Problem 1) Let

$$F(\omega) = \mathcal{F}\{f(t)\}$$

a. Show that

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

b. What is the restriction on $f(t)$ if the result give in (a) is valid ?

c. Show that

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (j\omega)^n F(\omega)$$

Solution. We use the defining integral and integrate by parts as follows

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-j\omega t} dt$$

- let $u = e^{-j\omega t}$, $dv = \frac{df(t)}{dt} dt \Rightarrow v = f(t)$,
$du = -j\omega e^{-j\omega t} dt$

- then

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = e^{-j\omega t} f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\left(-j\omega e^{-j\omega t}\right) dt$$

$$= \lim_{t\to\infty} e^{-j\omega t} f(t) - \lim_{t\to-\infty} e^{-j\omega t} f(t) + j\omega \underbrace{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}_{\mathcal{F}\{f(t)\}}$$

- assuming that

$$\lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) = 0$$

we obtain

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = 0-0+j\omega\, F(\omega)$$

$$= j\omega\, F(\omega)$$

b. The restriction on $f(t)$ is that Fourier transform of $f(t)$ exists, that $f(t)$ must be absolutely integrable on an infinite interval.

c. Note that

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = \frac{d^{n-1} f(t)}{dt^{n-1}}\, e^{-j\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d^{n-1} f(t)}{dt^{n-1}} (-j\omega) e^{-j\omega t}\, dt$$

$n=1$:

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega\, \mathcal{F}\{f(t)\}$$

$n=2$:

$$\mathcal{F}\left\{\frac{d^2 f(t)}{dt^2}\right\} = j\omega\, \mathcal{F}\left\{\frac{df(t)}{dt}\right\}$$

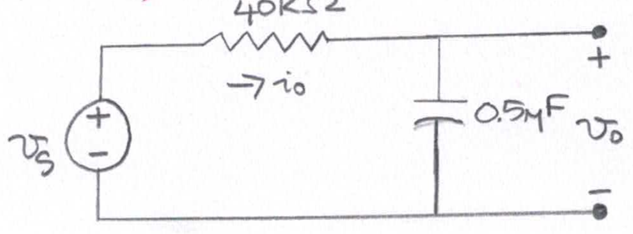
$$=(j\omega)(j\omega)\, \mathcal{F}\{f(t)\}$$

⋮

n is arbitrary :

$$\mathcal{F}\left\{\frac{d^n f(t)}{dt^n}\right\} = (j\omega)^n\, \mathcal{F}\{f(t)\}$$

Problem 2) Consider the following circuit



a. Use the Fourier transform method to find $v_o(t)$ if $v_g = 20\,\text{sgn}(t)$ V.

b. Does your solution make sense in terms of known circuit behavior? Explain.

Solution. We have

$$\mathcal{F}\{20\,\text{sgn}(t)\} = 20\,\frac{2}{j\omega}$$

$$= \frac{40}{j\omega} \triangleq V_g$$

$$I_o\left(40000 + \frac{1}{j\omega \cdot 0.5 \cdot 10^{-6}}\right) = V_g \qquad \text{"phasor-domain"}$$

$$\Rightarrow \quad I_o\left(40000 - j\,\frac{2\cdot 10^6}{\omega}\right) = \frac{40}{j\omega}$$

$$\Rightarrow \quad I_o\left(j40000\omega + 2\cdot 10^6\right) = 40$$

$$\Rightarrow \quad I_o\left(5\cdot 10^4 + j1000\omega\right) = 1$$

$$\Rightarrow \quad I_o = \frac{1}{5\cdot 10^4 + j1000\omega} = \frac{10^{-3}}{j\omega + 50}$$

Hence;

$$\mathcal{F}^{-1}\left\{\frac{10^{-3}}{j\omega+50}\right\} \triangleq i_o(t) = 10^{-3} e^{-50t}$$

Moreover;

$$\frac{V_o(s)}{V_g(s)} \triangleq H(s) = \frac{1/sC}{R + \frac{1}{sC}} = \frac{1}{1+sRC} = \frac{1}{1+s\,40\cdot 10^3\cdot 0.5\cdot 10^{-6}}$$

$$= \frac{1}{1+0.02s}$$

$$\Rightarrow H(\omega) = H(s)\Big|_{s=j\omega} = \frac{1}{1+0.02j\omega}$$

$$= \frac{50}{j\omega+50}$$

then

$$V_o(\omega) = H(\omega)\,V_s(\omega)$$

$$= \frac{50}{j\omega+50}\,\frac{40}{j\omega} = \frac{C_1}{j\omega} + \frac{C_2}{j\omega+50}$$

$$\Rightarrow C_1 = \frac{2000}{j\omega+50}\Big|_{j\omega=0} = 40$$

$$\Rightarrow C_2 = \frac{2000}{j\omega}\Big|_{j\omega=-50} = -40$$

$$\Rightarrow V_o(\omega) = \frac{40}{j\omega} - \frac{40}{j\omega+50}$$

$$\Rightarrow \mathcal{F}^{-1}\{V_o(\omega)\} = 20\,\mathrm{sgn}(t) - 40e^{-50t}u(t)\ \mathrm{V}$$

b. We have

$$v_o(0^-) = -20-0 = -20\mathrm{V}$$

$$v_o(0^+) = 20-40 = -20\mathrm{V}$$

$\Rightarrow$ the capacitor is charged to -20V when $t<0$ and shows NO instantaneous change, that is, $v_o(0^+) = v_o(0^-)$

$$v_o(\infty) = 20-0 = 20\mathrm{V}$$

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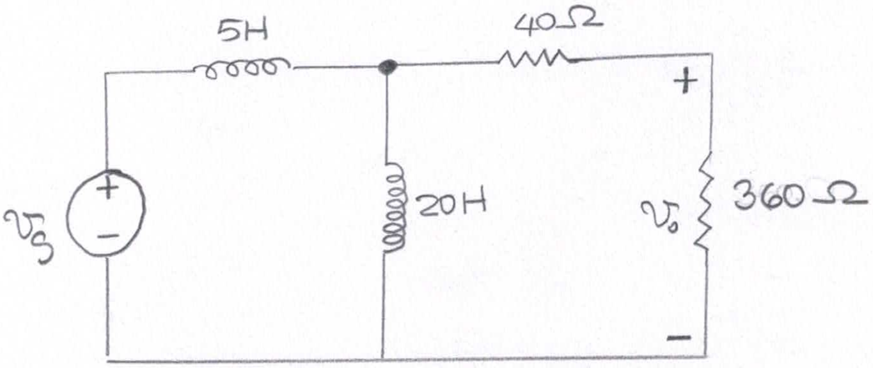
⇒ the capacitor behaves as a short circuit as $t \to \infty$

thus $v_o(\infty) = v_g = 20\ V$

- finally, the time constant, $RC = 40 \cdot 10^3 \cdot 0.5 \cdot 10^{-6} = 0.02\,s$

↳ $v_o(t)$ is of exponential form with a time constant of $1/50 = 0.02\,s$

Problem 3) Consider the following circuit shown as



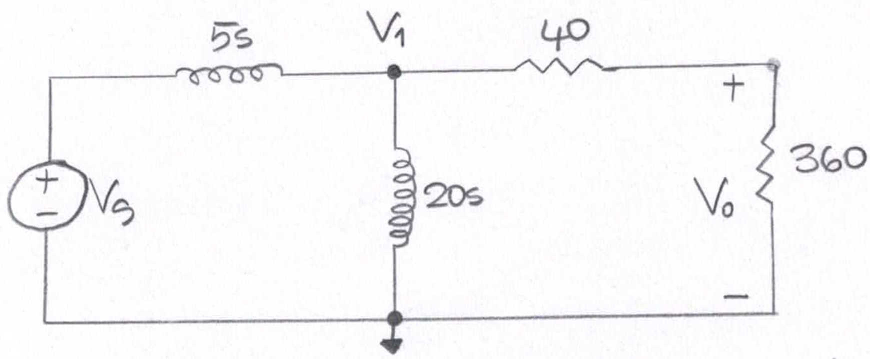
a. Use the Fourier transform method to find v_o if

$v_g = 125 \cos 75t$ V

b. Check the answer obtained in (a) by finding the steady-state expression for v_o using phasor domain analysis.

Solution.

a. We consider the s-domain equivalent circuit



- we consider the node-voltage equations:

$$\underset{(80)}{\frac{V_1 - V_g}{5s}} + \underset{(20)}{\frac{V_1}{20s}} + \underset{(s)}{\frac{V_1}{400}} = 0 \Rightarrow (s+100)V_1 = 80V_g$$

$$\Rightarrow V_1 = \frac{80}{s+100} V_g$$

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and

$$V_o = \frac{360}{40+360} V_1 = \frac{\cancel{360}}{\cancel{400}} \frac{\cancel{80}^{2}}{s+100} V_s$$

$$\Rightarrow \frac{V_o}{V_s} \triangleq H(s) = \frac{72}{s+100}$$

then

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{72}{j\omega+100}$$

$$V_s(\omega) = \mathcal{F}\{125 \cos 75t\} = 125\pi[\delta(\omega+75) + \delta(\omega-75)]$$

$$\Rightarrow V_o(\omega) = H(\omega) V_s(\omega)$$

$$= \frac{72}{j\omega+100} 125\pi[\delta(\omega+75)+\delta(\omega-75)]$$

$$= 90\pi\left[\frac{\delta(\omega+75)}{j\omega+100} + \frac{\delta(\omega-75)}{j\omega+100}\right]$$

Thus ;

$$v_o(t) = \mathcal{F}^{-1}\{V_o(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 90\pi\left[\frac{\delta(\omega+75)}{j\omega+100} + \frac{\delta(\omega-75)}{j\omega+100}\right] e^{j\omega t} d\omega$$

$$= 45\left(\frac{e^{-75t}}{100-j75} + \frac{e^{75t}}{100+j75}\right)$$

$$= \cancel{45}^{9}\left(\frac{e^{-75t}}{\cancel{125}_{25}\, e^{-j36.87°}} + \frac{e^{75t}}{125 e^{j36.87}}\right)$$

$$= \frac{9}{25}\left[e^{-j(75t-36.87°)} + e^{j(75t-36.87°)}\right]$$

$$= 0.72 \cos(75t - 36.87°)\ V$$

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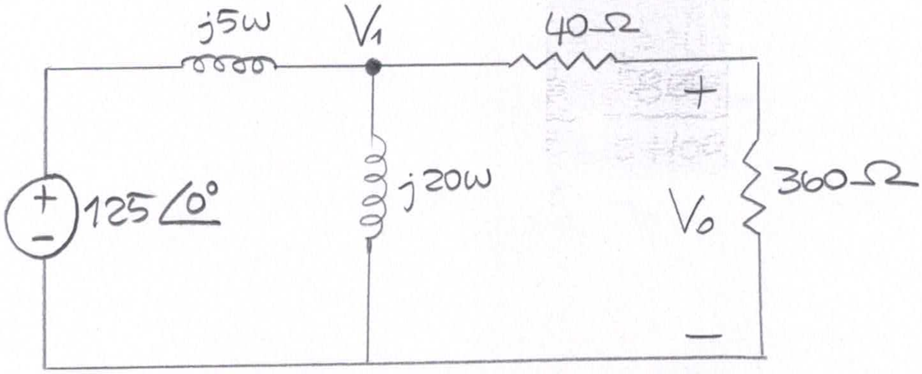
b. We now employ phasor domain analysis



where $\omega = 75$ rad/s

- we write the node-voltage equation as follows

$$\frac{V_1 - 125}{\underset{(80)}{j5\omega}} + \frac{V_1}{\underset{(20)}{j20\omega}} + \frac{V_1}{\underset{(j\omega)}{400}}$$

$$\Rightarrow \quad (j\omega + 100)V_1 = 10^4 \quad \Rightarrow \quad V_1 = \frac{10^4}{j\omega + 100}$$

and

$$V_o = \frac{360}{40+360} V_1 = \frac{\overset{9}{\cancel{360}}}{\underset{10}{\cancel{400}}} \frac{10^4}{j\omega+100} = \frac{900}{j\omega+100}$$

$$= \frac{9000}{j\omega+100}$$

- letting $\omega = 75$ rad/s gives

$$V_o = \frac{9000}{j75+100} = \frac{\overset{360}{\cancel{9000}}}{\underset{5}{\cancel{125}} e^{j36.87^\circ}} = 72 \angle -36.87^\circ$$

$$\Rightarrow \quad v_o(t) = 72 \cos(75t - 36.87^\circ) V$$

Problem 4) It is given that

$$F(\omega) = e^{\omega} u(-\omega) + e^{-\omega} u(\omega)$$

a. Find $f(t)$.

b. Find the 1Ω energy associated with $f(t)$ via time-domain

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c. Repeat part (b) using frequency-domain integration

d. Find the value of ω_1 if $f(t)$ has 90% of the energy in the frequency band $0 \leq \omega \leq \omega_1$.

Solution.

a.

$$f(t) = \mathcal{F}^{-1}\left\{F(\omega)\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{\omega}u(-\omega)+e^{-\omega}u(\omega)\right]e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{0}e^{\omega}e^{j\omega t}\,d\omega + \int_{0}^{\infty}e^{-\omega}e^{j\omega t}\,d\omega\right)$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0}e^{(1+jt)\omega}\,d\omega + \int_{0}^{\infty}e^{\omega(-1+jt)}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1+jt)\omega}}{1+jt}\Bigg|_{-\infty}^{0} + \frac{e^{(-1+jt)\omega}}{-1+jt}\Bigg|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{1+jt} - 0 + 0 - \frac{1}{-1+jt}\right)$$

$$= \frac{1}{2\pi}\,\frac{-1+\cancel{jt}-1-\cancel{jt}}{(1+jt)(-1+jt)}$$

$$= \frac{1}{\cancel{2}\pi}\,\frac{-\cancel{2}}{-t^2+1}$$

$$= \frac{1}{\pi(t^2+1)}$$

b. $W_{1\Omega} = 1 \cdot \int_{-\infty}^{\infty} f^2(t)\, dt$

$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{(t^2+1)^2}\, dt$

$= \frac{2}{\pi^2} \int_{0}^{\infty} \frac{1}{(t^2+1)^2}\, dt$

$= \frac{2}{\pi^2} \frac{1}{2} \left(\frac{t}{t^2+1} + \tan^{-1} t \right) \Big|_0^{\infty}$

$= \frac{1}{\pi^2} \left(0 + \frac{\pi}{2} - 0 - 0 \right)$

$= \frac{1}{2\pi}\ \mathrm{J}$

c. $W_{1\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [e^{\omega} u(-\omega) + e^{-\omega} u(\omega)]^2 d\omega$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [e^{2\omega} u^2(-\omega) + 2u(-\omega)u(\omega) + e^{-2\omega} u^2(\omega)]\, d\omega$

$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{2\omega} d\omega + \int_{0}^{\infty} e^{-2\omega} d\omega \right)$

$= \frac{1}{2\pi} \left(\frac{1}{2} e^{2\omega} \Big|_{-\infty}^{0} + \frac{1}{-2} e^{-2\omega} \Big|_{0}^{\infty} \right)$

$= \frac{1}{2\pi} \left(\frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 0 - \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 \right)$

$= \frac{1}{2\pi}\ \mathrm{J}$

d. We have

$$\frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} |F(\omega)|^2 d\omega = 0.9 \cdot \frac{1}{2\pi}$$

$$\Rightarrow \cancel{\frac{1}{2\pi}} \int_{-\omega_1}^{\omega_1} \left[e^{2\omega}u^2(-\omega) + 2u(-\omega)u(\omega) + e^{-2\omega}u^2(\omega)\right] d\omega = 0.9 \cancel{\frac{1}{2\pi}}$$

$$\Rightarrow \int_{-\omega_1}^{0} e^{2\omega} d\omega + \int_{0}^{\omega_1} e^{-2\omega} d\omega = 0.9$$

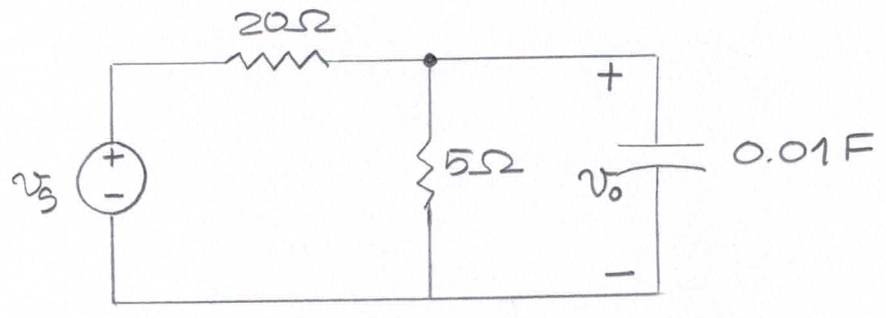
$$\Rightarrow \frac{1}{2} e^{2\omega}\Big|_{-\omega_1}^{0} + \frac{1}{-2} e^{-2\omega}\Big|_{0}^{\omega_1} = 0.9$$

$$\Rightarrow \frac{1}{2}\left(1 - e^{-2\omega_1}\right) - \frac{1}{2}\left(e^{-2\omega_1} - 1\right) = 0.9$$

$$\Rightarrow 1 - e^{-2\omega_1} = 0.9 \Rightarrow e^{-2\omega_1} = 0.1$$

$$\Rightarrow -2\omega_1 = \ln 0.1 \Rightarrow \omega_1 = \frac{\ln 0.1}{-2} = 1.1513 \text{ rad/s}$$

Problem 5) Consider the following circuit



$v_g = 60e^{-15|t|}$ V

a. Find $v_o(t)$.

b. Sketch $|V_g(\omega)|$ for $-10 \leq \omega \leq 10$ rad/s.

c. Sketch $|V_o(\omega)|$ for $-10 \leq \omega \leq 10$ rad/s.

d. Calculate the 1Ω energy content of v_g.

e. Calculate the 1Ω energy content of v_o.

f. What percentage of the 1Ω energy content in v_g lies in the frequency range $0 \leq \omega \leq 10$ rad/s?

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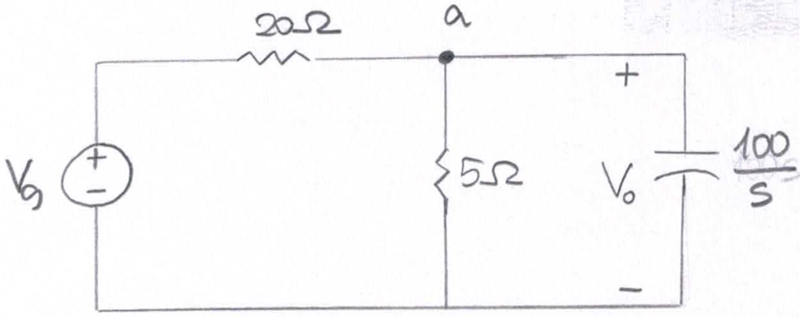
9. Repeat (f) for v_o.

Solution. We shall first draw the s-domain equivalent circuit as follows



- writing node-voltage equation at node a gives

$$\underset{(5)}{\frac{V_o - V_s}{20}} + \underset{(20)}{\frac{V_o}{5}} + \underset{(1)}{\frac{sV_o}{100}} = 0$$

$$\Rightarrow \quad (s+25)V_o - 5V_s = 0 \quad \Rightarrow \quad \frac{V_o}{V_s} \triangleq H(s) = \frac{5}{s+25}$$

a. We have

$$v_s = v_s^+ + v_s^-$$

where

$$v_s^+ = 60e^{-5t}, \quad t>0$$

$$v_s^- = 60e^{5t}, \quad t<0$$

then

$$\mathcal{F}\{v_s\} = \mathcal{L}\{60e^{-5t}\}\Big|_{s=j\omega} + \mathcal{L}\{60e^{-5t}\}\Big|_{s=-j\omega}$$

$$= \frac{60}{j\omega+5} + \frac{60}{-j\omega+5}$$

$$= \frac{600}{(j\omega+5)(-j\omega+5)}$$

Hence;

$$V_o(w) = H(w)\,V_s(w)$$

$$= H(s)\Big|_{s=jw} \; V_s(w)$$

$$= \frac{5}{jw+25}\;\frac{600}{(jw+5)(-jw+5)}$$

$$= \frac{C_1}{jw+5} + \frac{C_2}{-jw+5} + \frac{C_3}{jw+25}$$

$$\Rightarrow C_1 = \frac{3000}{(jw+25)(-jw+5)}\Big|_{jw=-5} = \frac{300\cancel{0}}{2\cancel{0}\cdot 1\cancel{0}} = 15$$

$$\Rightarrow C_2 = \frac{3000}{(jw+25)(jw+5)}\Big|_{jw=5} = \frac{30\cancel{00}}{3\cancel{0}\cdot 1\cancel{0}} = 10$$

$$\Rightarrow C_3 = \frac{3000}{(jw+5)(-jw+5)}\Big|_{jw=-25} = \frac{\overset{1}{\cancel{3}}0\cancel{00}}{(-2\cancel{0})\cdot \cancel{3}\cancel{0}} = -5$$

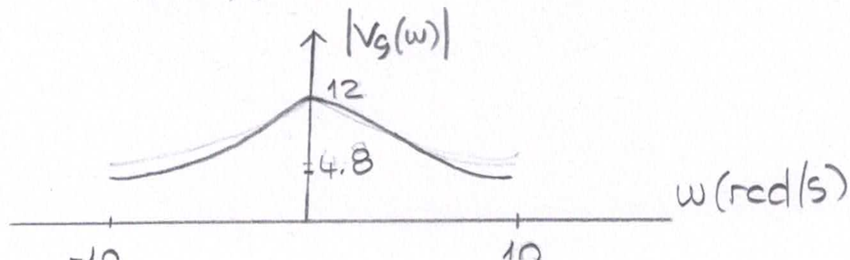
Thus;

$$\mathcal{F}^{-1}\{V_o(w)\} \triangleq v_o(t) = \mathcal{F}^{-1}\left\{\frac{15}{jw+5} + \frac{10}{-jw+5} - \frac{5}{jw+25}\right\}$$

$$= \left(15e^{-5t} - 5e^{-25t}\right)u(t) - 10e^{5t}u(-t)\ \text{V}$$

b. We have

$$|V_s(w)| = \left|\frac{600}{(jw+5)(-jw+5)}\right| = \frac{600}{\sqrt{w^2+25}\cdot\sqrt{w^2+25}} = \frac{600}{w^2+25}$$

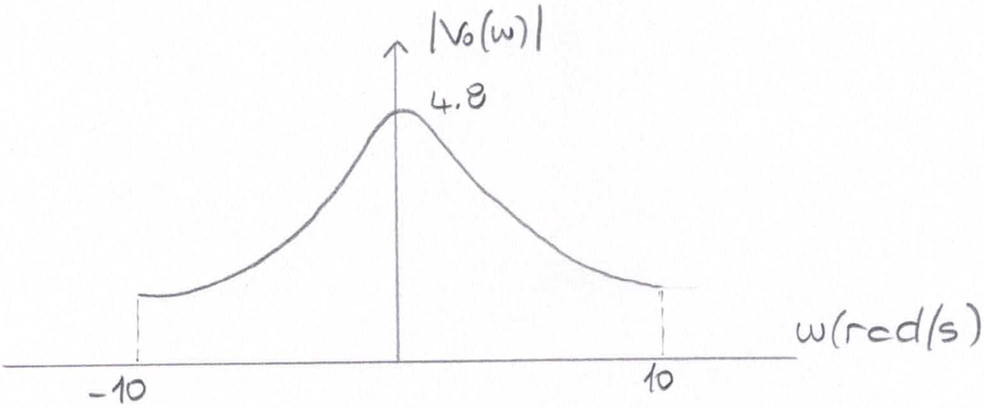


c. We calculate

$$|V_o(\omega)| = \left| \frac{3000}{(j\omega+5)(-j\omega+5)(j\omega+25)} \right|$$

$$= \frac{3000}{\sqrt{\omega^2+25}\cdot\sqrt{\omega^2+25}\cdot\sqrt{\omega^2+625}}$$

$$= \frac{3000}{\omega^2+25\ \sqrt{\omega^2+625}}$$



d.

$$W_s = \frac{1}{2\pi}\int_{-\infty}^{\infty} |V_s(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{36\cdot10^4}{(\omega^2+25)^2} d\omega$$

$$= \frac{36\cdot10^4}{2\pi}\, 2 \int_0^{\infty} \frac{1}{\omega^2+5^2} d\omega$$

$$= \frac{\overset{18}{\cancel{36}}\cdot\overset{400}{\cancel{10^4}}}{\cancel{2}\pi}\,\cancel{2}\,\frac{1}{\cancel{2}\cancel{5^2}}\left(\frac{\omega}{\omega^2+25}+\frac{1}{5}\tan^{-1}\frac{\omega}{5}\right)\Bigg|_0^{\infty}$$

$$= \frac{7200}{\cancel{\pi}}\left(0+\frac{1}{5}\,\frac{\cancel{\pi}}{2}-0-0\right)$$

$$= 720\ J$$

e.

$$W_o = \int_{-\infty}^{\infty} v_o^2(t)\,dt$$

$$= \int_{-\infty}^{\infty} \left[\left(15e^{-5t} - 5e^{-25t}\right)u(t) - 10e^{5t}u(-t) \right]^2 dt$$

$$= \int_{-\infty}^{\infty} \left[\left(15e^{-5t} - 5e^{-25t}\right)^2 u^2(t) - 20\left(15 - 5e^{-20t}\right)u(t)u(-t) + 100e^{10t}u^2(-t) \right] dt$$

$$= \int_{0}^{\infty} \left(225e^{-10t} - 150e^{-30t} + 25e^{-50t}\right) dt + \int_{-\infty}^{0} 100e^{10t}\,dt$$

$$= \left(-22.5e^{-10t} + 5e^{-30t} - 0.5e^{-5t}\right)\Big|_0^{\infty} + 10e^{10t}\Big|_{-\infty}^{0}$$

$$= (0 + 22.5 + 0 - 5 - 0 + 0.5) + 10 - 0$$

$$= 28\ J$$

f.

$$W_g' = \frac{1}{2\pi}\int_{-10}^{10} |V_g(w)|^2\,dw$$

$$= \frac{1}{2\pi}\int_{-10}^{10} \frac{36.10^4}{(w^2+5^2)^2}\,dw$$

$$= \frac{36.10^4}{2\pi}\, 2\int_{0}^{10} \frac{1}{(w^2+5^2)^2}\,dw$$

$$= \frac{\overset{18}{\cancel{36}}.\overset{400}{\cancel{10^4}}}{\cancel{2}\pi}\,\cancel{2}\,\frac{1}{\cancel{2}.\cancel{5^2}}\left(\frac{w}{w^2+5^2} + \frac{1}{5}\tan^{-1}\frac{w}{5}\right)\Big|_0^{10}$$

$$= \frac{7200}{\pi}\left(\frac{10}{125} + \frac{1}{5}\tan^{-1}2 - 0 - 0\right)$$

$$= \frac{7200}{\pi}\left(0.08 + \frac{1}{5}\,1.0171\right)$$

$$= 690.8038 \text{ J}$$

Hence;

$$\frac{690.8038}{720} \times 100 = \%\,95.9450$$

9. We calculate

$$W_o' = \frac{1}{2\pi}\int_{-10}^{10} |V_o(w)|^2\,dw$$

$$= \frac{1}{2\pi}\int_{-10}^{10} \frac{9.10^6}{(w^2+25)^2(w^2+625)}\,dw$$

Note that;

$$\frac{9.10^6}{(w^2+25)^2(w^2+625)} = \frac{15000}{(w^2+25)^2} - \frac{25}{w^2+25} + \frac{25}{w^2+625}$$

Hence;

$$W_o' = \frac{1}{2\pi}\,2\int_0^{10}\left[\frac{15000}{(w^2+25)^2} - \frac{1}{1+(w/5)^2} + \frac{1/25}{1+(w/25)^2}\right]dw$$

$$= \frac{1}{\pi}\left[\frac{1}{\cancel{2.25}}\,\overset{300}{\cancel{15000}}\left(\frac{w}{w^2+25} + \frac{1}{5}\tan^{-1}\frac{w}{5}\right) - 5\tan^{-1}\frac{w}{5} + \tan^{-1}\frac{w}{25}\right]_0^{10}$$

$$= \frac{1}{\pi}\left[300\left(\frac{10}{125} + \frac{1}{5}.1.0171\right) - 5.1.0171 + 0.3805 - 0 + 0 - 0\right]$$

$$= 27.1435 \text{ J}$$

As a result;

$$\frac{27.1435}{28} \times 100 = \% 96.9409$$