EEEN 322
Communication Engineering

İpek Şen

Spring 2019

Week 3

Last Week

- Classification of Signals
  - Continuous-time and discrete-time
  - Analog and digital
  - Periodic and aperiodic
  - Energy and power
  - Deterministic and random
- Size of a signal
  - Total energy
  - Total average power
- Some elementary signals
  - Unit impulse (dirac delta) – (multiplication of a function by unit impulse, sampling property of the unit impulse)
  - Unit step
- Analogy between vector space and signal space
- Orthogonal basis vectors, orthogonal basis signals
- Writing a vector in terms of orthogonal basis vectors
- Writing a signal in terms of orthogonal signals → Generalized Fourier Series
- Writing a **periodic** signal in terms of **complex exponentials** → (Complex Exponential) Fourier Series

Orthogonal Signal Space

$$g(t) = \sum_{n=1}^{\infty} c_n x_n(t), \quad t_1 \le t \le t_2$$

Generalized Fourier Series

where

$$c_n = \frac{1}{E_n}\int_{t_1}^{t_2} g(t) x_n^*(t)dt$$

Coefficients of the generalized Fourier Series

Since $\{x_n(t)\}$ are orthogonal, we have

$$E_g = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

Parseval's Theorem

For a periodic signal $g(t)$ with period T_0, complex exponentials $\{e^{jw_0 nt}\}_{n=-\infty}^{\infty}$ form an orthogonal basis set over any interval of duration T_0.
(Note that $w_0 = \frac{2\pi}{T_0}$)

Then

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0 nt}$$

Fourier Series Expansion of periodic signal *g(t)*

Synthesis equation

where

$$c_n = \frac{1}{T_0}\int_{T_0} g(t) e^{-jw_0 nt} dt$$

Fourier Series coefficients

Analysis equation

$$w_0 = \frac{2\pi}{T_0}$$

$$\frac{1}{T_0}\int_{T_0} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Parseval's Theorem
(note that this is the exact equivalent of the one on the left)

WHY?

FOR THE ANSWER OF "WHY"...

$$< e^{jw_0mt}, e^{jw_0nt} >= \int_{T_0} e^{jw_0mt}(e^{jw_0nt})^* dt$$

$$= \int_0^{T_0} e^{jw_0mt}e^{-jw_0nt}dt$$

$$= \int_0^{T_0} e^{jw_0(m-n)t}dt$$

$$= \frac{1}{jw_0(m-n)}[e^{jw_0(m-n)t}]|_0^{T_0} \qquad \text{if } m \neq n$$

$$= \frac{1}{jw_0(m-n)}[e^{jw_0(m-n)T_0} - 1]$$

$$= \frac{1}{jw_0(m-n)}[e^{j2\pi(m-n)} - 1]$$

$$= \frac{1}{jw_0(m-n)}[1 - 1]$$

$$= 0$$

and if $m = n$, then the integral becomes

$= \int_0^{T_0} 1dt = t|_0^{T_0} = T_0 - 0 = T_0$ (the squared norm, or, the energy, of e^{jw_0nt}, $\forall n$)

So we have $< e^{jw_0mt}, e^{jw_0nt} >= \begin{cases} 0, & m \neq n \\ T_0, & m = n \end{cases}$ $\Rightarrow$

- **Complex exponentials $e^{j\omega_0 nt}$ are orthogonal to each other over the period $T_0 = 2\pi/\omega_0$**
- **Their energy over period T_0 is equal to T_0**

$$\boxed{g(t) = \sum_{n=1}^{\infty} c_n x_n(t), \quad t_1 \le t \le t_2}$$

Generalized Fourier Series

$$\boxed{E_g = \sum_{n=1}^{\infty} |c_n|^2 E_n}$$

Parseval's Theorem

If $g(t)$ is periodic with T_0, $T_0 = \dfrac{2\pi}{w_0}$, $\{x_n(t) = e^{jw_0nt}\}_{n=-\infty}^{\infty}$ form an orthogonal basis set

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0nt}}$$

Complex Exponential Fourier Series

Then

$$\boxed{E_g = \sum_{n=1}^{\infty} |c_n|^2 E_n} \Rightarrow \int_{T_0} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 T_0 \implies \boxed{\frac{1}{T_0}\int_{T_0} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

Parseval's Theorem

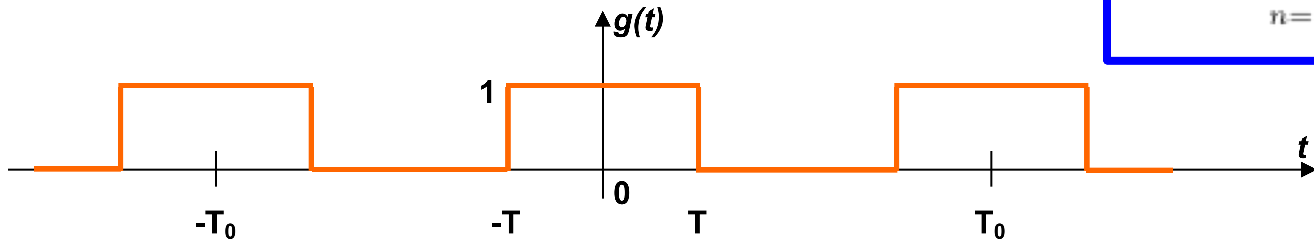
Parseval's Theorem
(for Complex Exponential Fourier Series)

THE ANSWER OF "WHY"

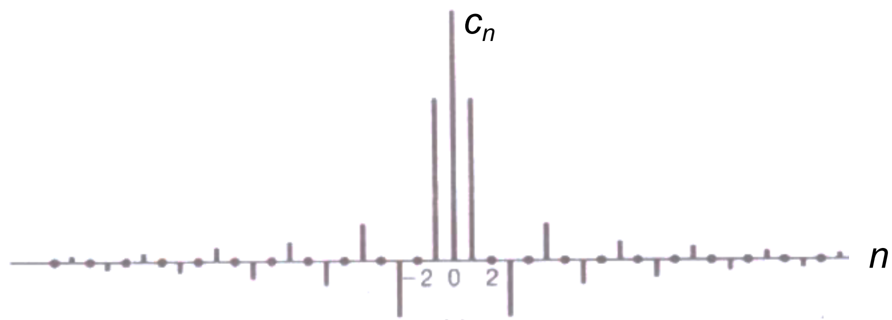
WEEK 3

Rectangular wave (periodic)

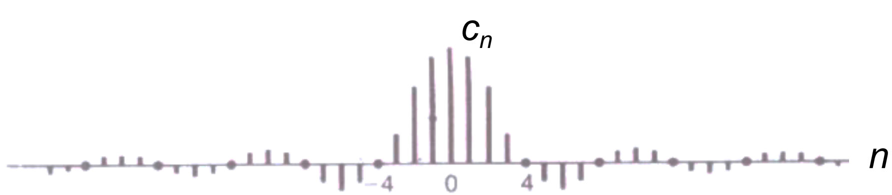
$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0 nt}$$



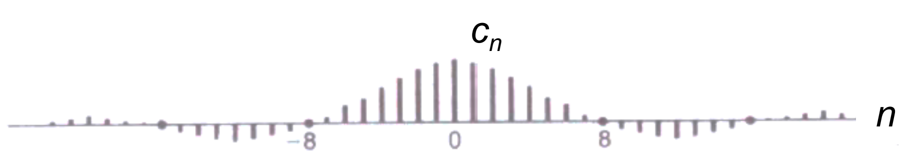
$T_0 = 4T$



$T_0 = 8T$



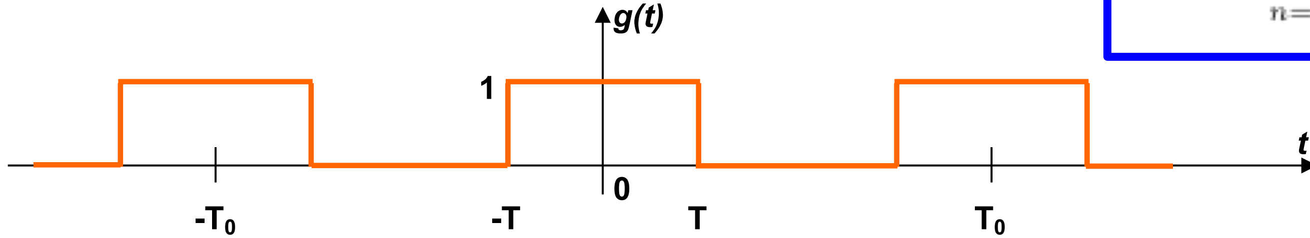
$T_0 = 16T$



(From EEEN 321)

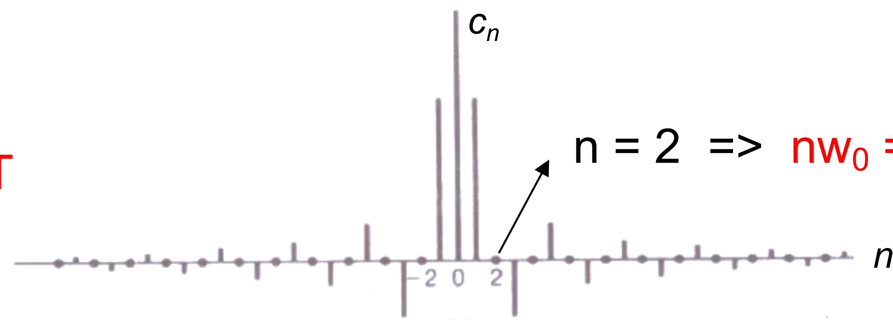
$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0 nt}$$

Rectangular wave (periodic)



$T_0 = 4T$

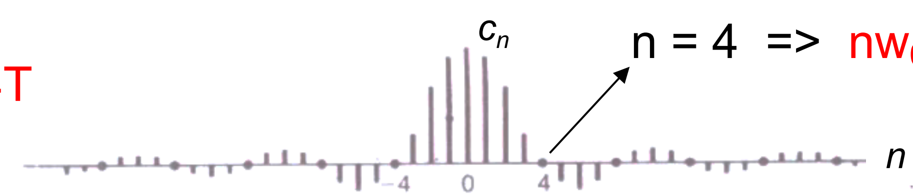
$\Rightarrow w_0 = 2\pi / T_0 = \pi/2T$



$n = 2 \Rightarrow nw_0 = \pi/T$

$T_0 = 8T$

$\Rightarrow w_0 = 2\pi / T_0 = \pi/4T$

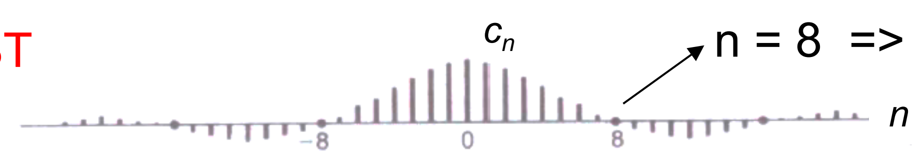


$n = 4 \Rightarrow nw_0 = \pi/T$

Same frequency!

$T_0 = 16T$

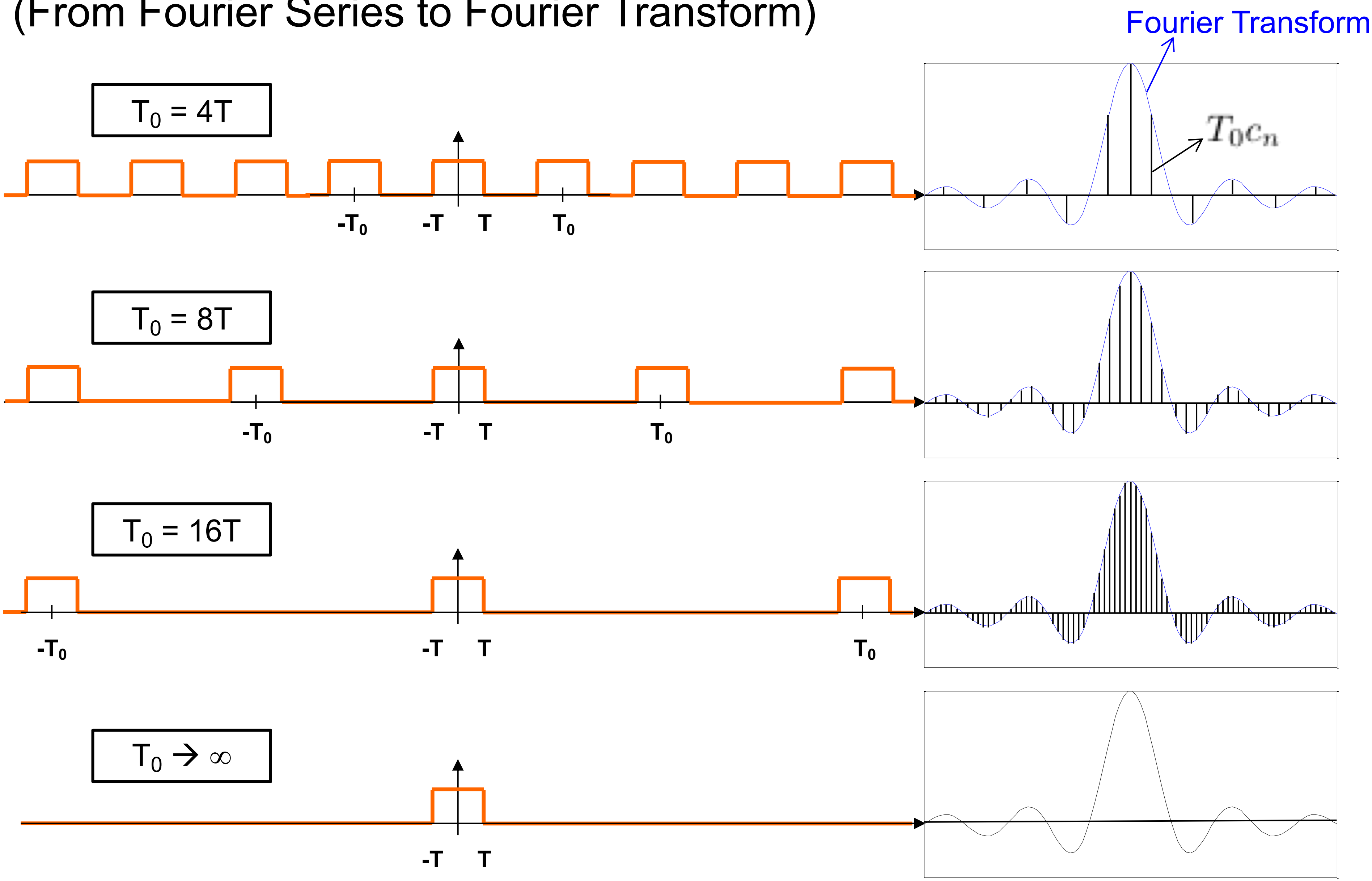
$\Rightarrow w_0 = 2\pi / T_0 = \pi/8T$



$n = 8 \Rightarrow nw_0 = \pi/T$

(From EEEN 321)

From Periodic to Aperiodic Signals
(From Fourier Series to Fourier Transform)



Remember: $w_0 = \dfrac{2\pi}{T_0}$

Now the x-axis shows $w = nw_0$

(Figures are from EEEN 321)

From Periodic to Aperiodic Signals
(From Fourier Series to Fourier Transform)

$$T_0 \to \infty$$

$$\Longrightarrow w_0 = \frac{2\pi}{T_0} \to 0$$ (nw_0 becomes continuous w)

$$\Longrightarrow \boxed{g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0 n t}} \quad \rightarrow \quad \boxed{g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w) e^{jwt}\, dw}$$

Inverse Fourier Transform

$$\boxed{c_n = \frac{1}{T_0}\int_{T_0} g(t) e^{-jw_0 n t}\, dt} \quad \rightarrow \quad \boxed{G(w) = \int_{-\infty}^{\infty} g(t) e^{-jwt}\, dt}$$

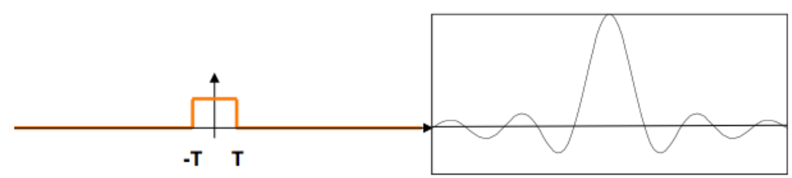
(Direct) Fourier Transform

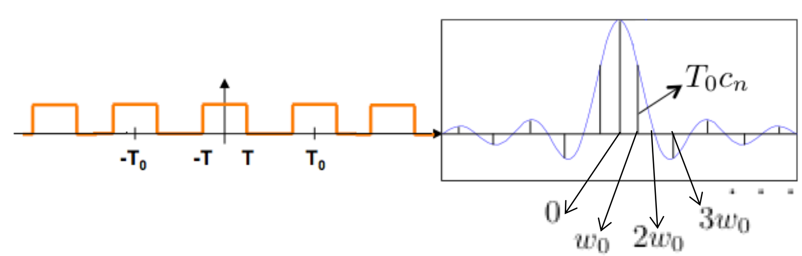
$$c_n = \frac{1}{T_0} G(nw_0)$$

The relationship between Fourier Series & Fourier Transform

To obtain the Fourier Series coefficients from the Fourier Transform

– Sample the Fourier Transform at frequencies nw_0
– Divide by T_0





Fourier Transform

$$\boxed{G(w) = \int_{-\infty}^{\infty} g(t)e^{-jwt}dt}$$

(Direct) Fourier Transform

$$\boxed{g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w)e^{jwt}dw}$$

Inverse Fourier Transform

$$\boxed{\int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(w)|^2 dw}$$

Parseval's Theorem *(For Fourier Transform)*

Dirichlet Conditions for the Fourier Transform

Let $g(t)$ satisfy the Dirichlet conditions:

1. $\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$
2. The number of maxima and minima of $g(t)$ on any finite interval on the real line is finite
3. The number of discontinuities of $g(t)$ on any finite interval on the real line is finite

then the Fourier transform of $g(t)$:

$$\boxed{G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt}$$

exists and the original signal can be obtained from its Fourier transform by

$$\boxed{g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega}$$

Dirichlet conditions are “sufficient” conditions, not “necessary” conditions

Since $G(\omega)$ is complex in general, it has a magnitude and phase:

$$G(\omega) = |G(\omega)| e^{j\theta_g(\omega)}$$

Magnitude: $|G(\omega)|$
Phase: $\theta_g(\omega)$

Conjugate Symmetry Property

If $g(t)$ is real, then we have:

$$G(-\omega) = G^*(\omega)$$

$G(\omega)$ and $G(-\omega)$ are complex conjugates

Therefore we have:

$$|G(-\omega)| = |G(\omega)|$$

$|G(\omega)|$ is an even function of ω

$$\theta_g(-\omega) = -\theta_g(\omega)$$

$\theta_g(\omega)$ is an odd functioin of ω

→ ***If $g(t)$ is real, then $G(\omega)$ and $G(-\omega)$ are complex conjugates, therefore $|G(\omega)|$ is even and $\theta_g(\omega)$ is odd***

Suppose that $g(t)$ is real:

$$G(\omega) = \int g(t)e^{-j\omega t}dt = \int g(t)\left(\cos\omega t - j\sin\omega t\right)dt = \int g(t)\cos\omega t\,dt - j\int g(t)\sin\omega t\,dt$$

If *g(t)* is **real**

… and **even** → *G(ω)* is **real and even**

… and **odd**→ *G(ω)* is **imaginary and odd**

Suppose *g*(*t*) is purely imaginary, i.e., $g(t) = jg_i(t)$, where $g_i(t)$ is real:

$$G(\omega) = j\int g_i(t)e^{-j\omega t}dt = j\int g_i(t)\left(\cos\omega t - j\sin\omega t\right)dt = j\int g_i(t)\cos\omega t\,dt + \int g_i(t)\sin\omega t\,dt$$

If *g(t)* is **imaginary**

… and **even** → *G(ω)* is **imaginary and even**

… and **odd**→ *G(ω)* is **real and odd**

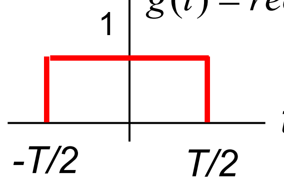
g(t)	G(ω)
real and even	real and even
real and odd	imaginary and odd
imaginary and even	imaginary and even
imaginary and odd	real and odd

$$G(-\omega) = G^*(\omega) \Rightarrow \begin{cases} |G(-\omega)| = |G(\omega)| \\ \theta_g(-\omega) = -\theta_g(\omega) \end{cases}$$

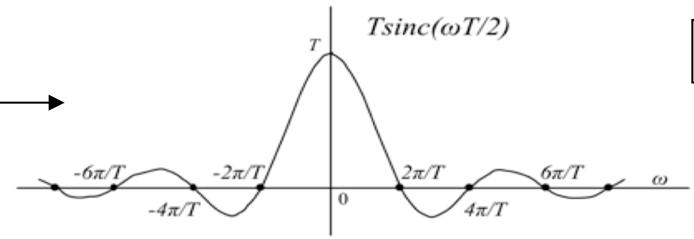
(Conjugate Symmetry Property)

Some Fourier Transform Pairs

$g(t) = rect(t/T)$

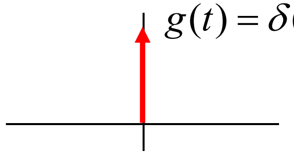


$\longleftrightarrow$



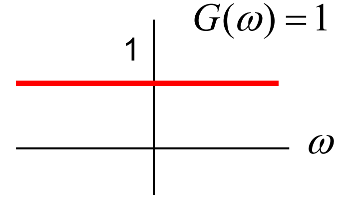
$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$

$g(t) = \delta(t)$

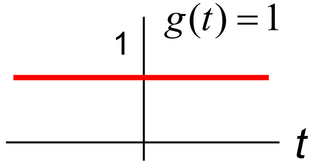


$\longleftrightarrow$

$G(\omega) = 1$

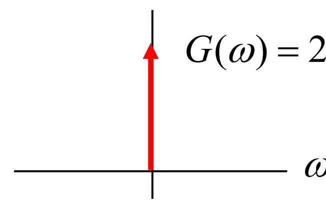


$g(t) = 1$

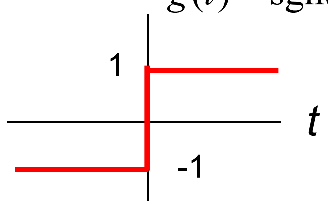


$\longleftrightarrow$

$G(\omega) = 2\pi\delta(\omega)$



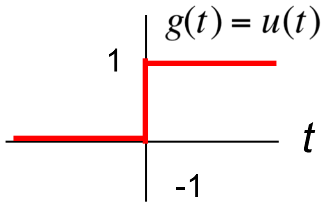
$g(t) = \mathrm{sgn}(t)$



$\longleftrightarrow$

$$\frac{2}{j\omega}$$

$g(t) = u(t)$

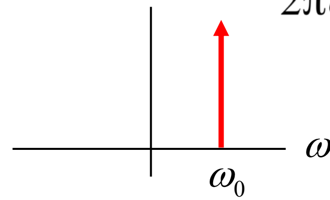


$\longleftrightarrow$

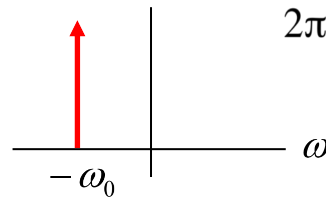
$$\frac{1}{j\omega} + \pi\delta(\omega)$$

Some Fourier Transform Pairs (cont'd)

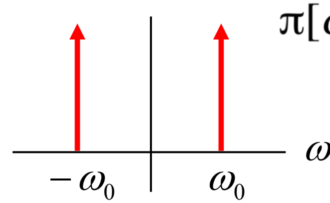
$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$$



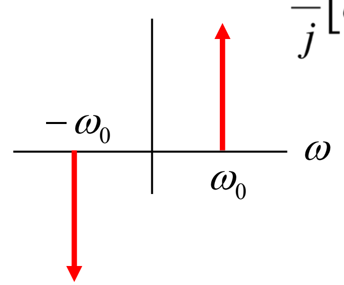
$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega+\omega_0)$$



$$\cos\omega_0 t = \left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right) \longleftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$



$$\sin\omega_0 t = \left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right) \longleftrightarrow \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$



Properties of the Fourier Transform

If $g(t) \leftrightarrow G(\omega)$ then

1. Duality $$G(t) \leftrightarrow 2\pi\, g(-\omega)$$

2. Time-Scaling $$g(at) \leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$$

3. Time-shifting $$g(t-t_0) \leftrightarrow G(\omega)e^{-j\omega t_0}$$

4. Frequency-shifting $$g(t)e^{j\omega_0 t} \leftrightarrow G(\omega-\omega_0)$$

5. Time Convolution

&

Frequency Convolution

$$\left.\begin{array}{l} g_1(t) \leftrightarrow G_1(\omega) \\ g_2(t) \leftrightarrow G_2(\omega) \end{array}\right\} \Rightarrow \quad g_1(t) * g_2(t) \leftrightarrow G_1(\omega) \cdot G_2(\omega)$$

$$\left.\begin{array}{l} g_1(t) \leftrightarrow G_1(\omega) \\ g_2(t) \leftrightarrow G_2(\omega) \end{array}\right\} \Rightarrow \quad g_1(t) g_2(t) \leftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$$

6. Time Differentiation

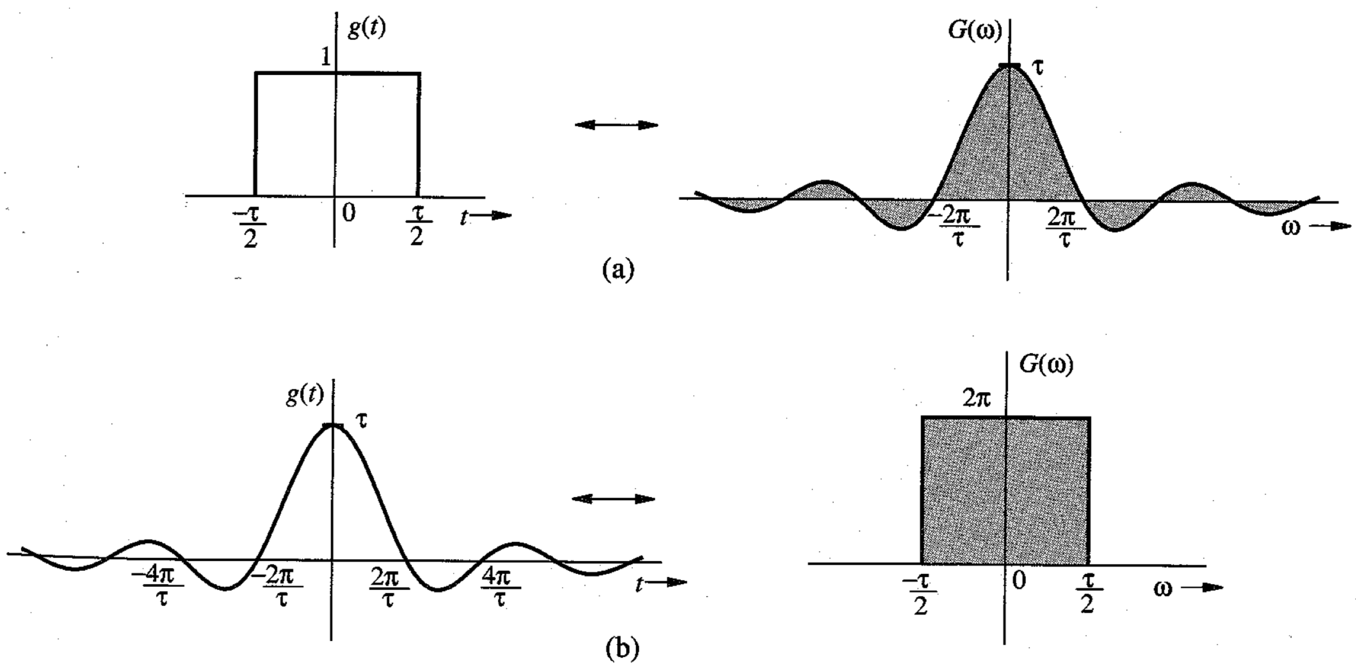
&

Time Integration

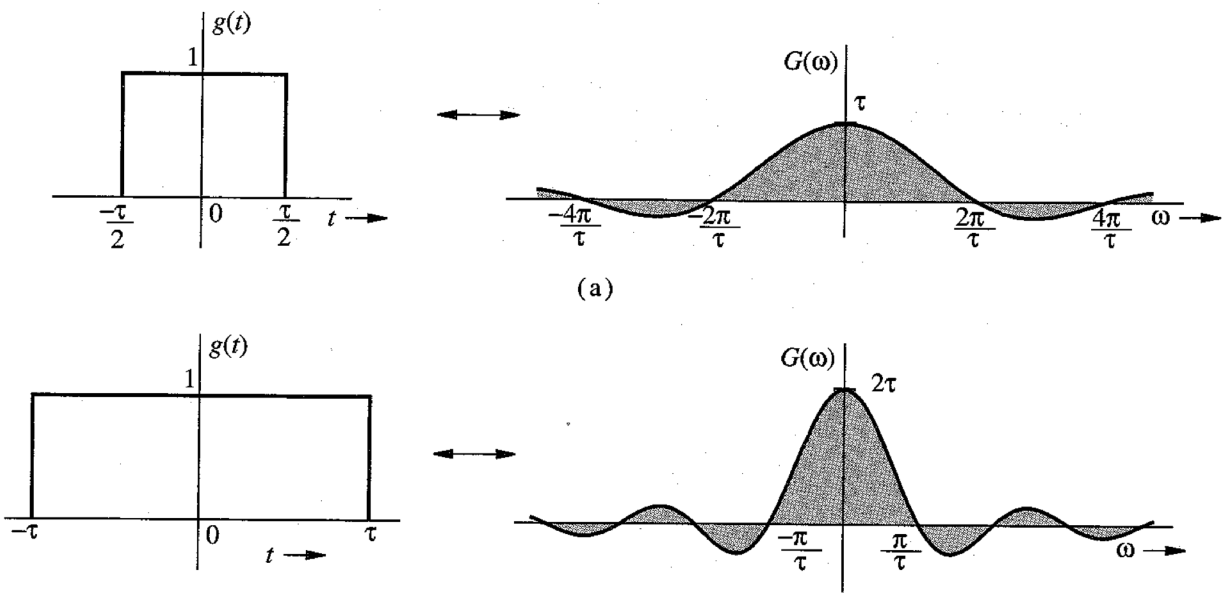
$$\frac{d}{dt} g(t) \leftrightarrow j\omega G(\omega)$$

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{G(\omega)}{j\omega} + \pi\, G(0)\,\delta(\omega)$$

1. Duality If $g(t) \leftrightarrow G(\omega)$ then $G(t) \leftrightarrow 2\pi\, g(-\omega)$

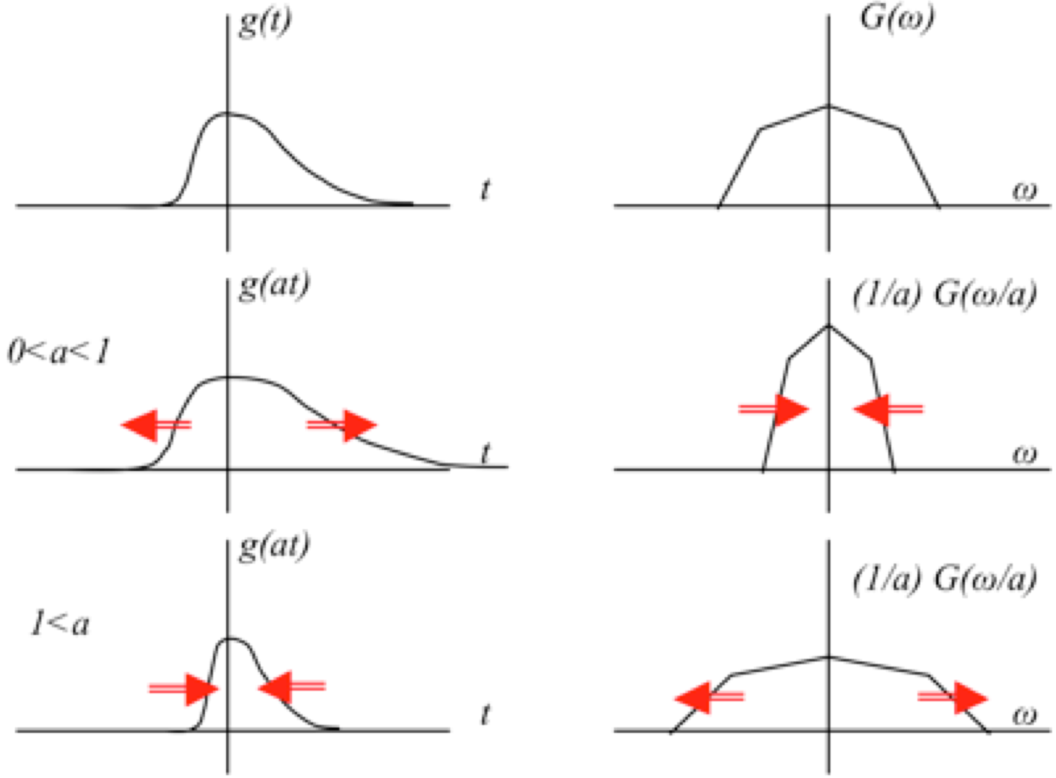


2. Time-Scaling If $g(t) \leftrightarrow G(\omega)$ then $g(at) \leftrightarrow \frac{1}{|a|} G(\frac{\omega}{a})$



example for a=1/2

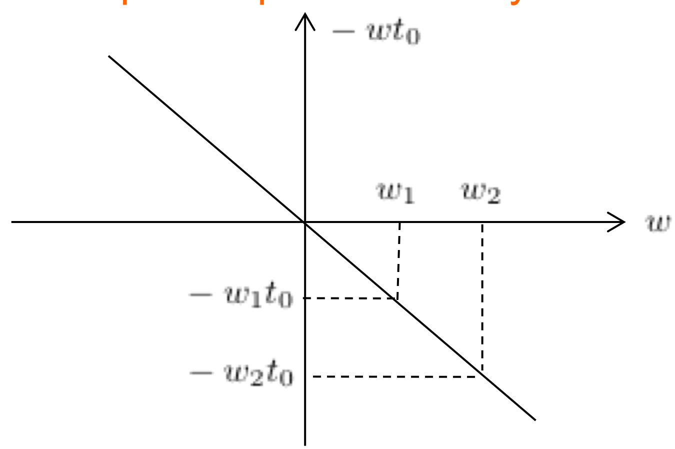
Reciprocity of signal duration and signal bandwidth



3. Time-shifting If $g(t) \leftrightarrow G(\omega)$ then $g(t-t_0) \leftrightarrow G(\omega)e^{-j\omega t_0} = |G(\omega)|e^{j\theta_g(\omega)}e^{-j\omega t_0} = |G(\omega)|e^{j[\theta_g(\omega)-\omega t_0]}$

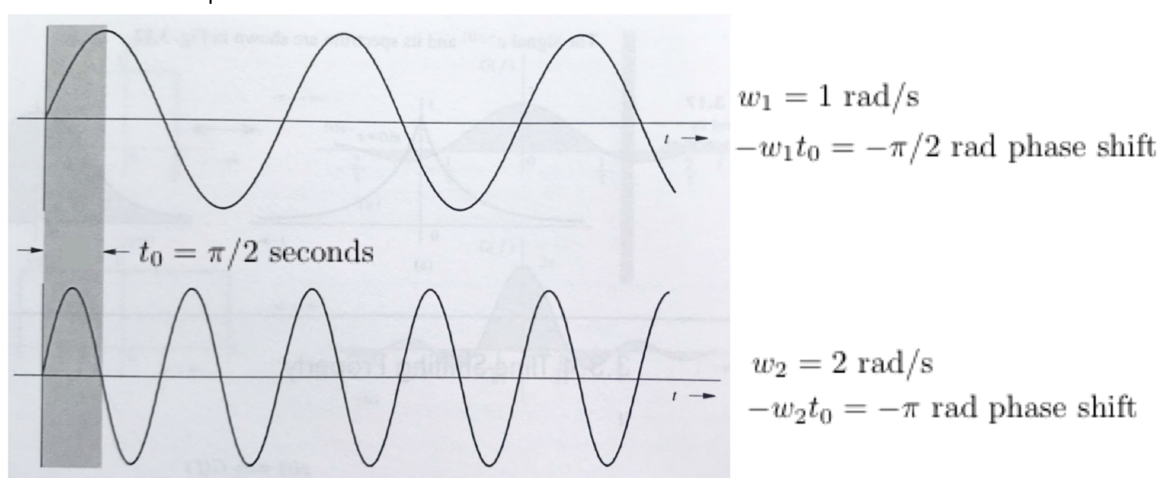
A constant time delay in a signal causes a linear phase shift in its frequency spectrum
Equivalently: a linear phase shift implies equal time delay for all frequency components

Linear shift in phase



Suppose that *g(t)* is sum of two sinusoids with angular frequencies 1 rad/s and 2 rad/s.

If *g(t)* undergoes a time delay of π/2 seconds, it corresponds to a phase delay of
- π/2 radians for the former sinusoid
- π radians for the latter sinusoid



(Higher frequency components must undergo proportionately higher phase shifts to achieve the same time delay)

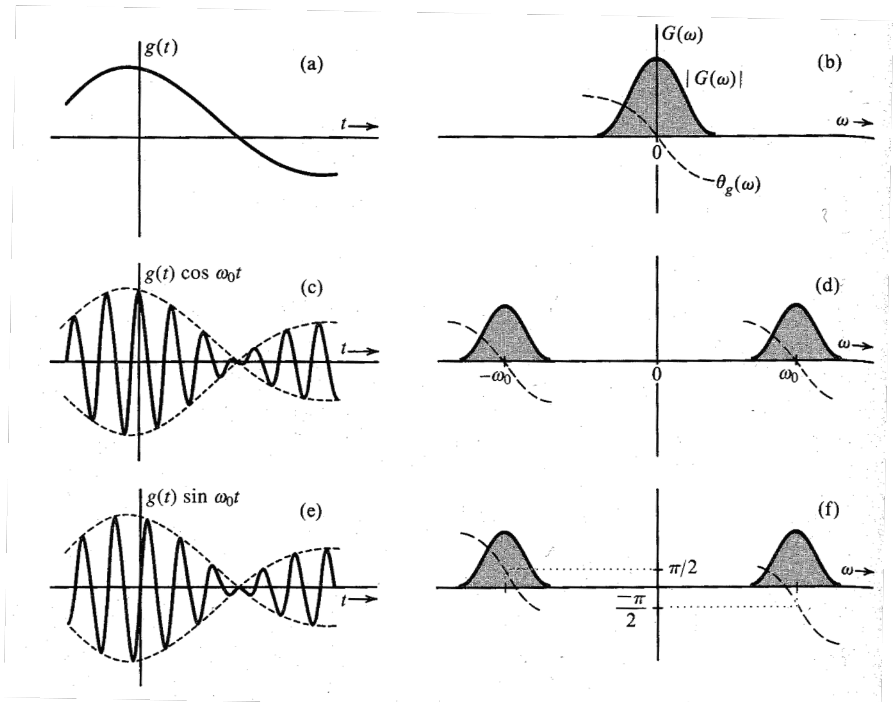
4. Frequency-shifting If $g(t) \leftrightarrow G(\omega)$ then $g(t)e^{j\omega_0 t} \leftrightarrow G(\omega - \omega_0)$

As a result of this property:

$$g(t)\cos\omega_0 t = g(t)\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] \leftrightarrow \frac{1}{2}\left[G(\omega - \omega_0) + G(\omega + \omega_0)\right]$$

and, similarly:

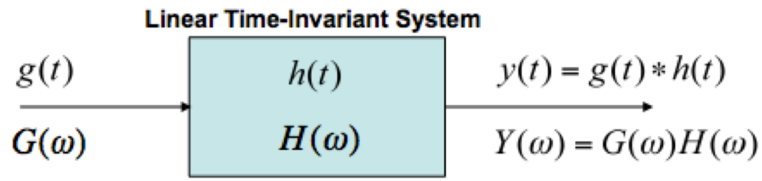
$$g(t)\sin\omega_0 t = g(t)\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] \leftrightarrow \frac{1}{2j}\left[G(\omega - \omega_0) - G(\omega + \omega_0)\right]$$



5.

Time Convolution

If $g_1(t) \leftrightarrow G_1(\omega)$, $g_2(t) \leftrightarrow G_2(\omega)$ then $g_1(t) * g_2(t) \leftrightarrow G_1(\omega) \cdot G_2(\omega)$



$h(t)$: **system impulse response**

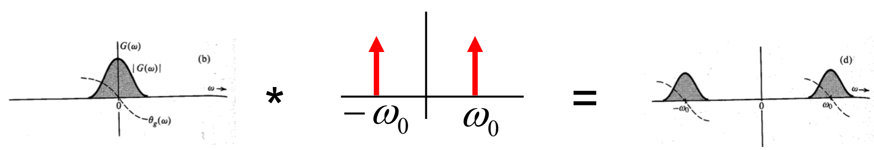
$H(\omega)$: **system frequency response (transfer function)**

Frequency Convolution

If $g_1(t) \leftrightarrow G_1(\omega)$, $g_2(t) \leftrightarrow G_2(\omega)$ then $g_1(t) g_2(t) \leftrightarrow \dfrac{1}{2\pi} G_1(\omega) * G_2(\omega)$

The Fourier Transform of $g(t)\cos\omega_0 t$ can also be found by using this property:

$$g(t)\cos\omega_0 t \leftrightarrow \frac{1}{2\pi}\left\{G(\omega) * \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\right\} = \frac{1}{2}\left[G(\omega-\omega_0)+G(\omega+\omega_0)\right]$$

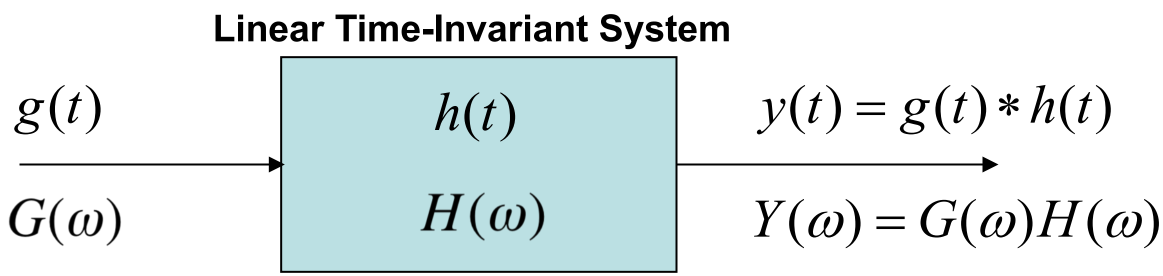


Similarly:

$$g(t)\sin\omega_0 t \leftrightarrow \frac{1}{2\pi}\left\{G(\omega) * \frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]\right\} = \frac{1}{2j}\left[G(\omega-\omega_0)-G(\omega+\omega_0)\right]$$

EEEN 322 – Some Basics

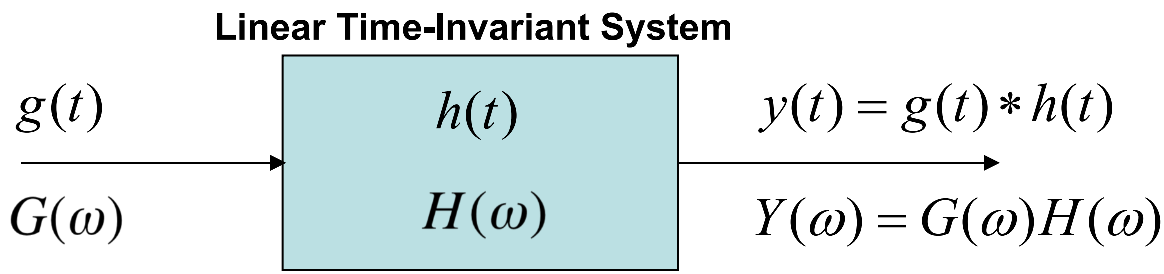
Signal Transmission Through a Linear System



$h(t)$ **: impulse response of the system**

$H(\omega)$**: frequency response (transfer function) of the system**

Distortionless Transmission



For distortionless transmission, the output is only allowed to be a scaled and shifted version of the input

$$y(t) = kg(t - t_d) \quad \textbf{OR} \quad Y(\omega) = ke^{-j\omega t_d}G(\omega)$$

Find the transfer function:

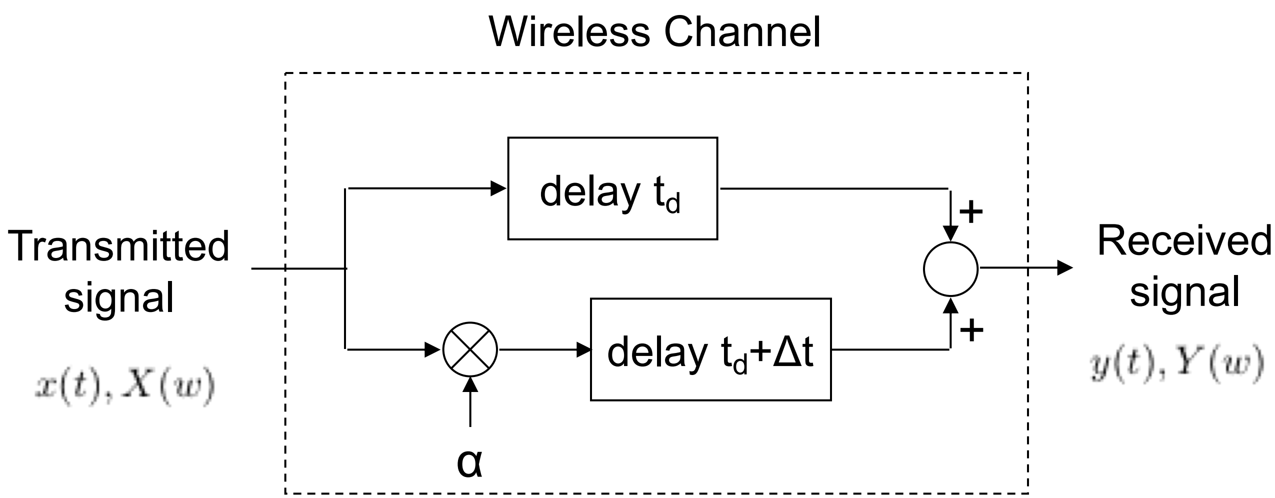
$$Y(\omega) = ke^{-j\omega t_d}G(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{G(\omega)} = ke^{-j\omega t_d}$$

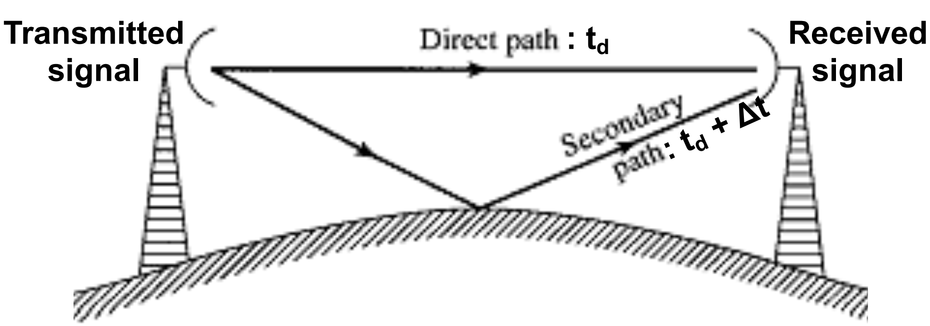
Therefore, for distortionless transmission, the transfer function should be in form:

$$\boxed{H(\omega) = k\, e^{-j\omega t_d}}$$

DISTORTIONLESS SYSTEM

Linear Distortion Due to Multipath Effects





$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = e^{-j\omega t_d} + \alpha e^{-j\omega(t_d+\Delta t)}$$

$$|H(\omega)|^2 = 1 + \alpha^2 + 2\alpha\cos(\omega\Delta t)$$

$$\theta_h(\omega) = -\omega t_d - \tan^{-1}\frac{\alpha\sin(\omega\Delta t)}{1+\alpha\cos(\omega\Delta t)}$$

Nonlinear Signal Distortion and Related Spectra

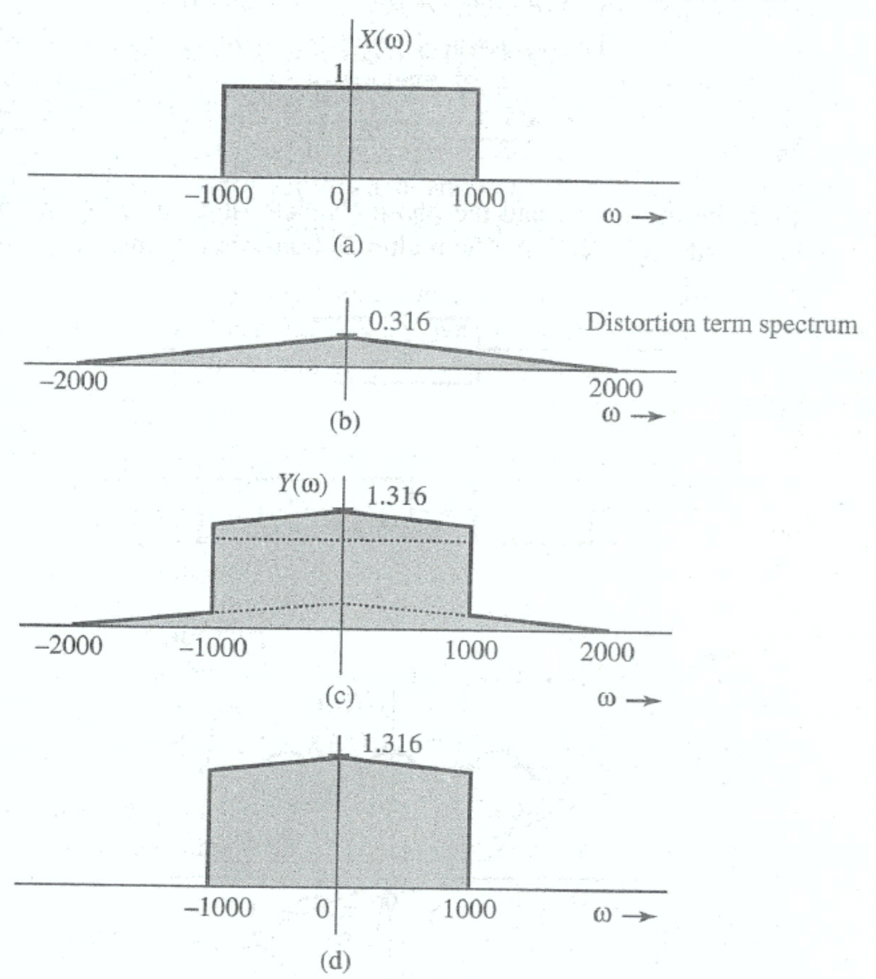
Example: $y(t) = x(t) + 0.001x^2(t)$ where $x(t) = \dfrac{1000}{\pi}\text{sinc}(1000t)$

$\mathcal{F}\{x(t)\}$

$\mathcal{F}\{0.001x^2(t)\}$

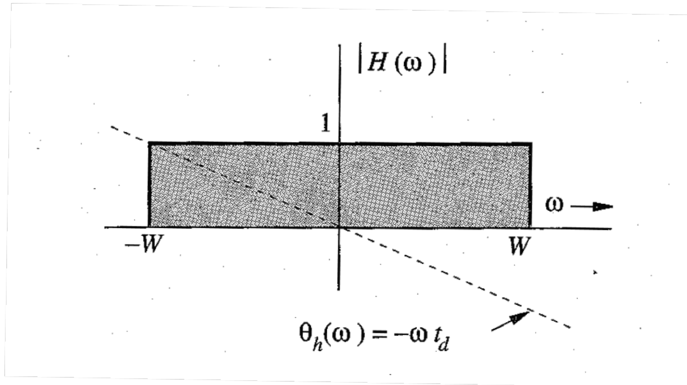
$\mathcal{F}\{x(t)\} + \mathcal{F}\{0.001x^2(t)\}$

Low-pass filtered

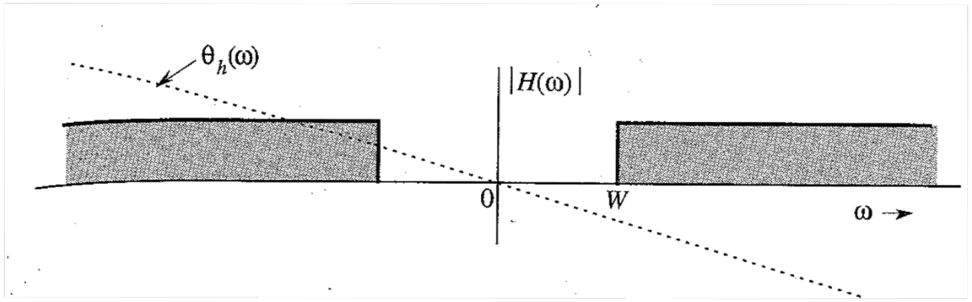


Ideal Filters

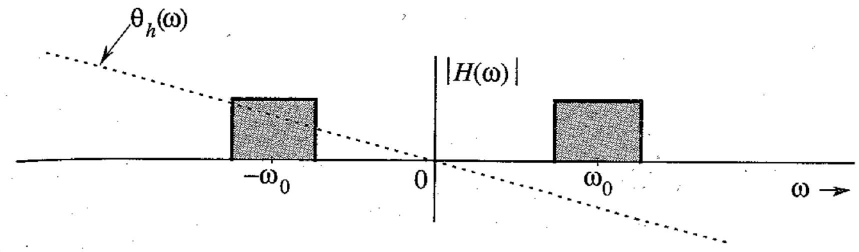
Ideal low-pass filter



Ideal high-pass filter

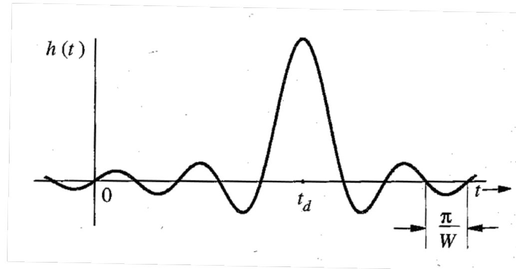


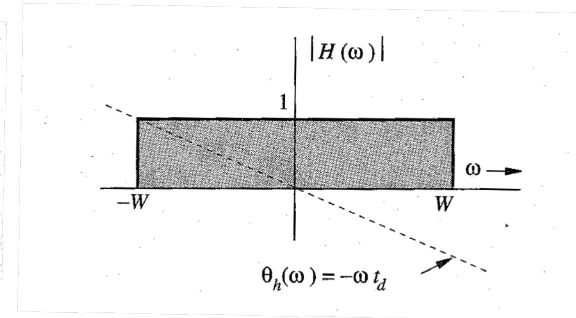
Ideal bandpass filter



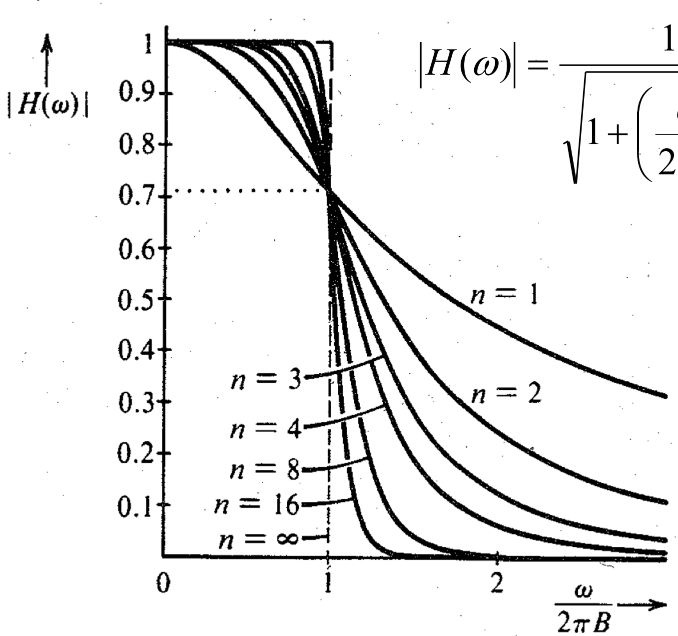
Ideal versus Practical Filters

An ideal filter has an impulse response with infinite duration!
(Not realizable)



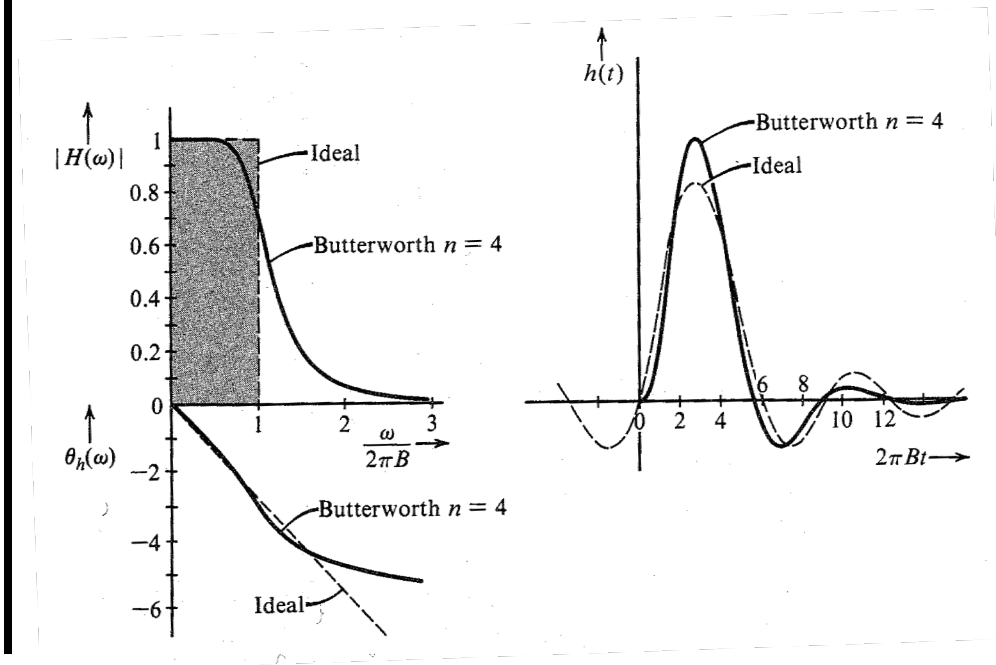


Butterworth filter characteristic:



$$|H(\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{2\pi B}\right)^{2n}}}$$

Comparison of Butterworth (*n=4*) and ideal low-pass filter:



Energy Spectral Density (ESD)

Energy
$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(w)|^2 dw$$
(Parseval)

This is called "Energy Spectral Density" (ESD)

Energy Spectral Density (ESD) $\boxed{\Psi_g(w) = |G(w)|^2}$

Note that you may also write: $E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_g(w) dw$

ESD of input / output:

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$Y(w) = H(w)G(w) \implies |Y(w)|^2 = |H(w)|^2|G(w)|^2$$

$$\boxed{\Psi_y(w) = |H(w)|^2\Psi_g(w)}$$

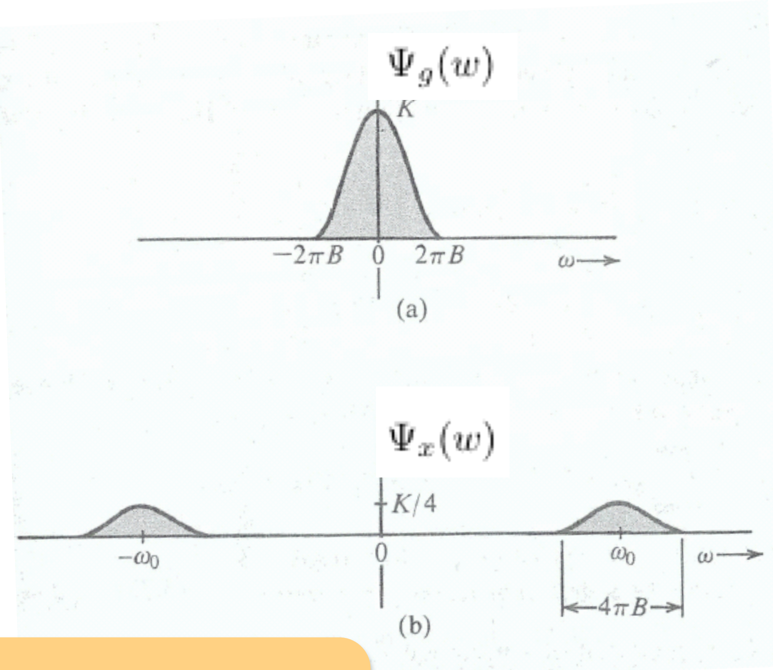
Energy Spectral Density (ESD) of *x(t)=g(t)cosω₀t*

$$x(t) = g(t)cosw_0t$$

$$\Longrightarrow \ X(w) = \frac{1}{2}[G(w-w_0) + G(w+w_0)]$$

$$\Longrightarrow \ |X(w)|^2 = \frac{1}{4}[|G(w-w_0)|^2 + |G(w+w_0)|^2]$$

$$\Longrightarrow \ \Psi_x(w) = \frac{1}{4}[\Psi_g(w-w_0) + \Psi_g(w+w_0)]$$



Note that: $E_x = \frac{1}{2}E_g$

because...

$$\begin{aligned} E_x &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_x(w)dw \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{1}{4}[\Psi_g(w-w_0) + \Psi_g(w+w_0)]\}dw \\ &= \frac{1}{8\pi}\int_{-\infty}^{\infty}\Psi_g(w-w_0)dw + \frac{1}{8\pi}\int_{-\infty}^{\infty}\Psi_g(w+w_0)dw \\ &= \frac{1}{4}E_g + \frac{1}{4}E_g \\ &= \frac{1}{2}E_g \end{aligned}$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_g(w)dw$$