Fourier Series

- Sinusoidal steady-state analysis is important

  because it leads to find the steady-state response to nonsinusoidal but periodic excitations.

- A periodic function is a function

  repeating itself every T seconds.

That is ;

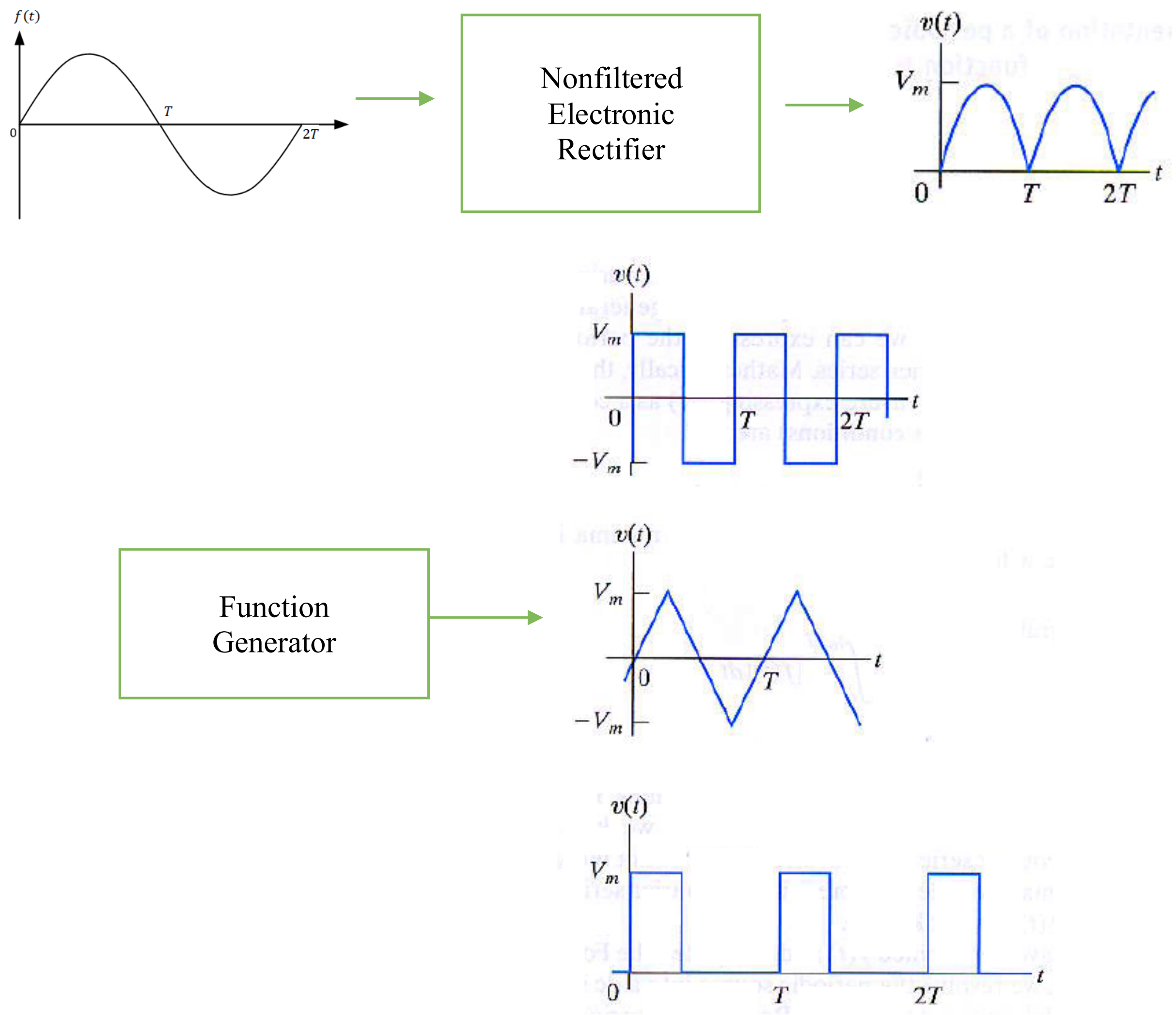
$$f(t) = f(t \pm nT)$$

where $n \in \mathbb{Z}$ and T represents the fundamental period.

Why interest in periodic functions ?

- Many electrical sources of practical value

  generate periodic waveforms.



- Power generators can produce only distorted sinusoidal wave

  but it is periodic.

- Any nonlinearity in a linear circuit generates a nonsinusoidal periodic function.

Nonsinusoidal periodic functions

- Important especially in the analysis of

  nonelectrical systems.

e.g. Mechanical vibration, fluid flow and heat flow..

Jean Baptiste Joseph Fourier (1768-1830)

- The study of heat flow in a metal rod

  led the French mathematician to the trigonometric series representation of a periodic function.

Fourier Series Analysis

- A periodic function can be represented

  by an infinite sum of sine or cosine functions that are harmonically related.

That is ;

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty}[a_n\, cosn\omega_0 t + b_n\, sinn\omega_0 t]$$

where

a_ϑ, a_n, b_n : Fourier coefficients
ω_0 : fundamental frequency of $f(t)$
$n\omega_0$: harmonic frequencies of $f(t)$

Existence of a convergent Fourier Series

- Mathematically known as "Dirichlet's conditions".

Namely ;

1. $f(t)$ must be single-valued.
2. $f(t)$ must have a finite number of discontinuities in the periodic interval.
3. $f(t)$ must have a finite number of maxima and minima in the periodic interval.

4. The integral

$$\int_{t_0}^{t_0+T} |f(t)|dt$$

exists.

- Any periodic function generated by a physically realizable source

  satisfies Dirichlet's conditions.

Note that ;

- These are sufficient conditions

  but NOT necessary.

- The necessary conditions on $f(t)$ are NOT known.

What is the purpose ?

- We resolve the periodic source into :

  a dc source (a_ϑ)

  PLUS

  a sum of sinusoidal sources (a_n, b_n)

- Since the periodic source is driving a linear circuit

  we may use the principle of superposition to find the steady-state response.

How to find the steady-state response ?

- We first calculate the response to each source

  generated by the Fourier series representation of $f(t)$.

- Then we add the individual responses

  to obtain the total response.

The Fourier coefficients

- We determine the Fourier coefficients as

$$a_\vartheta = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt \qquad \text{“average value of } f(t)\text{”}$$

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos k\omega_0 t\, dt$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin k\omega_0 t\, dt$$

where $k = 1, 2, 3, \ldots$

How to justify these relations ?

- Indeed, we have

$$\int_{t_0}^{t_0+T} f(t)\, dt = \int_{t_0}^{t_0+T}\left(a_\vartheta + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)dt$$

$$= \int_{t_0}^{t_0+T} a_\vartheta\, dt + \sum_{n=1}^{\infty}\underbrace{\int_{t_0}^{t_0+T}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)dt}_{\text{integrating a periodic sinusoidal function over one period is JUST ZERO.}}$$

$$= a_\vartheta T + 0$$

$$\Rightarrow \quad a_\vartheta = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, dt$$

- In a similar manner, we consider

$$\int_{t_0}^{t_0+T} f(t)\cos k\omega_0 t\, dt = \int_{t_0}^{t_0+T} a_\vartheta \cos k\omega_0 t\, dt$$
$$+ \int_{t_0}^{t_0+T}(a_n \cos n\omega_0 t \cos k\omega_0 t + b_n \sin n\omega_0 t \cos k\omega_0 t)\, dt$$

$$= 0 + \frac{1}{2}\sum_{n=1}^{\infty}\int_{t_0}^{t_0+T} a_n[\underbrace{\cos(n-k)\omega_0 t}_{1,\text{ for } n=k} + \underbrace{\cos(n+k)\omega_0 t]}_{0,\forall n,k}\, dt$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\underbrace{\int_{t_0}^{t_0+T} b_n[\sin(n+k)\omega_0 t + \sin(n-k)\omega_0 t]\, dt}_{0,\forall n,k}$$

$$= 0 + \frac{1}{2}\int_{t_0}^{t_0+T} a_n\, dt + 0 = \frac{1}{2}Ta_n$$

$$\Rightarrow \quad a_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos k\omega_0 t\, dt$$

- Finally for b_k we have

$$\int_{t_0}^{t_0+T} f(t)\sin k\omega_0 t\, dt = \int_{t_0}^{t_0+T} \underbrace{a_\vartheta \sin k\omega_0 t\, dt}_{0} + \int_{t_0}^{t_0+T} a_n \cos n\omega_0 t \sin k\omega_0 t\, dt$$

$$+ \int_{t_0}^{t_0+T} b_n \sin n\omega_0 t \sin k\omega_0 t\, dt$$

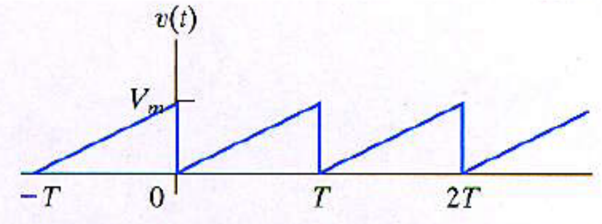
$$= 0 + \frac{1}{2}\underbrace{\int_{t_0}^{t_0+T} a_n[\sin(n+k)\omega_0 t - \sin(n-k)\omega_0 t]dt}_{0,\forall n,k}$$

$$+\frac{1}{2}\underbrace{\int_{t_0}^{t_0+T} b_n[\cos(n-k)\omega_0 t - \cos(n+k)\omega_0 t]dt}_{\substack{0,\forall n\neq k\\ 1,\text{ for } n=k}}$$

$$= 0 + 0 + \frac{1}{2}\int_{t_0}^{t_0+T} b_k dt = \frac{1}{2}Tb_k$$

$$\Rightarrow \quad b_k = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin k\omega_0 t\,dt$$

Ex. Find the Fourier series for the periodic voltage shown as



Solution. We may choose $t_0 = 0$ to find a_ϑ, a_k, b_k

$$\vartheta(t) = \left(\frac{V_m}{T}\right)t \qquad for \qquad 0 \le t \le T$$

- Then

$$a_\vartheta = \frac{1}{T}\int_0^T \left(\frac{V_m}{T}\right)t\,dt$$

$$= \frac{1}{T}\frac{V_m}{T}\int_0^T t\,dt$$

$$= \frac{V_m}{T^2}\frac{t^2}{2}\Bigg|_0^T = \frac{V_m}{T^2}\frac{T^2}{2} = \frac{V_m}{2}$$

- The equation for the k^{th} value of a_n is

$$a_k = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}\right)t\cos k\omega_0 t\,dt$$

$$= \frac{2V_m}{T^2}\left(t\frac{\sin k\omega_0 t}{k\omega_0}\Bigg|_0^T - \int_0^T \frac{\sin k\omega_0 t}{k\omega_0}dt\right)$$

$$= \frac{2V_m}{T^2}\left[T\frac{\sin k\omega_0 T}{k\omega_0} + \frac{\cos k\omega_0 t}{(k\omega_0)^2}\Bigg|_0^T\right]$$

$$= \frac{2V_m}{T^2}\left[T\frac{\sin k\omega_0 T}{k\omega_0} + \frac{\cos k\omega_0 T}{(k\omega_0)^2} - \frac{1}{(k\omega_0)^2}\right]$$

$$= \frac{2V_m}{T^2}\left[T\frac{\cancelto{0}{\sin 2\pi k}}{k\omega_0} + \frac{\cos 2\pi k}{(k\omega_0)^2} - \frac{1}{(k\omega_0)^2}\right]$$

$$= \frac{2V_m}{T^2}\frac{1}{(k\omega_0)^2}(\cancelto{1}{\cos 2\pi k} - 1)$$

$$= 0 \quad \text{for all } k$$

- Similarly we calculate b_k as follows ;

$$b_k = \frac{2}{T}\int_0^T \left(\frac{V_m}{T}\right) t \sin k\omega_0 t \, dt$$

$$= \frac{2V_m}{T^2}\left[-t\frac{\cos k\omega_0 t}{k\omega_0}\bigg|_0^T + \int_0^T \left(-\frac{\cos k\omega_0 t}{k\omega_0}\right) dt\right]$$

$$= \frac{2V_m}{T^2}\left[-T\frac{\cos 2\pi k}{k\omega_0} + \frac{\sin k\omega_0 t}{(k\omega_0)^2}\bigg|_0^T\right]$$

$$= \frac{2V_m}{T^2}\left[-\frac{T}{k\omega_0} + \frac{\cancelto{0}{\sin 2\pi k}}{(k\omega_0)^2}\right]$$

$$= -\frac{2V_m}{2\pi k}$$

$$= -\frac{V_m}{\pi k}$$

- The Fourier series for $\vartheta(t)$ is

$$\vartheta(t) = \frac{V_m}{2} - \frac{V_m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\omega_0 t$$

$$= \frac{V_m}{2} - \frac{V_m}{\pi}\sin \omega_0 t - \frac{V_m}{2\pi}\sin 2\omega_0 t - \cdots$$

The effect of symmetry on the Fourier coefficients

- We shall employ 4 types of symmetry

  to evaluate the Fourier coefficients in a Simpler manner.

a. even-function symmetry
b. odd-function symmetry
c. half-wave symmetry
d. quarter-wave symmetry

Even-function symmetry

- A function is said to be even if

$$f(t) = f(-t)$$

- For even periodic functions, the Fourier coefficients reduce to

$$a_\vartheta = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos k\omega_0 t$$

$$b_k = 0 \quad , \quad \forall k$$

justification

Indeed ;

$$a_\vartheta = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_0^{T/2} f(t)dt\right]$$

$$= \frac{1}{T}\left[\int_{T/2}^{0} f(-t)d(-t) + \int_0^{T/2} f(t)dt\right]$$

$$= \frac{1}{T}\left[-\int_0^{T/2} f(t)(-dt) + \int_0^{T/2} f(t)dt\right]$$

$$= \frac{2}{T}\int_0^{T/2} f(t)dt$$

- and for a_k we have

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{T} \left[\int_{-T/2}^{0} f(t) \cos k\omega_0 t \, dt + \int_{0}^{T/2} f(t) \cos k\omega_0 t \right]$$

$$= \frac{2}{T} \left[\int_{T/2}^{0} f(-t) \cos(-k\omega_0 t) \, d(-t) + \int_{0}^{T/2} f(t) \cos k\omega_0 t \right]$$

$$= \frac{2}{T} \left[- \int_{0}^{T/2} f(t) \cos k\omega_0 t \left(-d(t)\right) + \int_{0}^{T/2} f(t) \cos k\omega_0 t \right]$$

$$= \frac{4}{T} \int_{0}^{T/2} f(t) \cos k\omega_0 t$$

- Finally for b_k, we compute

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin k\omega_0 t \, dt$$

$$= \frac{2}{T} \left[\int_{-T/2}^{0} f(t) \sin k\omega_0 t \, dt + \int_{0}^{T/2} f(t) \sin k\omega_0 t \, dt \right]$$

$$= \frac{2}{T} \left[\int_{T/2}^{0} f(-t) \sin(-k\omega_0 t) \, d(-t) + \int_{0}^{T/2} f(t) \sin k\omega_0 t \, dt \right]$$

$$= \frac{2}{T} \left[- \int_{0}^{T/2} f(t) \sin k\omega_0 t \, dt + \int_{0}^{T/2} f(t) \sin k\omega_0 t \, dt \right]$$

$$= 0 \quad \text{for all k.}$$

Odd-function symmetry

- A function is said to be odd if

$$f(t) = -f(-t)$$

- The expressions of the Fourier coefficients reduce to

$$a_\vartheta = 0$$

$a_k = 0$ for all k.

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\omega_0 t\,dt$$

justification

Indeed ;

$$a_\vartheta = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} f(t)\,dt + \int_0^{T/2} f(t)dt\right]$$

$$= \frac{1}{T}\left[\int_{T/2}^{0} f(-t)d(-t) + \int_0^{T/2} f(t)dt\right]$$

$$= \frac{1}{T}\left[-\int_0^{T/2} -f(t)\big(-d(t)\big) + \int_0^{T/2} f(t)dt\right]$$

$$= 0$$

- We calculate a_k as

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-T/2}^{0} f(t)\cos k\omega_0 t\,dt + \int_0^{T/2} f(t)\cos k\omega_0 t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} f(-t)\cos(-k\omega_0 t)\, d(-t) + \int_{0}^{T/2} f(t)\cos k\omega_0 t\, dt\right]$$

$$= \frac{2}{T}\left[-\int_{0}^{T/2} \big(-f(t)\big)\cos k\omega_0 t\,\big(-d(t)\big) + \int_{0}^{T/2} f(t)\cos k\omega_0 t\, dt\right]$$

$= 0$ for all k.

- and then for b_k, we have

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\sin k\omega_0 t\, dt + \int_{0}^{T/2} f(t)\sin k\omega_0 t\, dt\right]$$

$$= \frac{2}{T}\left[\int_{T/2}^{0} f(-t)\sin(-k\omega_0 t)\, d(-t) + \int_{0}^{T/2} f(t)\sin k\omega_0 t\, dt\right]$$

$$= \frac{2}{T}\left[-\int_{0}^{T/2} \big(-f(t)\big)(-\sin k\omega_0 t)\big(-d(t)\big) + \int_{0}^{T/2} f(t)\sin k\omega_0 t\, dt\right]$$

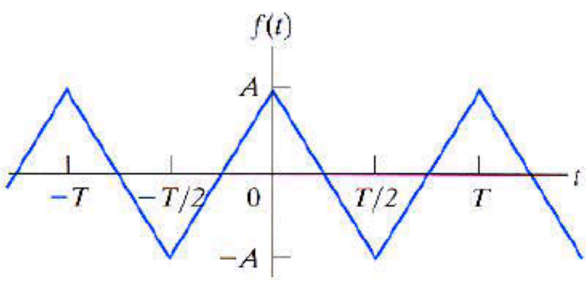
$$= \frac{4}{T}\int_{0}^{T/2} f(t)\sin k\omega_0 t\, dt$$

Half-wave symmetry

- A periodic function has half-wave symmetry if

$$f(t) = -f\left(t - \frac{T}{2}\right)$$

e.g.



- then the Fourier coefficients are given as

$$a_\vartheta = 0$$

$$a_k = \begin{cases} 0 \quad , \quad for\ even\ k \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos k\omega_0 t\, dt\,, \quad for\ odd\ k \end{cases}$$

$$b_k = \begin{cases} 0 \quad , \quad for\ even\ k \\ \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin k\omega_0 t\, dt \quad , \quad for\ odd\ k \end{cases}$$

How to justify then ?

- We have

$$a_\vartheta = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_{0}^{T/2} f(t)dt\right]$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_{-T/2}^{0} f\left(t-\frac{T}{2}\right)d\left(t-\frac{T}{2}\right)\right]$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_{-T/2}^{0} (-f(t))dt\right]$$

$$= 0$$

- In a similar manner, we consider a_k as

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega_0 t\, dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos k\omega_0 t\, dt + \int_{0}^{T/2} f(t)\cos k\omega_0 t\, dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} f\left(t+\frac{T}{2}\right)\cos(k\omega_0 t+\pi k)\,dt + \int_0^{T/2} f(t)\cos k\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} \left(-f(t)\right)\cos(k\omega_0 t+\pi k)\,dt + \int_0^{T/2} f(t)\cos k\omega_0 t\,dt\right]$$

- when k is even, we have

$$\cos(k\omega_0 t+\pi k) = \cos k\omega_0 t$$

- that is ;

$$a_k = \frac{2}{T}\left[-\int_0^{T/2} f(t)\cos k\omega_0 t\,dt + \int_0^{T/2} f(t)\cos k\omega_0 t\,dt\right]$$

$$= 0$$

- and when k is odd, we have

$$\cos(k\omega_0 t+\pi k) = -\cos k\omega_0 t$$

Therefore ;

$$a_k = \frac{2}{T}\left[-\int_0^{T/2} f(t)(-\cos k\omega_0 t)\,dt + \int_0^{T/2} f(t)\cos k\omega_0 t\,dt\right]$$

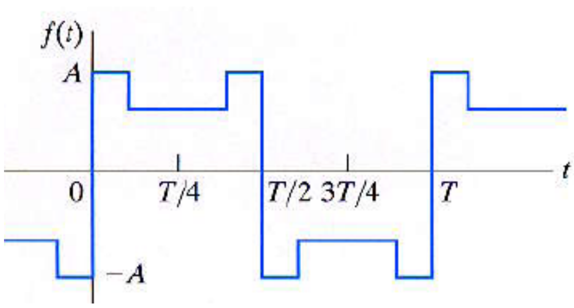
$$= \frac{4}{T}\int_0^{T/2} f(t)\cos k\omega_0 t\,dt$$

Quarter-wave symmetry

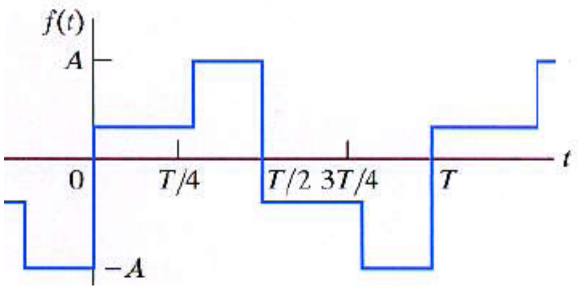
- A periodic function is said to be quarter-wave symmetric if

  a. It is half-wave symmetric
  b. It has symmetry (odd or even) at a quarter period point

e.g.



"A function that has quarter-wave symmetry."



"A function that does not have quarter-wave symmetry."

Note that ;

- A periodic function with quarter-wave symmetry can always be made either even (if it is odd) or odd (if it is even)

  by shifting the function $T/4$ units either right or left.

What happens to Fourier coefficients ?

- It follows from the half-wave symmetry that

$$a_{\vartheta} = 0$$

- If the function is made even, then

$$a_k = \begin{cases} 0 \quad , \quad for\ even\ k\ ; because\ of\ half-wave\ symmetry \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} f(t)\cos k\omega_0 t\,dt \quad , \quad for\ odd\ k \end{cases}$$

$$b_k = 0 \quad , \quad for\ all\ k\ ; because\ the\ function\ is\ even.$$

Moreover ;

- If the function is made odd, then

$$a_k = 0 \quad , \quad for\ all\ k\ ; because\ the\ function\ is\ odd.$$

$$b_k = \begin{cases} 0 \quad , \quad for\ even\ k\ ; because\ of\ half-wave\ symmetry \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} f(t)\sin k\omega_0 t\,dt \quad , \quad for\ odd\ k \end{cases}$$

justifying the equations

- If the function is made even, we have

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega_0 t\,dt$$

$$= \frac{2}{T} \int_{-T/2}^{-T/4} f(t) \cos k\omega_0 t \, dt + \int_{-T/4}^{0} f(t) \cos k\omega_0 t \, dt$$

$$+ \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt + \int_{T/4}^{T/2} f(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{T} \left[\int_{0}^{T/4} f\left(t - \frac{T}{2}\right) \cos(k\omega_0 t - \pi k) \, d\left(t - \frac{T}{2}\right) \right.$$

$$+ \int_{T/4}^{0} f(-t) \cos(-k\omega_0 t) \, d(-t) + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt$$

$$\left. + \int_{-T/4}^{0} f\left(t + \frac{T}{2}\right) \cos(k\omega_0 t + \pi k) \, d\left(t - \frac{T}{2}\right) \right]$$

- For even k, we get

$$a_k = \frac{2}{T} \left[-\int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt \right.$$

$$\left. + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt - \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt \right]$$

$$= 0$$

- and for odd k, we obtain

$$a_k = \left[\frac{2}{T} \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt \right.$$

$$\left. + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt + \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt \right]$$

$$= \frac{8}{T} \int_{0}^{T/4} f(t) \cos k\omega_0 t \, dt$$

- If the function is made odd, we then have

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin k\omega_0 t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{-T/4} f(t)\sin k\omega_0 t\,dt + \int_{-T/4}^{0} f(t)\sin k\omega_0 t\,dt\right.$$

$$\left. + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt + \int_{T/4}^{T/2} f(t)\sin k\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_{0}^{T/4} f\left(t-\frac{T}{2}\right)\sin(k\omega_0 t - \pi k)\,d\left(t-\frac{T}{2}\right)\right.$$

$$+ \int_{T/4}^{0} f(-t)\sin(-k\omega_0 t)\,d(-t) + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt$$

$$+ \int_{-T/4}^{0} f\left(t+\frac{T}{2}\right)\sin(k\omega_0 t + \pi k)\,d\left(t+\frac{T}{2}\right)$$

- For even k, we get

$$b_k = \frac{2}{T}\left[-\int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt\right.$$

$$\left. + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt - \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt\right]$$

$$= 0$$

- and for odd k, we obtain

$$b_k = \frac{2}{T}\left[\int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt\right.$$

$$+\int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt + \int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt\Bigg]$$

$$= \frac{8}{T}\int_{0}^{T/4} f(t)\sin k\omega_0 t\,dt$$

An alternative trigonometric form of the Fourier Series

- In circuit applications, we usually combine the cosine and sine terms

  into a single term for convenience.

- This allows to present $\vartheta(t)$ or $i(t)$ as a single phasor quantity.

- They can be merged in either a cosine or sine expression

  we choose the cosine expression for our analysis.

Therefore ;

- We shall write the Fourier series representation as

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

where A_n and θ_n are defined by

$$a_n - jb_n = \sqrt{a_n^2 + b_n^2}\,\angle - \theta_n = A_n\angle - \theta_n$$

How is it derived ?

- We use the phasor method to add the cosine and sine terms.

- We have ;

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= a_\vartheta + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \cos(n\omega_0 t - 90°)$$

Note that ;

- We shall write

$$\wp\{a_n \cos n\omega_0 t\} = a_n \angle 0°$$

$$\wp\{b_n \cos(n\omega_0 t - 90°)\} = b_n \angle -90°$$

$$= -jb_n$$

- then we obtain

$$\wp\{a_n \cos n\omega_0 t + b_n \cos(n\omega_0 t - 90°)\} = a_n - jb_n$$

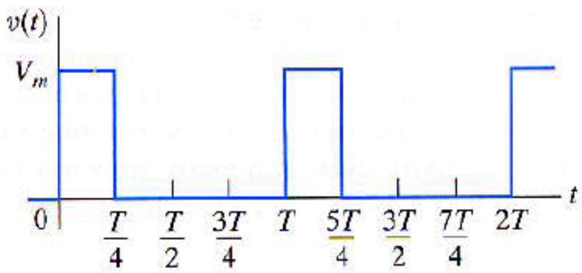
$$= \sqrt{a_n^2 + b_n^2}$$

$$= A_n \angle -\theta_n$$

- and using inverse transform, we get

$$a_n \cos n\omega_0 t + b_n \cos(n\omega_0 t - 90°) = A_n \cos(n\omega_0 t - \theta_n)$$

Ex. Consider the periodic function shown as



a. Derive the expressions for a_k and b_k of $\vartheta(t)$.
b. Write the first four terms of the Fourier series representation of $\vartheta(t)$ in trigonometric form.

Solution.

a. Note that $\vartheta(t)$ is neither even nor odd, nor does it have half-wave symmetry.

Hence ;

$$a_k = \frac{2}{T}\int_0^T \vartheta(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{T}\left[\int_{0}^{T/4} V_m \cos k\omega_0 t\, dt + \int_{T/4}^{T} 0.\cos k\omega_0 t\, dt\right.$$

$$= \left.\frac{2V_m \sin k\omega_0 t}{T.k\omega_0}\right|_{0}^{T/4}$$

$$= \frac{V_m}{k\pi}\sin\frac{k\pi}{2}$$

and

$$b_k = \frac{2}{T}\int_{0}^{T} \vartheta(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{T}\left[\int_{0}^{T/4} V_m \sin k\omega_0 t\, dt + \int_{T/4}^{T} 0.\sin k\omega_0 t\, dt\right] = \left.\frac{-2V_m \cos k\omega_0 t}{T.k\omega_0 t}\right|_{0}^{T/4}$$

$$= \frac{V_m}{k\pi}\left(1 - \cos\frac{k\pi}{2}\right)$$

b.

$$a_\vartheta = \frac{1}{T}\int_{0}^{T} \vartheta(t)dt$$

$$= \frac{1}{T}\left[\int_{0}^{T/4} V_m dt + \int_{T/4}^{T} 0.dt\right]$$

$$= \frac{1}{T} V_m \frac{T}{4} = \frac{V_m}{4}$$

- and for the values of $a_k - jb_k,\ \ j = 1, 2, 3,$ we have

$$a_1 - jb_1 = \frac{V_m}{\pi} - j\frac{V_m}{\pi} = \frac{V_m}{\pi}\ (1 - j) = \frac{V_m\sqrt{2}}{\pi}\ \angle -45°$$

$$a_2 - jb_2 = 0 - j\frac{V_m}{\pi} = \frac{V_m}{\pi}\ \angle -90°$$

$$a_3 - jb_3 = -\frac{V_m}{3\pi} - j\frac{V_m}{3\pi} = \frac{V_m}{3\pi}(-1 - j) = \frac{V_m\sqrt{2}}{3\pi}\ \angle -135°$$

- As a result ;

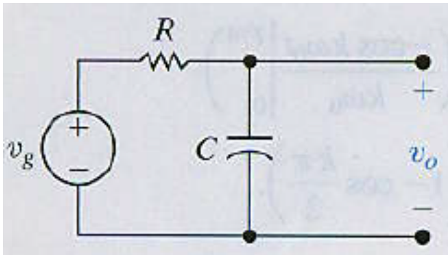
$$\vartheta(t) = \frac{V_m}{4} + \frac{V_m\sqrt{2}}{\pi}\cos(\omega t - 45°) + \frac{V_m}{\pi}\cos(\omega t - 90°) + \frac{V_m\sqrt{2}}{3\pi}\cos(\omega t - 135°) + \ldots$$

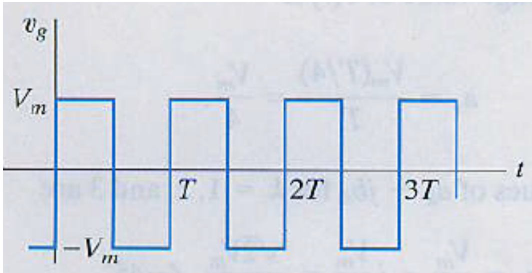
An application

- We shall employ the Fourier series representation of a periodic excitation function

  to find the steady-state response of a linear circuit.

- Let us consider the following circuit :





- We first represent the periodic excitation source with its Fourier series.

Note that ;

- The source ϑ_g has odd, half-wave and quarter-wave symmetry

  implying that the Fourier coefficients reduce to b_k with k taking odd integers.

Hence ;

$$b_k = \frac{8}{T}\int_0^{T/4} V_m \sin k\omega_0 t$$

$$= \frac{8V_m}{T.k\omega_0}\left(-\cos k\omega_0 t\right)\Big|_0^{T/4} = \frac{4V_m}{k\pi}\left(1 - \cancel{\cos k\frac{\pi}{2}}\right)$$

(since k is odd)

$$= \frac{4V_m}{k\pi}$$

- then the Fourier series representation of ϑ_g is

$$\vartheta_g = \frac{4V_m}{\pi}\sin\omega_0 t + \frac{4V_m}{3\pi}\sin 3\omega_0 t + \frac{4V_m}{5\pi}\sin 5\omega_0 t + \frac{4V_m}{7\pi}\sin 7\omega_0 t + \cdots$$

What does the representation mean ?

- It is the equivalent of

  ↳ infinitely many series-connected sinusoidal sources which have its own amplitude and frequency.

- To find the contribution of each source to the output voltage

  ↳ we just use the principle of superposition.

Hence ;

- For any of the sinusoidal sources, the phasor-domain expression for the output voltage is

$$V_0 = \frac{V_g}{1 + j\omega RC}$$

Note that ;

- All the voltage sources are expressed as sine functions

  ↳ thus we interpret a phasor in terms of the sine instead of the cosine.

- We then calculate

$$V_{01} = \frac{(4V_m/\pi)\angle 0°}{1 + j\omega_0 RC}$$

$$= \frac{4V_m \angle -\beta_1}{\pi\sqrt{1 + \omega_0^2 R^2 C^2}} \qquad \text{with } \beta_1 = \tan^{-1}\omega_0 RC$$

$$\Rightarrow \qquad V_{01}(t) = \frac{4V_m}{\pi\sqrt{1 + \omega_0^2 R^2 C^2}}\sin(\omega_0 t - \beta_1)$$

- In general, for the k^{th} harmonic, we obtain

$$\vartheta_{0k}(t) = \frac{4V_m}{k\pi\sqrt{1 + k^2\omega_0^2 R^2 C^2}}\sin(k\omega_0 t - \beta k)$$

with $\beta k = \tan^{-1} k\omega_0 RC$, (k is odd)

As a result ;

$$\vartheta_0(t) = \frac{4V_m}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{\sin(n\omega_0 t - \beta n)}{n\sqrt{1 + (n\omega_0 RC)^2}}$$

with $\beta n = \tan^{-1} n\omega_0 RC$

Observation

- We find that the output is

  ↳ a distorted replica of the input waveform.

- The distortion between the steady-state input and output can be seen as

  ↳ the change in the amplitude and phase of the harmonics when transmitted through the circuit.

A shortcoming of the Fourier series

- The principle shortcoming of the Fourier approach is

  ↳ the difficulty of ascertaining the waveform of the response.

Average-power calculations with periodic functions

- Given the Fourier series representation of the voltage and current at a pair of terminals in a linear lumped-parameter circuit

  ↳ the average power at the terminals is a function of the harmonic voltages and currents.

- We consider the trigonometric Fourier series representation for the current and voltage as

$$\vartheta = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{\vartheta n})$$

$$i = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in})$$

where

V_{dc} : the amplitude of the dc voltage component
V_n : the amplitude of the n^{th} harmonic voltage
$\theta_{\vartheta n}$: the phase angle of the n^{th} harmonic voltage
I_{dc} : the amplitude of the dc current component
I_n : the amplitude of the n^{th} harmonic current
θ_{in} : the phase angle of the n^{th} harmonic current

- We adopt to use the "passive sign convention".

Hence ;

- The average power is

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} \vartheta i \, dt$$

At first glance ;

- The product ϑi requires multiplying two infinite series.

However ;

- The only terms that survive the integration in one period of time are

  the products of the voltage and current having equal frequencies.

Therefore ;

- We only calculate

$$P = \frac{1}{T} V_{dc} I_{dc} t \Big|_{t_0}^{t_0 + T}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{T} \int_{t_0}^{t_0+T} V_n I_n \cos(n\omega_0 t - \theta_{\vartheta n}) \cos(n\omega_0 t - \theta_{in})\, dt$$

Now ;

- Using the trigonometric identity

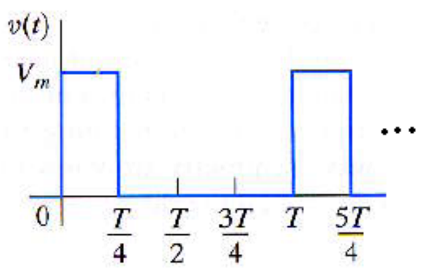
$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

- We can write

$$P = V_{dc} I_{dc} + \frac{1}{T} \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \int_{t_0}^{t_0+T} [\cos(\theta_{\vartheta n} - \theta_{in}) + \underbrace{\cos(2n\omega_0 t - \theta_{\vartheta n} - \theta_{in})}_{\text{integrates to zero in one period}}]\, dt$$

$$= V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\theta_{\vartheta n} - \theta_{in})$$

Ex. Assume that the periodic square-wave voltage



is applied across the terminals of a 15Ω resistor. The value of V_m $60V$, and that of T is $5ms$.

- **a.** Write the first five nonzero terms of the Fourier series representation of $\vartheta(t)$.
- **b.** Calculate the average power associated with each term in (a).
- **c.** Calculate the total average power delivered to the 15Ω resistor.
- **d.** What percentage of the total power is delivered by the 1^{st} five terms of the Fourier series ?

Solution.

a. The dc component of $\vartheta(t)$ is

$$a_\vartheta = \frac{1}{T}\int_0^T \vartheta(t)\,dt$$

$$= \frac{1}{T}\int_0^{T/4} 60\,dt$$

$$= \frac{1}{T}\,60t\,\Big|_0^{T/4}$$

$$= \frac{1}{T}\,60\frac{T}{4}$$

$$= 15\,V$$

- We have already calculated before that

$$A_1 = \frac{60\sqrt{2}}{\pi} = 27.01\,V\ ,\quad \theta_1 = -45°$$

$$A_2 = \frac{60}{\pi} = 19.10\,V\ ,\qquad \theta_2 = -90°$$

$$A_3 = \frac{60\sqrt{2}}{3\pi} = 9.00\,V\quad ,\quad \theta_3 = -135°$$

$$A_4 = 0\qquad ,\quad \theta_4 = 0°$$

$$A_5 = \frac{60\sqrt{2}}{5\pi} = 5.40\,V\quad ,\quad \theta_5 = -45°$$

$$\omega_0 = \frac{2\pi}{5.10^{-3}} = 400\pi\,rad/s$$

Thus ;

$$\vartheta(t) = 15 + 27.01\cos(400\pi t - 45°)$$
$$+19.1\cos(800\pi t - 90°)$$
$$+9\cos(1200\pi t - 135°)$$
$$+5.4\cos(2000\pi t - 45°) + \cdots \qquad V$$

b. The voltage is applied to the terminals of a resistor :

$$P_{dc} = \frac{15^2}{15} = 15\ W$$

$$P_1 = \frac{1}{2}\frac{27.01^2}{15} = 24.32\ W$$

$$P_2 = \frac{1}{2}\frac{19.10^2}{15} = 12.16\ W$$

$$P_3 = \frac{1}{2}\frac{9^2}{15} = 2.70\ W$$

$$P_5 = \frac{1}{2}\frac{5.4^2}{15} = 0.97\ W$$

c. We first calculate the rms value of $\vartheta(t)$

$$V_{rms} = \sqrt{\frac{(60)^2(T/4)}{T}} = \sqrt{900} = 30\ V$$

$$P_T = \frac{30^2}{15} = 60\ W$$

d.

$$P = P_{dc} + P_1 + P_2 + P_3 + P_5 = 55.15\ W$$

- This is $(55.15/60).100$, or 91.92% of the total.

The rms value of a periodic function

- We shall express the rms value of a periodic function

  in terms of the Fourier coefficients.

Indeed ;

$$F_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)\,dt}$$

$$= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}\left[a_\vartheta + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)\right]^2 dt}$$

Note that ;

- The only surviving terms in the integration are

  the product of the dc term and the harmonic products of the same frequency.

Hence ;

$$F_{rms} = \sqrt{\frac{1}{T}\left(a_\vartheta^2 T + \sum_{n=1}^{\infty}\frac{T}{2}A_n^2\right)}$$

$$= \sqrt{a_\vartheta^2 + \sum_{n=1}^{\infty}\frac{A_n^2}{2}}$$

$$= \sqrt{a_\vartheta^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2}$$

$\Downarrow$

square root of a sum obtained by adding the square of the rms value of each harmonic and the square of the dc value.

e.g. Let

$$\vartheta(t) = 10 + 30\cos(\omega_0 t - \theta_1) + 20\cos(2\omega_0 t - \theta_2)$$

$$+5\cos(3\omega_0 t - \theta_3) + 2\cos(5\omega_0 t - \theta_5)$$

- Then the rms value of $\vartheta(t)$ is calculated as

$$V = \sqrt{10^2 + \left(30/\sqrt{2}\right)^2 + \left(20\sqrt{2}\right)^2 + \left(5/\sqrt{2}\right)^2 + \left(2/\sqrt{2}\right)^2}$$

$$= \sqrt{764.5}$$

$$= 27.65\ V$$

The exponential form of the Fourier series

- Representing the Fourier series in exponential form

  yields a concise expression.

- The exponential form of the series is defined as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

How to justify it then ?

- It follows from trigonometric Fourier series that

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

- We know that

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Hence ;

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} \left[a_n . \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)\right.$$

$$\left. + b_n . \frac{1}{j2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)\right]$$

$$= a_\vartheta + \sum_{n=1}^{\infty} \left[\frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}\right]$$

- Let us define

$$C_n = \frac{1}{2}\,(a_n - jb_n) = \frac{A_n}{2}\,\angle - \theta_n$$

- Then we have

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + C_n^* e^{-jn\omega_0 t}$$

$$= a_\vartheta + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_{-n}^* e^{jn\omega_0 t}$$

Note that ;

$$C_{-n}^* = C_n$$

Therefore ;

$$f(t) = a_\vartheta + \sum_{n=1}^{\infty} C_n\, e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad \text{with} \quad C_0 \triangleq a_\vartheta \quad \text{“dc component”}$$

Moreover ;

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\overbrace{(\cos n\omega_0 t - j\sin n\omega_0 t)}^{e^{-jn\omega_0 t}}dt$$

$$= \frac{1}{2}\left[\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\,dt - j\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{1}{2}\,(a_n - jb_n)$$

which therefore justifies the formulation of the coefficients, C_n

What about the rms value ?

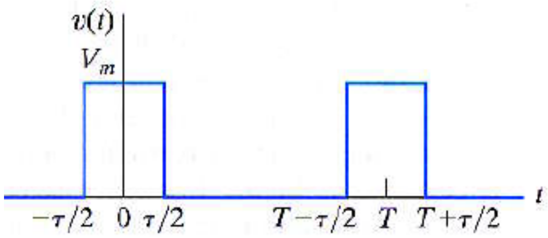
- We may express it in terms of the complex Fourier coefficients

$$F_{rms} = \sqrt{a_\vartheta^2 + \sum_{n=1}^{\infty} \frac{1}{2} A_n A_n^*}$$

$$= \sqrt{C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (2C_n)(2C_n^*)}$$

$$= \sqrt{C_0^2 + 2 \sum_{n=1}^{\infty} |C_n|^2}$$

Ex. Find the exponential Fourier series for the periodic voltage shown as



Solution. We calculate

$$C_n = \frac{1}{T} \int_{-\lambda/2}^{\lambda/2} V_m e^{-jn\omega_0 t}$$

$$= \frac{V_m}{T} \left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right) \Bigg|_{-\lambda/2}^{\lambda/2}$$

$$= \frac{jV_m}{n\omega_0 T} \left(e^{-jn\omega_0 \lambda/2} - e^{jn\omega_0 \lambda/2} \right)$$

$$= \frac{2V_m}{n\omega_0 T} \sin(n\omega_0 \lambda/2)$$

$$= \frac{V_m \lambda}{T} \frac{\sin(n\omega_0 \lambda/2)}{n\omega_0 \lambda/2}$$ “sinc function”

- Since $\vartheta(t)$ has even symmetry, $b_n = 0$, $\forall n$

  ↳ C_n is expected to be real.

Amplitude and phase spectra

- We shall define a periodic time function by

  ↳ its Fourier coefficients (a_ϑ, a_n, b_n) and its period (T).

- The knowledge of a_n and b_n yields the

  ↳ amplitude (A_n) and phase angle $(-\theta_n)$ of each harmonic.

Hence ;

- One can synthesize the time-domain waveform

  ↳ from the amplitude and phase angle data.

Moreover ;

- If the periodic function is exciting a highly frequency selective circuit

  ↳ then the description of the response in terms of amplitude and phase provides an understanding of the output waveform.

amplitude spectrum of $f(t)$

- It is the plot of the amplitude of each term in the Fourier series of $f(t)$

  ↳ versus the frequency.

phase spectrum of $f(t)$

- It is the plot of the phase angle of each term

  ↳ versus the frequency.

Note that ;

- The amplitude and phase angle data occur at discrete values of the frequency $(\omega_0, 2\omega_0, 3\omega_0, \ldots)$

  ↳ these plots are referred to as "line spectra".

An illustration of amplitude and phase spectra

- We shall consider either :

(i) A_n and $-\theta_n$, OR
(ii) C_n

for the amplitude and phase spectra.

Ex. Let us consider the periodic square function example for which

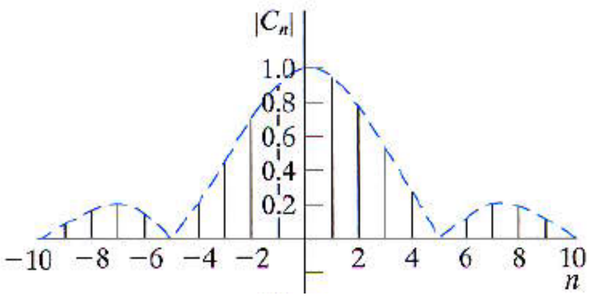
$$C_n = \frac{V_m \lambda}{T} \frac{\sin(n\omega_0\lambda/2)}{n\omega_0\lambda/2}$$

- Let $V_m = 5\,V$, $\lambda = T/5$, then

$$C_n = \frac{5.T/5}{T} \frac{\sin(n2\pi(T/5)/2.T)}{n.2\pi.(T/5)/2.T}$$

$$= \sin\frac{(n\pi/5)}{n\pi/5}$$

- The plot of the magnitude of C_n can be obtained as



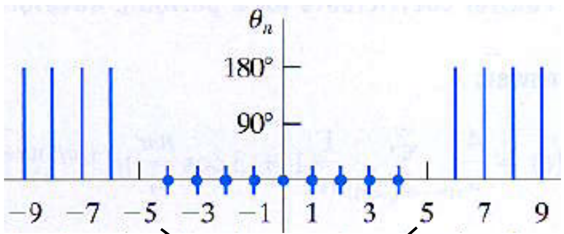
$\Rightarrow$ the amplitude spectrum is bounded by the envelope of $|\sin x/x|$ function.

- For the phase angle, we find that it is either 0° or 180°

because C_n is real.

Hence ;

- The phase plot can be drawn as



not defined as $C_5 = 0$

Note that ;

- The phase angle is simply determined by

the algebraic sign of $\sin(n\pi/5)\,/(n\pi/5)$ which gives $0°$ for $n = 0, \pm1, \pm2, \pm3, \pm4$ and $180°$ for $n = \pm6, \pm7, \pm8, \pm9$ and so on.