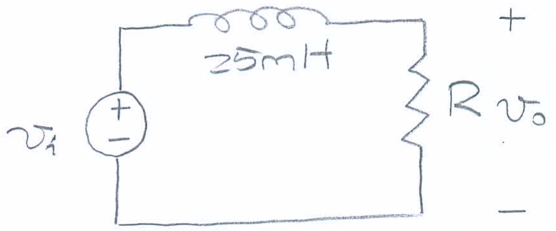
**Problem 1)** Use a 25mH inductor to design a low-pass, RL, passive filter with a cutoff frequency of 2.5kHz.

a. Specify the value of the resistor.

b. A load having a resistance of 750 $\Omega$ is connected across the output terminals of the filter. What is the corner, or cutoff, frequency of the loaded filter in Hertz?

**Solution.** We have



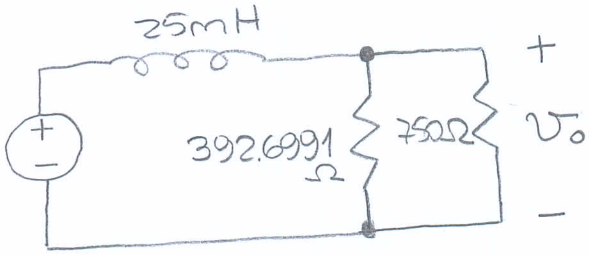

$$\frac{V_o(s)}{V_i(s)} = H(s) = \frac{R}{sL+R} = \frac{R/L}{s+R/L}$$

a.

$$\Rightarrow \quad w_c = \frac{R}{L} \quad \Rightarrow \quad 2\pi\,(2.5.10^3) = \frac{R}{25.10^{-3}}$$

$$\Rightarrow \quad R = 392.6991\ \Omega$$

b.

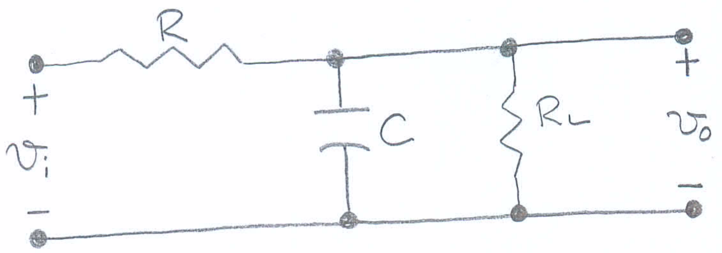



$$R_{eq} = \frac{392.6991 \cdot 750}{392.6991+750} = 257.7444\ \Omega$$

$$w_c = \frac{257.7444}{25.10^{-3}} = 10.3058 \cdot 10^3 \text{ rad/s}$$

$$f_c = \frac{w_c}{2\pi} = \frac{10.3058}{2\pi}10^3 = 1.6403 \text{ kHz} \quad \text{OR} \quad 1640.3 \text{ Hz}$$

**Problem 2)** A loaded low-pass filter circuit is shown as



a. Derive the expression for the voltage transfer function $V_o/V_i$.

b. At what frequency will the magnitude of $H(j\omega)$ be maximum?

c. What is the maximum value of the magnitude of $H(j\omega)$?

d. At what frequency will the magnitude of $H(j\omega)$ equal its maximum value divided by $\sqrt{2}$?

Solution.

a. $$\frac{1}{sC} \,//\, R_L \Rightarrow Z_{eq} = \frac{(1/sC)R_L}{\frac{1}{sC}+R_L} = \frac{R_L}{1+sR_LC}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_L/1+sR_LC}{R+\frac{R_L}{1+sR_LC}} = \frac{R_L}{sRR_LC+R+R_L}$$

$$= \frac{1/RC}{s+\frac{R+R_L}{RR_LC}}$$

b. $$|H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2+\left(\frac{R+R_L}{RR_LC}\right)^2}} = \frac{R_L/R+R_L}{\sqrt{1+\left[\frac{\omega}{(R+R_L)/RR_LC}\right]^2}}$$

$\Rightarrow |H(j\omega)|$ is maximum when $\omega = 0$

c. $$|H(j0)| = \frac{1/RC}{R+R_L/RR_LC} = \frac{R_L}{R+R_L} \triangleq H_{max}$$

d. $$|H(j\omega_c)| = \frac{1}{\sqrt{1+\left[\frac{\omega_c}{(R+R_L)/RR_LC}\right]^2}} H_{max} = \frac{1}{\sqrt{2}} H_{max}$$

$$\Rightarrow \omega_c = \frac{R+R_L}{RR_LC}$$

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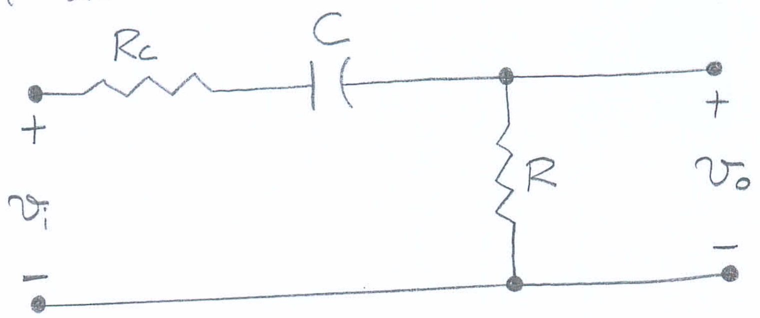
Problem 3) Consider a modified high-pass filter circuit shown as




a. Derive the expression for $H(s)$ where $H(s) = \frac{V_o}{V_i}$.

b. At what frequency will the magnitude of $H(j\omega)$ be maximum?

c. What is the maximum value of the magnitude of $H(j\omega)$?

d. At what frequency will the magnitude of $H(j\omega)$ equal its maximum value divided by $\sqrt{2}$?

Solution. We have

a.

$$H(s) = \frac{V_o}{V_i} = \frac{R}{R_c + \frac{1}{sC} + R} = \frac{sRC}{s(R+R_c)C+1}$$

$$= \frac{R}{R+R_c} \frac{s}{s + \frac{1}{(R+R_c)C}}$$

b.

$$H(j\omega) = \frac{R}{R+R_c} \frac{j\omega}{j\omega + \frac{1}{(R+R_c)C}}$$

$$|H(j\omega)| = \frac{R}{R+R_c} \frac{\omega}{\sqrt{\omega^2 + \frac{1}{(R+R_c)^2 C^2}}} = \frac{R}{R+R_c} \frac{1}{\sqrt{1 + \frac{1/(R+R_c)^2 C^2}{\omega^2}}}$$

when $\omega \to \infty$, $|H(j\omega)| \to H_{max}$

$$H_{max} = \lim_{\omega\to\infty} |H(j\omega)| = \lim_{\omega\to\infty} \frac{R}{R+R_c} \frac{1}{\sqrt{1+\frac{1/(R+R_c)^2C^2}{\omega^2}}}$$

$$= \frac{R}{R+R_c}$$

1. We have

$$|H(j\omega)| = \underbrace{\frac{R}{R+R_c}}_{H_{max}} \frac{1}{\sqrt{1+\frac{1/(R+R_c)^2C^2}{\omega_c^2}}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\frac{R}{R+R_c}}_{H_{max}}$$

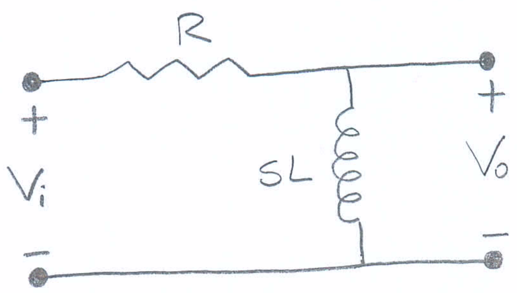
$$\Rightarrow \frac{1/(R+R_c)^2C^2}{\omega_c^2} = 1 \Rightarrow \omega_c = \frac{1}{(R+R_c)C}$$

Problem 4) Using a 25mH inductor, design a high-pass, RL, passive filter with a cutoff frequency of 160 krad/s.

a. Specify the value of the resistance.

b. Assume the filter is connected to a pure resistive load. The cutoff frequency is not to drop below 150krad/s. What is the smallest load resistor that can be connected across the output terminals of the filter?

Solution. We have



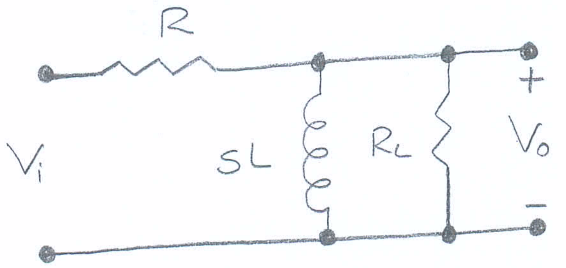

$$H(s) = \frac{V_o}{V_i} = \frac{sL}{sL+R}$$

$$= \frac{s}{s+(R/L)}$$

a.

$$\frac{R}{L} = \frac{R}{25.10^{-3}} = 160.10^{3} \Rightarrow R = 4k\Omega$$

b.




$$H(s) = \frac{V_o}{V_i} = \frac{\dfrac{(sL)R_L}{sL+R_L}}{R+\dfrac{(sL)R_L}{sL+R_L}} = \frac{sR_LL}{s(R+R_L)L+RR_L}$$

$$= \frac{R_L}{R+R_L}\,\frac{s}{s+\dfrac{RR_L}{(R+R_L)L}}$$

$$H(j\omega) = \frac{R_L}{R+R_L}\,\frac{j\omega}{j\omega+\dfrac{RR_L}{(R+R_L)L}}$$

$$|H(j\omega)| = \frac{R_L}{R+R_L}\,\frac{\omega}{\sqrt{\omega^2+\left[\dfrac{RR_L}{(R+R_L)L}\right]^2}}$$

$$= \frac{R_L}{R+R_L}\,\frac{1}{\sqrt{1+\left[\dfrac{RR_L}{\omega(R+R_L)L}\right]^2}}$$

Hence;

$$H_{max} = \frac{R_L}{R+R_L} \quad \text{and} \quad \omega_c = \frac{RR_L}{(R+R_L)L}$$

$$\frac{4\cdot \cancel{10^3}\cdot R_L}{(4\cdot 10^3+R_L)\cdot 25\cdot 10^{-3}} \geq 150\cdot \cancel{10^3}$$

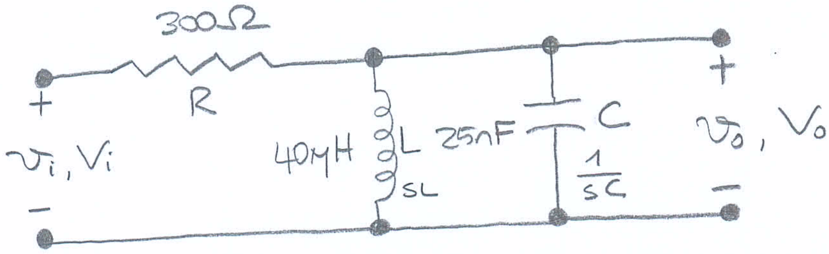
$$\Rightarrow \quad 4R_L \geq 15\cdot 10^3 + 3.75\,R_L$$

$$\Rightarrow \quad 0.25\,R_L \geq 15\cdot 10^3 \quad \Rightarrow \quad R_L \geq 60\,k\Omega$$

Thus;

$$[R_L]_{min} = 60\,k\Omega$$

Problem 5) Consider the following bandpass filter shown as




Find $\omega_o$, $Q$, $\omega_{c1}$, $\omega_{c2}$, and $\beta$.

Solution. We have

$$H(s) = \frac{V_o}{V_i} = \frac{\dfrac{sL(1/sC)}{sL+(1/sC)}}{R+\dfrac{sL(1/sC)}{sL+(1/sC)}} = \frac{L/C}{R\left(sL+\frac{1}{sC}\right)+\frac{L}{C}}$$

$$= \frac{sL}{s^2RLC+R+sL} = \frac{s(1/RC)}{s^2+s(1/RC)+(1/LC)} \triangleq \frac{\beta s}{s^2+\beta s+\omega_o^2}$$

Therefore;

$$\omega_o^2 = \frac{1}{LC} = \frac{1}{40\cdot 10^{-6}\cdot 25\cdot 10^{-9}} = 10^{12} \quad \Rightarrow \quad \omega_o = 10^6 \text{ rad/s}$$

$$\beta = \frac{1}{RC} = \frac{1}{300 \cdot 25 \cdot 10^{-9}} = 133.3333 \text{ krad/s}$$

$$Q = \frac{\omega_0}{\beta} = \frac{10^6}{133.3333 \cdot 10^3} = 7.5$$

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

$$= 935.55 \text{ krad/s}$$

$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

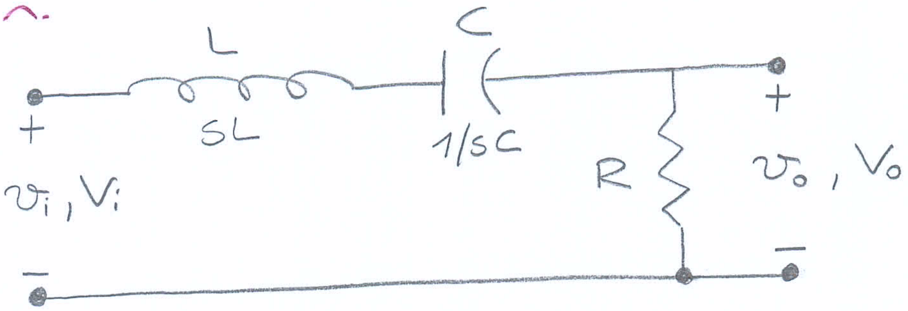
$$= 1068.5 \text{ krad/s}$$

Problem 6) Design a series RLC bandpass filter with a quality factor of 5 and a center frequency of 20 krad/s using a 0.05 μF capacitor

a. Draw your circuit by labeling the component values and output voltage.

b. Calculate $\beta$, $\omega_{c1}$ and $\omega_{c2}$.

Solution.




$$Q = \frac{\omega_0}{\beta} \Rightarrow 5 = \frac{20 \cdot 10^3}{\beta} \Rightarrow \beta = 4 \cdot 10^3 \text{ rad/s}$$

$$H(s) = \frac{V_o}{V_i} = \frac{R}{sL + \frac{1}{sC} + R} = \frac{sRC}{s^2LC + 1 + sRC} = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)}$$

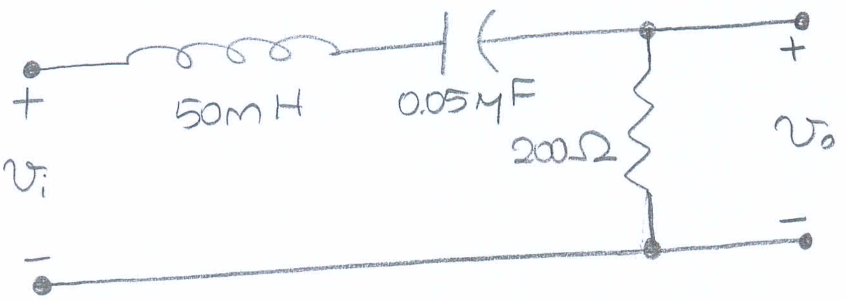
$$\triangleq \frac{\beta s}{s^2+\beta s+\omega_0^2}$$

$$\Rightarrow \quad \omega_0^2 = \frac{1}{LC} \Rightarrow (20\cdot 10^3)^2 = \frac{1}{L\cdot 0.05\, 10^{-6}}$$

$$\Rightarrow \quad L = \frac{1}{0.05\cdot 400} = 50\,mH$$

and

$$\beta = \frac{R}{L} \Rightarrow 4\cdot 10^3 = \frac{R}{50\cdot 10^{-3}} \Rightarrow R = 200\,\Omega$$




b. $\beta = 4\cdot 10^3$

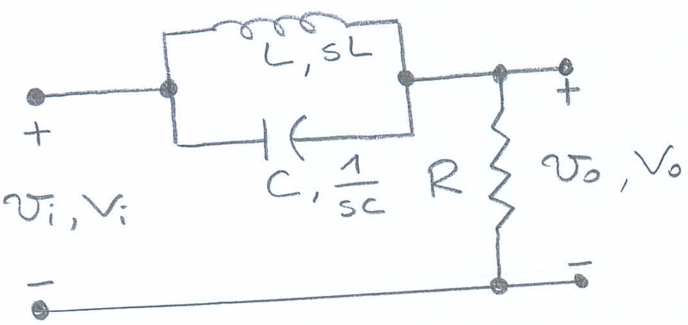
$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

$$\cong 18.1\, krad/s$$

$$\omega_{c2} \cong \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

$$= 22.1\, krad/s$$

Problem 7) Consider the following RLC circuit



a. Show that the circuit is a bandreject filter via a qualitative analysis.

b. Support the qualitative analysis by finding the voltage transfer function of the filter.

c. Derive an expression for the center frequency $\omega_o$.

d. Derive the expressions for $\omega_{c1}$ and $\omega_{c2}$.

e. What is the expression for the bandwidth of the filter?

f. What is the expression for the quality factor of the filter?

Solution.

a. when $\omega = 0$, inductor is short-circuit and capacitor is open-circuit

↳ that is, the output voltage is same as the input voltage both in magnitude and phase

When $\omega \to \infty$, inductor is open-circuit and capacitor is short-circuit

↳ therefore, the output voltage is again same as the input voltage both in magnitude and phase

For $0 < \omega < \infty$, there will be some voltage drop across L and C

↳ thus reducing the output voltage from the input voltage

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- At the resonant frequency of the parallel combination of L and C, the equivalent impedance is infinite, hence

↳ the output voltage is zero when $\omega = \omega_0$

- At frequencies lower or greater than $\omega_0$, the output voltage will be less than the input voltage

∴ The circuit behaves like a band reject filter.

b. We have

$$H(s) = \frac{V_o}{V_i} = \frac{R}{\dfrac{sL\left(\frac{1}{sC}\right)}{sL + \frac{1}{sC}} + R} = \frac{R}{\dfrac{sL}{s^2LC+1} + R}$$

$$= \frac{R(s^2LC+1)}{s^2LCR + R + sL} = \frac{s^2 + (1/LC)}{s^2 + (1/RC)s + (1/LC)}$$

$$\triangleq \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$ Band-reject filter characteristics

c. It follows that

$$H(j\omega) = \frac{-\omega^2 + \omega_0^2}{-\omega^2 + j\beta\omega + \omega_0} = 0$$ when $\omega = \omega_0$, $\omega_0^2 = \frac{1}{LC}$, $\omega_0 = \sqrt{\frac{1}{LC}}$

d. $$H(j\omega) = \frac{-\omega^2 + \omega_0^2}{-\omega^2 + j\beta\omega + \omega_0^2}, \quad |H(j\omega)| = \frac{\omega_0^2 - \omega^2}{\sqrt{\beta^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

$$\Rightarrow \quad |H(j\omega)| = \frac{1}{\sqrt{1+\dfrac{\beta^2\omega^2}{(\omega_0^2-\omega^2)^2}}}$$

$$|H(j\omega_c)| = \frac{1}{\sqrt{1+\underbrace{\dfrac{\beta^2\omega_c^2}{(\omega_0^2-\omega_c^2)^2}}_{1}}} = \frac{1}{\sqrt{2}}\cdot 1$$

$$\Rightarrow \quad \frac{\beta^2\omega_c^2}{(\omega_0^2-\omega_c^2)^2} = 1 \quad \Rightarrow \quad \frac{\beta\omega_c}{\omega_0^2-\omega_c^2} = \pm 1$$

$$\Rightarrow \quad \mp\omega_c^2 - \beta\omega_c \pm \omega_0^2 = 0 \quad \text{OR} \quad \omega_c^2 \pm \beta\omega_c \mp \omega_0^2 = 0$$

Hence;

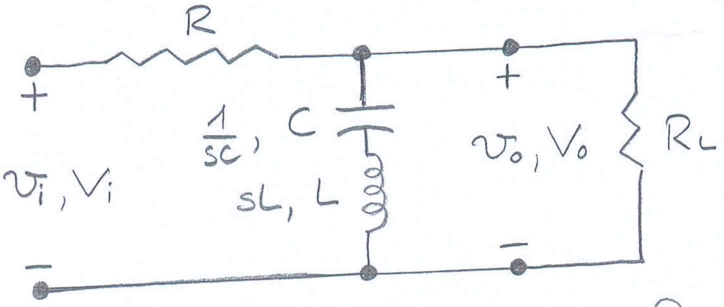
$$\omega_{c1} = \frac{-\beta+\sqrt{\beta^2+4\omega_0^2}}{2} = -\frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2+\omega_0^2}$$

$$\omega_{c2} = \frac{\beta+\sqrt{\beta^2+4\omega_0^2}}{2} = \frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2+\omega_0^2}$$

e. $\omega_{c2}-\omega_{c1} = \beta = \dfrac{1}{RC}$

f. $Q = \dfrac{\omega_0}{\beta} = \dfrac{\sqrt{1/LC}}{1/RC} = \sqrt{\dfrac{R^2C^{\cancel{2}}}{L\cancel{C}}} = R\sqrt{\dfrac{C}{L}}$

Problem 8) Consider a loaded bandreject filter circuit shown as




a. Find the voltage transfer function $\frac{V_o}{V_i}$.
b. What is the expression for the center frequency?
c. What is the expression for the bandwidth?
d. What is the expression for the quality factor?
e. Evaluate $H(j\omega_o)$, $H(j0)$, $H(j\infty)$.
f. What are the expressions for the corner frequencies, $\omega_{c1}$ and $\omega_{c2}$?

Solution. We have

$$H(s) = \frac{V_o}{V_i} = \frac{Z_{eq}}{R + Z_{eq}} \quad \text{where} \quad Z_{eq} = \frac{\left(\frac{1}{sC} + sL\right) R_L}{\frac{1}{sC} + sL + R_L}$$

$$\Rightarrow \quad Z_{eq} = \frac{(1 + s^2 LC) R_L}{1 + s^2 LC + sR_L C}$$

Therefore;

$$H(s) = \frac{R_L + s^2 R_L LC \,/\, 1 + s^2 LC + sR_L C}{R + \dfrac{R_L + s^2 R_L LC}{1 + s^2 LC + sR_L C}}$$

$$= \frac{R_L + s^2 R_L LC}{R + R_L + s^2 (R + R_L) LC + sRR_L C}$$

$$= \frac{R_L \cancel{LC}}{(R+R_L)\cancel{LC}} \frac{s^2 + (1/LC)}{s^2 + \frac{RR_L}{(R+R_L)} s + \frac{1}{LC}} \triangleq \frac{R_L}{R + R_L} \frac{s^2 + \omega_o^2}{s^2 + \beta s + \omega_o^2}$$

Bandreject filter characteristics

where

$$\omega_0^2 = \frac{1}{LC}, \quad \beta = \frac{RR_L}{(R+R_L)L}$$

b.

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

c.

$$\beta = \frac{RR_L}{(R+R_L)L}$$

d.

$$Q = \frac{\omega_0}{\beta} = \frac{\sqrt{1/LC}}{\frac{RR_L}{(R+R_L)L}} = \frac{R+R_L}{RR_L}\sqrt{\frac{L}{C}}$$

e.

$$H(j\omega_0) = \frac{R_L}{R+R_L}\,\frac{-\omega_0^2+\omega_0^2}{-\omega_0^2+j\beta\omega_0+\omega_0^2} = 0$$

$$H(j0) = \frac{R_L}{R+R_L}\,\frac{0+\omega_0^2}{0+0+\omega_0^2} = \frac{R_L}{R+R_L}$$

$$H(j\infty) = \lim_{\omega\to\infty}\frac{-\omega^2+\omega_0^2}{-\omega^2+j\beta\omega+\omega_0^2}\left(\frac{R_L}{R+R_L}\right)$$

$$= \frac{R_L}{R+R_L}\lim_{\omega\to\infty}\frac{-1+\left(\frac{\omega_0}{\omega}\right)^2}{-1+j\beta\frac{1}{\omega}+\left(\frac{\omega_0}{\omega}\right)^2}$$

$$= \frac{R_L}{R+R_L}\,\frac{-1}{-1}$$

$$= \frac{R_L}{R+R_L}$$

$$|H(j\omega_c)| = \frac{R_L}{R+R_L}\,\frac{-\omega_c^2+\omega_0^2}{\sqrt{(\omega_0^2-\omega_c^2)^2+\beta^2\omega_c^2}} = \frac{R_L}{R+R_L}\,\frac{1}{\sqrt{1+\frac{\beta^2\omega_c^2}{(\omega_0^2-\omega_c^2)^2}}}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{R_L}{R+R_L}$$

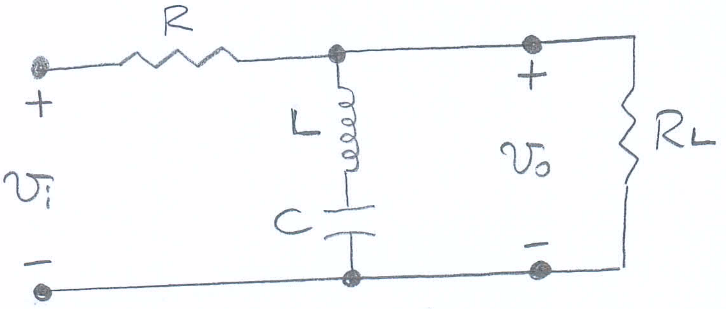
$$\Rightarrow \frac{\beta^2\omega_c^2}{(\omega_0^2-\omega_c^2)^2} = 1 \quad \Rightarrow \quad \frac{\beta\omega_c}{\omega_0^2-\omega_c^2} = \pm 1$$

yielding two positive roots as

$$\omega_{c1} = -\frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2+\omega_0^2}$$

$$\omega_{c2} = \frac{\beta}{2}+\sqrt{\left(\frac{\beta}{2}\right)^2+\omega_0^2}$$

Problem 9) Reconsider the loaded bandreject filter




$R_L = 36\,k\Omega$

$\omega_0 = 1\,M\,rad/s$

$C = 400\,pF$

At very low and very high frequencies, the amplitude of the sinusoidal output voltage should be at least 96% of the amplitude of the sinusoidal input voltage.

a. Specify the numerical values of R and L.

b. What is the quality factor of the circuit?

Solution. We have

$$|H(j0)| = |H(j\infty)| = \frac{36\cdot 10^3}{R+36\cdot 10^3} = \frac{96}{100}$$

$$\Rightarrow \quad 0{,}96R + 0{,}96 \cdot 36 \cdot 10^3 = 36 \cdot 10^3$$

$$\Rightarrow \quad \cancel{0{,}96}R = \cancel{0{,}04} \cdot \overset{3}{\cancel{36}} \cdot 10^3$$

$$\cancel{24}$$
$$2$$

$$\Rightarrow \quad R = 1{,}5\ k\Omega$$

$$10^6 = \sqrt{\frac{1}{L \cdot 400 \cdot 10^{-12}}} \quad \Rightarrow \quad 10^{12} = \frac{1}{L \cdot 400 \cdot 10^{-12}}$$

$$\Rightarrow \quad L = \frac{1}{400} = 2{,}5\ mH$$

b. Referring to Problem 8, we have

$$\beta = \frac{RR_L}{(R+R_L)L} = \frac{\cancel{1{,}5k} \cdot 36k}{\underbrace{(\cancel{1{,}5k+36k})}_{25} \cdot 2{,}5 \cdot 10^{-3}} = 576k\ rad/s$$

$$Q = \frac{\omega_0}{\beta} = \frac{10^6}{576 \cdot 10^3} = 1{,}7361$$