CMPE 352
Signal Processing & Algorithms
Spring 2019

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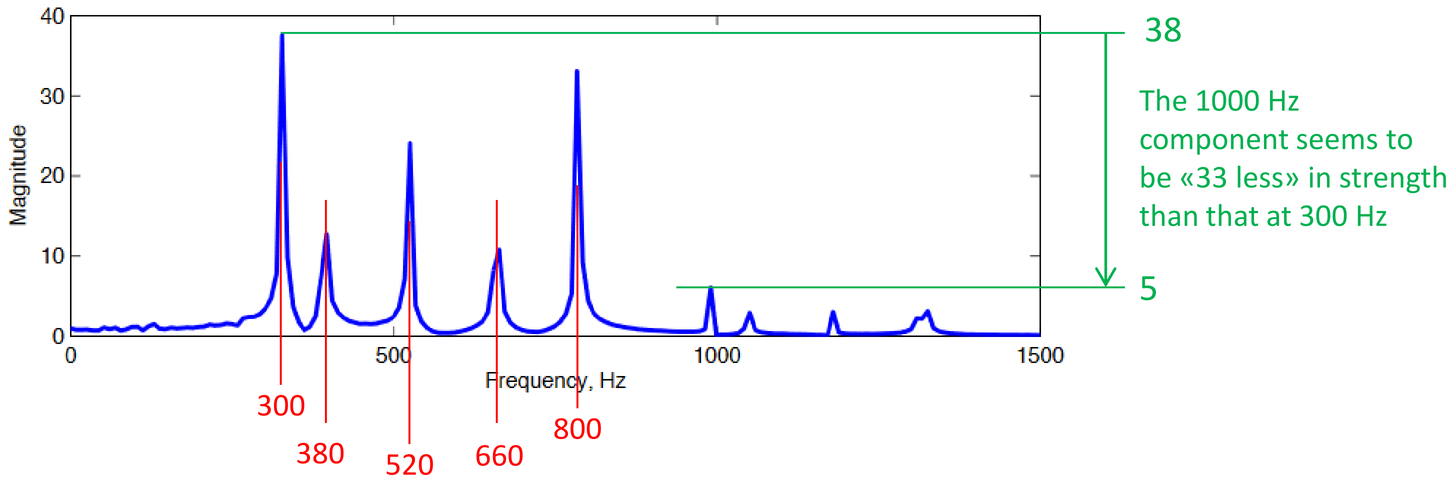
Review Questions (1)

- The sound signal obtained from hitting a piano key was recorded and analyzed. Its spectral analysis showed that the signal had the spectrum plotted below. What information does this plot give us?



The sound is composed of several tones (frequencies) or notes.

The strongest tone is a 300 Hz tone.

The next strongest tone is a 800 Hz tone.

The next strongest tone is a 520 Hz tone.

Tones at 380 Hz and 660 Hz also contribute to the sound heard, but the other remaining frequencies are not very significant

Review Questions (2)

- What is Euler's formula?

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

- $\theta \rightarrow \omega t$: Why is $e^{j\omega t}$ important?

  Because in signal processing and many disciplines of engineering, it is regarded as the <u>most elementary function</u>: it contains only one frequency (ω).

- What does $e^{j\omega t}$ describe in the complex plane?

  $e^{j\omega t}$ describes circular motion in the complex plane

Review Questions (3)

- What is the mathematical expression of the **trigonometric Fourier series**?

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

- What does this expression mean physically?
  - The signal $x(t)$ can be represented as the sum of sinusoids whose frequencies are harmonics of the fundamental frequency ω_0
  - The *k*th such sinusoid has frequency $k\omega_0$, amplitude A_k, and phase θ_k
  - a_0 is a constant term, independent of frequency (the ″DC content ″ or the ″mean value″ of the signal)

- Why is this useful?
  - If for a given signal $x(t)$ we are able to write the above sum, we know what frequencies ($k\omega_0$) this signal contains, the strength of each frequency component (A_k) and the relative shift in time of each frequency component (θ_k)

The Fourier Series

Assume that the signal $x(t)$ is known in the time domain. What is known & unknown in the right-hand side of the equation?

Given (known)

? ? $\omega_0 = \frac{2\pi}{T_0}$ (known) ?

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Fourier Series
(trigonometric form)

We determine the unknowns with the following formulas:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$\alpha_k = \frac{1}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\, dt$$

$$\beta_k = \frac{1}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\, dt$$

$$A_k = \sqrt{\alpha_k^2 + \beta_k^2}$$

$$\theta_k = \operatorname{atan}\frac{-\beta_k}{\alpha_k}$$

$$k = 1,2,\ldots$$

J. B. Joseph Fourier

Jean Baptiste Joseph Fourier

(1768 – 1830)

French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

Fourier went with Napoleon Bonaparte on his Egyptian expedition in 1798, and was made governor of Lower Egypt.

In 1801, Fourier returned from Egypt with many artifacts including an ink pressed copy of the Rosetta Stone (discovered in 1799).

J. B. Joseph Fourier

Jean Baptiste Joseph Fourier

(1768 – 1830)

In 1822 Fourier presented his work on heat flow in *Théorie analytique de la chaleur* (The Analytic Theory of Heat).

One of the important contributions in this work was Fourier's claim that any function of a variable (continuous or discontinuous) can be expanded in a series of sines of multiples of the variable. Though this result is not correct, Fourier's observation that some discontinuous functions are the sum of infinite series was a breakthrough.

His work enabled him to express the conduction of heat in two-dimensional objects (*i.e.,* very thin sheets of material) in terms of the differential equation

$$\frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

in which u is the temperature at any time t at a point (x, y) of the plane and k is a constant of proportionality called the diffusivity of the material.

Discovery of the greenhouse effect.

Fourier Series

https://www.youtube.com/watch?v=LznjC4Lo7lE

https://www.youtube.com/watch?v=Y9pYHDSxc7g

https://www.youtube.com/watch?v=ZRZIz81nXo4

Some Limitations of the Trigonometric Fourier Series

- Note that there are some shortcomings with the trigonometric Fourier Series

- For example, the trigonometric Fourier Series

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

is only valid when <u>$x(t)$ is real.</u> How about complex signals?

- Another point is this: we have seen that the complex exponential $e^{j\omega t}$ is the <u>most elementary function</u> in the complex plane. So it would make sense to express $x(t)$ as a weighted sum of these elementary exponential terms, that is, replace the cosine functions by complex exponentials!

The Exponential Fourier Series

- It is possible to satisfy both of these requirements if we use the complex exponential as basis function (instead of *cos*):

Fourier Series
(exponential form)

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}}$$

Notes:

- In general, the coefficients a_k are complex numbers
- this more general form can of course also be used for real signals
- there is obviously a relation between a_k and the earlier coefficients A_k and θ_k. (can you show it?)

Exponential Fourier Series - Example

Let $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$ with $a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}$

- What is the fundamental frequency and the period of this signal? $\omega_0 = 2\pi\ (\frac{rad}{s})$

  Because $\omega_0 = \frac{2\pi}{T}$, this implies that $T = 1\ (s)$. Also: $f_0 = \frac{1}{T} = 1\ (Hz)$.

- What frequencies does this signal contain?

  It contains the fundamental frequency $\omega_0 = 2\pi$ and its 2nd and 3rd harmonics ($2\omega_0$ and $3\omega_0$, respectively). It doesn't contain any other frequencies.

- What is the mean-value of this signal? The mean value is $a_0 = 1$.

- Is $x(t)$ a real-valued or a complex-valued signal?

$$k = \pm 1:\quad a_1 e^{j2\pi t} + a_{-1} e^{-j2\pi t} = \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) = \frac{1}{2}\frac{\left(e^{j2\pi t} + e^{-j2\pi t}\right)}{2} = \frac{1}{2}\cos(2\pi t)$$

$$k = \pm 2:\quad a_2 e^{j4\pi t} + a_{-2} e^{-j4\pi t} = \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) = \frac{\left(e^{j4\pi t} + e^{-j4\pi t}\right)}{2} = \cos(4\pi t)$$

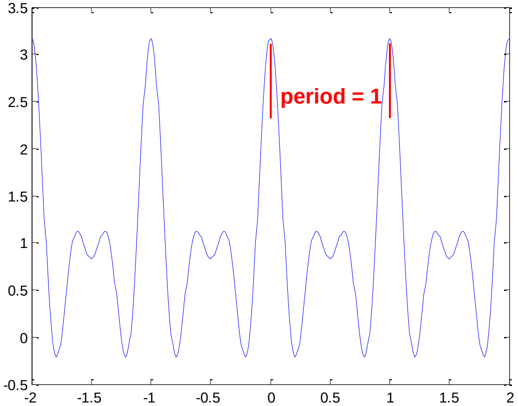
$$k = \pm 3:\quad a_3 e^{j6\pi t} + a_{-3} e^{-j6\pi t} = \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right) = \frac{2}{3}\frac{\left(e^{j6\pi t} + e^{-j6\pi t}\right)}{2} = \frac{2}{3}\cos(6\pi t)$$

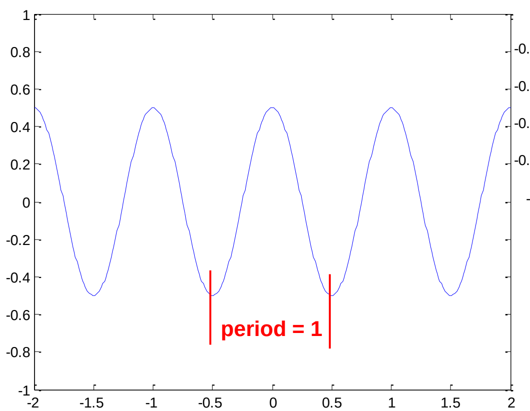
It is a <u>real-valued signal</u> because on the right-hand side of the Fourier Series equation the <u>addition of these 3 complex conjugate terms results in a real quantity</u>.

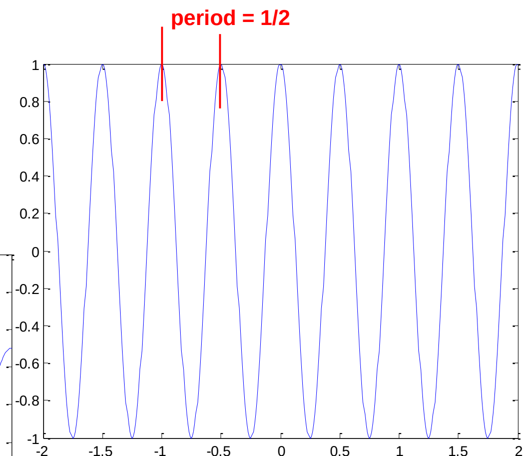
Exponential Fourier Series – Example (2)

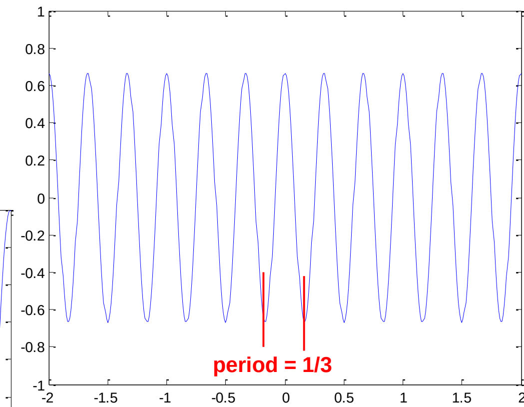
Hence: $$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t} = 1 + \frac{1}{2}\frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + \frac{e^{j4\pi t} + e^{-j4\pi t}}{2} + \frac{2}{3}\frac{e^{j6\pi t} + e^{-j6\pi t}}{2}$$

$$= 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$









The Fourier Series – Summary

Fourier Series
(exponential form)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

a_k : Fourier series coefficients (or spectral coefficients)

$\omega_0 = 2\pi \, f_0 = \dfrac{2\pi}{T}$: fundamental frequency

T : fundamental period

- For real periodic signals, assuming $a_k = A_k e^{j\theta_k}$

Fourier Series
(trigonometric form)

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

The Fourier Series – Graphical Plot

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

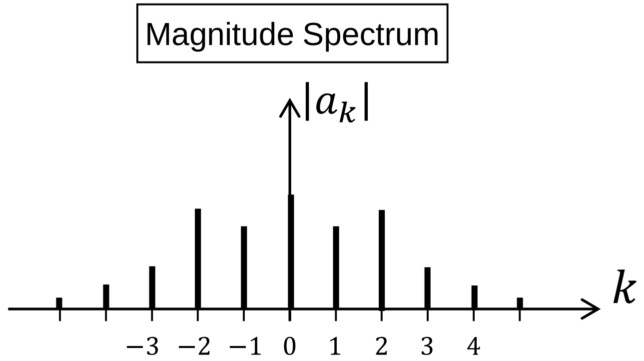
a_k is <u>in general a complex number</u> and corresponds to frequency $k\omega_0$.

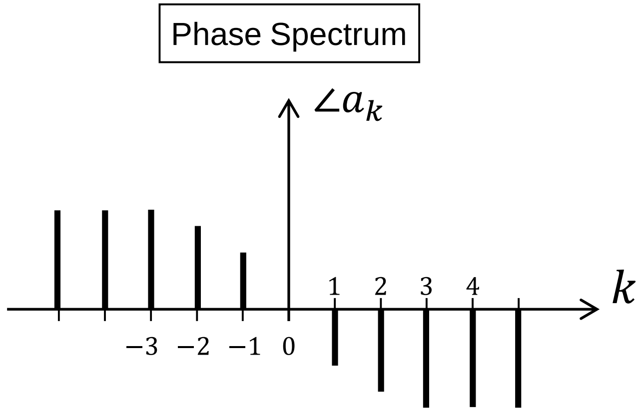
We can therefore show in a first graph the plot of $|a_k|$ versus frequency $k\omega_0$

➔ **Magnitude Spectrum**

We can show in a second graph the phase $\angle a_k$ versus frequency $k\omega_0$

➔ **Phase Spectrum**





Note: the frequency axis is <u>discrete</u> !

The Exponential Fourier Series & Real Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- If $x(t)$ is real, then the sum on the right-hand side must also be real
- For this, the k^{th}-term and the $(-k)^{\text{th}}$-term must form a complex-conjugate pair:

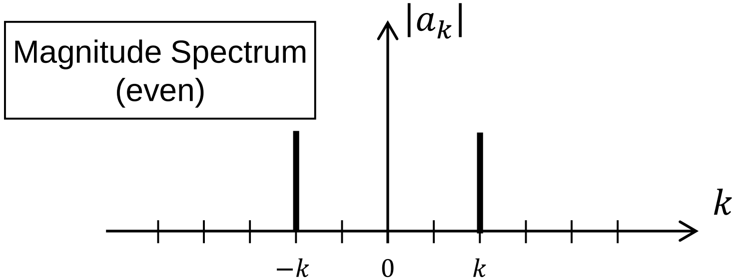
$$a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} = \left(a_k^R + ja_k^I\right)e^{jk\omega_0 t} + \left(a_k^R - ja_k^I\right)e^{-jk\omega_0 t}$$

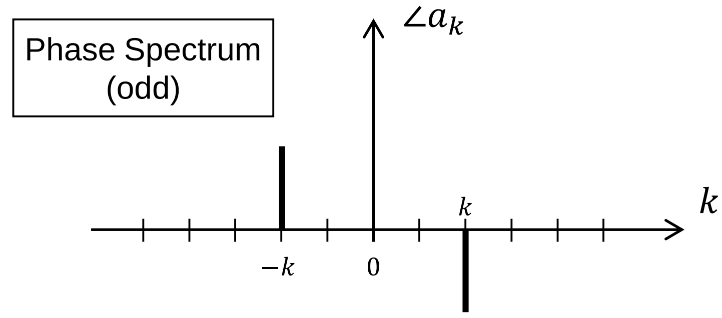
$$= a_k^R\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + ja_k^I\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= 2a_k^R \cos k\omega_0 t - 2a_k^I \sin k\omega_0 t \quad \rightarrow \text{real !}$$

- When the k^{th}-term and the $(-k)^{\text{th}}$-term are <u>complex conjugates for all k</u>, the <u>magnitude spectrum is an even function and the phase spectrum is an odd function</u> of frequency:

$$a_k = a_k^R + ja_k^I = \sqrt{(a_k^R)^2 + (a_k^I)^2}\; e^{j\,\mathrm{atan}(a_k^I/a_k^R)} \qquad a_{-k} = a_k^R - ja_k^I = \sqrt{(a_k^R)^2 + (a_k^I)^2}\; e^{-j\,\mathrm{atan}(a_k^I/a_k^R)}$$





The Fourier Series – Summary

Fourier Series
(exponential form)

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}}$$

How do we compute the a_k's ?

Determination of the Fourier Series Coefficients

$$x(t) = \sum_k a_k e^{jk\omega_0 t}$$

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_k a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$= \sum_k a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt = \sum_k a_k \underbrace{\int_0^T \{\cos[(k-n)\omega_0 t] + j \sin[(k-n)\omega_0 t]\}\, dt}_{\text{if } k \neq n \text{, this integral} = 0}$$

$$= a_n \int_0^T dt = a_n T$$

$$\boxed{\Rightarrow a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt}$$

The Fourier Series – Summary

Fourier Series
(exponential form)

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}}$$

Synthesis equation

$$\boxed{a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt}$$

Analysis equation

The Fourier series coefficient a_k specifies the characteristics (amplitude & phase) of the signal that is found at frequency $k\omega_0$

Also a_0 is the <u>average value</u> of $x(t)$:

$$\boxed{a_0 = \frac{1}{T}\int_T x(t) dt}$$

The Fourier Series Computation – Examples

Example 1

Find the Fourier Series coefficients for the signal $x(t) = \sin \omega_0 t$.

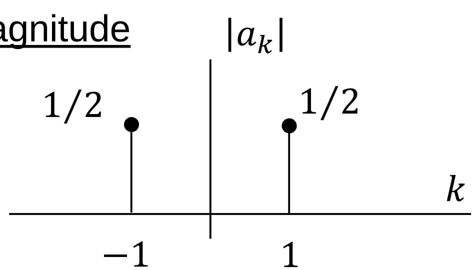
First possible approach: $$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T \sin(\omega_0 t)\ e^{-jn\omega_0 t}dt$$
$$= \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \sin(\omega_0 t)\ e^{-jn\omega_0 t}dt = \cdots$$

Second possible approach:

$$x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} = \textstyle\sum_k a_k e^{jk\omega_0 t}$$

$$\boxed{\Rightarrow a_k = \begin{cases} 1/2j & k = 1 \\ -1/2j & k = -1 \\ 0 & \text{otherwise} \end{cases}}$$

magnitude



phase