Selected Problems – XII

Problem 1) Find the Laplace transform of each of the following functions:

a. $f(t) = t\,u(t)$

b. $f(t) = \cos \omega t$

c. $f(t) = t e^{-at} u(t)$, $u(t)$: unit step fn.

Solution. We shall employ the definition of Laplace transform, that is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt \triangleq F(s)$$

a.
$$F(s) = \int_0^\infty t\,u(t) e^{-st}\, dt$$

$$= \int_0^\infty \underbrace{t}_{u}\, \underbrace{e^{-st}\, dt}_{dv}$$

$du = dt$, $v = \dfrac{e^{-st}}{-s}$

$$\Rightarrow \quad F(s) = + \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, dt \quad , \quad Re\{s\} > 0$$

$$= 0 - 0 + \frac{e^{-st}}{s.(-s)}\Big|_0^\infty$$

$$= 0 - \frac{1}{s.(-s)}$$

$$= \frac{1}{s^2}$$

b.

$$F(s) = \int_0^\infty \cos\omega t \, e^{-st} \, dt$$

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\Rightarrow F(s) = \int_0^\infty \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) e^{-st} \, dt$$

$$= \frac{1}{2} \int_0^\infty \left[e^{-(s-j\omega)t} + e^{-(s+j\omega)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)} \Bigg|_0^\infty + \frac{e^{-(s+j\omega)t}}{-(s+j\omega)} \Bigg|_0^\infty \right], \quad Re\{s\} > 0$$

$$= \frac{1}{2} \left[0 - \frac{1}{-(s-j\omega)} + 0 - \frac{1}{-(s+j\omega)} \right]$$

$$= \frac{1}{2} \left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega} \right)$$

$$(s+j\omega) \qquad (s-j\omega)$$

$$= \frac{1}{2} \, \frac{s + \cancel{j\omega} + s - \cancel{j\omega}}{s^2 + \omega^2}$$

$$= \frac{s}{s^2 + \omega^2}$$

c.

$$F(s) = \int_0^\infty t e^{-ct} u(t) e^{-st} \, dt$$

$$= \int_0^\infty \underbrace{t}_{u} \, \underbrace{e^{-(s+c)t} \, dt}_{dv}$$

$$du = dt, \quad v = \frac{e^{-(s+c)t}}{-(s+c)}$$

$$= + \frac{e^{-(s+a)t}}{-(s+a)}\Bigg|_0^\infty - \int_0^\infty \frac{e^{-(s+a)t}}{-(s+a)}\,dt \quad , \quad Re\{s\} > -a$$

$$= 0 - 0 + \frac{e^{-(s+a)t}}{-(s+a)^2}\Bigg|_0^\infty$$

$$= 0 - \frac{1}{-(s+a)^2}$$

$$= \frac{1}{(s+a)^2}$$

Problem 2) Find the Laplace transform for (a) and (b).

a. $f(t) = \frac{d}{dt}\left(e^{-at}\sin\omega t\right)$

b. $f(t) = \int_{0^-}^{t} e^{-ax}\cos(\omega x)\,dx$

Solution. We employ the operational transform techniques to find the Laplace transforms:

a.
$$F(s) = s\,\mathcal{L}\{e^{-at}\sin\omega t\} - \left(e^{-at}\sin\omega t\right)\Big|_{t=0^-}$$

$$= s\,\frac{\omega}{(s+a)^2+\omega^2} - 1 \cdot 0$$

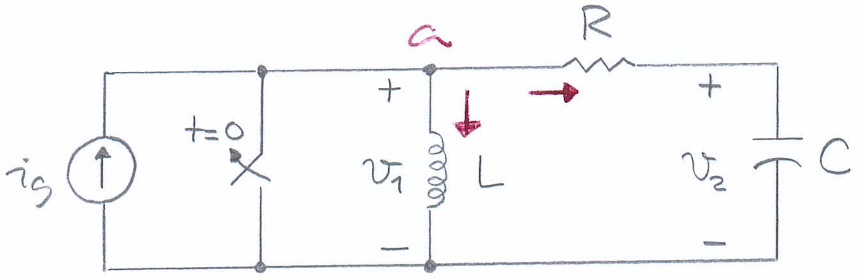
$$= \frac{\omega s}{(s+a)^2+\omega^2}$$

b.
$$F(s) = \frac{1}{s}\,\mathcal{L}\left\{e^{-at}\cos(\omega t)\right\}$$

$$= \frac{1}{s} \frac{s+a}{(s+a)^2+\omega^2}$$

$$= \frac{s+a}{s\left[(s+a)^2+\omega^2\right]}$$

Problem 3) There is no energy stored at the time the switch is opened in the circuit shown as



a. Derive the integrodifferential equations that govern the behavior of the node voltages v_1 and v_2.

b. Show that

$$V_2(s) = \frac{s\, I_g(s)}{C\left[s^2+(R/L)s+(1/LC)\right]}$$

Solution.

a. We have

$$i_L = \frac{1}{L}\int_0^t v_1(\tau)\,d\tau$$

$$i_c = C\,\frac{dv_2(t)}{dt} = \frac{v_1(t)-v_2(t)}{R}$$

Hence;

$$v_1(t) = v_2(t) + RC \frac{dv_2(t)}{dt} \qquad (1)$$

KCL at node a:

$$-i_s + i_L + i_c = 0$$

$$\Rightarrow -i_s + \frac{1}{L}\int_0^t v_1(\tau)\,d\tau + C\frac{dv_2(t)}{dt} = 0$$

$$\Rightarrow -i_s + \frac{1}{L}\int_0^t \left[v_2(\tau) + RC\frac{dv_2(\tau)}{dt}\right]d\tau$$

$$+ C\frac{dv_2(t)}{dt} = 0$$

$$\Rightarrow -i_s + \frac{1}{L}\int_0^t v_2(\tau)\,d\tau + \frac{RC}{L}\int_0^t \frac{dv_2(\tau)}{d\tau}\,d\tau$$

$$+ C\frac{dv_2(t)}{dt} = 0$$

$$\Rightarrow -i_s + \frac{1}{L}\int_0^t v_2(\tau)\,d\tau + \frac{RC}{L}v_2(t)$$

$$+ C\frac{dv_2(t)}{dt} = 0 \qquad (2)$$

Thus;

(1) and (2) are the integrodifferential equations governing $v_1(t)$ and $v_2(t)$

PS 12.5

b. Taking Laplace transform of both sides of (2) gives

$$-I_g(s) + \frac{1}{Ls} V_2(s) + \frac{RC}{L} V_2(s) + Cs\,V_2(s) = 0$$

$$\Rightarrow \quad V_2(s)\left(Cs + \frac{1}{Ls} + \frac{RC}{L}\right) = I_g(s)$$

$$\Rightarrow \quad V_2(s)\,\frac{LCs^2 + RCs + 1}{Ls} = I_g(s)$$

$$\Rightarrow \quad V_2(s) = \frac{Ls\,I_g(s)}{LCs^2 + RCs + 1}$$

$$= \frac{\cancel{L}s\,I_g(s)}{\cancel{L}C\left[s^2 + (R/L)s + (1/LC)\right]}$$

$$= \frac{s\,I_g(s)}{C\left[s^2 + (R/L)s + (1/LC)\right]}$$

Problem 4) The circuit parameters in the circuit drawn in Problem 3) are

$$R = 2500\,\Omega, \quad L = 500\,\text{mH}, \quad C = 0.5\,\mu\text{F}$$

if $i_g(t) = 15u(t)$ mA, then find $v_2(t)$.

Solution. We already know from the solution of

Problem 3 that

$$V_2(s) = \frac{s\, I_g(s)}{0.5 \cdot 10^{-6}\left[s^2 + \left(2500/500 \cdot 10^{-3}\right)s + \left(1/500 \cdot 10^{-3} \cdot 0.5 \cdot 10^{-6}\right)\right]}$$

where

$$I_g(s) = \mathcal{L}\{i_g(t)\} = \mathcal{L}\left\{15 \cdot 10^{-3} u(t)\right\} = \frac{15 \cdot 10^{-3}}{s}$$

Hence;

$$V_2(s) = \frac{\cancel{s} \cdot (15/\cancel{s}) \cdot 10^{-3}}{0.5 \cdot 10^{-6}\left(s^2 + 5000s + 4 \cdot 10^6\right)}$$

$$= \frac{3 \cdot 10^4}{s^2 + 5000s + 4 \cdot 10^6}$$

$$= \frac{3 \cdot 10^4}{(s+1000)\,(s+4000)}$$

$$= \frac{C_1}{s+1000} + \frac{C_2}{s+4000}$$

$$\Rightarrow \quad C_1 = (s+1000)V_2(s)\Big|_{s=-1000} = \frac{3 \cdot 10^4}{s+4000}\Big|_{s=-1000} = 10$$

$$\Rightarrow \quad C_2 = (s+4000)\,V_2(s)\Big|_{s=-4000} = \frac{3 \cdot 10^4}{s+1000}\Big|_{s=-4000} = -10$$

Thus ;

$$v_2(t) = \mathcal{L}^{-1}\{V_2(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{10}{s+1000} - \frac{10}{s+4000}\right\}$$

$$= \left[10e^{-1000t} - 10e^{-4000t}\right]u(t)$$

Problem 5) Find $f(t)$ for each of the following functions :

a. $$F(s) = \frac{100(s+1)}{s^2(s^2+2s+5)}$$

b. $$F(s) = \frac{40(s+2)}{s(s+1)^3}$$

Solution. We employ partial fraction expansion method to find $f(t)$:

a.

$$\frac{100(s+1)}{s^2\underbrace{(s^2+2s+5)}_{s_{1,2}=-1\pm j2}} = \frac{100(s+1)}{s^2(s+1-j2)(s+1+j2)}$$

$$= \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s+1-j2}$$

$$+ \frac{C_3^*}{s+1+j2}$$

$$\Rightarrow C_1 = \frac{100(s+1)}{s^2+2s+5}\Bigg|_{s=0} = \frac{100}{5} = 20$$

$$C_2 = \frac{d}{ds}\left[\frac{100(s+1)}{s^2+2s+5}\right]\Bigg|_{s=0} = 100\,\frac{1\cdot(s^2+2s+5)-(s+1)(2s+2)}{(s^2+2s+5)^2}\Bigg|_{s=0}$$

$$= \overset{4}{\cancel{100}}\,\frac{5-1\cdot 2}{\cancel{5\cdot 5}} = 12$$

$$C_3 = \frac{100(s+1)}{s^2(s+1+j2)}\Bigg|_{s=-1+j2} = \frac{\overset{50}{\cancel{100}}\,(\cancel{j2})}{(-1+j2)^2\,(\cancel{j4})}$$

$$= \frac{50}{(\sqrt{5}\,\angle 116.5651^\circ)^2} = 10\,\angle -233.1301^\circ = 10\,\angle 126.87^\circ$$

Hence;

$$f(t) = \mathcal{L}^{-1}\left\{\frac{20}{s^2} + \frac{12}{s} + \frac{10\,\angle 126.87^\circ}{s+1-j2} + \frac{10\,\angle -126.87^\circ}{s+1+j2}\right\}$$

$$= 20t + 12 + 10\,e^{j126.87}\,e^{-(1-j2)t} + 10\,e^{-j126.87}\,e^{-(1+j2)t}, \quad t>0$$

$$= 20t + 12 + 10\,e^{-t}\left[e^{j(2t+126.87^\circ)} + e^{-j(2t+126.87^\circ)}\right], \quad t>0$$

$$= \left[20t + 12 + 20e^{-t}\cos(2t+126.87^\circ)\right]u(t)$$

b.

$$\frac{40(s+2)}{s(s+1)^3} = \frac{C_1}{s} + \frac{C_2}{(s+1)^3} + \frac{C_3}{(s+1)^2} + \frac{C_4}{s+1}$$

$$\Rightarrow \quad C_1 = \left.\frac{40(s+2)}{(s+1)^3}\right|_{s=0} = \frac{40\cdot 2}{1^3} = 80$$

$$C_2 = \left.\frac{40(s+2)}{s}\right|_{s=-1} = \frac{40\cdot 1}{-1} = -40$$

$$C_3 = \frac{d}{ds}\left[\frac{40(s+2)}{s}\right]\Bigg|_{s=-1} = -\left.\frac{80}{s^2}\right|_{s=-1} = -80$$

$$C_4 = \frac{1}{2!}\,\frac{d^2}{ds^2}\left[\frac{40(s+2)}{s}\right]\Bigg|_{s=-1} = \left.\frac{80}{s^3}\right|_{s=-1} = -80$$

Hence;

$$f(t) = \mathcal{L}^{-1}\left\{\frac{80}{s} - \frac{40}{(s+1)^3} - \frac{80}{(s+1)^2} - \frac{80}{s+1}\right\}$$

$$= 80 - 20t^2e^{-t} - 80te^{-t} - 80e^{-t}, \quad t>0$$

$$= \left[80 - 20t^2e^{-t} - 80te^{-t} - 80e^{-t}\right]u(t)$$