$\mathcal{L}[e^{-at} \cdot f(t)] = F(s+a)$

1) $\mathcal{L}[t] = \frac{1}{s}$

$\mathcal{L}[t^n] = \frac{n!}{s^{n-1}}$

2) $u(t) \longrightarrow \frac{1}{s}$

$u(t-a) \rightarrow \frac{1}{s} \cdot e^{-as}$

3) $\sin wt \rightarrow \frac{w}{w^2+s^2}$

$t^2 g(t) \rightarrow \frac{d^2}{ds^2} G(s)$

$\cos wt \longrightarrow \frac{s}{s^2+w^2}$

PS-1

A2-2) Find the Laplace transform of $f(t)$ defined by

$$f(t)=0 \quad \text{for } t<0$$
$$=te^{-3t} \quad \text{for } t\geq 0$$

Eq. 2.6. $\longrightarrow$ $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}[e^{-at}.f(t)] = \int_0^\infty e^{-at}.f(t).e^{-st}\,dt = F(s+a)$$

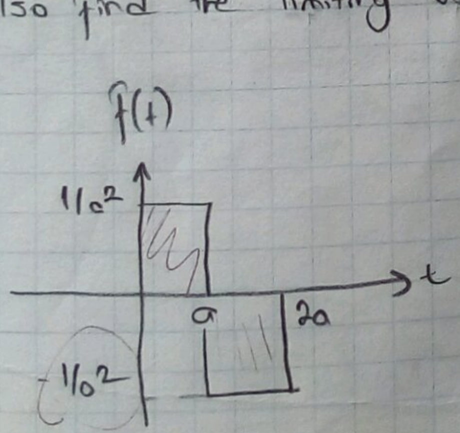
t,

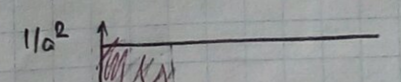
$$f(t) = t.e^{-3t} \longrightarrow +3$$

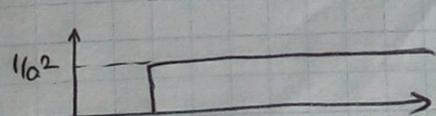
$$\mathcal{L}[t] = G(s) = \frac{1}{s^2} \quad (\text{LTP})$$

$$\mathcal{L}[t.e^{-3t}] = G(s+3) = \frac{1}{(s+3)^2}$$

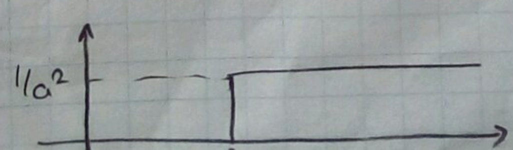
A2-4) Find the Laplace transform of $F(s)$ of the function $f(t)$ shown in the figure below where $f(t)=0$ for $t<0$ and $2a\leq t$. Also find the limiting value of $F(s)$ as a approaches zero.



① 

② 

-2x

③ 

$1/a^2$	$-1/a^2$	
0-a	a-2a	2a-∞

$f(t) = ① - 2\times ② + ③$

$$\frac{1}{a^2} - \frac{2}{a^2} + \frac{1}{a^2}$$

$$f(t) = \frac{1}{a^2}\,u(t) - \frac{2}{a^2}\cdot u(t-a) + \frac{1}{a^2}\cdot u(t-2a)$$

$$L\{f(t)\} = \frac{1}{a^2}\,L[u(t)] - \frac{2}{a^2}\cdot L[u(t-a)] + \frac{1}{a^2}\,L[u(t-2a)]$$

Laplace Transform Pairs

$$L[u(t)] = \frac{1}{s} \qquad L[u(t-a)] = \frac{1}{s}\,e^{-a.s}$$

$$f(s) = \frac{1}{a^2}\cdot\frac{1}{s} - \frac{2}{a^2}\cdot\frac{1}{s}\cdot e^{-as} + \frac{1}{a^2}\cdot\frac{1}{s}\,e^{-2as}$$

$$f(s) = \frac{1}{a^2.s}\left[1 - 2.e^{-as} + e^{-2as}\right]$$

indeterminate

$$\lim_{a\to 0} f(s)\;\frac{1-2.e^{-as}+e^{-2as}}{a^2.s} = \frac{0}{0} \quad \text{(l'Hôpital)}$$

$$\lim_{a\to 0}\frac{\frac{d}{da}\left(1-2e^{-as}+e^{-2as}\right)}{\frac{d}{da}\left(a^2.s\right)}$$

$$= \lim_{a\to 0}\frac{2.\cancel{s}.e^{-as} - 2\cancel{s}e^{-2as}}{2a\cancel{s}} = \lim_{a\to 0}\frac{\cancel{2}.e^{-as} - \cancel{2}.e^{-2as}}{2a\cancel{2}}$$

$$= \frac{0}{0}$$

$$= \lim_{a\to 0}\frac{\frac{d}{da}\left(\;.e^{-as} - e^{-2as}\right)}{\frac{d}{da}\left(\;a\right)} = \lim_{a\to 0}\frac{-s.e^{-as}+2s.e^{-as}}{1}$$

$$= -s+2s = \boxed{s}$$

$$F(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s+2}{s^2+2s+2}$$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{\overbrace{1}^{\omega}}{(s+1)^2 + \underbrace{1}_{\omega}{}^2} - \frac{1}{2} \cdot \frac{s+1}{(s+1)^2 + \underbrace{1}_{\omega}{}^2}$$

$$f(t) = \frac{1}{2} u(t) - \frac{1}{2} \cdot e^{-t} \cdot \sin t - \frac{1}{2} \cdot e^{-t} \cdot \cos t$$

A2-7) Find the Laplace transform of $f(t)$ defined by

$$f(t) = 0 \quad \text{for } t<0$$
$$= t^2 \cdot \sin\omega t \quad \text{for } t\geq 0$$

$$LTP \rightarrow \mathcal{L}[\underbrace{\sin\omega t}_{g(t)}] = \underbrace{\frac{\omega}{\omega^2+s^2}}_{G(s)}$$

$$\mathcal{L}[t^2 \cdot g(t)] = \frac{d^2}{ds^2}G(s)$$

complex differentiation theorem

$$\mathcal{L}[t^2 \cdot g(t)] = \frac{d^2}{ds^2}\left[\frac{\omega}{\omega^2+s^2}\right] = \boxed{\frac{-2\omega^2+6\omega s^2}{(s^2+\omega^2)^3}}$$

A2-11) Find the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{1}{s(s^2+2s+2)}$$

$$= \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+2s+2}$$

$$a_1(s^2+2s+2) + (a_2 s + a_3)s = 1$$

$a_1 + a_2 = 0$

$2a_1 + a_3 = 0$

$2a_1 = 1$

$$a_1 = 1/2 \qquad a_2 = -1/2 \qquad a_3 = -1$$

$$F(s) = \frac{1}{2}\cdot\frac{1}{s} + \underbrace{\frac{-1/2\, s - 1}{s^2+2s+2}}$$

$$-\frac{1}{2}\ \frac{s+1}{s^2+2s+2}$$

A2-12) Obtain the inverse Laplace transform of

$$f(s) = \frac{5(s+2)}{s^2(s+1).(s+3)}$$ partial-fractions decomposition

$$f(s) = \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{a_1}{(s+1)} + \frac{a_2}{(s+3)}$$

$$a_1 = \left.\frac{5(s+2)}{s^2.(s+3)}\right|_{s=-1} = \frac{5}{2}$$

$$t^n \rightarrow \frac{n!}{s^{n+1}}$$

$$a_2 = \left.\frac{5(s+2)}{s^2.(s+1)}\right|_{s=-3} = \frac{5}{18}$$

$$b_2 = \left.\frac{5.(s+2)}{(s+1).(s+3)}\right|_{s=0} = \frac{10}{3}$$

$$b_1 = \left.\frac{d}{ds}\left[\frac{5(s+2)}{(s+1).(s+3)}\right]\right|_{s=0}$$

$$b_1 = \left.\frac{5(s+1).(s+3) - 5(s+2).(2s+4)}{(s+1)^2.(s+3)^2}\right|_{s=0} = -\frac{25}{9}$$

$$f(s) = -\frac{25}{9}.\overset{1}{\frac{1}{s}} + \frac{10}{3}.\overset{t}{\frac{1}{s^2}} + \frac{5}{2}.\frac{1}{(s+1)} + \frac{5}{18}.\frac{1}{(s+3)}$$

$$\boxed{f(t) = -\frac{25}{9}u(t) + \frac{10}{3}.t\,u(t) + \frac{5}{2}.e^{-t}u(t) + \frac{5}{18}.e^{-3t}u(t), \text{ for } t \geq 0}$$

$$\left[\frac{1}{s}\right] \rightarrow u(t) \qquad \left[\frac{1}{s+1}\right] \rightarrow u(t).e^{-t}$$

A2-17) Solve the following differential equation:

$$\ddot{x} + 2\dot{x} + 10x = t^2, \quad x(0)=0 \quad \dot{x}(0)=0$$

$$s^2 X(s) + 2sX(s) + 10X(s) = \frac{2!}{s^{(2+1)}} = \frac{2}{s^3}$$

$$t^n = \frac{n!}{s^{n+1}}$$

$$X(s) = \frac{2}{s^3(s^2+2s+10)}$$

Partial fraction but difficult since we got triple pole

Matlab

$$\begin{array}{ccccccc} & s^5 & s^4 & s^3 & s^2 & s^1 & s^0 \end{array}$$

```
num = [0 0 0 0 0 2];
den = [1 2 10 0 0 0];

[r,p,k] = residue(num, den);
```

r → residue, p → poles, k → polynomial = 0

r = 0.0060 - 0.0087i
0.0060 + 0.0087i
-0.0120
-0.0400
0.2000

P = -1 + 3i
-1 - 3i
0
0
0

k = []

$$F(s) = \frac{0.0060-0.0087i}{\underset{(s+1)+3j}{s+1-3j}} + \frac{0.0060+0.0087j}{\underset{(s+1)-3j}{s+1+3j}} + \frac{-0.012}{s} + \frac{-0.0400}{s^2} + \frac{0.2000}{s^3}$$

$$X(s) = \frac{0.012(s+1) + 0.0522}{(s+1)^2 + 3^2} - \frac{0.012}{s} - \frac{0.04}{s^2} + \frac{0.2}{s^3}$$

?

$(2! \times 0.1)$

$$x(t) = -0.012 - 0.04t + \underline{\underline{0.1}}t^2$$

?

$$\frac{0.012(s+1)}{(s+1)^2 + \underset{\omega}{3^2}} \qquad \frac{0.0522}{(s+1)^2 + 3^2}$$

3 ? $\longrightarrow$ 0.0522

$$\cos\omega t \xrightarrow{L} \frac{s}{s^2+\omega^2} \qquad ? \times \sin\omega t \xrightarrow{L} \frac{\omega}{s^2+\omega^2}$$

$$e^{-t} \cdot 0.012 \cdot \cos 3t \qquad e^{-t} \cdot 0.0174 \cdot \sin 3t$$

?

$$x(t) = e^{-t} \cdot 0.012 \cdot \cos 3t + e^{-t} \cdot 0.0174 \cdot \sin 3t - 0.012 - 0.04t + \underline{0.1}t^2$$