EEEN 460
Optimal Control

Spring 2020

Lecture 8

Calculus of Variations

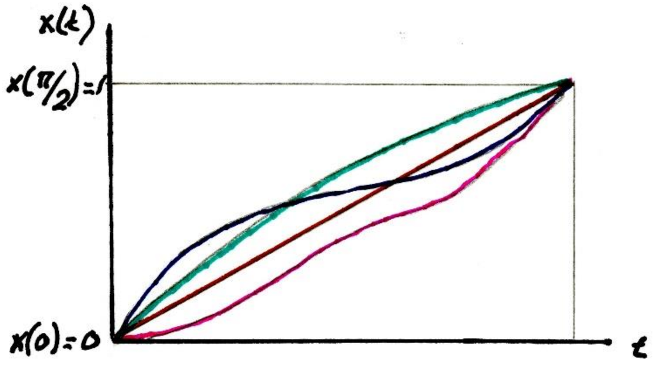
Euler-Lagrange Equations

Example:

Find the optimal solution for the following functional

$$J(x) = \int_0^{\pi/2} \left| \dot{x}^2(t) - x^2(t) \right| dt$$

which satisfies $x(0) = 0$ and $x(\pi/2) = 1$



$$g(x) = \left| \dot{x}^2(t) - x^2(t) \right|$$

$$0 = \frac{\partial g}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\left(x^*(t), \dot{x}^*(t), t\right)\right]$$

$$= -2x^*(t) - 2\ddot{x}^*(t) = 0$$

$$\Rightarrow \quad \ddot{x}^*(t) + x^*(t) = 0$$

$$\ddot{x}^*(t) + x^*(t) = 0$$

assume that the solution is of the form

$$x^*(t) = k e^{st}$$

$$\Rightarrow k s^2 e^{st} + k e^{st} = 0$$

$$\Rightarrow s^2 + 1 = 0$$

$$\Rightarrow s = \pm j1$$

This result tells us that the problem has got a sinusoidal solution.

$$x^*(t) = A \sin(t + \phi)$$

To determine A and ϕ, we should use the boundary conditions

$$x(0) = 0, \quad x(\pi/2) = 1$$

$$\left.\begin{array}{l} \text{at } t=0 \Rightarrow A \sin\phi = 0 \\ \text{at } t=\pi/2 \Rightarrow A \cos\phi = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \phi = 0 \\ A = 1 \end{array}$$

$\therefore$ The optimal solution is:

$$\boxed{x^* = \sin t}$$

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Let us try to verify this solution

$\dot{x}^* = \cos t$

$\dot{x}^{*2} = \cos^2 t$

$g(x) = |\cos^2 t - \sin^2 t|$

$= |2\cos^2 t - 1|$

$= |\cos 2t|$

$$J(x) = \int_0^{\pi/2} |\cos 2t| \, dt$$

call $u = 2dt$

$\frac{1}{2} du = dt$

$$= \frac{1}{2} \int_0^{\pi} |\cos u| \, du$$

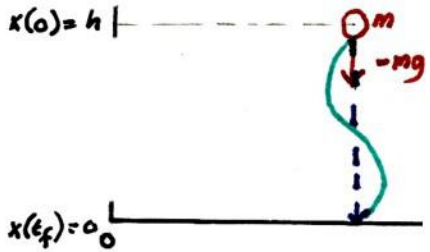
$$= \frac{1}{2} |\sin u| \Big]_0^{\pi}$$

$= 0$ ✓

Exercise

By using Euler-Lagrange equation show that a free falling object from a rest position at a height of h follow a trajectory

$$x^*(t) = h - \frac{1}{2} g t^2$$



$$\frac{\partial}{\partial x}\left(g(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(g(x^*(t), \dot{x}^*(t), t\right)\right] = 0$$

$$g(x^*(t), \dot{g}^*(t), t) = \frac{1}{2} m \dot{x}^{*2}(t) - mg\, x^*(t)$$

$$\frac{\partial g}{\partial x} = -mg, \quad \frac{\partial g}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) = m\ddot{x}$$

$$-mg - m\ddot{x}^*(t) = 0 \Rightarrow \ddot{x}^*(t) = -g \Rightarrow \dot{x}^*(t) = -gt + \dot{x}(0)^{=0}$$

$$\dot{x}^*(t) = -gt$$

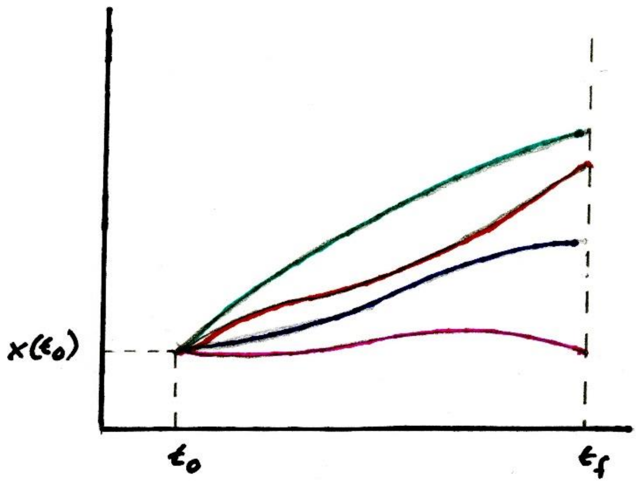
$$x^*(t) = -\frac{1}{2} g t^2 + x(0)^{=h}$$

$$\boxed{x^*(t) = h - \frac{1}{2} g t^2}$$

What are the necessary conditions for a function $g(x(t), \dot{x}(t), t)$ to make the functional J optimal

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

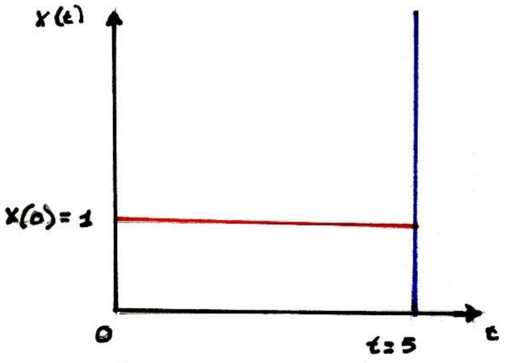
if t_0, $x(t_0)$ and t_f are specified but $x(t_f)$ is free



$$\boxed{\begin{array}{l} \frac{\partial}{\partial x}\left(g(x^*(t), \dot{x}^*(t), t)\right) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(g(x^*(t), \dot{x}^*(t), t)\right)\right] = 0 \\ \frac{\partial}{\partial \dot{x}}\left(g(x^*(t_f), \dot{x}^*(t_f), t_f)\right) = 0 \qquad \text{(boundary condition)} \end{array}}$$

Example :

Determine the smooth curve of the smallest length connecting the point $x(0) = 1$ to the line $t = 5$



$$J(x) = \int_0^5 [1 + \dot{x}^2(t)]^{1/2} dt$$

1. Necessary condition, Euler-Lagrange Equation

$$-\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}[(1 + \dot{x}^{*2}(t))^{1/2}]\right] = -\frac{d}{dt}\left[\frac{\dot{x}^*(t)}{[1 + \dot{x}^{*2}(t)]^{1/2}}\right] = 0$$

$\ddot{x}^*(t) = 0$

$x^*(t) = c_1 t + c_2$

from the initial conditions

$x(0) = 1 \Rightarrow c_2 = 1$

2. Necessary condition, $\frac{\partial g(x^*(t_f), \dot{x}^*(t_f), t_f)}{\partial \dot{x}} = 0 \Rightarrow \frac{\dot{x}^*(5)}{[1 + \dot{x}^{*2}(5)]^{1/2}} = 0 \Rightarrow \dot{x}^*(5) = 0$

Since $\dot{x}^*(5) = 0 \Rightarrow c_1 = 0$

$\therefore \boxed{x^*(t) = 1}$

Problem :

What is the optimal value of the functional

$$J(x) = \int_0^2 [\dot{x}^2(t) + 2x(t)\dot{x}(t) + 4x^2(t)]\, dt$$

given $x(0) = 1$ and $x(2) =$ free.

1. According to the first necessary condition

$$\frac{\partial}{\partial x}(g(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(g(x^*(t), \dot{x}^*(t), t)\right] = 0 \qquad g(x(t), \dot{x}(t), t) = \dot{x}^2(t) + 2x(t)\dot{x}(t) + 4x^2(t)$$

$$\frac{\partial}{\partial x}(g(x(t), \dot{x}(t), t) = 2\dot{x}(t) + 8x(t)$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(g(x(t), \dot{x}(t), t)\right] = \frac{d}{dt}(2\dot{x}(t) + 2x(t)) = 2\ddot{x}(t) + 2\dot{x}(t)$$

$$2\dot{x}^*(t) + 8x^*(t) - 2\ddot{x}^*(t) - 2\dot{x}^*(t) = 0 \Rightarrow \ddot{x}^*(t) - 4x^*(t) = 0$$

The solution has the form

$$x^*(t) = c_1 e^{-2t} + c_2 e^{2t}$$

2. To evaluate the coefficients we use

a) The boundary condition $x(0) = 1$

b) The boundary condition from the necessary condition 2.

$$\frac{\partial g}{\partial \dot{x}}(x^*(2), \dot{x}^*(2), 2) = 0 \Rightarrow 2\dot{x}(2) + 2x(2) = 0 \Rightarrow \dot{x}(2) + x(2) = 0$$

$$\Rightarrow -2c_1 e^{-4} + 2c_2 e^4 + c_1 e^{-4} + c_2 e^4 \Rightarrow -c_1 e^{-4} + 3c_2 e^4 = 0 \quad (1)$$

The boundary condition a) implies

$$c_1 + c_2 = 1 \quad (2)$$

Solving for c_1 and c_2

$$c_1 = \frac{3e^4}{e^{-4} + 3e^4}, \quad c_2 = \frac{e^{-4}}{e^{-4} + 3e^4} \Rightarrow \boxed{x^*(t) = \frac{3e^4}{e^{-4} + 3e^4} e^{-2t} + \frac{e^{-4}}{e^4 + 3e^4} e^{2t}}$$

Calculating the coefficients

$$-c_1 e^{-4} + 3(1 - c_1)e^4 = 0$$

$$(-e^{-4} - 3e^4)c_1 = -3e^4$$

$$(e^{-4} + 3e^4)c_1 = 3e^4$$

$$c_1 = \frac{3e^4}{e^{-4} + 3e^4}$$

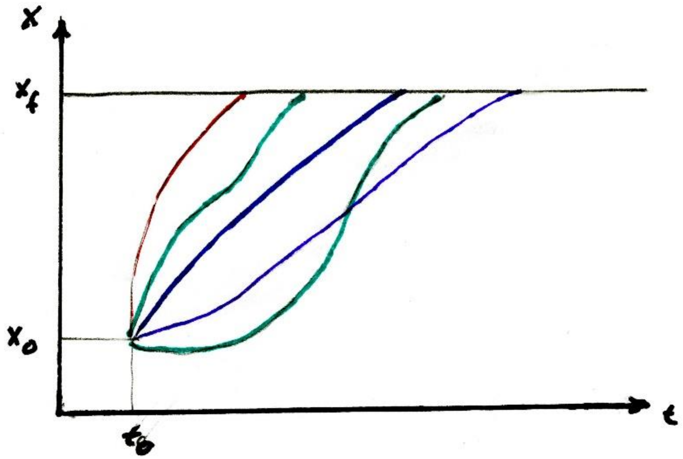
$$c_2 = \frac{e^{-4}}{e^{-4} + 3e^4}$$

What are the necessary conditions that must be satisfied for the optimization of the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

Given t_0, $x(t_0) = x_0$ and $x(t_f) = x_f$, t_f is free

Answer



$$\boxed{\begin{aligned} &\frac{\partial}{\partial x}\left(g(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(g(x^*(t), \dot{x}^*(t), t\right)\right] = 0 \\ &g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial}{\partial \dot{x}}(g(x^*(t_f), \dot{x}^*(t_f), t_f\right]^T \dot{x}^*(t_f) = 0 \end{aligned}}$$ boundary condition

Example :

Find the optimal value of t_f for the functional

$$J(x) = \int_1^{t_f} \left[2x(t) + \frac{1}{2}\dot{x}^2(t)\right] dt$$

the boundary conditions are $x(1) = 4$ $x(t_f) = 4$ and $t_f > 1$ free

$g(x(t), \dot{x}(t), t) = 2x(t) + \frac{1}{2}\dot{x}^2(t)$

$\frac{\partial}{\partial x}(g(x(t), \dot{x}(t), t) = 2$, $\frac{\partial}{\partial \dot{x}}(g(x(t), \dot{x}(t), t) = \dot{x}(t)$, $\frac{d}{dt}(g(x(t), \dot{x}(t), t) = \ddot{x}(t)$

$\Rightarrow 2 - \ddot{x}^*(t) = 0 \Rightarrow \ddot{x}^*(t) = 2$ $(\dot{x}^*(t) = 2t + c_1)$

$\Rightarrow x^*(t) = t^2 + c_1 t + c_2$

From the boundary condition

$0 = g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f\right]\dot{x}^*(t_f)$

$0 = 2x^*(t_f) + \frac{1}{2}\dot{x}^{*2}(t_f) - \dot{x}^{*2}(t_f)$

$0 = 2x^*(t_f) - \frac{1}{2}\dot{x}^2(t_f) \Rightarrow 2(t_f^2 + c_1 t_f + c_2) - \frac{1}{2}(2t_f + c_1)^2 = 0$

$\Rightarrow 2c_2 - \frac{1}{2}c_1^2 = 0$ 1

$x^*(1) = 4 = 1 + c_1 + c_2 \Rightarrow c_1 + c_2 = 3$ 2

$c_1 + \frac{c_1^2}{4} - 3 = 0$

$c_1^2 + 4c_1 - 12 = 0 \Rightarrow (c_1 + 6)(c_1 - 2) = 0$

If $c_1 = -6$, $c_2 = 9$

$x^*(t_f) = t_f^2 - 6t_f + 9$

$4 = t_f^2 - 6t_f + 9$

$0 = t_f^2 - 6t_f + 5$

$0 = (t_f - 5)\cancel{(t_f - 1)}$

$t_f = 5$

If $c_1 = 2$ $c_2 = 1$

$4 = t_f^2 + 2t_f + 1$

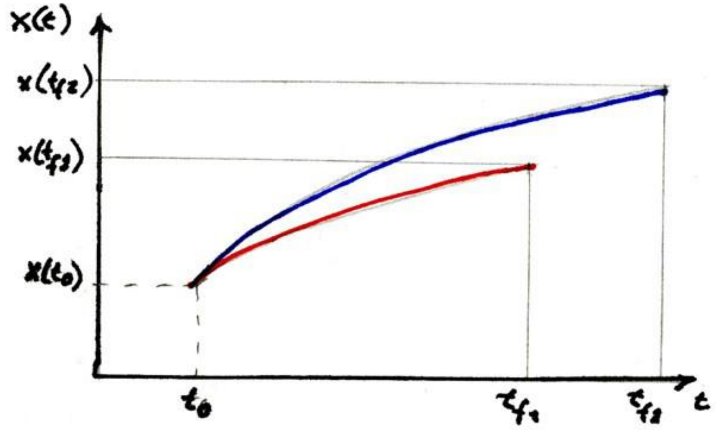
$0 = t_f^2 + 2t_f - 3$

$0 = \cancel{(t_f + 3)}\cancel{(t_f - 1)}$

What are the necessary conditions that must be satisfied for the optimization of the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) \, dt$$

Given t_0 and $x(t_0) = x_0$, and t_f and $x(t_f)$ are free.



a) if t_f and $x(t_f)$ are unrelated

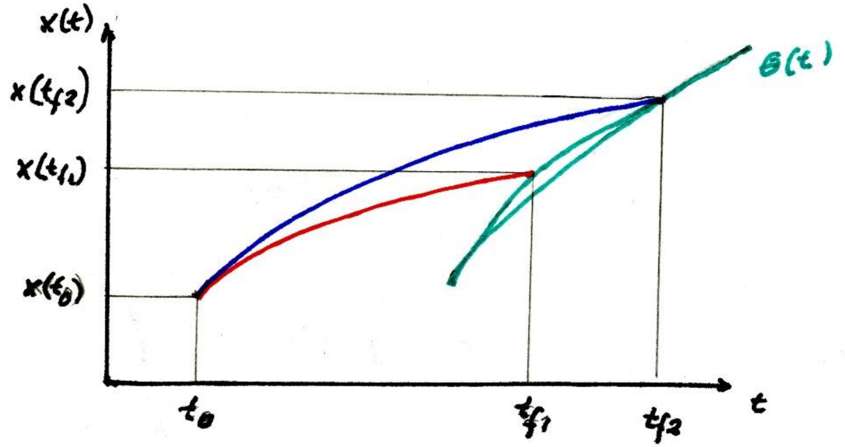
$$\frac{\partial}{\partial x}\left(g(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(g(x^*(t), \dot{x}^*(t), t\right)\right] = 0$$

$$\frac{\partial}{\partial \dot{x}}\left(g(x^*(t), \dot{x}^*(t), t\right) = 0$$

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$

Boundary condition

b) if t_f and $x(t_f)$ are related. e.g. $x(t_f)$ may lie on a specified cure $\theta(t)$.



$$\frac{\partial}{\partial x} g(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(g(x^*(t), \dot{x}^*(t), t\right] = 0$$

$$\left[\frac{\partial}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f\right]\left[\frac{d\theta}{dt}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$ (transversality condition)

Example :

Find the optimal curve for the functional

$$J(x) = \int_{t_0}^{t_f} [1 + \dot{x}^2(t)]^{1/2} \, dt$$

$t_0 = 0$, $x(0) = 0$ are specified, t_f and $x(t_f)$ are free, but $x(t_f)$ is required to lie on the line

$$\theta(t) = -5t + 15$$

Solution

We had found in earlier examples

$$x^*(t) = c_1 t + c_2$$

we know that $x^*(0) = 0$, so $c_2 = 0$, To evaluate the other coefficient we use the transversality condition

$$\left[\frac{\partial}{\partial \dot{x}} g(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\left[\frac{d\theta}{dt}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$

$$\frac{\dot{x}^*(t_f)}{[1 + \dot{x}^{*2}(t_f)]^{1/2}} \cdot [-5 - \dot{x}^*(t_f)] + [1 + \dot{x}^{*2}(t_f)]^{1/2} = 0$$

$$\dot{x}^*(t_f)[-5 - \dot{x}^*(t_f)] + [1 + \dot{x}^{*2}(t_f)] = 0$$

$$-5\dot{x}^*(t_f) + 1 = 0$$

$$-5c_1 + 1 = 0$$

$$c_1 = -1/5$$

$$x^*(t_f) = \theta(t_f)$$

$$-1/5 \, t_f = -5t_f + 15$$

$$\boxed{t_f = \frac{75}{26} = 2.88}$$

Problem description	*Boundary conditions*	*Remarks*
1. $\mathbf{x}(t_f)$, t_f both specified	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f

End of Lecture VIII