Problem 1) a. Write the transfer function for a 4^{th}-order Butterworth low-pass filter (prototype).

b. Write the transfer function for the scaled filter with a cutoff frequency of 1000 Hz.

c. Calculate the gain in decibels at 4000 Hz.

Solution.

a. We find the roots of the polynomial

$$1+(-s)^{2.4}=0$$

$$\Rightarrow \quad s^8=-1$$
$$=e^{j(\pi+2\pi n)}$$

Hence;

$$s=e^{j\left(\frac{\pi+2\pi n}{8}\right)} \quad , n=0,1,\dots,7$$

$s_0=e^{j\pi/8}=0.9239+j0.3827$ RHP

$s_1=e^{j3\pi/8}=0.3827+j0.9239$ RHP

$s_2=e^{j5\pi/8}=-0.3827+j0.9239$ LHP

$s_3=e^{j7\pi/8}=-0.9239+j0.3827$ LHP

$s_4=e^{j9\pi/8}=-0.9239-j0.3827$ LHP

$s_5=e^{j11\pi/8}=-0.3827-j0.9239$ LHP

$s_6=e^{j13\pi/8}=0.3827-j0.9239$ RHP

$s_7=e^{j15\pi/8}=0.9239-j0.3827$ RHP

use thes LHP poles to build $H(s)$

-we first calculate

$$(s+0.3827-j0.9239)(s+0.3827+j0.9239)$$
$$=s^2+0.7654+1.0001$$

-then we calculate

$$(s+0.9239-j0.3827)(s+0.9239+j0.3827)$$

$$= s^2+1.8478+1.0001$$

Therefore ;

$$H(s) = \frac{1}{(s^2+0.7654s+1.0001)(s^2+1.8478s+1.0001)}$$

b. We have

$$\omega_c = 2\pi \cdot 1000 \quad \Rightarrow \quad k_p = 2000\pi$$

– then we replace s by $s/2000\pi$ in $H(s)$ to get a magnitude and frequency scaled transfer fn. as

$$H'(s) = \frac{1}{\left[\left(\frac{s}{2000\pi}\right)^2+0.7654\left(\frac{s}{2000\pi}\right)+1.0001\right]\left[\left(\frac{s}{2000\pi}\right)^2+1.8478\left(\frac{s}{2000\pi}\right)+1.0001\right]}$$

$$= \frac{(4\cdot 10^6\pi^2)^2}{(s^2+4809.2s+4\cdot 10^6\pi^2)(s^2+11610s+4\cdot 10^6\pi^2)}$$

c.

$$H'(j8000\pi) = \frac{16}{(-60+j12.24)(-60+j29.568)} \quad , \quad \omega = 2\pi \cdot 4000 = 8000\pi$$

$$|H'(j8000\pi)| = \frac{16}{61.24\cdot 66.89} = 3.91\cdot 10^{-3}$$

Hence ;

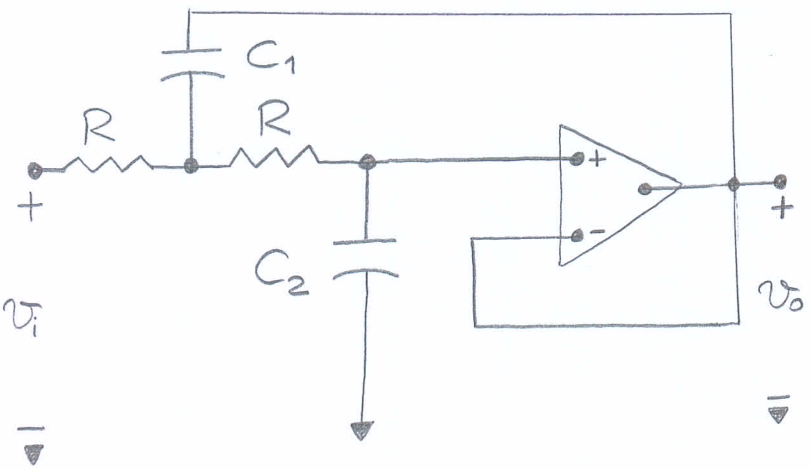
$$20\log_{10}|H'(j8000\pi)| = -48.16\ \text{dB}$$

PS6.2

Problem 2) a. Using 2kΩ resistors and ideal opamps, design a 4^{th}-order low-pass Butterworth filter with a cutoff frequency of 1000 Hz. The gain in the passband is one.

b. Construct the circuit diagram and label all component values.

Solution. The prototype 4^{th}-order low-pass Butterworth filter is a cascade of two 2^{nd}-order filters:



"2^{nd}-order filter circuit"

-we shall easily derive that

$$\frac{V_o}{V_i} = H(s) = \frac{\frac{1}{R^2C_1C_2}}{s^2 + \frac{2}{RC_1}s + \frac{1}{R^2C_1C_2}}$$

-for a prototype filter, we set $R = 1\,\Omega$

1^{st} stage : $s^2 + 0.765s + 1$

$$\frac{1}{C_1C_2} = 1 \quad , \quad \frac{2}{C_1} = 0.765 \quad \Rightarrow \quad C_1 = 2.6144 \text{ F}$$

$$\Rightarrow \; C_2 = \frac{1}{2.6144} = 0.3825 \text{ F}$$

2^{nd} stage : $s^2 + 1.848s + 1$

$$\frac{1}{C_3C_4} = 1 \quad , \quad \frac{2}{C_3} = 1.848 \quad \Rightarrow \quad C_3 = 1.0823$$

$$\Rightarrow \quad C_4 = \frac{1}{1.0823} = 0.9240$$

- we have $\quad k_f = 2000\pi \, , \quad k_m = 2000$

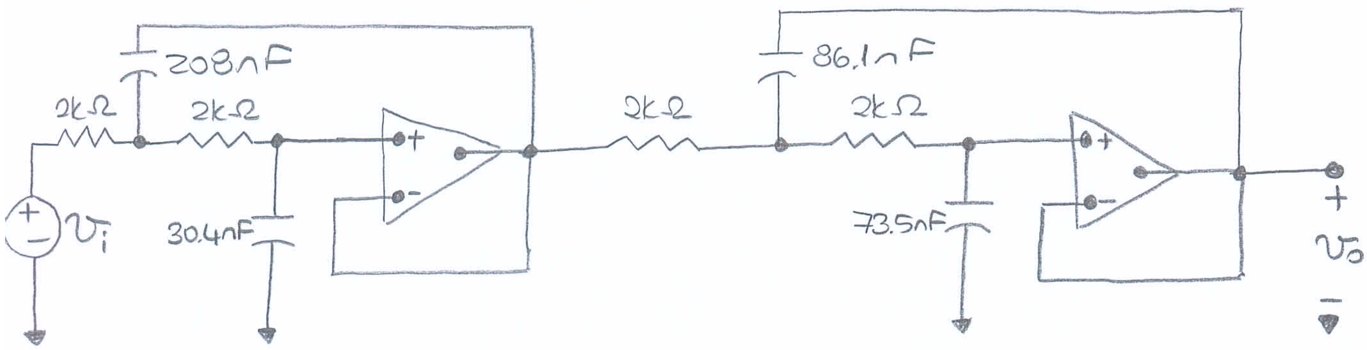
$$C_1' = \frac{2.6144}{2000\pi \cdot 2000} = 208\,nF$$

$$C_2' = \frac{0.3825}{2000\pi \cdot 2000} = 30.4\,nF$$

$$C_3' = \frac{1.0823}{2000\pi \cdot 2000} = 86.1\,nF$$

$$C_4' = \frac{0.9240}{2000\pi \cdot 2000} = 73.5\,nF$$

b.



4^{th}-order low-pass Butterworth filter

Problem 3) a. Using 25nF capacitors and ideal op amps, design a high-pass unity-gain Butterworth filter with a cutoff frequency of 5kHz and a gain of at least −25dB at 1kHz.

b. Draw a circuit diagram of the filter and label all component values.

Solution.

a. $\omega_c = 2\pi \cdot 5000 = 10000\pi \Rightarrow k_f = 10000\pi$

– we first determine the order of the filter:

$$|H(j\omega)| = \frac{(\omega/\omega_c)^n}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

$$-25\text{ dB} = 20\log_{10}|H(j2000\pi)|$$

$$\Rightarrow |H(j2000\pi)| = 10^{-1.25} = 0.0562$$

$$\Rightarrow \frac{(2000\pi/1000\pi)^n}{\sqrt{1+(2000\pi/10000\pi)^{2n}}} = 0.0562$$

$$\Rightarrow 1+(0.2)^{2n} = 316.2278\cdot(0.2)^{2n}$$

$$\Rightarrow (0.2)^{2n} = 0.0032 \Rightarrow 2n = \frac{\log 0.0032}{\log 0.2}$$

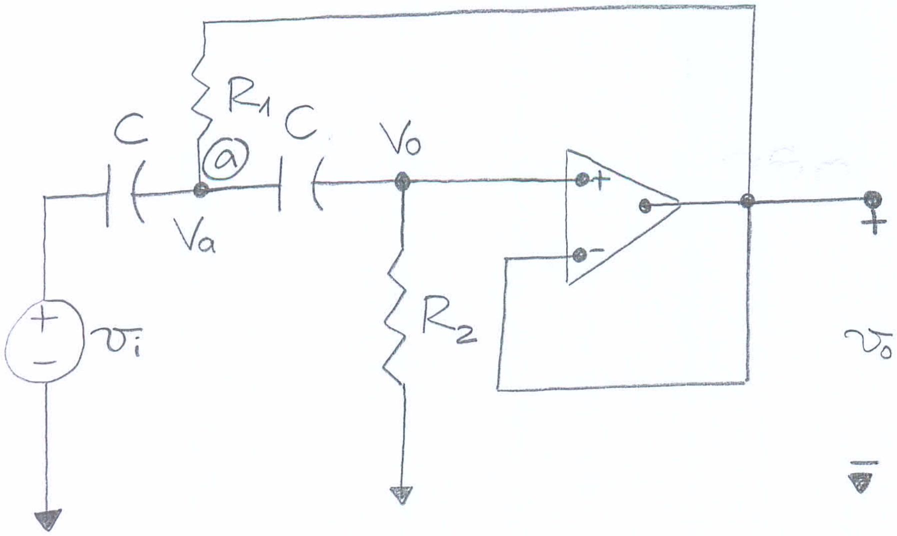
$$\Rightarrow n = 1.7874$$

$\therefore$ we choose the upper closest integer for n

$\Rightarrow$ a 2^{nd} order high-pass Butterworth filter is required

PS 6.5

-we have



$$\frac{V_o}{V_i} = H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

"2nd-order high-pass Butterworth filter transfer fn."

KCL at non-inverting input of the op amp:

$$\frac{V_o}{R_2} + \frac{V_o - V_a}{(1/sC)} = 0 \Rightarrow \left(\frac{1}{R_2} + sC\right)V_o = sC\,V_a$$

$$\Rightarrow V_a = \frac{1+sR_2C}{sR_2C}$$

KCL at node a:

$$\frac{V_a - V_i}{(1/sC)} + \frac{V_a - V_o}{(1/sC)} + \frac{V_a - V_o}{R_1} = 0$$

$$\Rightarrow \left(2sC + \frac{1}{R_1}\right)V_a - sC\,V_i - \left(sC + \frac{1}{R_1}\right)V_o = 0$$

$$\Rightarrow (2sR_1C+1)V_a - sR_1C\,V_i - (sR_1C+1)V_o = 0$$

$$\Rightarrow (2sR_1C+1)\frac{(1+sR_2C)}{sR_2C}V_o - sR_1C\,V_i - (sR_1C+1)V_o = 0$$

$$\Rightarrow \left[\frac{(2sR_1C+1)(sR_2C+1)}{sR_2C} - (sR_1C+1) \right] V_o = sR_1 C V_i$$

$$\Rightarrow \left[(2sR_1C+1)(sR_2C+1) - sR_2C(sR_1C+1) \right] V_o = sR_2C \cdot sR_1C \cdot V_i$$

$$\Rightarrow (s^2R_1R_2C^2 + 2sR_1C + 1) V_o = s^2 R_1R_2C^2 V_i$$

$$\Rightarrow \quad \frac{V_o}{V_i} \triangleq H(s) = \frac{(1/R_1R_2C^2)\, s^2}{s^2 + \frac{2}{R_2C} s + \frac{1}{R_1R_2C^2}}$$

- let us choose $C = 1\,F$, then

$$\frac{2}{R_2} = \sqrt{2} \quad \Rightarrow \quad R_2 = \sqrt{2}\,\Omega$$

$$\frac{1}{R_1R_2} = 1 \quad \Rightarrow \quad R_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\,\Omega$$

- for the magnitude scaling factor, we have

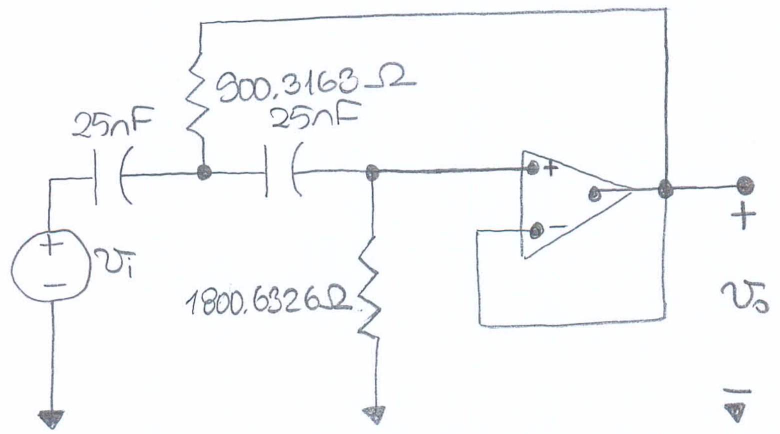
$$25.10^{-9} = \frac{1}{k_m \cdot 10^4 \pi} \quad \Rightarrow \quad k_m = 1273.2$$

Therefore;

$$R_1 = \frac{\sqrt{2}}{2} \cdot 1273.2 = 900.3163\,\Omega$$

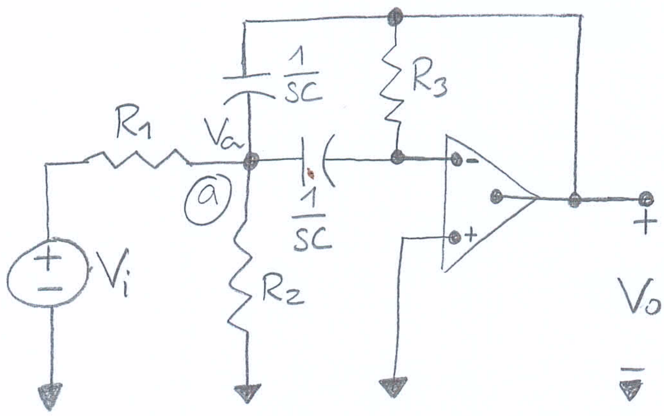
$$R_2 = \sqrt{2} \cdot 1273.2 = 1800.6326\,\Omega$$

b.



PS 6.7

Problem 4) Consider the following active high-Q bandpass filter



a. Use 300pF capacitors to design a bandpass filter with a quality factor of 20, a center frequency of 8 kHz, and a passband gain of 40dB.

b. Draw the circuit diagram of the filter and label all component values.

Solution. Let us first derive the transfer function of the above filter:

KCL at inverting input of the op amp:

$$\frac{0-V_a}{(1/sC)}+\frac{0-V_o}{R_3}=0 \implies V_a=-\frac{1}{sR_3C}V_o$$

KCL at node a:

$$\frac{V_a-V_i}{R_1}+\frac{V_a}{R_2}+\frac{V_a-V_o}{(1/sC)}+\frac{V_a}{(1/sC)}=0$$

$$\implies \left(\frac{1}{R_1}+\frac{1}{R_2}+2sC\right)V_a-\frac{1}{R_1}V_i-sCV_o=0$$

$$\implies (R_2+R_1+2sR_1R_2C)V_a-R_2V_i-sR_1R_2CV_o=0$$

$$\Rightarrow \left[-\frac{(R_1+R_2+2sR_1R_2C)}{sR_3C} - sR_1R_2C \right] V_o = R_2 V_i$$

$$\Rightarrow -\left(s^2 R_1R_2R_3C^2 + 2R_1R_2Cs + R_1 + R_2\right) V_o = sR_2R_3C\, V_i$$

$$\Rightarrow \frac{V_o}{V_i} \triangleq H(s) = -\frac{sR_2R_3C}{s^2R_1R_2R_3C^2 + 2R_1R_2Cs + R_1 + R_2}$$

$$-\frac{(1/R_1C)\,s}{s^2 + \frac{2}{R_3C}s + \frac{R_1+R_2}{R_1R_2R_3C^2}}$$

$$= -\frac{R_3}{2R_1}\frac{\left(\frac{2}{R_3C}\right)s}{s^2 + \frac{2}{R_3C}s + \frac{R_1+R_2}{R_1R_2R_3C^2}}$$

$$= -K\frac{\beta s}{s^2 + \beta s + \omega_o^2}$$

where

$$K = \frac{R_3}{2R_1}, \quad \beta = \frac{2}{R_3C}, \quad \omega_o^2 = \frac{R_1+R_2}{R_1R_2R_3C^2}$$

$$Q = \frac{\omega_o}{\beta} \Rightarrow \cancel{2}\cancel{0} = \frac{\cancel{2}\pi \cdot 800\cancel{0}}{\beta} \Rightarrow \beta = 800\pi$$

$$\Rightarrow R_3 = \frac{2}{\beta C} = \frac{\cancel{2}}{\cancel{800}_{4}\pi \cdot 300 \cdot 10^{-12}} = 2652.6\,k\Omega$$

$$40\,dB = 20\log_{10}|H(j\omega_o)|$$

$$\Rightarrow |H(j\omega_o)| = 100 = \frac{R_3}{2R_1} \Rightarrow R_1 = 13.263\,k\Omega$$

PS 6.9

and

$$\left(2\pi \cdot 8000\right)^2 = \frac{1}{R_{eq} \cdot 2652.6 \cdot 10^3 \cdot (300 \cdot 10^{-12})^2}$$

$$\Rightarrow R_{eq} = 1.6579 \cdot \cancel{10^3}\ \Omega = \frac{13.263 \cdot \cancel{10^3} R_2}{13.263 \cdot 10^3 + R_2}$$

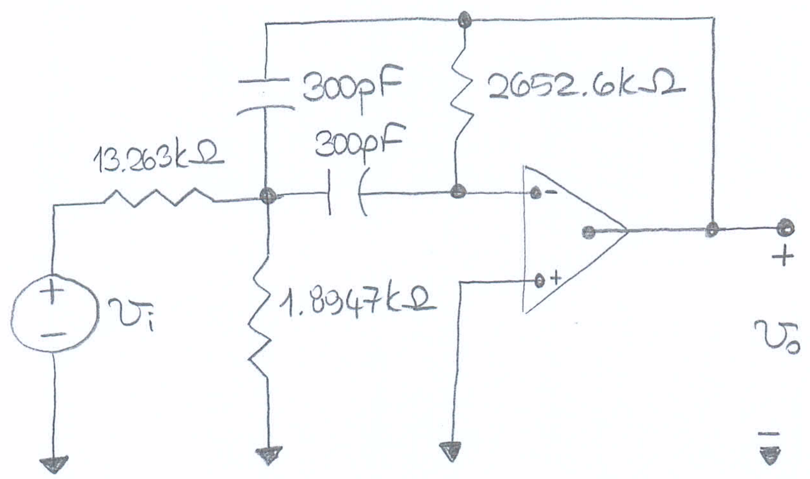
$$\Rightarrow 1.6579 \cdot 13.263 \cdot 10^3 = 11.6051\, R_2$$

$$\Rightarrow R_2 = 1.8947\, k\Omega$$

Note that;

- we could have used frequency scaling and magnitude scaling to find component values

b.



Problem 5) a. Design a broadband Butterworth bandpass filter with a lower cutoff frequency of 1000 Hz and an upper cutoff frequency of 8000 Hz. The passband gain of the filter is 10dB. The gain should be down at least 20dB at 400 Hz and 20kHz. Use 50nF capacitors in the high-pass circuit and 5 kΩ resistors in the low-pass circuit.
b. Draw the circuit and label all component values. PS 6.10

Solution. a. We consider the cascade of Butterworth low-pass and high-pass filters

Low-pass filter: $w_c = 2\pi . 8000$ rad/s

$$|H(jw)| = \frac{1}{\sqrt{1+(w/w_c)^{2n}}}$$

$$-20\,dB = 20\log \frac{1}{\sqrt{1+(\cancel{2\pi}.20.10^{\cancel{3}}/\cancel{2\pi}.8\cancel{000})^{2n}}}$$

$$\Rightarrow -2\cancel{0} = -1\cancel{0}\log[1+(2.5)^{2n}]$$

$$\Rightarrow 100 = 1+(2.5)^{2n} \Rightarrow (2.5)^{2n} = 99$$

$$\Rightarrow 2n\log 2.5 = \log 99 \Rightarrow n = \frac{\log 99}{2\log 2.5} = 2.5075$$

Hence;

- the order is 3

$$H_{LPF}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$\frac{1}{C_1C_2} = 1, \quad \frac{2}{C_1} = 1 \Rightarrow C_1 = 2, \quad C_2 = \frac{1}{2}, \quad R_1 = R_2 = 1$$

$k_f = 2\pi . 8000 = 16000\pi$, $k_m = 5000$

2nd order factor

- then we get

$$C_1 = \frac{2}{16000\pi . 5000} = 7.9577\,nF, \quad R_1 = R_2 = 5k\Omega$$

$$C_2 = \frac{1/2}{16000\pi . 5000} = 1.9894\,nF$$

- and for the 1st order factor

$$R = 5k\Omega, \quad C = \frac{1}{5000 . 16000\pi} = 3.9789\,nF$$

High-pass filter: $w_c = 2\pi \cdot 1000$

$$|H(jw)| = \frac{(\omega/\omega_c)^n}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

$$\cancel{-20}\,dB = \cancel{20}\log \frac{(\cancel{2\pi}\cdot 400/\cancel{2\pi}1000)^n}{\sqrt{1+(2\pi\cdot 400/2\pi\cdot 1000)^{2n}}}$$

$$\Rightarrow \quad -1 = \log \frac{0.4^n}{\sqrt{1+(0.4)^{2n}}}$$

$$\Rightarrow \quad 0.1 = \frac{0.4^n}{\sqrt{1+0.4^{2n}}} \quad \Rightarrow \quad 0.01 = \frac{0.4^{2n}}{1+0.4^{2n}}$$

$$\Rightarrow \quad 0.99 \cdot 0.4^{2n} = 0.01 \quad \Rightarrow \quad 0.4^{2n} = \frac{1}{99}$$

$$\Rightarrow \quad 2n = \frac{\log(1/99)}{\log(0.4)} \quad \Rightarrow \quad n = 2.5075$$

Hence;

- the order is 3

$$H_{HPF}(s) = \frac{1}{(s+1)(s^2+s+1)}$$

1^{st}-order factor: $R = 1\Omega$, $C = 1\Omega$ (prototype version)

$$k_f = 2\pi \cdot 1000 = 2000\pi, \quad 50 \cdot 10^{-9} = \frac{1}{2000\pi \cdot k_m}$$

$$\Rightarrow \quad k_m = 3183.1$$

$$R = 3183.1 \cdot 1 = 3.1831\ k\Omega$$

2^{nd}-order factor: $R_2 = 0.5\,\Omega$, $R_1 = 2\,\Omega$, $C = 1F$ (prototype)

$$R_1 = 3183.1 \cdot 2 = 6.3662\ k\Omega$$

$$R_2 = 3183.1 \cdot 0.5 = 1.5915\ k\Omega$$

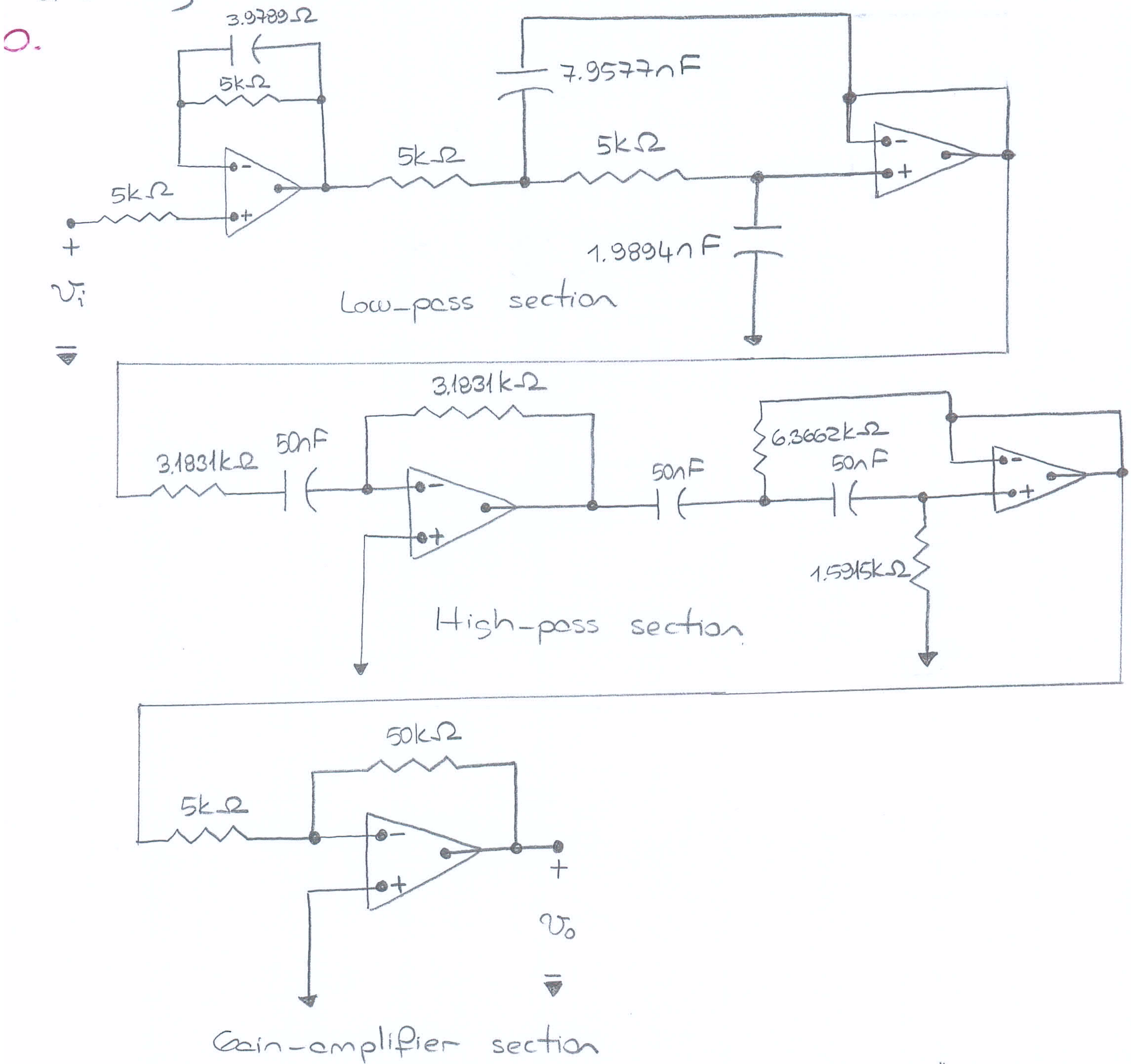
PS 6.12

- and for the passband gain, we have

$$20\,dB = 20 \log_{10} K \Rightarrow K = 10 = \frac{R_f}{R_i}$$

- choosing $R_i = 5\,k\Omega$, $R_f = 50\,k\Omega$

a.



Problem 6) Using the "twin-T notch filter" circuit diagram,

a. design a narrow-band bandreject filter having a center frequency of 4kHz, and a quality factor of 15. Base the design on C=150nF.

b. draw the circuit diagram of the filter and label all component values on the diagram.

c. what is the scaled transfer function of the filter?

PS 6.13

Solution. Note that we have

$$H(s) = \frac{s^2 + (1/R^2C^2)}{s^2 + \frac{4(1-\sigma)}{RC}s + \frac{1}{R^2C^2}}$$

$$\triangleq \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

where

$$\omega_0^2 = \frac{1}{R^2C^2}, \quad \beta = \frac{4(1-\sigma)}{RC}$$

a.

Hence;

- we start with prototype version, i.e. $R = 1\,\Omega$, $C = 1\,F$

$$k_f = 2\pi \cdot 4000 = 8000\pi, \quad 150 \cdot 10^{-9} = \frac{1}{8000\pi \cdot k_m}$$

$$\Rightarrow k_m = 265.2582$$

$$\Rightarrow R = 265.82 \cdot 1 = 265.82\,\Omega$$

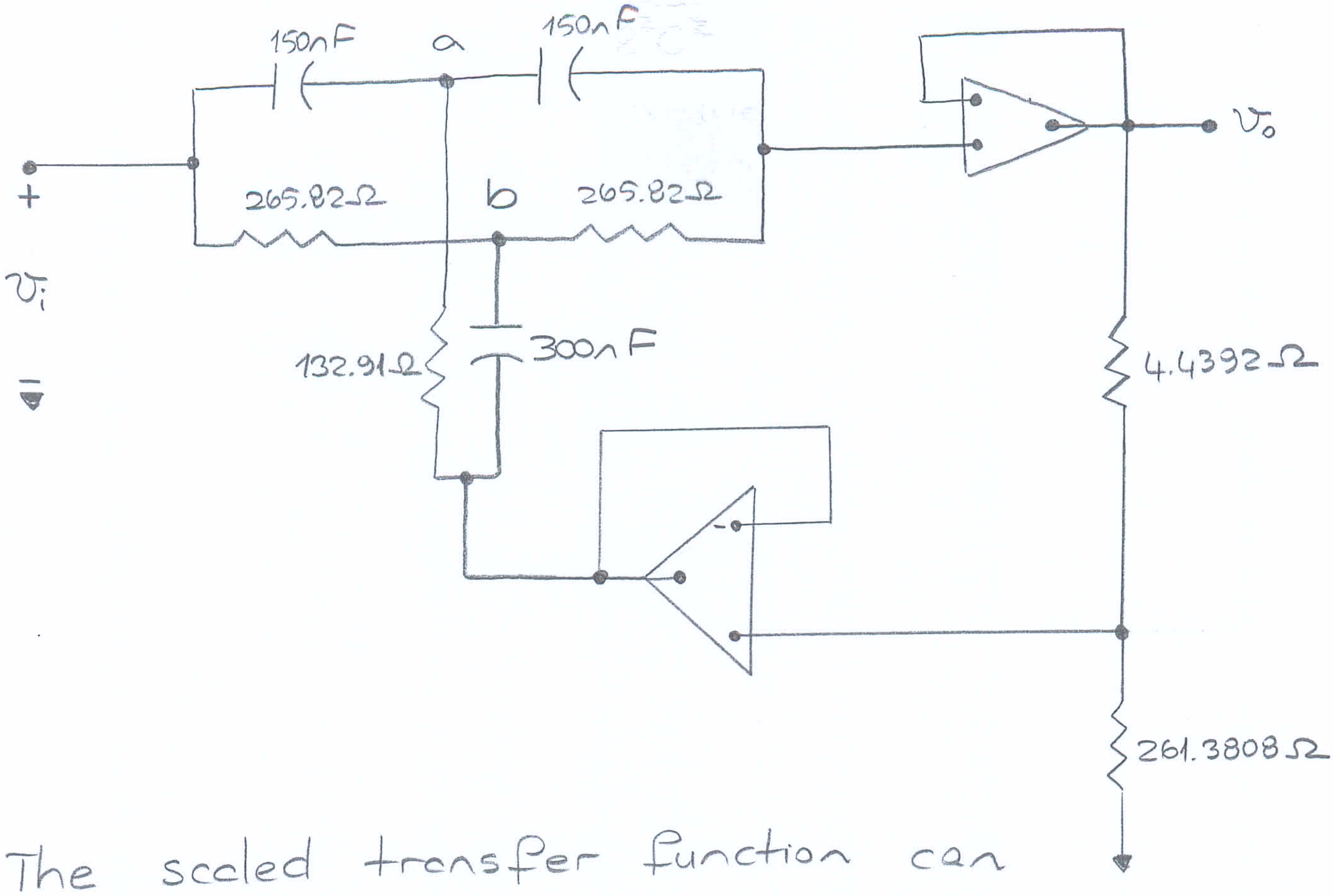
- we also have

$$\frac{4(1-\sigma)}{\underbrace{RC}_{(1/\omega_0)}} = \beta \Rightarrow 4(1-\sigma) = \frac{\beta}{\omega_0} = \frac{1}{Q}$$

$$\Rightarrow 1-\sigma = \frac{1}{4Q} \Rightarrow \sigma = 1 - \frac{1}{4Q} = 1 - \frac{1}{4 \cdot 15} = \frac{59}{60}$$

$$\Rightarrow \sigma = 0.9833$$

b. We shall draw the circuit diagram of the filter as follows:



c. The scaled transfer function can be written as

$$\omega_o = 2.5133 \cdot 10^4 \text{ rad/s}$$

$$\beta = \frac{\omega_o}{Q} = \frac{2.5133 \cdot 10^4}{15} = 1.6755 \cdot 10^3$$

$$H(s) = \frac{s^2 + 6.3165 \cdot 10^8}{s^2 + 1.6755 \cdot 10^3 s + 6.3165 \cdot 10^8}$$