EEEN 460

Optimal Control

2020 Spring

Lecture IV

State transition Matrix
State Controllability
Output Controllability
Observability

Solution of Nonhomogeneous State Equations

We shall begin by considering the scalar case

$$\dot{x} = ax + bu$$

Let us rewrite Equation as

$$\dot{x} - ax = bu$$

Multiplying both sides of this equation by e^{-at}, we obtain

$$e^{-at}[\dot{x}(t) - ax(t)] = \frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t)$$

Integrating this equation between 0 and t gives

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)\,d\tau$$

or

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)\,d\tau$$

The first term on the right-hand side is the response to the initial condition and the second term is the response to the input $u(t)$.

Let us now consider the nonhomogeneous state equation described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x} = n$-vector

$\mathbf{u} = r$-vector

$\mathbf{A} = n \times n$ constant matrix

$\mathbf{B} = n \times r$ constant matrix

By writing Equation (9–40) as

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and premultiplying both sides of this equation by $e^{-\mathbf{A}t}$, we obtain

$$e^{-\mathbf{A}t}\left[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right] = \frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating the preceding equation between 0 and t gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Equation can also be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Laplace Transform Approach to the Solution of Nonhomogeneous State Equations. The solution of the nonhomogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

can also be obtained by the Laplace transform approach. The Laplace transform of this last equation yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Premultiplying both sides of this last equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Using the relationship given above

$$\mathbf{X}(s) = \mathcal{L}[e^{\mathbf{A}t}]\mathbf{x}(0) + \mathcal{L}[e^{\mathbf{A}t}]\mathbf{B}\mathbf{U}(s)$$

The inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Solution in Terms of $\mathbf{x}(t_0)$. Thus far we have assumed the initial time to be zero. If, however, the initial time is given by t_0 instead of 0, then the solution to Equation (9–40) must be modified to

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

EXAMPLE

Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where $u(t)$ is the unit-step function occurring at $t = 0$, or

$$u(t) = 1(t)$$

For this system,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state-transition matrix $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ was obtained in Example as

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The response to the unit-step input is then obtained as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1]\, d\tau$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or $\mathbf{x}(0) = \mathbf{0}$, then $\mathbf{x}(t)$ can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Cayley–Hamilton Theorem

Consider an $n \times n$ matrix $\mathbf{A}$ and its characteristic equation:

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

The Cayley–Hamilton theorem states that the matrix $\mathbf{A}$ satisfies its own characteristic equation, or that

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n \mathbf{I} = \mathbf{0}$$

EXAMPLE

```
%Calculation of
%state transition matrix phi
%enter A
A=[1 0;0 1 ];
t=sym('t');
phi=expm(A*t)
```

```
phi =

[ exp(t),      0]
[      0, exp(t)]
```

EXAMPLE

```
%Calculation of
%state transition matrix phi
%enter A
A=[0 1;0 -2];
t=sym('t');
phi=expm(A*t)

phi =

[ 1, 1/2 - 1/(2*exp(2*t))]
[ 0,          1/exp(2*t)]
```

EXAMPLE

```
%Calculation of
%state transition matrix phi
%enter A
A=[0 1;-2 -3];
t=sym('t');
phi=expm(A*t)
```

```
phi =

[ 2/exp(t) - 1/exp(2*t), 1/exp(t) - 1/exp(2*t)]
[ 2/exp(2*t) - 2/exp(t), 2/exp(2*t) - 1/exp(t)]
```

CONTROLLABILITY

A system is said to be **controllable** at time **t0** if it is possible by means of an unconstrained control vector to transfer the system from any initial state **x(t0)** to any other state in a finite interval of time.
Uncontrollable System. An uncontrollable system has a subsystem that is physically disconnected from the input.

It is necessary to know the conditions under which a system is **controllable.** Consider the continuous-time system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where $\mathbf{x}$ = state vector (n-vector)
u = control signal (scalar)
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{B}$ = $n \times 1$ matrix

If the system is completely state controllable, then, this requires that the rank of the *nxn* matrix

$$\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$$

is of rank *n*.

EXAMPLE

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Since

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \right] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{singular}$$

the system is not completely state controllable.

OBSERVABILITY

In this section we discuss the observability of linear systems. Consider the unforced system described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{y}$ = output vector (m-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{C}$ = $m \times n$ matrix

The system is said to be completely observable if every state $\mathbf{x}(t_0)$ can be determined from the observation of ***y(t)*** over a finite time interval, The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector.

We can state the condition for complete observability as follows:
The system described by Equations aboveis completely observable if and only if the ***nxnm*** matrix is of rank ***n***.

$$\begin{bmatrix} \mathbf{C} \\ \hdashline \mathbf{CA} \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \mathbf{CA}^{n-1} \end{bmatrix}$$

$$\left[\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is this system observable?

To test the observability condition, examine the rank of $[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*]$. Since

$$[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

the rank of $[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^*]$ is 2. Hence, the system is completely observable.

MATLAB commands for the computation of controllability and observability matrices.

for the system defined by matrices **A**, **B**, **C**, and **D**.

```
CONT = ctrb(A,B)
OBSER = obsv(A,C)

rank(OBSER)
rank(CONT)
```

If rank (CONT) is less than n
the system is not controllable

If rank (OBSER) is less than n
the system is not observable.

EXAMPLE

Examine the controllability and observability of the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = [5 \quad 6 \quad 1]$$

```
>> A = [0   1   0;0   0   1;-6   -11   -6];
>> B = [0;0;1];
>> C = [5   6   1];
>> D = [0];
>> CONT = ctrb(A,B)

CONT =

     0     0     1
     0     1    -6
     1    -6    25

>> rank(CONT)

ans =

     3

>> OBSER = obsv(A,C)

OBSER =

     5     6     1
    -6    -6     0
     0    -6    -6

>> rank(OBSER)

ans =

     2
```

End of Lecture IV