EEEN 322 PS 2 QUESTIONS

Q1

3.3-8 A signal $g(t)$ is band-limited to B Hz. Show that the signal $g^n(t)$ is band-limited to nB Hz. Hint: $g^2(t) \Longleftrightarrow [G(\omega) * G(\omega)]/2\pi$, and so on. Use the width property of convolution.

* *

Q2

3.4-1 Signals $g_1(t) = 10^4 \text{rect}\,(10^4 t)$ and $g_2(t) = \delta(t)$ are applied at the inputs of the ideal low-pass filters $H_1(\omega) = \text{rect}\,(\omega/40{,}000\pi)$ and $H_2(\omega) = \text{rect}\,(\omega/20{,}000\pi)$ (Fig. P3.4-1). The outputs $y_1(t)$ and $y_2(t)$ of these filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$.

(a) Sketch $G_1(\omega)$ and $G_2(\omega)$.

(b) Sketch $H_1(\omega)$ and $H_2(\omega)$.

(c) Sketch $Y_1(\omega)$ and $Y_2(\omega)$.

(d) Find the bandwidths of $y_1(t)$, $y_2(t)$, and $y(t)$.

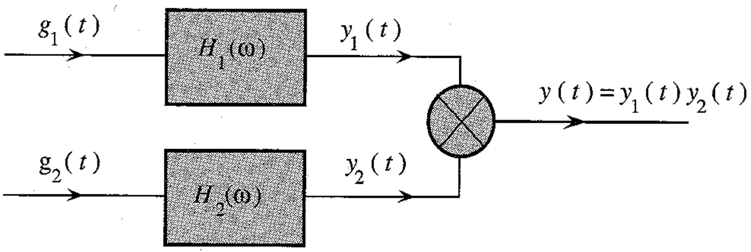

Figure P3.4-1

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Q3

EXAMPLE 3.19 Verify Parseval's theorem for the signal $g(t) = e^{-at}u(t)$ $(a > 0)$.

* *

Q4

EXAMPLE 3.20 Estimate the essential bandwidth W rad/s of the signal $e^{-at}u(t)$ if the essential band is required to contain 95% of the signal energy.

* *

Q5

EXAMPLE 3.24 A noise signal $n_i(t)$ with PSD $S_{n_i}(\omega) = K$ is applied at the input of an ideal differentiator (Fig. 3.43a). Determine the PSD and the power of the output noise signal $n_o(t)$.

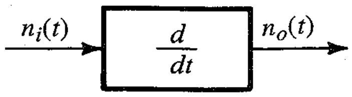

Figure 3.43 (a)

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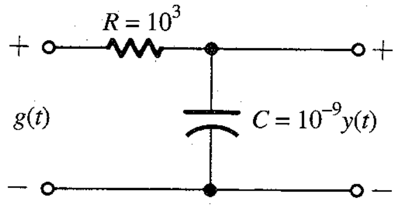
Q6

3.5-3 Determine the maximum bandwidth of a signal that can be transmitted through the low-pass RC filter in Fig. 3.27a with $R = 1000$ and $C = 10^{-9}$ if, over this bandwidth, the amplitude response (gain) variation is to be within 5% and the time delay variation is to be within 2%.



(a)

Figure 3.27 Simple RC filter,

EEEN 322 PS 2 SOLUTIONS

Q1

3.3-8 From the frequency convolution property, we obtain

$$g^2(t) \Longleftrightarrow \frac{1}{2\pi} G(\omega) * G(\omega)$$

The width property of convolution states that if $c_1(x) * c_2(x) = y(x)$, then the width of $y(x)$ is equal to the sum of the widths of $c_1(x)$ and $c_2(x)$. Hence, the width of $G(\omega) * G(\omega)$ is twice the width of $G(\omega)$. Repeated application of this argument shows that the bandwidth of $g^n(t)$ is nB Hz (n times the bandwidth of $g(t)$).

*

Q2

3.4-1

$$G_1(\omega) = \text{sinc}\left(\frac{\omega}{20000}\right) \quad \text{and} \quad G_2(\omega) = 1$$

Figure S3.4-1 shows $G_1(\omega)$, $G_2(\omega)$, $H_1(\omega)$ and $H_2(\omega)$. Now

$$Y_1(\omega) = G_1(\omega)H_1(\omega)$$
$$Y_2(\omega) = G_2(\omega)H_2(\omega)$$

The spectra $Y_1(\omega)$ and $Y_2(\omega)$ are also shown in Fig. S3.4-1. Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 kHz, 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.

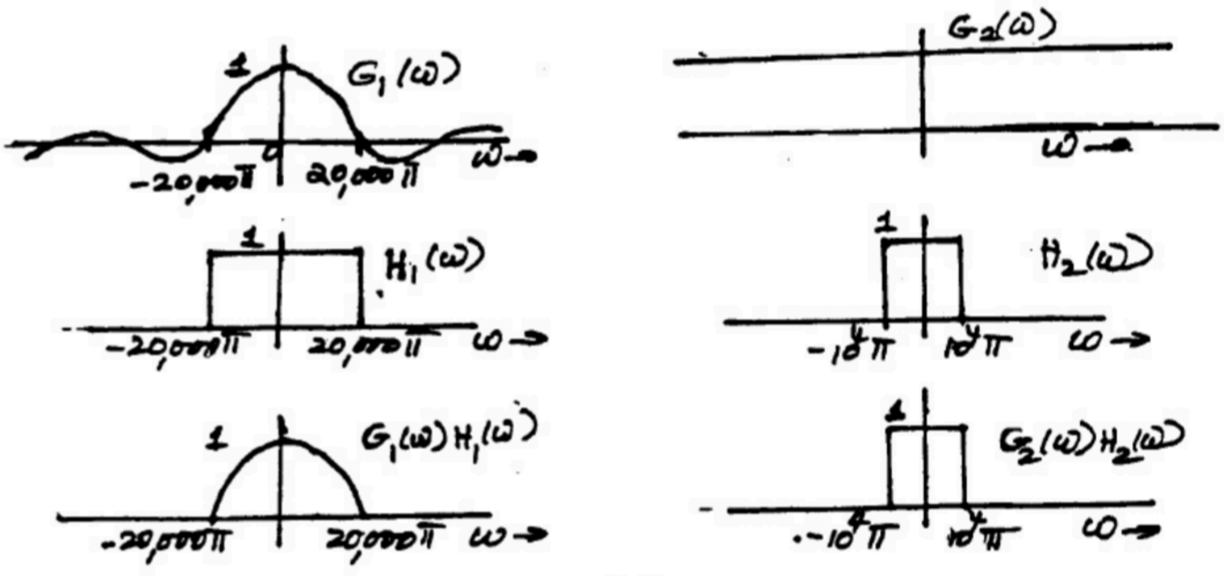


Fig. S3.4-1

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Q3

EXAMPLE 3.19

We have

$$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt = \int_0^{\infty} e^{-2at}\, dt = \frac{1}{2a} \tag{3.65}$$

We now determine E_g from the signal spectrum $G(\omega)$ given by

$$G(\omega) = \frac{1}{j\omega + a}$$

and from Eq. (3.64),

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\, d\omega = \frac{1}{2\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_{-\infty}^{\infty} = \frac{1}{2a}$$

which verifies Parseval's theorem.

* *

Q4

EXAMPLE 3.20

In this case,

$$G(\omega) = \frac{1}{j\omega + a}$$

and the ESD is

$$|G(\omega)^2| = \frac{1}{\omega^2 + a^2}$$

This ESD is shown in Fig. 3.37. Moreover, the signal energy E_g is $1/2\pi$ times the area under this ESD, which has already been found to be $1/2a$. Let W rad/s be the essential bandwidth, which contains 95% of the total signal energy E_g. This means $1/2\pi$ times the shaded area in Fig. 3.37 is $0.95/2a$, that is,

$$\frac{0.95}{2a} = \frac{1}{2\pi}\int_{-W}^{W} \frac{d\omega}{\omega^2 + a^2}$$

$$= \frac{1}{2\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_{-W}^{W} = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$$

or

$$\frac{0.95\pi}{2} = \tan^{-1}\frac{W}{a} \Longrightarrow W = 12.706a \text{ rad/s}$$

This means that the spectral components of $g(t)$ in the band from 0 (dc) to 12.706 rad/s (2.02 Hz) contribute 95% of the total signal energy; all the remaining spectral components (in the band from 12.706 rad/s to ∞) contribute only 5% of the signal energy.*

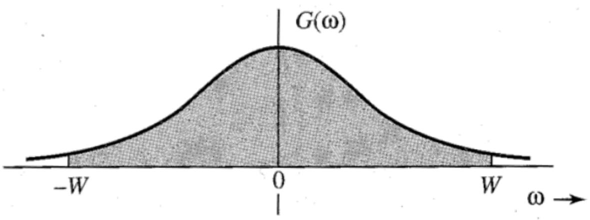


Figure 3.37 Estimating the essential bandwidth of a signal.

* Although the ESD exists over the range from -∞ to ∞, the concept of bandwidth is only meaningful for positive frequencies. Therefore, the essential bandwidth is not from –W to W, but from 0 to W.

* *

Q5

EXAMPLE 3.24

The transfer function of an ideal differentiator is $H(\omega) = j\omega$. If the noise at the demodulator output is $n_o(t)$, then from Eq. (3.90),

$$S_{n_o}(\omega) = |H(\omega)|^2\, S_{n_i}(\omega) = |j\omega|^2 K$$

The output PSD $S_{n_o}(\omega)$ is parabolic, as shown in Fig. 3.43c. The output noise power N_o is $1/2\pi$ times the area under the output PSD. Therefore,

$$N_o = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} K\omega^2\, d\omega = K\int_{-2\pi B}^{2\pi B} \omega^2\, d\omega = \frac{8\pi^2 B^3 K}{3}$$

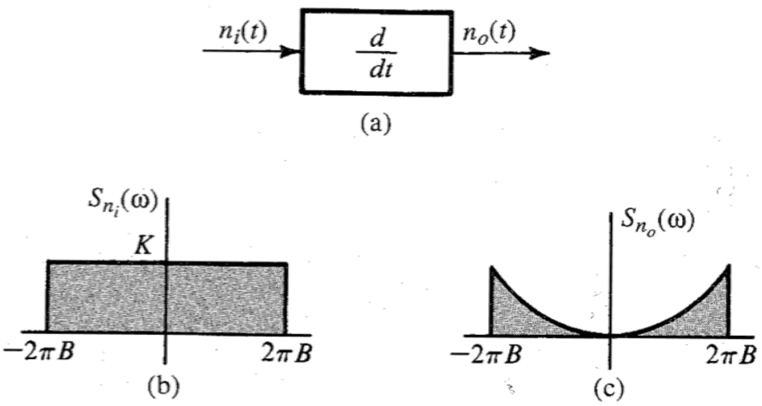


Figure 3.43 Power spectral densities at the input and the output of an ideal differentiator.

* *

Q6

3.5-3 From the results in Example 3.16

$$|H(\omega)| = \frac{a}{\sqrt{\omega^2 + a^2}} \qquad a = \frac{1}{RC} = 10^6$$

Also $H(0) = 1$. Hence if ω_1 is the frequency where the amplitude response drops to 0.95, then

$$|H(\omega_1)| = \frac{10^6}{\sqrt{\omega_1^2 + 10^{12}}} = 0.95 \Longrightarrow \omega_1 = 328,684$$

Moreover, the time delay is given by (see Example 3.16)

$$t_d(\omega) = \frac{a}{\omega^2 + a^2} \Longrightarrow t_d(0) = \frac{1}{a} = 10^{-6}$$

If ω_2 is the frequency where the time delay drops to 0.98% of its value at $\omega = 0$, then

$$t_d(\omega_2) = \frac{10^6}{\omega_2^2 + 10^{12}} = 0.98 \times 10^{-6} \Longrightarrow \omega_2 = 142,857$$

We select the smaller of ω_1 and ω_2, that is $\omega = 142,857$, where both the specifications are satisfied. This yields a frequency of 22,736.4 Hz.

Table 3.1

Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\,\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\,\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\; u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\; u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\; u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\; u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$

Energy Spectral Density (ESD) and Power Spectral Density (PSD)

Energy

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(\omega)|^2 d\omega \quad \textbf{(Parseval)}$$

Energy Spectral Density (ESD)

$$\Psi_g(\omega) = |G(\omega)|^2$$

$$\Rightarrow E_g = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_g(\omega) d\omega$$

Relationship with time autocorrelation

$$\psi_g(\tau) = \int_{-\infty}^{+\infty} g(t)g(t+\tau)dt$$

$$\psi_g(\tau) \leftrightarrow \Psi_g(\omega) \qquad \textbf{(FT pair)}$$

$$E_g = \psi_g(0)$$

ESD of Input / Output

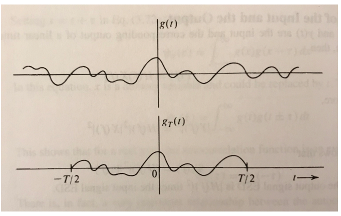
$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\Psi_y(\omega) = |H(\omega)|^2 \Psi_g(\omega)$$

Energy Spectrum of $x(t)=g(t)\cos\omega_0 t$

$$\Psi_x(\omega) = \frac{1}{4}\left[\Psi_g(\omega+\omega_0) + \Psi_g(\omega-\omega_0)\right]$$

$$E_x = \frac{1}{2}E_g$$



Power

$$P_g = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} |g(t)|^2 dt$$

Power Spectral Density (PSD)

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T} \quad (*)$$

The power spectrum can be computed this way as long as the limit exists

$$\Rightarrow P_g = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_g(\omega) d\omega$$

Relationship with time autocorrelation

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} g(t)g(t+\tau)dt$$

$$R_g(\tau) \leftrightarrow S_g(\omega) \qquad \textbf{(FT pair)}$$

$$P_g = R_g(0) \quad \text{and} \quad \text{RMS} = \sqrt{P_g}$$

PSD of Input / Output

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$S_y(\omega) = |H(\omega)|^2 S_g(\omega)$$

Power Spectrum of $x(t)=g(t)\cos\omega_0 t$

$$S_x(\omega) = \frac{1}{4}\left[S_g(\omega+\omega_0) + S_g(\omega-\omega_0)\right]$$

$$P_x = \frac{1}{2}P_g$$

$(*)\, g_T(t)$ is $g(t)$ truncated to $\left[-\frac{T}{2}, \frac{T}{2}\right]$