Laplace Transform

- A powerful analytic technique

  ⮡ widely used to study the behavior of linear, lumped-parameter circuits.

- Allows a systematic way of

  ⮡ relating the time-domain behavior of a circuit to its frequency-domain behavior.

Definition of Laplace transform

- The Laplace transform of a function is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

$$\triangleq F(s)$$

Some general comments

- In linear circuit analysis, we excite circuits with sources

  ⮡ that have Laplace transforms.

- Frequently referred to as the one-sided, or unilateral Laplace transform

  ⮡ we do NOT use the two-sided or bilateral Laplace transform.

Two types of Laplace transform

- We consider
  1. Functional transforms

     ⮡ transform of a specific function. **e.g.** $sin\omega t, t, e^{-at}$

  2. Operational transforms

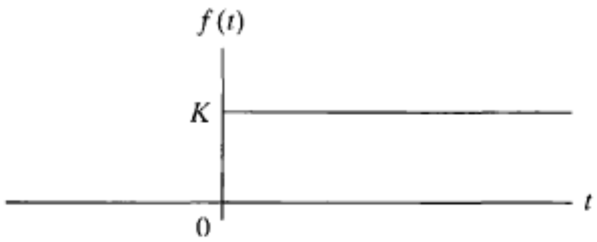
     ⮡ transforms involving mathematical operations of $f(t)$

     e.g.

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\}$$

Step function

- Illustrated as



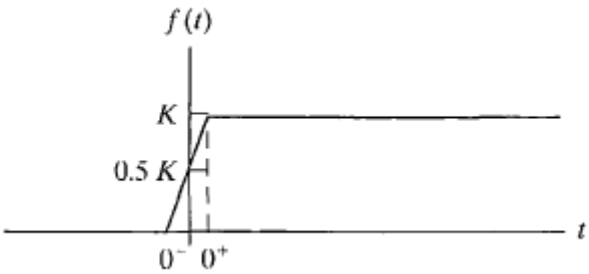
$$f(t) = \begin{cases} 0 & , \ t < 0 \\ K & , \ t > 0 \end{cases}$$

$$\triangleq Ku(t)$$

- If $K = 1$ then it is referred to as unit step, $u(t)$

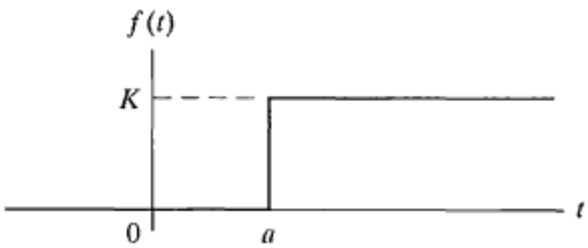
- NOT defined at $t = 0$, i.e. discontinuity at , i.e. discontinuity at $t = 0$

only a linear approximation is used.



- A step occurring at $t = a$ is expressed as

$$Ku(t-a) = \begin{cases} 0 & , \ t < a \\ K & , \ t > a \end{cases}$$



Impulse function

- If we have a finite discontinuity in a function then

the derivative of the function is NOT defined at the point of discontinuity.

- The concept of an impulse function enables

to define the derivative at a discontinuity.

- Generated by defining a function in terms of a variable parameter and then allowing this parameter approach zero.

- The variable-parameter function generates an impulse

  if it exhibits 3 characteristics as the parameter approaches zero.

1. The amplitude approaches infinity.

2. The duration of the function approaches zero.

3. The area under the variable-parameter function is constant as the parameter changes.

- An impulse of strength K is denoted as $K\delta(t)$

  where K is the area under the impulse-generating function.

- Also known as the "Dirac delta function".

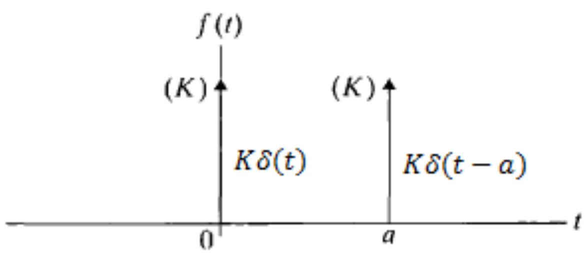
Mathematically;

- We define impulse function as

$$\int_{-\infty}^{+\infty} K\delta(t)dt = K$$

$$\delta(t) = 0, \; t \neq 0$$

- Graphically represented as



Sifting property

- Expressed as

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$$

where

$f(t)$: continuous at $t = 0$

Proof.

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = \int_{a-\mathcal{E}}^{a+\mathcal{E}} f(t)\delta(t-a)dt$$

$$= \int_{a-\mathcal{E}}^{a+\mathcal{E}} f(a)\delta(t-a)dt$$

$$= f(a)\underbrace{\int_{a-\mathcal{E}}^{a+\mathcal{E}} \delta(t-a)dt}_{=1}$$

$$= f(a)$$

Laplace transform of the impulse function

- We use the sifting property to find

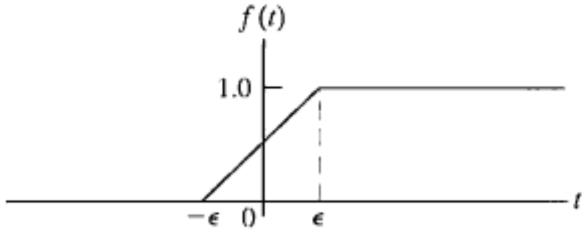
$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \overbrace{e^{-st}|_{t=0^-}}^{1} \overbrace{\int_{0^-}^{\infty} \delta(t)}^{1}$$

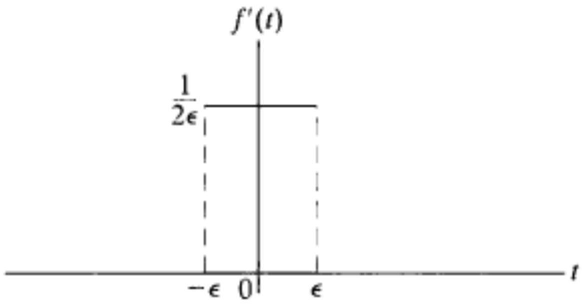
$$= 1$$

Impulse function and step function

- The relation is given by

$$\delta(t) = \frac{du(t)}{dt}$$





$f(t) \to u(t) \quad as \;\; \mathcal{E} \to 0$

$f'(t) \to \delta(t) \;\; as \;\; \mathcal{E} \to 0$

Functional transforms

- We consider the Laplace transform of a specified function.
- We illustrate Laplace transform of a unit step function.

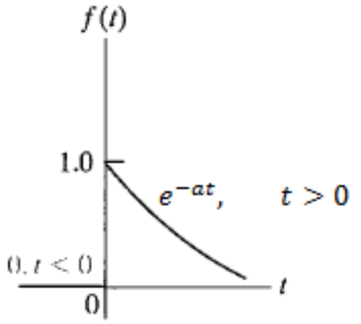
$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} u(t)e^{-st}\,dt = \int_{0^+}^{\infty} 1.e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = \lim_{t\to\infty}\frac{e^{-st}}{-s} - \left(\frac{1}{-s}\right)$$

$$= 0 + \frac{1}{s}$$

$$= \frac{1}{s}$$

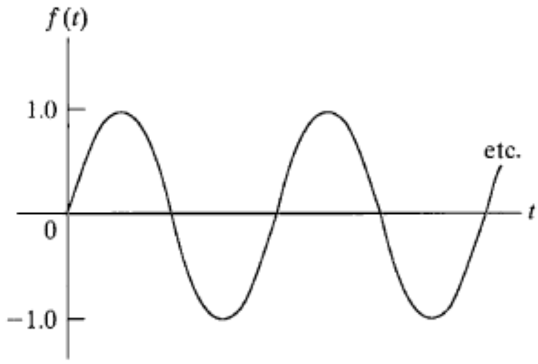
- Now we consider a decaying exponential function



$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}dt = \int_{0^+}^{\infty} e^{-(a+s)t}dt$$

$$= \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_{0^+}^{\infty} = \frac{1}{s+a}$$

- Finally we take into account the sinusoidal function



$$\mathcal{L}\{sin\omega t\} = \int_{0}^{\infty} \sin(\omega t)\, e^{-st} dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{j2}\right) e^{-st} dt$$

$$= \int_{0^-}^{\infty} \frac{e^{-(s-j\omega)t}}{j2} dt - \int_{0^-}^{\infty} \frac{e^{-(s+j\omega)t}}{j2} dt$$

$$= \frac{1}{j2}\left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)} + \frac{e^{-(s+j\omega)t}}{s+j\omega}\right]\Bigg|_{0^-}^{\infty}$$

$$= \frac{1}{j2}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) \quad , \quad ROC : Re\{s\} > 0$$

$$= \frac{j2\omega}{j2(s^2+\omega^2)} = \frac{\omega}{s^2+\omega^2}$$

Some Laplace transform pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Operational transforms

- Indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are

  ↳ translated into the opposite domain.

multiplication by a constant

- We have

$$\mathcal{L}\{Kf(t)\} = K\mathcal{L}\{f(t)\}$$

$$\triangleq KF(s)$$

where K is some constant.

addition(substraction)

- We have

$$\mathcal{L}\{f_1(t) \pm f_2(t)\} = \mathcal{L}\{f_1(t)\} \pm \mathcal{L}\{f_2(t)\}$$

$$\triangleq F_1(s) + F_2(s)$$

Note that ;

- These properties can be easily justified via direct substitution defining integral.

differentiation

- We have

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\mathcal{L}\{f(t)\} - f(0^-)$$

$$\triangleq sF(s) - f(0^-)$$

Proof.

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \underbrace{\left[\frac{df(t)}{dt}\right]}_{du} \underbrace{e^{-st}}_{u} dt$$

$$= f(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-s)e^{-st}dt$$

$$= \lim_{t\to\infty} f(t)e^{-st} - f(0^-) + s\underbrace{\int_{0^-}^{\infty} f(t)e^{-st}dt}_{F(s)}$$

$$= sF(s) - f(0^-)$$

$\therefore Q.E.D$

In general ;

- We also have

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \cdots$$
$$- \cdots - s\frac{d^{n-2}f(0^-)}{dt^{n-2}} - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

integration

- We have

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)d\tau\right\} = \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$\triangleq \frac{1}{s}F(s)$$

Proof.

$$\mathcal{L}\left\{\int_{0^-}^{t} f(\tau)d\tau\right\} = \int_{0^-}^{\infty} \underbrace{\left[\int_{0^-}^{t} f(\tau)d\tau\right]}_{u} \underbrace{e^{-st}dt}_{d\vartheta}$$

$$du = f(t)dt \quad , \quad \vartheta = -\frac{e^{-st}}{s}$$

$$= \left[-\frac{e^{-st}}{s}\int_{0^-}^{t} f(\tau)d\tau\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-\frac{e^{-st}}{s}\right)dt$$

$$= 0 - 0 + \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s) \qquad \therefore Q.E.D$$

Translation in the time-domain

- We have

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$

$$\triangleq e^{-as}\mathcal{L}\{f(t)\} \quad , \quad a > 0$$

Proof.

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} f(t-a)u(t-a)e^{-st}dt$$

$$= \int_{a}^{\infty} f(t-a)e^{-st}dt$$

$$= \int_{a}^{\infty} \underbrace{f(t-a)}_{\tau} e^{-\underbrace{s(t-a)}_{\tau}} e^{-as} \underbrace{dt}_{d\tau}$$

$$= e^{-as}\int_{0}^{\infty} f(\tau)e^{-s\tau}d\tau$$

$$= e^{-st}F(s) \qquad Q.E.D$$

Translation in the frequency domain

- We have

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Proof.

$$\mathcal{L}\{e^{-at}f(t)\} = \int_{0^-}^{\infty} e^{-at}f(t)e^{-st}dt$$

$$= \int_{0^-}^{\infty} f(t)e^{-(s+a)t}dt$$

$$= \int_{0^-}^{\infty} f(t) e^{-st} dt \Big|_{s = s + a}$$

$$= \mathcal{L}\{f(t)\}_{s = s + a}$$

$$= F(s + a) \qquad\qquad Q.E.D$$

Scale changing

- We have

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) \quad , \quad a > 0$$

Proof.

$$\mathcal{L}\{f(at)\} = \int_{0^-}^{\infty} f(at) e^{-st} dt$$

$$= \int_{0^-}^{\infty} f(at) e^{-\frac{s}{a}(at)} \frac{1}{a} d(at)$$

$$= \int_{0^-}^{\infty} f(\tau) e^{-\frac{s}{a}\tau} \frac{1}{a} d(\tau)$$

$$= \frac{1}{a} \int_{0^-}^{\infty} f(\tau) e^{-\frac{s}{a}\tau} d\tau$$

$$= \frac{1}{a} \mathcal{L}\{f(t)\} \Big|_{s = \frac{s}{a}}$$

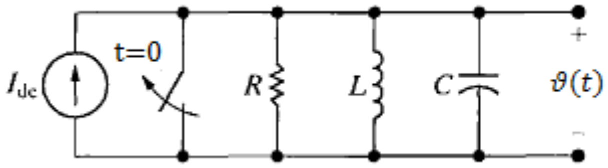
$$= \frac{1}{a} F\left(\frac{s}{a}\right) \qquad\qquad Q.E.D$$

Laplace transform and integro-differential equations

- We now illustrate how to use Laplace transform

  ↳ to solve integro-differential equations that describe the behavior of lumped-parameter circuits.

- Consider a parallel RLC circuit



- We assume that no initial energy is stored in the circuit at the instant

  ⮡ when the switch is opened.

- Then find $\vartheta(t)$ for $t \geq 0$.

Integro-differential equation

- A single node-voltage equation can be written as

$$\frac{\vartheta(t)}{R} + \frac{1}{L}\int_0^t \vartheta(\tau)d\tau + C\frac{d\vartheta(t)}{dt} = I_{dc}u(t)$$

- We now take the Laplace transform of both sides

$$\frac{V(s)}{R} + \frac{1}{L}\frac{1}{s}V(s) + CsV(s) = I_{dc}\left(\frac{1}{s}\right)$$

⮡ an algebraic equation in s where $V(s)$ is the unknown variable.

$$\Rightarrow \quad V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s}$$

$$\Rightarrow \quad V(s) = \frac{I_{dc}}{\frac{s}{R} + \frac{1}{L} + s^2C} = \frac{\frac{I_{dc}}{C}}{s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}}$$

- To find $\vartheta(t)$, we must find inverse transform of $V(s)$

i.e.

$$\vartheta(t) = \mathcal{L}^{-1}\{V(s)\}$$

Inverse transforms

- The s-domain expressions for the unknown voltages and currents for linear, lumped-parameter, time-invariant circuits are

  always rational functions of s , **i.e.** that can be expressed in the form of a ratio of two polynomials in s.

- We thus consider

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}$$

where a_i, b_j are real constants, $i = 1, \ldots, n$; $j = 1, \ldots, m$ and m, n are positive integers

$$F(s) \;:\; \begin{cases} proper \quad , \;\; m > n \\ improper \, , \;\; m \le n \end{cases}$$

- Only a proper rational function can be expanded as a sum of partial fractions.

Partial fraction expansion : proper rational functions

- We consider 3 different cases ; $D(s)$ has

  a. real and distinct roots

  b. distinct and complex roots

  c. real and repeated OR complex and repeated roots

a. real and distinct roots

$$F(s) = \frac{N(s)}{\prod_{i=1}^{n}(s + p_i)} = \frac{C_1}{s + p_1} + \cdots + \frac{C_n}{s + p_n}$$

where p_i's are real numbers.

- Then the partial fraction can be written as

$$F(s) = \frac{N(s)}{\prod_{i=1}^{n}(s + p_i)} = \frac{C_1}{s + p_1} + \cdots + \frac{C_n}{s + p_n}$$

- and C_j's are calculated as

$$C_j = (s+p_j)\frac{N(s)}{\prod_{i=1}^{n}(s+p_i)}\Bigg|_{s=-p_j}$$

e.g. Let

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{C_1}{s} + \frac{C_2}{s+8} + \frac{C_3}{s+6}$$

$$\Rightarrow C_1 = sF(s)\Big|_{s=0} = \frac{96(s+5)(s+12)}{(s+8)(s+6)}\Bigg|_{s=0} = \frac{96.5.12}{8.6} = 120$$

$$\Rightarrow C_2 = (s+8)F(s)\Big|_{s=-8} = \frac{96(s+5)(s+12)}{s(s+6)}\Bigg|_{s=-8} = \frac{96.(-3).(4)}{(-8).(-2)} = -72$$

$$\Rightarrow C_3 = (s+6)F(s)\Big|_{s=-6} = \frac{96(s+5)(s+12)}{s(s+8)}\Bigg|_{s=-6} = \frac{96.(-1).6}{(-6).2} = 48$$

Hence ;

$$F(s) = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}\right\} \triangleq f(t)$$

$$\Rightarrow f(t) = 120 + 48e^{-6t} - 72e^{-8t} \quad , \quad t > 0$$

$$= (120 + 48e^{-6t} - 72e^{-8t})u(t)$$

b. distinct and complex roots

- The procedure for finding coefficients associated with distinct roots

  ↳ same as that given for the case of distinct real roots.

e.g. Let

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$s^2+6s+25 = (s+3-j4)(s+3+j4)$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{C_1}{s+6} + \frac{C_2}{s+3-j4} + \frac{C_2^*}{s+3+j4}$$

- Notice that coefficients associated with complex conjugate roots are ALSO conjugates.

$$C_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100\,(-3)}{36-36+25} = -12$$

$$C_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100\,(j4)}{(3+j4)(j8)} = \frac{50}{5\angle 53.13^\circ} = 10\angle 53.13^\circ$$

Hence ;

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4} + \frac{10\angle 53.13^\circ}{s+3+j4}$$

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} \triangleq f(t)$$

$$= -12e^{-6t} + 10e^{-j53.13}e^{-(3-j4)t} + 10e^{j53.13}e^{-(3+4j)t} \quad , \quad t>0$$

$$= -12e^{-6t} + 20e^{-3t}\cos(4t-53.13^\circ) \quad , \quad t>0$$

$$= [-12e^{-6t} + 20e^{-3t}\cos(4t-53.13^\circ)]u(t)$$

c. real and repeated OR complex and repeated roots

- To find coefficients associated with the terms generated by a multiple root of multiplicity r

  we multiply both sides by the multiple root raised to its r^{th} power.

- And to find the remaining $(r-1)$ coefficients associated with the multiple root

  we differentiate both sides $(r-1)$ times, at the end of each differentiation, we evaluate both sides at the multiple root.

e.g.

$$\frac{180(s+30)}{s(s+5)(s+3)^2} = \frac{C_1}{s} + \frac{C_2}{s+5} + \frac{C_3}{(s+3)^2} + \frac{C_4}{s+3}$$

$$C_1 = \left.\frac{180(s+30)}{(s+5)(s+3)^2}\right|_{s\,=\,0} = \frac{180.30}{5.3^2} = 120$$

$$C_2 = \left.\frac{180(s+30)}{s(s+3)^2}\right|_{s\,=\,-5} = \frac{180.25}{(-5).2^2} = -225$$

$$C_3 = \left.\frac{180(s+30)}{s(s+5)}\right|_{s\,=\,-3} = \frac{180.27}{(-3).2} = -810$$

$$C_4 = \frac{d}{ds}\left[\frac{180(s+30)}{s(s+5)}\right]_{s\,=\,-3} = 180\left.\frac{s(s+5)-(s+30)(2s+5)}{s^2(s+5)^2}\right|_{s\,=\,-3}$$

$$= 180\frac{(-3).2-27.(-1)}{9.4} = 180\frac{21}{36} = 105$$

$$\frac{180}{s(s+5)(s+3)^2} = \frac{120}{s} - \frac{225}{s+5} - \frac{810}{(s+3)^2} + \frac{105}{s+3}$$

Therefore ;

$$\mathcal{L}^{-1}\left\{\frac{180(s+30)}{s(s+5)(s+3)^2}\right\} = (120 - 225e^{-5t} - 810te^{-3t} + 105e^{-3t})u(t)$$

e.g.

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$= \frac{C_1}{(s+3-j4)^2} + \frac{C_2}{s+3-j4} + \frac{C_1^*}{(s+3+j4)^2} + \frac{C_2^*}{s+3+j4}$$

$$C_1 = \frac{768}{(s+3+j4)^2}\Bigg|_{s=-3+j4} = \frac{768}{(j8)^2} = \frac{768}{-64} = -12$$

$$C_2 = \frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right]\Bigg|_{s=-3+j4} = 768\frac{-2}{(s+3+j4)^3}\Bigg|_{s=-3+j4}$$

$$= 768\frac{-2}{(j8)^3} = \frac{768.(-2)}{-j64.8} = -j3 = 3\angle{-90^{\circ}}$$

$$F(s) = \frac{-12}{(s+3-j4)^2} + \frac{3\angle{-90^{\circ}}}{s+3-j4} + \frac{-12}{(s+3+j4)^2} + \frac{3\angle 90^{\circ}}{s+3+j4}$$

Hence ;

$$\mathcal{L}^{-1}\{F(s)\} \triangleq f(t)$$

$$= -12te^{-(3-j4)t} - 12te^{-(3+j4)t} + 3e^{-j90}e^{-(3-j4)t} + 3e^{j90}e^{-(3+j4)t}$$

$$= [-24te^{-3t}cos4t + 6e^{-3t}\cos(4t - 90°)]u(t)$$

Initial-and final-value theorems

- They are useful because they enable us

  to determine from $F(s)$ the behavior of $f(t)$ at 0 and ∞.

- The initial-value theorem states that

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

proof.

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty}\frac{df(t)}{dt}e^{-st}dt$$

$$\Rightarrow \lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}\int_{0^-}^{\infty}\frac{df(t)}{dt}e^{-st}dt$$

$$= \lim_{s\to\infty}\left[\int_{0^-}^{0^+}\frac{df(t)}{dt}e^{-st}dt + \int_{0^+}^{\infty}\frac{df(t)}{dt}e^{-st}dt\right]$$

$$= \int_{0^-}^{0^+}\frac{df(t)}{dt}dt + 0$$

$$= f(0^+) - f(0^-)$$

$$\lim_{s\to\infty} sF(s) = f(0^+) = \lim_{t\to 0^+} f(t)$$

$Q.E.D$

- The final-value theorem states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

proof.

$$\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0} \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$= \int_{0^-}^{\infty} \frac{df(t)}{dt} dt$$

$$= \lim_{t\to\infty} f(t) - f(0^-)$$

$$\lim_{s\to 0} sF(s) - f(0^-) = \lim_{t\to\infty} f(t) - f(0^-)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$Q.E.D$

e.g. We have

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t)$$

- The initial-value theorem gives

$$\lim_{s\to\infty} sF(s) = \frac{100s^2\left[1 + \left(\frac{3}{s}\right)\right]}{s^3\left[1 + \left(\frac{6}{s}\right)\right]\left[1 + \left(\frac{6}{s}\right) + \left(\frac{25}{5^2}\right)\right]} = 0$$

$$\lim_{t\to 0^+} f(t) = [-12 + 20\cos(53.13^\circ)](1) = -12 + 12 = 0$$

- and the final-value theorem yields

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{100s(s+3)}{(s+6)(s^2+6s+25)} = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[-12e^{-6t} + 20e^{-3t} + \cos(4t - 53.13^{\circ})]u(t) = 0$$

Remark. In applying the theorems, we already had the time-domain expression

and were merely testing our understanding.

However;

- these theorems have the real value of being able to test the s-domain expressions

  before working out the inverse transform.