STEADY-STATE ERRORS

e_{ss}

(Textbook Chap. 7)

Definition of Steady-State Errors, e_{ss}

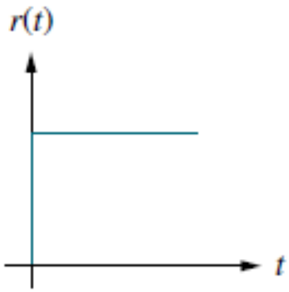
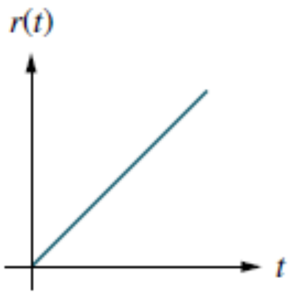
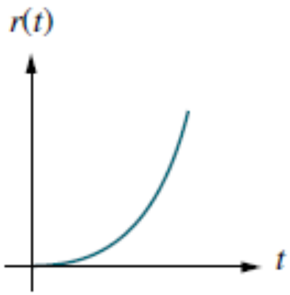
- Steady-state error is the difference between the input and the output for a prescribed test input as $t \rightarrow \infty$. In order to explain how these test signals are used, let us assume a position control system, where the output position follows the input commanded position.

- Step inputs represent constant position and thus are useful in determining the ability of the control system to position itself with respect to a stationary target.

- Ramp inputs represent constant-velocity inputs to a position control system by their linearly increasing amplitude. These waveforms can be used to test a system's ability to follow a linearly increasing input or, equivalently, to track a constant velocity target.

- Parabolas, whose second derivatives are constant, represent constant acceleration inputs to position control systems and can be used to represent accelerating targets.

Test Waveforms for Evaluating Steady-state Errors

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

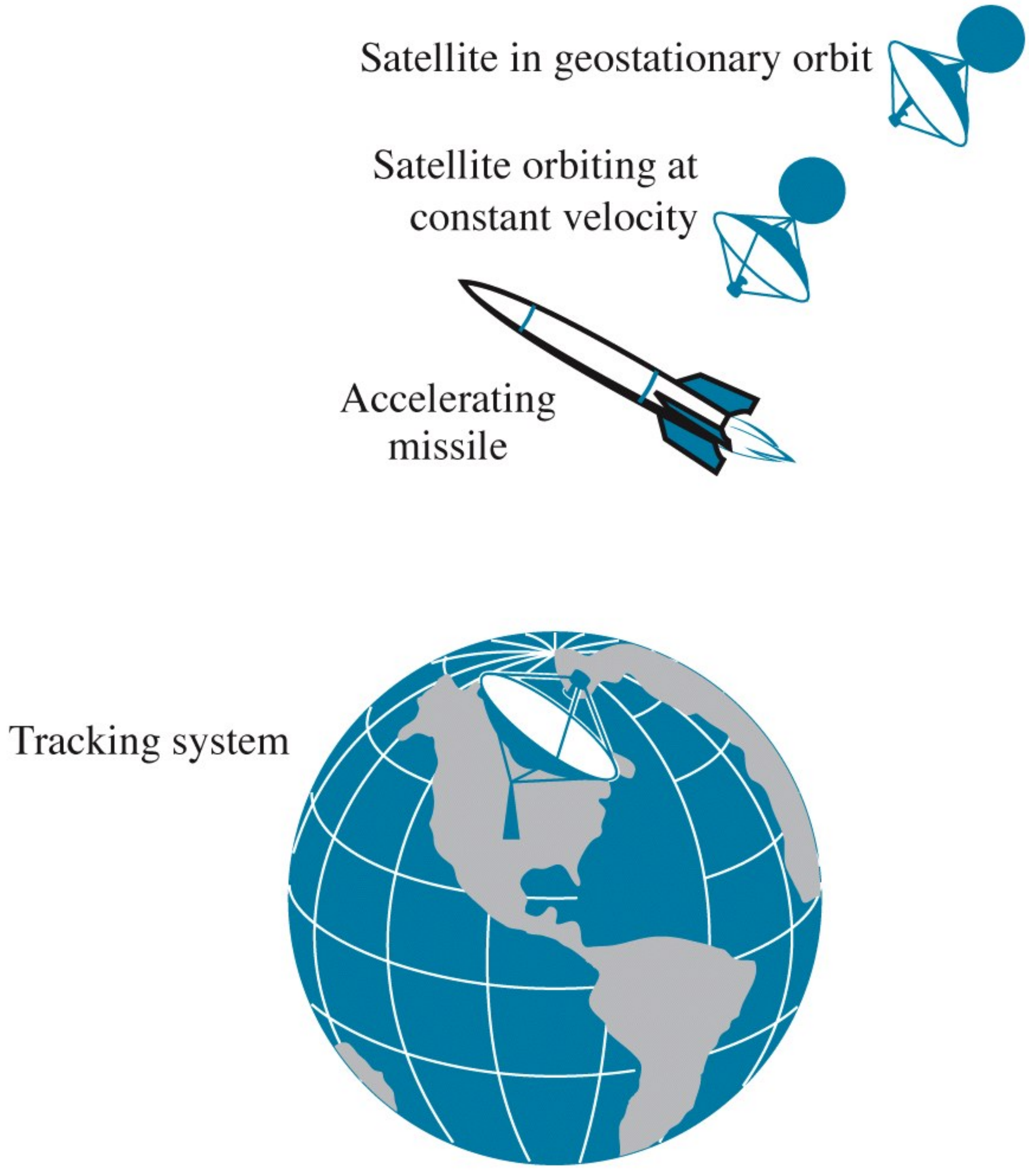
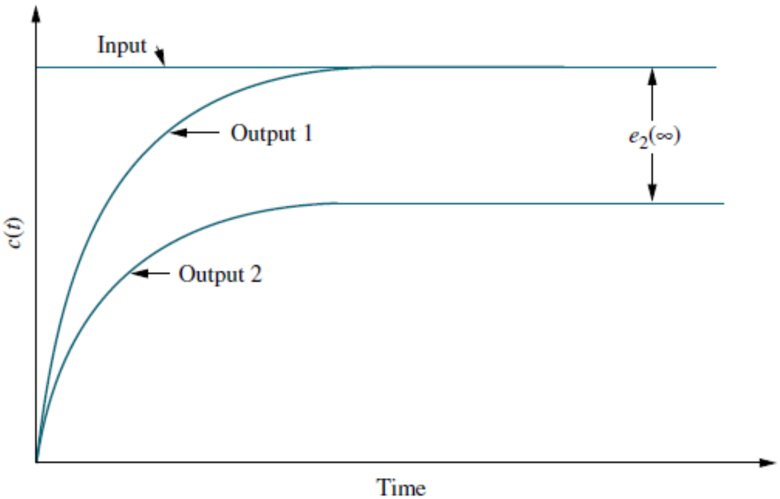
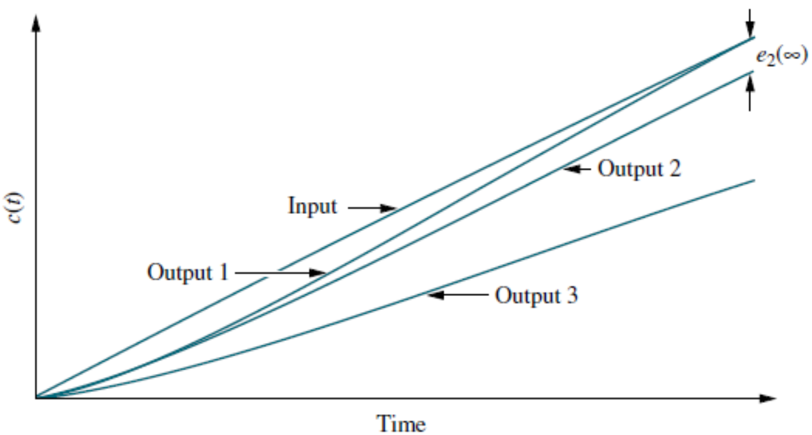

Satellite in geostationary orbit
Satellite orbiting at constant velocity
Accelerating missile
Tracking system

Figure 7.1

Evaluating Steady-State Errors



(*a*)



(*b*)

- Output 1 has zero steady-state error, and output 2 has a finite steady-state error, $e_2(\infty)$

- A similar example is shown where a ramp input is compared with output 1, which has zero steady-state error, and output 2, which has a finite steady-state error, $e_2(\infty)$ as measured vertically between the input and output 2 after the transients have died down.

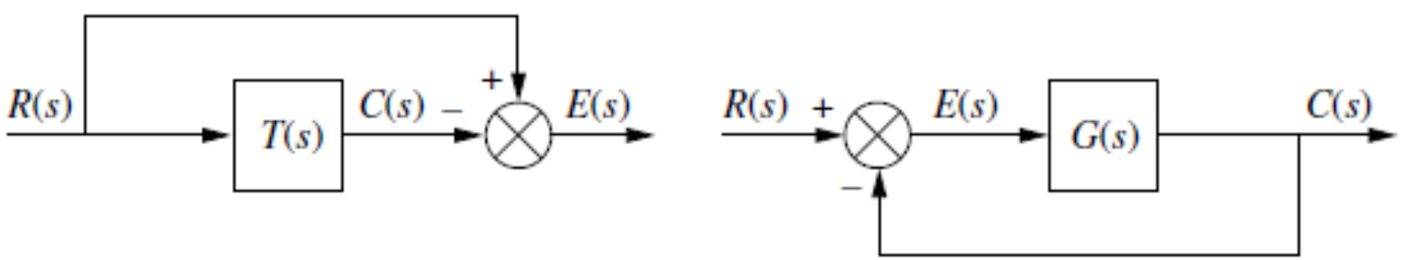
- For the ramp input another possibility exists. If the output's slope is different from that of the input, then output 3 results.

- Here the steady-state error is infinite as measured vertically between the input and output 3 after the transients have died down, and t approaches infinity.

Evaluating Steady-State Errors

- Since the error is the difference between the input and the output of a system, we assume a closed-loop transfer function, *T*(*s*), and form the error, *E*(*s*), by taking the difference between the input and the output.

- Here we are interested in the steady-state, or final, value of $e(t)$.

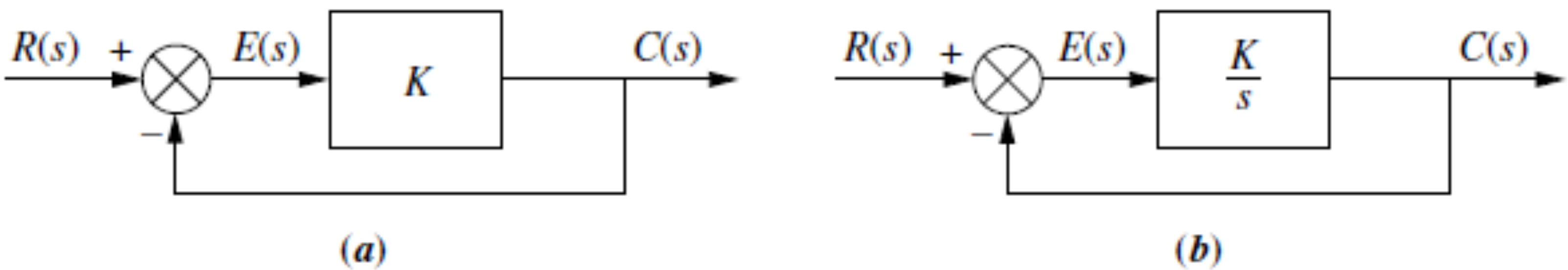


(a)

(b)

Sources of Steady-State Error

- The steady state errors we study here are errors that arise from the configuration of the system itself and the type of applied input.
- For example, look at the system below, where R(s) is the input, C(s) is the output, and $E(s)=R(s)-C(s)$ is the error.



- Consider a step input. In the steady state, if $c(t)$ equals $r(t)$, $e(t)$ will be zero. But with a pure gain, K, the error, $e(t)$, cannot be zero if $c(t)$ is to be finite and nonzero.
- By virtue of the configuration of the system (a pure gain of K in the forward path), an error must exist. If we call $c_{steady\text{-}state}$ the steady-state value of the output and $e_{steady\text{-}state}$ the steady-state value of the error, then

$$e_{steady-state} = \frac{1}{K} c_{steady-state}$$

Sources of Steady-State Error

- If the forward-path gain is replaced by an integrator, there will be zero error in the steady state for a step input.

- The reasoning is as follows:

  - As $c(t)$ increases, $e(t)$ will decrease, since $e(t) = r(t) - c(t)$.

  - This decrease will continue until there is zero error, but there will still be a value for $c(t)$ since an integrator can have a constant output without any input.

Steady-State Error in Terms of T(s)

➢ To find E(s), the error between the input, R(s), and the output, C(s), we write

$$E(s) = R(s) - C(s)$$

$$C(s) = R(s)T(s)$$

➢ Substituting, simplifying, and solving for E(s) yields

$$E(s) = R(s)[1 - T(s)]$$

➢ Although the previous equation allows us to solve for e(t) at any time, t, we are interested in the final value of the error. Applying the final value theorem, which allows us to use the final value of e(t) without taking the inverse Laplace transform of E(s), and then letting t approach infinity, we obtain

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

which yields

$$e(\infty) = \lim_{s\to\infty} sR(s)[1 - T(s)]$$

Final Value Theorem

- The final value theorem is derived from the Laplace transform of the derivative.

$$\mathcal{L}[\dot{f}(t)] = \int_{0-}^{\infty} \dot{f}(t)e^{st}dt = sF(s) - f(0-)$$

- As $s \to 0$

$$\int_{0^-}^{\infty} \dot{f}(t)dt = f(\infty) - f(0\,-) = \lim_{s\to 0} sF(s) - f(0-)$$

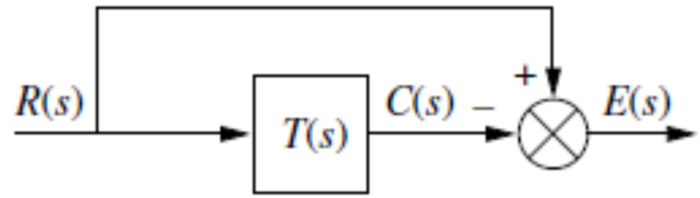
$$f(\infty) = \lim_{s\to 0} sF(s)$$

Example

Find the steady-state error for the system if

$$T(s) = \frac{5}{s^2 + 7s + 10}$$

and the input is a unit step.



Solution:

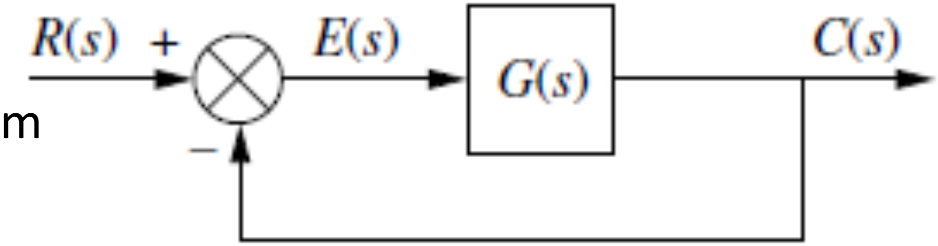
From the problem statement, $R(s) = \frac{1}{s}$ and $T(s) = \frac{5}{s^2 + 7s + 10}$

$$E(s) = R(s)[1 - T(s)] \quad \rightarrow \quad E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

Since $T(s)$ is stable and, subsequently, $E(s)$ does not have right–half-plane poles or $j\omega$ poles other than at the origin, we can apply the final value theorem which gives $e(\infty) = \frac{1}{2}$.

Steady-State Error in Terms of $G(s)$

- Consider the feedback control system



(b)

- Since the feedback, $H(s)$, equals 1, the system has unity feedback.
- The implication is that $E(s)$ is actually the error between the input, $R(s)$, and the output, $C(s)$.
- Thus, if we solve for $E(s)$, we will have an expression for the error. We will then apply the final value theorem, to evaluate the steady-state error.
- Writing $E(s)$, we obtain

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s)$$

- Finally, substituting the previous equation for $C(s)$ into $E(s)$ and solving for $E(s)$ yields

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Steady-State Error in Terms of $G(s)$, *cnt.*

- We now apply the final value theorem.

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

- We now substitute several inputs for R(s) and then draw conclusions about the relationships that exist between the open-loop system, G(s) and the nature of the steady-state error.

Step Input:

$$e(\infty) = e_{\text{step}}(\infty) = \lim_{s \to 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

- For a step input to a unity feedback system, the steady-state error will be zero if there is at least one pure integration in the forward path. If there are no integrations, then there will be a nonzero finite error.

Steady-State Error in Terms of *G*(*s*), *cnt.*

Ramp Input:

$$e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0}\frac{s(1/s^2)}{1+G(s)} = \lim_{s\to 0}\frac{1}{s+sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)}$$

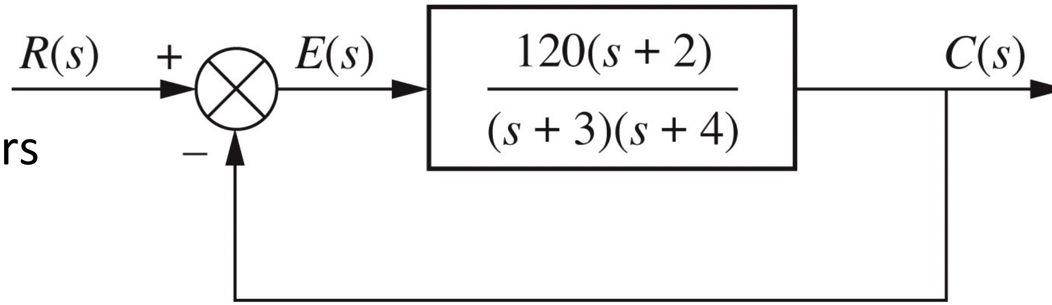
To have zero steady-state error for a ramp input, there must be at least two integrations in the forward path.

Parabolic Input:

$$e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0}\frac{s(1/s^3)}{1+G(s)} = \lim_{s\to 0}\frac{1}{s^2+s^2G(s)} = \frac{1}{\lim\limits_{s\to 0} s^2G(s)}$$

In order to have zero steady-state error for a parabolic input, there must be at least three integrations in the forward path.

Example



- Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown in the fig. where the function $u(t)$ is the unit step.

- For the input $5u(t)$, whose Laplace transform is $5/s$, the steady-state error will be

$$e(\infty) = e_{step}(\infty) = \frac{5}{1 + \lim_{s \to 0} G(s)} = \frac{5}{1 + 20} = \frac{5}{21}$$

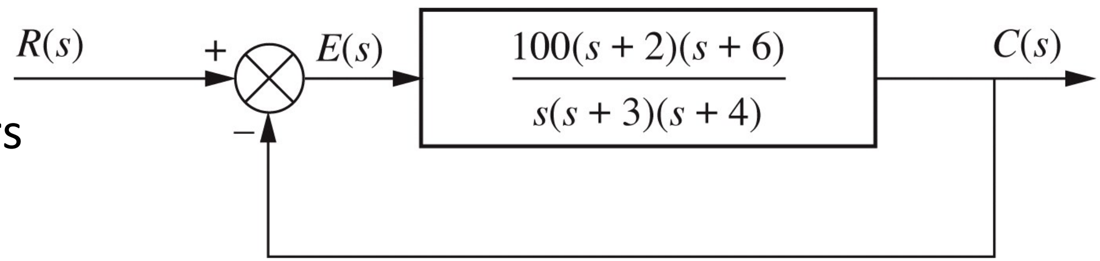
- For the input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error will be

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s \to 0} sG(s)} = \frac{5}{0} = \infty$$

- For the input $5t^2u(t)$, whose Laplace transform is $10/s^3$, the steady-state error will be

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s \to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Example



- Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown in the fig. where the function $u(t)$ is the unit step, where the function u(t) is the unit step.

- For the input $5u(t)$, whose Laplace transform is $5/s$, the steady-state error will be

$$e(\infty) = e_{step}(\infty) = \frac{5}{1 + \lim_{s \to 0} G(s)} = \frac{5}{\infty} = 0$$

- For the input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error will be

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s \to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

- For the input $5t^2u(t)$, whose Laplace transform is $10/s^3$, the steady-state error will be

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s \to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Static Error Constants

- The three terms in the denominator that are taken to the limit determine the steady-state error.

- We call these limits static error constants. Individually, their names are position constant, K_p, where

$$\boxed{K_p = \lim_{s \to 0} G(s)}$$

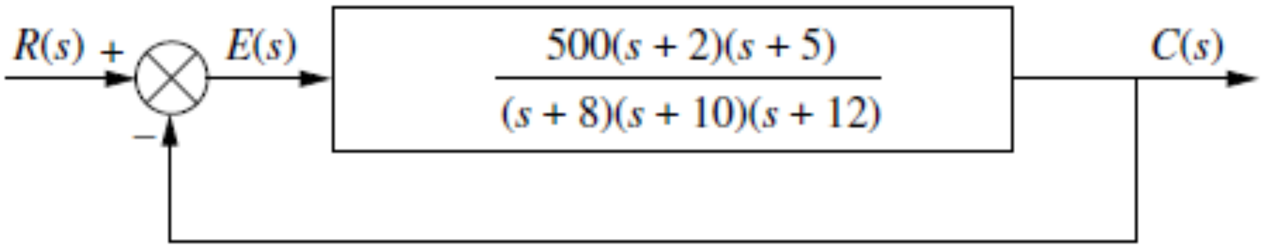
velocity constant, K_v, where

$$\boxed{K_v = \lim_{s \to 0} sG(s)}$$

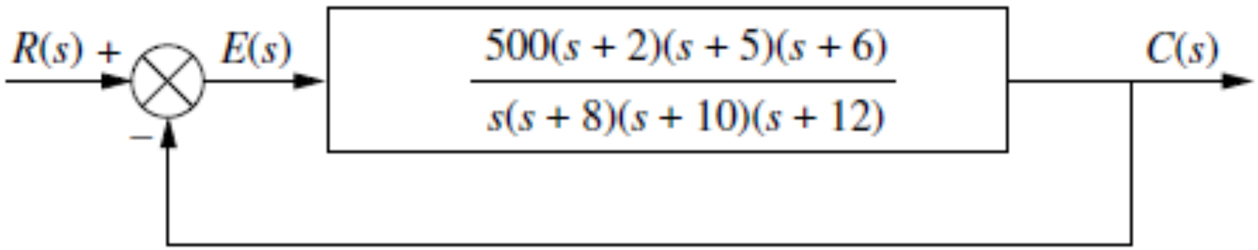
acceleration constant, K_a, where

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s)}$$

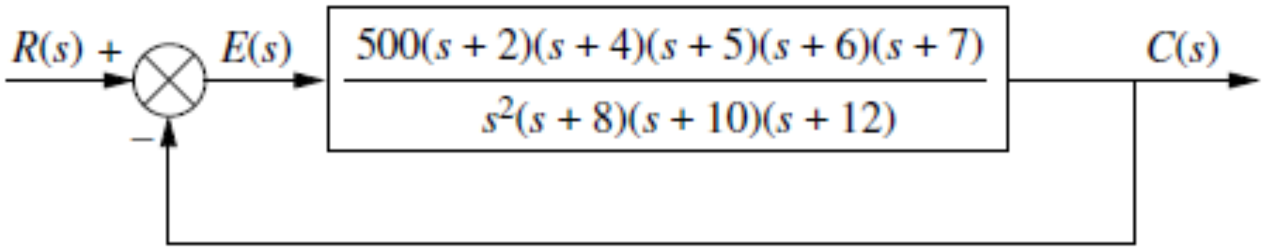
Example



(a)



(b)



(c)

Evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs.

Example

For the system (a)

$$K_p = \lim_{s \to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = 5.208$$

$$K_v = \lim_{s \to 0} sG(s) = 0$$

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

$$e(\infty) = \frac{1}{1 + K_p} = 0.161$$

$$e(\infty) = \frac{1}{K_v} = \infty$$

$$e(\infty) = \frac{1}{K_a} = \infty$$

For the system (b)

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \frac{500 \times 2 \times 5 \times 6}{8 \times 10 \times 12} = 31$$

$$K_a = \lim_{s \to 0} s^2 G(s) = 0$$

$$e(\infty) = \frac{1}{1 + K_p} = 0$$

$$e(\infty) = \frac{1}{K_v} = \frac{1}{31.25} = 0.032$$

$$e(\infty) = \frac{1}{K_a} = \infty$$

For the system (c)

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s) = \frac{500 \times 2 \times 4 \times 5 \times 6 \times 7}{8 \times 10 \times 12} = 875$$

$$e(\infty) = \frac{1}{1 + K_p} = 0$$

$$e(\infty) = \frac{1}{K_v} = 0$$

$$e(\infty) = \frac{1}{K_a} = \frac{1}{875} = 1.14 \times 10^{-3}$$

System Type

- The values of the static error constants, again, depend upon the form of *G*(*s*), especially the number of pure integrations in the forward path.

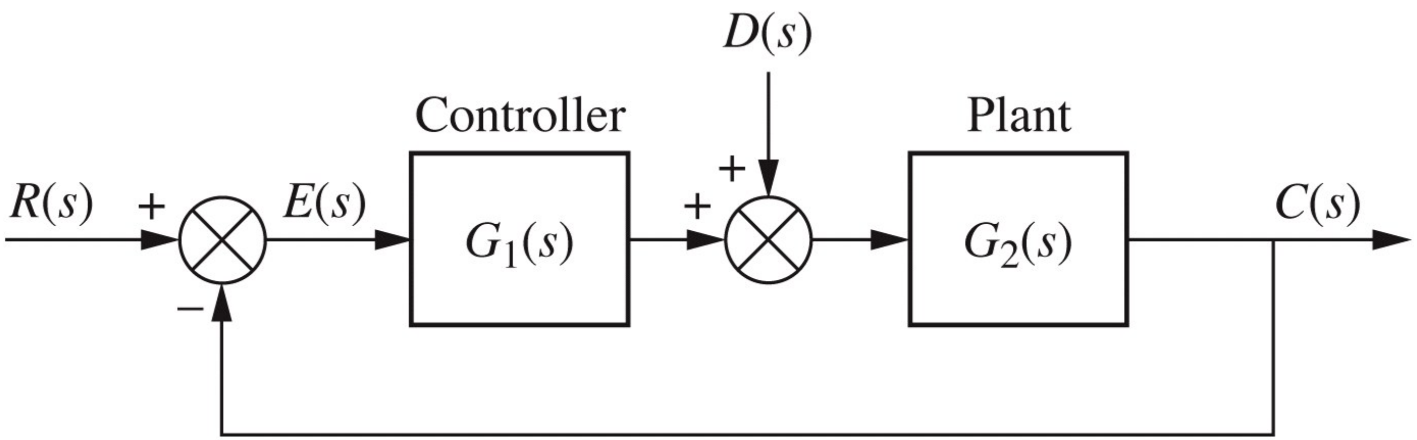
- Since steady-state errors are dependent upon the number of integrations in the forward path, we give a name to this system attribute.

- We define system type to be the value of *n* in the denominator or, equivalently, the number of pure integrations in the forward path. Therefore, a system with *n* = 0 is a Type 0 system. If *n* = 1 or *n* = 2, the corresponding system is a Type 1 or Type 2 system, respectively.

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Steady-State Errors due to Disturbances

- Feedback control systems are used to compensate for disturbances or unwanted inputs that enter a system.
- The advantage of using feedback is that regardless of these disturbances, the system can be designed to follow the input with small or zero error.
- The figure shows a feedback control system with a disturbance, $D(s)$, between the controller and the plant.



- We now re-derive the expression for steady-state error with the disturbance included. The transform of the output is given by,

Steady-State Errors due to Disturbances…

The transform of the output is given by

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

and

$$C(s) = R(s) - E(s)$$

Substituting the second equation into the first,

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

As seen, the error expression has two terms, which can be considered as two transfer functions; the first one relating $E(s)$ to $R(s)$ and the second to $D(s)$.

In order to find the steady-state error, we can use the Final Value Theorem (FVT) as we did before. FVT is defined as

$$e_{ss} = e(\infty) = \lim_{s \to \infty} sE(s)$$

Steady-State Errors due to Disturbances…

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)} R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)} D(s)$$
$$= e_R(\infty) + e_D(\infty)$$

As seen, the rhs consists of two terms: $e_R(\infty)$ and $e_D(\infty)$

$$e_R(\infty) = \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)} R(s)$$

and

$$e_D(\infty) = -\lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)} D(s)$$

The first one is the error component due to the input $R(s)$, which is already obtained. And the second one is the error due to the disturbance input $D(s)$.

Now let us evaluate this second error component, $e_D(\infty)$ for unit step input,

$$D(s) = \frac{1}{s}$$

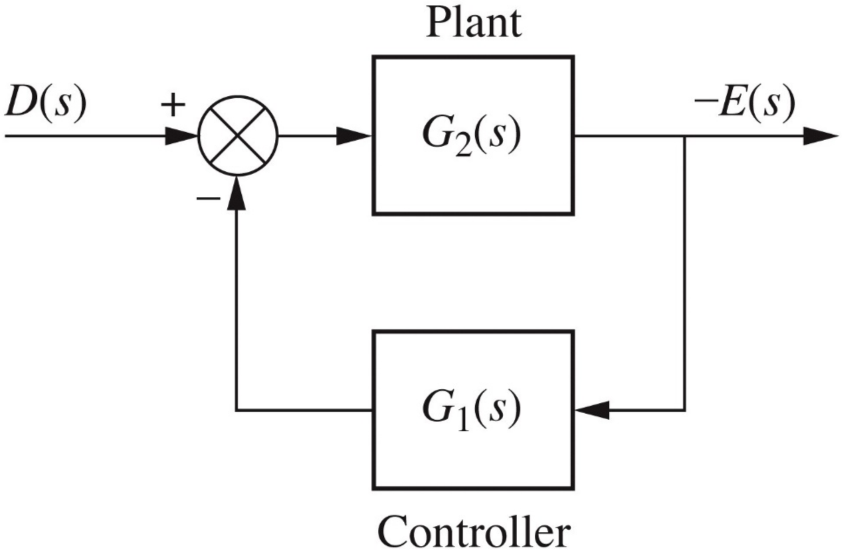
Steady-State Errors due to Disturbances…

The steady-state error component due to a step disturbance is found to be

$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\dfrac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

This equation shows that the steady-state error produced by a step disturbance can be reduced by increasing the dc gain of $G_1(s)$ or decreasing the dc gain of $G_2(s)$.

The previous block diagram, where the disturbance input enters the system between the controller and the plant can be converted to the block diagram on the right for $R(s) = 0$. So that we can analyze e_{ss} due to the disturbance input, $D(s)$.



Steady-State Errors due to Disturbances…

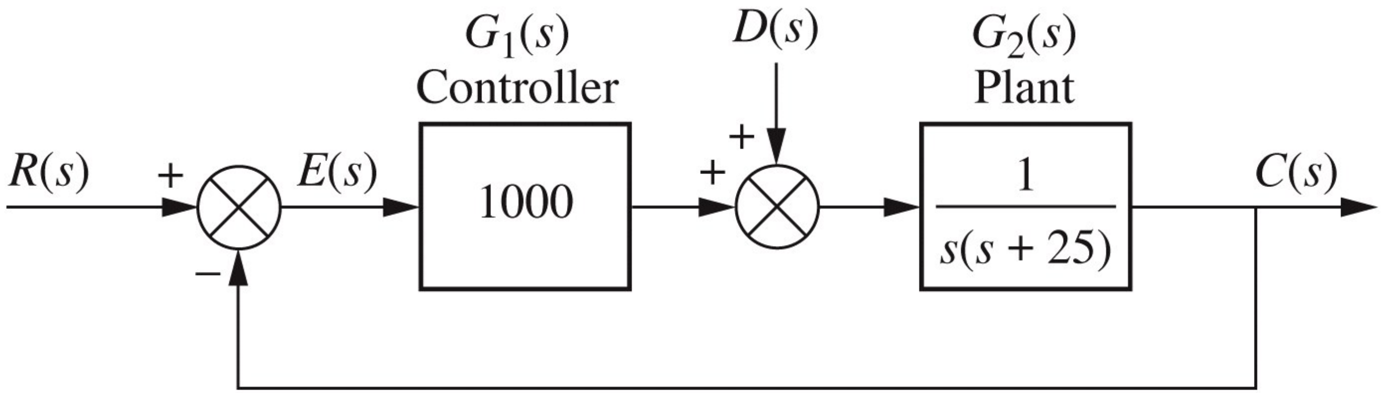
If we want to minimize the steady-state value of $E(s)$, now it is the output in the last figure, we must either increase the dc gain of $G_1(s)$ so that a lower value of $E(s)$ will be fed back to match the steady-state value of $D(s)$, or decrease the dc value of $G_2(s)$, which then yields a smaller value of e_{ss} as predicted by the feedback formula.

Let us look at an example and calculate the numerical value of the steady-state error that results from a disturbance.

Problem: Find the steady-state error component due to a step disturbance for the system given below.

Steady-State Errors due to Disturbances…

Problem: Find the steady-state error component due to a step disturbance for the system given below.



Solution:

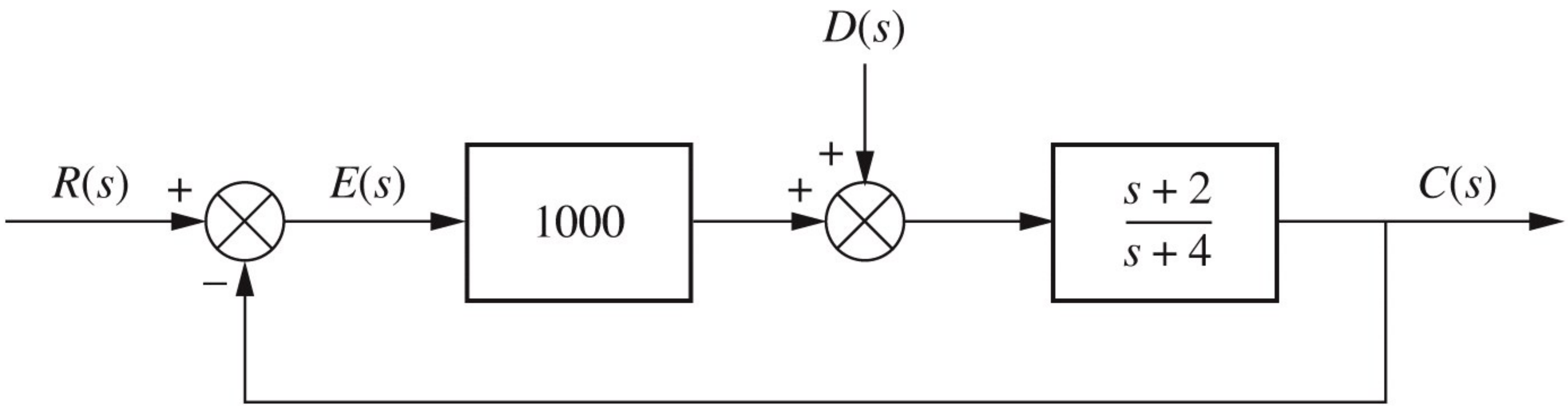
$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\frac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)} = -\frac{1}{\lim\limits_{s\to 0}\big(s(s+25)\big) + \lim\limits_{s\to 0} 1000}$$

$$= -\frac{1}{0 + 1000} = -0.001$$

- The result shows that the steady-state error produced by the step disturbance is inversely proportional to the dc gain of $G_1(s)$.
- The dc gain of $G_2(s)$ is infinite in this example.

Steady-State Errors due to Disturbances…

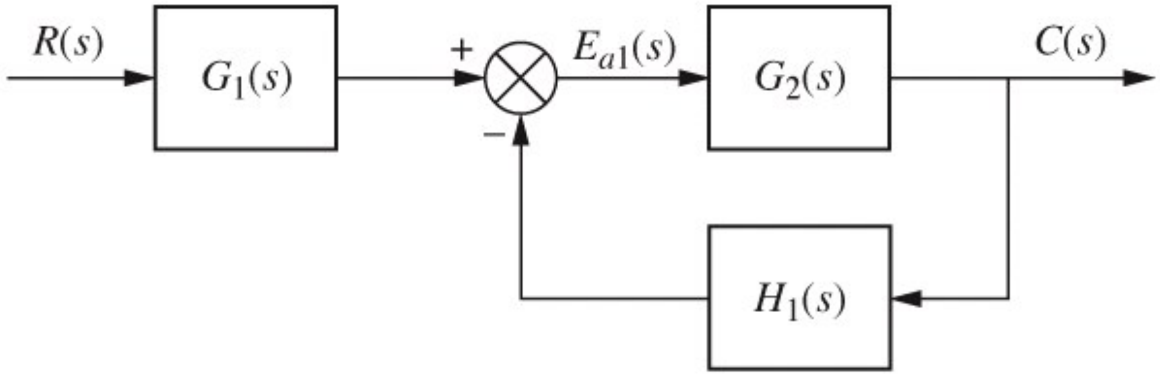
Problem: Evaluate the steady-state error component due to a step disturbance for the system given below.

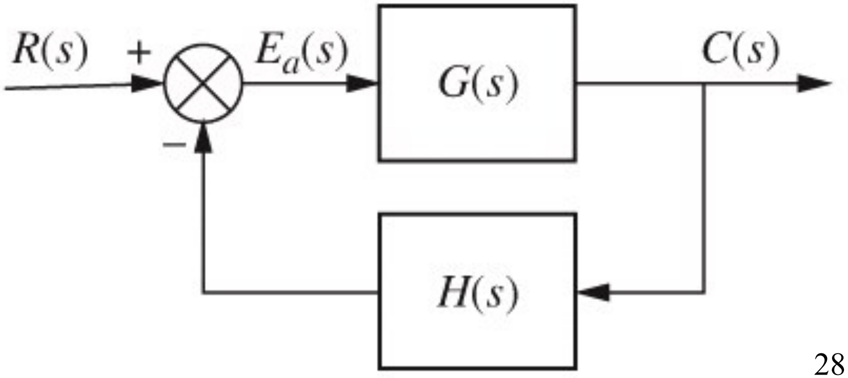


Answer: $e_D(\infty) = -9.98 \times 10^{-4}$

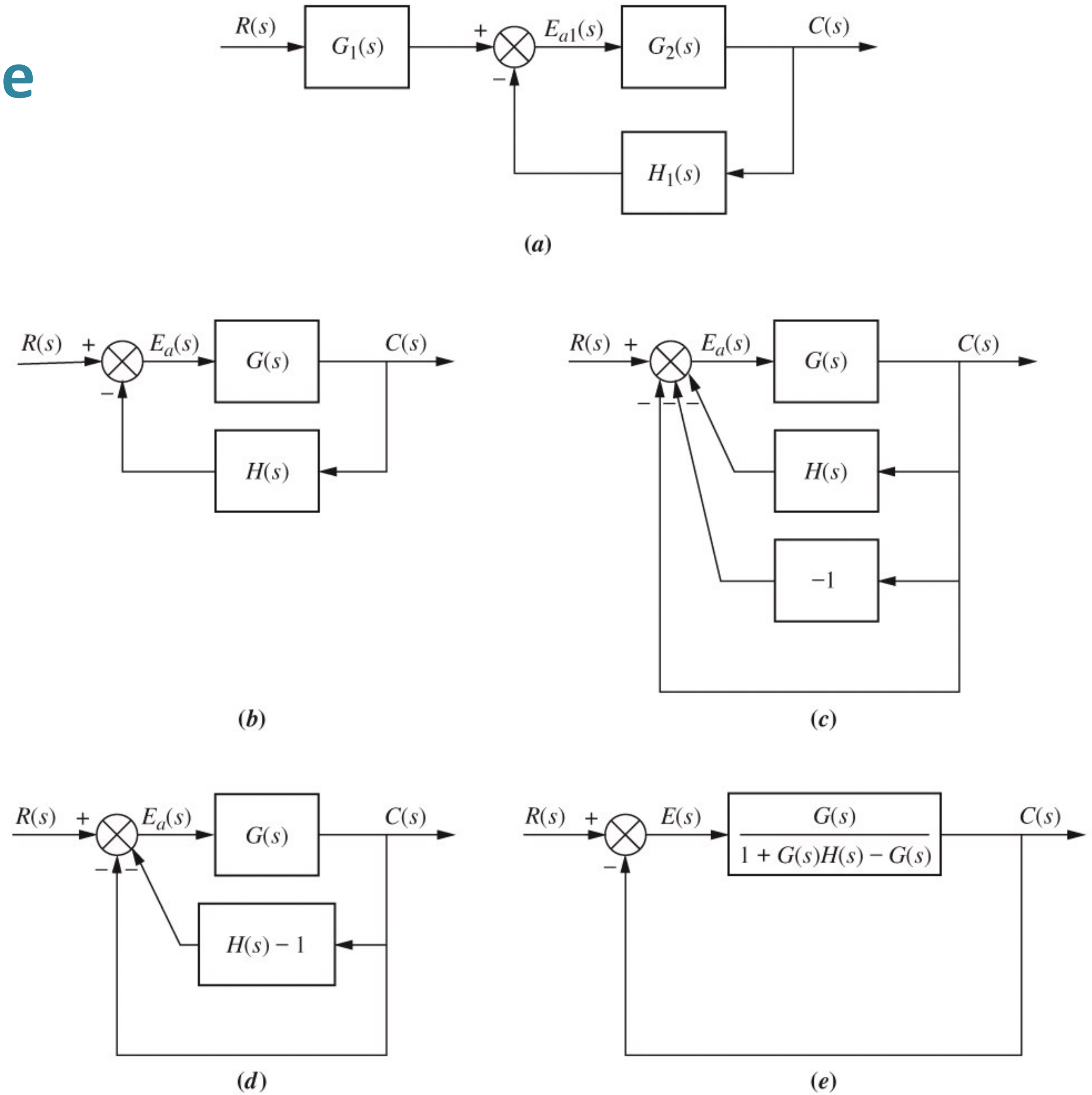
Steady-State Error for Non-unity Feedback Systems

Control systems often do not have unity feedback because of the compensation used to improve performance or because of the physical model for the system. The feedback path can be a pure gain other than unity or have some dynamic representation.



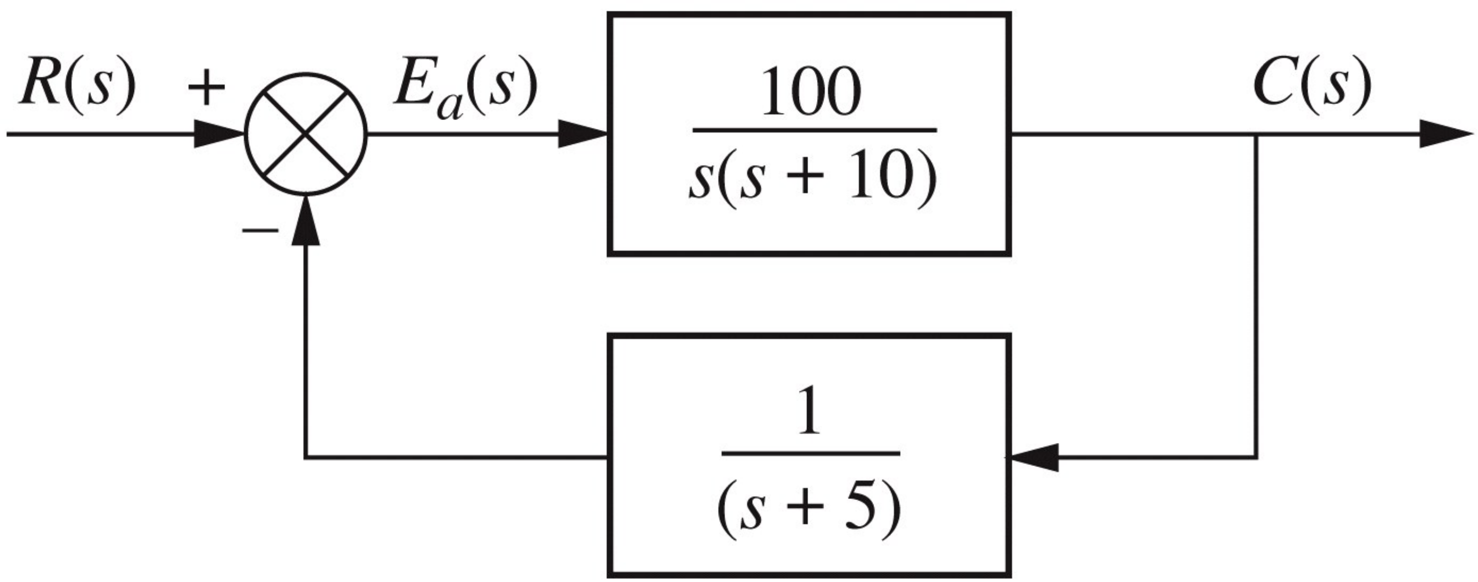


Steady-State Error for Nonunity Feedback Systems



Steady-State Error for Nonunity Feedback Systems

Problem: For the system shown below, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same.



Steady-State Error for Nonunity Feedback Systems

After determining that the system is indeed stable, one may impulsively declare the system to be Type 1. This may not be the case, since there is a nonunity feedback element, and the plant's actuating signal is not the difference between the input and the output.

The first step in solving the problem is to convert the system into an equivalent unity feedback system. Using the equivalent forward transfer function along with

$$G(s) = \frac{100}{s(s+10)} \qquad H(s) = \frac{1}{(s+5)}$$

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

Thus, the system is Type 0, since there are no pure integrations. The appropriate static error constant is then K_p, whose value is

$$K_p = \lim_{s \to 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

Steady-State Error for Nonunity Feedback Systems

The steady-state error is

$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step.