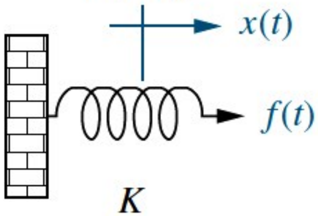
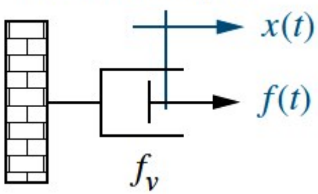
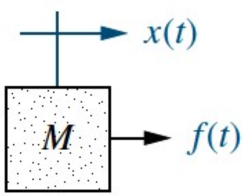
Mathematical Models
Modeling in the Frequency Domain
Mechanical Systems
Nise's Textbook Chap. 2-2

Translational Mechanical System Transfer Functions

- There are analogies between electrical and mechanical components and variables.
- Mechanical systems, like electrical networks, have three passive, linear components.
  - Two of them, the **spring** and the **mass**, are *energy-storage* elements;
  - One of them, the **viscous damper**, dissipates energy.
- The two energy-storage elements are analogous to the two electrical energy-storage elements, the **inductor** and **capacitor**. The energy dissipater is analogous to electrical **resistance**.
- Mechanical elements are shown in Table 2.4. In the table, K, f_v (b), and M are called spring constant, coefficient of viscous friction, and mass, respectively.

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t) = \text{N}$ (newtons), $x(t) = \text{m}$ (meters), $v(t) = \text{m/s}$ (meters/second), $K = \text{N/m}$ (newtons/meter), $f_v = \text{N-s/m}$(newton-seconds/meter), $M = \text{kg}$ (kilograms = newton-seconds^2/meter).

Table 2.4

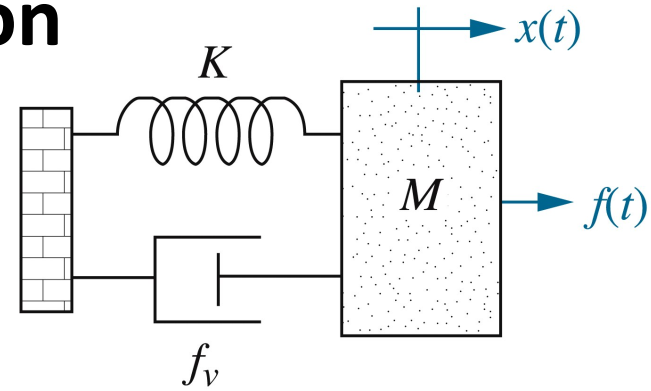
Translational Mechanical System Transfer Functions – *cnt…*

- Comparing the voltage-current column of Table 2.3 and the force-velocity column of Table2.4 leads that,
  - Mechanical force is analogous to electrical voltage,
- Also, comparing the voltage-charge column of Table 2.3 and the force-displacement column of Table2.4 leads that,
  - Mechanical displacement is analogous to electrical charge,
  - The spring is analogous to the capacitor,
  - The viscous damper is analogous to the resistor.
- Another analogy can be drawn by comparing the force-velocity column of Table 2.4 to the current-voltage column of Table 2.3 in reverse order:
  - Mechanical force is analogous to electrical current,
  - Mechanical velocity is analogous to electrical voltage,
  - The spring is analogous to the inductor,
  - The viscous damper is analogous to the capacitor.

Translational Mechanical System Transfer Functions – *cnt…*

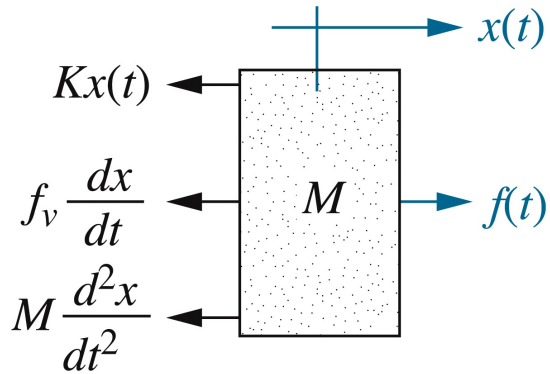
- The mechanical system requires just one differential equation, called the **equation of motion**, to describe it. To obtain the transfer function of a mechanical system;
  - We will begin by assuming a positive direction of motion, for example, to the right,
  - Using our assumed direction of positive motion, we first draw **a free-body diagram** (for each mass), placing on the body all forces that act on the body either in the direction of motion or opposite to it,
  - Next we use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero.
  - Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function.

EXAMPLE (2.16, pp.63) Transfer function of one equation of motion



Find the transfer function, X(s)/F(s).

The free-body diagram:



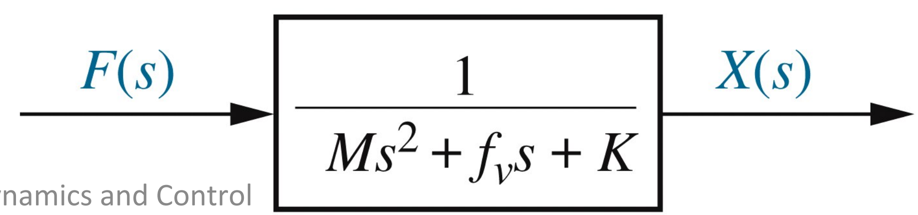
Note that,

- All the forces felt by the mass are placed on the mass, and we assumed the mass is traveling toward the right.
- Thus only the applied force points to the right, and all other forces (the spring, viscous damper and the force due to acceleration) act to oppose it.

$$Ms^2X(s) + f_v sX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$



Translational Mechanical System Transfer Functions – *cnt…*

- Taking the Laplace transform of the force-displacement column in Table 2.4, we obtain

  for the spring $F(s)=KX(s)$

  for the viscous damper, $F(s)=f_v sX(s)$

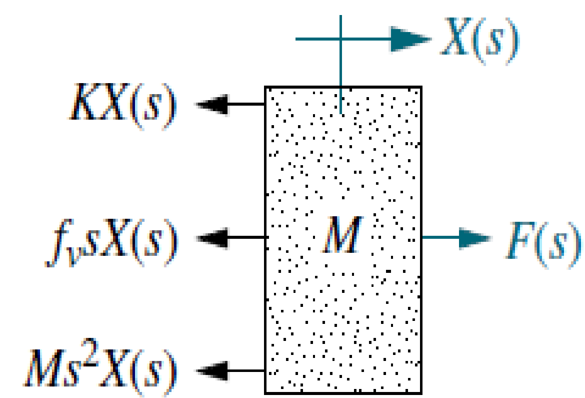
  and for the mass $F(s)=Ms^2 X(s)$

If we define impedance for mechanical components as $Z_M(s)=F(s)/X(s)$

Replacing each force in Figure 2.16(a) by its Laplace transform, which is in the format, we obtain from which we could have obtained

$$\left(Ms^2+f_v s+K\right)X(s)=F(s)$$

$$[\text{Sum of impedances}]X(s)=[\text{Sum of applied forces}]$$

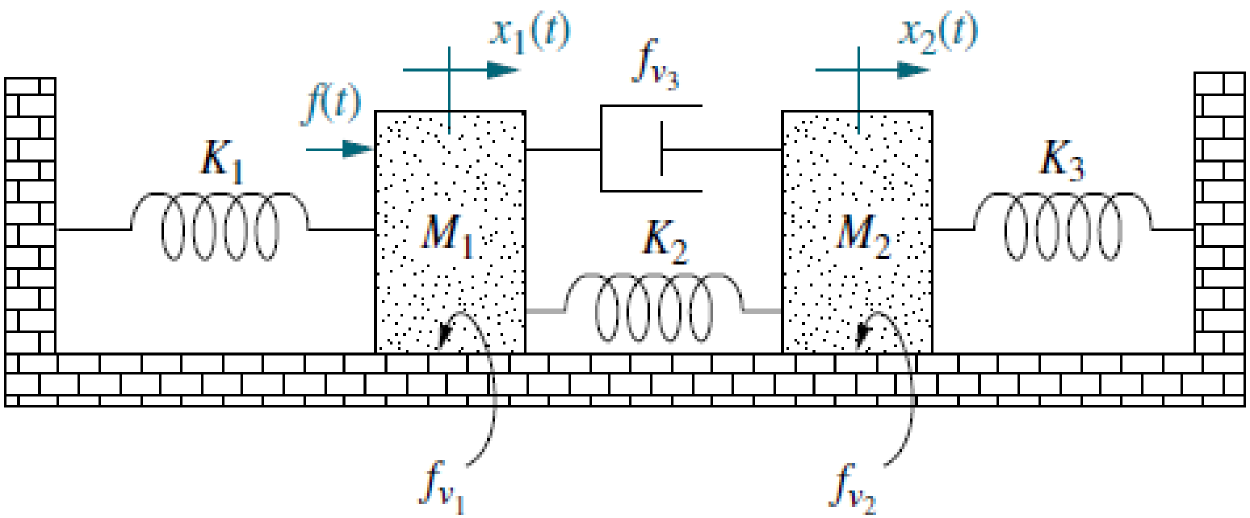


(b)

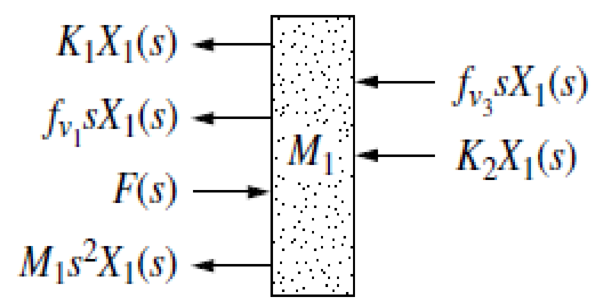
EXAMPLE (2.17, pp.65) Transfer function of 2DOF

Find the transfer function, $X_2(s)/F(s)$.

- The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system.
- The two equations come from free-body diagrams of each mass. Superposition is used to draw the free body diagrams. For example, the forces on $M\downarrow 1$ are due to (1) its own motion and (2) the motion of $M\downarrow 2$ transmitted to $M\downarrow 1$ through the system. We will consider these two sources separately.



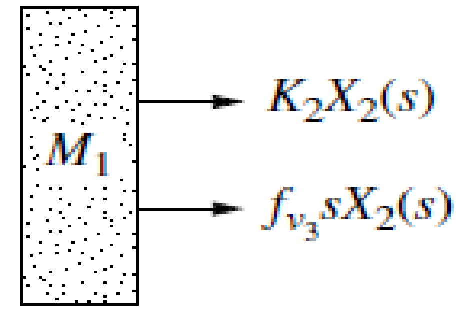
If we hold $M\downarrow 2$ still and move $M\downarrow 1$ to the right, we see the forces shown on the right.



(a)

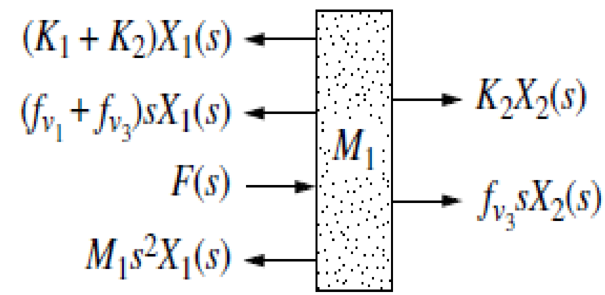
EXAMPLE (2.17, pp.65) Transfer function of 2DOF...

- If we hold M1 still and move M2 to the right, we see the forces shown below



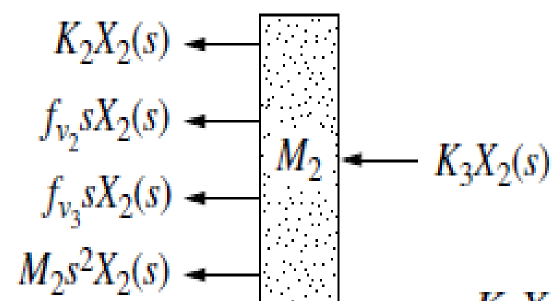
(*b*)

- The total force on M1 is the superposition, or sum, of the forces



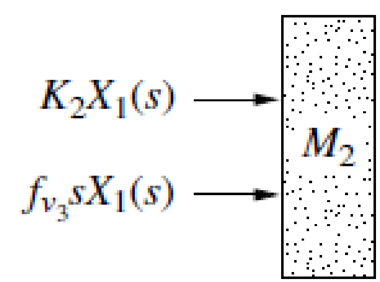
(*c*)

- If we move M2 to the right while holding M1 still



(*a*)

- If we move M1 to the right and hold M2 still



(*b*)

EXAMPLE (2.17, pp.65) Transfer function 2DOF

- Then, the total force on M1 is the superposition, or sum, of the forces
- The Laplace transform of the equations of motion can now be written

$$\left[M_1 s^2 (f_{v_1} + f_{v_3})s + (K_1 + K_2)\right]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s)$$

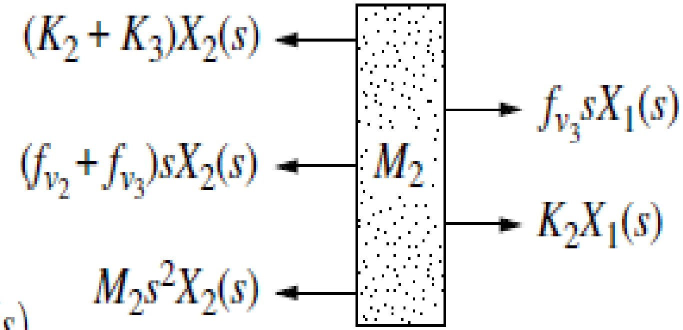
$$-(f_{v_3}s + K_2)X_1(s) + \left[M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\right]X_2(s) = 0$$

- From this, the transfer function, $X_2(s)/F(s)$, is

$$\Delta = \begin{vmatrix} \left[M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)\right] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & \left[M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\right] \end{vmatrix}$$

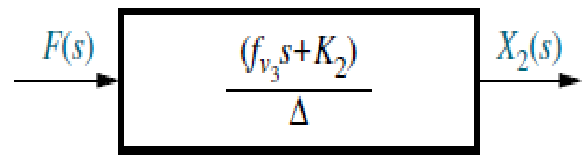
$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{bmatrix} X_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{bmatrix} \qquad (2.120a)$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{bmatrix} X_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_2 \end{bmatrix} X_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_2 \end{bmatrix} \qquad (2.120b)$$



(c)

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta}$$

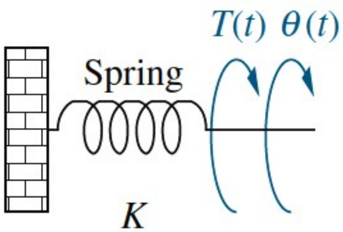
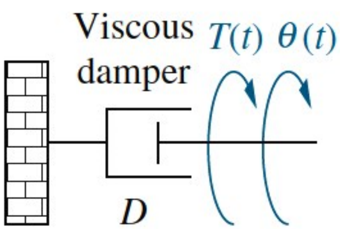
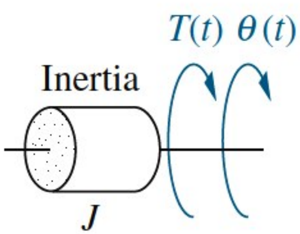


(b)

Rotational Mechanical System Transfer Functions

- Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement.
- The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation.
- The table below shows the components along with the relationships between torque and angular velocity, as well as angular displacement.

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Table 2.5

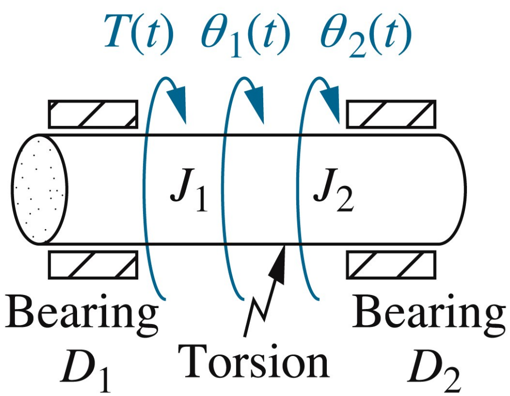
Rotational Mechanical System Transfer Functions

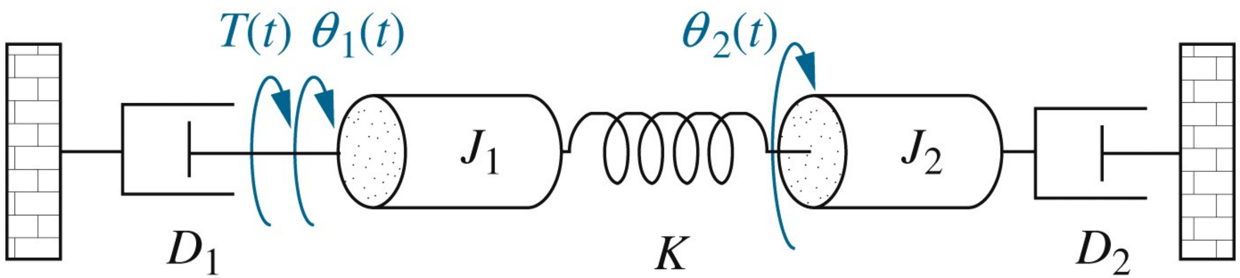
- Notice that the term associated with the mass is replaced by inertia. The values of *K, D* and *J* are called spring constant, coefficient of viscous friction, and moment of inertia, respectively.
- Similar to the translational systems, we test a point of motion by rotating it while holding still all other points of motion. The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.
- Writing the equations of motion for rotational systems is similar to writing them for translational systems; the only difference is that the free-body diagram consists of torques rather than forces. We obtain these torques using superposition.

  - First, we rotate a body while holding all other points still and place on its free-body diagram all torques due to the body's own motion.
  - Then, holding the body still, we rotate adjacent points of motion one at a time and add the torques due to the adjacent motion to the free-body diagram.
  - The process is repeated for each point of motion.
  - For each free-body diagram, these torques are summed and set equal to zero to form the equations of motion.

EXAMPLE (2.19, pp.71) Transfer function of two equations of motion

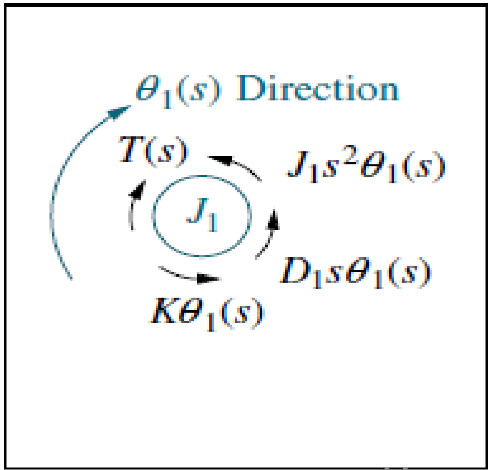
- Find the transfer function, $\theta_2(s)/T(s)$. The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.





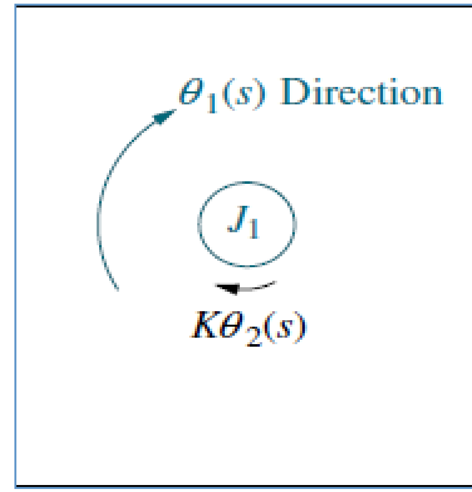
- There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system.

- Now we can draw a free-body diagram of J_1, using superposition. The figure below shows the torques on J_1 if J_2 is held still and J_1 rotated.

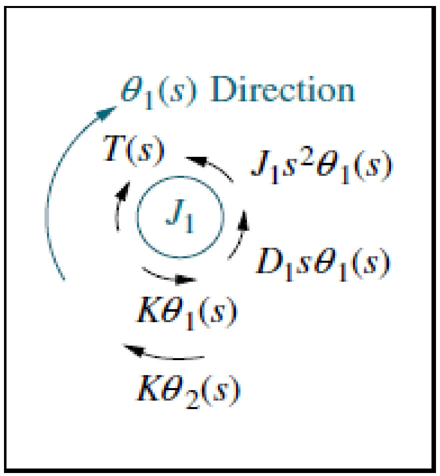


EXAMPLE (2.19, pp.71) Transfer function of two equations of motion

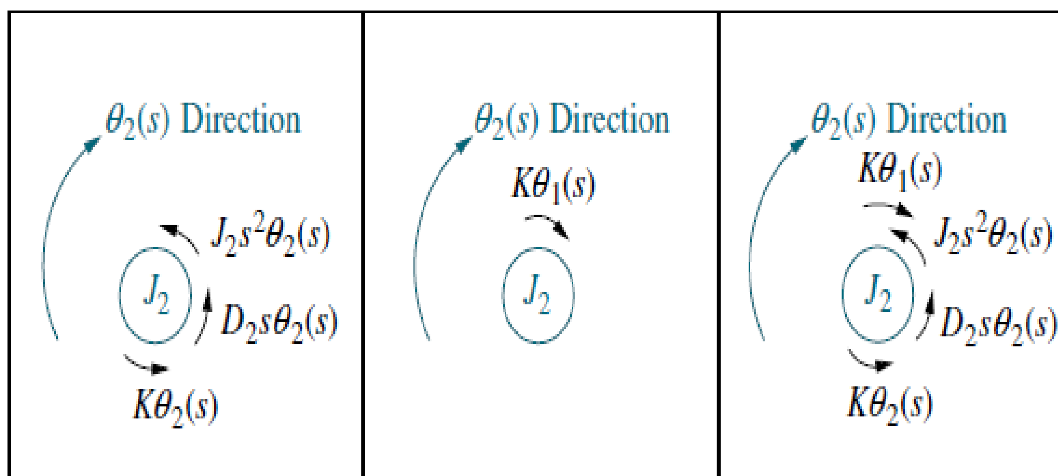
- The figure below shows the torques on J_1 if J_1 is held still and J_2 rotated.



- So, the final (sum) free-body diagram is given below.



- Repeating the same process for J_2 gives:



(a) (b) (c)

EXAMPLE (2.19, pp.71) Transfer function of two equations of motion

Summing torques respectively from free-body diagrams, we obtain the equations of motion,

$$(J_1s^2 + D_1s + K)\theta_1(s) - K\theta_2(s) = T(s) \qquad (2.127a)$$

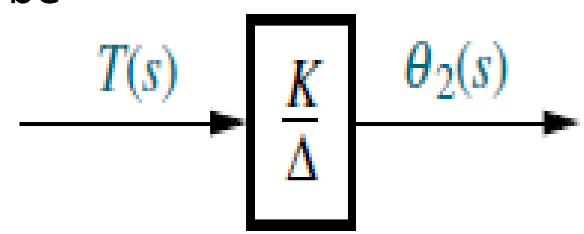
$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0 \qquad (2.127b)$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

where

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

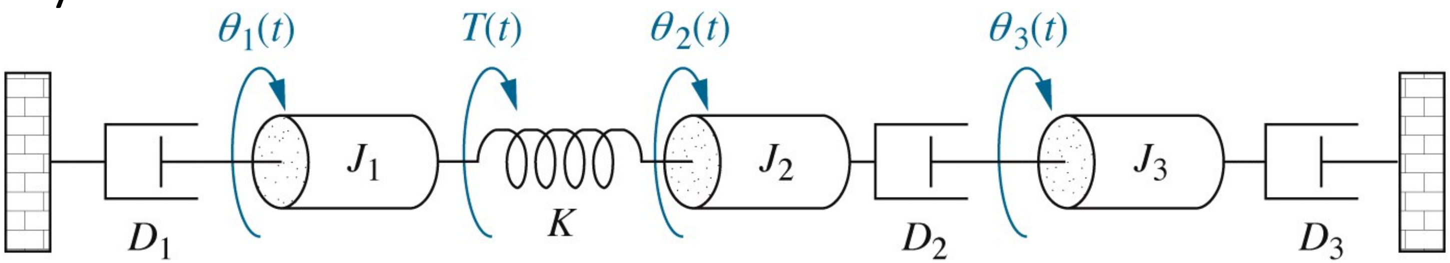


$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix} \qquad (2.129a)$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix} \qquad (2.129b)$$

EXAMPLE (2.20, pp.72) Equations of motion by inspection

Write, but do not solve, the Laplace transform of the equations of motion for the system below.



$$\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{array}\right]\theta_1(s) - \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{array}\right]\theta_2(s) - \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{array}\right]\theta_3(s) = \left[\begin{array}{c}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{array}\right]$$

$$-\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{array}\right]\theta_1(s) + \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{array}\right]\theta_2(s) - \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{array}\right]\theta_3(s) = \left[\begin{array}{c}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{array}\right]$$

$$-\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{array}\right]\theta_1(s) - \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{array}\right]\theta_2(s) + \left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_3\end{array}\right]\theta_3(s) = \left[\begin{array}{c}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_3\end{array}\right]$$

The equations will take on the following form, similar to electrical mesh equations:

$$(J_1s^2 + D_1s + K)\theta_1(s) \quad -K\theta_2(s) \quad -0\theta_3(s) = T(s)$$

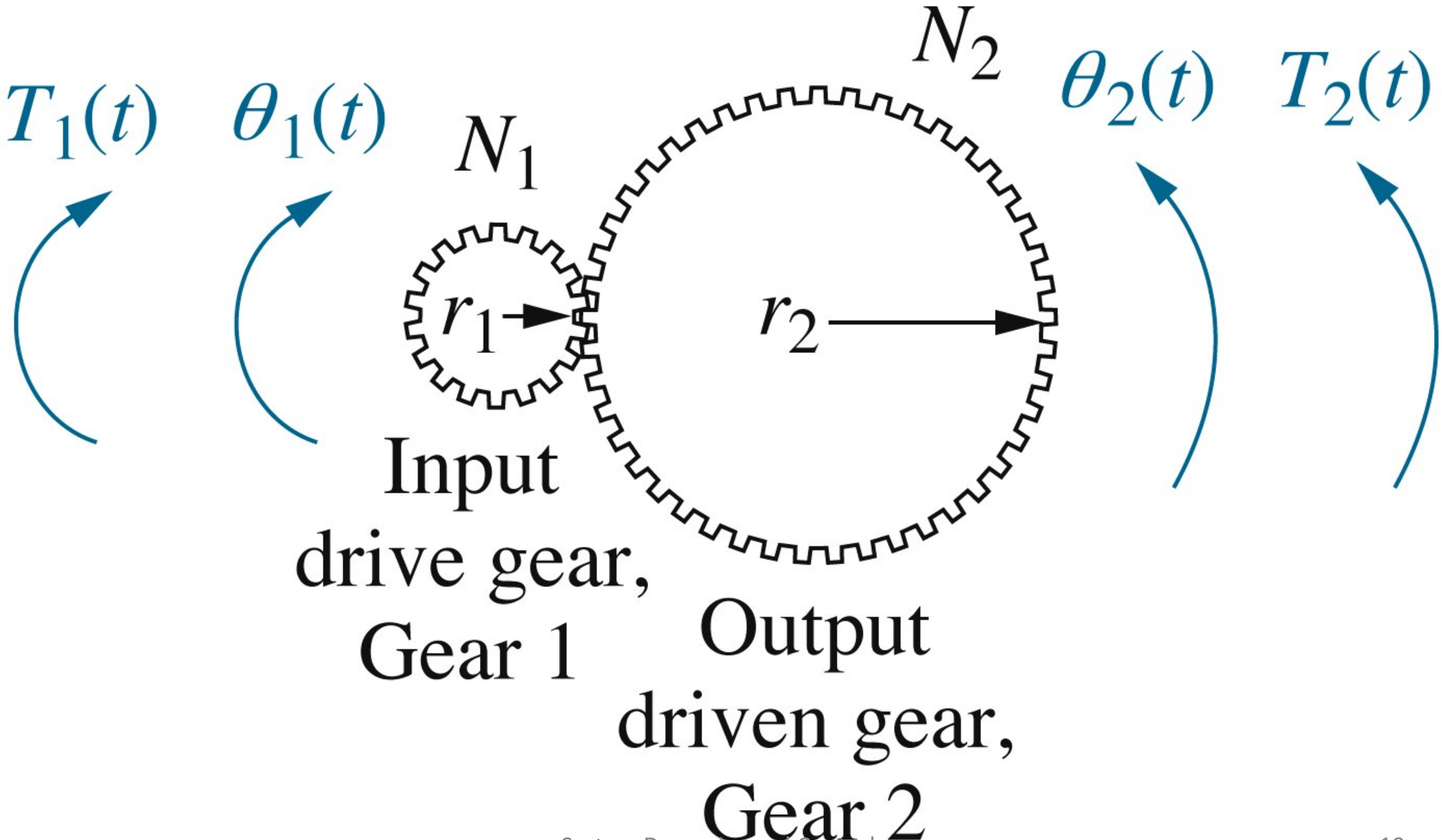
$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) \quad -D_2s\theta_3(s) = 0$$

$$-0\theta_1(s) \quad -D_2s\theta_2(s) + (J_3s^2 + D_3s + D_2s)\theta_3(s) = 0$$

Transfer Functions for Systems with Gears

- Rotational systems, especially those driven by motors, are rarely seen without associated gear trains driving the load. Gears provide mechanical advantage to rotational systems.
- Anyone who has ridden a 10-speed bicycle knows the effect of gearing. Going uphill, you shift to provide <u>more torque and less speed</u>. On the straightaway, you shift to obtain <u>more speed and less torque</u>. Thus, gears allow you to match the drive system and the <u>load—a trade-off between speed and torque</u>.
- For many applications, gears exhibit *backlash*, which occurs because of the loose fit between two meshed gears. The drive gear rotates through a small angle before making contact with the meshed gear.
- The result is that the angular rotation of the output gear does not occur until a small angular rotation of the input gear has occurred. In this section, we idealize the behavior of gears and assume that there is no backlash.
- The linearized interaction between two gears is depicted in the figure:

Two meshed gears



Transfer Functions for Systems with Gears

- An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$. From the figure, as the gears turn, the distance traveled along each gear's circumference is the same. Thus,

$$\boxed{r_1\theta_1 = r_2\theta_2} \rightarrow \boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$

- If we assume the gears are lossless, that is they do not absorb or store energy, the energy into Gear 1 equals the energy out of Gear 2. Since the translational energy of force times displacement becomes the rotational energy of torque times angular displacement,

$$\boxed{T_1\theta_1 = T_2\theta_2} \rightarrow \boxed{\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}}$$

- Thus, the torques are directly proportional to the ratio of the number of teeth. All results can be summarized in

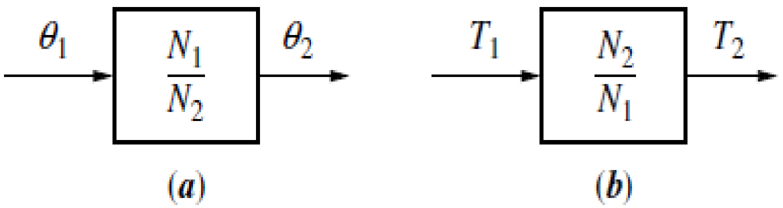
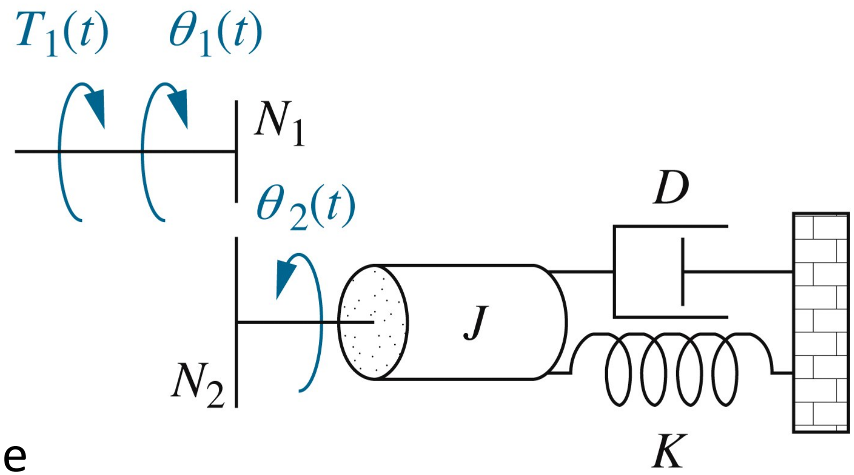


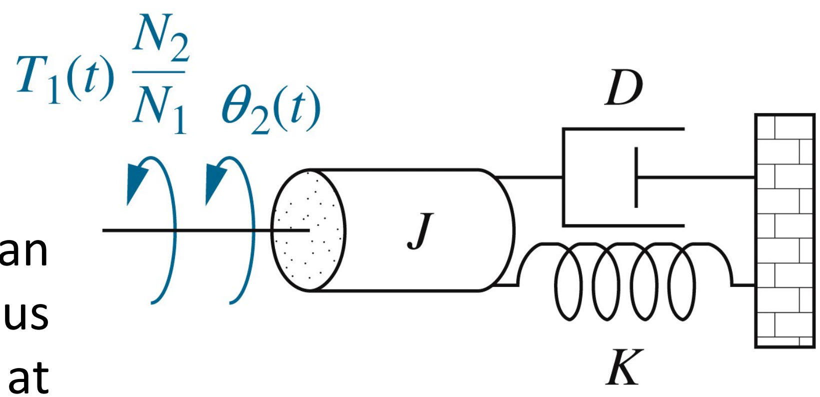
FIGURE 2.28 Transfer functions for **a.** angular displacement in lossless gears and **b.** torque in lossless gears

Transfer Functions for Systems with Gears…

- The figure shows gears driving a rotational inertia, spring, and viscous damper.
- We want to represent the figure as an equivalent system at θ_1 without the gears.
- In this system, T_1 can be reflected to the output by multiplying



$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$



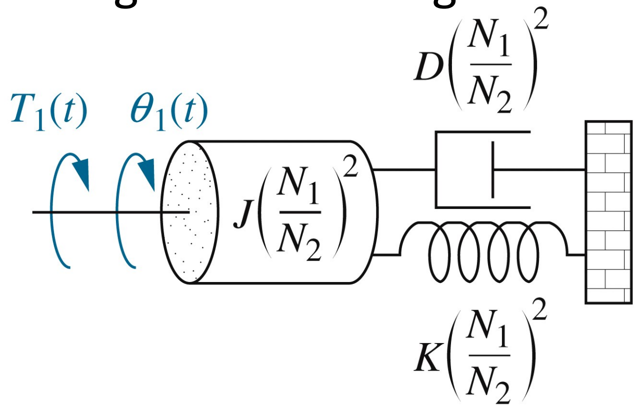
- Now, we can convert $\theta_2(s)$ into an equivalent $\theta_1(s)$, so that the previous equation will look as if it were written at the input. Using the first figure (a) to obtain $\theta_2(s)$ in terms of $\theta_1(s)$, we get

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

Transfer Functions for Systems with Gears

- Thus, the load can be thought of as having been reflected from the output to the input.

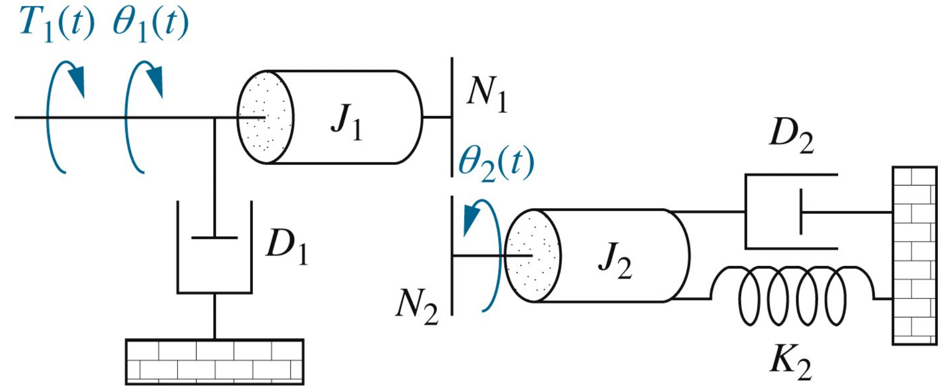


- Generalizing the results, we can make the following statement:
Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio

$$\boxed{\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2}$$

where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft.

EXAMPLE 2.21, System with lossless gears: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system given.

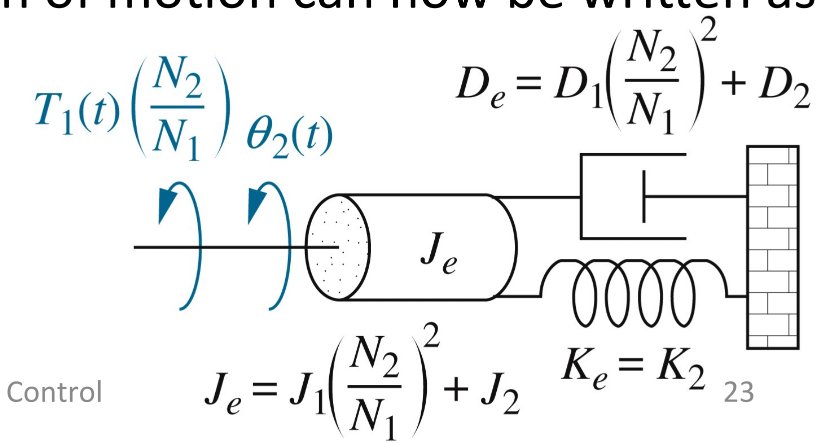


Solution: It may be tempting at this point to search for two simultaneous equations corresponding to each inertia. The inertias, however, do not undergo linearly independent motion, since they are tied together by the gears. Thus, there is only **one degree of freedom** and hence **one equation of motion**.

We first reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1). The equation of motion can now be written as

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2; \quad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2; \quad K_e = K_2$$

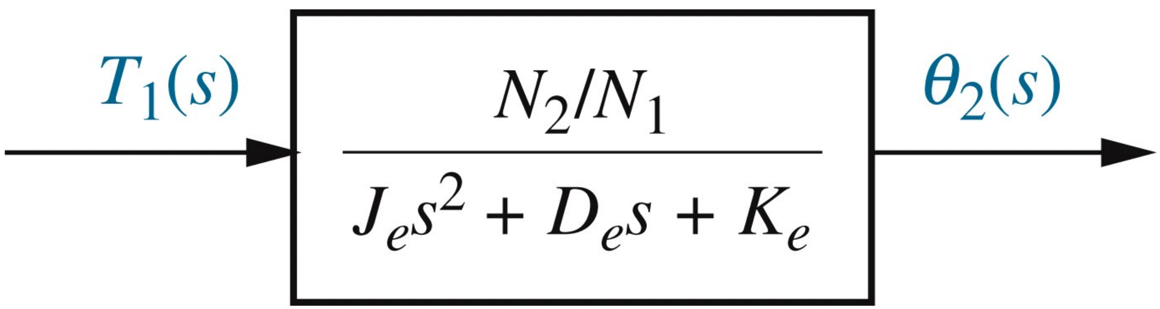
$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$



EXAMPLE (2.21, p.76) System with lossless gears (last part)

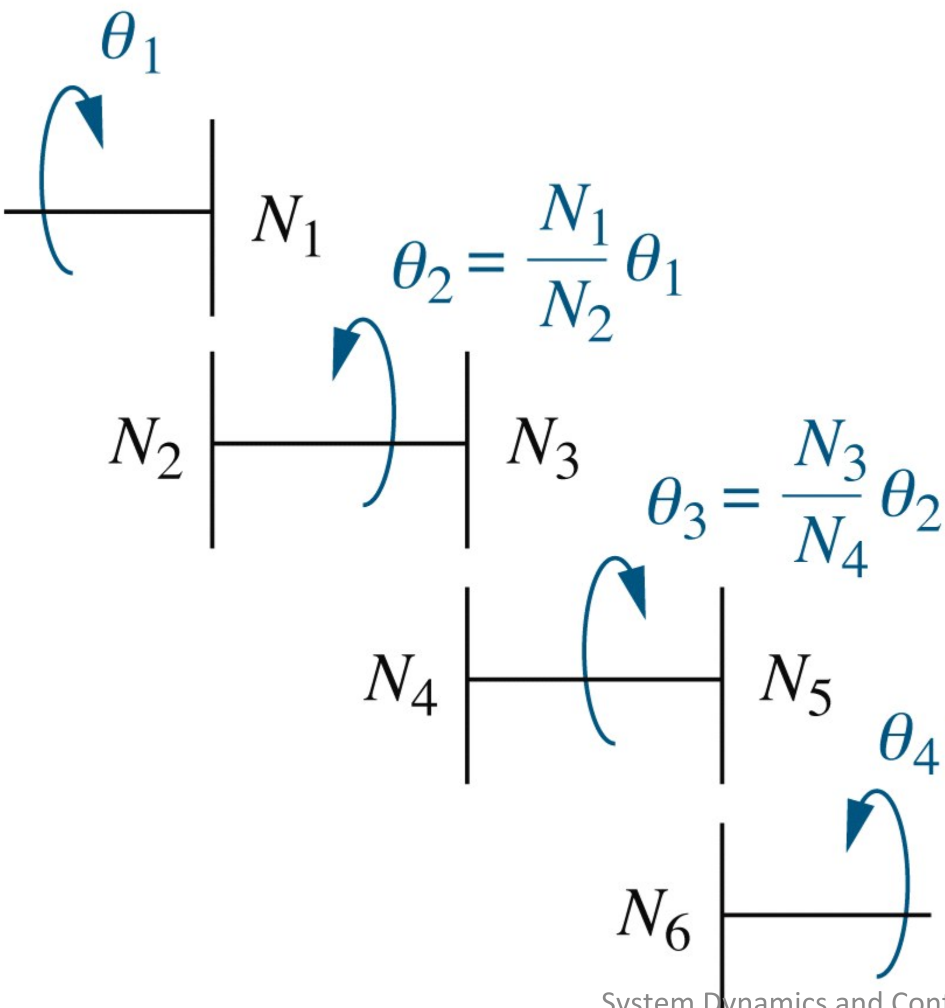
- Solving for $\theta_2(s)\,/T_1(s)$, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e}$$



Gear Trains

In order to eliminate gears with large radii, a gear train is used to implement large gear ratios by cascading smaller gear ratios.



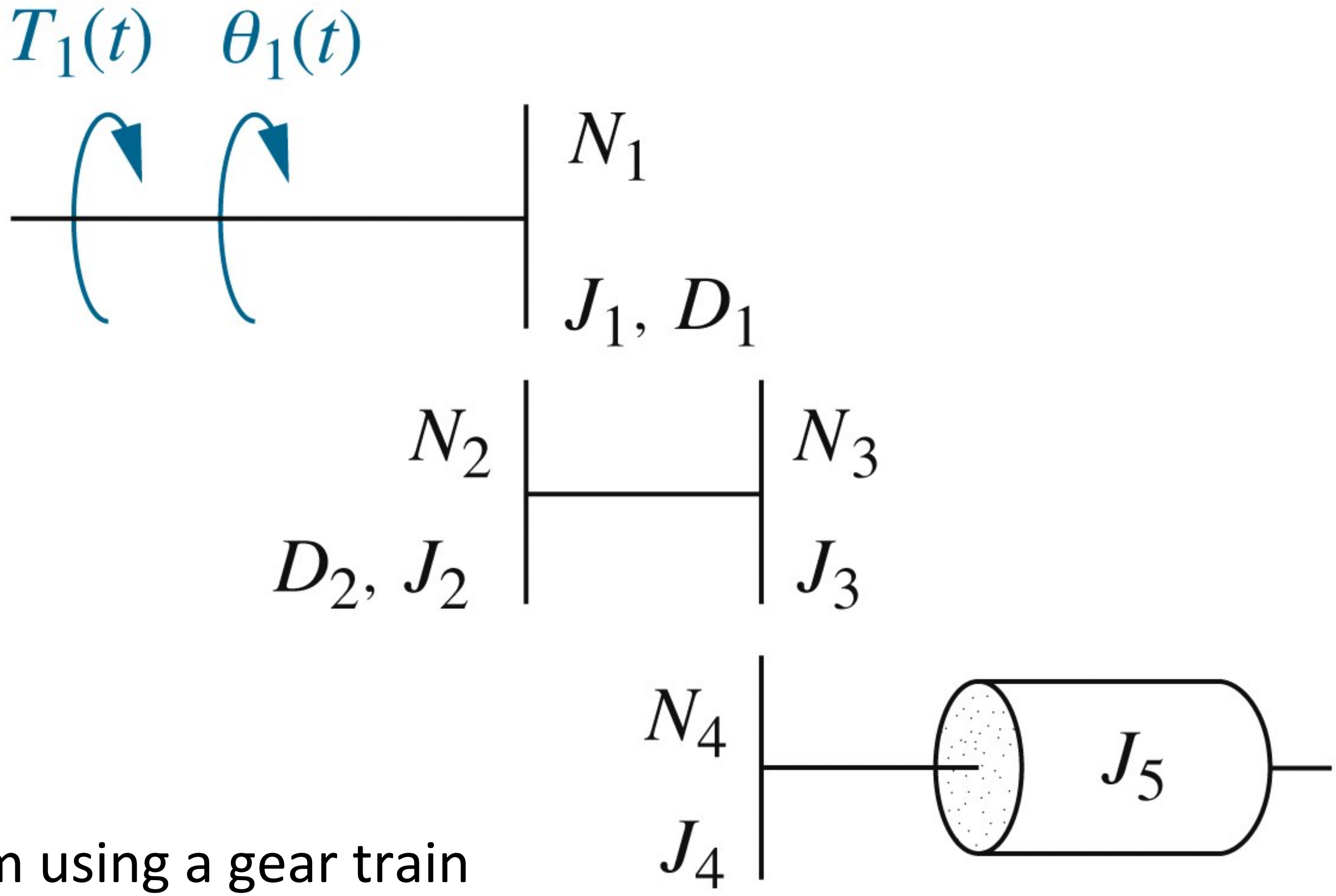
$$\theta_2 = \frac{N_1}{N_2}\theta_1$$

$$\theta_3 = \frac{N_3}{N_4}\theta_2 = \frac{N_1 N_3}{N_2 N_4}\theta_1$$

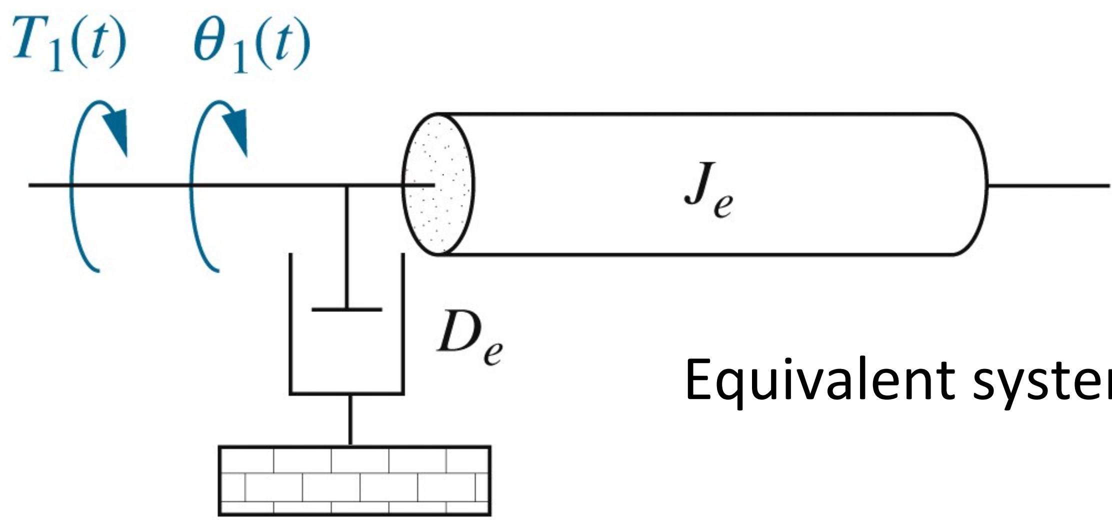
$$\theta_4 = \frac{N_5}{N_6}\theta_3 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

Example for Transfer Function—Gears with Loss

Problem: Find the transfer function, $\theta_1(s)\,/T_1(s)$, for the system given below.



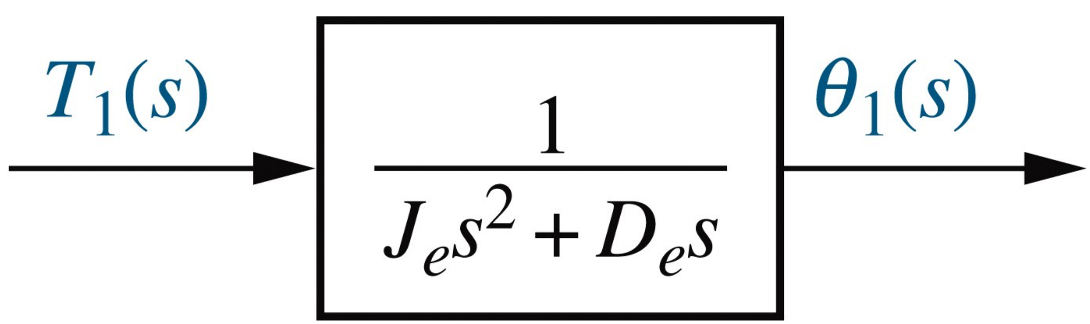
System using a gear train



Equivalent system at the input.

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$



The block diagram for the equivalent system

PROBLEM:

Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for mechanical system with gears shown in the figure below.

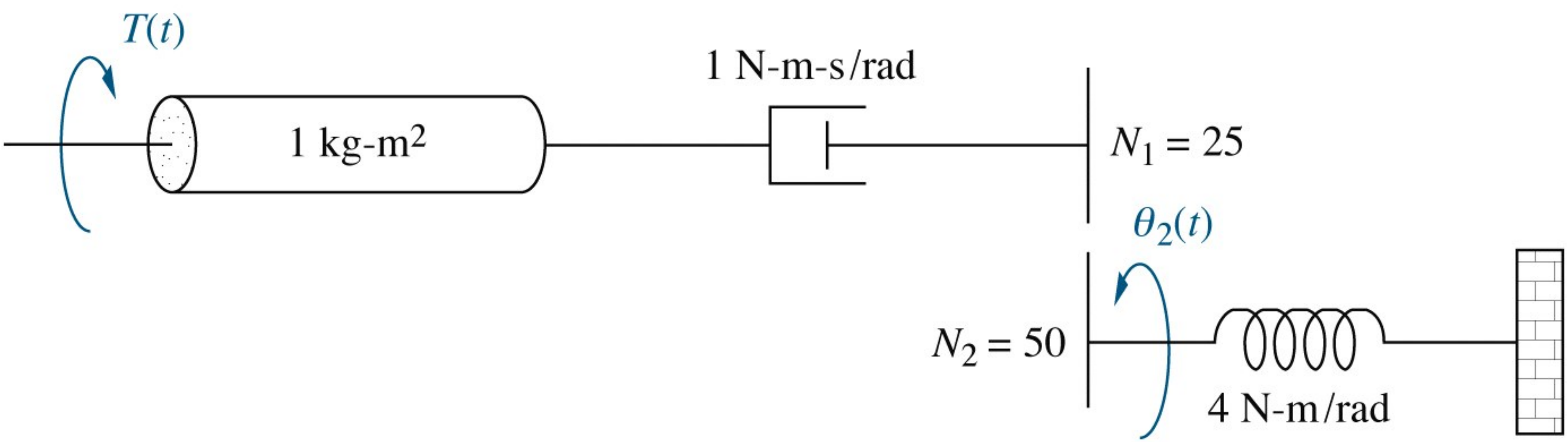


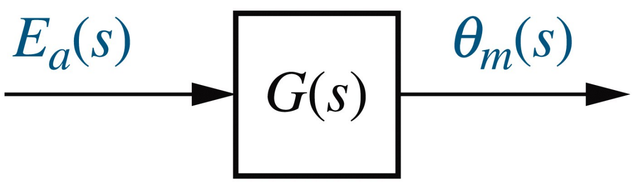
Figure 2.33

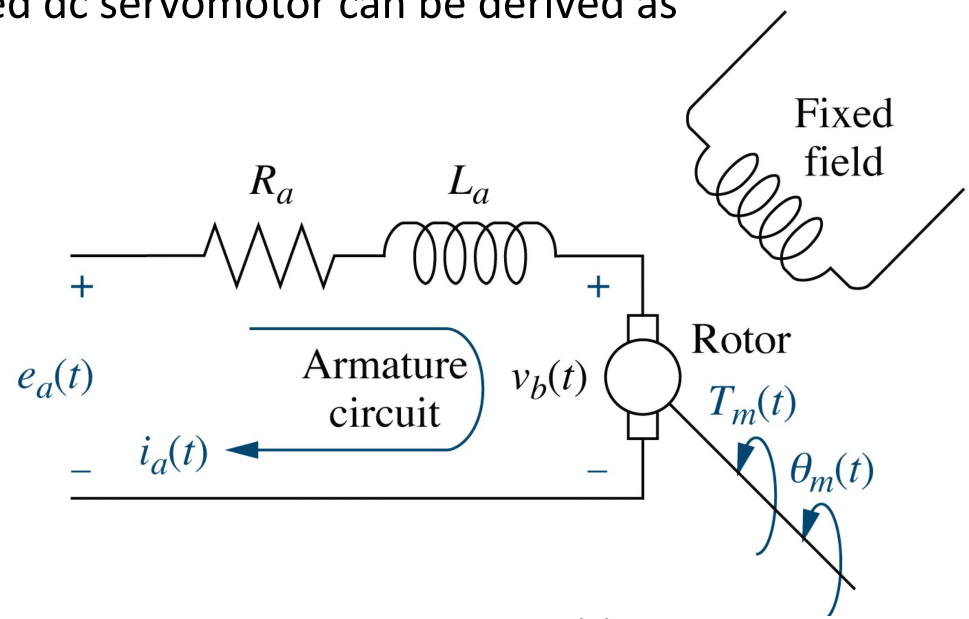
ANSWER:

$$G(s) = \frac{1/2}{s^2 + s + 1}$$

Electromechanical System Transfer Functions

- Now, we move to systems that are hybrids of electrical and mechanical variables, the electromechanical systems.
- A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input.
- The schematic of an armature-controlled dc servomotor can be derived as





In this schematic,

- A magnetic field is developed by stationary permanent magnets or a stationary electromagnet called the fixed field.
- A rotating circuit called the armature, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $F = B\, l\, i_a(t)$, where B is the magnetic field strength and l is the length of the conductor.
- The resulting torque turns the rotor, the rotating member of the motor.

Electromechanical System Transfer Functions

- A conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to $e = Blv$, where e is the voltage and v is the velocity of the conductor normal to the magnetic field. Since the current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed. Thus,

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt} \qquad (2.144)$$

- We call $v_b(t)$ the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant; and $d\theta_m(t)/dt = \omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s\theta_m(s) \qquad (2.145)$$

- The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$, is found by writing a loop equation around the Laplace transformed armature circuit

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s) \qquad (2.146)$$

- The torque developed by the motor is proportional to the armature current; thus,

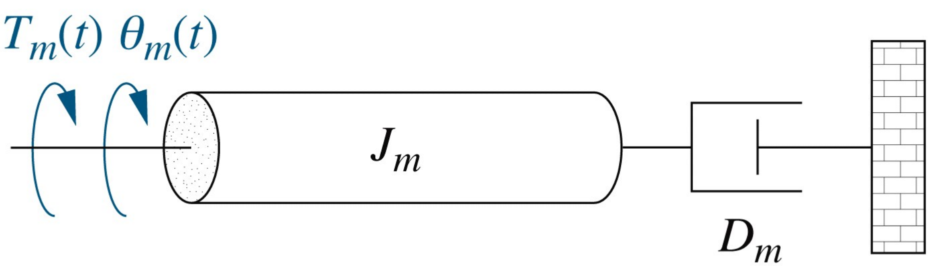
$$T_m(s) = K_t I_a(s) \qquad (2.147)$$

Electromechanical System Transfer Functions

- where T_m is the torque developed by the motor, and K_t is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of K_t is equal to the value of K_b.

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \qquad (2.149)$$

- Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to separate the input and output variables and obtain the transfer function, $\theta_m(s)/E_a(s)$.



$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

- If we assume that the armature inductance, L_a is small compared to the armature resistance, R_a, which is usual for a dc motor, Eq. (2.151) becomes

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s)$$

$$\boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}}$$

Now we need to evaluate the mechanical and electrical constants:

Let us first discuss the mechanical constants, J_m and D_m. Consider Figure 2.37, which shows a motor with inertia J_a and damping D_a at the armature driving a load consisting of inertia J_L and damping D_L. Assuming that all inertia and damping values shown are known, J_L and D_L can be reflected back to the armature as some equivalent inertia and damping to be added to J_a and D_a, respectively. Thus, the equivalent inertia, J_m, and equivalent damping, D_m, at the armature are

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \quad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

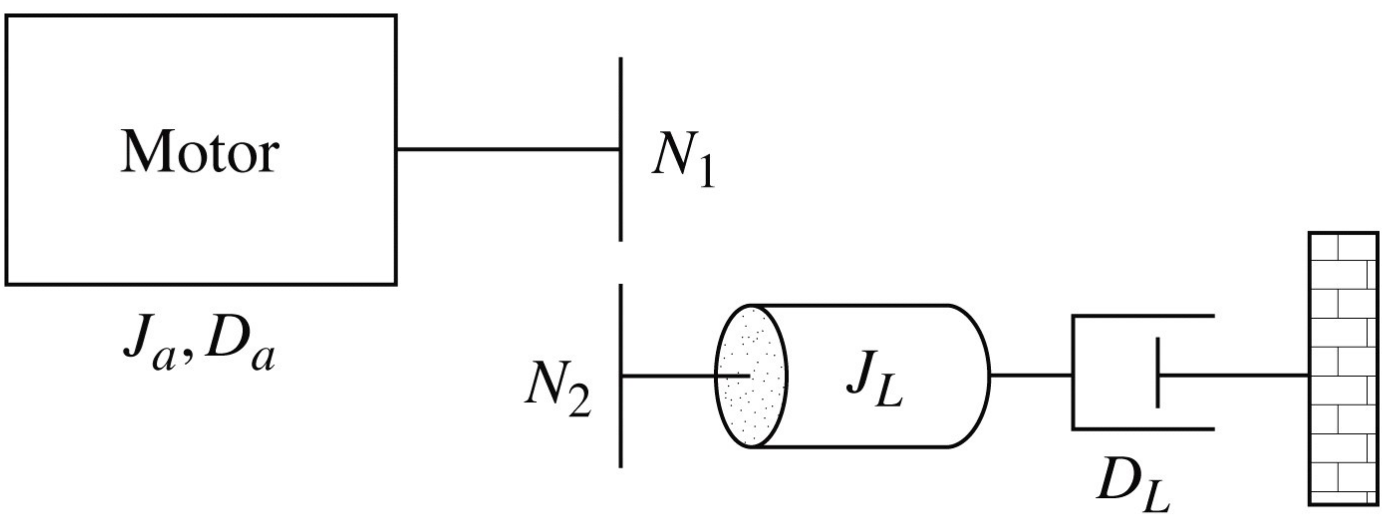


Figure 2.37

Fig. 2.37 DC motor driving a rotational mechanical load

Evaluation of the electrical constants:

The electrical constants can be obtained through a dynamometer test of the motor, where a dynamometer measures the torque and speed of a motor under the condition of a constant applied voltage. Let us first develop the relationships that dictate the use of a dynamometer.

From the armature loop equation:

$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s)$$

Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t)$$

When the motor is operating at steady state with a dc voltage input:

$$\frac{R_a}{K_t}T_m + K_b\omega_m = e_a \Longrightarrow \boxed{T_m = -\frac{K_bK_t}{R_a}\omega_m + \frac{K_t}{R_a}e_a} \quad \text{Eqn. (2.159)}$$

Equation (2.159) is a straight line, T_m vs. ω_m, and is shown in Figure 2.38. This plot is called the *torque-speed curve*. The torque axis intercept occurs when the angular velocity reaches zero. That value of torque is called the *stall torque*, T_{stall}. Thus,

$$T_{\text{stall}} = \frac{K_t}{R_a} e_a \tag{2.160}$$

The angular velocity occurring when the torque is zero is called the *no-load speed*, $\omega_{\text{no-load}}$. Thus,

$$\omega_{\text{no-load}} = \frac{e_a}{K_b} \tag{2.161}$$

The electrical constants of the motor's transfer function can now be found from Eqs. (2.160) and (2.161) as

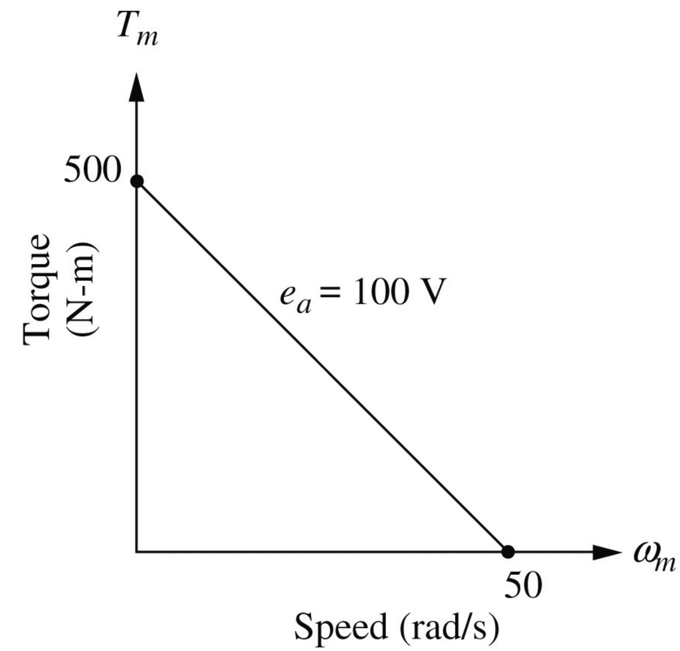
$$\boxed{\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a}} \tag{2.162}$$

$$\boxed{K_b = \frac{e_a}{\omega_{\text{no-load}}}} \tag{2.163}$$

The electrical constants, K_t/R_a and K_b, can be found from a dynamometer test of the motor, which would yield T_{stall} and $\omega_{\text{no-load}}$ for a given e_a.

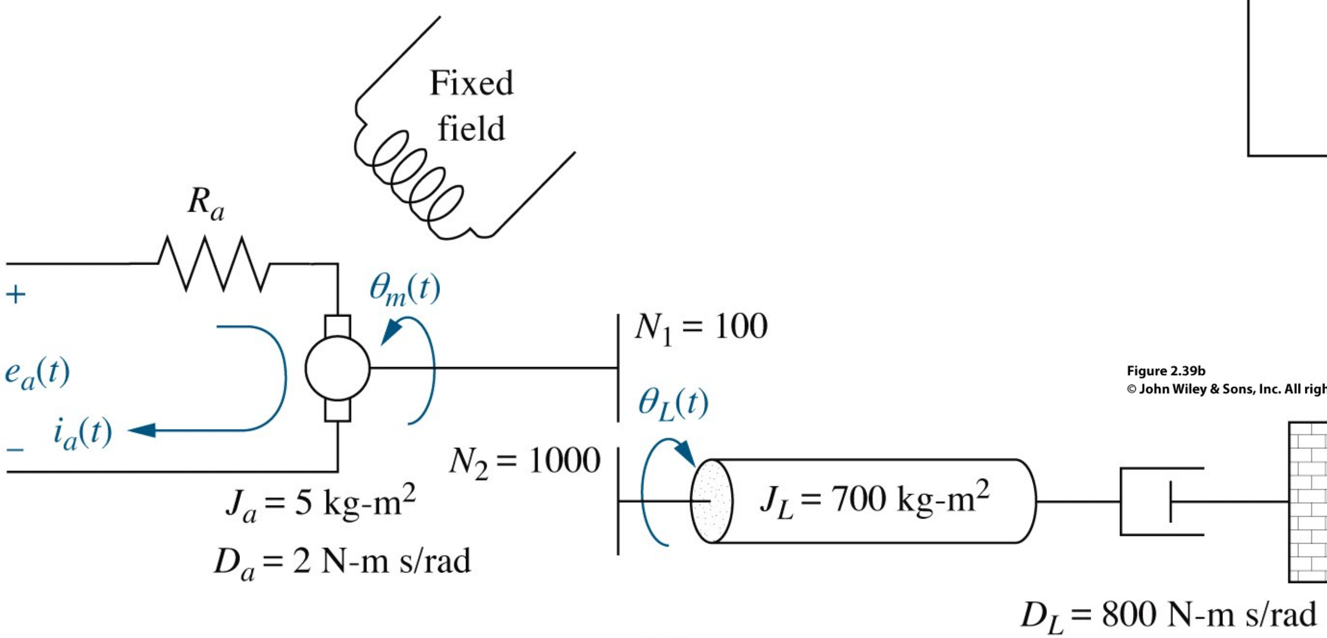
Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.



(b)

Figure 2.39b



(a)

Figure 2.39a

Transfer Function—DC Motor and Load…

SOLUTION: Begin by finding the mechanical constants, J_m and D_m, in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

$$J_m = J_a + J_L \left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \tag{2.164}$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L \left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \tag{2.165}$$

Now we will find the electrical constants, K_t/R_a and K_b. From the torque-speed curve of Figure 2.39(b),

$$T_{\text{stall}} = 500 \tag{2.166}$$

$$\omega_{\text{no-load}} = 50 \tag{2.167}$$

$$e_a = 100 \tag{2.168}$$

Transfer Function—DC Motor and Load…

Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a} = \frac{500}{100} = 5 \tag{2.169}$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{100}{50} = 2 \tag{2.170}$$

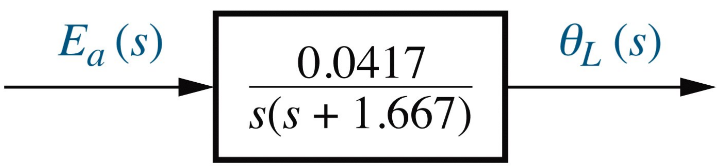
Substituting Eqs. (2.164), (2.165), (2.169), and (2.170) into Eq. (2.153) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s\left\{s + \frac{1}{12}[10 + (5)(2)]\right\}} = \frac{0.417}{s(s+1.667)} \tag{2.171}$$

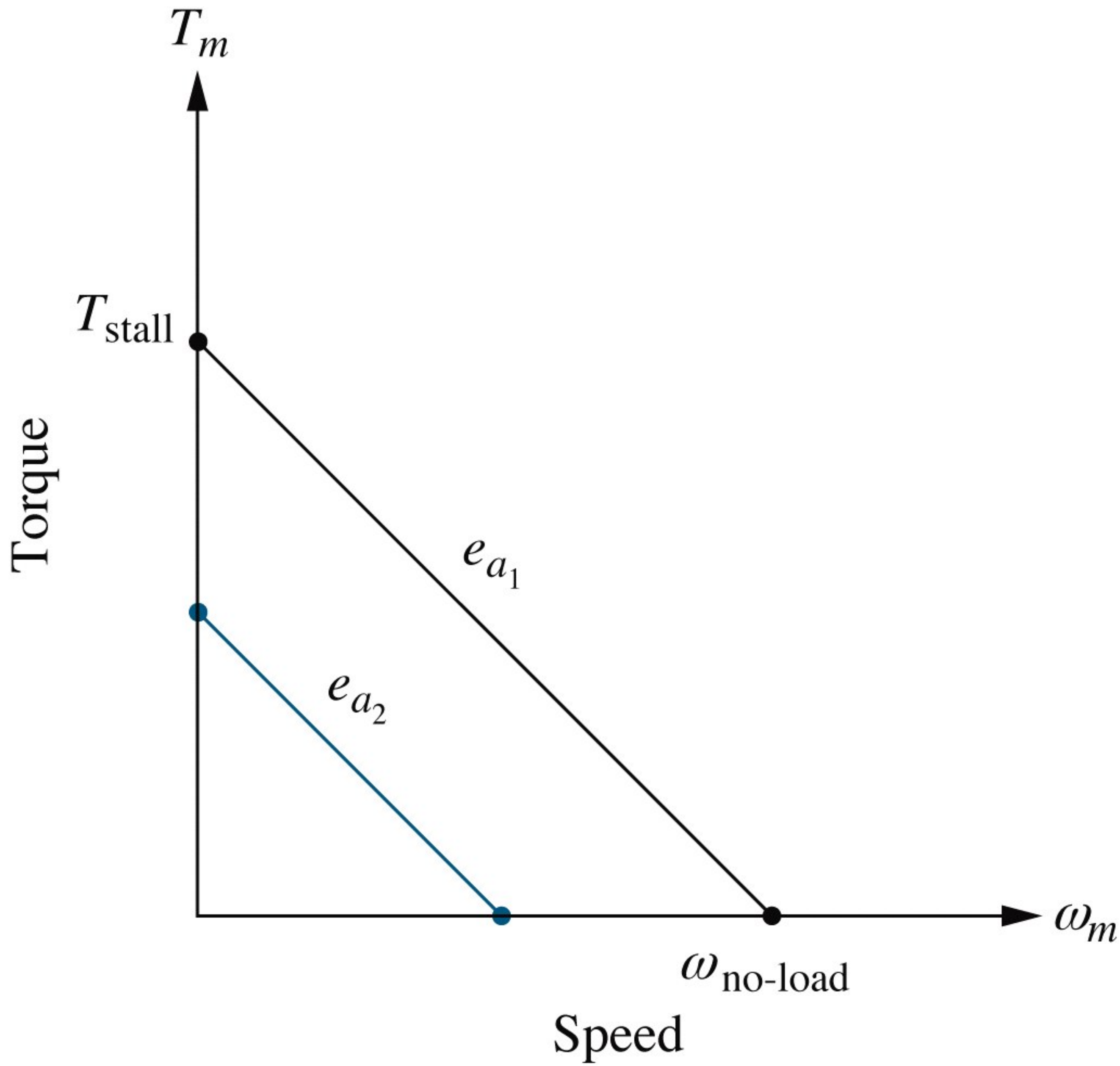
In order to find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)} \tag{2.172}$$

as shown in Figure 2.39(c).



(c)



T_m
T_{stall}
Torque
e_{a_1}
e_{a_2}
ω_m
$\omega_{\text{no-load}}$
Speed

Figure 2.38

PROBLEM: Find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, for the motor and load shown in Figure 2.40. The torque-speed curve is given by $T_m = -8\omega_m + 200$ when the input voltage is 100 volts.

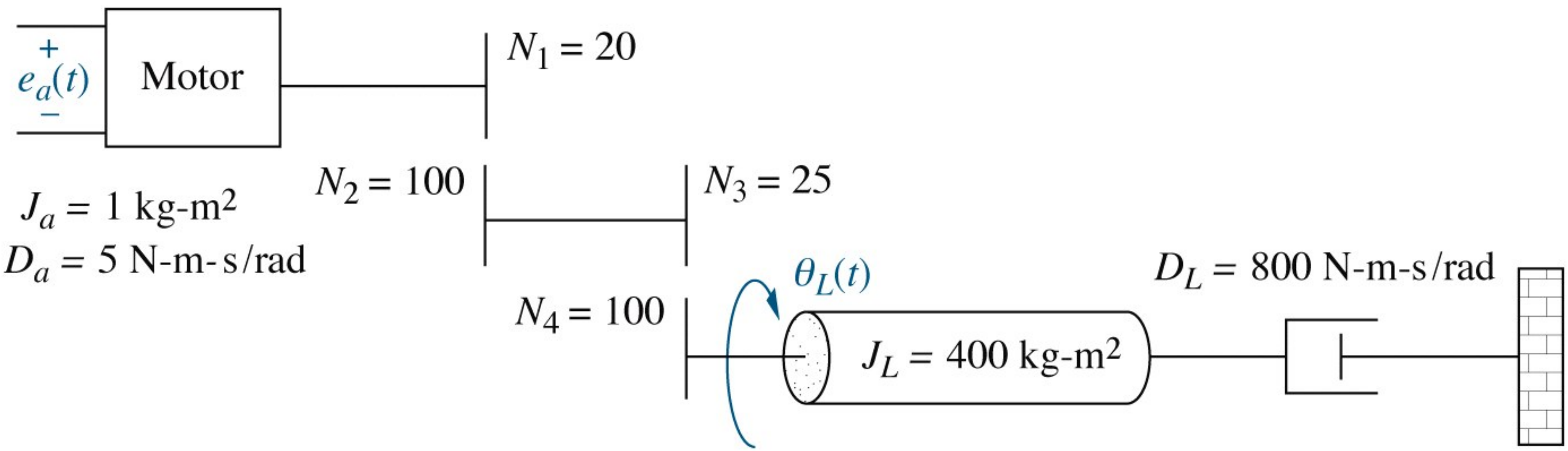


Figure 2.40

ANSWER: $$G(s) = \frac{1/20}{s[s + (15/2)]}$$

Electric Circuit Analogs

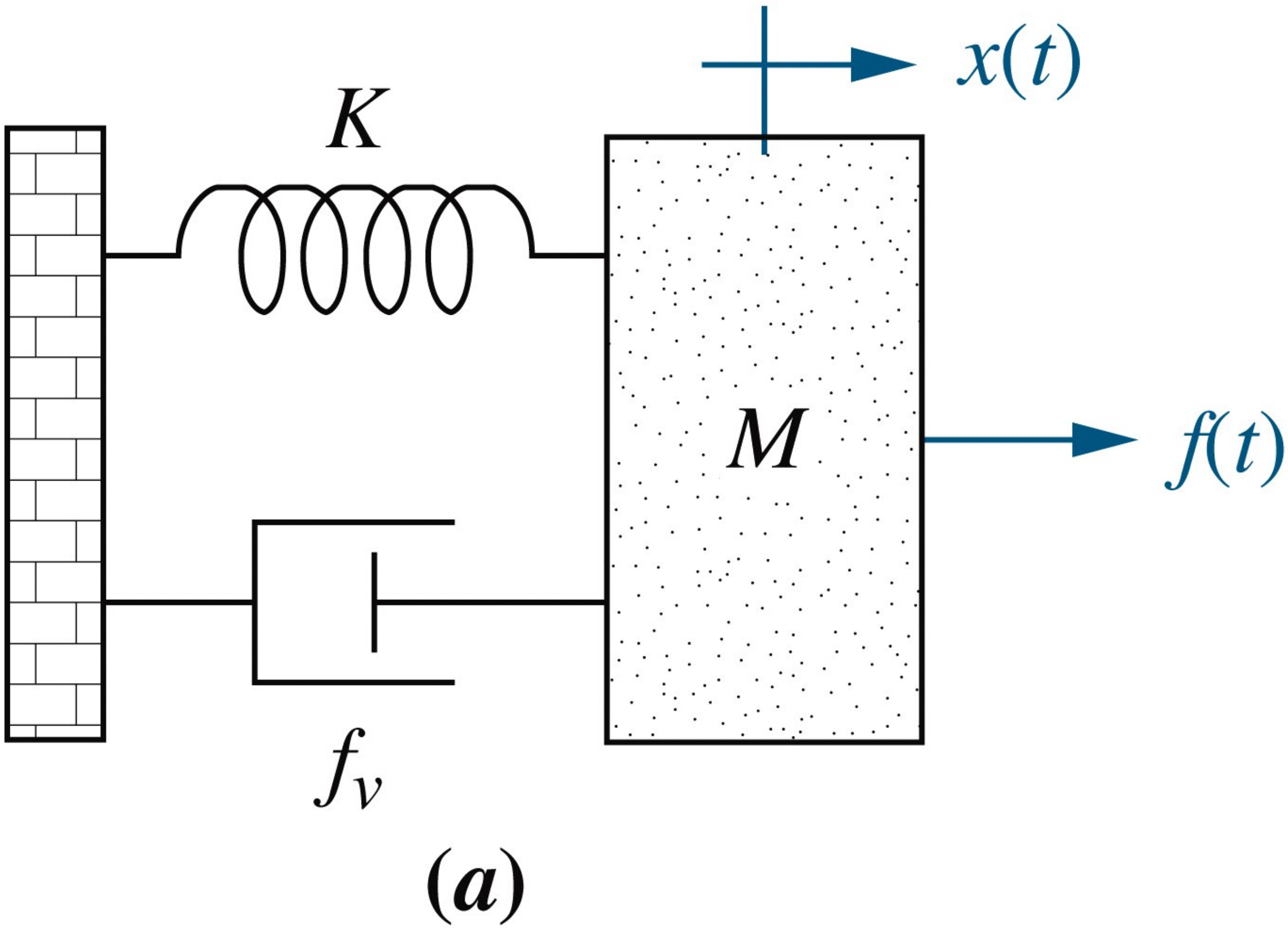


Figure 2.41a

Electric Circuit Analogs…

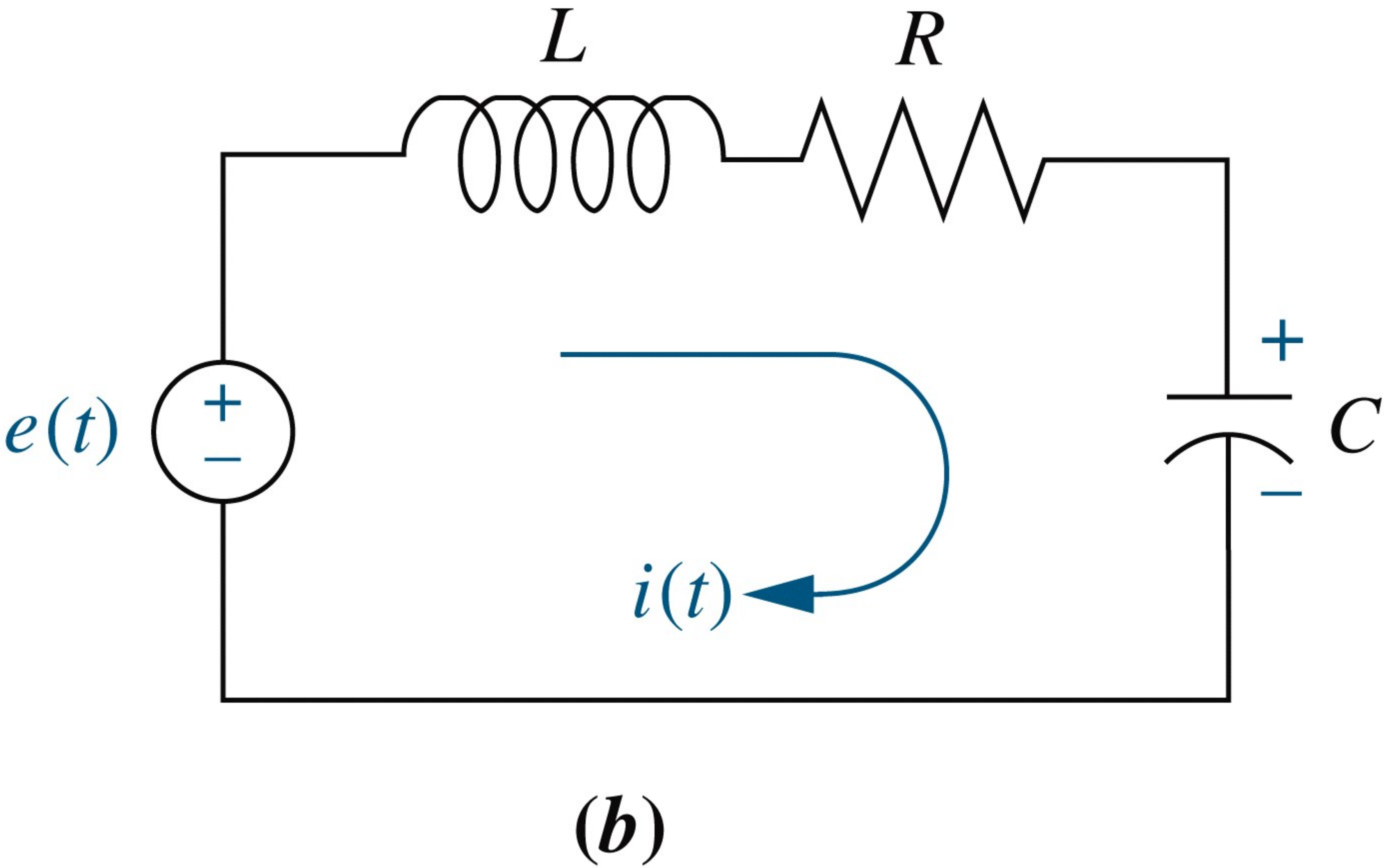
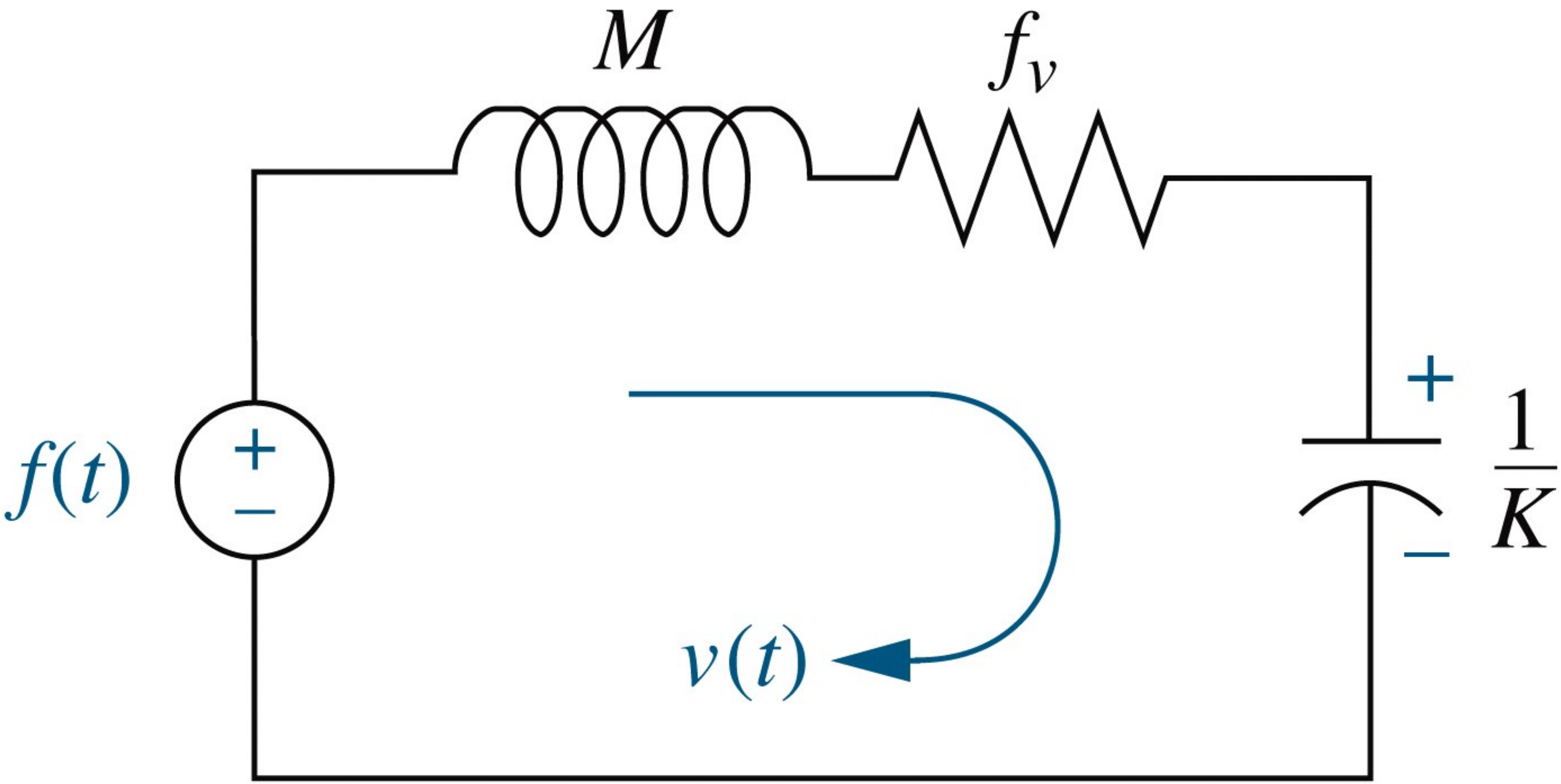


Figure 2.41b

Electric Circuit Analogs…



(c)

Figure 2.41c

mass $= M$	$\longrightarrow$	inductor	$=$	M henries
viscous damper $= f_v$	$\longrightarrow$	resistor	$=$	f_v ohms
spring $= K$	$\longrightarrow$	capacitor	$=$	$\frac{1}{K}$ farads
applied force $= f(t)$	$\longrightarrow$	voltage source	$=$	$f(t)$
velocity $= v(t)$	$\longrightarrow$	mesh current	$=$	$v(t)$

(*d*)

Figure 2.41d

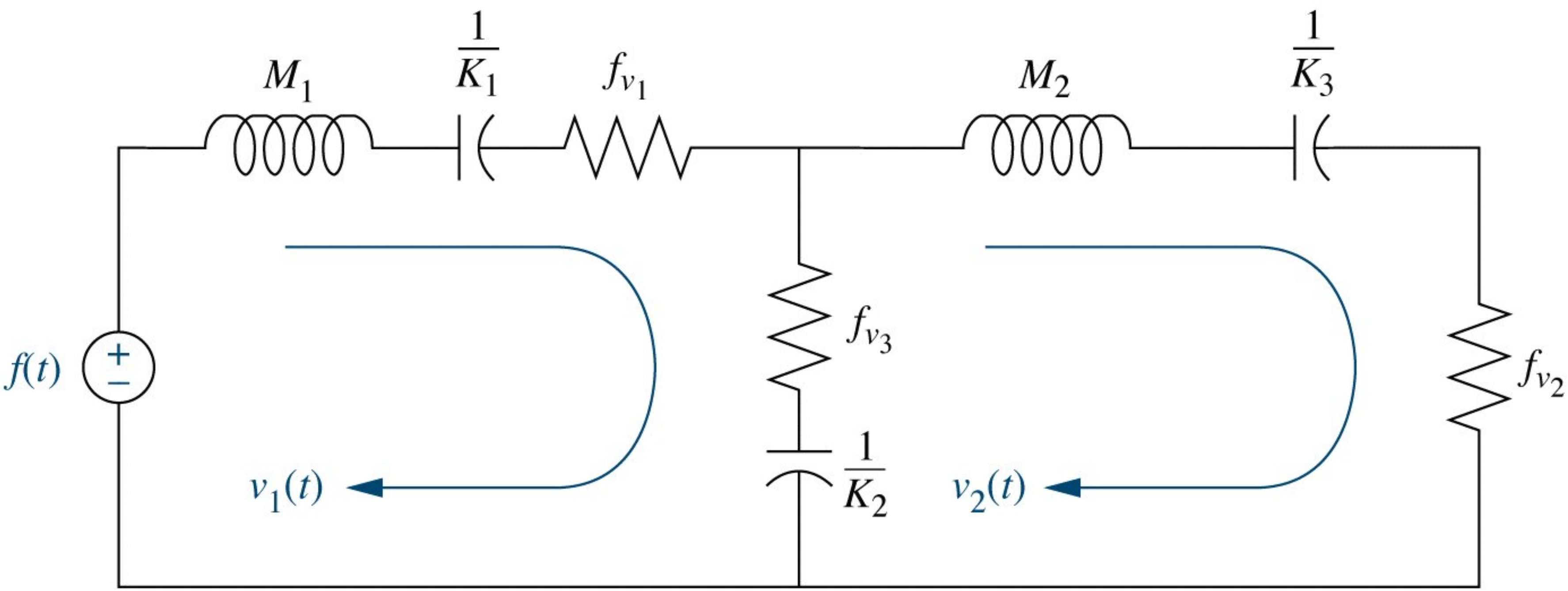


Figure 2.42

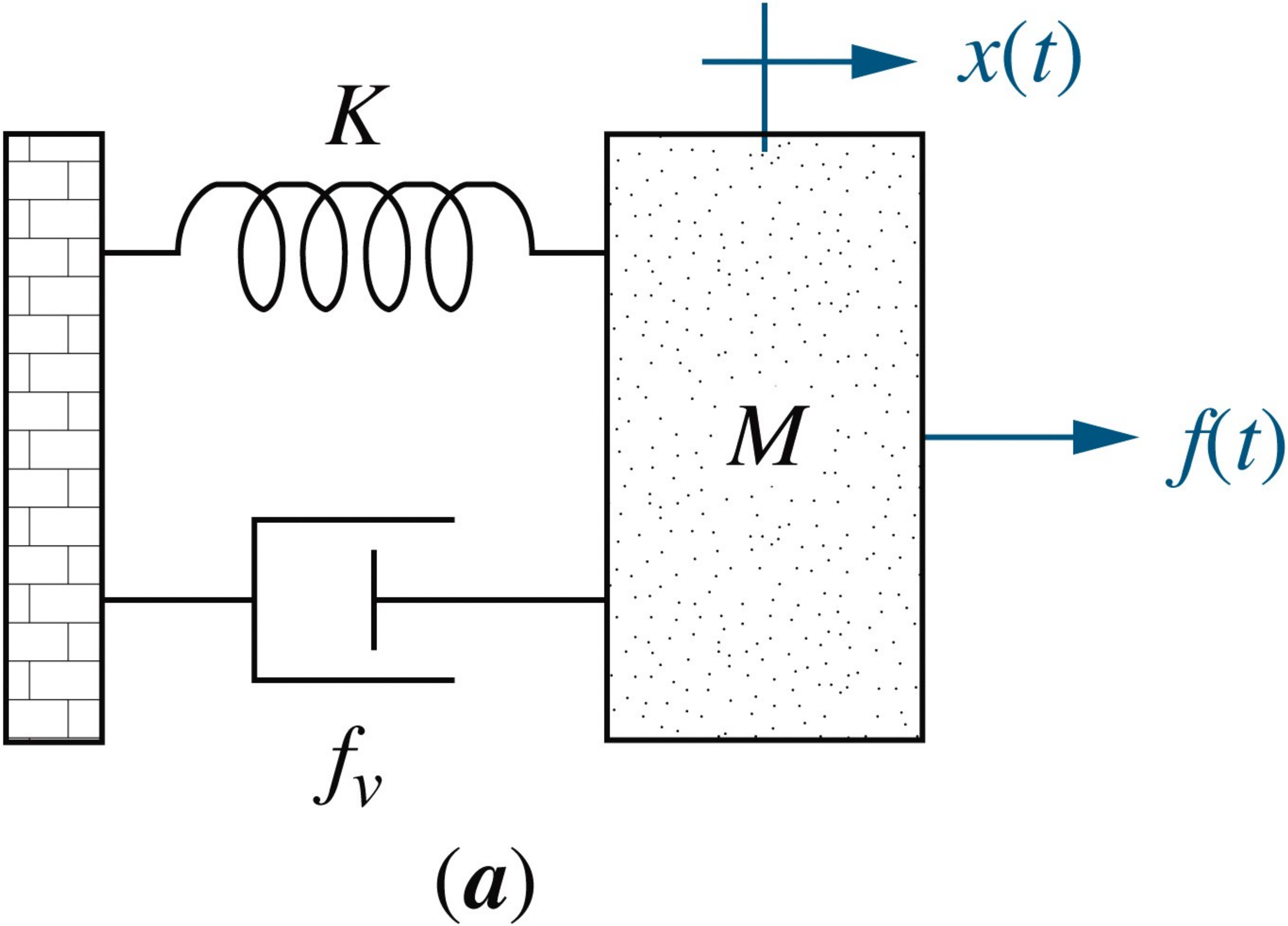


Figure 2.43a

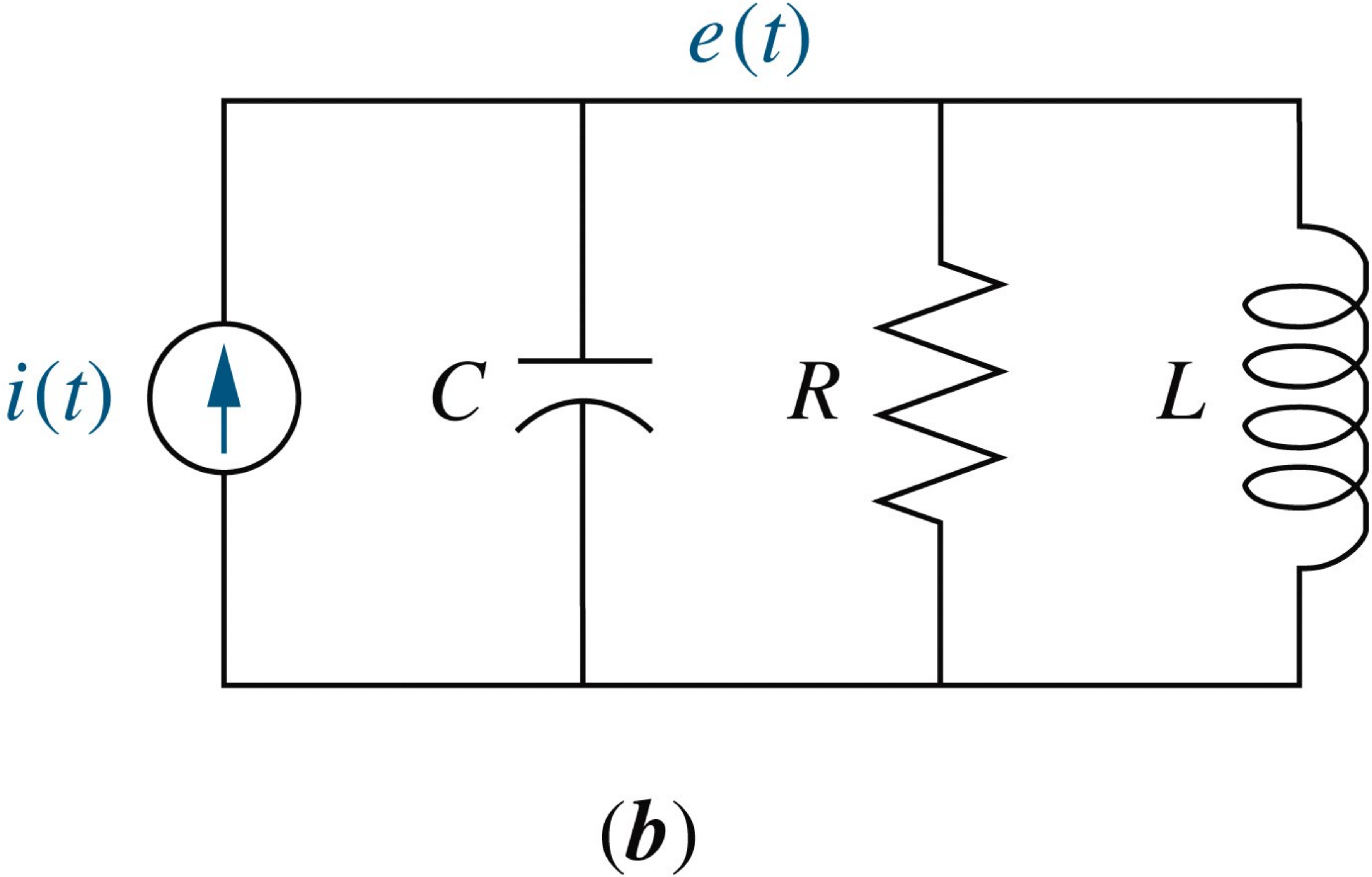


Figure 2.43b

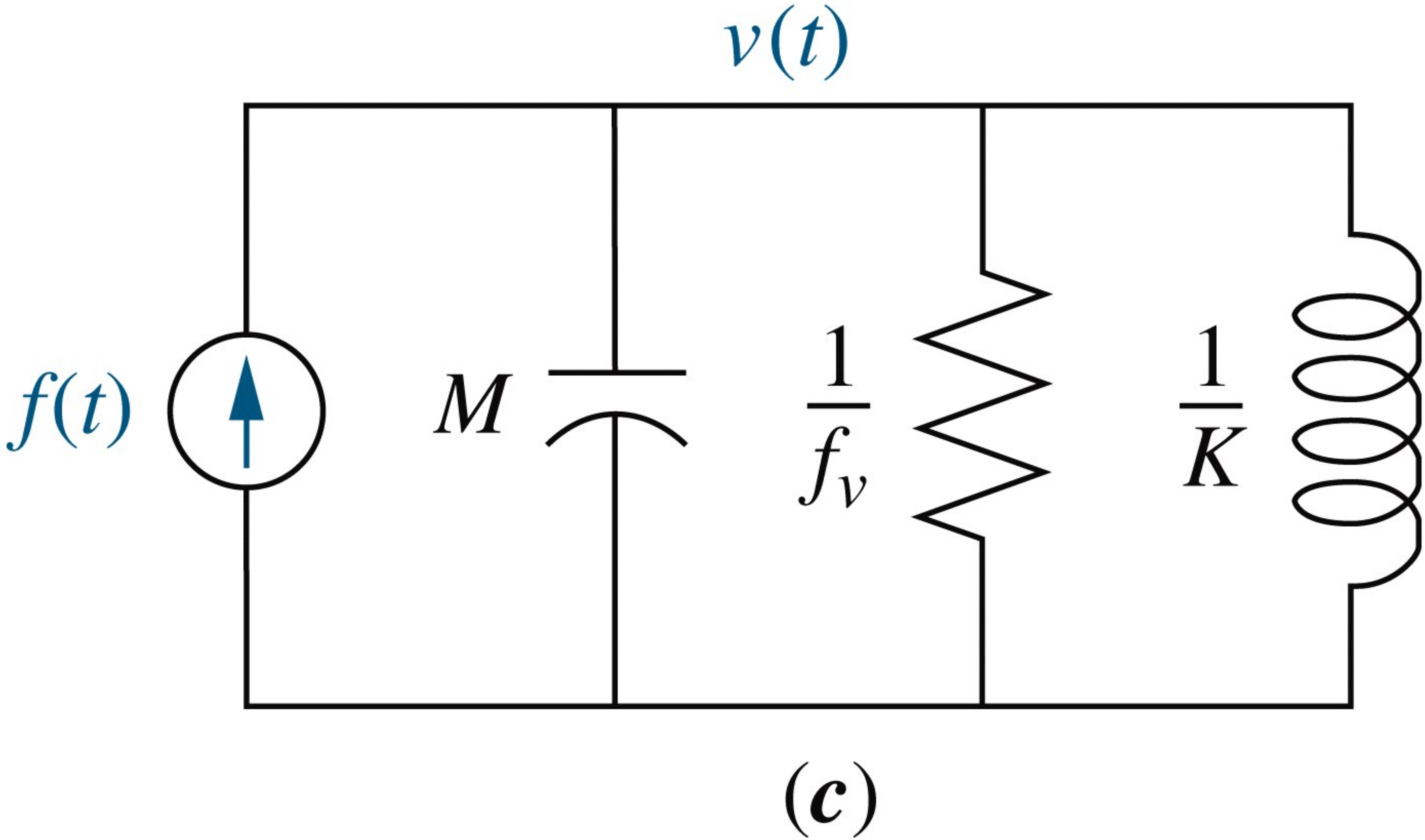


Figure 2.43c

$$\text{mass} = M \longrightarrow \text{capacitor} = M \text{ farads}$$

$$\text{viscous damper} = f_v \longrightarrow \text{resistor} = \frac{1}{f_v} \text{ ohms}$$

$$\text{spring} = K \longrightarrow \text{inductor} = \frac{1}{K} \text{ henries}$$

$$\text{applied force} = f(t) \longrightarrow \text{current source} = f(t)$$

$$\text{velocity} = v(t) \longrightarrow \text{node voltage} = v(t)$$

(***d***)

Figure 2.43d

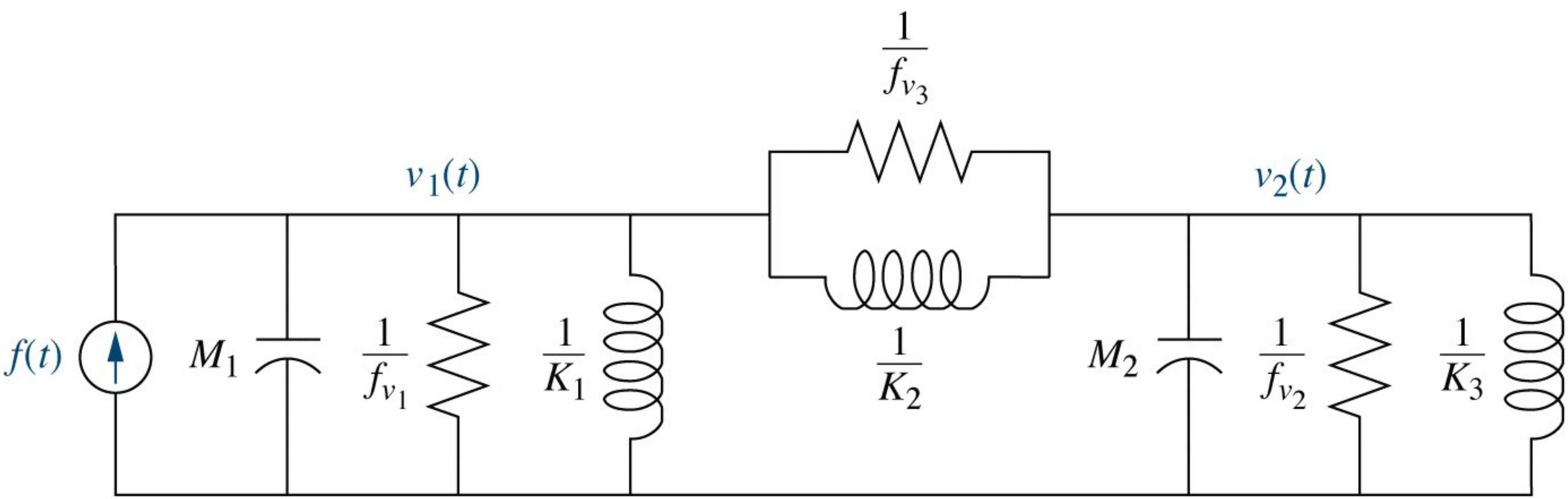

$f(t)$
$v_1(t)$
$v_2(t)$
M_1
$\frac{1}{f_{v_1}}$
$\frac{1}{K_1}$
$\frac{1}{f_{v_3}}$
$\frac{1}{K_2}$
M_2
$\frac{1}{f_{v_2}}$
$\frac{1}{K_3}$

Figure 2.44