Lecture Notes for MATH 233
Fall 2018

Four ways of distributing objects into boxes

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1. Distinguishable objects into distinguishable boxes:

Consider the problem of assigning 8 MS students at thesis stage to 6 different professors in the Electrical Engineering department of a university where there is no uppuer or lower limit to the number of MS students a professors can have. That is to say, a professor can have no thesis student or all 8 thesis students of any number in between.

There are two ways to think about this:

a) Students are: S1, S2, …. S8
Professors are: P1, P2, … P6

Assg #1: (S1, P2) (S2,P2) (S3, P2) (S4, P2) (S5, P2) (S6, P2) (S7, P2) (S8, P2)
All 8 students are assigned to P2

Assg #2: (S1, P1) (S2, P2) (S3, P3) (S4, P4) (S5, P5) (S6, P6) (S7, P6) (S8, P1)
Students S1, S2 are assigned to P1 and S2 to P2, S3 to P3, … S5 to P5 and S6, S7 both to P6.

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After writing some sample assignments we see that S1 can be assigned to 6 different professors. Thus, there are 6 ways that this task can be done. Similarly S2 can be assigned to 8 different professors. Thus, there are 6 ways that this task can be done as well.

Thus, there are 6x6x6x6x6x6 = 6^8 = 2^8 x 3^8 = 256 x 81 x 81 = 1,679,616 ways the whole assignment can be done.

<u>Note that this is also the number of functions from a set with 8 elements to a set with 6 elements.</u>

b) If you try to make assignments starting from professors. The task is hard. For example, for the Assgn#1 above:

Assg #1 : (P2, {S1,S2,….,S8})
Assg #2 : (P1, {S1,S8}) (P2, {S2}) (P3, {S3}) (P4, {S4}) (P5, {S5,S6}) (P6, {S7})

P1 can take any subset of the set {S1, S2, …. S8}, thus there are 2^8 different possibilities. And depending on what students she is assigned, P2 will choose any subset from the remaining set.

Quite difficult to tackle…
Thus the way in a) works effortlessly.

2. Undistinguishable objects into distinguishable boxes:

How many ways are there to put 6 candies to 4 children?

There are two ways to handle this:

a) Assume the children's candies are ordered from C1's candies to C4's candies and 'I' seperates two children's candies. Thus,

Possible distributions are:

Distr #1: oo I oo I oo I
C1, C2, C3 take two candies and C4 none.

Distr #2: I ooo I ooo I
C2, C3 take three candies and C1 and C4 none.

Distr #3: o I o I o I ooo
C1, C2, C3 take one candy and C4 three.

Thus, the problem is, in how many ways can we place 3 I's into a string of 3 !'s and 6 o's?

And this is C(9,3) = 9! / (3! 6!) = 9.8.7 / (3.2) = 84

b) An alternative way to think about this problem is looking at the solution of the problem 3), i.e. ways of partitioning 6 items into 4 identical boxes and then for each partition, multiply the number with the number of possible reorderings. Reorderings for 6000 is 4! / (3! 1!) = 4 and reorderings of 5100 is 4! / (1! 1! 2!) = 6

identical boxes	**distinct boxes**	**number of reorderings**
{6,0,0,0}	6000	4
{5,1,0,0}	5100	12
{4,2,0,0}	4200	12
{4,1,1,0}	4110	12
{3,3,0,0}	3300	6
{3,2,1,0}	3210	24
{3,1,1,1}	3111	4
{2,2,2,0}	2220	4
{2,2,1,1}	2211	6
+ 9		84

3. Undistinguishable objects into undistinguishable boxes:

How many ways are there to put 6 candies to 4 identical urns?

U U U U

Think about the distributions. Any urn can have one candy and another can have 5 and the other two none.

Distr#1 = {1,5,0,0}

See that there is only one such distribution since each urn is identical and so does each candy

Distr#1 = {6,0,0,0}

which means that one urn has all candies and the remaining three have none.

We see that this problem is a partitioning problem. I.e., in haow many ways can you partition the integer 6 into 4 parts?

There is no closed formula that gives partitioning integer n into r parts. Therefore, enlist all partitionnings. They are:

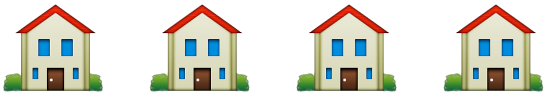
{6,0,0,0}
{5,1,0,0}
{4,2,0,0}
{4,1,1,0}
{3,3,0,0}
{3,2,1,0}
{3,1,1,1}
{2,2,2,0}
{2,2,1,1}

4. Distinguishable objects into undistinguishable boxes:

There are six different animals: cat, dog, mouse, horse, goat, sheep one from each.

In how many ways can we place them into 4 identical farm houses?

C D M H G S



How is this problem different from the previous (3) problem?

Think about the following distribution in 3)

{1,5,0,0}

Now, in this case, for this distribution there will be 6 different distributions because the single animal could be any of the six animals. That is to say,

Distr #1: {C, {D,M,H,G,S}, {}, {}}
Distr #2: {D, {C,M,H,G,S}, {}, {}}
Distr #3: {M, {C,D,H,G,S}, {}, {}}
Distr #4: {H, {C,M,D,G,S}, {}, {}}
Distr #5: {G, {C,M,H,D,S}, {}, {}}
Distr #6: {S, {C,M,H,G,D}, {}, {}}

For this distribution however,

{6,0,0,0}

There will still be only one distribution (all animals in the same farm). That is to say,

Distr #7: {{S,C,M,H,G,D}, {}, {}, {}}

Therefore, considering all possibilities gives us the following table:

	identical animals	**different animals**
	{6,0,0,0}	1
	{5,1,0,0}	C(6,1) = 6
	{4,2,0,0}	C(6,4) = 15
	{4,1,1,0}	C(6,4) = 15
	{3,3,0,0}	C(6,3) = 20
	{3,2,1,0}	C(6,3)xC(3,2) = 60
	{3,1,1,1}	C(6,3) = 20
	{2,2,2,0}	C(6,4)xC(4,2) = 90
	{2,2,1,1}	C(6,2)xC(4,2) = 90
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