CMPE 352
Signal Processing & Algorithms

Spring 2019

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The Fourier Series – Summary Chart

- For general periodic signals

Fourier Series
(exponential form)

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}}$$

a_k : Fourier series coefficients (or spectral coefficients)

$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$: fundamental frequency

T_0 : fundamental period

$$\boxed{a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt}$$

- For real periodic signals, assuming $a_k = A_k e^{j\theta_k}$

Fourier Series
(trigonometric form)

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t) dt$$

$$\alpha_k = \frac{1}{T_0}\int_{T_0} x(t) \cos k\omega_0 t \, dt$$

$$\beta_k = \frac{1}{T_0}\int_{T_0} x(t) \sin k\omega_0 t \, dt$$

$$A_k = \sqrt{\alpha_k^2 + \beta_k^2}$$

$$\theta_k = \operatorname{atan}\frac{-\beta_k}{\alpha_k}$$

Review Questions (1)

- In the exponential Fourier Series, what does a_0 represent?

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt \qquad \Rightarrow a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

Hence a_0 is the <u>mean value</u> of $x(t)$. It is also called the <u>DC component</u> of the signal $x(t)$.

- What is the advantage of the exponential Fourier Series compared with the trigonometric Fourier Series?
  - It is more compact mathematically
  - The exponential Fourier Series can also be used for complex periodic signals (the trigonometric Fourier Series can be used for real periodic signals only)
  - Note: a_k can in general be a complex number whereas A_k (in the trigonometric Fourier Series) is a real number

Review Questions (2)

- Why are negative frequencies ($k = \cdots, -3, -2, -1$) used in the exponential Fourier Series?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

We have seen that to be able to represent a cosine (or sine) function in the complex plane, we need both negative and positive frequencies, because

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

The Fourier Series – Graphical Plot

Fourier Series
(exponential form)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

a_k is in general a complex number and corresponds to frequency $k\omega_0$

How is the plot of $|a_k|$ versus frequency $k\omega_0$ called?

➔ Amplitude Spectrum (or: Magnitude Spectrum)

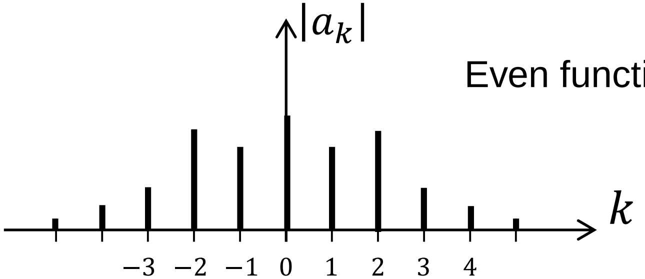
How is the plot of the phase $\angle a_k$ versus frequency $k\omega_0$ called?

➔ Phase Spectrum

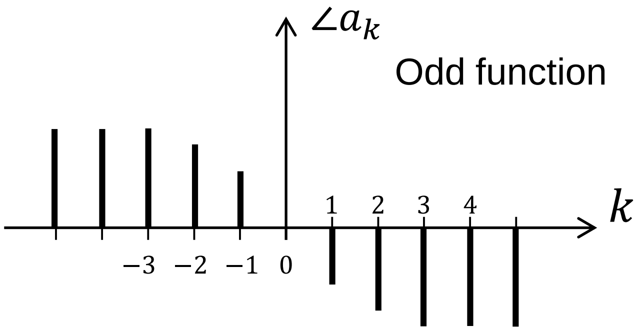
What is the special characteristic of the spectrum of a periodic signal? It is a line spectrum.

For $x(t)$ real:

Magnitude Spectrum



Phase Spectrum



Note: the frequency axis is discrete !

Note on the Computation of the Fourier Series Coefficients

If the signal is a sum of sines and cosines, there is no need to compute the Fourier Series coefficients by using the integral expression

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

because the a_k's can then directly be obtained by replacing the sine and cosine terms with complex exponentials (Euler).

For example $x(t) = 1 + \sin(\omega_0 t) + 3\cos(2\omega_0 t + \frac{\pi}{3})$

can be equivalently written as

$$x(t) = 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + 3\,\frac{e^{j(2\omega_0 t+\pi/3)} + e^{-j(2\omega_0 t+\pi/3)}}{2}$$

and hence we have directly the solution:

$$a_0 = 1 \qquad a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_2 = \frac{3}{2}e^{j\pi/3} \qquad a_{-2} = \frac{3}{2}e^{-j\pi/3}$$

The Fourier Series Computation – Example

Another example (the signal $x(t)$ is a <u>sum of sines and cosines</u>):

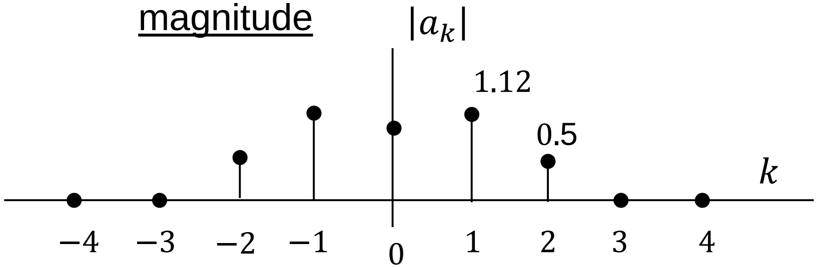
Find the Fourier Series coefficients for the signal $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$.

$$x(t) = 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + 2\,\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{e^{j(2\omega_0 t+\pi/4)} - e^{-j(2\omega_0 t+\pi/4)}}{2}$$

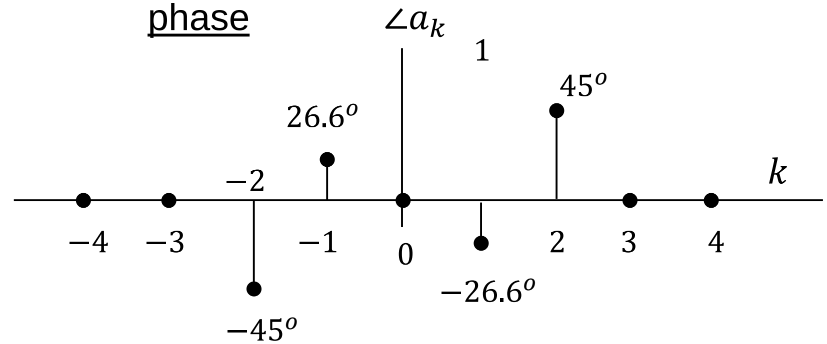
$$= \underbrace{1}_{a_0} + \underbrace{\left(1 + \frac{1}{2j}\right)}_{a_1} e^{j\omega_0 t} + \underbrace{\left(1 - \frac{1}{2j}\right)}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\left(\frac{1}{2} e^{j\pi/4}\right)}_{a_2} e^{j2\omega_0 t} + \underbrace{\left(\frac{1}{2} e^{-j\pi/4}\right)}_{a_{-2}} e^{-j2\omega_0 t}$$

$$a_1 = 1 + \frac{1}{2j} = 1 - \frac{1}{2} j = \sqrt{5/4}\, e^{-j\,\mathrm{atan}\frac{1}{2}} = 1.12 e^{-j26.6^o} \qquad a_{-1} = a_1^* = 1.12 e^{+j26.6^o}$$

<u>magnitude</u>

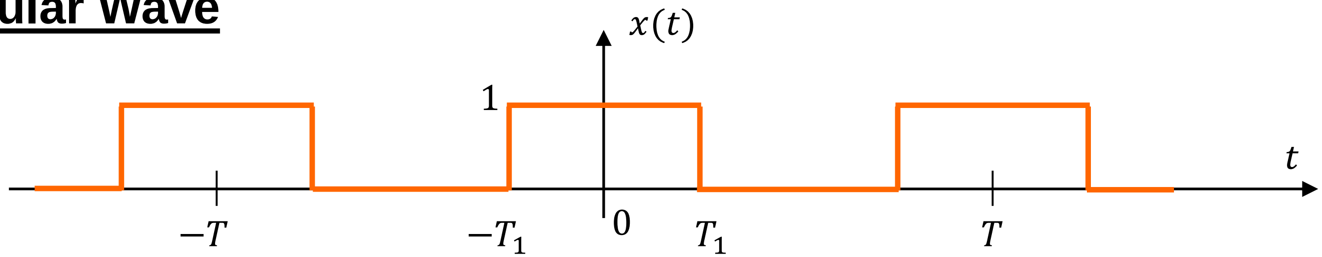


<u>phase</u>



The Fourier Series Computation – Example

Rectangular Wave



For $k = 0$: $$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = 2T_1/T$$

For $k \neq 0$: $$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{1}{jk\omega_0 T}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)$$

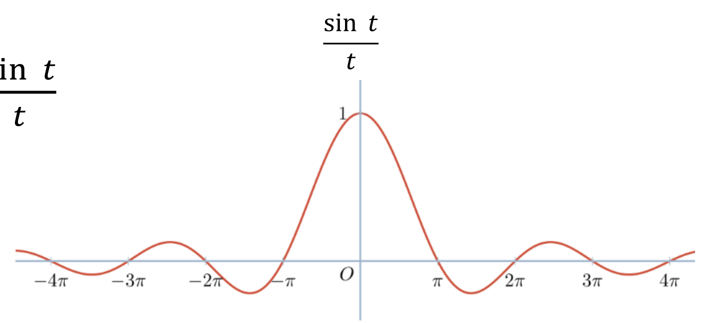
$$= \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right) \qquad \boxed{\Rightarrow \quad a_k = 2\frac{\sin k\omega_0 T_1}{k\omega_0 T}}$$

Note that for $T_1 = \frac{T}{4}$ (square wave):

$$a_0 = 1/2$$

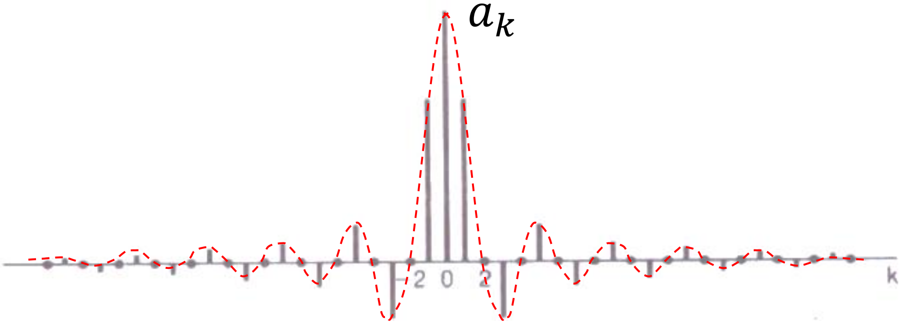
$$a_k = 2\frac{\sin(k\omega_0 T/4)}{k\omega_0 T} \overset{\omega_0 = 2\pi/T}{=} 2\frac{\sin(k\pi/2)}{2\pi k} = \frac{1}{2}\frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{2}k}$$

like $\frac{\sin\ t}{t}$



The Fourier Series Computation – Example

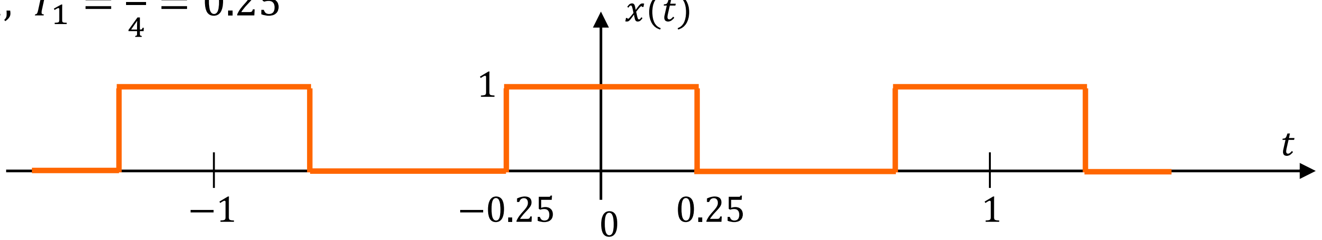
For $T_1 = \frac{T}{4}$ (square wave): $$a_k = \frac{1}{2}\frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{2}k}$$



The Fourier Series – Example

Square Wave

Assume $T = 1,\ T_1 = \frac{T}{4} = 0.25$



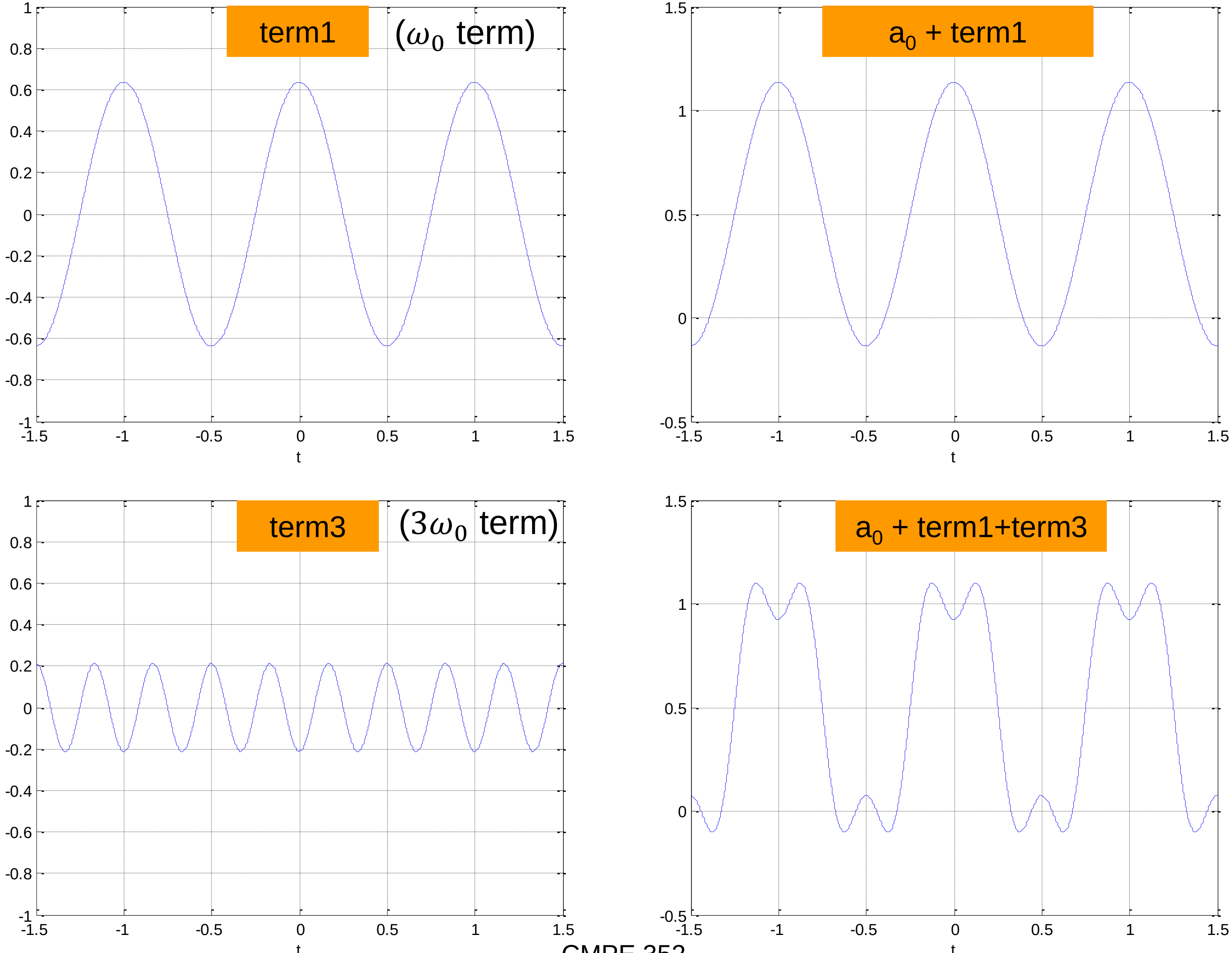
$$a_0 = 1/2$$

$$a_k = \frac{1}{2}\frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{2}k} \quad \Rightarrow \text{ For } k > 0 \text{, even, } a_k = 0$$

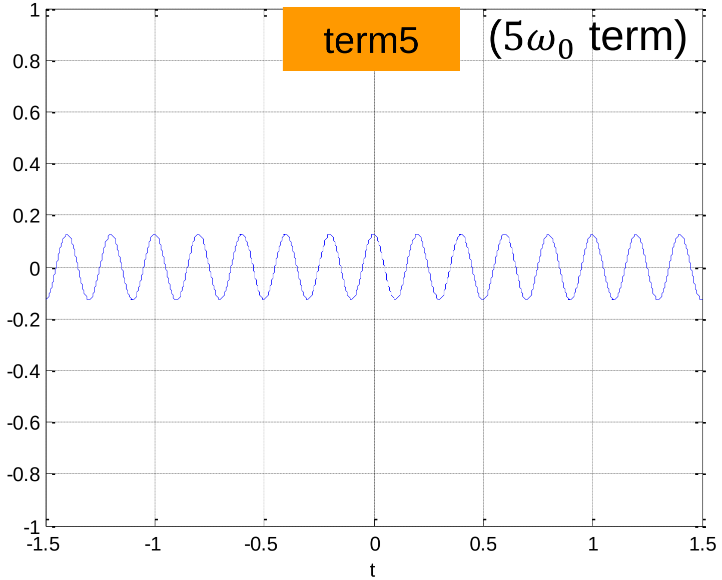
$$\left.\begin{aligned} &\text{For } k \text{ odd, } a_1 = a_{-1} = \frac{1}{\pi} \\ &a_3 = a_{-3} = -\frac{1}{3\pi} \\ &a_5 = a_{-5} = \frac{1}{5\pi} \\ &\ldots \end{aligned}\right\}$$

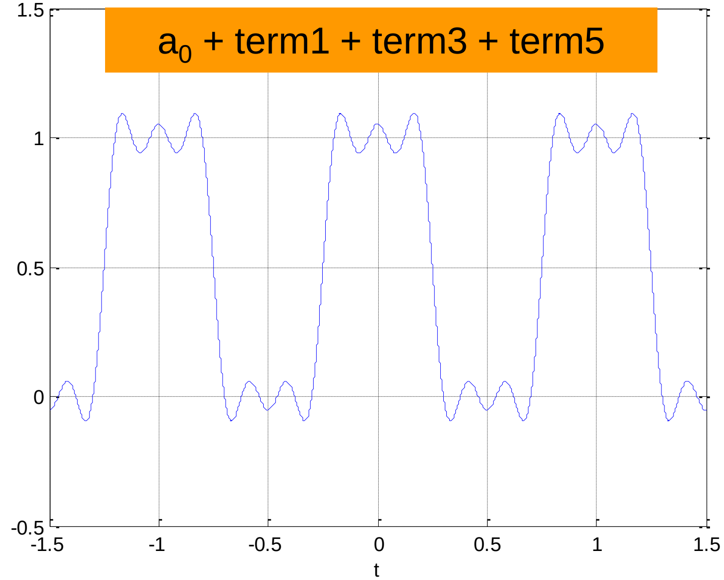
Each one of these terms for $k = \pm 1, \pm 3, \pm 5$, etc. corresponds to a sinusoidal wave. Let us look at the contribution of each term ...

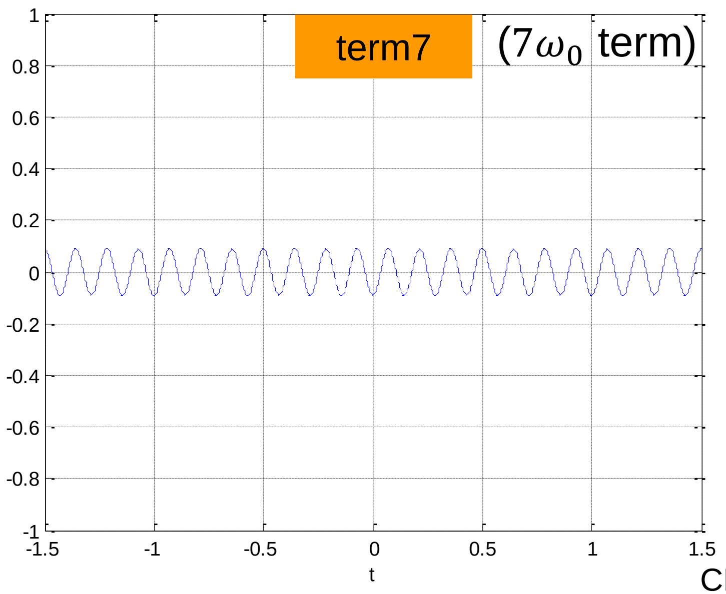
The Fourier Series – Example (cntd)

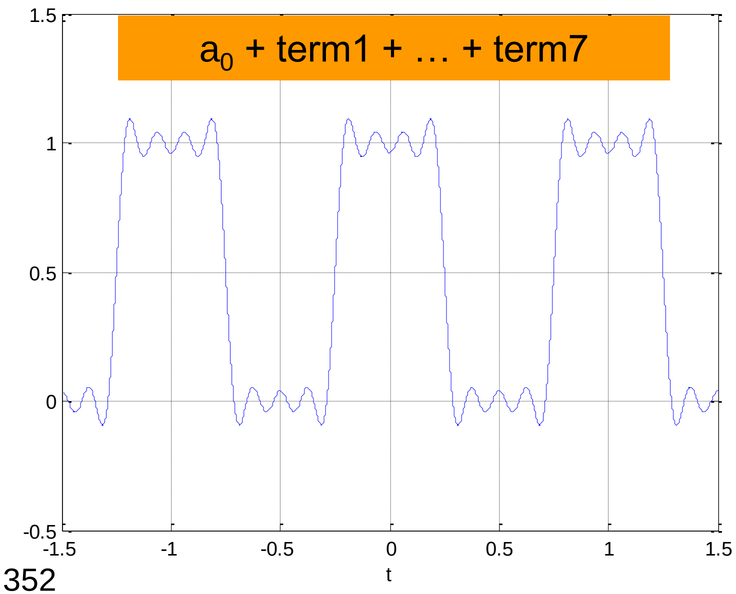


The Fourier Series – Example (cntd)

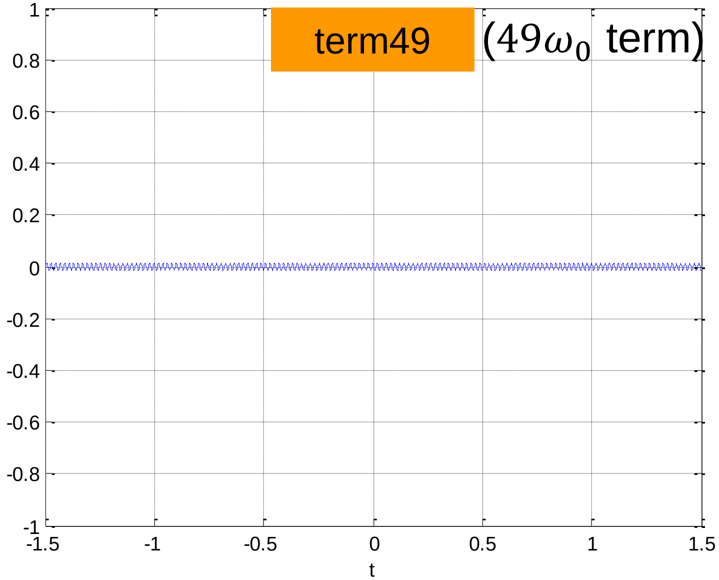


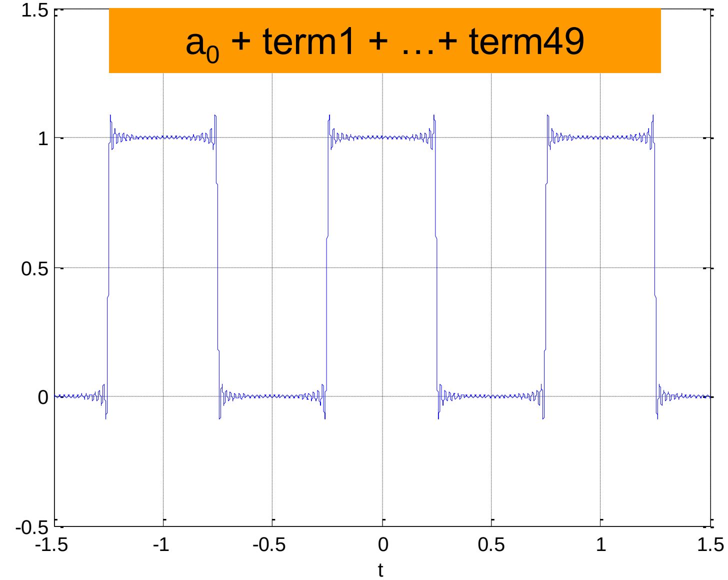


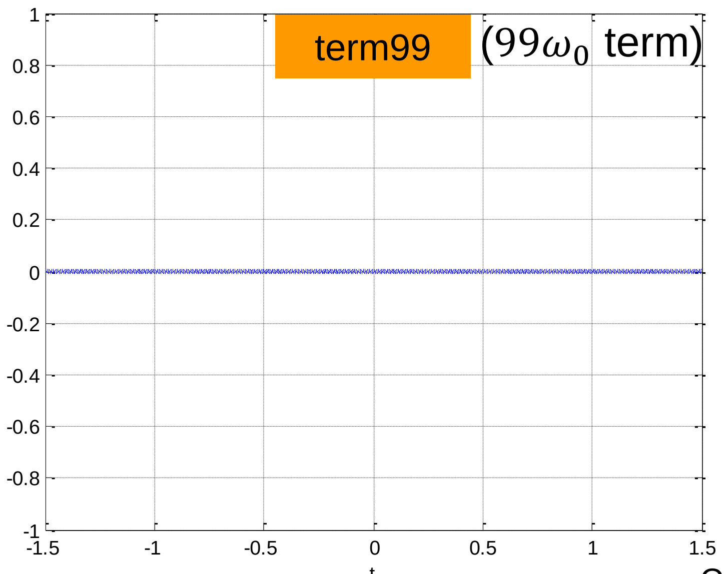


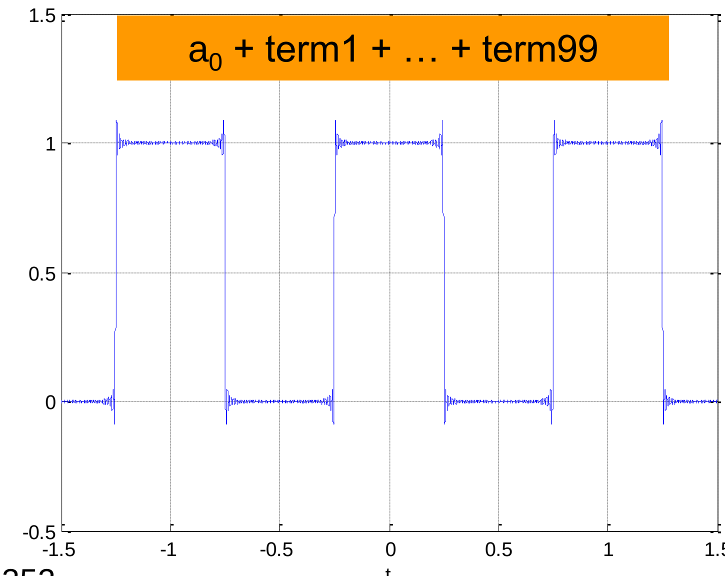


The Fourier Series – Example (cntd)



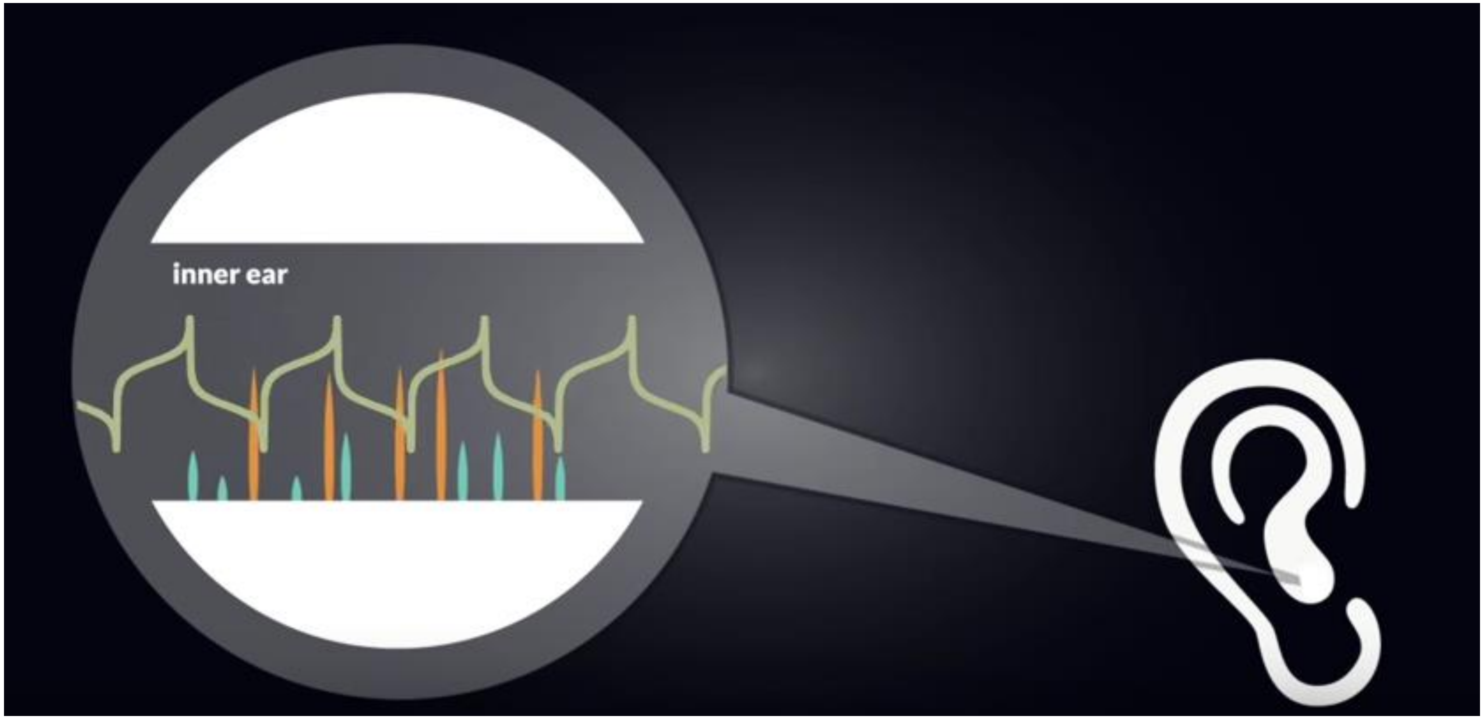






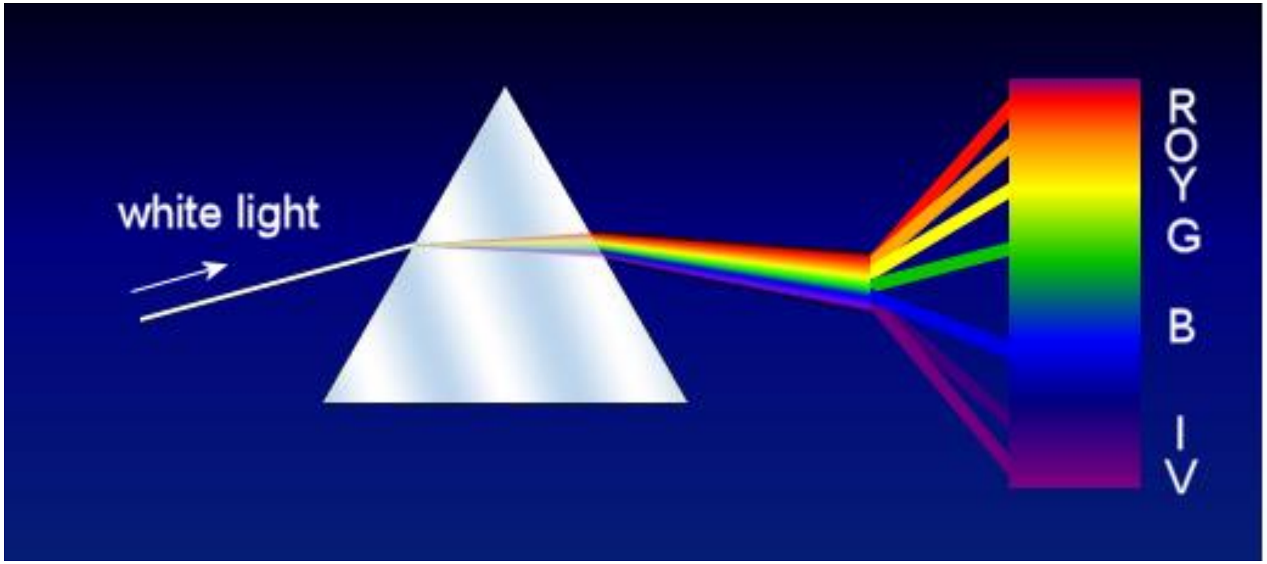
Fourier Decomposition in Nature (1)

The inner ear is a device that naturally performs Fourier Series decomposition



Fourier Decomposition in Nature (2)

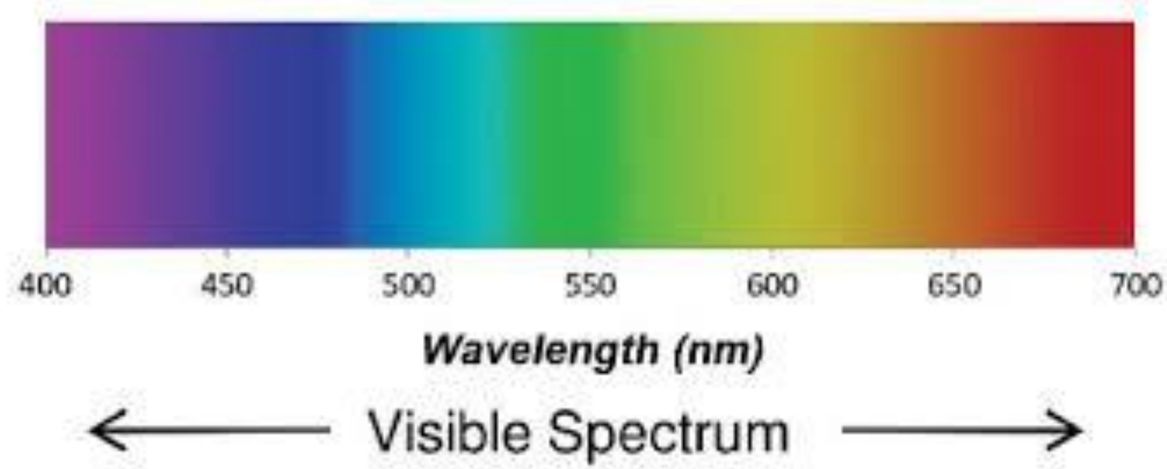
- Is there another device/object that naturally performs spectral decomposition in nature?
  - The prism



White light is split into individual frequencies by a prism

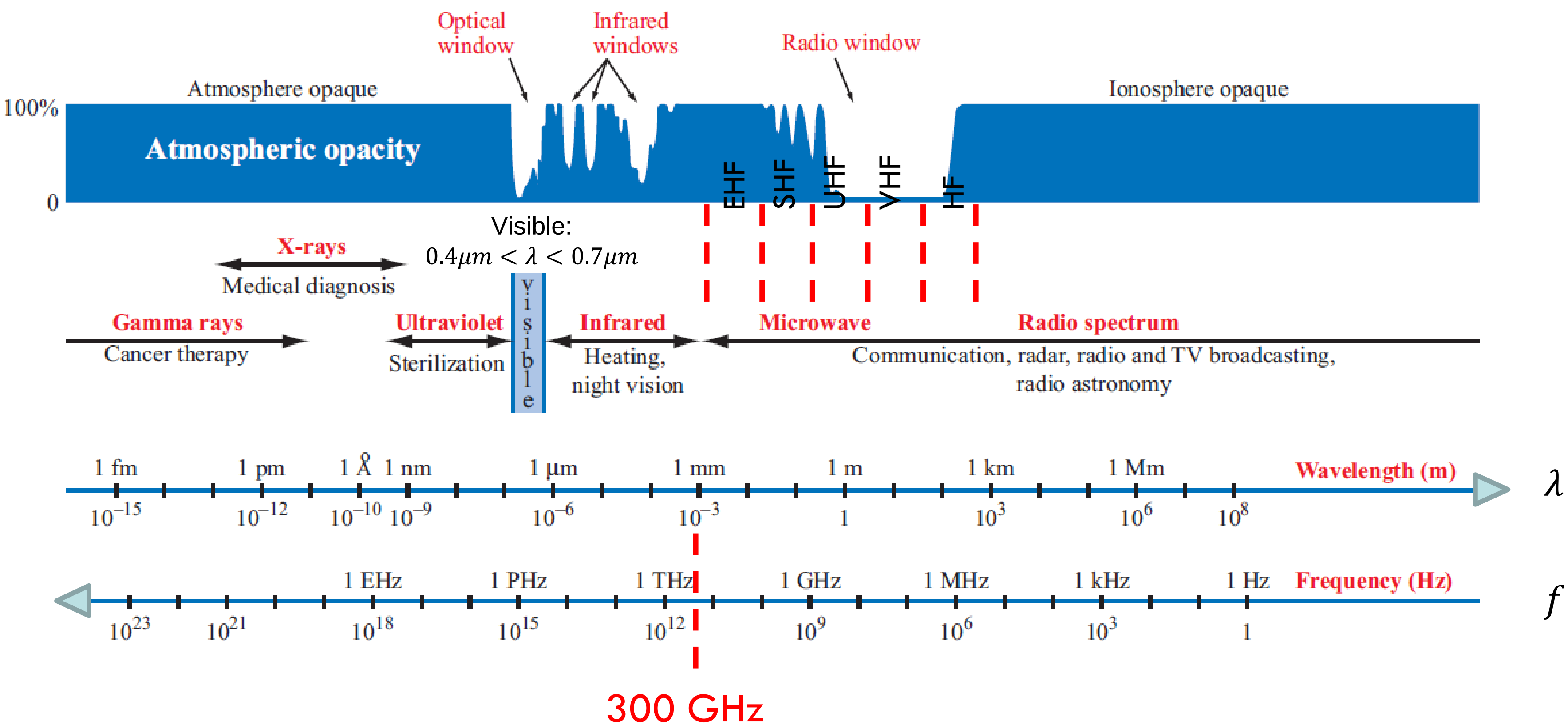
White light consists of all visible frequencies (red, orange, yellow, green, blue, indigo and violet) mixed together and the prism breaks them apart (does a Fourier decomposition) so we can see the separate frequencies

Fourier Decomposition in Nature (3)



The Electromagnetic (EM) Spectrum

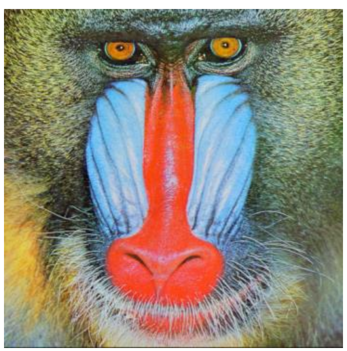
Wavelength in vacuum: $\lambda = c/f$



Applications of the Fourier Series (1)

Embedded sytems

voice



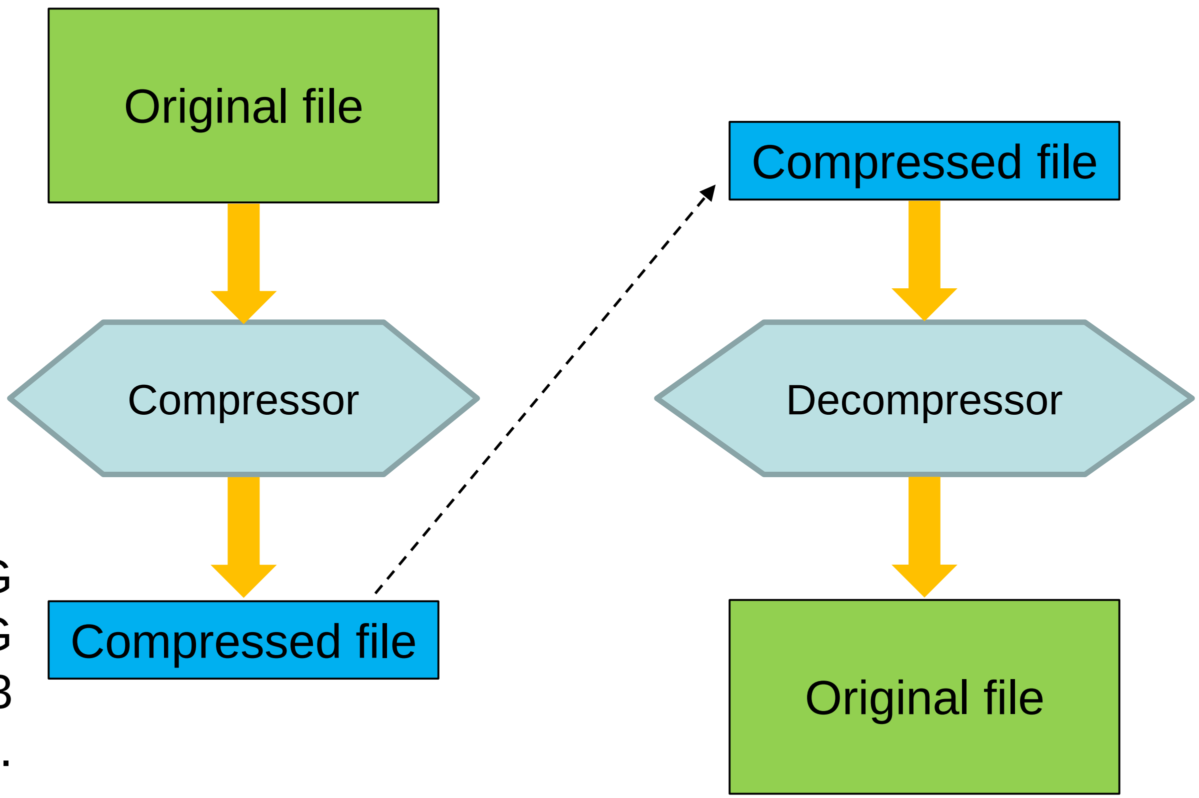
image

Microprocessor
System

Clock frequency

Applications of the Fourier Series (2)

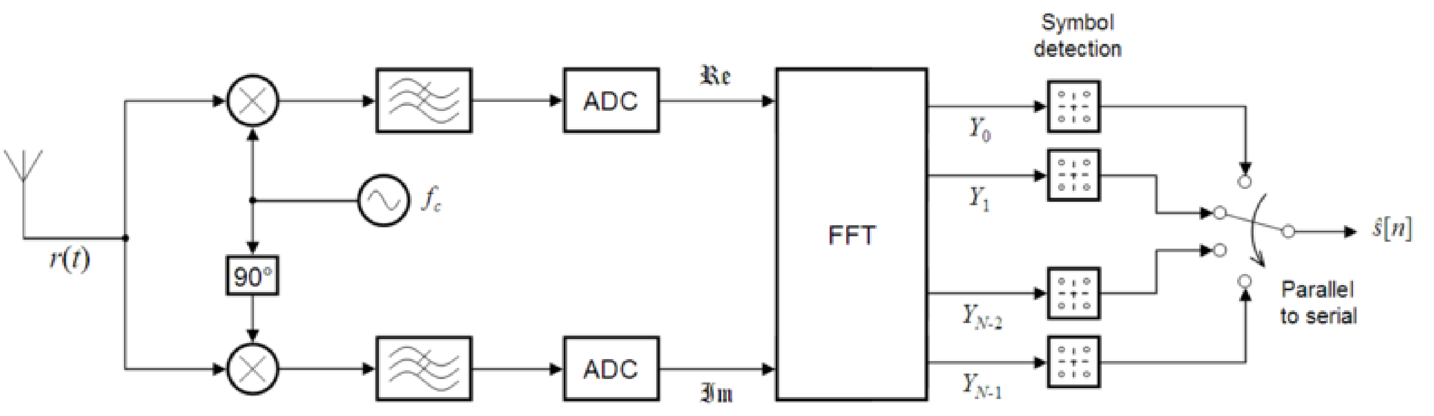
Data Compression

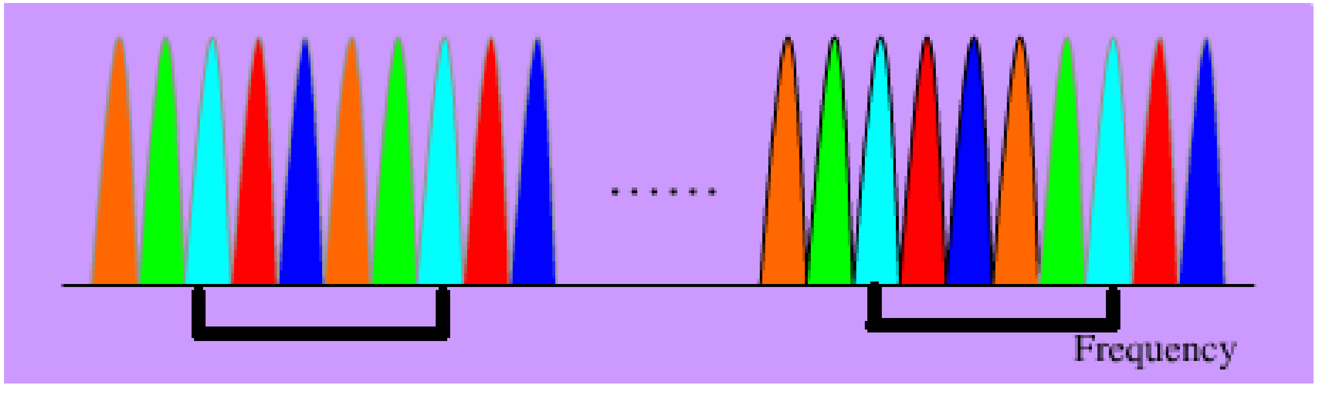


MPEG
JPEG
MP3
...

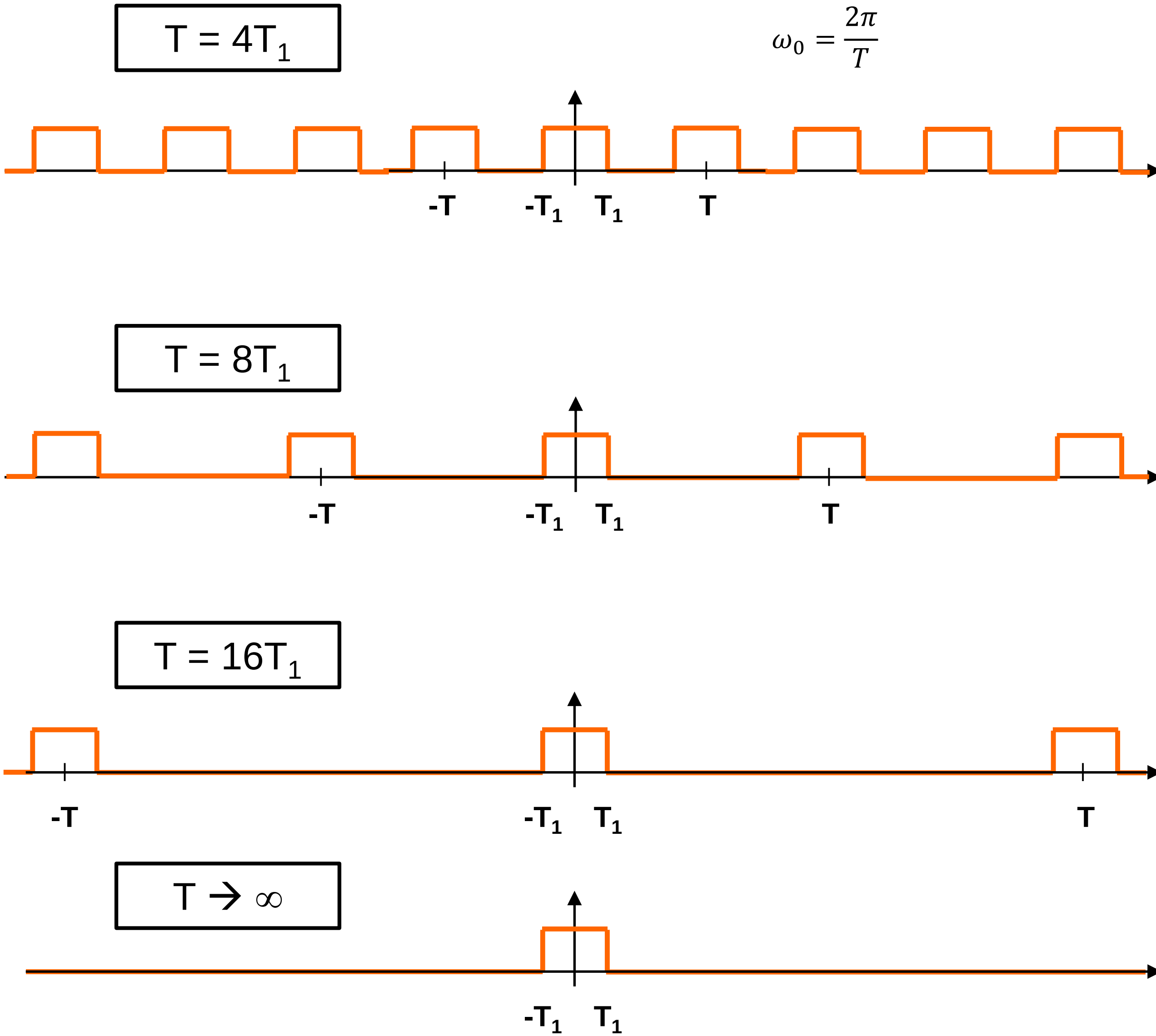
Applications of the Fourier Series (3)

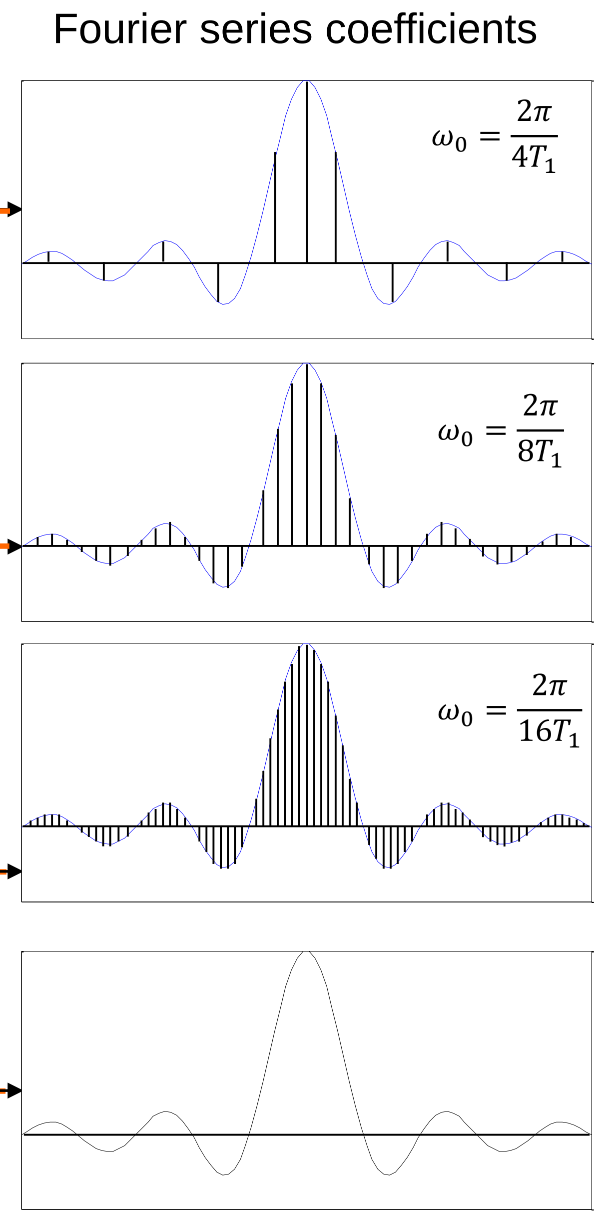
Data Communications (e.g., 3G, 4G, 4.5 G, 5G, ADSL,...)





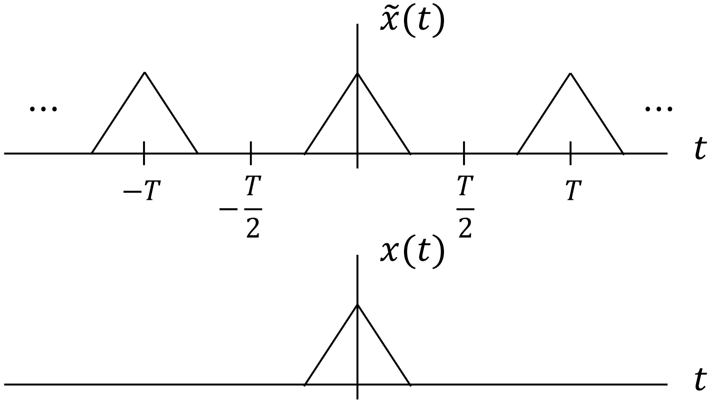
From Periodic to Aperiodic Signals





From Periodic to Aperiodic Signals (2)

Let $\tilde{x}(t)$ be the periodic repetition of $x(t)$ with period T:



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t} dt \overset{\text{def.}}{=} \frac{1}{T} X(jk\omega_0)$$

$$\Rightarrow X(jk\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-jk\omega t} dt$$

$$\Rightarrow \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \frac{2\pi}{T} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \omega_0$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t} \omega_0$$

$T \rightarrow \infty$

$$x(t) \qquad \int \qquad X(jk\omega)\, e^{jk\omega t} \qquad d\omega$$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega}$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt}$$

The Fourier Transform

- Let $x(t)$ be a nonperiodic continuous-time function
- The Fourier transform of $x(t)$ is

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\ dt}$$

- The inverse Fourier transform of $X(j\omega)$ is

$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\ d\omega}$$

- $x(t)$ and $X(j\omega)$ form a **Fourier-transform pair**
- $X(j\omega)$ is called the **spectrum** of $x(t)$
- For given ω, $X(j\omega)$ is in general a complex number
  - $|X(j\omega)|$ is called the ***magnitude spectrum***
  - $\arg(X(j\omega))$ is called the ***phase spectrum***

The Fourier Transform – Examples

Example 1

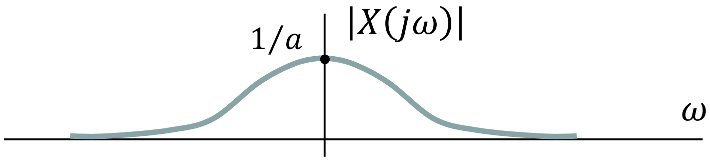
Determine and plot the Fourier transform of the signal $x(t) = e^{-at}u(t),\ a > 0$

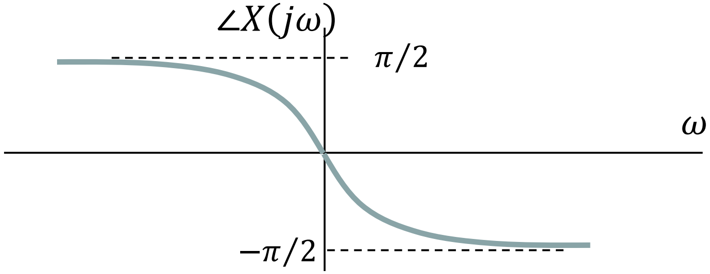
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty} = -\frac{1}{a+j\omega}\,(0-1) = \frac{1}{a+j\omega}$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$\boxed{|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(j\omega) = -\operatorname{atan}\frac{\omega}{a}}$$





The Fourier Transform – Examples

Example 2

Determine and plot the Fourier transform of the signal $x(t) = e^{-a|t|}$, $a > 0$

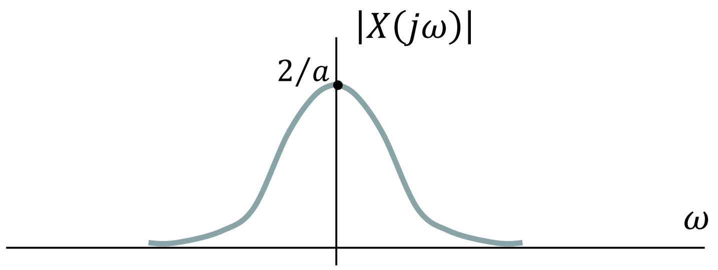
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$|X(j\omega)| = \frac{2a}{a^2+\omega^2}$$

$$\angle X(j\omega) = 0$$



The Fourier Transform – Examples

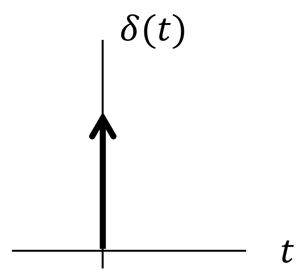
Example 3

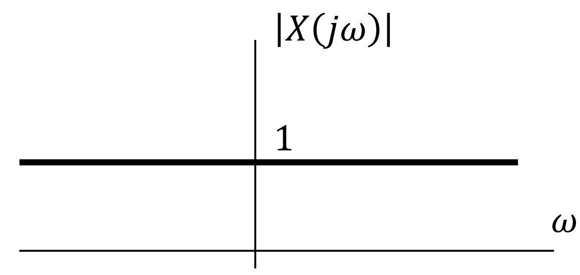
Determine and plot the Fourier transform of the signal $x(t) = \delta(t)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt \overset{\text{def.}}{=} 1$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$\boxed{\begin{aligned} |X(j\omega)| &= 1 \\ \angle X(j\omega) &= 0 \end{aligned}}$$





Makes sense?