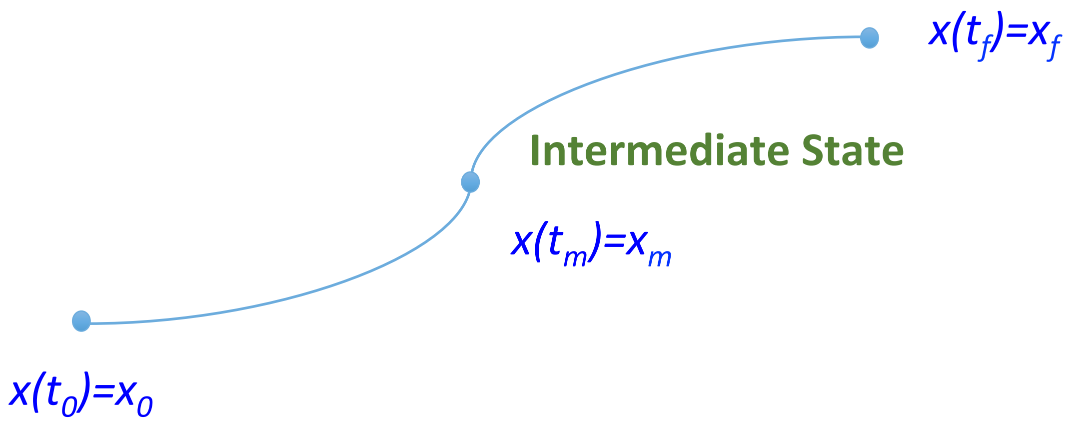
EEEN 460
Optimal Control

2020 Spring

Lecture XI
Application of Dynamic Programming
for Finding the Optimal Control Strategy

Optimal Control Policy

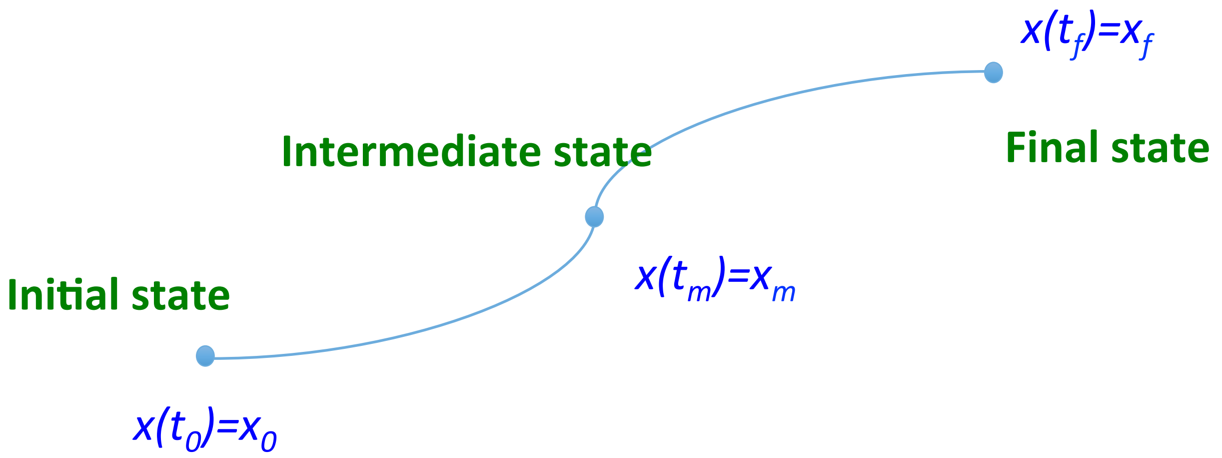
has the property that staring with initial state and optimal initial decision all the subsequent decisions must also contain optimal control policies.



Assume $x(t_m)=x_m$ be an intermediate state on the optimal trajectory an let $J^*(t_m, x(t_m))$ be the optimal cost of going from
x_m to x_f

Optimal Control Policy

We had said that initial decision at the initial state must contain optimal control policy.



That means the cost function at $x(t_0)=x_0$ must be optimal $J^*(t_o, x(t_o))$

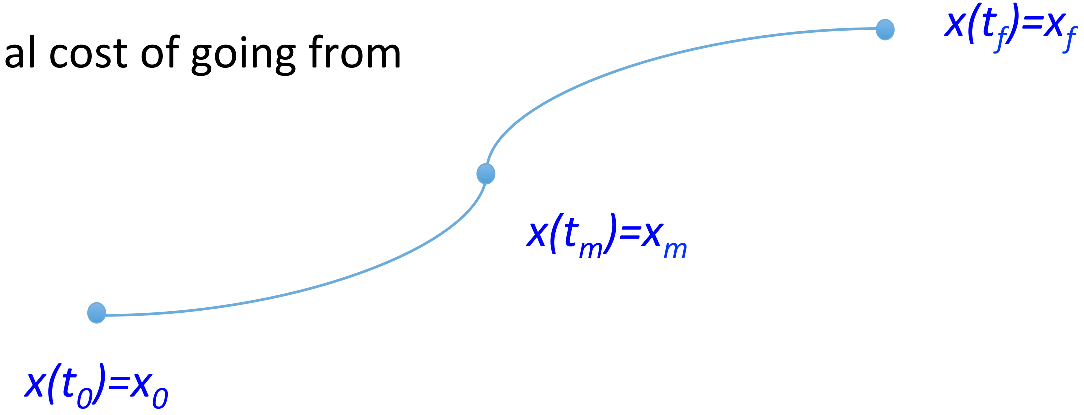
Where $J^*(t_o, x(t_o))$ is the cost of going from $x(t_0)=x_0$ to $x(t_f)=x_f$

Optimal Control Policy

We had learned in **Lecture X** that if the decision at the initial state $x(t_0)=x_0$ is optimal,
i.e. the cost function is $J^*(t_o, x(t_o))$, then the decision at $x(t_m)=x_m$ also must be optimal
i.e. the cost function is $J^*(t_m, x(t_m))$

$J^*(t_m, x(t_m))$ Is the optimal cost of going from

$x(t_m)=x_m$ to $x(t_f)=x_f$

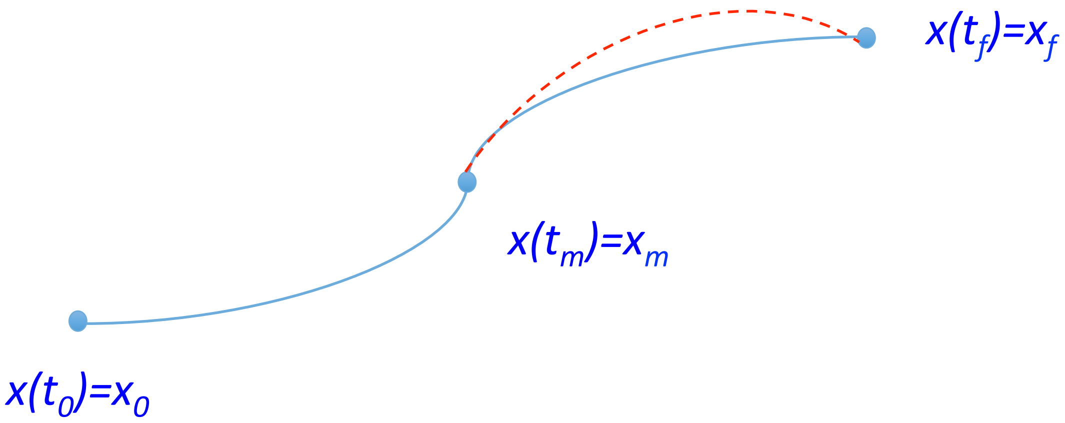


That means if the total path from $x(t_0)=x_0$ to $x(t_f)=x_f$ is optimal then the portion of the path from $x(t_m)=x_m$ to $x(t_f)=x_f$ must also be optimal .

What did we call this principle ? Ans: Principle of optimality

Optimal Policy

Remember how we demonstrated it:
Assume that the dashed curve (path or trajectory) represents a smaller cost than the solid path.



Then this dashed path would provide a less costly path from $x(t_0)=X_0$ to $x(t_f)=x_f$,
and it would contradict that the solid path is the optimal path.

Principle of Optimality

is the backbone of dynamical programming . The optimal control strategy can be obtained by backward propagation from final stage to the earlier stage, and so on, iteratively by using the **Principle of Optimality**.

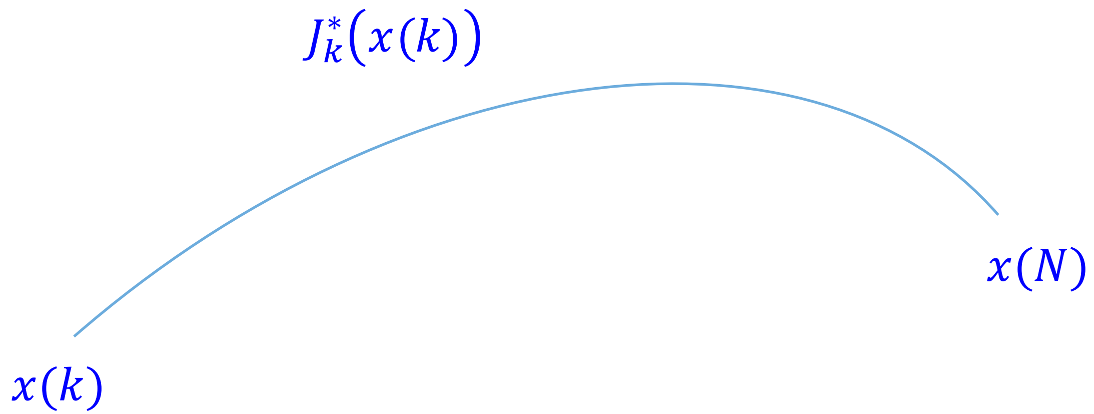
We will demonstrate this with application to a discrete- time dynamical system.

Discrete-time Dynamical System:

Model: $\mathrm{x}(\mathrm{k}+1)=\mathrm{f}(\mathrm{k},\mathrm{x}(\mathrm{k}),\mathrm{u}(\mathrm{k}))$ $\quad\quad \mathrm{x}(t_0)=\mathrm{x}_0$

Cost Function or Performance index (PI) is given by

$$\mathrm{PI} \Rightarrow \mathrm{J} = \emptyset(\mathrm{N},\mathrm{x}(\mathrm{N})) + \sum_{k=0}^{N-1} F(k,x(k),u(k))$$



Let

$$J_k^*(x(k))$$

Be the cost of going from $x(k)$ to $x(N)$

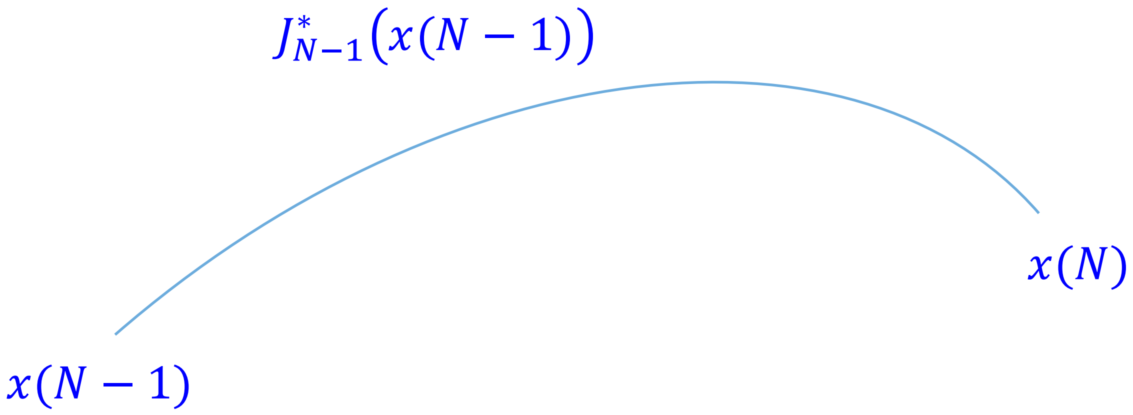
Since $x(N)$ is the terminal state, we will have the cost at this state:

$$J_N^*(\mathrm{x(N)}) = \emptyset(\mathrm{N}, \mathrm{x(N)})$$

Remember

$$\mathrm{J} = \emptyset(\mathrm{N}, \mathrm{x(N)}) + \sum_{k=0}^{N-1} F(x, x(k), u(k))$$

Using Principle of Optimality the cost of going from $x(N-1)$ to $x(N)$



By subsituting k with N-1

$$= \text{Ø}(\mathrm{N}, \mathrm{x}(\mathrm{N})) + \sum_{k}^{N-1} F(k, x(k), u(k))$$

$$J_{N-1}^{*}\big(x(N-1)\big) = \text{Ø}(\mathrm{N}, \mathrm{x}(\mathrm{N})) + F(N-1, x(N-1), u^{*}(N-1))$$

$$J^*_{N-1}\big(x(N-1)\big) = \text{Ø}(\mathrm{N}, \mathrm{x}(\mathrm{N})) + F(N-1, x(N-1), u^*(N-1))$$

$$= J^*_N(\mathrm{x}(\mathrm{N})) + F(N-1, x(N-1), u^*(N-1)$$

Similarly,

$$J^*_{N-2}\big(\mathrm{x}(\mathrm{N}-2)\big) = J^*_{N-1}\big(\mathrm{x}(\mathrm{N}-1)\big) + F(N-2, x(N-2), u^*(N-2))$$

In general

$$J^*_{N-k}\big(\mathrm{x}(\mathrm{N}-\mathrm{k})\big) = J^*_{N-k+1}\big(\mathrm{x}(\mathrm{N}-\mathrm{k}+1)\big) + F(N-k, x(N-k), u^*(N-k))$$

This gives us the back propagation algorithm

We will illustate this with an example

Example:

$$x(k+1) = 4x(k) - 6u(k) \qquad\qquad x(0) = 0$$

$$\text{PI} \Rightarrow J = \emptyset(N, x(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

For this problem

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

Find u*(k) which minimizes this performance index
i.e. Find u*(0) and u*(1)

$$x(k+1) = 4x(k) - 6u(k) \qquad x(0) = 8$$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

Solution

Start with backward pass

$$J_2^*\big(x(2)\big) = (x(2) - 20)^2 \ldots\ldots\ldots\ldots\ldots (1)$$

$$J_1^*\big(x(1)\big) = \min_{u(1)}\{F(1, x(1), u(1) + J_2^*(x(2))\}$$

$$= \min_{u(1)}\left\{\frac{1}{2}\big(2x^2(1) + 4u^2(1)\big) + (x(2) - 20)^2\right\}$$

$x(k+1) = 4x(k) - 6u(k)$ $x(0) = 8$ $J_2^*\big(x(2)\big) = (x(2)-20)^2 = [4x(1) - 6u(1) - 20]^2$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2)-20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$J_1^*\big(x(1)\big) = \min_{u(1)} \left\{\frac{1}{2}\big(2x^2(1) + 4u^2(1)\big) + (x(2)-20)^2\right\}$$

Considering

$$x(k+1) = 4x(k) - 6u(k)$$

$$x(2) = 4x(1) - 6u(1)$$

$$\Rightarrow \min_{u(1)} \left\{\frac{1}{2}[2x^2(1) + 4u^2(1)] + [4x(1) - 6u(1) - 20]^2\right\}$$

$x(k+1) = 4x(k) - 6u(k) \qquad x(0) = 8$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$\Rightarrow \min_{u(1)} \left\{\frac{1}{2}[2x^2(1) + 4u^2(1)] + [4x(1) - 6u(1) - 20]^2\right\}$$

i. $u_1^*(1)$ (optimal u) is a function of $x(1)$

$$\frac{\partial}{\partial u_1}\{[x^2(1) + 2u^2(1)] + [4x(1) - 6u(1) - 20]^2\} = 0$$

$$x(k+1) = 4x(k) - 6u(k) \qquad x(0) = 8$$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$\frac{\partial}{\partial u_1}\{[x^2(1) + 2u^2(1)] + [4x(1) - 6u(1) - 20]^2\} = 0$$

$$4u(1) + 2[4x(1) - 6u(1) - 20].(-6) = 0$$

$$u(1) - 3[4x(1) - 6u(1) - 20] = 0$$

$$19u(1) - 12x(1) + 60 = 0$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$x(k+1) = 4x(k) - 6u(k)$ $\quad x(0) = 8$ $\quad J_2^*(x(2)) = (x(2)-20)^2 = [4x(1) - 6u(1) - 20]^2$

$$J = \varnothing(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2)-20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$u(1) = \frac{12x(1) - 60}{19}$$

But from $x(k+1) = 4x(k) - 6u(k)$

$$x(1) = 4x(0) - 6u(0)$$

$$J_1^*[x(1)] = \frac{1}{2}[2x^2(1) + 4u^2(1)] + J_2^*[x(2)]$$

Substituting $u(1)$

$$= \frac{1}{2}\left\{2x^2(1) + 4\left[\frac{12x(1) - 60}{19}\right]^2\right\} + [4x(1) - 6u(1) - 20]^2$$

$x(k+1) = 4x(k) - 6u(k) \qquad x(0) = 8$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$J_2^*\big(x(2)\big) = (x(2) - 20)^2 = [4x(1) - 6u(1) - 20]^2$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$$J_1^*[x(1)] = \frac{1}{2}\left\{2x^2(1) + 4\left[\frac{12x(1) - 60}{19}\right]^2\right\} + [4x(1) - 6u(1) - 20]^2$$

Substituting $u(1)$

$$= x^2(1) + 2\left[\frac{12x(1) - 60}{19}\right]^2 + [4x(1) - 6\left[\frac{12x(1) - 60}{19}\right] - 20]^2$$

$$= x^2(1) + 2\left[\frac{12x(1) - 60}{19}\right]^2 + \left[\frac{4x(1) - 20}{19}\right]^2$$

$x(k+1) = 4x(k) - 6u(k) \qquad x(0) = 8$

$$J = \varnothing(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$J_2^*\big(x(2)\big) = (x(2) - 20)^2 = [4x(1) - 6u(1) - 20]^2$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$$J_1^*[x(1)] = x^2(1) + 2\left[\frac{12x(1) - 60}{19}\right]^2 + \left[\frac{4x(1) - 20}{19}\right]^2$$

$$J_0^*[x(0)] = \min_{u(0)} \left\{\frac{1}{2}[2x^2(0) + 4u^2(0)] + J_1^* x(1)\right\}$$

$$= \min_{u(0)} \left\{\begin{array}{c} \frac{1}{2}[2x^2(0) + 4u^2(0)] + [4x(0) - 6u(0)]^2 \\ +2\left[\frac{12[4x(0)-6u(0)]-60}{19}\right]^2 + \left[\frac{4[4x(0)-6u(0)]-20}{19}\right]^2 \end{array}\right\}$$

$x(k+1) = 4x(k) - 6u(k)$ $\quad x(0) = 8$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$J_2^*(x(2)) = (x(2) - 20)^2 = [4x(1) - 6u(1) - 20]^2$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$$J_0^*[x(0)] = \min_{u(0)} \left\{ \begin{array}{c} \frac{1}{2}[2x^2(0) + 4u^2(0)] + [4x(0) - 6u(0)]^2 \\ +2\left[\frac{12[4x(0) - 6u(0)] - 60}{19}\right]^2 + \left[\frac{4[4x(0) - 6u(0)] - 20}{19}\right]^2 \end{array} \right\}$$

$$\frac{\partial J}{\partial u(0)} = 0$$

$$\Rightarrow 4u(0) + 2[4x(0) - 6u(0)].(-6)$$

$$+\frac{2}{19^2}\{2.[12(4x(0) - 6u(0) - 60].(-72)\}$$

$$+\frac{1}{19^2}\{2.[4(4x(0) - 6u(0) - 20].(-24)\} = 0$$

Using $x(0) = 8$

$$u(0) = 4.81$$

$x(k+1) = 4x(k) - 6u(k)$ $\quad x(0) = 8$

$$J = \emptyset(N, K(N)) + \sum_{k=0}^{N-1} F(k, x(k), u(k))$$

$$J = (x(2) - 20)^2 + \frac{1}{2}\sum_{k=0}^{1} 2x^2(k) + 4u^2(k)$$

$$J_2^*(x(2)) = (x(2) - 20)^2 = [4x(1) - 6u(1) - 20]^2$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$$u(0) = 4.81$$

$$x(1) = 4x(0) - 6u(0) \qquad [since\ x(k+1) = 4x(k) - 6u(0)]$$

$$= 4(8) - 6(4.81) = 3.41$$

$$u(1) = \frac{12x(1) - 60}{19}$$

$$= \frac{12(3.41) - 60}{19} = -1.175$$

$$x(2) = 4x(1) - 6u(1)$$

$$= 4(3.41) - 6(-1.175)$$

$$= 19.61$$

Which completes the forward pass.

End of Lecture XI