The Fourier Transform

- We have shown that a periodic function can be described
  - by means of a Fourier series.
- The Fourier transform extends this frequency-domain description to functions
  - that are NOT periodic.

Note that ;

- The idea of transforming an aperiodic function from the time-domain to the frequency-domain
  - was already introduced through the Laplace transform.

NOT a new transform

- The Fourier transform is just a special case of the bilateral Laplace transform ;
  - real part of the complex frequency is set equal to zero.

Physical interpretation/application

- It can be viewed as a limiting case of the Fourier series.
- More useful than the Laplace transform
  - especially in communications theory and signal processing issues.

How to derive the Fourier transform ?

- We first consider the exponential Fourier series representation of a periodic function

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

Now ;

- If the fundamental period, T is allowed to go to ∞
  - implying that the function never repeats itself and hence is aperiodic.

What happens when T increases ?

- The separation between adjacent harmonic frequencies becomes smaller and smaller.

- We have

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T}$$

Therefore ;

$$\frac{1}{T} \to \frac{d\omega}{2\pi} \qquad as \qquad T \to \infty$$

Moreover ;

- As the period increases, the frequency moves form being a discrete variable

  ↳ to becoming a continuous variable.

That is ;

$$n\omega_0 \to \omega \qquad as \qquad T \to \infty$$

- and

$$C_n T \to \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} (C_n T) e^{jn\omega_0 t} \left(\frac{1}{T}\right)$$

$$= \sum_{n=-\infty}^{\infty} (C_n T) e^{j\underbrace{n\omega_0}_{\omega} t} \frac{d\omega}{2\pi}$$

as $T \to \infty$, the summation approaches integration

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

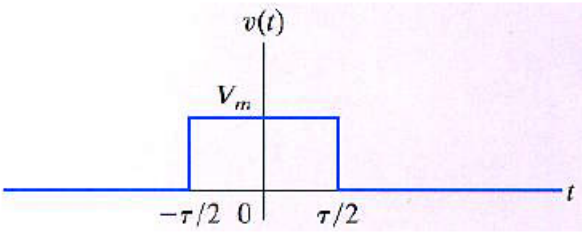
Fourier transform pair

- defined by the following pair of equations

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ "Fourier transform"

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{j\omega t}d\omega$$ "Inverse Fourier transform"

Ex. Let us derive the Fourier transform of a voltage pulse shown as



$$V(\omega) = \int_{-\infty}^{\infty} \vartheta(t)e^{-j\omega t}dt$$

$$= \int_{-\lambda/2}^{\lambda/2} V_m e^{-j\omega t}dt$$

$$= \frac{V_m}{-j\omega}e^{-j\omega t}\Bigg|_{-\lambda/2}^{\lambda/2}$$

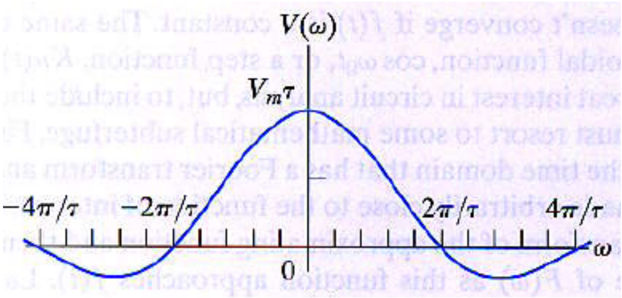
$$= \frac{V_m}{-j\omega}\left(e^{-j\omega\lambda/2} - e^{j\omega\lambda/2}\right)$$

$$= \frac{V_m}{-j\omega}(-j2)\sin\omega\lambda/2$$

$$= \frac{2V_m}{\omega}\sin(\omega\lambda/2)$$

$$= \frac{2V_m}{(\omega\lambda/2)(2/\lambda)}\sin(\omega\lambda/2)$$

$$= V_m\lambda\frac{\sin(\omega\lambda/2)}{\omega\lambda/2}$$ " *sinc* function, $\sin(x)/x$"



The convergence of the Fourier transform

- A function of time $f(t)$ has a Fourier transform if

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

converges.

- If $f(t)$ is single valued and encloses a finite area over the range of integration

  → $f(t)$ is a well-behaved function.

Note that ;

- If $f(t)$ is a well-behaved function that differs from zero over a finite interval of time

  → convergence is NO problem.

However ;

- If $f(t)$ is different from zero over an infinite interval

  → then it is has a Fourier transform.

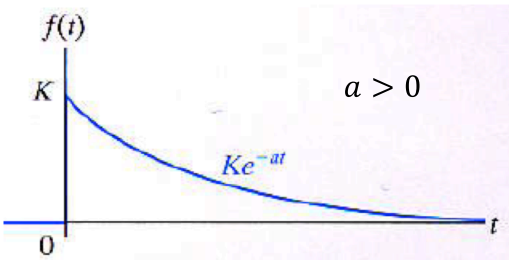
(i) if the integral

$$\int_{-\infty}^{\infty} |f(t)|dt$$

exists.

(ii) If any discontinuities in $f(t)$ are finite.

Ex. We compute the Fourier transform of a decaying exponential function shown as



$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{0}^{\infty} Ke^{at}e^{-j\omega t}dt$$

$$= \frac{K\,e^{-(a+j\omega)t}}{-(a+j\omega)}\Bigg|_0^{\infty} = \frac{K}{-(a+j\omega)}\,(0-1)$$

$$= \frac{K}{a+j\omega} \quad , \quad a > 0$$

Using Laplace transforms to find Fourier transforms

- We shall use a table of unilateral Laplace transform pairs

  ↳ to find the Fourier transform of functions for which the Fourier integral converges.

- The Fourier integral converges when

  ↳ all the poles of $F(s)$ lie in the left half of the s-plane.

Why on LHP ?

- Because if $F(s)$ has poles in RHP or on the imaginary axis

  ↳ the integral $\int_{-\infty}^{\infty} |f(t)|dt$ does NOT exist.

How to find $F(\omega)$?

- We have several rules to apply :

1. If $f(t) = 0$ for $t \leq 0^-$, we have

$$\mathcal{F}\{f(t)\} \triangleq F(\omega) = \mathcal{L}\{f(t)\}\Big|_{s\,=\,j\omega}$$

e.g. Let

$$f(t) = \begin{cases} 0 \quad , \quad t \leq 0^- \\ e^{-at}\cos\omega_0 t \quad , \quad t \geq 0^+ \end{cases}$$

then

$$\mathcal{F}\{f(t)\} = \frac{s+a}{(s+a)^2+\omega_0^2}\Bigg|_{s\,=\,j\omega} = \frac{j\omega+a}{(j\omega+a)^2+\omega_0^2}$$

2. A negative-time function is nonzero for negative values of time and

zero for positive values of time.

- To find the Fourier transform of a negative-time function

  we reflect it over to the positive-time domain and find its one-sided Laplace transform.

- Then we replace s with $-j\omega$

  to obtain the Fourier transform of the original time function.

That is ;

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{F(-t)\}_{\,s\,=\,-j\omega}$$

e.g. Let

$$f(t) = \begin{cases} 0 & , \quad t \geq 0^+ \\ e^{at}\cos\omega_0 t & , \quad t \geq 0^- \end{cases}$$

then

$$f(-t) = \begin{cases} 0 & , \quad t \leq 0^- \\ e^{-at}\cos\omega_0 t & , \quad t \geq 0^+ \end{cases}$$

Hence ;

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{f(-t)\} = \left.\frac{s+a}{(s+a)^2+\omega_0^2}\right|_{s\,=\,-j\omega}$$

$$= \frac{-j\omega + a}{(-j\omega + a)^2 + \omega_0^2}$$

3. Functions that are nonzero over all time

can be resolved into positive and negative-time functions.

- If we let

$$f^+(t) \triangleq f(t) \quad , \quad for\ \ t > 0$$
$$f^-(t) \triangleq f(t) \quad , \quad for\ \ t < 0$$

then

$$f(t) = f^{+}(t) + f^{-}(t)$$

Thus ;

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{\,f^{+}(t)\} + \mathcal{F}\{f^{-}(t)\}$$

$$= \mathcal{L}\{f^{+}(t)\}_{\,s\,=\,j\omega} + \mathcal{L}\{f^{-}(t)\}_{\,s\,=\,-j\omega}$$

e.g. Let us consider

$$f(t) = e^{-a|t|}$$

- We have

$$f^{+}(t) = e^{-at} \quad , \quad f^{-}(t) = e^{at}$$

$\Rightarrow$

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{e^{-at}\}_{s=j\omega} + \mathcal{L}\{e^{-at}\}_{s=-j\omega}$$

$$= \frac{1}{s+a}\bigg|_{s\,=\,j\omega} + \frac{1}{s+a}\bigg|_{s\,=\,-j\omega}$$

$$= \frac{1}{j\omega + a} + \frac{1}{-j\omega + a}$$

$$= \frac{2a}{\omega^2 + a^2}$$

Fourier transforms in the limit

- There exists a group of functions which have great practical interest

  ↳ but do NOT have a Fourier transform in a strict sense.

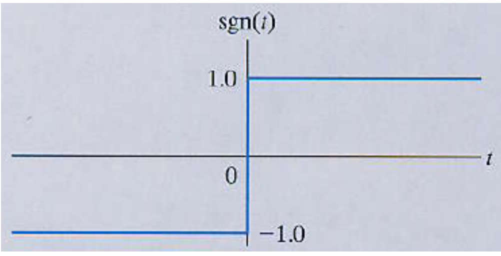
- We need to define the Fourier transform

  ↳ through a limit process.

Signum function

- Defined as

$$f(t) = \begin{cases} 1 & , \quad for\ t > 0 \\ -1 & , \quad for\ t < 0 \end{cases}$$



$$\triangleq sgn\ (t)$$

$$= u(t) - u(-t)$$

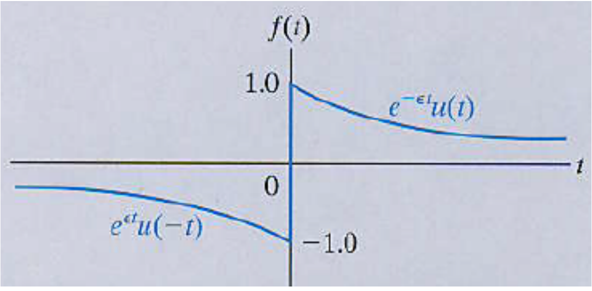
where $u(t)$ represents the unit-step function.

A general approach

- We first generate a function that approaches the signum function in the limit :

$$sgn\ (t) = \lim_{\varepsilon \to 0} [e^{-\varepsilon t} u(t) - e^{\varepsilon t} u(-t)]$$

which can be plotted as



- We calculate

$$\mathcal{F}\{f(t)\} = \left.\frac{1}{s+\varepsilon}\right|_{s\, =\, j\omega} - \left.\frac{1}{s+\varepsilon}\right|_{s\, =\, -j\omega}$$

$$= \frac{1}{j\omega + \varepsilon} - \frac{1}{-j\omega + \varepsilon}$$

$$= \frac{-j2\omega}{\omega^2 + \varepsilon^2}$$

- As $\varepsilon \to 0$, $\quad f(t) \to sgn\ (t)$ and

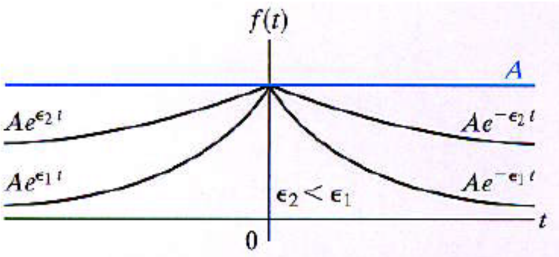
$$\mathcal{F}\{f(t)\} \to \frac{2}{j\omega}$$

a constant function

- We can approximate a constant with the exponential function

$$f(t) = Ae^{-\varepsilon|t|} \quad , \quad \varepsilon > 0$$

- As $\varepsilon \to 0$, $f(t) \to A$



- then we calculate

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{Ae^{-\varepsilon t}\}\Big|_{s\,=\,j\omega} + \mathcal{L}\{Ae^{-\varepsilon t}\}\Big|_{s\,=\,-j\omega}$$

$$= \frac{A}{j\omega + \varepsilon} + \frac{A}{-j\omega + \varepsilon}$$

$$= \frac{2\varepsilon A}{\omega^2 + \varepsilon^2}$$

$$\triangleq F(\omega)$$

Note that ;

- The function $F(\omega)$ generates an impulse function at $\omega = 0$ as $\varepsilon \to 0$

Because ;

(i)

$$F(0) = \lim_{\varepsilon \to 0} \frac{2\varepsilon A}{\varepsilon^2} = \infty$$

(ii) the duration of $F(\omega) \to 0$ as $\varepsilon \to 0$

(iii) the area under $F(\omega)$ is independent of ε

That is ;

$$\int_{-\infty}^{\infty} \frac{2\varepsilon A}{\varepsilon^2 + \omega^2} d\omega = 4\varepsilon A \int_{0}^{\infty} \frac{d\omega}{\varepsilon^2 + \omega^2}$$

$$= 4\varepsilon A\left(\frac{1}{\varepsilon}\right)\int_{0}^{\infty}\frac{d(\omega/\varepsilon)}{1+(\omega/\varepsilon)^{\wedge}2}$$

$$= 4A\arctan(\omega/\varepsilon)\Big|_{0}^{\infty}$$

$$= 4A\left(\frac{\pi}{2}-0\right)$$

$$= 2\pi A$$

As a result ;

$$\mathcal{F}\{A\} = 2\pi A\,\delta(\omega) \leftarrow F(\omega) \quad \text{approaches an impulse function}$$

in the limit $f(t)$ approaches a constant A

Unit step function

- The unit-step function can be expressed as

$$u(t) = \frac{1}{2}+\frac{1}{2}sgn\,(t)$$

Thus ;

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2}\right\}+\mathcal{F}\left\{\frac{1}{2}sgn\,(t)\right\}$$

$$= 2\pi\left(\frac{1}{2}\right)\delta(\omega)+\frac{1}{2}\frac{2}{j\omega}$$

$$= \pi\delta(\omega)+\frac{1}{j\omega}$$

Cosine function

- We observe that if

$$\mathcal{F}(\omega) = 2\pi\delta(\omega-\omega_0)$$

- Then from Inverse Laplace transform equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\,\delta(\omega-\omega_0)e^{j\omega t}\,d\omega$$

$$= e^{j\omega_0 t}$$

Note that ;

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\,[2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)]$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

Fourier transforms of elementary functions

Type	$f(t)$	$F(\omega)$
impulse	$\delta(t)$	1
constant	A	$2\pi A\delta(\omega)$
signum	$\mathrm{sgn}(t)$	$2/j\omega$
step	$u(t)$	$\pi\delta(\omega) + 1/j\omega$
positive-time exponential	$e^{-at}u(t)$	$1/(a+j\omega),\ a>0$
negative-time exponential	$e^{at}u(-t)$	$1/(a-j\omega),\ a>0$
positive- and negative-time exponential	$e^{-a\|t\|}$	$2a/(a^2+\omega^2),\ a>0$
complex exponential	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
cosine	$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
sine	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

Alternative representation of Fourier transform

- Since $F(\omega)$ is a complex quantity, it can be expressed in either

  rectangular or polar form.

Thus ;

- from the defining integral, we have

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f(t)(\cos\omega t - j\sin\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$\triangleq A(\omega) - jB(\omega) = |F(\omega)|e^{j\theta(\omega)}$$

where

$$A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

$$|F(\omega)| = \sqrt{A^2(\omega) + B^2(\omega)} \quad , \quad \theta(\omega) = \arctan\frac{B(\omega)}{A(\omega)}$$

Some observations about $F(\omega)$

- $A(\omega)$ is an even function

  $\Rightarrow$ $A(\omega) = A(-\omega)$

- $B(\omega)$ is an odd function

  $\Rightarrow$ $B(\omega) = -B(-\omega)$

- $|F(\omega)|$ is an even function

  $\Rightarrow$ while $\theta(\omega)$ is an odd function.

- $F(-\omega) = F^*(\omega)$

Hence ;

- If $f(t)$ is an even function

  $\Rightarrow$ $F(\omega)$ is real, i.e. $B(\omega) = 0$

- and if $f(t)$ is an odd function

  $\Rightarrow$ $F(\omega)$ is imaginary, i.e. $A(\omega) = 0$

Indeed ;

$$f(t) = f(-t) \qquad \Rightarrow \qquad A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{-\infty}^{0} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{\infty}^{0} f(-t)\cos(-\omega t)\, d(-t) + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= -\int_{\infty}^{0} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{0}^{\infty} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= 2\int_{0}^{\infty} f(t)\cos\omega t\, dt$$

- and in a similar manner, we calculate

$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt$$

$$= \int_{-\infty}^{0} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= \int_{\infty}^{0} f(-t)\sin(-\omega t)\, d(-t) + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= \int_{\infty}^{0} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= -\int_{0}^{\infty} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= 0$$

Moreover ;

$$f(t) = -f(-t) \qquad \Rightarrow \qquad A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{-\infty}^{0} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{\infty}^{0} f(-t)\cos(-\omega t)\, d(-t) + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= \int_{\infty}^{0} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= -\int_{0}^{\infty} f(t)\cos\omega t\, dt + \int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$$= 0$$

Similarly ;

- We calculate

$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

$$= \int_{-\infty}^{0} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= \int_{\infty}^{0} f(-t)\sin(-\omega t)\, d(-t) + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= -\int_{\infty}^{0} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= \int_{0}^{\infty} f(t)\sin\omega t\, dt + \int_{0}^{\infty} f(t)\sin\omega t\, dt$$

$$= 2\int_{0}^{\infty} f(t)\sin\omega t\, dt$$

As a result,

- If $f(t)$ is an even function

  $\Rightarrow$ $F(\omega)$ is real and even.

- If $f(t)$ is an odd function

  $\Rightarrow$ $F(\omega)$ is imaginary and odd.

Operational transforms

- We introduce some of the important operational transforms

  → without their proof (left as an exercise)

Multiplication by a constant

- If

$$\mathcal{F}\{f(t)\} = F(\omega)$$

then

$$\mathcal{F}\{Kf(t)\} = KF(\omega)$$

Addition (Subtraction)

- If

$$\mathcal{F}\{f_1(t)\} = F_1(\omega) \quad , \quad \mathcal{F}\{f_2(t)\} = F_2(\omega)$$

then

$$\mathcal{F}\{f_1(t) \pm f_2(t)\} = F_1(\omega) \pm F_2(\omega)$$

Differentiation

- The Fourier transform of the 1^{st} derivative of $f(t)$ is

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

and

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (j\omega)^n F(\omega)$$

Integration

- If

$$g(t) = \int_{-\infty}^{t} f(\tau)\, d\tau$$

then

$$\mathcal{F}\{g(t)\} = \frac{F(\omega)}{j\omega}$$

which is valid if $\int_{-\infty}^{\infty} f(\tau)\, d\tau = 0$

Scale change

- Time and frequency are reciprocals, hence

  when time is stretched out, frequency is compressed (and vice versa)

That is ;

$$\mathcal{F}\{f(at)\} = \frac{1}{a} F\left(\frac{\omega}{a}\right) \quad , \quad a > 0$$

Note that ;

$$0 < a < 1.0 \quad \Rightarrow \quad \text{time is stretched out}$$
$$a > 1.0 \quad \Rightarrow \quad \text{time is compressed}$$

Translation in the time-domain

- Translating a function in the time-domain

  changes the phase spectrum while the amplitude spectrum keeps untouched.

That is ;

$$\mathcal{F}\{f(t-a)\} = e^{-jwa}F(\omega)$$

Translation in the frequency domain

- Corresponds to multiplication by the complex exponential in the time-domain :

$$\mathcal{F}\left\{e^{j\omega_0 t}f(t)\right\} = F(\omega - \omega_0)$$

Modulation

- Amplitude modulation is the process of changing the amplitude of a sinusoidal carrier:

$$\mathcal{F}\{f(t)\cos\omega_0 t\} = \frac{1}{2}[F(\omega - \omega_0) + F(\omega + \omega_0)]$$

where

$f(t)$: modulating signal

Convolution in the time-domain

- Corresponds to multiplication in the frequency-domain

- If

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t)$$

then

$$\mathcal{F}\{y(t)\} = Y(\omega) = X(\omega)H(\omega)$$

Convolution in the frequency domain

- Corresponds to finding the Fourier transform of the product of two time functions :

- If

$$f(t) = f_1(t)f_2(t)$$

then

$$\mathcal{F}\{f(t)\} = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)F_2(\omega-u)\,du = \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

Circuit applications

- The Laplace transform is used more widely

  to find the response of a circuit than is the Fourier transform.

Why ?

- Two reasons appear :

  (i) the Laplace transform integral converges for a wider range of driving functions
  (ii) it accommodates initial conditions.

However ;

- we can still use the Fourier transform

  to find the response.

- The fundamental relationship is

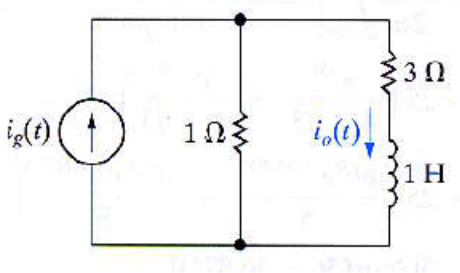
$$Y(\omega) = X(\omega)H(\omega)$$

where

$Y(\omega)$: transform of the response
$X(\omega)$: transform of the input
$H(\omega)$: transfer function of the circuit

which is $H(s)$ with s replaced by $j\omega$

Ex. Use the Fourier transform to find $i_0(t)$ in the circuit shown as



$$i_g(t) = 20sgn(t)\ A$$

Solution. The Fourier transform of the driving source is

$$I_g(\omega) = \mathcal{F}\{20sgn(t)\}$$

$$= 20\left(\frac{2}{j\omega}\right)$$

$$= \frac{40}{j\omega}$$

- The transfer function of the circuit is

$$H(\omega) = \frac{I_0}{I_g} = \frac{1}{4+s}\bigg|_{s\,=\,j\omega} = \frac{1}{4+j\omega}$$

$$\Rightarrow \quad I_0(\omega) = H(\omega)I_g(\omega)$$

$$= \frac{1}{4+j\omega}\cdot\frac{40}{j\omega}$$

$$= \frac{40}{j\omega(4+j\omega)}$$

- Using partial fraction expansion gives

$$\frac{40}{j\omega(4+j\omega)} = \frac{c_1}{j\omega} + \frac{c_2}{4+j\omega}$$

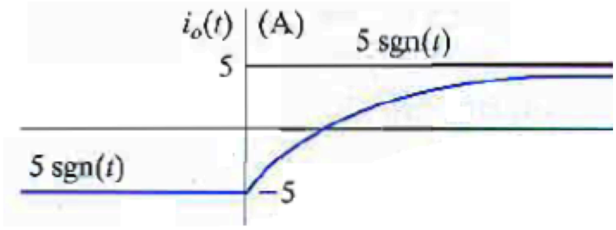
$$\Rightarrow \quad c_1 = \frac{40}{4+j\omega}\bigg|_{j\omega\,=\,0} = 10 \quad , \qquad c_2 = \frac{40}{j\omega}\bigg|_{j\omega\,=\,-4} = -10$$

Hence ;

$$I_0(\omega) = \frac{10}{j\omega} - \frac{10}{4+j\omega}$$

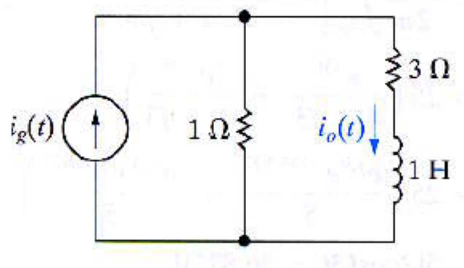
$$\Rightarrow \quad i_0(t) = \mathcal{F}^{-}\{I_0(\omega)\}$$

$$= 5sgn\,(t) - 10e^{-4t}u(t)$$



Using Fourier transform to find the sinusoidal steady-state response

- We shall reconsider the former example



$$i_g(t) = 50\cos 3t \;\; A$$

$$I_g(\omega) = 50\pi[\delta(\omega-3) + \delta(\omega+3)]$$

- as before the transfer function of the circuit is

$$H(\omega) = \frac{1}{4+j\omega}$$

$$\Rightarrow \quad I_0(\omega) = 50\pi\frac{\delta(\omega-3)+\delta(\omega+3)}{4+j\omega}$$

Hence ;

$$i_0(t) = \mathcal{F}^{-1}\{I_0(\omega)\}$$

$$= \frac{50\pi}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\delta(\omega-3)+\delta(\omega+3)}{4+j\omega}\right]e^{j\omega t}\,d\omega$$

$$= 25\left(\frac{e^{j3t}}{4+j3} + \frac{e^{-j3t}}{4-j3}\right)$$

$$= 25\left(\frac{e^{j3t}e^{-j36.87°}}{5} + \frac{e^{-j3t}e^{j36.87°}}{5}\right)$$

$$= 5\left[e^{j(3t-36.87°)} + e^{-j(3t-36.87°)}\right]$$

$$= 10\cos(3t - 36.87°)$$

Parseval's theorem

- It relates the energy associated with a time-domain function of finite energy

  to the Fourier transform of the function.

- Assume that $f(t)$ is either the voltage across or the current in a 1Ω resistor.

- then the energy associated with this function is

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt$$

- Parseval's theorem states that this same energy can be calculated

  by an integration in the frequency domain.

That is ;

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

How to justify it ?

- We have

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} f(t)f(t)\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\int_{-\infty}^{\infty} f(t)e^{j\omega t} dt\,d\omega$$

$$= \frac{1}{2\pi}\int_{\infty}^{-\infty} F(-\omega)\,\underbrace{\int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt}_{F(\omega)}\,d(-\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Ex. Let

$$f(t) = e^{-a|t|}$$

- We calculate

$$\int_{-\infty}^{\infty} e^{-2a|t|}\,dt = \int_{-\infty}^{0} e^{2at}\,dt + \int_{0}^{\infty} e^{-2at}\,dt$$

$$= \frac{1}{2a}\,e^{2at}\Big|_{-\infty}^{0} - \frac{1}{2a}e^{-2at}\Big|_{0}^{\infty}$$

$$= \frac{1}{2a}\,(1-0) - \frac{1}{2a}\,(0-1)$$

$$= \frac{1}{a}$$

- We have already calculated before that

$$\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}$$

- Then we compute

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2a}{a^2+\omega^2}\right)^2 d\omega = \frac{4a^2}{2\pi}\,2\int_{0}^{\infty}\frac{1}{(a^2+\omega^2)^2}\,d\omega$$

$$= \frac{4a^2}{\pi}\,\frac{1}{2a^2}\left(\frac{\omega}{\omega^2+a^2} + \frac{1}{a}\tan^{-1}\frac{\omega}{a}\right)\Big|_{0}^{\infty}$$

$$= \frac{2}{\pi}\left(0 + \frac{\pi}{2a} - 0 - 0\right)$$

$$= \frac{1}{a}$$

Ex. The current in a 40Ω resistor is

$$i = 20e^{-2t}u(t)\ \ A$$

What percentage of the total energy dissipated in the resistor can be associated with the frequency band $0 \le \omega \le 2\sqrt{3}\ \ rad/s$?

Solution. The total energy dissipated in the 40Ω resistor is

$$W_{40\Omega} = \int_0^\infty (20e^{-2t})^2 . 40\, dt$$

$$= 16000 \int_0^\infty e^{-4t}\, dt$$

$$= 16000 \frac{e^{-4t}}{-4} \Big|_0^\infty$$

$$= -4000(0 - 1)$$

$$= 4000\, J$$

- to use Parseval's theorem, we calculate first

$$F(\omega) = \frac{20}{j\omega + 2} \qquad \Rightarrow \qquad |F(\omega)| = \frac{20}{\sqrt{\omega^2 + 4}}$$

Then

$$40 . \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{20}{\sqrt{\omega^2 + 4}}\right)^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{16000}{\omega^2 + 4} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{16000/4}{1 + (\omega/2)^2}\, 2\, d(\omega/2)$$

$$= \frac{4000}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + (\omega/2)^2} d(\omega/2)$$

$$= \frac{4000}{\pi} \arctan\left(\frac{\omega}{2}\right) \Big|_{-\infty}^{\infty}$$

$$= \frac{4000}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= 4000\, J$$

- for the energy associated with the frequency band $0 \leq \omega \leq 2\sqrt{3}\ \ rad/s$

$$W_{40\Omega} = 40.\frac{1}{2\pi}\int_{0}^{2\sqrt{3}} \frac{400}{4+\omega^2}d\omega$$

$$= \frac{20}{\pi}\int_{0}^{2\sqrt{3}} \frac{400/4}{1+(\omega/2)^2}\ 2\ d(\omega/2)$$

$$= \frac{8000}{\pi}\arctan\left(\frac{\omega}{2}\right)\Big|_{0}^{2\sqrt{3}}$$

$$= \frac{8000}{\pi}\ \left(\frac{\pi}{3} - 0\right)$$

$$= \frac{8000}{3}\ J$$

Hence ;

$$\frac{8000/3}{4000}x\ 100 = 66.67\ \%$$

Ex. The input voltage to an ideal bandpass filter is

$$V(t) = 120e^{-24t}u(t) \quad V$$

The filter passes all frequencies that lie between 24 and 48 rad/s without attenuation, and completely rejects all frequencies outside this passband.

a. Sketch $|V(\omega)|^2$ for the filter input voltage.

b. Sketch $|V_0(\omega)|^2$ for the filter output voltage.

c. What percentage of the total 1Ω energy content of the signal at the input of the filter is available at the output ?

Solution.

a.

$$V(\omega) = \int_{-\infty}^{\infty} 120e^{-24t}u(t)e^{-j\omega t}\,dt$$

$$= 120 \int_{0}^{\infty} e^{-(j\omega+24)t}\,dt$$

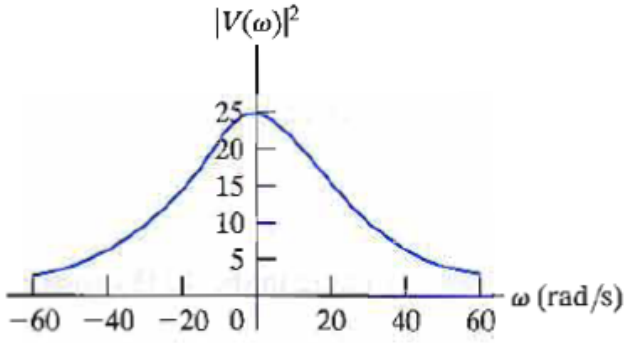
$$= 120 \left. \frac{e^{-(j\omega+24)t}}{-(j\omega+24)} \right|_{0}^{\infty}$$

$$= -\frac{120}{j\omega+24}\,(0-1)$$

$$= \frac{120}{24+j\omega}$$

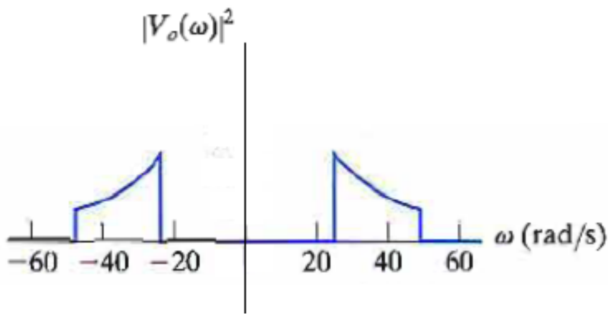
$$\Rightarrow \quad |V(\omega)|^2 = V(\omega)V^*(\omega)$$

$$= \frac{14400}{\omega^2+576}$$



b. The ideal bandpass filter rejects all frequencies outside the passband

then the plot of $|V_0(\omega)|^2$ can be given as follows



c.

$$W_0 = 1.2.\frac{1}{2\pi}\int_{24}^{48}\frac{14400}{\omega^2+576}d\omega$$

$$= \frac{1}{\pi}\int_{24}^{48}\frac{25}{1+(\omega/24)^2}24\,d(\omega/24)$$

$$= \frac{600}{\pi}\arctan\left(\frac{\omega}{24}\right)\Bigg|_{24}^{48}$$

$$= \frac{600}{\pi}\ (\arctan 2 - \arctan 1)$$

$$= \frac{600}{\pi}\ \left(\frac{\pi}{284} - \frac{\pi}{4}\right)$$

$$= 61.45\ \ J$$

- and the total energy at the input of the filter is

$$W_i = 1.\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{14400}{\omega^2+576}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{25}{1+(\omega/24)^2}24\,d(\omega/24)$$

$$= \frac{300}{\pi}\arctan(\omega/24)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{300}{\pi}\ \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{300}{\pi}\ \pi$$

$$= 300\ \ J$$

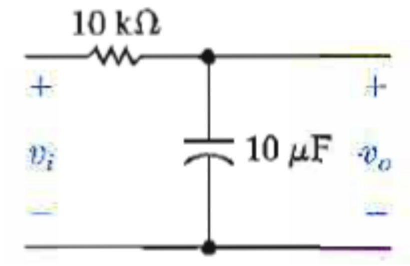
Hence ;

$$\frac{61.45}{300}x\ 100 = 20.48\ \%$$

Remark. Parseval's theorem makes it possible to calculate the energy available at the output of the filter

even if we do NOT know the time-domain expression for $V_0(t)$.

Ex. Consider the following low-pass filter circuit



$$V_i(t) = 15e^{-5t}\, u(t)\ V$$

a. What percentage of the 1Ω energy available in the input signal is available at the output signal ?

b. What percentage of the output energy is associated with the frequency range $0 \leq \omega \leq 10\, rad/s$?

Solution.

a.

$$W_i = \int_0^\infty (15e^{-5t})^2\, dt$$

$$= \int_0^\infty 225e^{-10t}\, dt$$

$$= \frac{225}{-10}\, e^{-10t}\Big|_0^\infty$$

$$= -22.5\,(0-1)$$

$$= 22.5\ \ J$$

$$V_i(\omega) = \frac{15}{j\omega + 5}$$

$$H(\omega) = \frac{1/RC}{j\omega + (1/RC)} = \frac{1/10.10^3.10.10^{-6}}{j\omega + (1/10.10^3.10.10^{-6})} = \frac{10}{j\omega + 10}$$

Hence ;

$$V_0(\omega) = H(\omega)\, V_i(\omega)$$

$$= \frac{10}{j\omega + 10}\,\frac{15}{j\omega + 5}$$

$$= \frac{150}{(j\omega + 10)(j\omega + 5)}$$

then

$$|V_0(\omega)|^2 = \frac{22500}{(\omega^2 + 100)\quad(\omega^2 + 25)} = \frac{300}{\omega^2 + 25} - \frac{300}{\omega^2 + 100}$$

and

$$W_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{300}{\omega^2 + 25} - \frac{300}{\omega^2 + 100}\right)d\omega$$

$$= \frac{1}{\pi}\left[\int_0^{\infty}\frac{12}{1 + (\omega/5)^2}5\,d(\omega/5) - \int_0^{\infty}\frac{3}{1 + (\omega/10)^2}10\,d(\omega/10)\right]$$

$$= \frac{1}{\pi}\left[60 \arctan\left(\frac{\omega}{5}\right) - 30 \arctan\left(\frac{\omega}{10}\right)\right]\Bigg|_0^{\infty}$$

$$= \frac{1}{\pi}\left(60.\frac{\pi}{2} - 0 - 30.\frac{\pi}{2} + 0\right)$$

$$= \frac{1}{\pi}\,30\,\frac{\pi}{2}$$

$$= 15 \quad J$$

Hence ;

$$\frac{15}{22.5}\,x\,100 = 66.67\ \%$$

b. The output energy associated with the frequency range $0 \leq \omega \leq 10 \ rad/s$ is calculated as

$$W_0' = \frac{1}{\pi} \int_0^{10} \left(\frac{300}{\omega^2 + 25} - \frac{300}{\omega^2 + 100} \right) d\omega$$

$$= \frac{1}{\pi} \left[60 \arctan\left(\frac{\omega}{5}\right) - 30 \arctan\left(\frac{\omega}{10}\right) \right]_0^{10}$$

$$= \frac{1}{\pi} \left[60 \arctan(2) - 30 \arctan(1) - 0 + 0 \right]$$

$$= \frac{1}{\pi} \left(60 \frac{\pi}{2.84} - 30 \frac{\pi}{4} \right)$$

$$= 13.64 \ J$$

Therefore ;

$$\frac{13.64}{15} \ x \ 100 = 90.57 \ \%$$