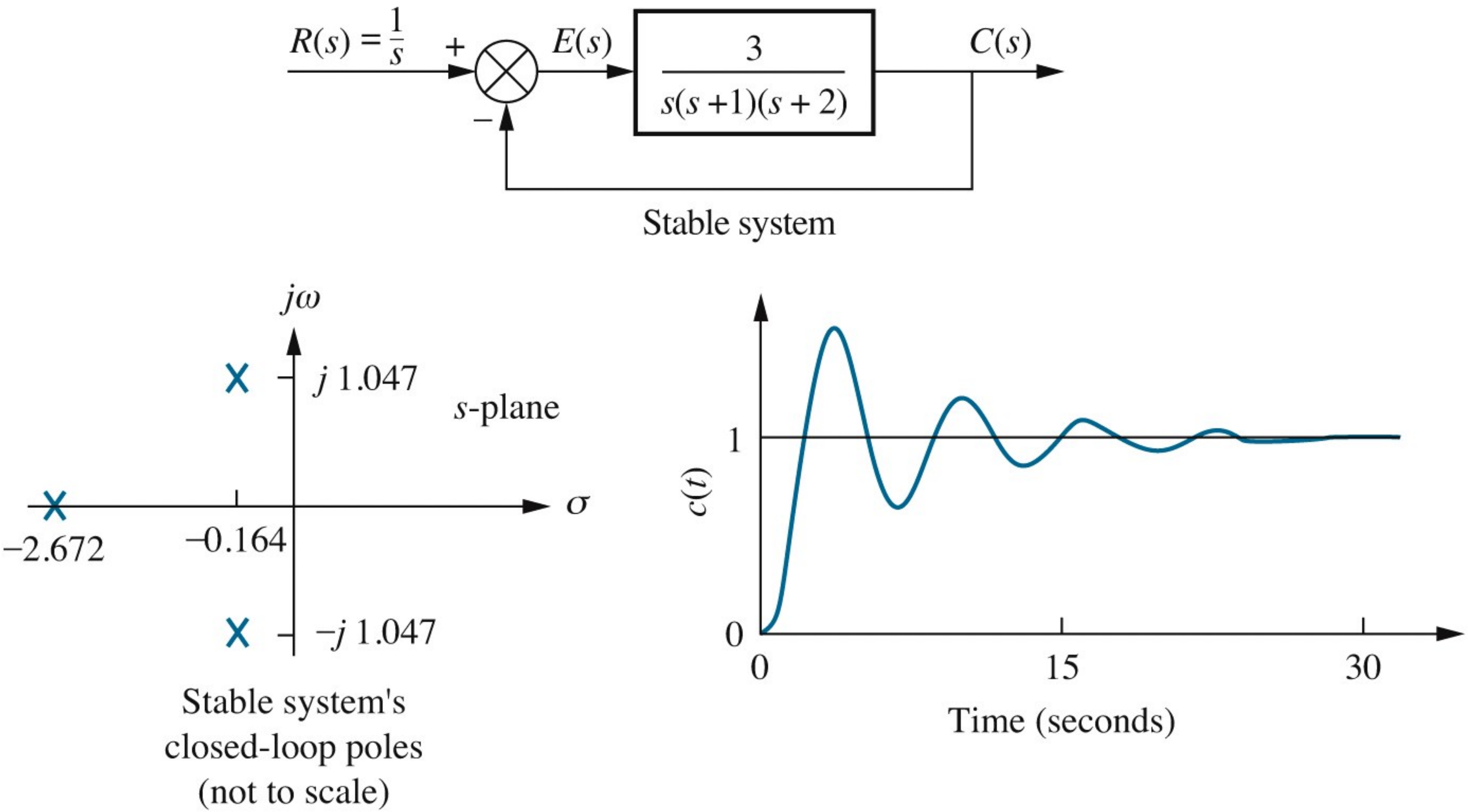
- It is not always a simple matter to determine if a feedback control system is stable. Unfortunately, a typical problem that arises is shown in the figures below.

- Although we know the poles of the forward transfer function, we do not know the location of the poles of the equivalent closed-loop system without factoring or otherwise solving for the roots.





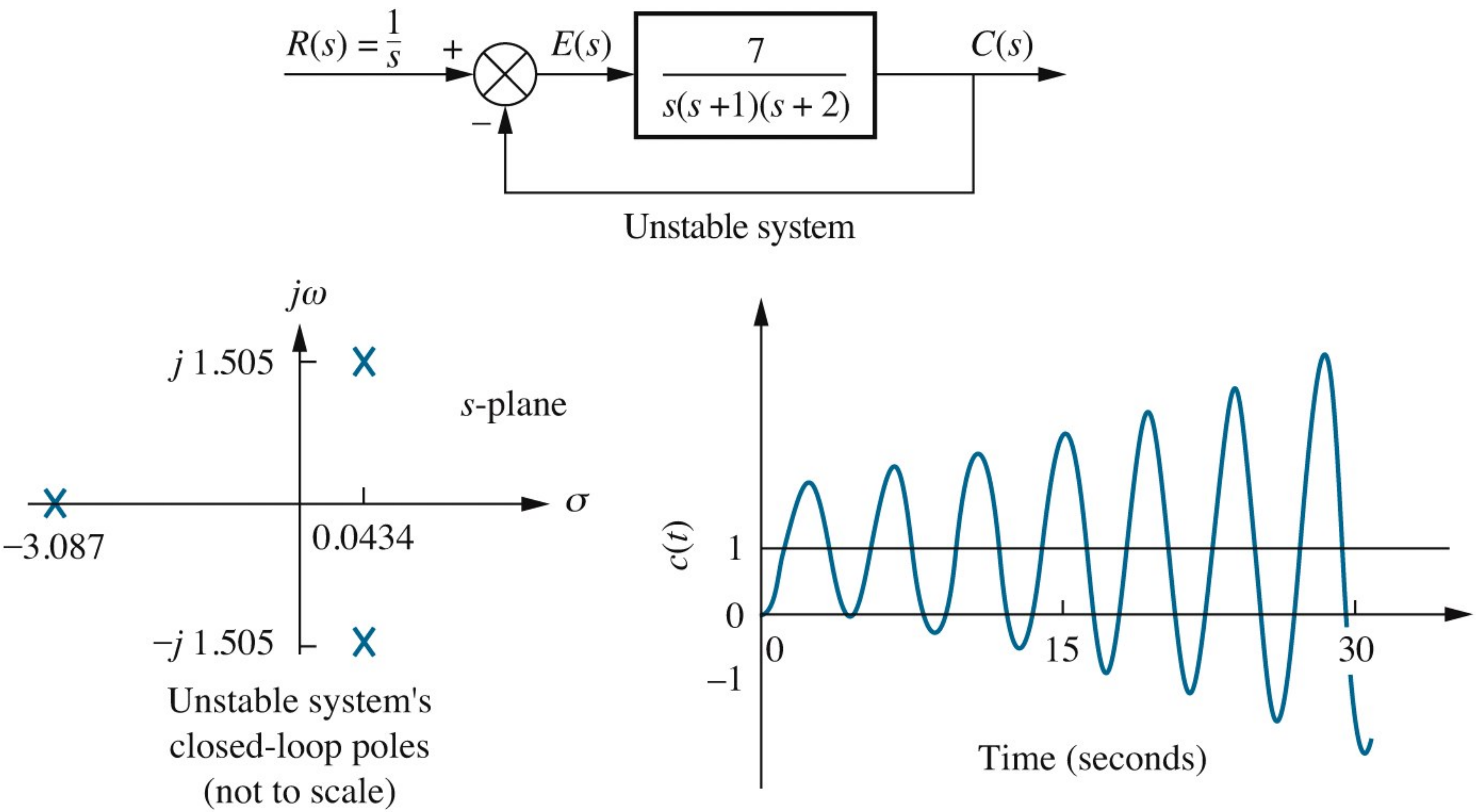
(b)



$R(s) = \frac{1}{s}$
+
−
$E(s)$
$\frac{3}{s(s+1)(s+2)}$
$C(s)$
Stable system
$j\omega$
$j\,1.047$
s-plane
σ
−2.672
−0.164
$-j\,1.047$
Stable system's closed-loop poles (not to scale)
$c(t)$
1
0
0
15
30
Time (seconds)

(*a*)

Figure 6.1a



$R(s) = \frac{1}{s}$
+
−
$E(s)$
$\frac{7}{s(s+1)(s+2)}$
$C(s)$
Unstable system
$j\omega$
$j\,1.505$
s-plane
σ
−3.087
0.0434
$-j\,1.505$
Unstable system's
closed-loop poles
(not to scale)
$c(t)$
1
0
−1
0
15
30
Time (seconds)

(*b*)

Figure 6.1b

Routh-Hurwitz Criterion

- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$-axis.

- Notice that we say how many, not where.

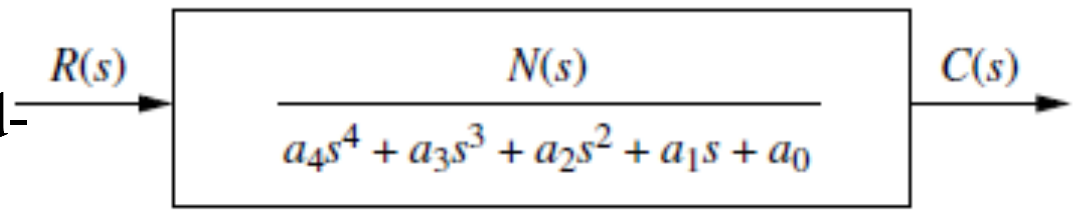
- We can find the number of poles in each section of the s-plane, but we cannot find their coordinates.

The method requires two steps:

- Generate a data table called a Routh table
- Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$-axis.

Generating Basic Routh Table

- Begin by labeling the rows with powers of s from the highest power of the denominator of the closed-loop transfer fn. to s^0.
- Start with the coefficient of the highest power of s in the denominator and list, horizontally in the first row, every other coefficient.
- In the second row, list horizontally, starting with the next highest power of s, every coefficient that was skipped in the first row.



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

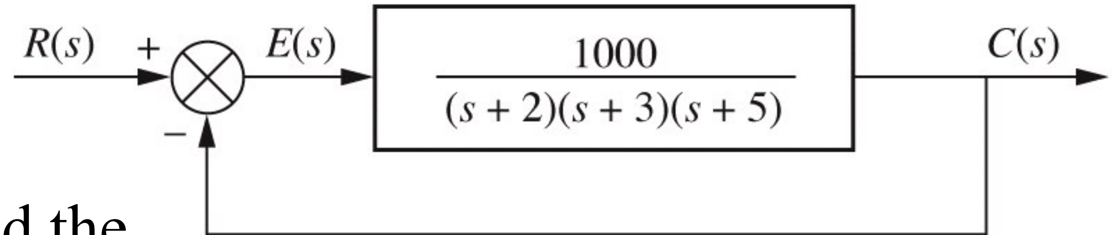
Generating Basic Routh Table

- The remaining entries are filled in as follows.
  - Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row.
  - The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right.
  - The table is complete when all of the rows are completed down to s^0.

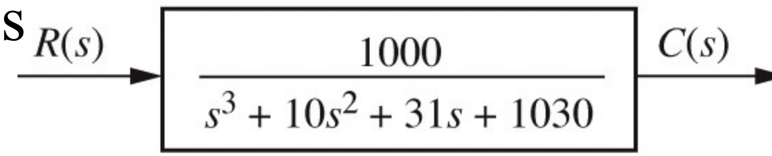
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example 6.1

Make the Routh table for the system given



Solution: We need to find the equivalent closed-loop system because we want to test the denominator of this function, not the given forward transfer function, for pole location.



The Routh-Hurwitz criterion will be applied to this denominator.

s^3	1	31	0
s^2	$\cancel{10}\ \ 1$	$\cancel{1030}\ \ 103$	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\ 1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\ -72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$

Example 6.1…

- Form the first row of the table, using the coefficients of the denominator of the closed-loop transfer function.
- Start with the coefficient of the highest power and skip every other power of s.
- Form the second row with the coefficients of the denominator skipped in the previous step.
- For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below.
- This can be proved by examining the expressions for the entries and verifying that any multiplicative constant from a previous row cancels out. In the second row, for example, the row was multiplied by 1/10.

Interpreting Basic Routh Table

- The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

- If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

Routh-Hurwitz Criterion: Special Cases:

Two special cases can occur:

- The Routh table sometimes will have a zero only in the first column of a row.

- The Routh table sometimes will have an entire row that consists of zeros.

Special Case: 1 Zero Only in the First Column

- If the first element of a row is zero, division by zero would be required to form the next row.

- To avoid this phenomenon, an epsilon, ε, is assigned to replace the zero in the first column.

- The value ε is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

<u>Example 6.2:</u>

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Example 6.2…

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

- Begin by assembling the Routh table down to the row where a zero appears only in the first column (the s^3 row).
- Next replace the zero by a small number, ε, and complete the table.
- To begin the interpretation, we must first assume a sign, positive or negative, for the quantity ε.

Example 6.2...

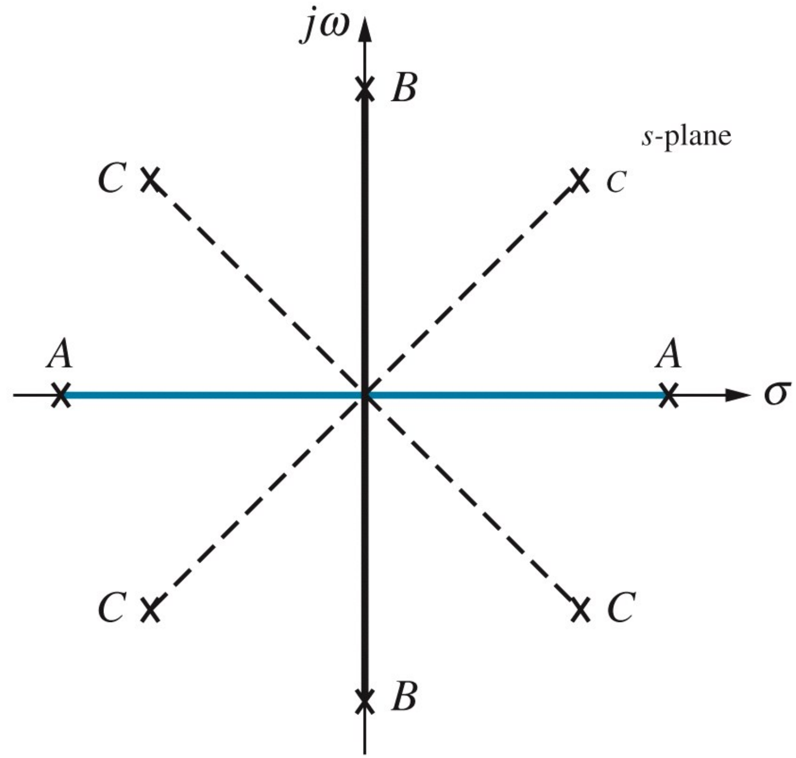
Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}\ \ \epsilon$	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

The table above shows the first column of the previous table along with the resulting signs for choices of ε positive and ε negative.

- If ε is chosen positive, the Routh table will show a sign change from the s^3 row to the s^2 row, and there will be another sign change from the s^2 row to the s^1 row.
- Hence, the system is unstable and has two poles in the right half-plane.

Special Case: 2 Entire Row is Zero

- Sometimes while making a Routh table, we find that an entire row consists of zeros because there is an *even polynomial* that is a factor of the original polynomial.
- Even polynomials only have roots that are symmetrical about the origin.
- This symmetry can occur
  1. The roots are symmetrical and real,
  2. the roots are symmetrical and imaginary, or
  3. the roots are quadrantal.



A: Real and symmetrical about the origin
B: Imaginary and symmetrical about the origin
C: Quadrantal and symmetrical about the origin

Figure 6.5

Examples for Special Case: 2

Example 6.4:

Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

s^5			1			6			8
s^4		$\cancel{7}$	1		$\cancel{42}$	6		$\cancel{56}$	8
s^3	$\cancel{0}$	$\cancel{4}$	1	$\cancel{0}$	$\cancel{12}$	3	$\cancel{0}$	$\cancel{0}$	0
s^2			3			8			0
s^1			$\frac{1}{3}$			0			0
s^0			8			0			0

Example 6.4 (*cnt*...)

- At the second row we multiply through by 1/7 for convenience.
- We stop at the third row, since the entire row consists of zeros, and use the following procedure.
  - We return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients.
  - The polynomial will start with the power of s in the label column and continue by skipping every other power of s.
  - The polynomial formed for this example is $P(s) = s^4 + 6s^2 + 8$
  - We differentiate the polynomial with respect to *s* and obtain

$$\frac{dP(s)}{ds} = 4s^3 + 12s$$

- Finally, we use the coefficients $\dot{P}$ to replace the row of zeros. For convenience, the third row is multiplied by 1/4 after replacing the zeros.
- The Routh table shows that all entries in the first column are positive. Hence, there are no right–half-plane poles.

Special Case: 2 - Example 6.5

Pole Distribution via Routh Table with Row of Zeros

PROBLEM: For the transfer function

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20} \tag{6.11}$$

tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis.

TABLE 6.8 Routh table for Example 6.5

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ \ 1$	$\cancel{20}\ \ 2$	0
s^5	$\cancel{20}\ \ 1$	$\cancel{60}\ \ 3$	$\cancel{40}\ \ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \ \cancel{4}\ \ 2$	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ \ 3$	$\cancel{2}\ \ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Special Case: 2 - Example 6.5 *cnt…*

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

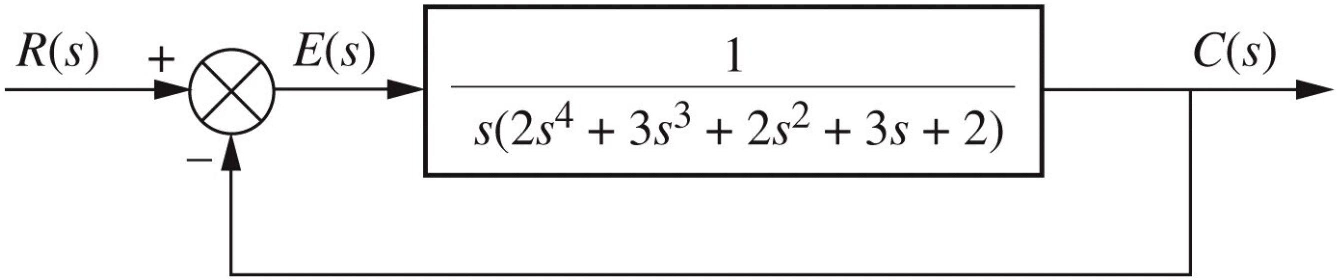
TABLE 6.9 Summary of pole locations for Example 6.5

	Polynomial		
Location	**Even (fourth-order)**	**Other (fourth-order)**	**Total (eighth-order)**
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

Table 6.9

Example 6.7

Find the number of poles in the left half-plane, the right half-plane, and on the imaginary-axis for the system of



Solution: The closed-loop transfer function is

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

Example 6.7…

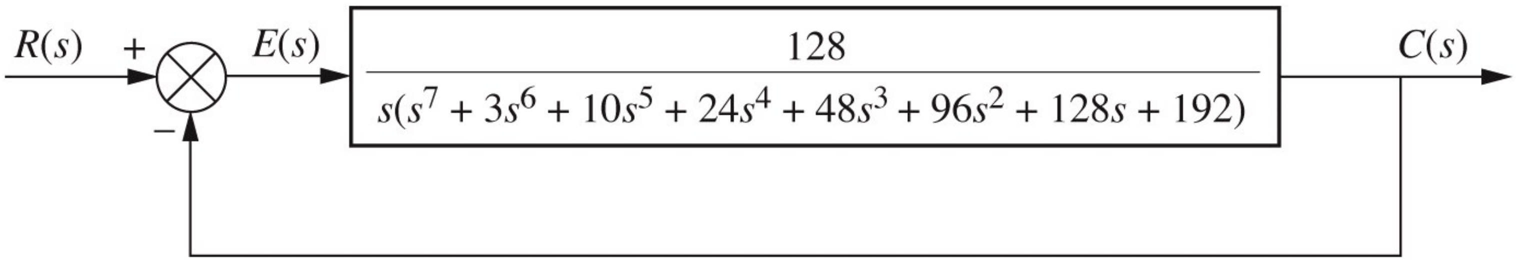
$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}\quad \epsilon$	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

- A zero appears in the first column of the s^3 row. Replace the zero with a small quantity, ε, and continue the table.
- Permitting ε to be a small, positive quantity, we find that the first term of the s^2 row is negative.
- There are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining poles are in the left half-plane.

Example 6.8

Find the number of poles in the left half-plane, the right half-plane, and on the imaginary-axis for the system



Solution: The closed-loop transfer function for the system is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

s^8	1	10	48	128	128
s^7	$\cancel{3}$ 1	$\cancel{24}$ 8	$\cancel{96}$ 32	$\cancel{192}$ 64	
s^6	$\cancel{2}$ 1	$\cancel{16}$ 8	$\cancel{64}$ 32	$\cancel{128}$ 64	
s^5	$\cancel{0}$ $\cancel{6}$ 3	$\cancel{0}$ $\cancel{32}$ 16	$\cancel{0}$ $\cancel{64}$ 32	$\cancel{0}$ $\cancel{0}$ 0	
s^4	$\cancel{\frac{8}{3}}$ 1	$\cancel{\frac{64}{3}}$ 8	$\cancel{64}$ 24		
s^3	$\cancel{-8}$ -1	$\cancel{-40}$ -5			
s^2	$\cancel{3}$ 1	$\cancel{24}$ 8			
s^1	3				
s^0	8				

Example 6.8…

- A row of zeros appears in the s^5 row. Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64$$

- Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros:

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s$$

- Replace the row of zeros at the s^5 row by the coefficients and multiply through by 1/2 for convenience.
- There are two sign changes from the even polynomial at the s^6 row down to the end of the table. Hence, the even polynomial has two right–half- plane poles.
- Because of the symmetry about the origin, the even polynomial must have an equal number of left–half-plane poles. Therefore, the even polynomial has two left–half-plane poles.
- Since the even polynomial is of sixth order, the two remaining poles must be on the jω-axis.
- There are no sign changes from the beginning of the table down to the even polynomial at the s^6 row, the rest of the polynomial has no right–half plane poles.

Example 6.9

Find the range of gain, K, for the system given that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.

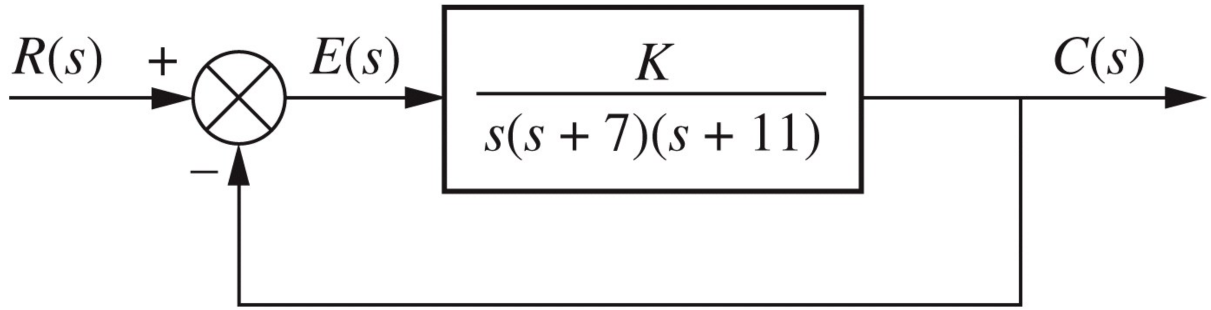


Figure 6.10

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

- **Solution:** First find the closed-loop transfer function
- Next form the Routh table shown as
- Since K is assumed positive, we see that all elements in the first column are always positive except the s^1 row.

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

- This entry can be positive, zero, or negative, depending upon the value of K.

Example 6.9…

- If $K < 1386$, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles in the left half-plane and be stable.

- If $K > 1386$, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right–half-plane poles and one left–half-plane pole, which makes the system unstable.

- If $K = 1386$, we have an entire row of zeros, which could signify jw poles. Returning to the s^2 row and replacing K with 1386, we form the even polynomial $P(s) = 18s^2 + 1386$

- Differentiating with respect to s, we have $\frac{dP(s)}{ds} = 36s$

Example 6.9…

- Replacing the row of zeros with the coefficients, we obtain the Routh-Hurwitz table shown as

s^3	1	77
s^2	18	1386
s^1	$\cancel{0}$ 36	
s^0	1386	

for the case of $K = 1386$.

- Since there are no sign changes from the even polynomial (s^2 row) down to the bottom of the table, the even polynomial has its two roots on the $j\omega$-axis of unit multiplicity.
- Since there are no sign changes above the even polynomial, the remaining root is in the left half-plane. Therefore the system is marginally stable.