

Introduction to Frequency Selective Circuits

- We wish to analyze the effect of changing frequency on circuit voltages and currents

↳ which leads to the frequency response of a circuit.

Note that ;

- Varying the frequency of a sinusoidal source

↳ alters the impedance of capacitors and inductors because their impedances are functions of frequency.

Therefore ;

- Choosing appropriate values for the circuit elements and their connections to other elements

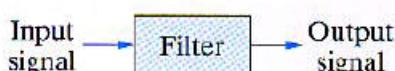
↳ allows us to construct circuits that pass to the output only those input signals that reside in a desired range of frequencies.

- Such circuits are called “frequency selective circuits”

↳ often used by phones, radios, televisions and satellites.

- Also referred to as filters

↳ due to their ability to filter out certain input signals on the basis of frequency.



- No practical filter can filter out selected frequencies

↳ they can only attenuate any input signals with frequencies outside a particular frequency band.

Some preliminaries

- The transfer function of a circuit provides an easy way

↳ to compute the steady-state response to a sinusoidal input.

- To study the frequency response of a circuit,

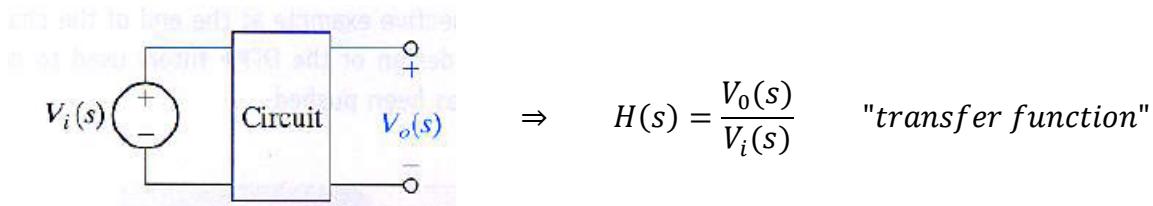
↳ we replace a fixed-frequency sinusoidal source with a varying-frequency sinusoidal source.

- The transfer function is still useful

↳ because the magnitude and phase of the output signal depend only on the magnitude and phase of the transfer function $H(j\omega)$

- Consider the following circuit in which

↳ both the input and output signals are sinusoidal voltages.



Some terminology

Passband \Rightarrow a band of frequencies such that the signals passed from the input to the output fall within.

Stopband \Rightarrow the frequencies which are NOT in a circuit's passband.

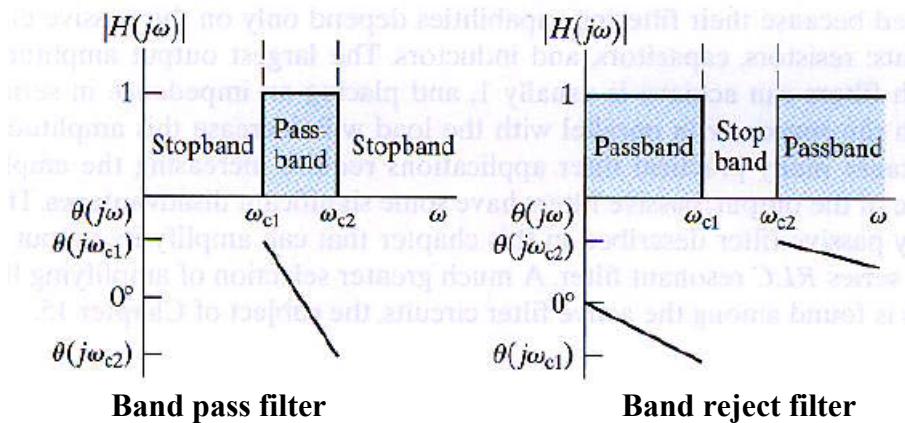
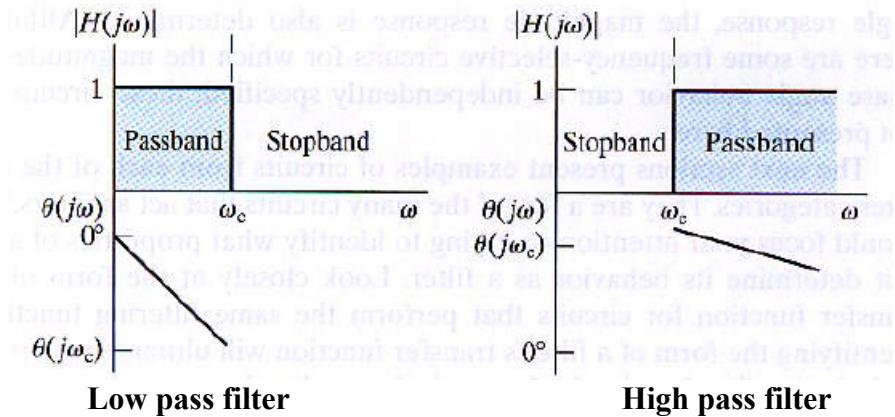
Frequency response plot \Rightarrow shows how a circuit's transfer function changes wrt source frequency.

Magnitude plot, $|H(j\omega)|$

phase angle plot, $\theta(j\omega)$

Major categories of filters

- | | |
|--------------|----------------|
| 1. Low pass | 3. Band pass |
| 2. High pass | 4. Band reject |
- The ideal plots of these filters are shown as :



cutoff frequency

- The frequency that separates the passband and stopband.
- All of these filters we will consider are passive filters
 - ↳ their filtering capabilities depend on the passive elements.
- The largest output amplitude such filters can achieve is usually 1
 - ↳ which is a significant disadvantage of the passive filters.

Because ;

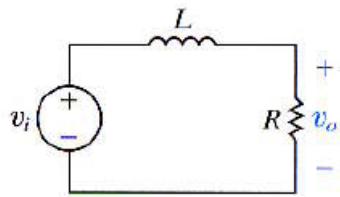
- Many practical filter applications require increasing the amplitude of the output
 - ↳ leading to the use of active filters.

Low pass filters

- We investigate two filters that behave as low-pass filters
 - ↳ series RL and series RC circuit.

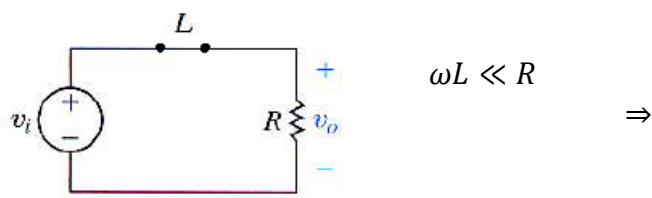
Series RL circuit-qualitative analysis

- We consider



- At low frequencies, the inductor's impedance is very small compared with the resistor's impedance

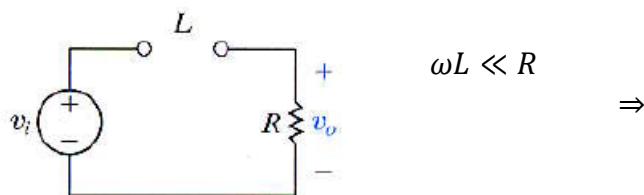
↳ i.e. the inductor effectively functions as a short-circuit.



- The output voltage and input voltage are equal both in magnitude and in phase angle.

- At high frequencies, the inductor's impedance is very large compared with the resistor's impedance

↳ i.e. the inductor functions as an open-circuit.

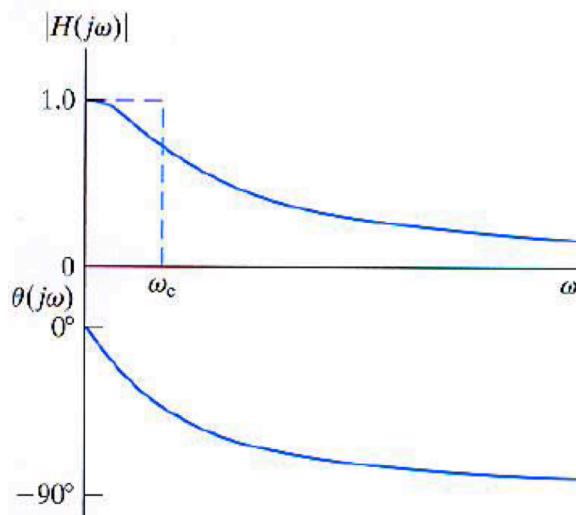


- The phase angle of the output voltage is 90° more negative than that of the input voltage.

Hence ;

- This series RL circuit selectively passes low-frequency inputs to the output

↳ and blocks high-frequency inputs from reaching the output.



Note that ;

- Circuits acting as low-pass filters have a magnitude response

 that changes gradually from the passband to the stopband.

Defining the cutoff frequency

- The frequency for which the transfer function magnitude is decreased by the factor $1/\sqrt{2}$ from its maximum value :

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{max}$$

why “ $1/\sqrt{2}$ ” ?

- Define P_{max} as the value of the average power

 delivered to a load when its voltage is maximum

$$P_{max} = \frac{1}{2} \frac{V_{Lmax}^2}{R}$$

- If we vary the frequency of the sinusoidal voltage source, $V_i(j\omega)$

 then $V_{Lmax} = H_{max}|V_i|$

- Now when $\omega = \omega_c$ we have

$$\begin{aligned} |V_L(j\omega_c)| &= |H(j\omega_c)||V_i| \\ &= \frac{1}{\sqrt{2}} H_{max} |V_i| \\ &= \frac{1}{\sqrt{2}} V_{Lmax} \end{aligned}$$

$$\Rightarrow P(j\omega_c) = \frac{1}{2} \frac{|V_L(j\omega_c)|^2}{R}$$

$$= \frac{1}{2} \frac{\left| \frac{1}{\sqrt{2}} V_{Lmax} \right|^2}{R} = \frac{P_{max}}{2}$$

Hence ;

- At the cutoff frequency ω_c , the average power delivered by the circuit is

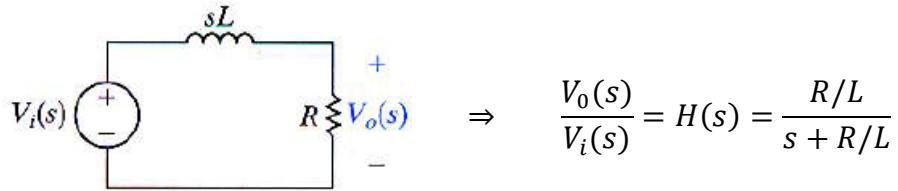
 one half the maximum average power.

- ω_c is also called the half-power frequency

 in the passband, the average power delivered to a load is at least 50% of the maximum average power.

Quantitative analysis

- Consider the s-domain equivalent of the series RL circuit



- Let $s = j\omega$ for the frequency response

$$H(j\omega) = \frac{R/L}{j\omega + R/L}$$

$$\Rightarrow |H(j\omega)| = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}}, \quad \theta(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$\omega \rightarrow \infty \Rightarrow |H(j\omega)|$ decreases, $\theta(j\omega)$ becomes more negative.

$\omega \rightarrow 0 \Rightarrow |H(j\omega)| = 0, \theta(j\infty) = -90^\circ$

Moreover ;

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{max}$$

$$= \frac{1}{\sqrt{2}} \cdot 1 = \frac{R/L}{\sqrt{\omega_c^2 + (R/L)^2}} \Rightarrow \omega_c^2 + \left(\frac{R}{L}\right)^2 = 2 \left(\frac{R}{L}\right)^2 \Rightarrow \omega_c = R/L$$

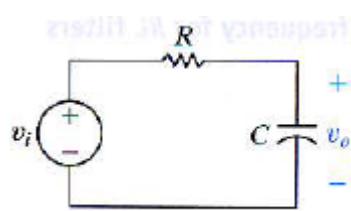
Therefore ;

- We can design a low pass filter

→ with whatever cutoff frequency is needed.

A series RC circuit-qualitative analysis

- The following series RC circuit also behaves as a low-pass filter



- We consider three frequency regions
1. Zero frequency, $\omega = 0$: the impedance of the capacitor $\rightarrow \infty$ acting as an open-circuit

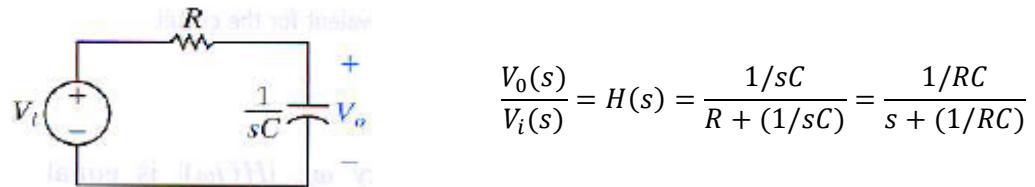
↳ the input and output voltages are same.
 2. Increasing frequency, $\omega \uparrow \infty$: the impedance of the capacitor decreases relative to the resistor

↳ the output voltage gets smaller than the source voltage.
 3. Infinite frequency, $\omega = \infty$: the impedance of the capacitor is zero acting as a short circuit

↳ the output voltage is thus zero.

Quantitative analysis

- We consider the s-domain equivalent of the series RC circuit



- and substituting $s = j\omega$ gives

$$\begin{aligned}
 |H(j\omega_c)| &= \frac{1}{\sqrt{2}} |H(j0)| \\
 &\quad \text{1} \\
 &= \frac{1/RC}{\sqrt{\omega_c^2 + (1/RC)^2}} \\
 &\Rightarrow \omega_c^2 + \left(\frac{1}{RC}\right)^2 = 2\left(\frac{1}{RC}\right)^2 \\
 &\Rightarrow \omega_c = \frac{1}{RC}
 \end{aligned}$$

General form of transfer function of low pass filters

- We shall write for both RL and RC low pass filters

$$H(s) = \frac{\omega}{s + \omega_c}$$

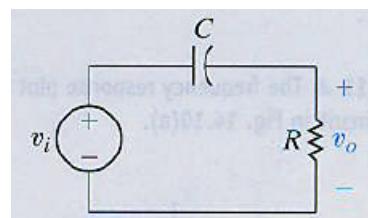
- Any circuit with $H(s)$ behaves as a low pass filter having cut off frequency, ω_c

High pass filters

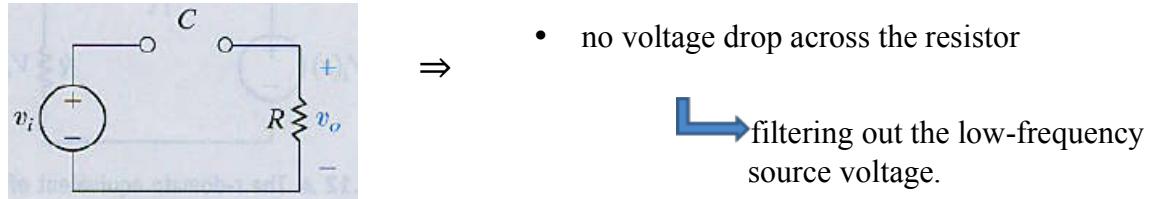
- We consider series RL and RC circuits
 - ↳ that function as high pass filters.
- The same circuit can act as either a low-pass or a high-pass filter
 - ↳ depending on where the output voltage is defined.

The series RC circuit-qualitative analysis

- We consider



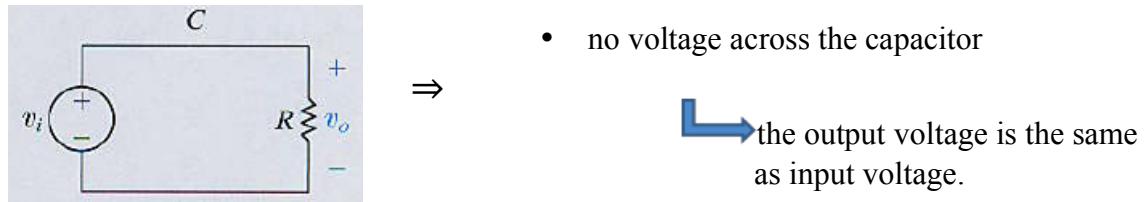
1. At $\omega = 0$: the capacitor behaves like an open-circuit.



2. Increasing frequency, $\omega \uparrow$: the impedance of the capacitor decreases relative to the resistor

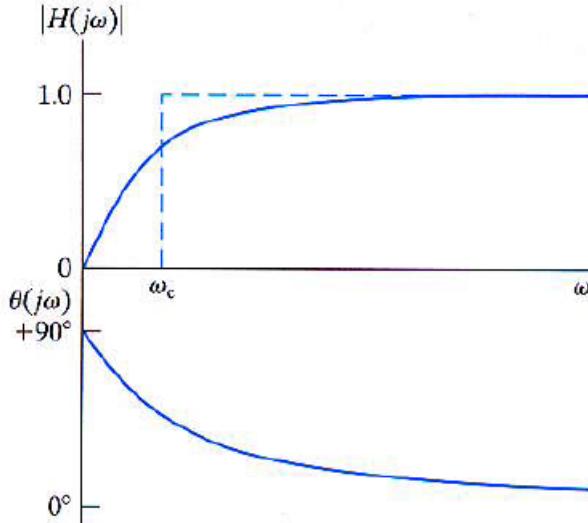
↳ the output voltage starts to increase.

3. When $\omega = \infty$: the capacitor behaves like a short-circuit



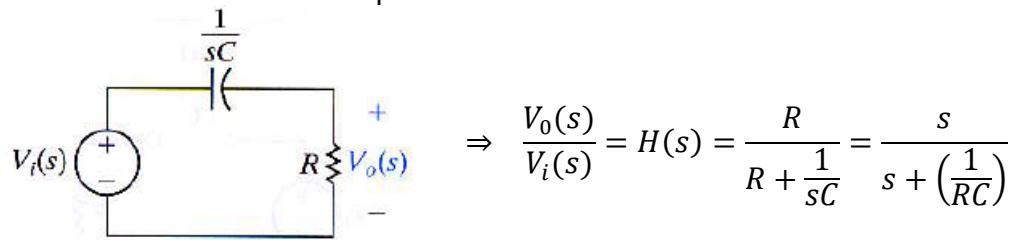
Phase angle difference

- For $\omega = \infty$, phase angle difference is zero.
- As ω decreases, the phase angle difference between the source and output voltages increases.
- When $\omega = 0$, this phase angle difference reaches its maximum of $+90^\circ$



The series RC circuit-quantitative analysis

- The series the s-domain equivalent of series RC circuit



- and letting $s = j\omega$ gives

$$H(j\omega) = \frac{j\omega}{j\omega + (1/RC)}$$

$$\Rightarrow |H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (1/RC)^2}} , \quad \theta(j\omega) = 90^\circ - \tan^{-1} \omega RC$$

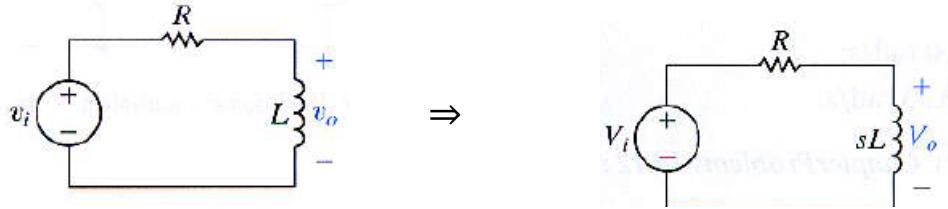
$$H_{max} = |H(j\infty)| = 1$$

$$\Rightarrow |H(j\omega_c)| = \frac{1}{\sqrt{2}} \cdot 1$$

$$= \frac{\omega_c}{\sqrt{\omega_c^2 + (1/RC)^2}} \quad \Rightarrow 2\omega_c^2 = \omega_c^2 + (1/RC)^2 \\ \Rightarrow \omega_c = 1/RC$$

A series RL high-pass filter

- We consider



$$\frac{V_0(s)}{V_i(s)} = H(s) = \frac{sL}{R + sL} = \frac{s}{s + (R/L)}$$

$$\Rightarrow H(j\omega) = \frac{j\omega}{j\omega + (R/L)} \Rightarrow |H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}, \quad \theta(j\omega) = 90^\circ - \tan^{-1} \frac{wL}{R}$$

$$H_{max} = |H(j\infty)| = 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cdot 1 = \frac{\omega_c}{\sqrt{\omega_c^2 + (R/L)^2}} \Rightarrow \omega_c = \frac{R}{L}$$

$|H(j\omega_c)|$

↳ same as the cutoff frequency for the series RL low-pass filter.

General form of a high-pass filter transfer function

- We shall represent $H(s)$ as

$$H(s) = \frac{s}{s + \omega_c}$$

- Any circuit with the above $H(s)$ acts like a high-pass filter

↳ having the cut-off frequency, ω_c

Bandpass filters

- Pass voltages within a band of frequencies to the output

↳ while filtering out voltages at frequencies outside this band.

- Ideal bandpass filters have two cutoff frequencies, ω_{c1} and ω_{c2}

↳ which identify the passband.

Center frequency, ω_0

- Defined as the frequency for which a circuit's transfer function is purely real

↳ so called as “resonant frequency”.

- Calculated as
- $$\omega_0 = \sqrt{\omega_{c_1} \omega_{c_2}}$$
- The magnitude of the transfer function is maximum at the center frequency

$$H_{max} = |H(j\omega_0)|$$

Bandwidth, β

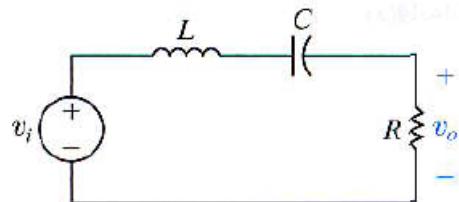
- Described as the width of the passband.

Quality factor

- Defined as the ratio of the center frequency to the bandwidth.

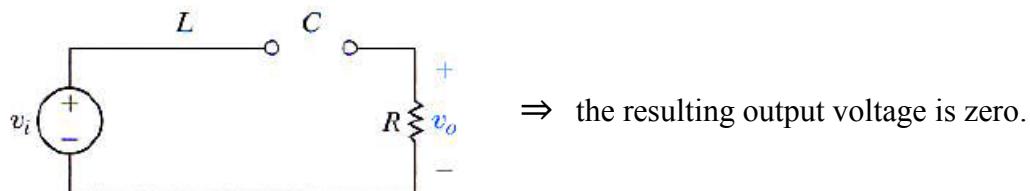
The series RLC circuit-qualitative analysis

- Depicted as



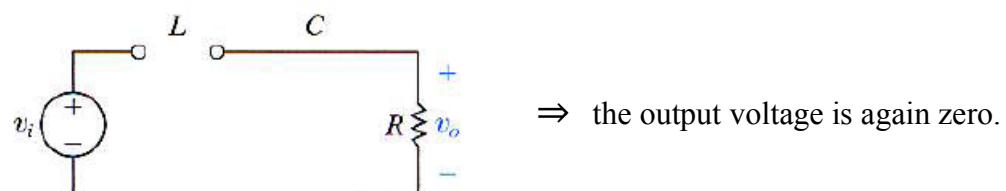
- At $\omega = 0$, the capacitor behaves like an open circuit

→ and the inductor behaves like a short circuit.



- At $\omega = \infty$, the capacitor behaves like a short circuit,

→ and the inductor behaves like an open circuit.



- $0 < \omega < \infty$, both the capacitor and inductor have finite impedances

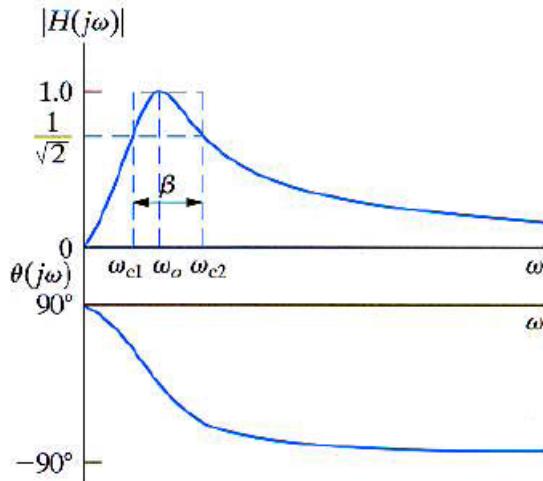
→ some voltage will be reach the resistor.

- At some frequency, the impedance of the capacitor and that of inductor cancel out each other

↳ the output voltage becomes equal to the source voltage.

Hence ;

- We have



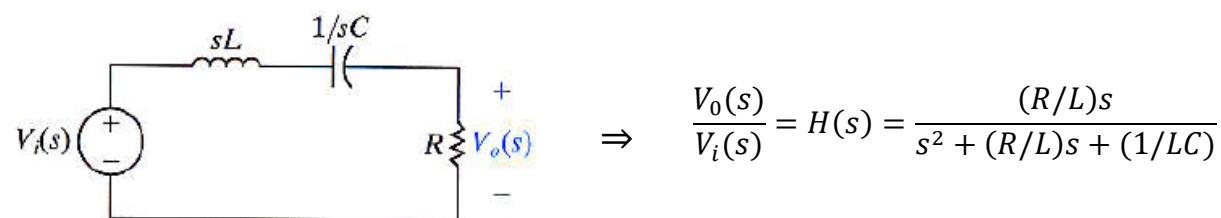
Phase angle difference

- At $\omega = \omega_0$, phase angle difference is zero.
 - As the frequency decreases, the phase angle contribution from the capacitor is larger than that of inductor
- ↳ the net phase angle is positive approaching to $+90^\circ$
- If the frequency increases from ω_0 , then the phase angle contribution of the inductor dominates that of capacitor

↳ the net phase angle is negative reaching to -90°

Quantitative analysis

- We first draw the s-domain equivalent



- Substituting $s = j\omega$ gives

$$|H(j\omega)| = \frac{\omega(R/L)}{\sqrt{[(1/LC) - \omega^2]^2 + [\omega(R/L)]^2}}$$

$$\theta(j\omega) = 90^\circ - \tan^{-1} \left[\frac{\omega(R/L)}{(1/LC) - \omega^2} \right]$$

Calculating the filter characteristics

- For the center frequency, ω_0 , we have

$$j\omega_0 L + \frac{1}{j\omega_0 L} = 0$$

$$\Rightarrow \omega_0 = \sqrt{\frac{1}{LC}}$$

- Then for the cutoff frequencies

$$H_{max} = |H(j\omega_0)|$$

$$= \frac{\omega_0(R/L)}{\sqrt{[(1/LC) - \omega_0^2] + [\omega_0(R/L)]^2}}$$

$$= 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{\omega_c(R/L)}{\sqrt{[(1/LC) - \omega_c^2]^2 + (\omega_c R/L)^2}}$$

$$= \frac{1}{\sqrt{[(\omega_c L/R) - (1/\omega_c RC)]^2 + 1}}$$

$$\Rightarrow \pm 1 = \omega_c \frac{L}{R} - \frac{1}{\omega_c RC}$$

$$\Rightarrow \omega_c^2 L \pm \omega_c R - 1/C = 0 \quad \begin{cases} \omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \\ \omega_{c_2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \end{cases}$$

- The bandwidth is calculated as

$$\beta = \omega_{c_2} - \omega_{c_1}$$

$$= \frac{R}{L}$$

- and finally the quality factor is computed as

$$Q = \omega_0/\beta$$

$$= \frac{\sqrt{1/LC}}{R/L}$$

$$= \sqrt{\frac{L}{CR^2}}$$

Alternative forms of these characteristics

$$\omega_{c_1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

$$\omega_{c_2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$$

OR

$$\omega_{c_1} = \omega_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

$$\omega_{c_2} = \omega_0 \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

Note that ;

- only two of the parameters (β, ω_0, Q) are sufficient

 to design the filter independently.

General form of a bandpass filter transfer function

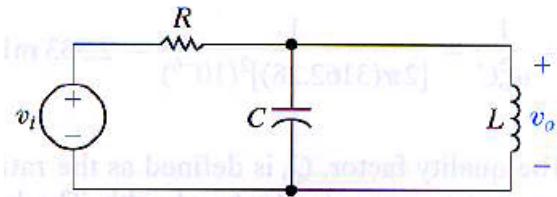
- Can be represented as

$$H(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

- Thus any circuit with the transfer function $H(s)$

 acts as a bandpass filter with a center frequency, ω_0 and a bandwidth, β

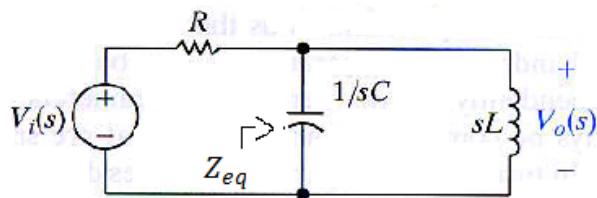
Example. Consider the following RLC circuit



- Show that this circuit is also a bandpass filter by deriving an expression for $H(s)$.
- Compute the center frequency, ω_0 .
- Calculate the cutoff frequencies, ω_{c1} and ω_{c2} , the bandwidth, β and the quality factor, Q .
- Compute the values for R and L to yield a bandpass filter with a center frequency of 5kHz and a bandwidth of 200Hz using a $5\mu F$ capacitor.

Solution.

- We first draw the s-domain equivalent



$$Z_{eq} = \frac{(1/sC)sL}{sL + \frac{1}{sC}} = \frac{L/C}{sL + \frac{1}{sC}}$$

- and

a.

$$\begin{aligned} \frac{V_0(s)}{V_i(s)} &= \frac{\frac{L/C}{sL + (1/sC)}}{R + \frac{L/C}{sL + (1/sC)}} = \frac{L/C}{sRL + \frac{R}{sC} + \frac{L}{C}} \\ &= \frac{sL}{s^2RLC + R + sL} = \frac{s/RC}{s^2 + \frac{R}{LC} + \frac{1}{LC}} \end{aligned}$$

- b.** We need to calculate where the transfer function magnitude is maximum

$$|H(j\omega)| = \frac{\frac{\omega}{RC}}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{\omega}{RC}\right)^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\omega RC - \frac{1}{\omega \frac{L}{R}} \right)^2}}$$

- and

$$H_{max} = |H(j\omega_0)| = 1$$

c. At the cutoff frequencies, $|H(j\omega_c)| = \frac{1}{\sqrt{2}}$

$$\Rightarrow \left(\omega_c R C - \frac{1}{\omega_c \frac{L}{R}} \right)^2 = 1$$

$$\Rightarrow \omega_c RC - \frac{1}{\omega_c \frac{L}{R}} = \pm 1 \quad \begin{cases} \omega_{c_1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \\ \omega_{c_2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \end{cases}$$

d. We have

$$\beta = \omega_{c_1} - \omega_{c_2} = \frac{1}{RC} \quad \Rightarrow \quad R = \frac{1}{\beta C} = \frac{1}{(2\pi)(200)5.10^{-6}} = 159.15 \Omega$$

$$L = \frac{1}{\omega_0^2 C} = \frac{1}{[(2\pi).5000]^2.5.10^{-6}} = 202.64 \mu H$$

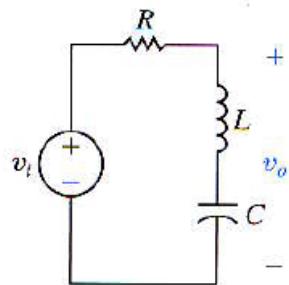
Bandreject filters

- The filter passes source voltages outside the band between the two cutoff frequencies to the output.
 - Characterized by the same five parameters as bandpass filters

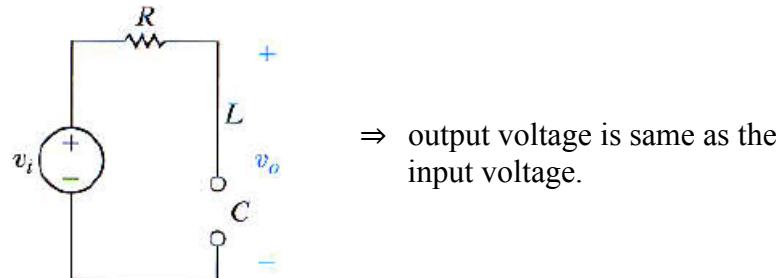
only two of these five parameters can be specified independently.

The series RLC circuit-qualitative analysis

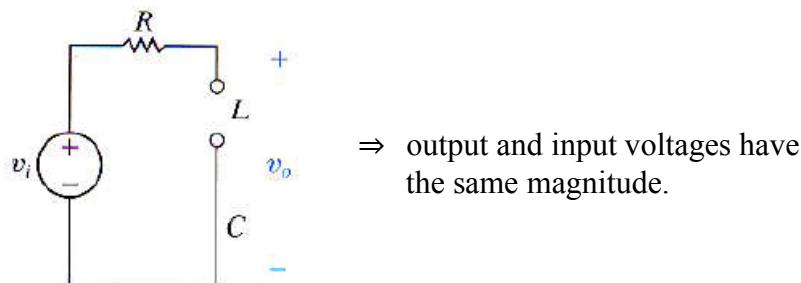
- We consider the following circuit



- At $\omega = 0$, the inductor behaves like a short circuit and a capacitor acts as an open circuit.



- At $\omega = \infty$, the roles of inductor and capacitor switch



- This series RLC circuit has two passbands

↳ one below a lower cutoff frequency and the other above an upper cutoff frequency.

phase angle difference

- As the frequency is increased from zero

↳ the impedance of the inductor increases and that of the capacitor decreases.

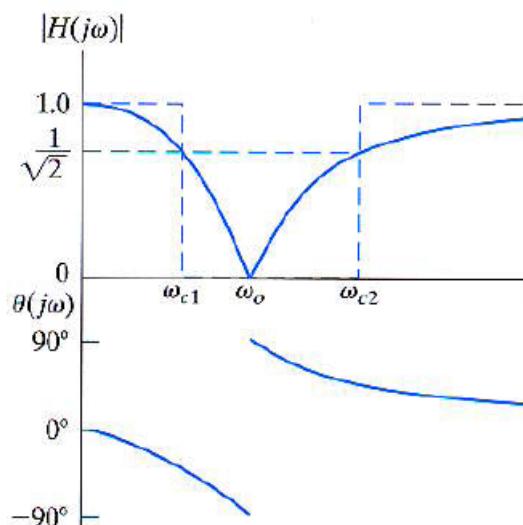
- The phase shift between input and output approaches -90°

↳ as ωL approaches $1/\omega C$

- As ωL exceeds $1/\omega C$, the phase shift jumps to $+90^\circ$

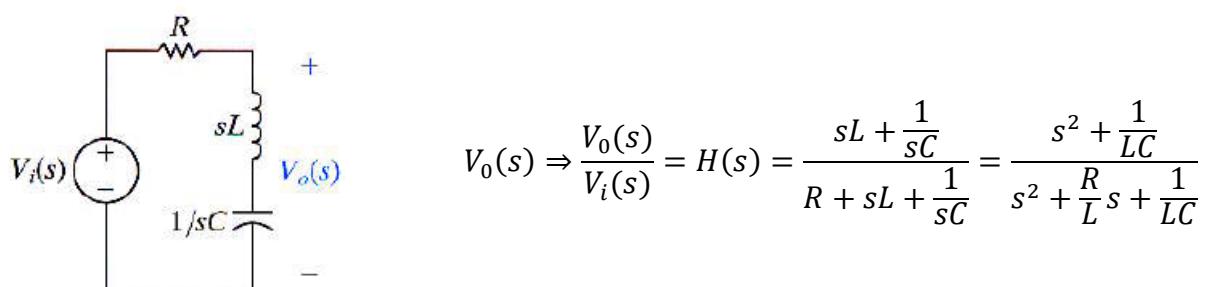
↳ then approaches zero as ω continues to increase.
- At some frequency the impedances of inductor and capacitor are equal but of opposite sign.

↳ the output voltage is just zero at this center frequency.



Quantitative analysis

- We take into account the s-domain equivalent of the series RLC circuit



- Substituting $s = j\omega$ gives

$$|H(j\omega)| = \frac{\left| \frac{1}{LC} - \omega^2 \right|}{\sqrt{\left(\frac{1}{LC} - \omega^2 \right)^2 + \left(\frac{\omega R}{L} \right)^2}}$$

$$\theta(j\omega) = -\tan^{-1} \left(\frac{\frac{\omega R}{L}}{\frac{1}{LC} - \omega^2} \right)$$

- For the center frequency, we have

$$j\omega_0L + \frac{1}{j\omega_0C} = 0 \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

- At the center frequency, $|H(j\omega_0)| = 0$

- For the cutoff frequencies, we have

$$\frac{1}{\sqrt{2}} = \frac{\left| \frac{1}{LC} - \omega_c^2 \right|}{\sqrt{\left(\frac{1}{LC} - \omega_c^2 \right)^2 + \left(\frac{\omega_c R}{L} \right)^2}} \quad \begin{array}{c} \swarrow \\ \omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 + \frac{1}{LC}} \end{array} \quad \begin{array}{c} \searrow \\ \omega_{c_2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L} \right)^2 + \frac{1}{LC}} \end{array}$$

- and the bandwidth is

$$\beta = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}$$

- and the quality factor is

$$Q = \sqrt{\frac{L}{R^2 C}}$$

- As an alternative representation we obtain

$$\omega_{c_1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2} \right)^2 + \omega_0^2}$$

$$\omega_{c_2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2} \right)^2 + \omega_0^2}$$

OR

$$\omega_{c_1} = \omega_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

$$\omega_{c_2} = \omega_0 \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q} \right)^2} \right]$$

Example. Using the series RLC circuit, compute the component values that yield a bandreject filter with a bandwidth of 250 Hz and a center frequency of 750 Hz. Use a 100 nF capacitor. Compute values for R , L , ω_{c_1} , ω_{c_2} and Q .

Solution. We calculate

$$Q = \frac{\omega_0}{\beta} = \frac{750}{250} = 3$$

$$\Rightarrow L = \frac{1}{\omega_0^2 C} = \frac{1}{(2\pi \cdot 750)^2 \cdot 100 \cdot 10^{-9}} = 450 \text{ mH}$$

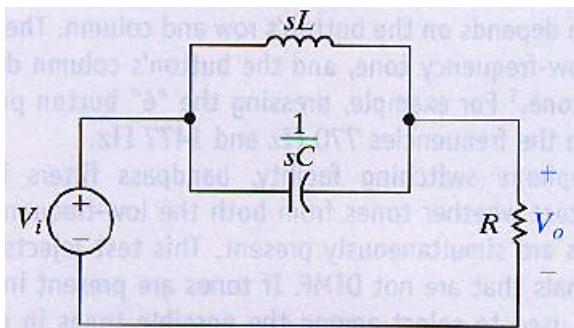
$$\Rightarrow R = \beta L = 2\pi \cdot 250 \cdot (450 \cdot 10^{-3}) = 707 \Omega$$

$$\omega_{c_1} = -\frac{2\pi \cdot 250}{2} + \sqrt{\left(\frac{2\pi \cdot 250}{2}\right)^2 + (2\pi \cdot 750)^2} = 3992.0 \text{ rad/s} \rightarrow 653.3 \text{ Hz}$$

$$\omega_{c_2} = \frac{2\pi \cdot 250}{2} + \sqrt{\left(\frac{2\pi \cdot 250}{2}\right)^2 + (2\pi \cdot 750)^2} = 5562.8 \text{ rad/s} \rightarrow 885.3 \text{ Hz}$$

General form of a band reject filter transfer function

- A parallel RLC circuit also acts as a bandreject filter



$$H(s) = \frac{s^2 + 1/LC}{s^2 + s/RC + 1/LC} , \quad \omega_0 = \sqrt{1/LC} , \quad \beta = 1/RC$$

- We can state a general form of the transfer function of bandreject filters

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

Hence ;

- any circuit with the above transfer function

→ can be used as a bandreject filter.