Mathematical Models

Modeling in the Time Domain

(Nise's Textbook Ch. 3)

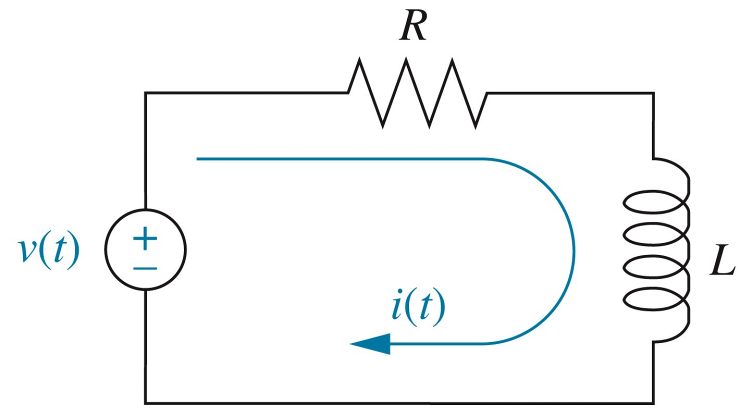
Modeling in the Time Domain

- Two approaches are available for the analysis and design of feedback control systems.
- The first is known as the **classical** or **frequency-domain technique** which is based on converting a system's differential equation to a **transfer function**, thus generating a mathematical model of the system that algebraically relates a representation of the output to a representation of the input.
- The second is known as the **state-space** approach (also referred to as the **modern**, or **time-domain**, approach) which is a unified method for modeling, analyzing, and designing a wide range of systems.
- The state-space approach can be used to represent nonlinear systems that have backlash, saturation, and dead zone.
- Also, it can handle systems with *nonzero initial conditions*. *Time-varying systems* (for example, missiles with varying fuel levels or lift in an aircraft flying through a wide range of altitudes) can be represented in state space.

Modeling in the Time Domain…

- Multiple-input, multiple-output systems (MIMO systems, such as a vehicle with input direction and input velocity yielding an output direction and an output velocity) can be compactly represented in state space with a model similar in form and complexity to that used for single-input, single-output systems.
- The time-domain approach can be used to represent systems with a digital computer in the loop or to model systems for digital simulation. With a simulated system, system response can be obtained for changes in system parameters - an important design tool.
- The time-domain approach can also be used for the same class of systems modeled by the classical approach. This alternate model gives the control systems designer another perspective from which to create a design.
- The designer has to engage in several calculations before the physical interpretation of the model is apparent, whereas in classical control a few quick calculations or a graphic presentation of data rapidly yields the physical interpretation.

Modeling in the Time Domain



Consider the RL network shown in with an initial current of $i(0)$.

1. We select the current, i(t), for which we will write and solve a differential equation using Laplace transforms. We write the loop equation (3.1),

$$L\frac{di}{dt} + Ri = v(t)$$

2. Taking the Laplace transform, using Table 2.2, Item 7, and including the initial conditions, yields

$$L[sI(s) - i(0)] + RI(s) = V(s)$$

3. Assuming the input, $v(t)$, to be a unit step, u(t), whose Laplace transform is, $1/s$, we solve for $I(s)$ and get

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0)}{s+\frac{R}{L}} \qquad i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t}$$

Modeling in the Time Domain

The function $i(t)$ is a subset of all possible network variables that we are able to find from Eq. (3.4) if we know its initial condition, $i(0)$, and the input, $v(t)$. Thus, $i(t)$ is a state variable, and the differential equation (3.1) is a state equation.

4. We can now solve for all of the other network variables algebraically in terms of $i(t)$ and the applied voltage, $v(t)$.

   For example, the voltage across the resistor is (3.5), $v_R(t) = Ri(t)$

   The voltage across the inductor is (3.6) $v_L(t) = v(t) - Ri(t)$

   The derivative of the current is (3.7) $\frac{di}{dt} = \frac{1}{L}[v(t) - Ri(t)]$

   Thus, knowing the state variable, $i(t)$, and the input, $i(t)$, we can find the value, or state, of any network variable at any time, $t \geq t_0$. Hence, the algebraic equations, Eqs. (3.5) through (3.7), are the *output equations*.

5. Since the variables of interest are completely described by Eq. (3.1) and Eqs. (3.5) through (3.7), we say that the combined state equation (3.1) and the output equations (3.5 through 3.7) form a viable representation of the network, which we call a state-space representation.

Modeling in the Time Domain

6. Equation (3.1), which describes the dynamics of the network, is not unique. This equation could be written in terms of any other network variable. For example, substituting $i = v_R/R$ into Eq. (3.1) yields (3.8)

$$\frac{L}{R}\frac{dv_R}{dt} + v_R = v(t)$$

which can be solved knowing that the initial condition $v_R(0) = Ri(0)$ and knowing $v(t)$. In this case, the state variable is $v_R(t)$. Similarly, all other network variables can now be written in terms of the state variable, $v_R(t)$, and the input, $v(t)$.

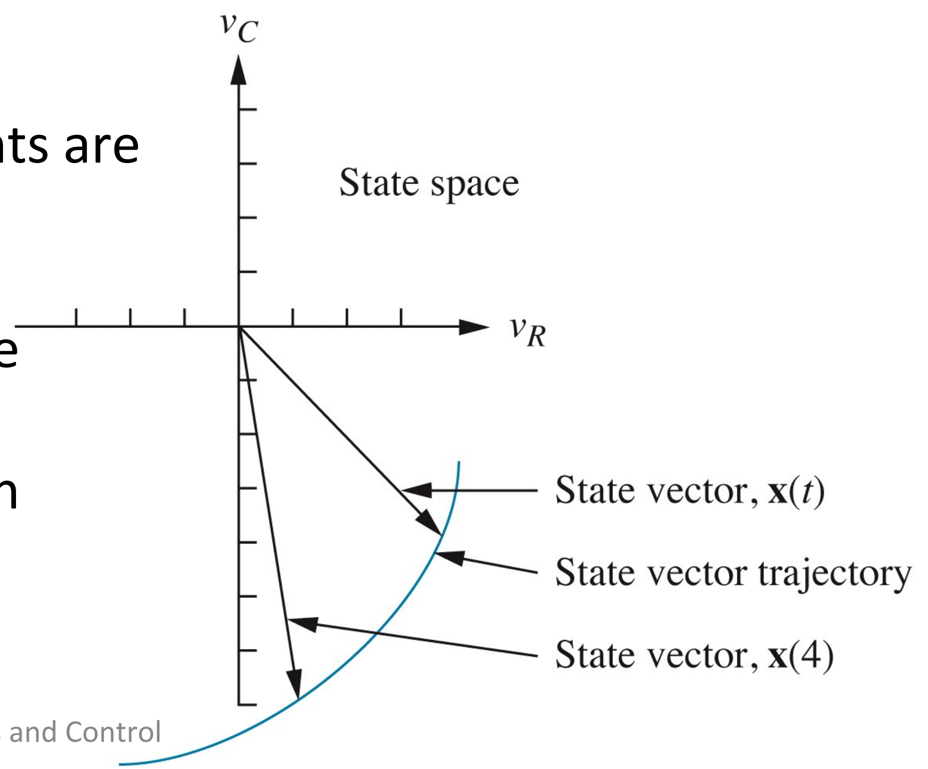
The General State-Space Representation

System variable: Any variable that responds to an input or initial conditions in a system.

State variables: The smallest set of linearly independent system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$.

State vector: A vector whose elements are the state variables.

State space: The n-dimensional space whose axes are the state variables. This is a new term and is illustrated in the figure, where



The General State-Space Representation…

where the state variables are assumed to be a resistor voltage, V_R, and a capacitor voltage, V_C. These variables form the axes of the state space. A trajectory can be thought of as being mapped out by the state vector, $\mathbf{x}(t)$, for a range of t. Also shown is the state vector at the particular time t =4.

State equations: A set of n simultaneous, first-order differential equations with n variables, where the n variables to be solved are the state variables.

Output equation: The algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

We define the state-space representation of a system. A system is represented in state space by the following equations:

$\mathbf{x}$ = state vector

$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time

$\mathbf{y}$ = output vector

$\mathbf{u}$ = input or control vector

$\mathbf{A}$ = system matrix

$\mathbf{B}$ = input matrix

$\mathbf{C}$ = output matrix

$\mathbf{D}$ = feed-forward matrix

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}} \tag{3.18}$$

$$\boxed{\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}} \tag{3.19}$$

Applying the State-Space Representation

In order to apply the state-space formulation to the representation of more complicated physical systems. The first step in representing a system is to **select the state vector**, which must be chosen according to the following,

- A **minimum number of state variables must be selected** as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system.
- The components of the state vector (that is, this minimum number of state variables) must be linearly independent.

<u>**Linearly Independent State Variables:**</u>

The components of the state vector must be **linearly independent**.

- For example, if x_1, x_2, and x_3 are chosen as state variables, then x_3 is not linearly independent of x_1 and x_2, since knowledge of the values of x_1 and x_2 will yield the value of x_3.
- **Variables and their successive derivatives are linearly independent**. For example, the voltage across an inductor, v_L, is linearly independent of the current through the inductor, i_L. Thus, v_L cannot be evaluated as a linear combination of the current, i_L.

Applying the State-Space Representation…

Minimum Number of State Variables:

How do we know the minimum number of state variables to select?

- Typically, **the minimum number required equals the order of the differential equation** describing the system.
- For example, if a third-order differential equation describes the system, then three simultaneous, first-order differential equations are required along with three state variables.
- From the perspective of the transfer function, the order of the differential equation is **the order of the denominator of the transfer function** after canceling common factors in the numerator and denominator.
- **In most cases, another way to determine the number of state variables is to count the number of independent energy-storage elements in the system.** The number of these energy-storage elements equals the order of the differential equation and the number of state variables.

Applying the State-Space Representation…

Our approach for selecting state variables and representing a system in state space:

- First, we **write the simple derivative equation for each energy-storage element** and solve for each derivative term as a linear combination of any of the system variables and the input that are present in the equation.

- Next we **select each differentiated variable as a state variable**.

- Then we express all other system variables in the equations in terms of the state variables and the input.

- Finally, we write the output variables as linear combinations of the state variables and the input.

Example 3.1 - Representing an Electrical Network

Given the electrical network in the figure, obtain a state-space representation if the output is the current through the resistor.

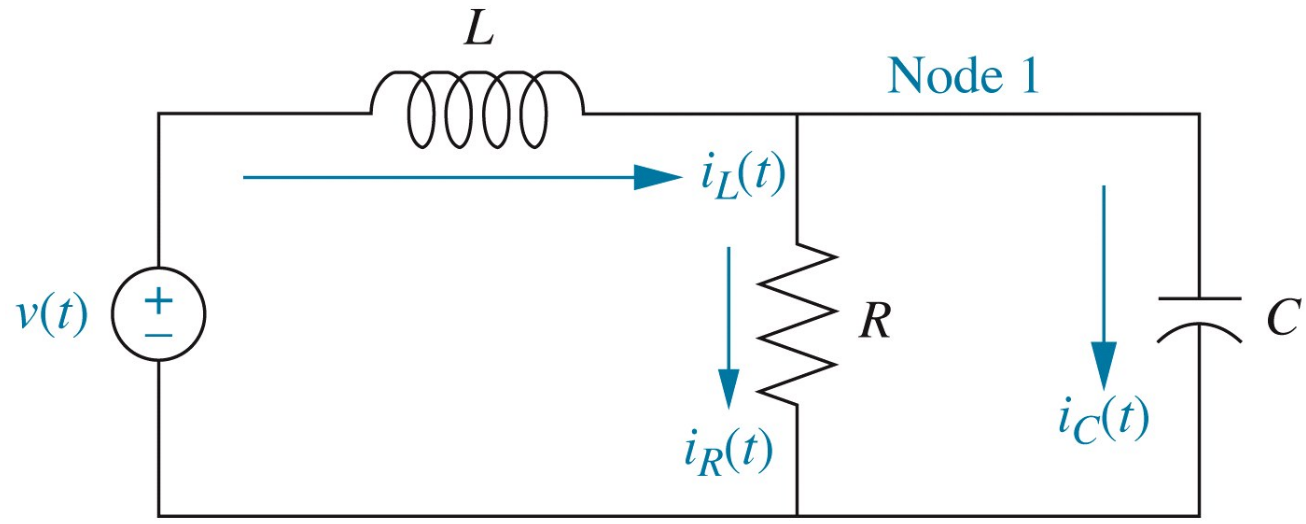


Figure 3.5

Solution:

The following steps will yield a viable representation of the network in ss.

1. Label all of the branch currents in the network. These include i_L, i_R, and i_C, as shown in Figure 3.5.
2. Select the state variables by writing the derivative equation for all energy storage elements, that is, the inductor and the capacitor. Thus,

$$C\frac{dv_C}{dt} = i_C \qquad (3.22)$$

$$L\frac{di_L}{dt} = v_L \qquad (3.23)$$

3. From Eqs. (3.22) and (3.23), choose the state variables as the quantities that are differentiated, namely v_C and i_L. Using Eq. (3.20) as a guide,

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1v(t) \tag{3.20a}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2v(t) \tag{3.20b}$$

we see that the state-space representation is complete if the right-hand sides of Eqs. (3.22) and (3.23) can be written as linear combinations of the state variables and the input. Since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$.

4. Apply network theory, such as Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L. At node 1,

$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R}v_C + i_L \end{aligned} \tag{3.24}$$

which yields i_C in terms of the state variables, v_C and i_L.

Around the outer loop,

$$v_L = -v_C + v(t) \tag{3.25}$$

which yields v_L in terms of the state variable, v_C, and the source, v(t).

5. Substitute the results of Eqs. (3.24) and (3.25) into Eqs. (3.22) and (3.23) to obtain the following state equations:

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L \tag{3.26a}$$

$$L\frac{di_L}{dt} = \quad -v_C + v(t) \tag{3.26b}$$

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \tag{3.27a}$$

$$\frac{di_L}{dt} = \quad -\frac{1}{L}v_C + \frac{1}{L}v(t) \tag{3.27b}$$

6. Find the output equation. Since the output is $i_R(t)$,

$$i_R = \frac{1}{R}v_C \tag{3.28}$$

The final result for the state-space representation is found by representing Eqs. (3.27) and (3.28) in vector-matrix form as follows:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t) \tag{3.29a}$$

$$i_R = \begin{bmatrix} 1/R & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \tag{3.29b}$$

Example 3.3 - Representing a Translational Mechanical System

Find the state equations for the translational mechanical system shown below

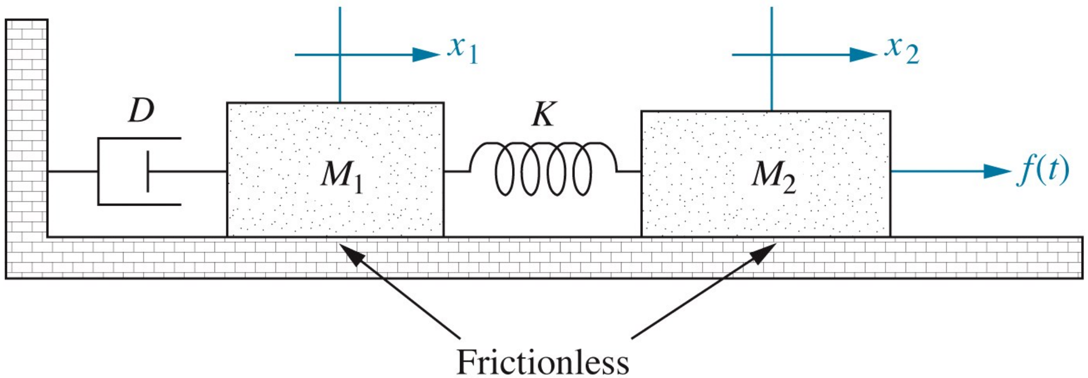


Figure 3.7

Solution:

- First write the differential equations for the network to find the Laplace-transformed equations of motion.
- Next take the inverse Laplace transform of these equations, assuming zero initial conditions, and obtain

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0 \qquad (3.44)$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t) \qquad (3.45)$$

Now let $d^2x_1/dt^2 = dv_1/dt$, and $d^2x_2/dt^2 = dv_2/dt$, and then select x_1, v_1, x_2, and v_2 as state variables. Next form two of the state equations by solving Eq. (3.44) for dv_1/dt and Eq. (3.45) for dv_2/dt. Finally, add $dx_1/dt = v_1$ and $dx_2/dt = v_2$ to complete the set of state equations. Hence,

$$\frac{dx_1}{dt} = +v_1 \tag{3.46a}$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \tag{3.46b}$$

$$\frac{dx_2}{dt} = +v_2 \tag{3.46c}$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t) \tag{3.46d}$$

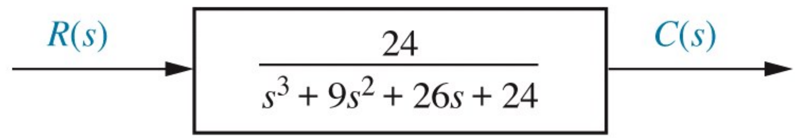
In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t) \tag{3.47}$$

where the dot indicates differentiation with respect to time. What is the output equation if the output is $x(t)$?

Example 3.4 - Converting a Transfer Function with Constant Term in Numerator

- Find the state-space representation in phase-variable form for the transfer function shown in Fig. 3.10(a)



(*a*)

SOLUTION:

Step 1 Find the associated differential equation. Since

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \tag{3.54}$$

cross-multiplying yields

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \tag{3.55}$$

The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \tag{3.56}$$

$$x_1 = c \tag{3.57a}$$

$$x_2 = \dot{c} \tag{3.57b}$$

$$x_3 = \ddot{c} \tag{3.57c}$$

Differentiating both sides and making use of Eq. (3.57) to find $\dot{x}_1$ and $\dot{x}_2$, and Eq. (3.56) to find $\dddot{c} = \dot{x}_3$, we obtain the state equations. Since the output is $c = x_1$, the combined state and output equations are

$$\dot{x}_1 = x_2 \tag{3.58a}$$

$$\dot{x}_2 = x_3 \tag{3.58b}$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \tag{3.58c}$$

$$y = c = x_1 \tag{3.58d}$$

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \tag{3.59a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.59b}$$

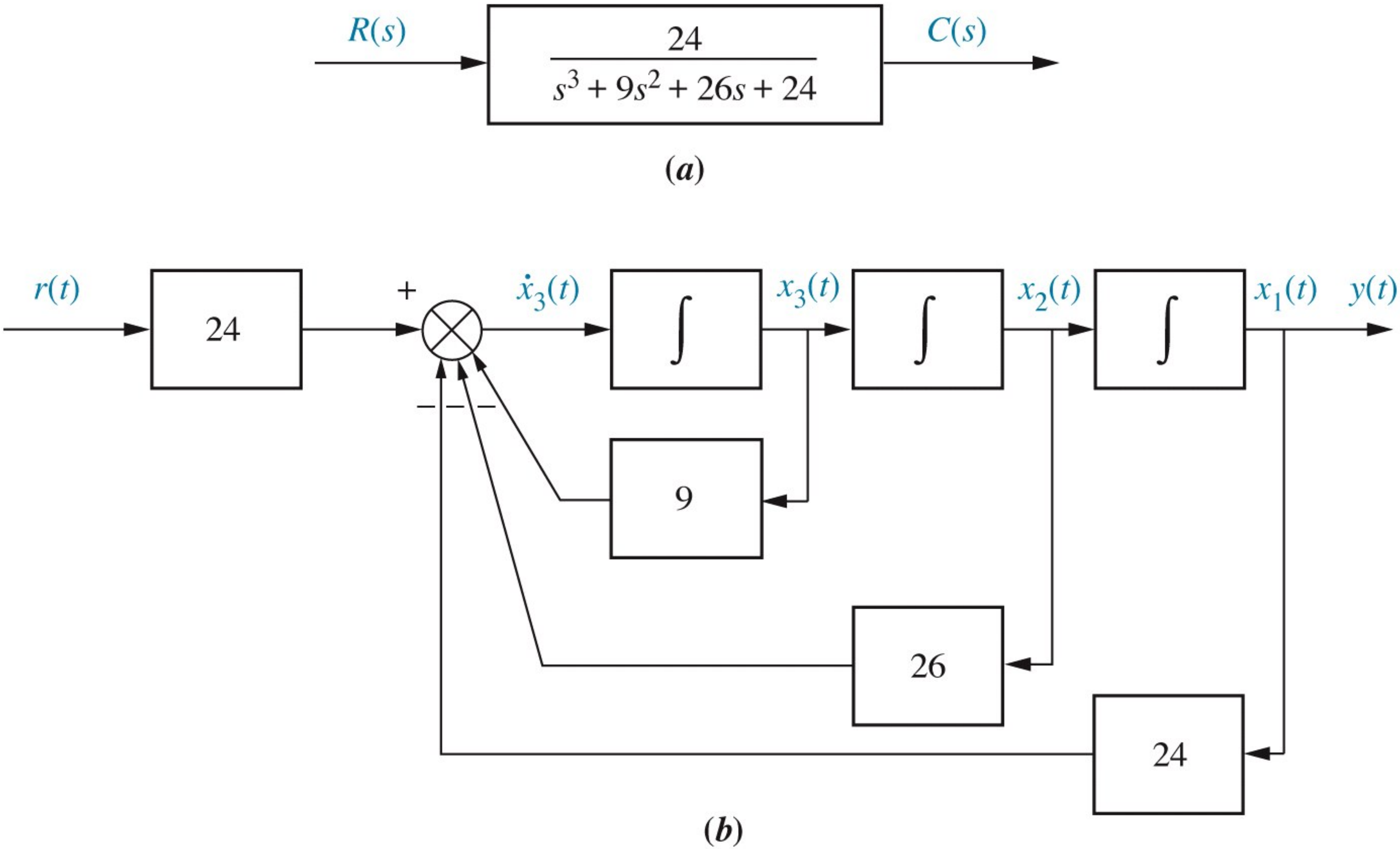


Figure 3.10

FIGURE 3.10 **a.** Transfer function; **b.** equivalent block diagram showing phase variables. Note: $y(t) = c(t)$.

Example 3.5 - Converting a Transfer Function with Polynomial in Numerator

Find the state-space representation of the transfer function shown in block diagram given.

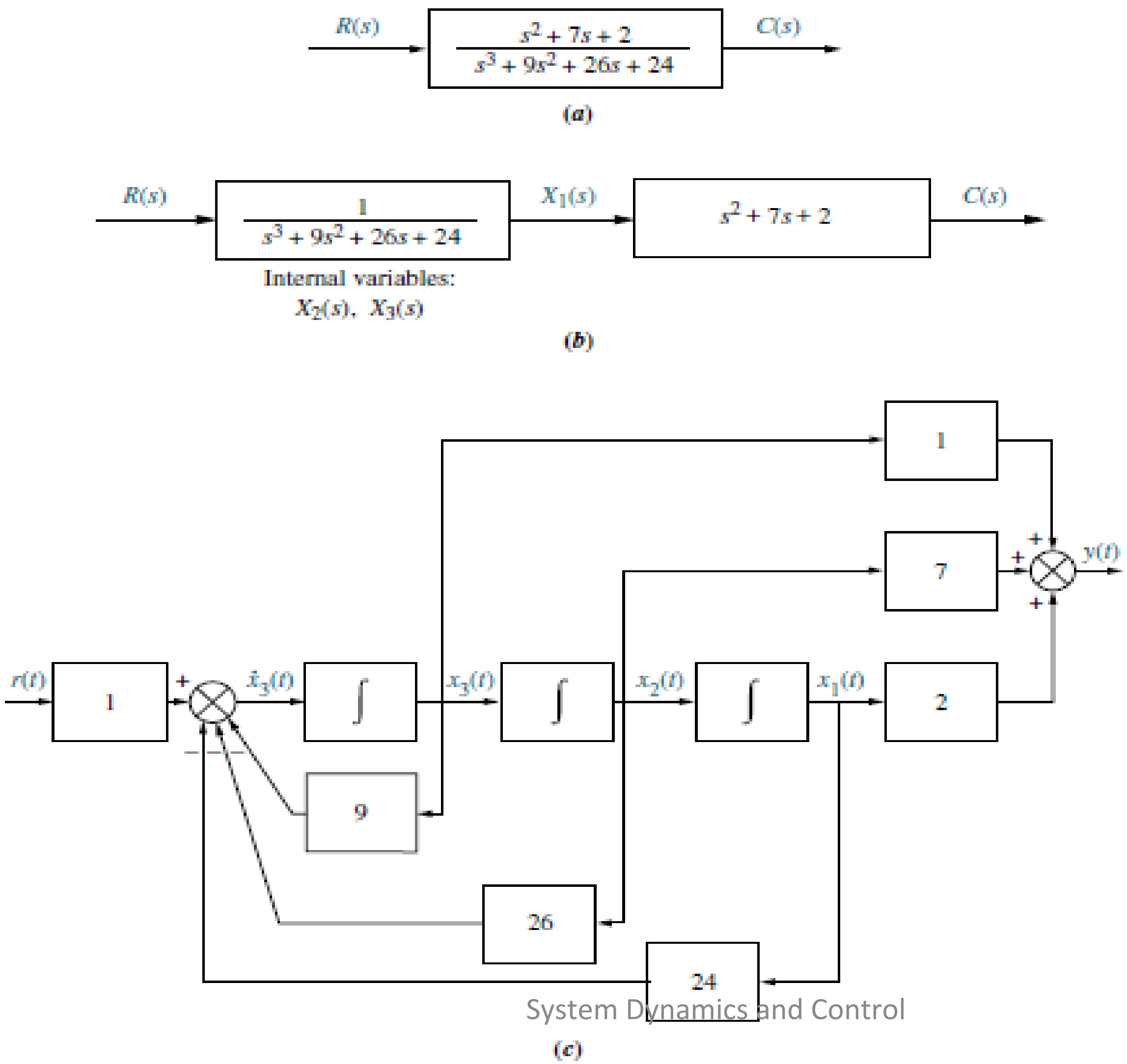
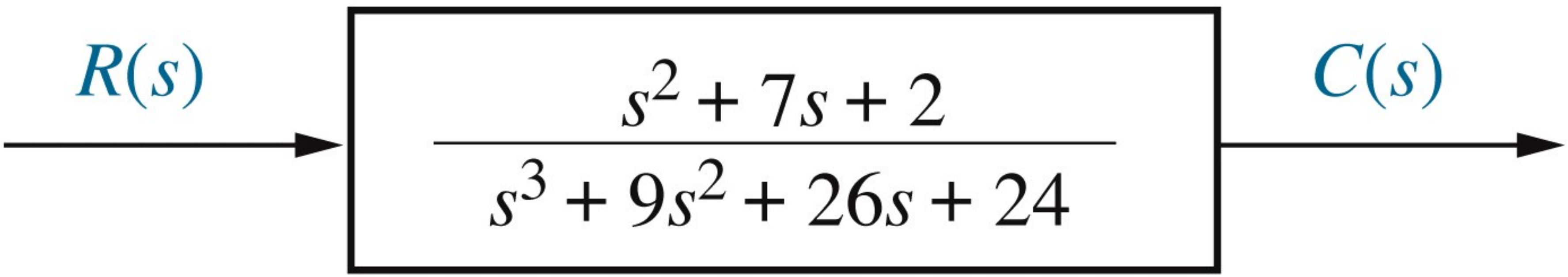


FIGURE 3.12 a. Transfer function; b. decomposed transfer function; c. equivalent block diagram Note: $y(t) = c(t)$.



(*a*)

Figure 3.12a

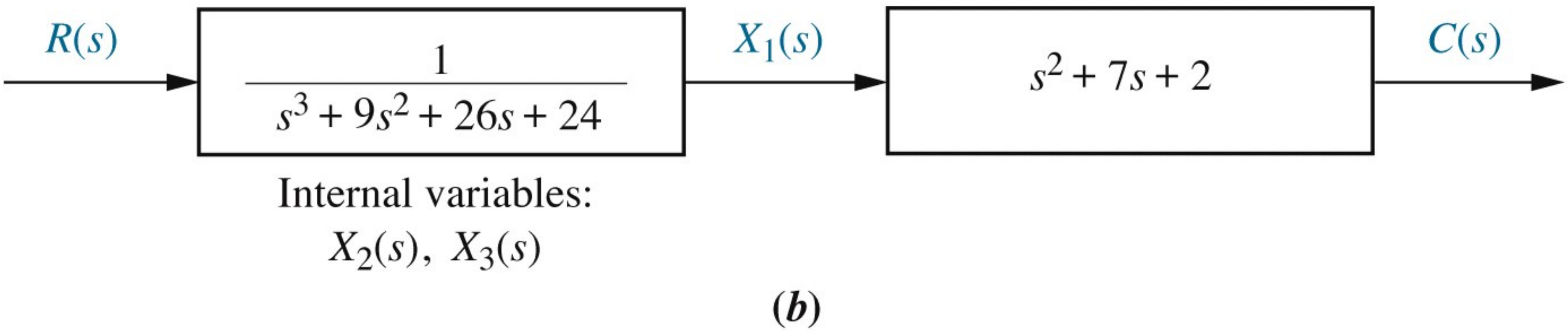

$R(s)$
$\frac{1}{s^3 + 9s^2 + 26s + 24}$
$X_1(s)$
$s^2 + 7s + 2$
$C(s)$
Internal variables:
$X_2(s),\ X_3(s)$
(*b*)

Figure 3.12b

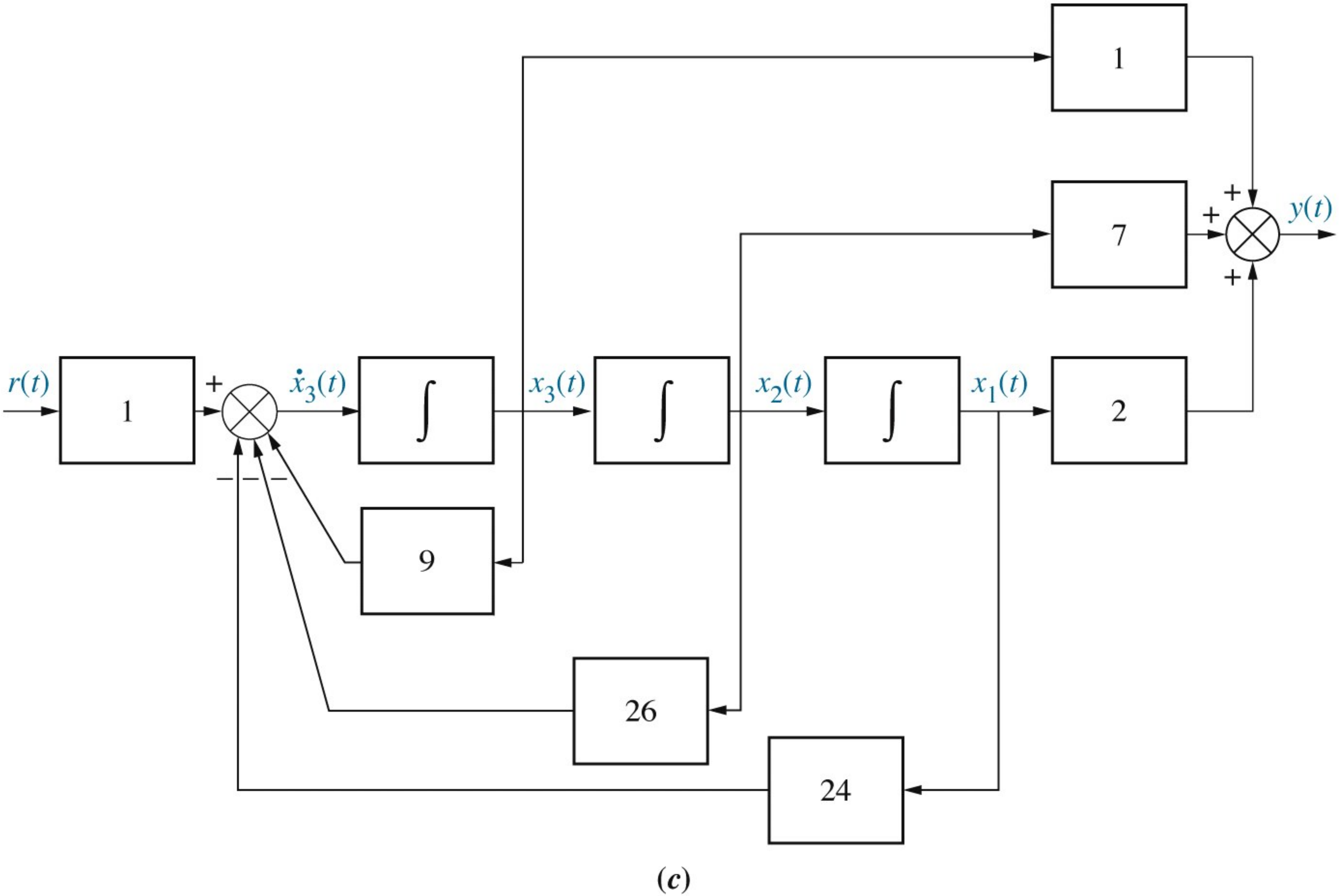


Figure 3.12c

Example 3.5 - Converting a Transfer Function with Polynomial in Numerator

Solution:

Separate the system into two cascaded blocks, as shown in Figure 3.12(b).The first block contains the denominator and the second block contains the numerator.

Find the state equations for the block containing the denominator. We notice that the first block's numerator is 1/24 that of Example 3.4. Thus, the state equations are the same except that this system's input matrix is 1/24 that of Example 3.4. Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (3.63)$$

Introduce the effect of the block with the numerator. The second block of Figure 3.12(b), where $b_2=1$; $b_1=7$, and $b_0=2$, states that

$$C(s) = (b_2 s^2 + b_1 s + b_0)X_1(s) = (s^2 + 7s + 2)X_1(s) \qquad (3.64)$$

Taking the inverse Laplace transform with zero initial conditions, we get

$$x_1 = x_1$$

$$\dot{x}_1 = x_2$$

$$\ddot{x}_1 = x_3$$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \qquad (3.65)$$

$$y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + x_2 + 2x_1$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.67)$$

Converting from State Space to a Transfer Function

- Now we move in the opposite direction and convert the state-space representation into a transfer function.
- Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.68a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.68b}$$

take the Laplace transform assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.69a}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{3.69b}$$

- Solving for X(s) in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{3.70}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{3.71}$$

where I is the identity matrix. Substituting Eq. (3.71) into Eq. (3.69b) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

- We call the matrix $[C(sI - A)^{-1}B + D]$ the transfer function matrix, since it relates the output vector, Y(s), to the input vector, U(s). we can find the transfer function,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

Example 3.6 - State-Space Representation to Transfer Function

Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \qquad (3.74a)$$

$$y = [1 \quad 0 \quad 0]\mathbf{x} \qquad (3.74b)$$

Solution:

The solution revolves around finding the term $(s\mathbf{I} - \mathbf{A})^{-1}$. All other terms are already defined. Hence, first find $(s\mathbf{I} - \mathbf{A})$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \qquad (3.75)$$

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1} \qquad (3.76)$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0]$$

$$\mathbf{D} = 0$$

we obtain the final result for the transfer function:

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1} \qquad (3.77)$$