CMPE 352
Signal Processing & Algorithms
Spring 2019

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Review Questions (1)

- How is the energy of a signal $x(t)$ computed? $E = \int_{-\infty}^{+\infty} x^2(t)dt$

- What is the (average) power of the signal $x(t)$? $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x^2(t)dt$

- What is an energy signal? A signal for which $0 < E < \infty$

- What is a power signal? A signal for which $0 < P < \infty$

- Can a signal be both an energy and a power signal? No, if it is one, it cannot be the other

Review Questions (2)

- What is the energy of a power signal?

  Its energy is infinite

- What is the power of an energy signal?

  Its power is zero

$$E = \int_{-\infty}^{+\infty} x^2(t)dt \qquad P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x^2(t)dt$$

- Is $x(t) = t\,u(t)$ and energy or a power signal?

$$E = \int_{-\infty}^{+\infty} x^2(t)dt = \int_{0}^{+\infty} t^2\,dt = \frac{1}{3}t^3\Big|_0^{\infty} = \infty$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x^2(t)dt = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{+T} t^2 dt = \lim_{T \to \infty} \frac{1}{2T}\frac{1}{3}t^3\Big|_0^{T} = \lim_{T \to \infty} \frac{1}{2T}\frac{1}{3}T^3 = \lim_{T \to \infty} \frac{1}{6}T^2 = \infty$$

$\Rightarrow$ neither an energy nor a power signal

Review Questions (3)

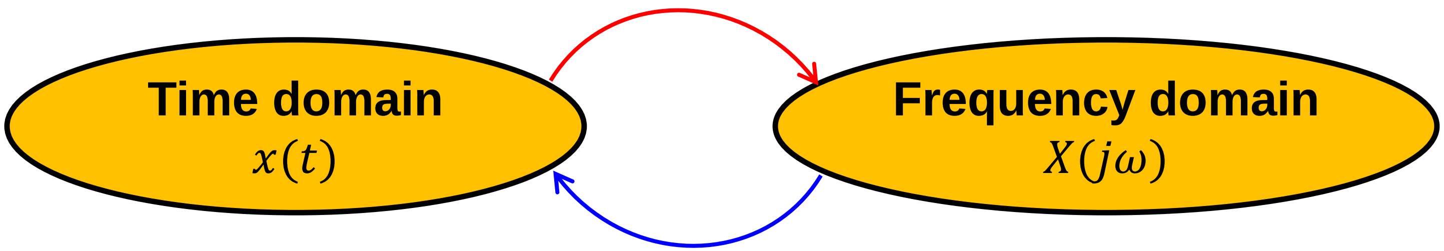
- What is the definition of the decibel? $10\ \log_{10}\dfrac{P_1}{P_2}$

- What is dBm? If $P_2 = 1\ mW$ the decibel is written as: dBm

- What is a power ratio of 8 in dB? $10\log_{10}8 = 10\ \log_{10}2^3 =$
$3 \times 10\ \log_{10}2 = 3 \times 3\ dB = 9$ dB

The Fourier Transform

- Let $x(t)$ be a nonperiodic continuous-time function
- The Fourier transform of $x(t)$ is $$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt}$$
- The inverse Fourier transform of $X(j\omega)$ is $$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega}$$



- $X(j\omega)$ is called the **spectrum** of $x(t)$
  - $|X(j\omega)|$ is called the ***magnitude spectrum***
  - $\arg(X(j\omega))$ is called the ***phase spectrum***
- $x(t)$ and $X(j\omega)$ form a **Fourier-transform pair**

The Fourier Transform – Examples

Example 1

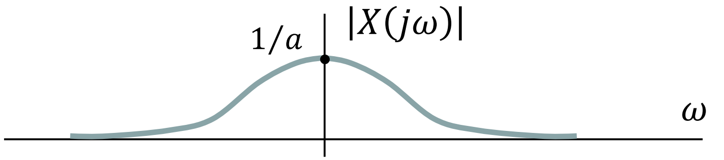
Determine and plot the Fourier transform of the signal $x(t) = e^{-at}u(t),\ a > 0$

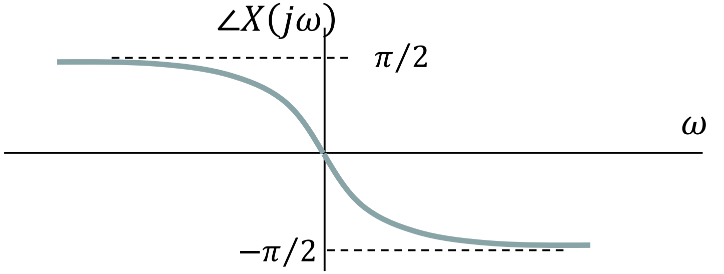
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = -\frac{1}{a+j\omega}\,(0-1) = \frac{1}{a+j\omega}$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$\boxed{|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \angle X(j\omega) = -\operatorname{atan}\frac{\omega}{a}}$$





The Fourier Transform – Examples

Example 2

Determine and plot the Fourier transform of the signal $x(t) = e^{-a|t|}, \ a > 0$

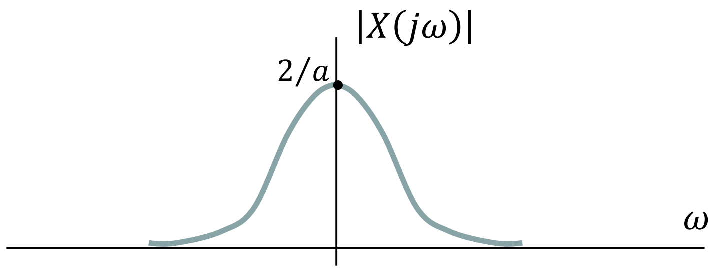
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} \, dt \ = \int_{-\infty}^{0} e^{at} e^{-j\omega t} \, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} \, dt + \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$|X(j\omega)| = \frac{2a}{a^2+\omega^2}$$

$$\angle X(j\omega) = 0$$



The Fourier Transform – Examples

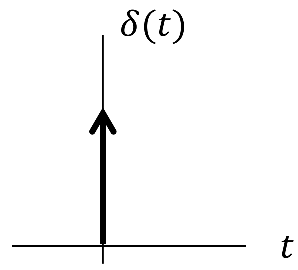
<u>Example 3</u>

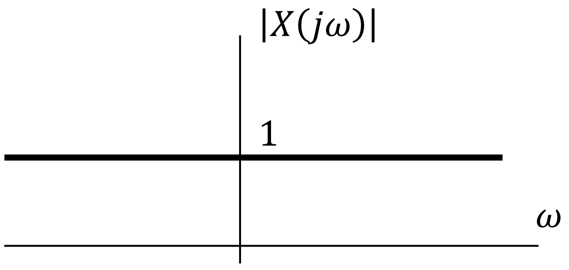
Determine and plot the Fourier transform of the signal $x(t) = \delta(t)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\; dt \;=\; \int_{-\infty}^{\infty} \delta(t)\; dt \overset{\text{def.}}{=} 1$$

What are $|X(j\omega)|$ and $\angle X(j\omega)$?

$$\boxed{\begin{aligned} |X(j\omega)| &= 1 \\ \angle X(j\omega) &= 0 \end{aligned}}$$

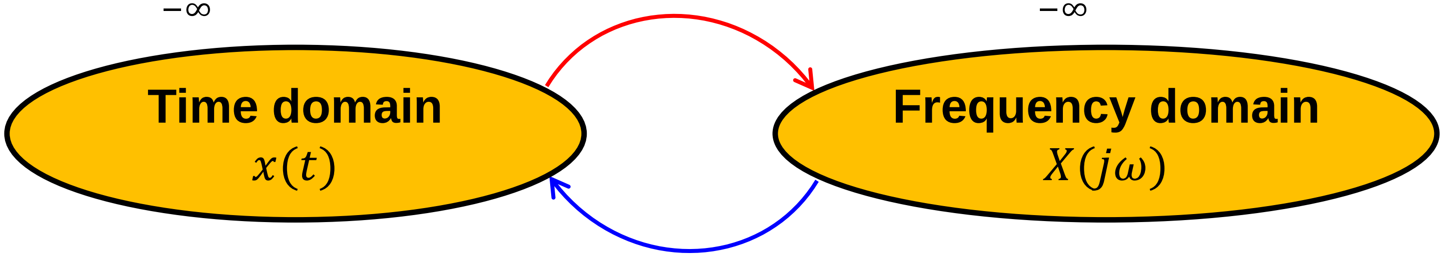




Properties of the Fourier Transform (1)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\ d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\ dt$$



We will use the notation $\boxed{x(t) \ \leftrightarrow \ X(j\omega)}$ (**Fourier Transform pair**)

For example $$e^{-at}u(t) \ \leftrightarrow \frac{1}{a + j\omega}$$

$$\delta(t) \ \leftrightarrow 1$$

Properties of the Fourier Transform (2)

Linearity

$$\left.\begin{aligned} x(t) &\leftrightarrow X(j\omega) \\ y(t) &\leftrightarrow Y(j\omega) \end{aligned}\right\} \qquad \boxed{ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)}$$

Differentiation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega\, X(j\omega)]e^{j\omega t}\, d\omega$$

$$\boxed{\frac{d}{dt}x(t) \leftrightarrow j\omega\; X(j\omega)}$$

Integration

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j\omega}\; X(j\omega)}$$

(assumes $x(0) = 0$)

Time & frequency scaling

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|}\; X(j\frac{\omega}{a})}$$

in particular: $x(-t) \leftrightarrow X(-j\omega)$

Properties of the Fourier Transform (3)

Time Shifting

$$x(t) \leftrightarrow X(j\omega)$$

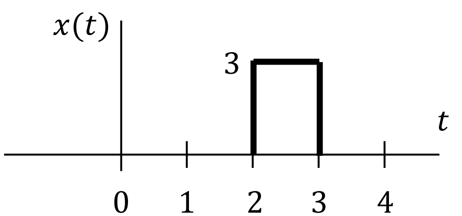
$$x(t - t_0) \leftrightarrow \ ?$$

$$x(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t - t_0)}\, d\omega \ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[e^{-j\omega t_0} X(j\omega)\right]}\, e^{j\omega t}\, d\omega$$

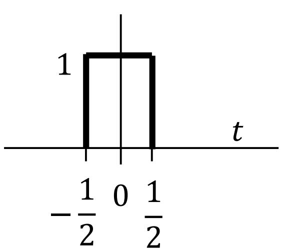
hence this term represents the Fourier transform of $x(t - t_0)$

$$\boxed{x(t - t_0) \ \leftrightarrow \ e^{-j\omega t_0} X(j\omega)}$$

Example: Compute the Fourier transform of the signal:



We know that



$$\leftrightarrow \quad 2\frac{\sin(\omega/2)}{\omega}$$

Hence: $x(t) \leftrightarrow 6\, e^{-j2.5\omega} \dfrac{\sin(\omega/2)}{\omega}$

Properties of the Fourier Transform

Let x(t) and y(t) be aperiodic signals with Fourier Transform representations

$$x(t) \leftrightarrow X(j\omega) \qquad\qquad y(t) \leftrightarrow Y(j\omega)$$

Then the following properties hold:

Linearity $Ax(t) + By(t) \leftrightarrow AX(j\omega) + BY(j\omega)$

Time shift $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

Frequency shift $e^{j\omega_0 t} x(t) \leftrightarrow X[j(\omega - \omega_0)]$

Scaling $x(\alpha t) \leftrightarrow \dfrac{1}{|\alpha|} X\left(\dfrac{j\omega}{\alpha}\right)$

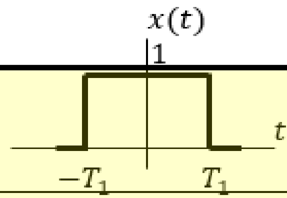
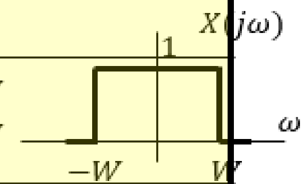
Time reversal $x(-t) \leftrightarrow X(-j\omega)$

Differentiation in time $\dfrac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$

Inte-gration $\displaystyle\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Differentiation in frequency $tx(t) \leftrightarrow j\dfrac{d}{d\omega} X(j\omega)$

Fourier Transform Pairs

Signal	Fourier Transform
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ 	$\dfrac{2\sin\omega T_1}{\omega}$
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ 
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
$e^{-at}u(t), \quad Re\{a\} > 0$	$\dfrac{1}{a + j\omega}$
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad Re\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$

Fourier Transform Computation – Problems (1)

Problem 1

Determine the inverse Fourier transform of

$$X(j\omega) = \underbrace{2\pi\delta(\omega)}_{\downarrow \; 1} + \underbrace{\pi\delta(\omega - 4\pi)}_{\downarrow \; \frac{1}{2\pi} e^{j4\pi t}} + \underbrace{\pi\delta(\omega + 4\pi)}_{\downarrow \; \frac{1}{2\pi} e^{-j4\pi t}}$$

$\boxed{1 \leftrightarrow 2\pi\delta(\omega)}$

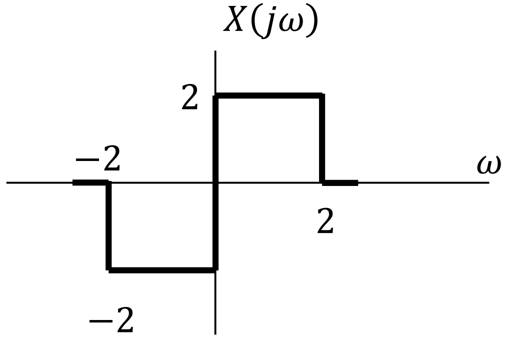
$\boxed{e^{j\omega_0 t} x(t) \leftrightarrow X[j(\omega - \omega_0)]}$

$$x(t) = 1 + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) = 1 + \cos 4\pi t$$

Fourier Transform Computation – Problems (2)

Problem 2

Determine the inverse Fourier transform of $X(j\omega)$:



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-2}^{0}(-2)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{2}(2)e^{j\omega t}d\omega$$

$$= -\frac{1}{\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-2}^{0} + \frac{1}{\pi}\frac{1}{jt}e^{j\omega t}\Big|_{0}^{2} = -\frac{1}{\pi jt}\left(1 - e^{-j2t}\right) + \frac{1}{\pi jt}\left(e^{j2t} - 1\right)$$

$$= -\frac{2}{\pi t}e^{-jt}\left(\frac{e^{jt} - e^{-jt}}{2j}\right) + \frac{2}{\pi t}e^{jt}\left(\frac{e^{jt} - e^{-jt}}{2j}\right)$$

$$= \frac{2}{\pi t}\sin t\left(e^{jt} - e^{-jt}\right) = \frac{4j}{\pi t}\sin t\left(\frac{e^{jt} - e^{-jt}}{2j}\right) = \frac{4j\, sin^2 t}{\pi t}$$

Fourier Transform Computation – Problems (3)

Problem 3

Find the Fourier transform of $x(t) = u(-t)$

$$\boxed{u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)}$$

$$\boxed{x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\omega}{\alpha}\right)} \Rightarrow x(-t) \leftrightarrow X(-j\omega)$$

$$\Rightarrow X(j\omega) = -\frac{1}{j\omega} + \pi\delta(\omega)$$

Problem 4

Find the Fourier transform of $x(t) = e^{at}u(-t) \quad (a > 0)$

$$\boxed{e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega}}$$

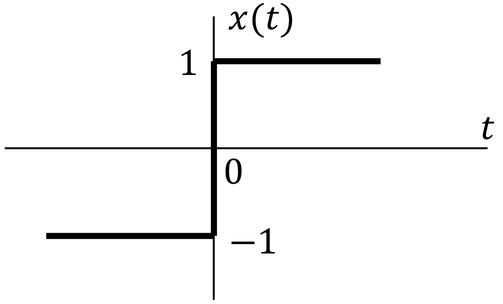
$$\boxed{x(-t) \leftrightarrow X(-j\omega)}$$

$$\Rightarrow X(j\omega) = \frac{1}{a - j\omega}$$

Fourier Transform Computation – Problems (4)

Problem 5

Determine the inverse Fourier transform of $x(t) = \mathrm{sgn}(t)$



$$x(t) = 2u(t) - 1$$

$$\boxed{u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)}$$

$$\boxed{1 \leftrightarrow 2\pi\delta(\omega)}$$

$$\frac{1}{j\omega} + \pi\delta(\omega) \qquad 2\pi\delta(\omega)$$

$$\Rightarrow X(j\omega) = \frac{2}{j\omega}$$

Fourier Transform Computation – Problems (5)

Problem 6

Find the Fourier transform of $x(t) = \cos(\omega_0 t)$

Note: $\cos(\omega_0 t) = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} \Rightarrow e^{j\omega_0 t} \leftrightarrow \ ?$

$$\left.\begin{array}{l} \boxed{1 \leftrightarrow 2\pi\delta(\omega)} \\ \boxed{e^{j\omega_0 t}x(t) \leftrightarrow X[j(\omega-\omega_0)]} \end{array}\right\} e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$\Rightarrow X(j\omega) = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\ ]$$

Problem 7

Show that $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\{X[j(\omega-\omega_0)] + X[j(\omega+\omega_0)]\}$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \quad \Rightarrow x(t)\cos\omega_0 t = \frac{1}{2}\underbrace{x(t)e^{j\omega_0 t}}_{\to X[j(\omega-\omega_0)]} + \frac{1}{2}\underbrace{x(t)e^{-j\omega_0 t}}_{\to X[j(\omega+\omega_0)]}$$

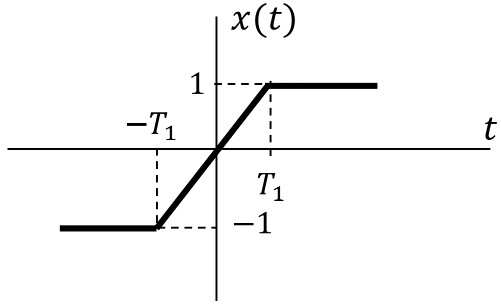
$$\boxed{e^{j\omega_0 t}x(t) \leftrightarrow X[j(\omega-\omega_0)]}$$

$$\Rightarrow x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\{X[j(\omega-\omega_0)] + X[j(\omega+\omega_0)]\}$$

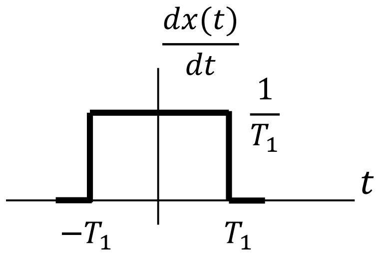
Fourier Transform Computation – Problems (6)

Problem 8

Find the Fourier transform of $x(t)$



We know that $\frac{dx(t)}{dt}$ is the function



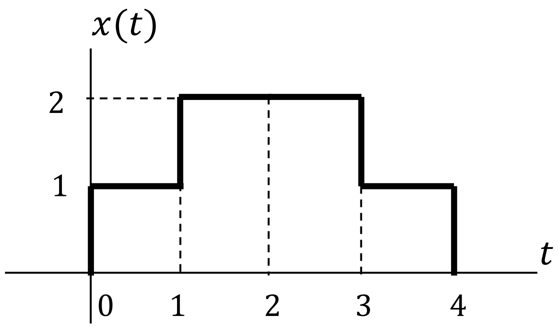
$$\left.\begin{aligned} &\frac{dx(t)}{dt} \leftrightarrow 2\frac{\sin \omega T_1}{T_1} \\ &\boxed{\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)} \end{aligned}\right\} \Rightarrow j\omega X(j\omega) = 2\frac{\sin \omega T_1}{T_1}$$

$$X(j\omega) = \frac{2}{j\omega}\frac{\sin \omega T_1}{T_1}$$

Fourier Transform Computation – Problems (7)

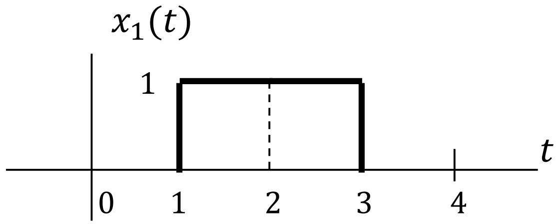
Problem 9

Find the Fourier transform of $x(t)$

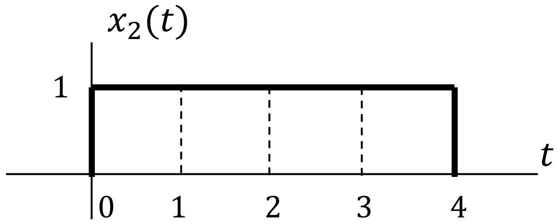


We can write $x(t)$ as the sum of $x_1(t)$ and $x_2(t)$:

$$y(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \leftrightarrow \frac{2 \sin \omega T_1}{\omega}$$



$$x_1(t) \leftrightarrow 2 \frac{\sin \omega}{\omega} e^{-j2\omega} \quad (T_1 = 1)$$



$$x_2(t) \leftrightarrow 2 \frac{\sin 2\omega}{\omega} e^{-j2\omega} \quad (T_1 = 2)$$

$$\Rightarrow x(t) \leftrightarrow \frac{2}{\omega} e^{-j2\omega} (\sin 2\omega + \sin \omega)$$