NOTES

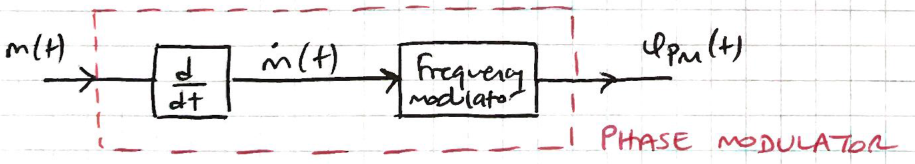
$$\varphi_{PM}(t) = A\cos[w_c t + k_p m(t)]$$

replace $m(t)$ with $\int m(t) \rightarrow$ FM

$$\varphi_{FM}(t) = A\cos[w_c t + k_f \int_{-\infty}^{t} m(\alpha)\,d\alpha]$$

replace $\int m(t)$ with $m(t) \rightarrow$ PM

1. An FM wave corresponding to $m(t)$ is the PM wave corresponding to $\int m(\alpha)\,d\alpha$
2. A PM wave corresponding to $m(t)$ is the FM wave corresponding to $\dot{m}(t)$
3. By looking at an angle modulated signal, you cannot tell whether it is FM or PM
4. An FM wave can be generated by a PM modulator, and a PM wave can be generated by an FM modulator.



Additional Note:

In PM and FM, the angle $\theta(t)$ of a carrier is varied in proportion to some measure of $m(t)$

In PM $\rightarrow$ it is directly proportional to $m(t)$

In FM $\rightarrow$ it is proportional to the integral of $m(t)$

Actually, there is an infinite number of possible ways of generating a measure of $m(t)$.

$$\varphi_{EM}(t) = A\cos[\omega_c t + \psi(t)]$$

$\psi(t)$ → some measure of $m(t)$

For example, if we choose to use a linear operator:

$$\varphi_{EM}(t) = A\cos\left[\omega_c t + \int_{-\infty}^{t} m(\alpha)\, h(t-\alpha)\, d\alpha\right]$$

where $h(t)$ is the impulse response of the linear operator/system.
(Note that the integral is the convolution integral and the upper limit is t here, implying that $h(t)$ is causal)

$h(t) = k_p\,\delta(t)$ corresponds to PM
$h(t) = k_f\, u(t)$ " " FM

} and these are just two possibilities out of an infinite number

Power of an Angle-Modulated wave

Always $A^2/2$, regardless of the value of k_f or k_p

Bandwidth of Angle-Modulated waves

let us define $a(t) = \int_{-\infty}^{t} m(\alpha)\, d\alpha$

and $\hat{\varphi}_{FM}(t) = A e^{j[\omega_c t + k_f a(t)]} = A e^{jk_f a(t)} e^{j\omega_c t}$ $\quad (*)$

Now, $\varphi_{FM}(t) = Re\{\hat{\varphi}_{FM}(t)\}$

Expanding $e^{jk_f a(t)}$ in $(*)$ power series

we have $\hat{\varphi}_{FM}(t) = A\left[1 + jk_f a(t) - \frac{k_f^2 a^2(t)}{2!} - \frac{jk_f^3 a^3(t)}{3!} + \dots\right] e^{j\omega_c t}$

$$= A\left(1 + jk_f a(t) - \frac{k_f^2 a^2(t)}{2!} - \frac{jk_f^3 a^3(t)}{3!} + \dots\right](\cos\omega_c t + j\sin\omega_c t)$$

$$= A\Big(\cos\omega_c t + j\sin\omega_c t + jk_f a(t)\cos\omega_c t - k_f a(t)\sin\omega_c t$$
$$- \frac{k_f^2 a^2(t)}{2!}\cos\omega_c t - \frac{jk_f^2 a^2(t)}{2!}\sin\omega_c t$$
$$- \frac{k_f^3 a^3(t)}{3!}\cos\omega_c t + \frac{k_f^3 a^3(t)}{3!}\sin\omega_c t + \dots\Big]$$

$$\Rightarrow \varphi_{FM}(t) = Re\{\hat{\varphi}_{FM}(t)\} = A\Big[\cos\omega_c t - k_f a(t)\sin\omega_c t - \frac{k_f^2 a^2(t)}{2!}\cos\omega_c t$$
$$+ \frac{k_f^3 a^3(t)}{3!}\sin\omega_c t + \dots\Big]$$

= unmodulated carrier + some amplitude-modulated terms

If $m(t)$ is band-limited to B, $a(t) = \int_{-\infty} m(\alpha)\,d\alpha$ is also band-limited to B (because $A(\omega) = \frac{1}{j\omega} M(\omega)$), $a^2(t)$ is band-limited to $2B$ (because its spectrum is $\frac{1}{2\pi} A(\omega) * A(\omega)$) and similarly, $a^n(t)$ is band-limited to nB (i.e., it is not band-limited but has infinite bandwidth).

In Summary:

- The spectrum of $\varphi_{FM}(t)$ is centered at ω_c
- Theoretically, $\varphi_{FM}(t)$ is not band-limited, it has an infinite bandwidth nB
- The bandwidth is not related to the modulating signal spectrum in any simple way as in AM.

Important Note:

- Although the theoretical bandwidth of an FM wave is infinite, most of the modulated signal power is in a finite bandwidth
- Two distinct possibilities in terms of bandwidth are narrow-band FM (NBFM) and wide-band FM (WBFM)

Narrow-Band Angle Modulation

If k_f is very small, i.e., if $|k_f a(t)| \ll 1$, then all terms after the first two in the expansion of $\varphi_{FM}(t)$ are negligible, and we have $\boxed{\varphi_{FM}(t) \approx A[\cos\omega_c t - k_f\, a(t) \sin\omega_c t]}$ (NBFM)

Although FM is nonlinear, this is a linear expression similar to that of AM

1. Both AM & NBFM have carrier & sidebands at $\pm\omega_c$
2. Both have the same bandwidth, $2B$ (B is the bandwidth of $m(t)$)
3. In NBFM, sideband spectrum has a phase shift of $\pi/2$ w.r.t. carrier
4. Keep in mind: Despite the similarities in the expressions, AM and FM/PM have very different waveforms

Similarly, NBPM is $\boxed{\varphi_{PM}(t) \approx A[\cos\omega_c t - k_p m(t) \sin\omega_c t]}$ and all these are also valid here.

Expressions of NBFM & NBPM suggest that we may generate them by using DSB-SC modulators (see Fig 5.6)

General expression for the bandwidth of FM signals :

$$B_{FM} = 2(\Delta f + B) \qquad \text{(Carson's Rule)}$$

B: bandwidth of $m(t)$

Δf: $\frac{k_f m_p}{2\pi}$

m_p : the peak amplitude of $m(t)$

NBFM corresponds to $\Delta f \ll B$ since k_f is very small $\Rightarrow B_{FM} \approx 2B$

Wide-Band FM (WBFM)

$|k_f a(t)| \ll 1$ is not satisfied, $\Delta f \gg B \Rightarrow B_{FM} \approx 2\Delta f$

(Theory: Infinite bandwidth, practice: frequencies outside this bandwidth can be neglected)

Note: If we define a deviation ratio β as $\beta = \frac{\Delta f}{B}$, Carson's Rule becomes $B_{FM} = 2B(\beta + 1)$

Note that if $\beta \ll 1$ $(\Delta f \ll B)$ $\Rightarrow B_{FM} \approx 2B$ (NBFM)

and if $\beta \gg 1$ $(\Delta f \gg B)$ $\Rightarrow B_{FM} \approx 2\Delta f$ (WBFM)

Therefore, β controls the amount of modulation, plays a role similar to the modulation index in AM. For tone modulated FM, β is called the modulation index.

PM All the results derived for FM can be directly applied to PM.

$$\omega_i = \omega_c + k_p \dot{m}(t)$$

$\Rightarrow$ The frequency deviation is $\Delta\omega = k_p m_p'$ $\Rightarrow \Delta f = \frac{\Delta\omega}{2\pi} = \frac{k_p m_p'}{2\pi}$

where m_p' is the peak amplitude of $\dot{m}(t)$

And, $B_{PM} = 2(\Delta f + B) = 2\left(\frac{k_p m_p'}{2\pi} + B\right)$

Note: In FM, $\Delta\omega = k_f m_p$ depends only on the peak value of $m(t)$ and is independent of the spectrum of $m(t)$. In PM, on the other hand, $\Delta\omega = k_p m_p'$ depends on the peak value of $\dot{m}(t)$, which strongly depends on the spectrum of $m(t)$.

How?

Higher frequency components in the spectrum of $m(t)$ $\rightarrow$ rapid time variations $\rightarrow$ a higher value of m_p'

Lower frequency components in the spectrum of $m(t)$ $\rightarrow$ slow time variations $\rightarrow$ a lower value of m_p'

In Summary:

- While WBFM bandwidth is independent of the spectrum of $m(t)$
  WBPM " strongly depends on " " " "
- For $m(t)$ concentrated at lower frequencies, B_{PM} will be smaller than when the spectrum of $m(t)$ is concentrated at higher frequencies

FM Tone Modulation

If $m(t) = \alpha \cos \omega_m t$

$$B = f_m, \quad \beta = \frac{\Delta f}{B} = \frac{\Delta f}{f_m} \quad \text{where} \quad \Delta f = \frac{\Delta\omega}{2\pi} = \frac{k_f m_p}{2\pi} = \frac{\alpha k_f}{2\pi}$$

The spectrum looks like Fig 5.8(b).

There are infinite number of sidebands at frequencies

$\omega_c \pm \omega_m, \omega_c \pm 2\omega_m, \ldots, \omega_c \pm n\omega_m$.

The strength of the nth sideband at $\omega = \omega_c \pm n\omega_m$ is given by Bessel function $J_n(\beta)$ (see Fig 5.8(a)).

$J_n(\beta)$ is negligible for $n > \beta + 1 \Rightarrow$ the number of significant sidebands is $\beta + 1 \Rightarrow B_{FM} = 2(\beta+1) f_m = 2(\beta f_m + f_m)$

$$= 2(\Delta f + B)$$

(Carson's Rule verified ✓)

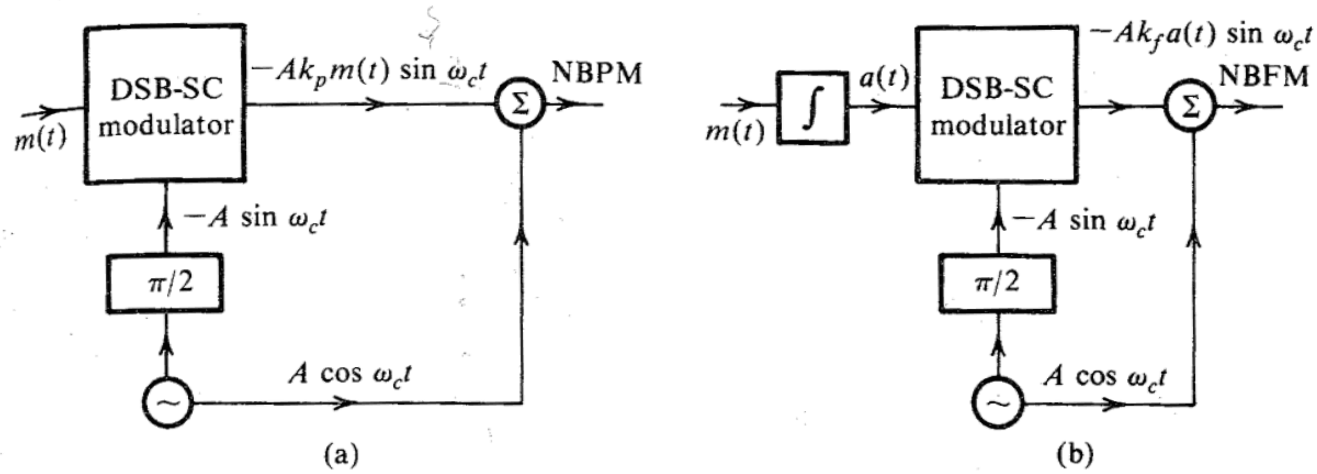


Figure 5.6 Narrow-band PM and FM wave generation.

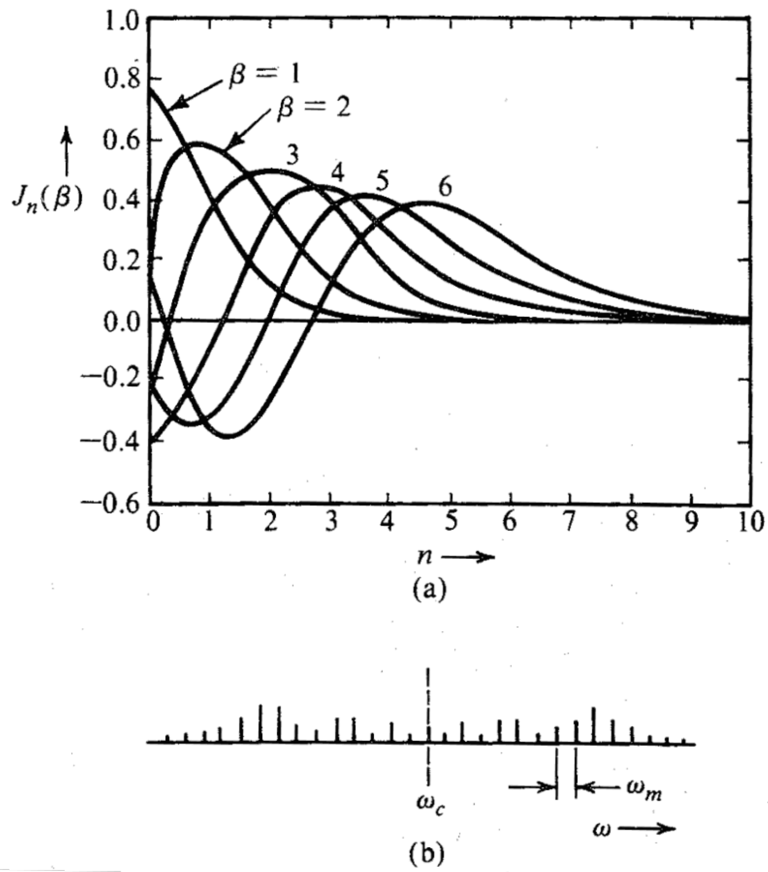


Figure 5.8 (a) Variations of $J_n(\beta)$ as a function of n for various values of β. (b) Tone-modulated FM wave spectrum.