Active Filter Circuits

- They are active circuits that employ opamps

  → having certain advantages over passive circuits.

- For example, active circuits can produce bandpass and bandreject filters

  → without using inductors which are large, heavy, costly.

- Passive filters are in general incapable of amplification

  → the output magnitude does not exceed the input magnitude.

However ;

- Active filters provide a control over amplification

  → NOT available in passive filter circuits.

Recall that ;

- Both the cutoff frequencies and the passband magnitude of passive filters change

  → with the addition of resistive load at the output of the filter.

- This is NOT the case with active filters

  → due to the properties of opamps.

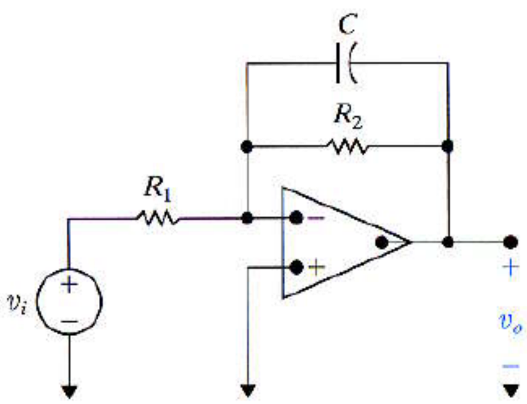
Hence ;

- We use active circuits to implement filter designs

  → when gain, load variation and physical size are important parameters in the design specifications.

First-order-low-pass and high-pass filters

- Let us consider the following circuit.



- At very low frequencies, the capacitor acts like an open circuit

  → the op amp circuit behaves like an amplifier with a gain of $-R_2/R_1$

- At very high frequencies, the capacitor acts like a short circuit

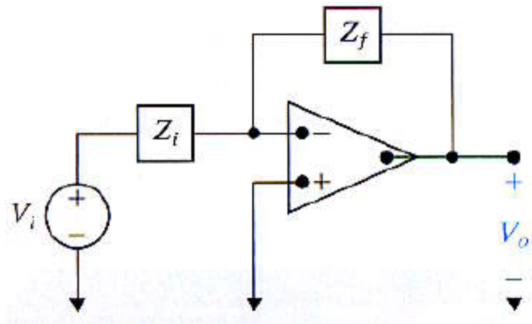
  → the output of the op amp circuit is connected to the ground.

Hence ;

- The above op amp circuit functions as a low pass filter.

qualitative analysis

- We have



$$Z_i = R_1$$

$$Z_f \equiv R_2 // \left(\frac{1}{sC}\right)$$

$$= \frac{R_2/sC}{R_2 + \frac{1}{sC}} = \frac{R_2}{sR_2C + 1}$$

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{-Z_f}{Z_i} = \frac{-R_2}{sR_1R_2C + R_1} = -K\frac{\omega_c}{s + \omega_c}$$

where $K = R_2/R_1$, $\omega_c = 1/R_2C$

Note that ;

- Active low-pass filter equation has the same form

  → as the general equation for passive low pass filter.

However ;

- the gain in the passband, K is set by R_2/R_1

  → the pass band gain and the cutoff frequency are specified independently.

Bode plots

- A special type of frequency response plots

  → which differ from the frequency response plots in two important ways.

1. A Bode plot uses a logarithmic axis for the frequency values

   → allowing to plot a wider range of frequencies.

2. The Bode magnitude is plotted in decibels (dB) versus the log of the frequency

$$20\log_{10}|H(j\omega)| \equiv A_{dB}$$

Thus ;

$|H(j\omega)| = 1 \quad \Rightarrow \quad A_{dB} = 0$

$0 < |H(j\omega)| < 1 \quad \Rightarrow \quad A_{dB} < 0$

$|H(j\omega)| > 1 \quad \Rightarrow \quad A_{dB} > 0$

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad A_{dB} = 20\log_{10}\frac{1}{\sqrt{2}} = -3dB$$

What does $-3dB$ imply ?

- We define the cutoff frequency of a filter

  by determining frequency at which the maximum magnitude of the transfer function in dB has been reduced by $-3dB$.

Ex. Calculate the values for C and R_2 with $R_1 = 1\,\Omega$ to produce a low-pass filter having

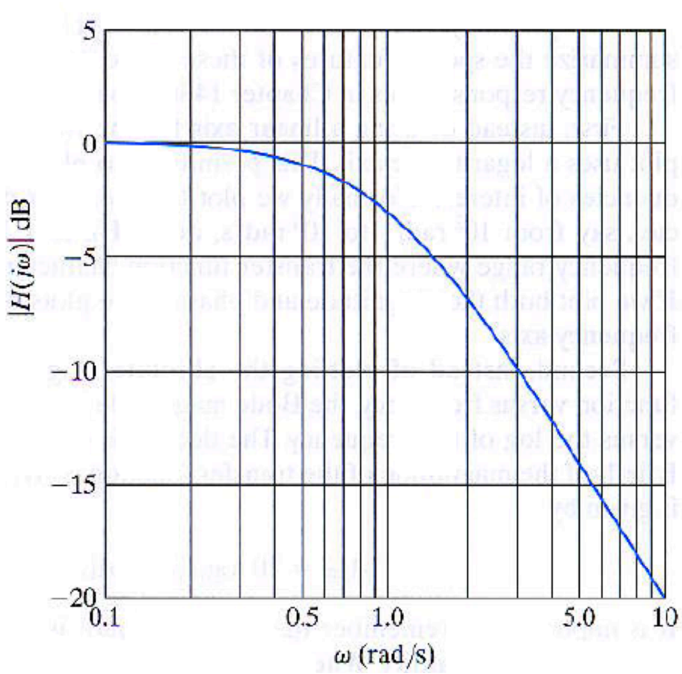
$$K = 1 \quad , \quad \omega_c = 1\,rad/sec$$

Sketch the Bode magnitude plot.

$$R_2 = KR_1 = 1.1 = 1\,\Omega$$

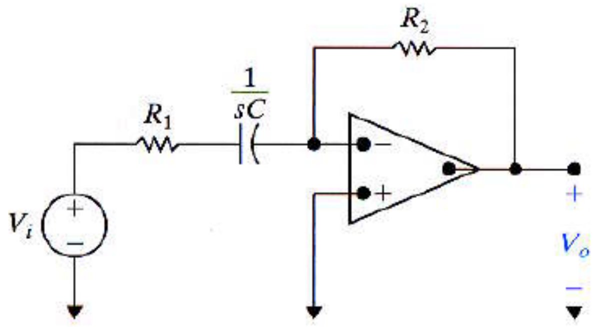
$$C = \frac{1}{R_2\omega_C} = \frac{1}{1.1} = 1\,F$$

$$\Rightarrow \quad H(s) = -K\frac{\omega_C}{s+\omega_c} = -\frac{1}{s+1} \quad , \quad |H(j\omega_C)| = \frac{1}{\sqrt{\omega^2+1}}$$



High-pass filter

- Consider now



- We have

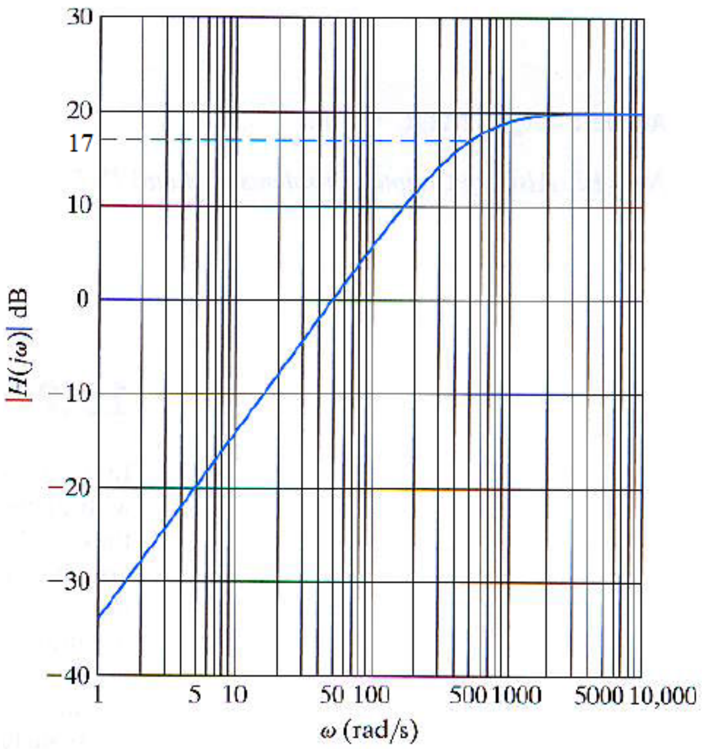
$$H(s) = \frac{-Z_f}{Z_i} = \frac{-R_2}{R_1 + \frac{1}{sC}}$$

$$= -K\frac{s}{s+\omega_c}$$ “same as the eqn. for passive high-pass filter”

where

$$K = \frac{R_2}{R_1} \quad , \quad \omega_c = \frac{1}{R_1 C}$$

Ex. Given the Bode magnitude plot and using the active high-pass filter, calculate R_1 and R_2 with $C = 0.1\ \mu F$



- The gain in the passband is 20 dB then $K = 10$
- and the $3dB$ point is 500 rad/sec

Hence ;

$$H(s) = \frac{-10s}{s+500} = \frac{-(R_2/R_1)s}{s+(1/R_1C)}$$

$$\Rightarrow \quad \frac{1}{R_1C} = 500 \quad \Rightarrow \quad R_1 = \frac{1}{0.1.10^{-6}.500} = 20\ k\Omega$$

$$\Rightarrow \quad \frac{R_2}{R_1} = 10 \quad \Rightarrow \quad R_2 = 200\ k\Omega$$

Scaling

- We usually make computations for filter circuits

  using convenient values of R, L and C such as 1 Ω, 1 H and 1 F.

- A designer can then transform the convenient values into realistic values

  using "scaling".

magnitude scaling

- We scale a circuit in magnitude

  by multiplying the impedance at a given frequency with the scale factor, $K_m > 0$

$$R' = k_m R \quad , \quad L' = k_m L \quad \text{and} \quad C' = C/k_m$$

frequency scaling

- We change the circuit parameters so that

  at the new frequency, the impedance of each element is the same as it was at the original frequency.

Hence ;

- If we let $K_f > 0$ denote the scale factor, then

$$R' = R \quad , \quad L' = L/k_f \quad \text{and} \quad C' = C/k_f$$

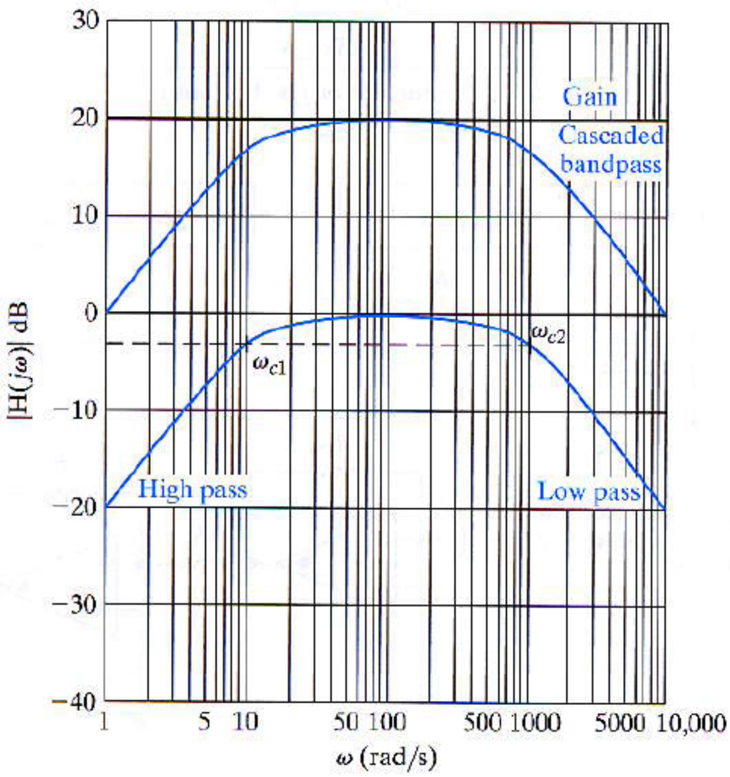
Simultaneous scaling

- A circuit can be scaled simultaneously in magnitude and frequency

$$R' = k_m R \quad , \quad L' = \frac{k_m}{k_f} L \quad , \quad C' = \frac{1}{k_m k_f} C$$

Op amp bandpass filter

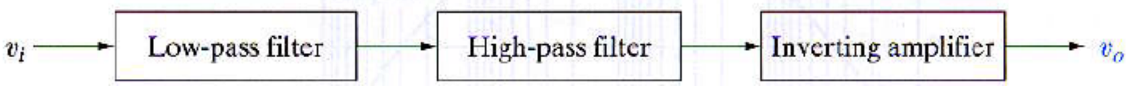
- An initial approach is motivated by the following Bode plot



- The bandpass filter consists of three separate components :

  1. A unity-gain low-pass filter whose cut off frequency is ω_{c_2}
  2. A unity-gain high-pass filter whose cut off frequency is ω_{c_1}
  3. A gain component to provide the desired level of gain in the pass band.

- These three components are cascaded in series :



Our goal

- We aim to establish the relationship between ω_{c_1} and ω_{c_2}

  that will permit each subcircuit to be designed independently.

Transfer function

- Can be obtained as the product of the transfer functions of the cascaded units :

$$H(s)=\frac{V_0}{V_i}=\left(\frac{-\omega_{c_2}}{s+\omega_{c_2}}\right)\left(\frac{-s}{s+\omega_{c_1}}\right)\left(\frac{-R_f}{R_i}\right)$$

$$=\frac{-K\omega_{c_2}s}{(s+\omega_{c_2})(s+\omega_{c_1})}=\frac{-K\omega_{c_2}s}{s^2+\underbrace{(\omega_{c_1}+\omega_{c_2})}_{\to\ \omega_{c_2}}s+\omega_{c_1}\omega_{c_2}}$$

$$\triangleq\frac{\beta s}{s^2+\beta s+\omega_0^2}$$

Therefore ;

- We require that

$$\omega_{c_2}\gg\omega_{c_1}\qquad\Rightarrow\qquad\omega_{c_1}+\omega_{c_2}\approx\omega_{c_2}$$

- Then resistor and capacitor values are computed by

$$\omega_{c_2}=\frac{1}{R_LC_L}\qquad,\qquad\omega_{c_1}=\frac{1}{R_HC_H}$$

- and for R_f and R_i

$$|H(j\omega_0)|=\left|\frac{-K\omega_{c_2}j\omega_0}{(j\omega_0)^2+\omega_{c_2}(j\omega_0)+\omega_{c_1}\omega_{c_2}}\right|$$

$$=\frac{K\omega_{c_2}}{\omega_{c_2}}$$

$$=K=\frac{R_f}{R_i}$$

Ex. Design a bandpass filter for a graphic equalizer to provide an amplification of 2 within the band of frequencies between 100 and $10^4\ Hz$. Use 0.2 μF capacitors.

Solution. Note that $\omega_{c_2}=100\omega_{c_1}$, $i.e.$ $\omega_{c_2}\gg\omega_{c_1}$

$$\omega_{c_2}=\frac{1}{R_L.0.2.10^{-6}}=2\pi.10\,000\qquad\Rightarrow\qquad R_L\approx 80\ \Omega$$

Next ;

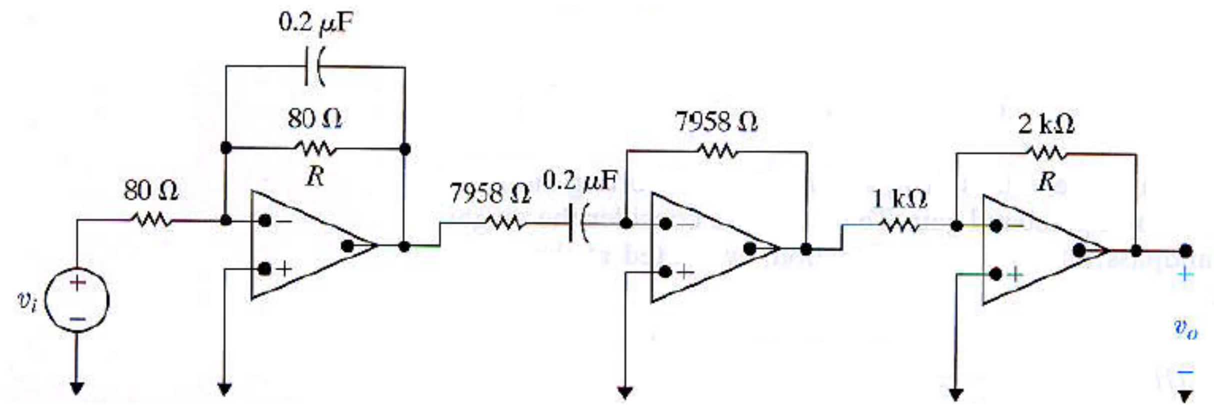
$$\omega_{c_1}=\frac{1}{R_H.0.2.10^{-6}}=2\pi.100\qquad\Rightarrow\qquad R_H\approx 7958\ \Omega$$

- and for the gain stage, we have

$$R_f = 2R_i$$

- selecting a 1 $k\Omega$ resistor for R_i gives

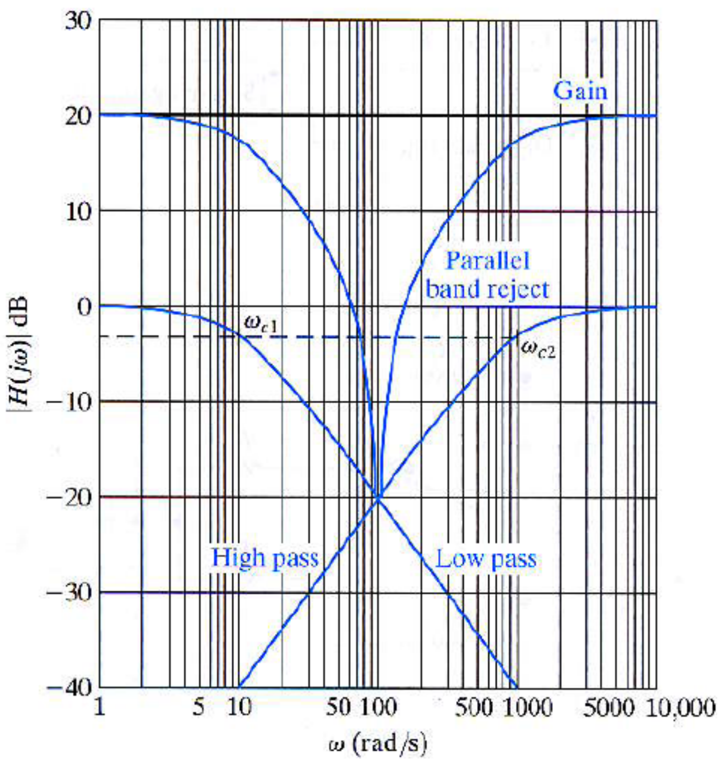
$$R_f = 2(100) = 2\ k\Omega$$



Op amp bandreject filter

- The bandreject filter also consists of 3 separate components :

  1. The unity-gain low-pass filter with a cut off frequency of ω_{c_1}
  2. The unity-gain high-pass filter with a cut off frequency of ω_{c_2}
  3. The gain component to provide the desired level of gain in the passbands.



"Bode magnitude plot"

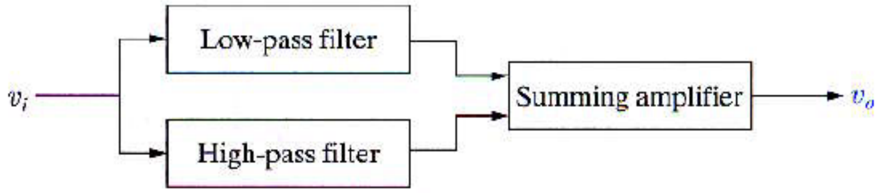
Note that ;

- These 3 components can NOT be cascaded

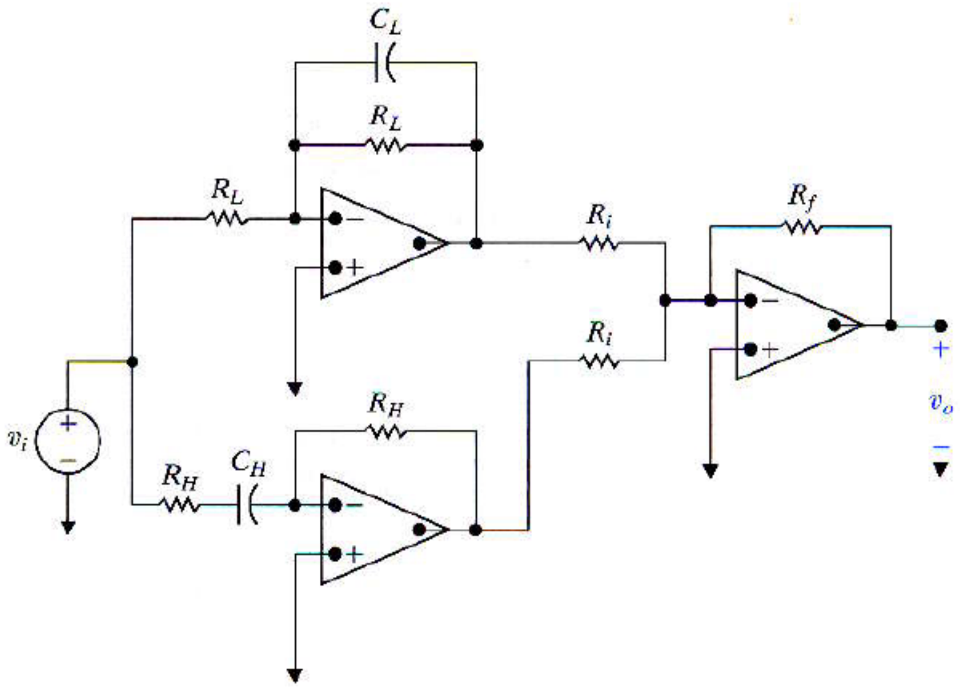
because they do NOT combine additively on the Bode plot.

Instead ;

- We use a parallel connection and a summing amplifier.



That is ;



- The resulting circuit has a transfer function of

$$H(s) = \left(-\frac{R_f}{R_i}\right)\left(\frac{-\omega_{c_1}}{s+\omega_{c_1}} + \frac{-s}{s+\omega_{c_2}}\right)$$

$$= \frac{R_f}{R_i}\left[\frac{s^2 + 2\omega_{c_1}s + \omega_{c_1}\omega_{c_2}}{(s+\omega_{c_1})(s+\omega_{c_2})}\right] \triangleq K\frac{s^2+\omega_0^2}{s^2+\beta s+\omega_0^2}$$

Similarly ;

- Assuming that $\omega_{c_2} \gg \omega_{c_1}$, the two cut off frequencies for the transfer function are ω_{c_1} and ω_{c_2}

$$\omega_{c_1} = \frac{1}{R_L C_L} \quad , \quad \omega_{c_2} = \frac{1}{R_H C_H}$$

- In the two passbands ($as\ s \to 0\ and\ s \to \infty$), the gain is R_f/R_i

that is, $K = R_f/R_i$

Design consideration

- We have 6 unknowns and 3 equations
- Typically we choose a commercially available capacitor value for C_L and C_H

then R_L and R_H are calculated accordingly.

- Finally we choose a value for either R_F or R_i

to compute the other resistance.

Moreover ;

- At the center frequency, $\omega_0 = \sqrt{\omega_{c_1}\omega_{c_2}}$

$$|H(j\omega_0)| = \left|\frac{R_f}{R_i}\frac{(j\omega_0)^2 + 2\omega_{c_1}(j\omega_0) + \omega_{c_1}\omega_{c_2}}{(j\omega_0)^2 + (\omega_{c_1} + \omega_{c_2})(j\omega_0) + \underbrace{\omega_{c_1}\omega_{c_2}}_{-(j\omega_0)^2}}\right|$$

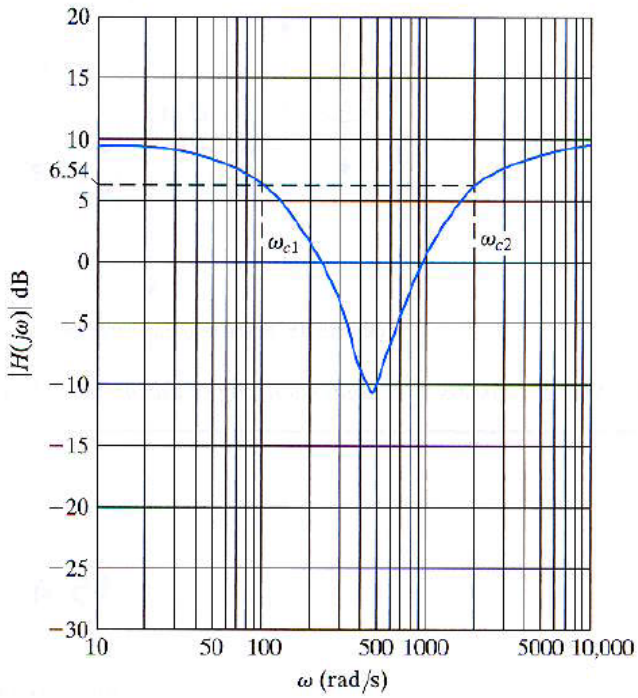
$$= \frac{R_f}{R_i}\frac{2\omega_{c_1}}{\omega_{c_1} + \omega_{c_2}}$$

$$\approx \frac{R_f}{R_i}\frac{2\omega_{c_1}}{\omega_{c_2}}$$

- If $\omega_{c_2} \gg \omega_{c_1}$ then $|H(j\omega_0)| \ll 2R_f/R_i$, that is

the magnitude at the center frequency is much smaller than the passband magnitude.

Ex. Design an op amp bandreject filter with the following Bode plot. Use 0.5 μF capacitors in your design.



- We find that

$$\omega_{c_1} = 100\ rad/s$$
$$\omega_{c_2} = 2000\ rad/s$$
$$K = 10^{(10/20)} \approx 3$$

$$\Rightarrow \quad \omega_{c_2} = 20\omega_{c_1} \quad \Rightarrow \quad \omega_{c_2} \gg \omega_{c_1}$$

- We design a prototype LPF

  and use a scaling to meet the specifications.

- The frequency scale factor, $k_f = 100$

  shifting the cut off frequency from 1 rad/s to 100 rad/s

- The magnitude scale factor, $k_m = 20\ 000$

  because $0.5.10^{-6} = \frac{1}{k_m.\underbrace{100}_{k_f}}$

- Then we obtain

$$R_L = 20\ k\Omega \quad , \quad C_L = 0.5.10^{-6}$$

- For the high-pass filter, we have

$$k_f = 2000 \qquad \text{and} \qquad 0.5.10^{-6} = \frac{1}{k_m.2000} \Rightarrow k_m = 1000$$

- We thus get

$$R_H = 1\ k\Omega \quad , \quad C_H = 0.5\ \mu F$$

- Let us choose $R_i = 1\ k\Omega$ then

$$\frac{R_f}{1\ k\Omega} = 3 \quad \Rightarrow \quad R_f = 3\ k\Omega$$

Higher order op amp filters

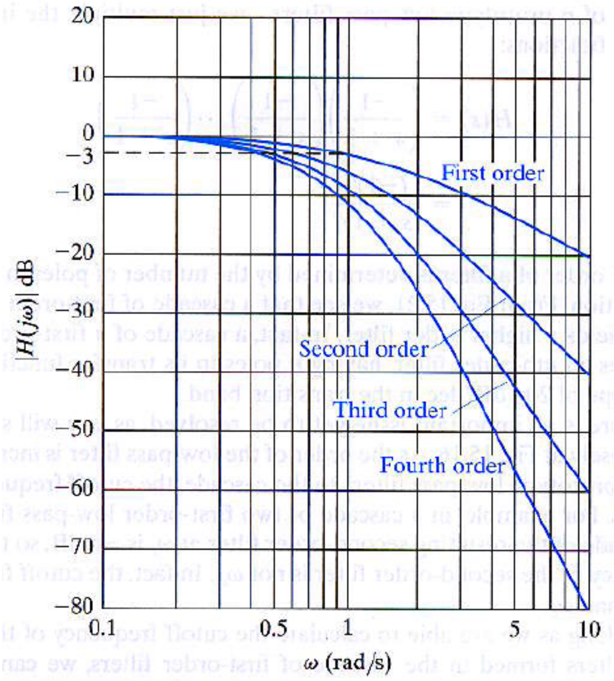
- All filters covered so far are all nonideal

  → which can NOT sharply divide the passband and stopband.

- Thus we can construct circuits with a sharper transition at the cut off frequency.

Cascading identical filters

- Consider the following Bode plots :



- It shows the Bode magnitude plots of a cascade of identical prototype low-pass filters.

- As more filters are added to the cascade

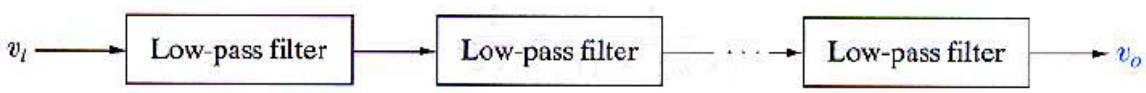
  → the transition from passband to stopband becomes sharper.

- Using one filter, the transition occurs with

  → an asymptotic slope of 20 dB/dec

In general ;

- An n^{th} element cascade of identical low-pass filters



  → a transition from the passband to the stopband with a slope of 20 ndB/dec.

What about the transfer function

- We just multiply the individual transfer functions :

$$H(s) = \left(\frac{-1}{s+1}\right)\left(\frac{-1}{s+1}\right)\cdot\cdot\cdot\left(\frac{-1}{s+1}\right)$$

$$= \frac{(-1)^n}{(s+1)^n}$$

- The order of a filter is determined by

  ↳ the # of poles in its transfer function.

Cut off frequency of higher order filters

- We shall solve for ω_{cn} that results in

$$|H(j\omega_{cn})| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left|\frac{(-1)^n}{(j\omega_{cn}+1)^n}\right| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{\left(\sqrt{\omega_{cn}^2+1}\right)^n} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad (\omega_{cn}+1)^n = 2 \qquad \Rightarrow \quad \omega_{cn} = \sqrt{\sqrt[n]{2}-1}$$

e.g. If $N = 4$ then $\omega_{c4} = \sqrt{\sqrt[4]{2}-1} = 0.435\, rad/s$

Ex. Design a 4^{th} order low-pass op amp filter with $\omega_c = 500\ Hz$ and a passband gain of 10. Use $1\ \mu F$ capacitors.

Solution. We first calculate

$$k_f = \frac{2\pi.500}{0.435} = 7222.39$$

$$1.10^{-6} = \frac{1}{k_m.7222.39} \qquad \Rightarrow \quad k_m = 138.46$$

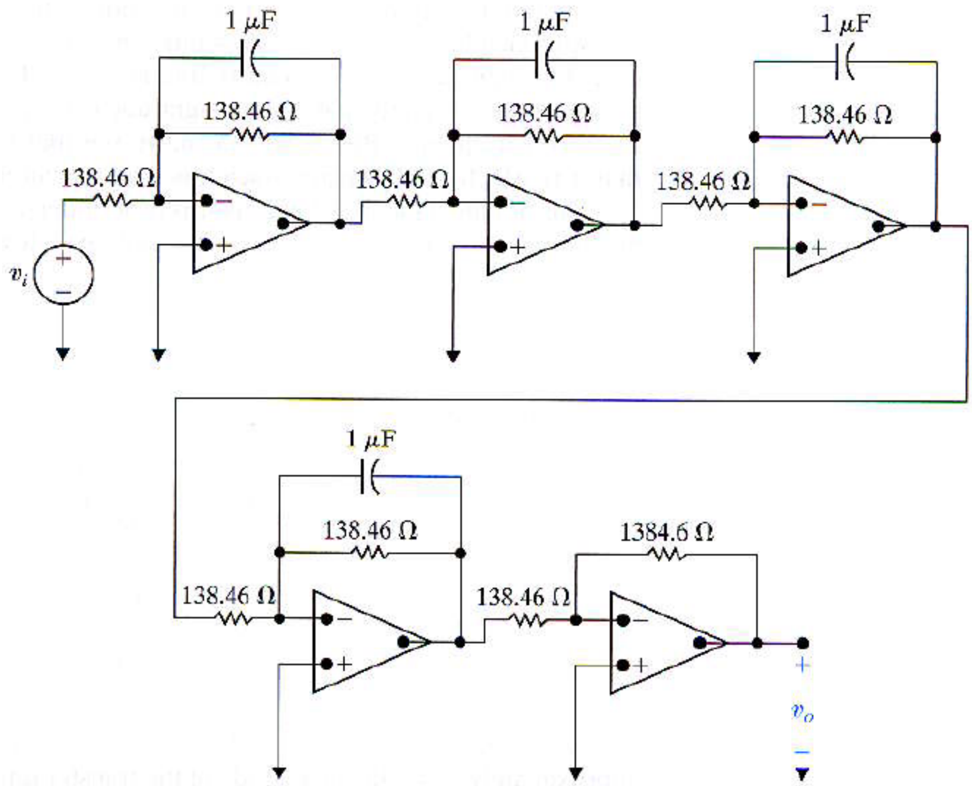
$$\Rightarrow \quad R = 138.46\,\Omega \quad , \quad C = 1\ \mu F$$

- For the inverting amplifier stage, we have

$$\frac{R_f}{R_i} = 10 \qquad \Rightarrow \qquad \text{selecting} \quad R_i = 138.46\,\Omega \;\; gives \;\; R_f = 1384.6\,\Omega$$

Hence ;

$$H(s) = -10\left(\frac{7222.39}{s + 7222.39}\right)^4$$



A serious shortcoming

- The gain of the cascaded identical filters is NOT constant

  between the zero and the cut off frequency, ω_c.

- This gain is equal to 1 for an ideal filter.

Indeed ;

- A unity-gain low-pass n^{th} order cascade has

$$H(s) = \frac{\omega_{cn}^n}{(s + \omega_{cn})^n}$$

$$\Rightarrow \quad H(j\omega) = \frac{\omega_{cn}^n}{\left(\sqrt{\omega^2 + \omega_{cn}^2}\right)^n} = \frac{1}{\left(\sqrt{(\omega/\omega_{cn})^2 + 1}\right)^n}$$

- For $\omega \ll \omega_{cn}$, we have $|H(j\omega) = 1|$

However ;

- The magnitude decreases when ω approaches ω_c.

Hence ;

- We need to employ other approaches

  because of this non-ideal behavior of the cascade of low-pass filters in the passband.

Butterworth filters (Maximally flat magnitude filter)

- A unity-gain Butterworth low-pass filter has a transfer function with

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^{2n}}}$$

Note : First described in 1930 by the British engineer Stephen Butterworth.

where n is the order of the filter.

Observations

- We notice that ;
  1. The cut off frequency is $\omega_c\ rad/s$ for all n
  2. If $n \gg 1$, the denominater is close to 1 when $\omega < \omega_c$
  3. The exponent of (ω/ω_c) is always even

  required for a physically realizable circuit.

how to find $H(s)$ then ?

- We first consider a prototype filter, that is

$$\omega_c = 1\ rad/s$$

$$|H(j\omega)|^2 = H(j\omega)\,H(-j\omega)$$

$$= H(s)H(-s) \quad , \qquad \text{because} \quad s = j\omega \quad \Rightarrow \quad s^2 = -\omega^2$$

$$= \frac{1}{1 + \omega^{2n}}$$

$$= \frac{1}{1 + (\omega^2)^n}$$

$$= \frac{1}{1+(-s^2)^n}$$

Hence ;

$$H(s)H(-s) = \frac{1}{1+(-1)^n s^{2n}}$$

A procedure to find $H(s)$

1. Find the roots of the polynomial

$$1+(-1)^n s^{2n} = 0$$

2. Assign the LHP roots to $H(s)$ and RHP roots to $H(-s)$
3. Combine terms in the denominator of $H(s)$ to form 1^{st} and 2^{nd} order factors.

Ex. Find the Butterworth transfer functions for $n = 2$.

Solution. For $= 2$, we find the roots of

$$1+(-1)^2 s^4 = 0$$

$$\Rightarrow \quad s^4 = -1 = 1\,\angle 180^\circ$$

$$s_1 = 1\,\angle 45^\circ = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$s_2 = 1\,\angle 135^\circ = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$s_3 = 1\,\angle 225^\circ = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$s_4 = 1\,\angle 315^\circ = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

- Roots s_2 and s_3 are in the LHP.

Thus ;

$$H(s) = \frac{1}{\left(s+\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\left(s+\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)} = \frac{1}{s^2+\sqrt{2}s+1}$$

Butterworth filter circuits

- We now consider the problem of designing a circuit

with the transfer function of a Butterworth filter.

- For n^{th}order Butterworth polynomial, we have

1	$(s+1)$
2	$(s^2+\sqrt{2}s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.518s+1)(s^2+\sqrt{2}+1)(s^2+1.932s+1)$
7	$(s+1)(s^2+0.445s+1)(s^2+1.247s+1)(s^2+1.802s+1)$
8	$(s^2+0.390s+1)(s^2+1.111s+1)(s^2+1.6663s+1)(s^2+1.962s+1)$

Note that ;

- The Butterworth polynomials are the product of 1^{st} and 2^{nd} order factors.

- All odd-order Butterworth polynomials include the factor $(s+1)$

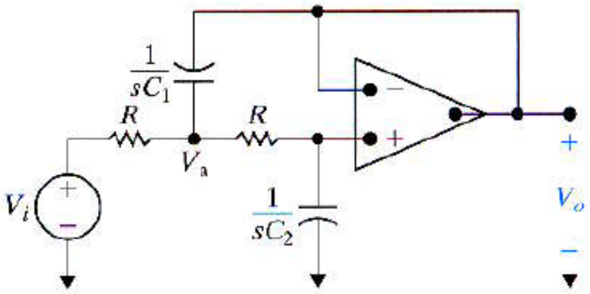
$$H(s)=\frac{1}{s+1} \qquad \Rightarrow \qquad \text{a prototype low-pass filter.}$$

Moreover ;

- One needs to find a circuit that provides a transfer function of the form

$$H(s)=\frac{1}{s^2+b_1 s+1}$$

- Consider the following circuit



- Writing node-voltage equations give

$$\frac{V_a-V_i}{R}+\frac{V_a-V_0}{(1/sC_1)}+\frac{V_a-V_0}{R}=0$$

$$\frac{V_0-V_a}{R}+\frac{V_0}{(1/sC_2)}=0$$

- Solving for V_0 yields

$$V_0 = \frac{V_i}{R^2 C_1 C_2 s^2 + 2RC_2 s + 1}$$

$$\Rightarrow \quad \frac{V_0}{V_i} = H(s) = \frac{\frac{1}{R^2 C_1 C_2}}{s^2 + \frac{2}{RC_1}s + \frac{1}{R^2 C_1 C_2}}$$

- We set $R = 1\ \Omega$ and obtain

$$\frac{\frac{1}{C_1 C_2}}{s^2 + \frac{2}{C_1}s + \frac{1}{C_1 C_2}} = \frac{1}{s^2 + b_1 s + 1}$$

$$\Rightarrow \qquad C_2 = \frac{1}{C_1} \qquad \text{and} \qquad b_1 = \frac{2}{C_1}$$

Ex. Design a 4^{th}order Butterworth low-pass filter with $\omega_c = 500\ Hz$ and a passband gain of 10. Use as many $1\ k\Omega$ resistors as possible.

Solution. The 4^{th}order Butterworth polynomial is

$$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$$

- We thus need a cascade of two 2^{nd}order filters

  plus an inverting amplifier for the passband gain of 10.

- For the 1^{st} stage, we have

$$\frac{2}{C_1} = 0.765 \quad \Rightarrow \quad C_1 = 0.38\ F \quad , \quad C_2 = \frac{1}{0.38} = 2.61\ F$$

- Similarly for the 2^{nd} stage, we have

$$\frac{2}{C_3} = 1.848 \quad \Rightarrow \quad C_3 = 1.08\ F \quad , \quad C_4 = \frac{1}{1.08} = 0.924\ F$$

Note that ;

$$(C_1,\ C_2,\ C_3,\ C_4) \quad \Rightarrow \quad \text{a } 4^{th}\text{order Butterworth filter with } \omega_c = 1\ rad/s$$

$$k_f = \frac{2\pi(500)}{1} \quad \Rightarrow \quad k_f = 3141.6$$

$$k_m = 1000$$

Hence ;

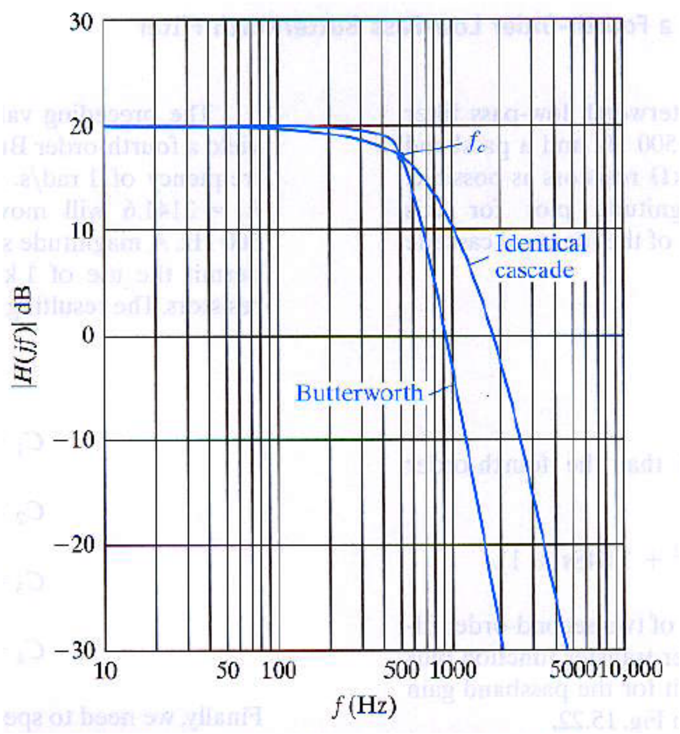
$$C_1' = \frac{0.38}{3141.6.1000} = 120.96\, nF \quad , \quad C_2' = \frac{2.61}{3141.6.1000} = 830.79\, nF$$

$$C_3' = \frac{1.08}{3141.6.1000} = 343.77\, nF \quad , \quad C_4' = \frac{0.924}{3141.6.1000} = 294.12\, nF$$

- Letting

$$R_i = 1\, k\Omega \quad \Rightarrow \quad R_f = 10.1\, k\Omega = 10\, k\Omega$$

- Consider the Bode magnitude plots of 4^{th} order Butterworth and identical cascade filters



- Both filters have a passband gain of 10 ($20dB$) and a cutoff frequency of 500 Hz.

However ;

- The Butterworth filter is closer to an ideal low pass filter

  due to its flatter passband and steeper rolloff at the cutoff frequency.

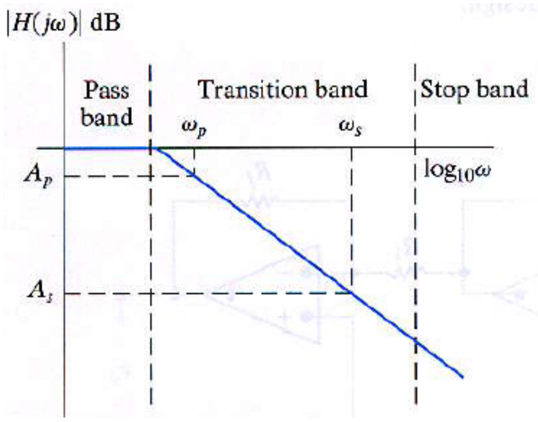
Finding the order of a Butterworth filter

- We wish to determine the smallest possible n

  to meet the filtering specifications.

- The filtering specifications are usually given

  in terms of the abruptness of the transition region.



- For the Butterworth filter,

$$A_p = 20\log_{10}\frac{1}{\sqrt{1+\omega_p^{2n}}} = -10\log_{10}\left(1+\omega_p^{2n}\right)$$

$$A_s = 20\log_{10}\frac{1}{\sqrt{1+\omega_s^{2n}}} = -\,10\log_{10}(1+\omega_s^{2n})$$

$$\Rightarrow \quad 1+\omega_p^{2n} = 10^{-0.1A_p} \quad , \quad 1+\omega_s^{2n} = 10^{-0.1A_s}$$

- Then we get

$$\left(\frac{\omega_s}{\omega_p}\right)^n = \frac{\sqrt{10^{-0.1A_s}-1}}{\sqrt{10^{-0.1A_p}-1}}$$

$$\Rightarrow \quad n = \frac{\log\sqrt{10^{-0.1A_s}-1} - \log\sqrt{10^{-0.1A_p}-1}}{\log\sqrt{\omega_s/\omega_p}}$$

- Choosing $\omega_p \triangleq \omega_c = 1\,rad/s$ yields

$$A_p = -20\log_{10}\sqrt{2} \quad \Rightarrow \quad 10^{-0.1A_p} = 2$$

- Since $10^{-0.1A_s} \gg 1$ to achieve a steep transition

$$n = \frac{-0.05A_s}{\log_{10} \omega_s} \quad \Rightarrow \quad \text{Select the nearest possible integer}$$

Ex.

a. Determine the order of a Butterworth filter whose magnitude is 10 dB less than the passband magnitude at 500 Hz and 60 dB less than the passband magnitude at 5000 Hz.

b. Determine the cutoff frequency of the filter.

c. What is the actual gain of the filter at 5000 Hz ?

Solution.

a.

$$\left(\frac{\omega_s}{\omega_p}\right)^n = \frac{\sqrt{10^{-0.1A_s} - 1}}{\sqrt{10^{-0.1A_p} - 1}}$$

$$\Rightarrow \quad \left(\frac{5000}{500}\right)^n = \frac{\sqrt{10^{-0.1(-60)} - 1}}{\sqrt{10^{-0.1(-10)} - 1}} \quad \Rightarrow \quad 10^n \cong \frac{1000}{3}$$

$$n \cong 2.52$$

$\therefore$ We need a 3^{rd} order Butterworth filter.

b. We know that

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^6}}$$

$$\Rightarrow \quad 20\log_{10}\frac{1}{\sqrt{1 + (500/f_c)^6}} = -10$$

$$\Rightarrow \quad \log_{10}[1 + (500/f_c)^6] = 1 \quad \Rightarrow \quad 1 + (500/f_c)^6 = 10$$

$$\Rightarrow \quad (500/f_c)^3 \quad \Rightarrow \quad f_c = 346.6806 \quad \Rightarrow \quad \omega_c = 2\pi f_c = 2178.33\, rad/s$$

c.

$$K = 20\log_{10}\frac{1}{\sqrt{1 + (5000/346.6806)^6}} = -69.54\, dB$$

Butterworth high-pass filter

- An n^{th}-order Butterworth filter has a transfer function

  with the n^{th} order polynomial in the denominator and s^n in the numerator.

- We pursue a cascade approach in its design.

first-order factor

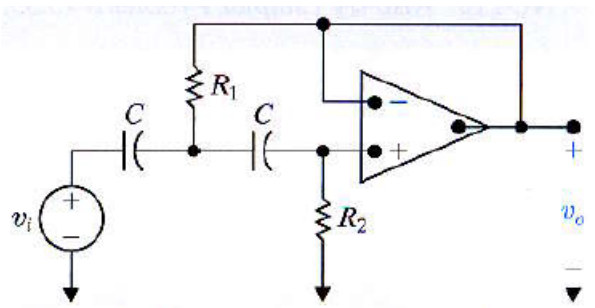
- Achieved by including a prototype high-pass filter in the cascade

  with $R_1 = R_2 = 1\,\Omega$ and $C = 1\,F$

second-order factors

- We need a circuit with a transfer function of the form

$$H(s) = \frac{s^2}{s^2 + b_1 s + 1}$$



$$H(s) = \frac{s^2}{s^2 + \frac{2}{R_2 C} s + \frac{1}{R_1 R_2 C^2}} \triangleq \frac{s^2}{s^2 + b_1 s + 1}$$

- Setting $C = 1\,F$ yields

$$2/R_2 = b_1 \quad \text{and} \quad \frac{1}{R_1 R_2} = 1$$

an important observation

- Note that the high-pass circuit was obtained from the low-pass circuit

  by simply interchanging resistors and capacitors.

- The prototype transfer function of a high-pass filter can be obtained

  by replacing s in the low-pass transfer function with $(1/s)$

Finally ;

- We can use magnitude and frequency scaling

  ↳ to design a Butterworth high-pass filter with practical component values and cutoff frequency other than 1 rad/s.

Butterworth bandpass filter

- We can combine n^{th}-order low-pass and high-pass Butterworth filters in cascade

  ↳ to produce n^{th}-order Butterworth bandpass filters.

Butterworth bandreject filter

- We shall combine these filters in parallel with a summing amplifier

  ↳ to produce n^{th}-order Butterworth bandreject filters.

Narrowband bandpass and bandreject filters

- The cascade and parallel component design methods for bandpass and bandreject filters

  ↳ lead to only broadband or low-Q filters.

- This limitation is due to the discrete poles of the transfer functions.

- The largest quality factor arises when the discrete pole locations are the same.

Hence ;

- We have

$$H(s) = \left(\frac{-\omega_c}{s+\omega_c}\right)\left(\frac{-s}{s+\omega_c}\right)$$

$$= \frac{s\omega_c}{s^2 + 2\omega_c s + \omega_c^2}$$

$$= \frac{0.5\beta s}{s^2 + \beta s + \omega_c^2}$$ "standard form of bandpass filter transfer function"

$$\Rightarrow \quad \beta = 2\omega_c \quad , \quad \omega_0^2 = \omega_c^2$$

- Then we find that

$$Q = \frac{\omega_0}{\beta} = \frac{\omega_c}{2\omega_c} = \frac{1}{2}$$

Thus ;

- With discrete-poles, the highest quality bandpass filter (or bandreject filter) we can achieve

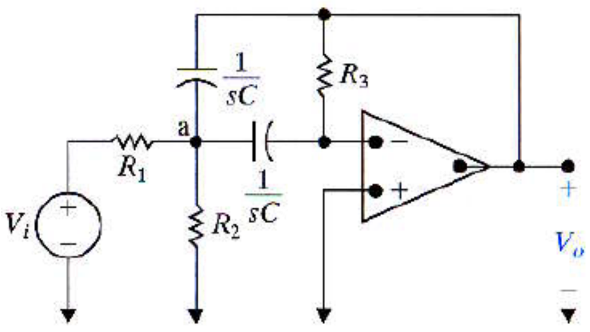
  has $Q = \frac{1}{2}$

How to achieve high quality factors ?

- We need an op amp circuit that can produce

  a transfer function with complex conjugate poles.

- Let us consider the following circuit :



- KCL at the inverting input :

$$\frac{-V_a}{(1/sC)} + \frac{-V_0}{R_3} = 0 \quad \Rightarrow \quad V_a = \frac{V_0}{sR_3C}$$

- KCL at node a :

$$\frac{V_a - V_i}{R_1} + \frac{V_a}{(1/sC)} + \frac{V_a - V_0}{(1/sC)} = 0$$

$$\Rightarrow \quad V_i = (1 + 2sR_1C + R_1/R_2)V_a - sR_1CV_0$$

- Substituting V_a expression into V_i gives

$$H(s) = \frac{V_0}{V_i} = \frac{-s/R_1C}{s^2 + \frac{2}{R_3C}s + \frac{1}{R_{eq}R_3C^2}} \triangleq \frac{-K\beta s}{s^2 + \beta s + \omega_0^2}$$

where $R_{eq} = R_1//R_2$ and

$$\beta = \frac{2}{R_3C} \quad , \quad K = \frac{R_3}{2R_1} \quad , \quad \omega_0 = \frac{1}{\sqrt{R_{eq}R_3}C}$$

design of a prototype version

- We let $\omega_0 = 1\ rad/s$ and $C = 1\ F$, then

$$Q = \frac{\omega_0}{\beta} = \frac{1}{\beta}$$

$$\Rightarrow \quad R_3 = \frac{2}{\beta C} = 2Q \quad , \quad K = \frac{2Q}{2R_1} \quad \Rightarrow \quad R_1 = \frac{Q}{K}$$

- and

$$\frac{R_1 R_2}{R_1 + R_2} R_3 = 1 \quad \Rightarrow \quad \frac{Q}{K} R_2 (2Q) = \frac{Q}{K} + R_2$$

$$\Rightarrow \quad (2Q^2 - K)^2 = Q \quad \Rightarrow \quad R_2 = \frac{Q}{2Q^2 - K}$$

Therefore ;

- R_1, R_2 and R_3 are given in terms of the desired quality factor, Q and passband gain, K :

$$R_1 = \frac{Q}{K} \quad , \quad R_2 = \frac{Q}{2Q^2 - K} \quad , \quad R_3 = 2Q$$

- Then we shall use "scaling"

to specify practical values of the circuit components.

Ex. Design a bandpass filter with $\omega_0 = 3\ kHz, Q = 10$ and $K = 2$. Use $0.01\ \mu F$ capacitors. Find the transfer function of your circuit and draw Bode diagram.

Solution. We first calculate prototype circuit components

$$R_1 = 10/2 = 5\ \Omega \quad , \quad R_2 = 10/(2.10^2 - 2) = 10/198 \quad , \quad R_3 = 2.10 = 20\ \Omega$$

$$k_f = 2\pi.3000 = 6000\pi$$

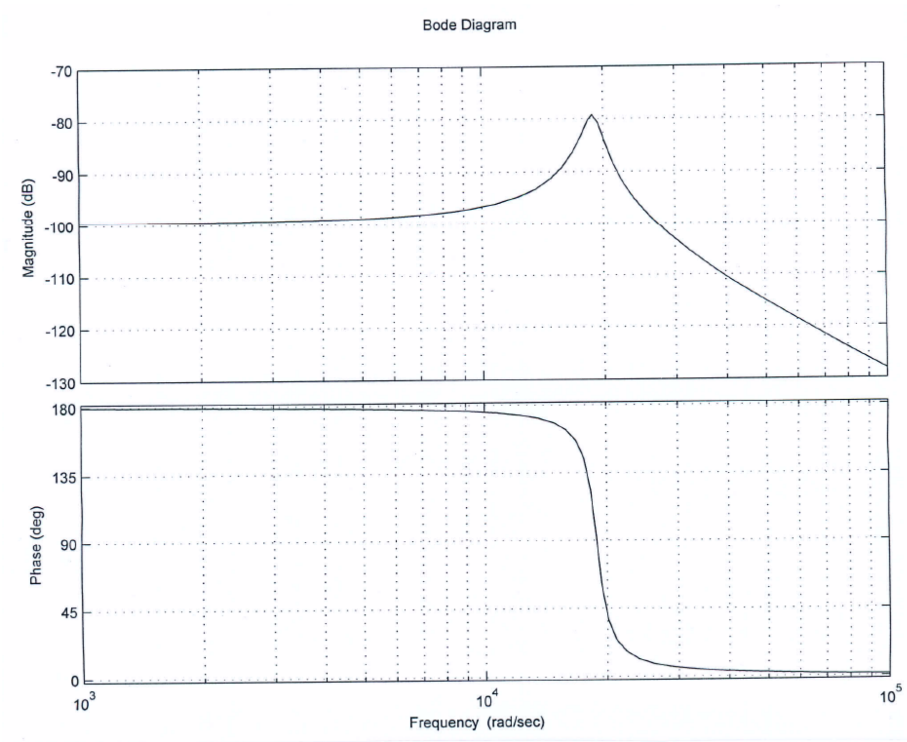
$$0.01.10^{-6} = \frac{1}{k_m.6000\pi} = k_m = 5300$$

Hence ;

$$R_1' = 5300.5 \cong 26.5\ \Omega$$
$$R_2' = 5300.10/198 \cong 267.6768\ \Omega$$
$$R_3' = 5300.20 \cong 106\ k\Omega$$

- and

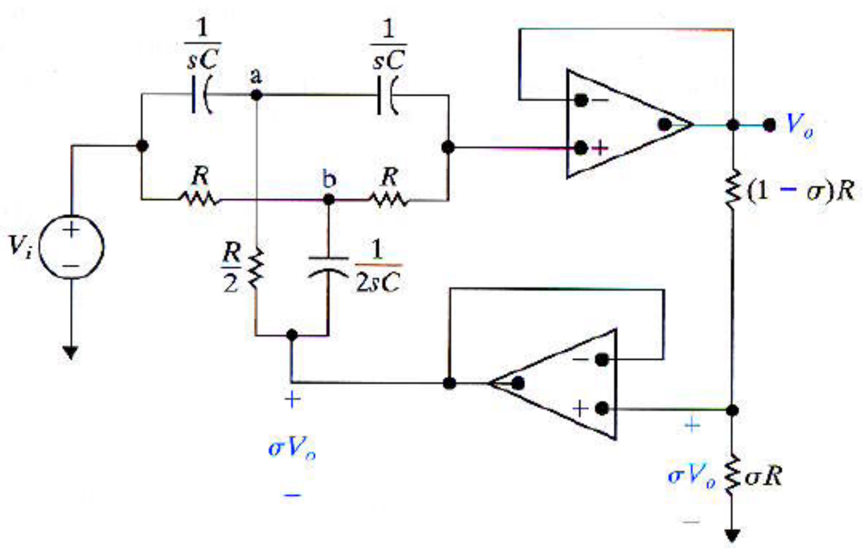
$$H(s) = -\frac{2.600\pi}{s^2 + 600\pi s + (6000\pi)^2} = -\frac{3769.9}{s^2 + 1850.0s + 3.5531.10^8}$$



Active high-Q bandreject filter

- We consider an active high-Q bandreject filter circuit

  known as the "twin-T notch filter" because of the two T-shaped parts of the circuit at nodes a and b.



- KCL at node a :

$$\frac{V_a - V_i}{(1/sC)} + \frac{V_a - V_0}{(1/sC)} + \frac{V_a - \sigma V_0}{(R/2)} = 0$$

- KCL at node b :

$$\frac{V_b - V_i}{R} + \frac{V_b - V_0}{R} + \frac{V_b - \sigma V_0}{(1/2sC)} = 0$$

- KCL at noninverting terminal of the top op amp :

$$\frac{V_0 - V_b}{R} + \frac{V_0 - V_a}{(1/sC)} = 0$$

- Making algebraic rearrangements allows to get

$$H(s) = \frac{V_0}{V_i} = \frac{s^2 + \frac{1}{R^2C^2}}{s^2 + \frac{4(1-\sigma)}{RC}s + \frac{1}{R^2C^2}}$$

$$\triangleq \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

where

$$\omega_0^2 = \frac{1}{R^2C^2} \quad , \quad \beta = \frac{4(1-\sigma)}{RC}$$

- We have 3 parameters $(R,\ C,\ \sigma)$ and two design constraints $(\omega_0\ and\ \beta)$.

Thus ;

- We choose one parameter, usually, the capacitor value arbitrarily.

- Then we shall choose

$$R = \frac{1}{\omega_0 C}$$

and

$$\beta = \frac{4(1-\sigma)}{(1/\omega_0)} \quad \Rightarrow \quad 1 - \sigma = \frac{\beta}{4\omega_0} \quad \Rightarrow \quad \sigma = 1 - \frac{\beta}{4\omega_0}$$

OR

$$\sigma = 1 - \frac{1}{4Q}$$

Ex. Design a high-Q active bandreject filter with $\omega_0 = 5000\, rad/s$, $\beta = 1000\, rad/s$. Use 1 μF capacitors in your design.

Solution. We first consider a bandreject prototype filter, that is

$$R = 1\,\Omega \quad , \quad C = 1\,F \quad , \quad and \quad \omega_0 = 1\, rad/s$$

- We calculate

$$Q = \frac{\omega_0}{\beta} = \frac{5000}{1000} = 5$$

$$\sigma = 1 - \frac{1}{4.5} = 0.95$$

$$k_f = 5000 \qquad , \qquad 10^{-6} = \frac{1}{k_m.5000} \qquad \Rightarrow \qquad k_m = 200$$

$$\Rightarrow \quad R' = 200.1 = 200\,\Omega$$

$$H(s) = \frac{s^2 + 25.10^6}{s^2 + 1000s + 25.10^6}$$