Selected Problems – IX

Problem 1) Use the defining integral to find the Fourier transform of the following function shown as

$$f(t) = \begin{cases} A \sin \frac{\pi}{2} t, & -2 \leq t \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Solution. We calculate

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-2}^{2} A \sin \frac{\pi}{2} t\, e^{-j\omega t}\, dt$$

$$= \frac{A}{j2} \int_{-2}^{2} \left(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t} \right) e^{-j\omega t}\, dt$$

$$= \frac{A}{j2} \left(\int_{-2}^{2} e^{-j(\omega - \frac{\pi}{2})t}\, dt - \int_{-2}^{2} e^{-j(\omega + \frac{\pi}{2})t}\, dt \right)$$

$$= \frac{A}{j2} \left(\frac{e^{-j(\omega - \frac{\pi}{2})t}}{-j(\omega - \frac{\pi}{2})} - \frac{e^{-j(\omega + \frac{\pi}{2})t}}{-j(\omega + \frac{\pi}{2})} \right) \Bigg|_{-2}^{2}$$

$$= \frac{A}{2} \left[\frac{1}{\omega - \frac{\pi}{2}} \left(e^{-j(2\omega - \pi)} - e^{j(2\omega - \pi)} \right) - \frac{1}{\omega + \frac{\pi}{2}} \left(e^{-j(2\omega + \pi)} - e^{j(2\omega + \pi)} \right) \right]$$

$$= A \left[\frac{1}{2\omega - \pi} (-j2) \sin(2\omega - \pi) - \frac{1}{2\omega + \pi} (-j2) \sin(2\omega + \pi) \right]$$

$$= -j2A \left(\frac{-\sin 2\omega}{2\omega - \pi} + \frac{\sin 2\omega}{2\omega + \pi} \right)$$

$$= j2A\sin 2w\left(\frac{1}{2w-\pi}-\frac{1}{2w+\pi}\right)$$

$$= j2A\sin 2w\,\frac{4w}{4w^2+\pi^2}$$

$$= j\,\frac{8A\sin 2w}{4w^2-\pi^2}$$

Problem 2) Find the Fourier transform of the following function

$$f(t) = |t|e^{-a|t|}, \quad a>0, \quad -\infty < t \leq \infty$$

Solution. We shall reexpress $f(t)$ as

$$f(t) = f^{+}(t) + f^{-}(t)$$

where

$$f^{+}(t) = te^{-at}, \quad t>0$$

$$f^{-}(t) = -te^{at}, \quad t<0$$

then

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{f^{+}(t)\} + \mathcal{L}\{f^{-}(t)\}$$

$$= \mathcal{L}\{f^{+}(t)\}\Big|_{s=jw} + \mathcal{L}\{f^{-}(-t)\}_{s=-jw}$$

and

$$\mathcal{L}\{te^{-at}\} = \int_0^\infty te^{-at}e^{-st}\,dt$$

$$= \int_0^\infty \underbrace{t}_{u}\,\underbrace{e^{-(a+s)t}\,dt}_{dv}$$

$$du = dt \qquad v = \frac{e^{-(a+s)}}{-(a+s)}$$

$$= -t\,\frac{e^{-(a+s)t}}{a+s}\Big|_0^\infty - \int_0^\infty \frac{e^{-(a+s)t}}{-(a+s)}\,dt$$

$$= 0 + \frac{e^{-(a+s)t}}{-(a+s)^2}\Bigg|_0^{\infty}$$

$$= 0 + 0 + \frac{1}{(a+s)^2}$$

$$= \frac{1}{(a+s)^2}$$

Hence;

$$\mathcal{F}\{f(t)\} = \frac{1}{(a+s)^2}\Bigg|_{s=j\omega} + \frac{1}{(a+s)^2}\Bigg|_{s=-j\omega}$$

$$= \frac{1}{(a+j\omega)^2} + \frac{1}{(a-j\omega)^2}$$

$$= \frac{(a-j\omega)^2 + (a+j\omega)^2}{(a+j\omega)^2(a-j\omega)^2}$$

$$= \frac{a^2 - j2a\omega - \omega^2 + a^2 + j2a\omega - \omega^2}{(a^2+\omega^2)^2}$$

$$= 2\,\frac{a^2 - \omega^2}{(a^2+\omega^2)^2}$$

Problem 3) If $f(t)$ is a real, odd function of t, show that the inversion integral reduces to

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\sin\omega t \, d\omega$$

Solution. The inversion integral is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad , \quad F(\omega) = A(\omega) - jB(\omega)$$

– as $f(t)$ is real and odd, $F(\omega)$ reduces to

$$F(\omega) = 0 - jB(\omega)$$
$$= -jB(\omega)$$

Therefore;

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -jB(\omega)\left(\cos\omega t + j\sin\omega t\right) d\omega$$

$$= \frac{1}{2\pi}\left[-j\int_{-\infty}^{\infty} B(\omega)\cos\omega t \, d\omega + \int_{-\infty}^{\infty} B(\omega)\sin\omega t \, d\omega\right.$$

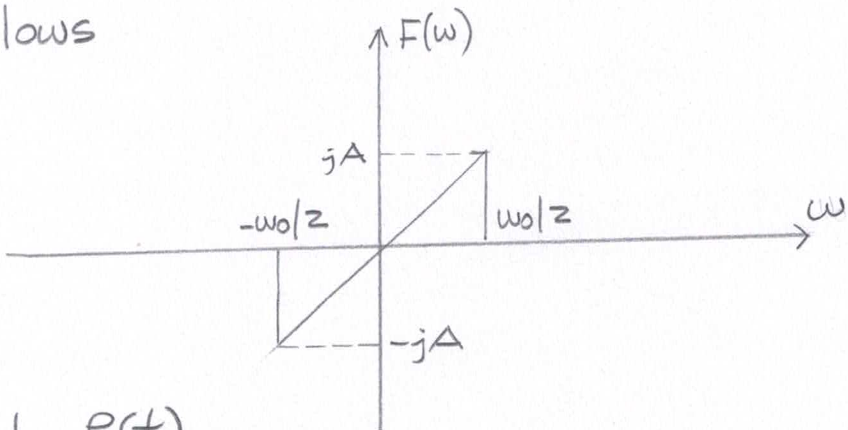
-since $B(\omega)$ is an odd function,

↳ $$\int_{-\infty}^{\infty} B(\omega)\cos\omega t \, d\omega = 0$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\sin\omega t \, d\omega$$

Problem 4) The Fourier transform of $f(t)$ is shown as follows



a. Find $f(t)$.

b. Evaluate $f(0)$.

c. Sketch $f(t)$ for $-10 \le t \le 10$ when $A = 2\pi$ and $\omega_0 = 2$ rad/s

Solution. We have

$$F(\omega) = j2\frac{A}{\omega_0}\omega \quad \text{for} \quad -\omega_0/2 \le \omega \le \omega_0/2$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0/2}^{\omega_0/2} j2\frac{A}{\omega_0}\omega e^{j\omega t} d\omega$$

$$= \frac{jA}{\pi\omega_0}\int_{-\omega_0/2} \underbrace{\omega}_{u}\underbrace{e^{j\omega t}d\omega}_{dv}$$

$$du = d\omega,\ v = \frac{e^{j\omega t}}{jt}$$

$$= \frac{jA}{\pi\omega_0}\left(\omega\frac{e^{j\omega t}}{jt}\Big|_{-\omega_0/2}^{\omega_0/2} - \int_{-\omega_0/2}^{\omega_0/2}\frac{e^{j\omega t}}{jt}d\omega\right)$$

$$= \frac{jA}{\pi\omega_0}\left[\frac{\omega_0}{j2t}\left(e^{j\omega_0 t/2} + e^{-j\omega_0 t/2}\right) - \frac{1}{(jt)^2}\left(e^{j\omega_0 t/2} - e^{-j\omega_0 t/2}\right)\right]$$

$$= \frac{jA}{\pi\omega_0}\left(\frac{\omega_0}{j\cancel{2}t}\cancel{2}\cos\frac{\omega_0 t}{2} - \frac{1}{(jt)^2}j2\sin\frac{\omega_0 t}{2}\right)$$

$$= \frac{A}{\pi t}\cos\frac{\omega_0 t}{2} - \frac{2A}{\pi\omega_0 t^2}\sin\frac{\omega_0 t}{2}$$

$$= \frac{A}{\pi\omega_0 t^2}\left(\omega_0 t\cos\frac{\omega_0 t}{2} - 2\sin\frac{\omega_0 t}{2}\right)$$

b.

$$f(0) = \lim_{t\to 0}\frac{A}{\pi\omega_0 t^2}\left(\omega_0 t\cos\frac{\omega_0 t}{2} - 2\sin\frac{\omega_0 t}{2}\right)$$

-since we have $\frac{0}{0}$ uncertainty, we shall apply L'hospital rule to get

$$f(0) = \lim_{t\to 0}\frac{A}{\pi\omega_0}\frac{\left(\cancel{\omega_0\cos\frac{\omega_0 t}{2}} - \omega_0 t\frac{\omega_0}{2}\sin\frac{\omega_0 t}{2} - \cancel{2\frac{\omega_0}{2}\cos\frac{\omega_0 t}{2}}\right)}{2t}$$

$$= \lim_{t\to 0} -\frac{A}{\pi\omega_0}\frac{\omega_0^2}{4}\sin\frac{\omega_0 t}{2}$$

$$= 0$$

c. when $A = 2\pi$ and $\omega_0 = 2\,rad/s$, we obtain

$$f(t) = \frac{\cancel{2\pi}}{\cancel{\pi 2}t^2}\left(2t\cos\frac{2t}{2} - 2\sin\frac{2t}{2}\right)$$

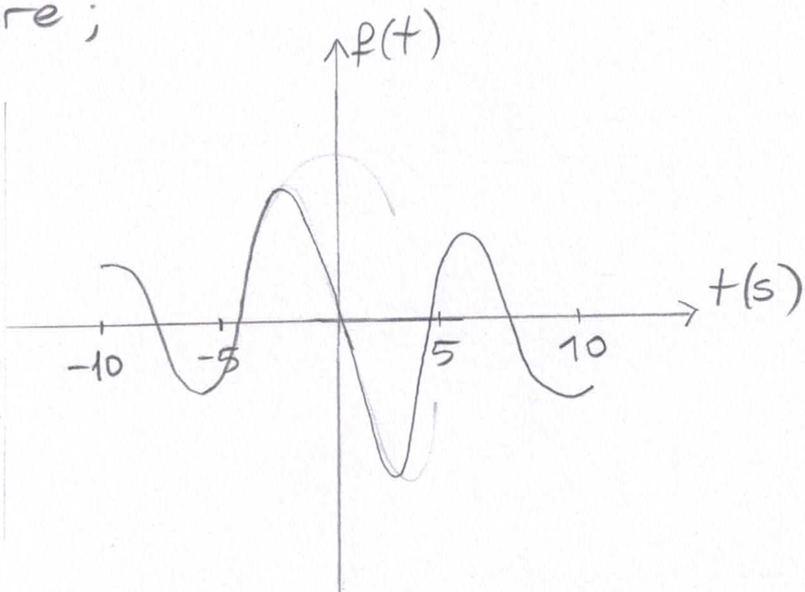
$$= \frac{1}{t^2}\left(2t\cos t - 2\sin t\right)$$

$$= \frac{2}{t}\cos t - \frac{2}{t^2}\sin t$$

Note that;

$f(t) = -f(-t)$, that is $f(t)$ is ODD

Therefore;



Problem 5) Find $\mathcal{F}\{\cos w_0 t\}$ by using the approximating function

$$f(t) = e^{-\epsilon|t|}\cos w_0 t$$

where ϵ is a positive real constant.

Solution. We first reexpress $f(t)$ as

$$f(t) = f^+(t) + f^-(t)$$

where

$$f^+(t) = e^{-\epsilon t}\cos w_0 t\,, \quad t>0$$

$$f^-(t) = e^{\epsilon t}\cos w_0 t\,, \quad t<0$$

then

$$\mathcal{F}\{f(t)\} = \mathcal{L}\{e^{-\epsilon t}\cos\omega_0 t\}\Big|_{s=j\omega} + \mathcal{L}\{e^{-\epsilon t}\cos\omega_0 t\}\Big|_{s=-j\omega}$$

-it follows from Laplace transform lookup table that we have

$$\mathcal{F}\{f(t)\} = \frac{s+\epsilon}{(s+\epsilon)^2+\omega_0^2}\Big|_{s=j\omega} + \frac{s+\epsilon}{(s+\epsilon)^2+\omega_0^2}\Big|_{s=-j\omega}$$

$$= \frac{0.5}{(s+\epsilon)-j\omega_0} + \frac{0.5}{(s+\epsilon)+j\omega_0}\Big|_{s=j\omega}$$

$$+ \frac{0.5}{(s+\epsilon)-j\omega_0} + \frac{0.5}{(s+\epsilon)+j\omega_0}\Big|_{s=-j\omega}$$

$$= \frac{0.5}{\epsilon+j(\omega-\omega_0)} + \frac{0.5}{\epsilon+j(\omega+\omega_0)}$$

$$+ \frac{0.5}{\epsilon-j(\omega+\omega_0)} + \frac{0.5}{\epsilon-j(\omega-\omega_0)}$$

$$= \frac{\epsilon}{\epsilon^2+(\omega-\omega_0)^2} + \frac{\epsilon}{\epsilon^2+(\omega+\omega_0)^2}$$

Note that;

-as $\epsilon \to 0$, $F(\omega) \to 0$ everywhere except at $\omega = \pm\omega_0$ and for $\omega = \pm\omega_0$, we have $F(\omega) = \frac{1}{\epsilon}$ yielding

$F(\omega) \to \infty$ as $\epsilon \to 0$

Moreover;

$$\int_{-\infty}^{\infty} \frac{\epsilon\, dw}{\epsilon^2 + (w-w_0)^2} = \int_{-\infty}^{\infty} \frac{d(w/\epsilon)}{1+\left(\frac{w-w_0}{\epsilon}\right)^2}$$

$$= \arctan\left(\frac{w-w_0}{\epsilon}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

(L) the area under each part of $F(w)$ is fixed and independent of ϵ

Hence;

$$F(w) = \pi\,\delta(w-w_0) + \pi\,\delta(w+w_0) \quad \text{as } \epsilon \to 0$$