PROBABILITY:

The concept of chance, certainty/uncertainty, possibility .. existed since ancient Greeks.

A calculus of probability (a calculus of chance, a theory of chan
started w/ gambling.

<u>Cardano</u> (an Italian renaissance man) formulated probability concepts: sample space, outcom. And first calculations of probability (he was a successful gambler), mathematician, etc...) An eccentric man (The Book of My Life, Book on Games of Chance, w/ superstition beliefs.

$\boxed{1}$ Experiment: A procedure that results in a finite set of equiprobable outco

$\boxed{2}$ Sample Space: The set of all possible outcomes in an experim

$\boxed{3}$ Event: A subset of the sample space.

Example (De Méré's Paradox:)

As a result of extensive observation of dice games, the French gambler de Méré noticed that the total # of spots showing on three dice thrown simultaneously turns out to be 11 more often than it turns out to be 12, although from his point of view both events should occur equally often.

De Mere reasoned as follows:

6 6 3 6 3 3

E_{11} occurs in just six ways (6:4:1, 6:3:2, 5:5:1, 5:4:2, 5:3:3, 4:4:3)

and E_{12} " " " " " (6:5:1, 6:4:2, 6:3:3, 5:5:2, 5:4:3, 4:4:4)

6 6 3 3 6 1

Therefore $P(E_{11}) = P(E_{12})$

Where's the fallacy of this argument?

$|E_{11}| = 27$ $\quad P(E_{11}) = \frac{|E_{11}|}{|S.S.|} = \frac{27}{216}$ $\quad P(E_{11}) > P(E_{12})$

$|E_{12}| = 25$ $\quad P(E_{12}) = \frac{|E_{12}|}{|S.S.|} = \frac{25}{216}$

<u>Example 2:</u> Rolling three dice, what's the probability of getting a six in one of the three dice?

Again Sample Space $= \{(1,1,1), \dots, (6,6,6)\}$

E_6: The set of outcomes with at least one 6 in one of the three dice.

$E_6 = \{(1,1,6), (6,6,2), \dots (6,6,6)\}$

Apparently $P(E_6) = \frac{|E_6|}{|S.S.|} = \frac{|E_6|}{216}$

~~Let~~ $\overline{E_6}$: The set of outcomes with no 6 in any of the dice

$\overline{E_6} = \{(1,1,1), (1,1,5), \dots, (5,5,5)\}$ = ~~&~~ $|\overline{E_6}| = 125$

$P(\overline{E_6}) = \frac{125}{216}$ $\quad \therefore P(E_6) = 1 - \frac{125}{216} = \frac{91}{216}$

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~~Example: What's the prob. that two playing cards picked at random from a full deck are both aces?~~ $\frac{1}{13}\cdot\frac{\cancel{4}}{52}\cdot\frac{\cancel{3}}{51}$ $\frac{3}{13.17}$

~~S.S. {(A,A),(A,2),...,(A,K),(2,A),(2,2),(2,3),...,(K,K)}~~

~~|S.S.| = 13^2 = 169~~

Example: Find the prob. that <u>a hand of five cards</u> in poker contain four cards of the same kind.

↳ (1,2,...13 kinds and 4 suits for each kind in a deck of 52 cards

E_s: 4 out of 5 cards are of the same kind

~~S.S.~~

Experiment: Select 5 cards from a deck of 52 cards.

Sample Space: $\{(1,1,1,1,2),\ldots (1,1,1,1,2s)\ldots,(13,13,13,13,12)\}$

$|\text{Sample Space}| = C(52,5) = \frac{52.51.50.49.48^{4}}{120} = 52.51.49.4$

$\downarrow$ x4 (4 suits for each kind)

$$E_s: \left.\begin{matrix}\{\{1,1,1,1,2\},\{1,1,1,1,3\},\ldots\{1,1,1,1,13\},\\ \{2,2,2,2,1\},\{2,2,2,2,3\},\ldots\{2,2,2,2,13\},\\ \}\end{matrix}\right\} 13$$

48

$|E_s| = 12\times13\times4 = 48\times13$

$$P(E_s) = \frac{|E_s|}{|\text{Sample Space}|} = \frac{48\times13}{52.51.49.4} \cong 0.00024$$

Example: What's the prob. that a poker hand contain a full house, that is three of one kind, and two of the other kind?

Sample Space: $\{\{1_A,1_S,1_C,1_H,2_A\},\{1_A,1_S,1_C,1_H,2_S\},\ldots\}$

E_F: $\{\{1_A,1_S,1_C,2_A,2_S\},\{1_A,1_S,1_C,2_H,2_C\},\ldots\}$

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$|\text{Sample Space}| = C(52,5)$

E_F : ~~Event of selecting five cards~~
3 cards are of one kind and 2 are of another.

$E_F = \{ \{1_A, 1_H, 1_C, 2_A, 2_H\}, \ldots\ldots \{13_A, 13_H, 13_S, 12_A, 12_H\} \}$

$|E_F| = ?$ Selecting

4C2 is the # of way of selecting two 2's out of 4.

4C3 is the # of way of selecting three 1's out of 4

13C2 is the # of ways of selecting two types.

$\therefore |E_F| = C(13,2)\,C(4,3)\,C(4,2) = 3744$

$P(E_F) = \dfrac{|E_F|}{|S.S.|} = \dfrac{3744}{2{,}598{,}960} \cong 0.0014$

↓Week 8 - Lecture

Example: What's the prob. that the numbers 11, 4, 17, 39 and 23 are drawn in that order from a bin containing 50 balls labeled with numbers 1, 2, ..., 50 if

a) the ball selected is NOT returned to the bin before the next ball is selected.

b) the ball selected is returned to the bin before the next ball is selected.

a) Experiment: Select 5 balls from 50 balls.

$S.S = \{ \{1,2,3,4,5\}, \{1,2,3,4,6\}, \ldots, \{46,47,48,49,50\} \}$
$\{2,1,3,4,5\}, \ldots$

~~|S.S.|~~

ball 1 can be choosen in 50 ways
ball 2 " " in 49 ways
⋮
ball 5 " " " 46 ways.

$|S.S.| = 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$

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$E_{11,4,17,39,23}$: Selecting ~~balls~~ 11, 4, 17, 39, 23 in that o...

$|E_{11,\dots}| = 1$

$$P(E_{11,\dots,23}) = \frac{1}{50 \cdot 49 \cdots 46} = \frac{1}{254{,}251{,}200}$$

b) $|S.S.| = 50^5$

$$P(E_{11,\dots,23}) = \frac{1}{50^5} = \frac{1}{312{,}500{,}000}$$

~

Probabilities of Union of Events:

$\bar{E}$: The complementary event of E

$\bar{E} = S.S. - E$

Therefore $P(\bar{E}) = 1 - P(E)$

Let E_1, E_2 be two events.

if E_1, E_2 are disjoint $E_1 \cap E_2 = \phi$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof: $P(E_1) = \dfrac{|E_1|}{|S.S.|}$ $\qquad P(E_2) = \dfrac{|E_2|}{|S.S.|}$

$$P(E_1 \cup E_2) = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S.S.|}$$

because

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

Example: What's the prob. that a positive integer selected at random from the set of integers [1,100] is divisable by 2 or 5.

integers divisable by 2 betw. [1,100] = 50

" 5 " [1,100] = 20

~~E₂ event~~

Experiment: A positive integer betw. [1,100] is selected at random

E_2 : event that the integer is divisable by 2

E_5 : event that " 5.

$E_2 = \{2, 4, 6, 8, \underline{10}, \ldots, \underline{100}\}$ $|E_2| = 50$

$E_5 = \{5, \underline{10}, 15, 20, \ldots, \underline{100}\}$ $|E_5| = 20$

E_{2or5} : event that integer is divisable by 2 or 5.
$= \{2, 4, 5, 6, \underline{10}, 12, 14, 15, \ldots\}$

$|E_{2,5}| = ?$

$P(E_2) = \frac{50}{100}$ $P(E_5) = \frac{20}{100}$

$E_{2and5} = \{10, 20, 30, \ldots, 100\}$ $|E_{2and5}| = 10$

$$P(E_{2or5}) = \frac{50+20-10}{100} = \frac{60}{100} = 0{,}6$$

RESULT : $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$ when $E_1, E_2, \ldots$ are disjoint events.

CONDITIONAL PROBABILITY:

Finding the probability of an event when more about the outcome is known about the experiment

Example: Suppose a fair coin is tossed three times, and it's known that the first flip is tails. What's the prob. that an odd number of tails appears?

Experiment : 3 coins are tossed
Sample Space $= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$|S.S.| = 8$
$\text{Event}_{\text{odd-tail}} = \{HHT, HTH, THH, TTT\}$
$\text{Event}_{\text{first-flip tail}} = \{THH, THT, TTH, TTT\}$

Now, $\text{Event}_{\text{first-flip-tail}}$ is <u>like</u> the new sample space.

$$\text{Event}_{\substack{\text{odd-tail when} \\ \text{S.S. is Event}_{\text{fr-f-t}}}} = \{THH, TTT\}$$

$$\left|\text{Event}_{\text{odd-tai}}\right| = 2$$

$$\left|\text{Event}_{\text{fir-fli-tail}}\right| = 4$$

$$P\left(\substack{\text{odd tail} \\ \text{when}}\right) = \frac{2}{4} = \frac{1}{2}$$

DEFINITION: Let E and F be events with $P(F) > 0$.
The conditional probability of E given F, denoted by $P(E|F)$ is defined as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Rethink the previous question:

$$P(\text{odd-\#-of-tail} \mid \text{first-is-tail}) = \frac{P(\text{odd-\#-of-tail} \cap \text{first-is-tail})}{P(\text{first-is-tail})}$$

$$\cancel{P(\text{odd\# of tail}) = \frac{4}{8} = \frac{1}{2}}$$

$$\cancel{P(\text{first-is-tail}) = \frac{4}{8} = \frac{1}{2}}$$

$$P(\text{odd\# of tails} \cap \text{first-is-tail}) = \frac{2}{8} = \frac{1}{4}$$

$$P(\text{first-is-tail}) = \frac{4}{8} = \frac{1}{2} \qquad \therefore \frac{1/4}{1/2} = \frac{1}{2}$$

Example: A bitstring of length four is generated randomly. What's the probability that ~~it~~ the generated bitstring contains at least two consecutive 0s given that its first bit is a 0.

Experiment = Generate a 4-bitstring randomly where 0's and 1's for every bit is equally likely.

Sample Space = $\{0000, 0001, 0010, 0011, 0100, 0101, \dots, 1111\}$

$|\text{Sample Space}| = 2^4 = 16$

$\text{Event}_{\text{2-consec-0s}}$ = the event that two consecutive zeros exist in the bitstring.

$= \{0000, 0001, 0010, 0011, 0100, 1000, 1001\}$

~~Event~~

F = the event that first bit is a 0. $= \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111\}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

~~$P(E \cap E) =$~~

$E \cap F = \{0000, 0001, 0010, 0010, 0100\}$

$|E \cap F| = 5$

$P(E \cap F) = \frac{5}{16}$

$\therefore P(E/F) = \frac{5/16}{1/2} = \frac{5}{8}$

$P(F) = \frac{|F|}{|S.S.|} = \frac{8}{16} = \frac{1}{2}$

INDEPENDENCE:

When two events are independent, the occurence of one of the events gives no information about the probability of the other event that occurs.

Example a fair coin is tossed 60 times: the outcome is all tails, What's the prob. that 61th outcome is a head?

Answer: If the coin is fail, for every toss $P(T) = P(H) = \frac{1}{2}$

$$P(E_{61th\ is\ head} \mid E_{first\ 60\ is\ tails}) = \frac{P(E_{61} \cap E_{60})}{P(E_{60})} = \frac{1/2^{61}}{1/2^{60}} = \frac{1}{2}$$

$P(E_{60}) = \left(\frac{1}{2^{60}}\right)$ $\quad E_{61} \cap E_{60} = \{HT\ldots T\} = \frac{1}{2^{61}}$

$P(E_{61}) = \frac{1}{2}$ $\quad P(E_{6}$

$P(E_{6}$

$$P(E_{61} \cap E_{60}) = P(E_{61}) \cdot P(E_{60}) \text{ since } E_{61} \text{ and } E_{60} \text{ are independent events.}$$
$$= \frac{1}{2} \cdot \frac{1}{2^{60}} = \frac{1}{2^{61}}$$

$$P(E_{61} \mid E_{60}) = P(E_{61}) = \frac{1}{2}$$

When E and F are independent events: $P(E|F)$ =

$P(E \cap F) = P(E) \cdot P(F)$ therefore

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \cdot \cancel{P(F)}}{\cancel{P(F)}} = P(E)$$

~~gives me new info~~ does not diminish the sample space

The events E and F are <u>independent</u> if and only if

$$P(E \cap F) = P(E) \cdot P(F)$$

<u>Example:</u> Let E be the event that an randomly generated bitstring of length four begins with a 1 and F is the event that this bitstring contains an <u>even number of 1s.</u>
Are E and F independent, if the 16 bit strings of length four are equally likely (i.e. each bit being 0 or 1 has prob. of 1/2)

S.S. = {0000, - - - -, 1111}

E = {1000, 1001, - - -, <u>1111</u>}

F = {0000, 0011, 0101, 0110, <u>1001</u>, <u>1010</u>, <u>1100</u>, <u>1111</u>}

$$P(E) = \frac{|E|}{|S.S.|} = \frac{8}{16} = \frac{1}{2}$$

$$P(F) = \frac{|F|}{|S.S.|} = \frac{8}{16} = \frac{1}{2}$$

$$P(E \cap F) = \frac{|E \cap F|}{|S.S.|} = \frac{4}{16} = \frac{1}{4}$$

$$P(E \cap F) \stackrel{?}{=} P(E) \cdot P(F)$$

$$\frac{1}{4} \stackrel{?}{=} \frac{1}{2} \cdot \frac{1}{2} \checkmark$$

∴ E and F are independent.

<u>Example:</u> Are the events E, that a family with three children has children of both sexes, and F, that this family has at least <u>one boy</u>, independent? Assume that the eight ways a family can have three children are equally likely.

S.S. = {GGG, - - -, BBB} $|S.S.| = 2^3 = 8$

E = {GBB, - - -} = S.S. − {BBB, GGG}

$|E| = 6$

F = {BGG, BGB, BBG, GBB, GBG, GGB, BBB} = S.S. − {GGG}

$|F| = 7$

$E \cap F =$ both sexes and at least one boy

$= |S.S.| - \{GGG\}$

$|E \cap F| = 7$

$P(E) = \frac{6}{8} = \frac{3}{4}$

$P(F) = \frac{7}{8}$

$P(E \cap F) = \frac{7}{8}$ $\quad \frac{7}{8} \stackrel{?}{=} \frac{3}{4} \cdot \frac{7}{8}$ No! $\therefore$ E and F are not independent

b) What if F is the event that the family has at most one boy?

$F = \{GGG, GBG, GGB, BGG\}$ $\quad P(F) = \frac{|F|}{|S.S.|} = \frac{4}{8} = \frac{1}{2}$

$E \cap F =$ both sexes & at most one boy $= \{GBG, GGB, BGG\} \Rightarrow P(E \cap F) = \frac{3}{8}$

$P(E) \cdot P(F) \stackrel{?}{=} P(E \cap F)$

$\frac{3}{4} \cdot \frac{1}{2} \stackrel{?}{=} \frac{3}{8}$ YES $\therefore$ E and F are independent.

BAYES' THEOREM (18th cc.)