CMPE 352
Signal Processing & Algorithms

Spring 2019

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Review Questions (1)

- What is the form of the even/odd decomposition of a signal?

$$x(t) = x_e(t) + x_o(t)$$

- Let $s(t) = e^{\alpha t}$. What is the condition on α to obtain a decaying (growing) exponential?

  $\alpha < 0$ ($\alpha > 0$)

- How are ″frequency″ f and ″radian frequency″ ω related? What are their units?

  $\omega = 2\pi f$ f: 1/s (Hz)
  ω: rad/s

Review Questions (2)

- The signal $x(t)$ (where t is measured in seconds) is delayed by 1ms. What is the expression of the delayed signal?

$$x(t - 0.001)$$

- The signal $x(t)$ is expanded in time by a factor of 4. What is the expression of the time-expanded signal?

$$x(t/4)$$

- We add the time-reversed signal of $x(t)$ to the signal $x(t)$ itself. What property does the resulting signal have?

$x(t) + x(-t)$ is an even signal

Review Questions (3)

- What are the three basic signal operations?

  Time shifting
  Time scaling
  Time reversal

- What are the four elementary signals defined during last week’s lecture?

  Unit impulse
  Unit step
  Exponential
  Sinusoid

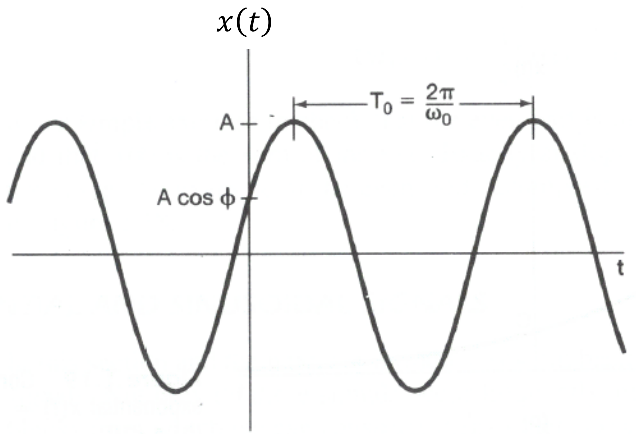
- What are harmonic signals?

  Sinusoidal signals whose frequencies are integer multiples of a basic (fundamental) frequency

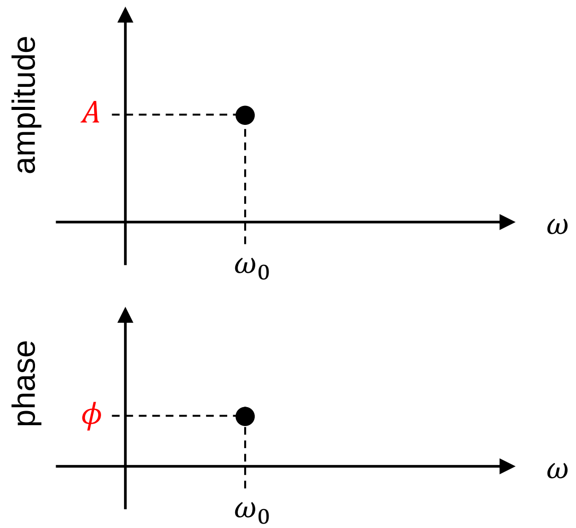
Frequency Representation of a Sinusoid

Time-domain graphical representation

$$x(t) = A\cos(\omega_0 t + \phi)$$



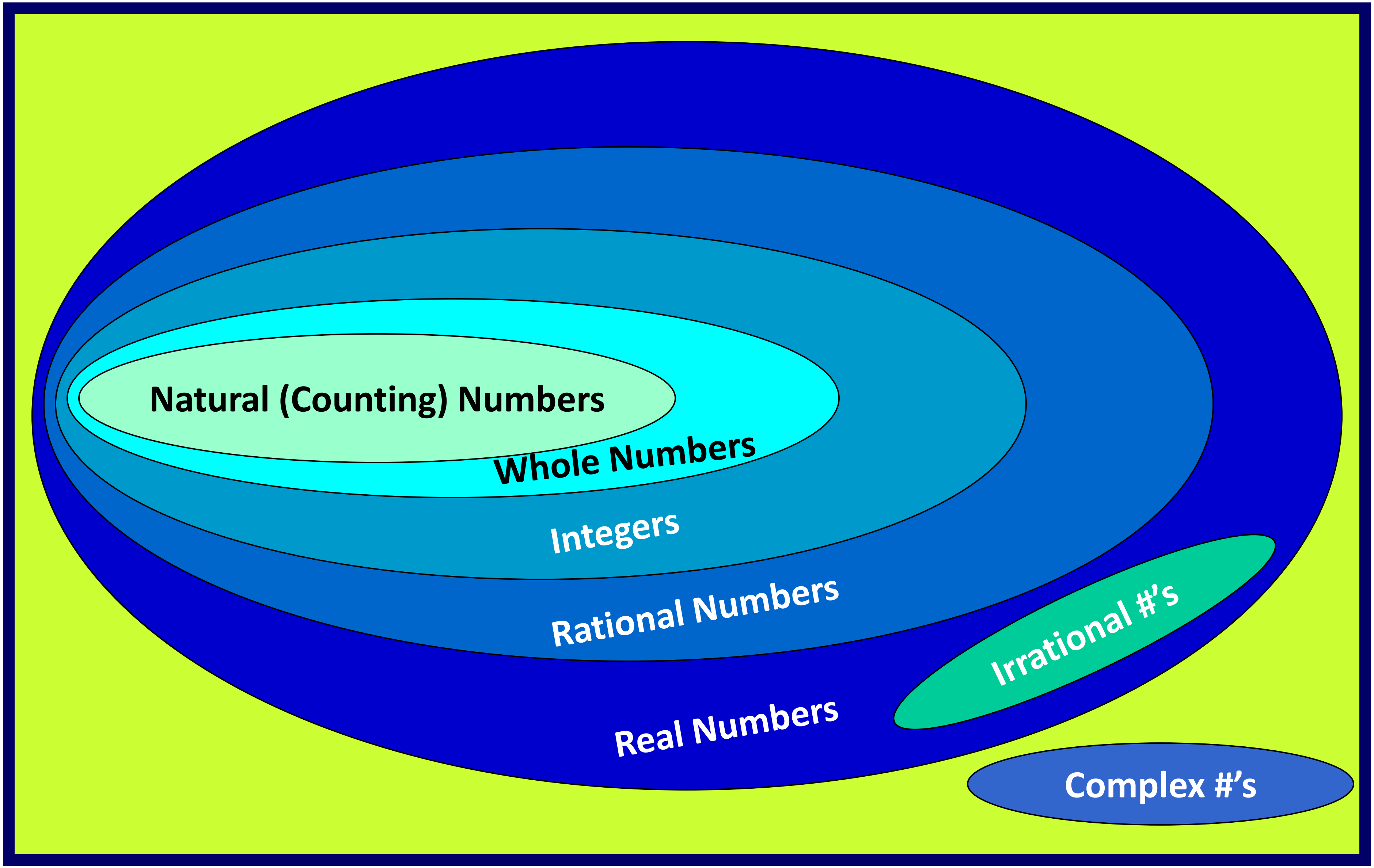
Frequency-domain graphical representation



→ Idea: represent A and ϕ as a single number: $Ae^{j\phi}$

→ Complex number

Complex Numbers



Complex Numbers

Complex numbers are written in the form $a + bj$, where a is the real part and b is the imaginary part

$$\underbrace{a}_{\text{real part}} + \underbrace{bj}_{\text{imaginary part}}$$

$\boxed{\text{Note: } j^2 = -1}$

Complex Numbers: Addition

When adding complex numbers, add the real parts together and add the imaginary parts together

imaginary parts

$$(3 + 7j) + (8 + 11j)$$

real parts

$$\boxed{11 + 18j}$$

Complex Numbers: Subtraction

When subtracting complex numbers, be sure to distribute the subtraction sign; then add like parts

$$(5 + 10j) - (15 - 2j)$$

$$5 + 10j - 15 + 2j$$

$$\boxed{-10 + 12j}$$

Complex Numbers: Multiplication

When multiplying complex numbers, use the distributive property and simplify.

$$(3 - 8j)(5 + 7j)$$

$$15 + 21j - 40j - 56j^2$$

Remember
$j^2 = -1$

$$15 - 19j + 56$$

$$\boxed{71 - 19j}$$

Complex Numbers: Division

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the complex number in the denominator of the fraction.

$$\frac{7+2j}{3-5j}$$

The complex conjugate of $3-5j$ is $3+5j$

Complex Numbers (6)

$$\frac{7+2j}{3-5j}\,\frac{3+5j}{3+5j} \Rightarrow \frac{21+35j+6j+10j^2}{9+15j-15j-25j^2}$$

$$\Rightarrow \frac{21+41j-10}{9+25} \qquad \boxed{\Rightarrow \frac{11+41j}{34}}$$

Complex Numbers (7)

More examples:

$$(3+5j)-(11-9j) \qquad -8+14j$$

$$(5-6j)(2+7j) \qquad 52+23j$$

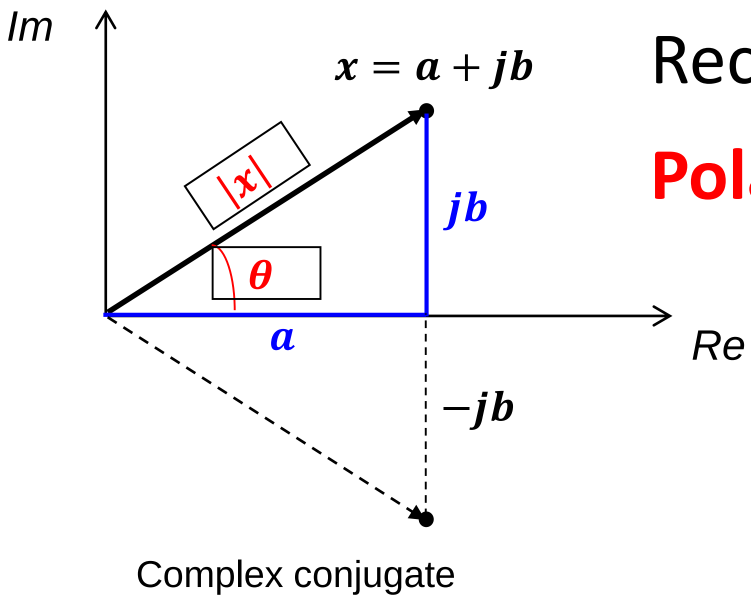
$$\frac{2-3j}{5+8j} \qquad \frac{-14-31j}{89}$$

Complex Numbers: powers of *j*

Investigate the powers of j:

Power	Exponential form	Simplified
1	j	j
2	j^2	-1
3	j^3	$-j$
4	j^4	1
27	j^{27}	$-j$
-1	j^{-1}	$-j$
-10	j^{-10}	-1

Complex Numbers – Polar Coordinates (1)



Rectangular coordinates: $x = a + jb$

Polar coordinates: $x = |x|e^{j\theta}$

Magnitude: $|x| = \sqrt{a^2 + b^2}$

Phase: $\theta = atan\frac{b}{a}$

Note: The name ″polar coordinates″ comes from thinking of the origin of the plane at (0,0) as being the pole of the coordinate system.

Complex Numbers – Polar Coordinates (2)

Complex conjugate of $x = a + jb = |x|e^{j\theta}$:

$$x^* = a - jb = |x|e^{-j\theta}$$

Note: $|x|^2 = x \cdot x^*$

Complex rational number:

$$z = \frac{x}{y} = \frac{a + jb}{c + jd} = \frac{|x|e^{j\theta_x}}{|y|e^{j\theta_y}} = |z|e^{j\theta_z}$$

$$|z| = \frac{|x|}{|y|} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

$$\theta_z = \theta_x - \theta_y = atan\frac{b}{a} - atan\frac{d}{c}$$

Complex Numbers – Polar Coordinates (3)

Example:

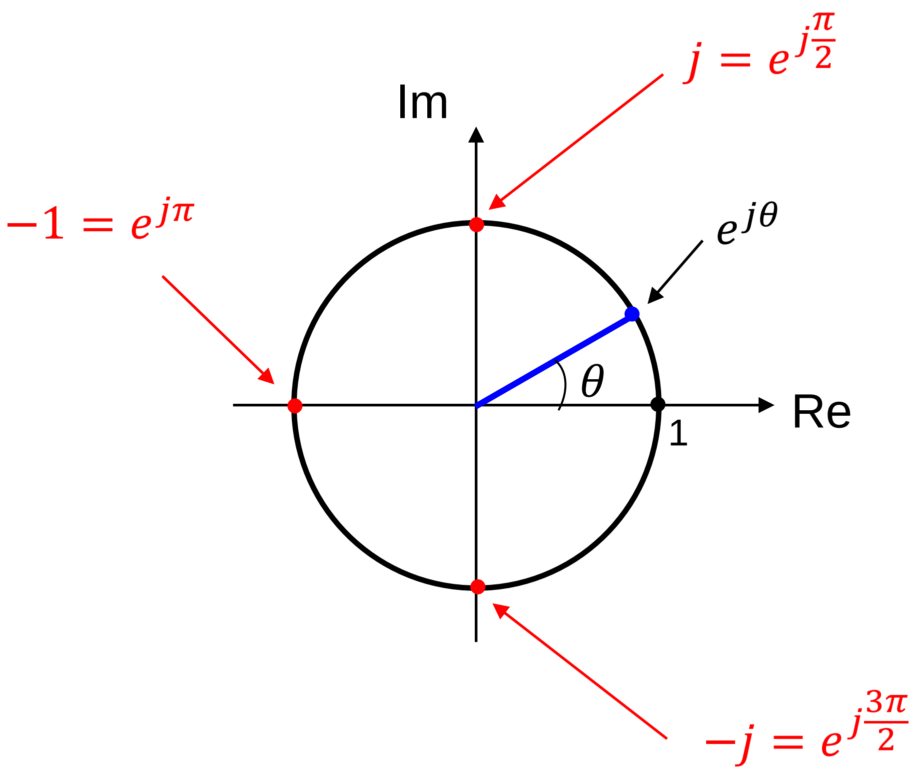
$$\frac{3+4j}{1-2j} = \frac{(3+4j)(1+2j)}{(1-2j)(1+2j)} = \frac{-5+10j}{5} = -1 + j2$$

Alternatively:

$$\frac{3+4j}{1-2j} = \frac{5e^{jatan(4/3)}}{\sqrt{5}e^{-jatan(2/1)}} = \sqrt{5}e^{j\left[atan\frac{4}{3}+atan2\right]}$$

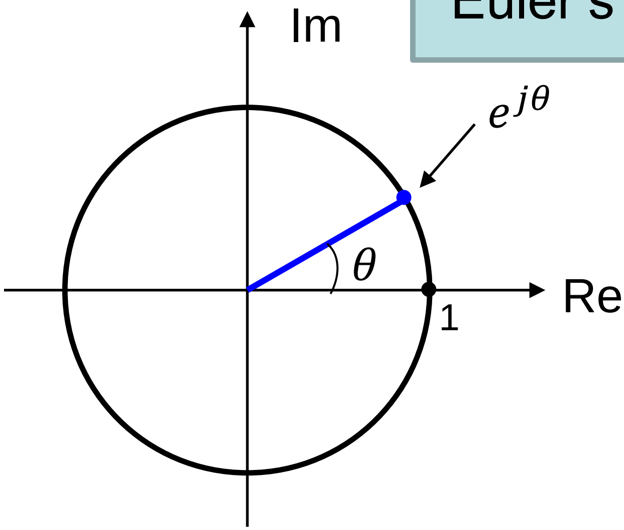
$$= \sqrt{5}e^{j[53.13^o+63.43^o]} = \sqrt{5}e^{j116.56^o} = -1 + j2$$

The Unit Circle

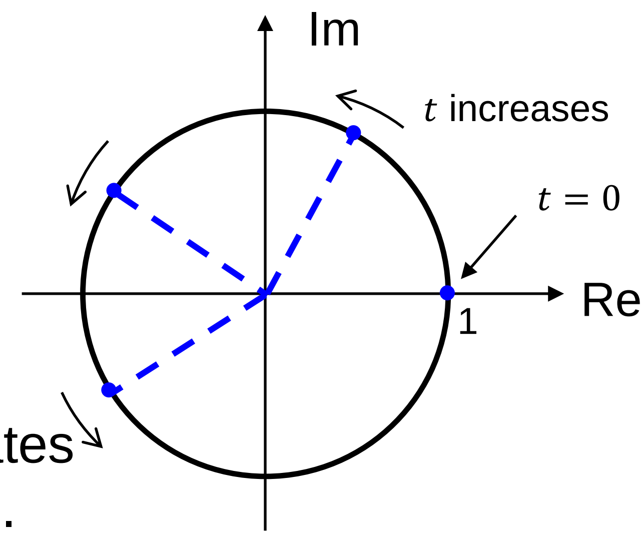


The Complex Exponential (1)

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$



Let θ be a function of time: $\theta = \omega t$
What happens?



As time goes by, we progress along the unit circle !

We can think of $e^{j\omega t}$ as a function that rotates (counterclockwise) in the complex plane.

Note that e^{t} (real) is nothing like that !

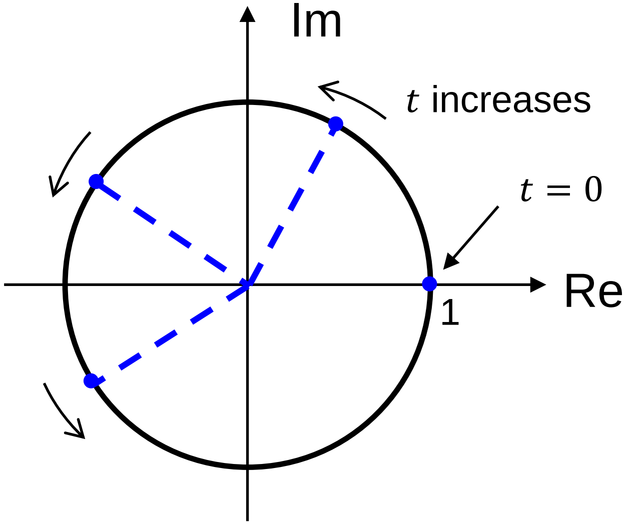
$e^{j\omega t}$ describes circular motion in the complex plane

Note: Leonhard Euler, Swiss mathematician (1707-1783)

The Complex Exponential (2)

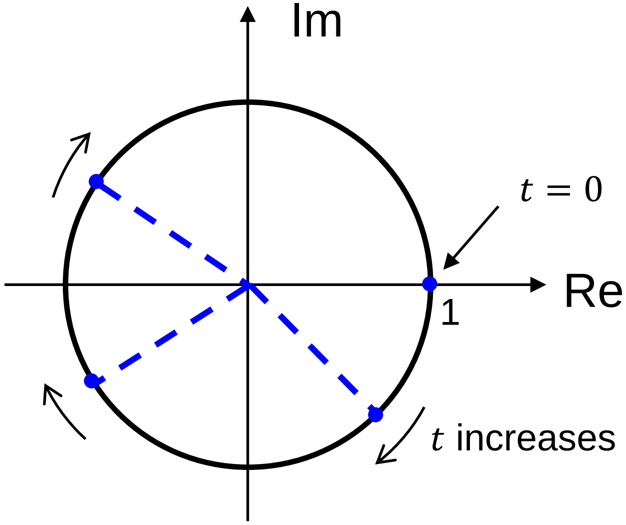
Let now: $\theta = -\omega t$ What happens? Clockwise rotation !

Hence:



$\boxed{e^{j\theta} = e^{j\omega t}}$

Counterclockwise rotation



$\boxed{e^{j\theta} = e^{-j\omega t}}$

Clockwise rotation

The Complex Exponential (3)

- $\cos \omega t$ is simply the projection of the circular motion along the Real axis
- $\sin \omega t$ is simply the projection of the circular motion along the Imaginary axis

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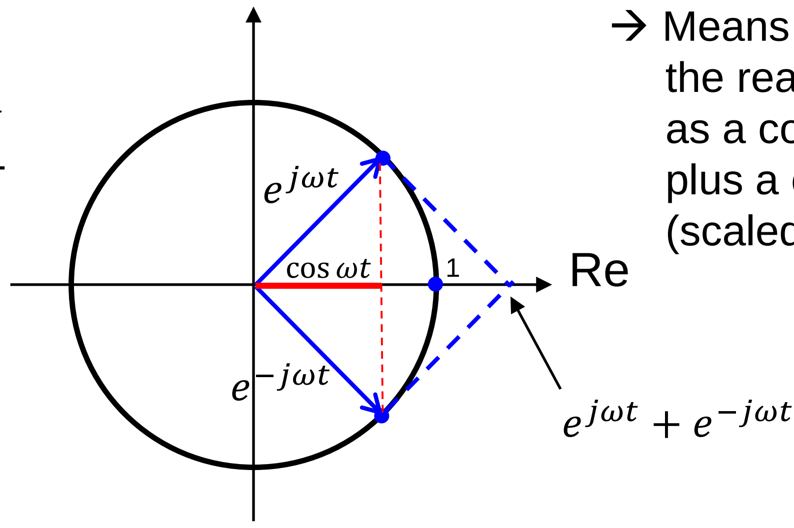
The Complex Exponential (4)

Euler's formula:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \Rightarrow \begin{cases} \cos\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2} \\[2ex] \sin\omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j} \end{cases}$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$



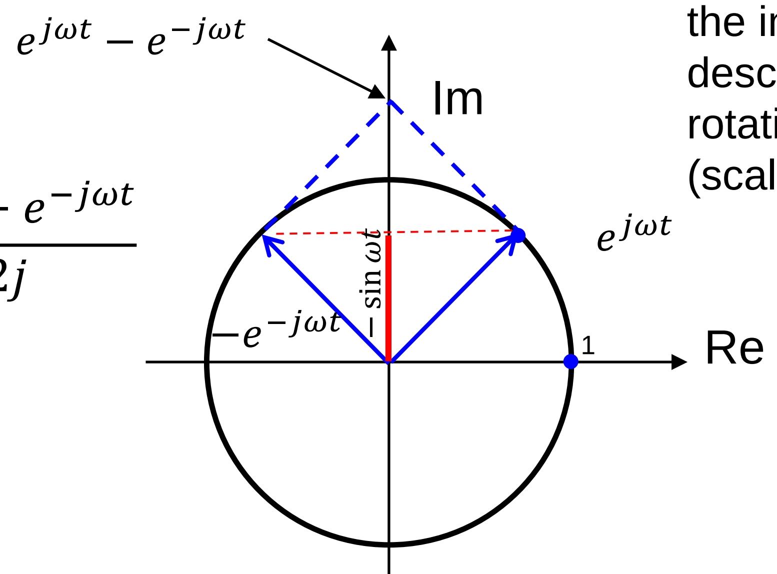
→ Means that the projection along the real axis can be described as a counterclockwise rotation plus a clockwise rotation (scaled by 2)

The Complex Exponential (5)

Euler's formula:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \Rightarrow \begin{cases} \cos\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2} \\ \sin\omega t = \dfrac{e^{j\omega t} - e^{-j\omega t}}{2j} \end{cases}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$



→ Means that the projection along the imaginary axis can be described as a counterclockwise rotation minus a clockwise rotation (scaled by $2j$)

The Complex Exponential (6)

In signal processing and many disciplines of engineering, $e^{j\omega t}$ is regarded as the <u>most elementary function</u>: it contains only one frequency (ω).

By contrast, in the complex plane, $\cos \omega t$ and $\sin \omega t$ contain two frequency components: ω and $-\omega$ (see Euler's formula).

We need these two components to stay on the real axis (or on the imaginary axis).

Sum of sinusoidal functions

Take a number of sinusoidal functions and add them. What do you get?

For example let us take these 3 particular cosine functions:

$$x_1(t) = A_1 \cos \omega_1 t = A_1 \cos 2\pi f_1 t = \frac{1}{2}\cos 2\pi t \qquad \Rightarrow A_1 = \frac{1}{2}, \quad \omega_1 = 2\pi \text{ (rad/s)}, \quad f_1 = 1\,\text{Hz}$$

$$x_2(t) = A_2 \cos \omega_2 t = A_2 \cos 2\pi f_2 t = \cos(4\pi t) \qquad \Rightarrow A_2 = 1, \quad \omega_2 = 4\pi \text{ (rad/s)}, \quad f_2 = 2\,\text{Hz}$$

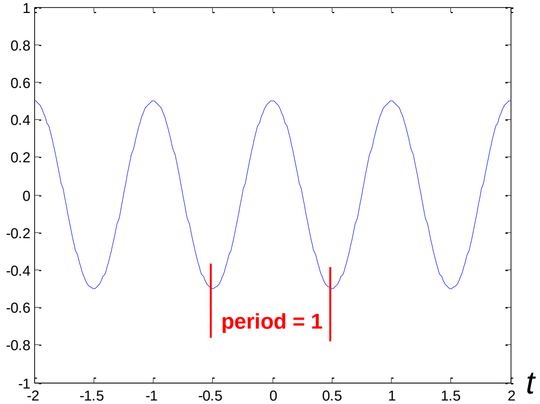
$$x_3(t) = A_3 \cos \omega_3 t = A_3 \cos 2\pi f_3 t = \frac{2}{3}\cos(6\pi t) \qquad \Rightarrow A_3 = \frac{2}{3}, \quad \omega_3 = 6\pi \text{ (rad/s)}, \quad f_3 = 3\,\text{Hz}$$

and let us add them:

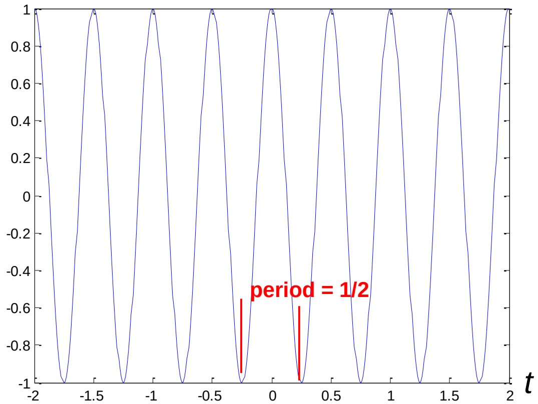
$$x(t) = x_1(t) + x_2(t) + x_3(t) = \frac{1}{2}\cos 2\pi t + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

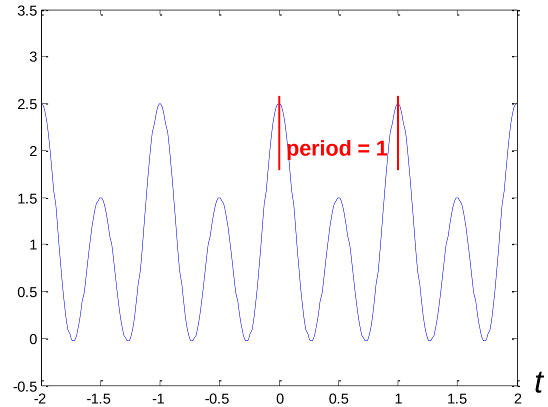
Sum of sinusoidal functions (cntd)

$$x_1(t) = \frac{1}{2}\cos 2\pi t$$



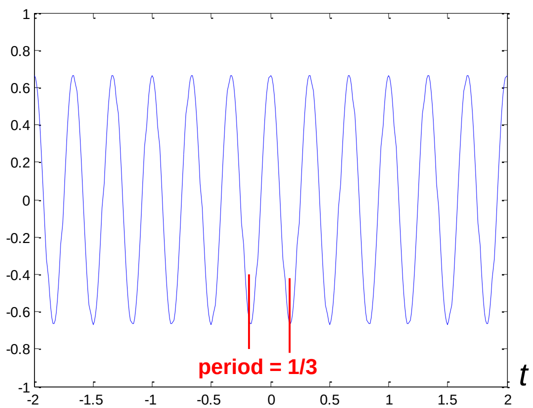
$$x_2(t) = \cos 4\pi t$$

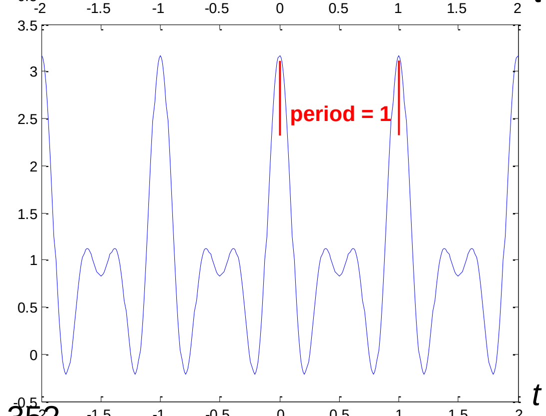




$$x_1(t) + x_2(t)$$

$$x_3(t) = \frac{2}{3}\cos 6\pi t$$





$$x_1(t) + x_2(t) + x_3(t)$$

Sum of sinusoidal functions (cntd)

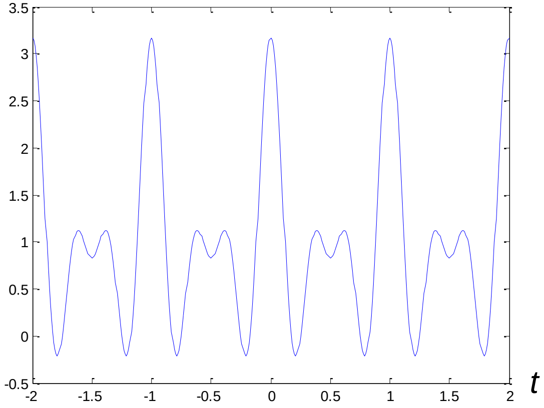
So by adding....

$$x_1(t) = \frac{1}{2}\cos 2\pi t$$

$$x_2(t) = \cos 4\pi t$$

$$x_3(t) = \frac{2}{3}\cos 6\pi t$$

we've got:



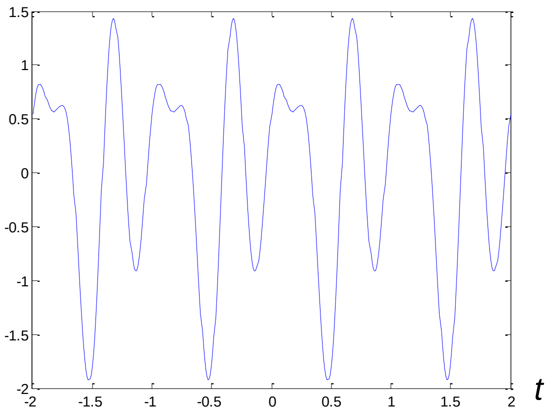
$$x_1(t) + x_2(t) + x_3(t)$$

Another parameter that we can vary is the phase shift of each sinusoid. For example let add a phase shift to the 2nd sinusoid:

$$x_1(t) = \frac{1}{2}\cos 2\pi t$$

$$x_2(t) = \cos(4\pi t - 2.25)$$

$$x_3(t) = \frac{2}{3}\cos 6\pi t$$



$$x_1(t) + x_2(t) + x_3(t)$$

Varying the phase shift of a single sinusoid had a significant effect on the shape of the total signal !

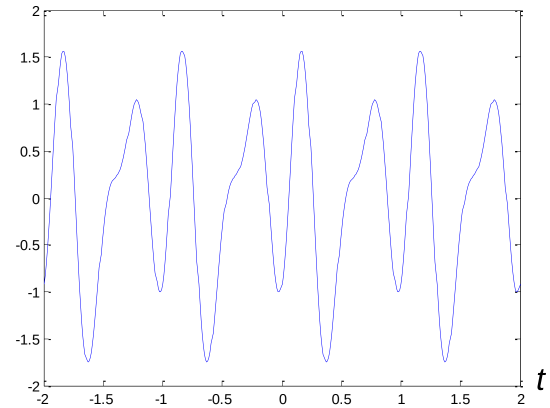
Sum of sinusoidal functions (cntd)

Another example:

$$x_1(t) = \frac{1}{2}\cos(2\pi t + 0.7)$$

$$x_2(t) = \cos(4\pi t - 2.25)$$

$$x_3(t) = \frac{2}{3}\cos(6\pi t - \pi)$$



$$x_1(t) + x_2(t) + x_3(t)$$

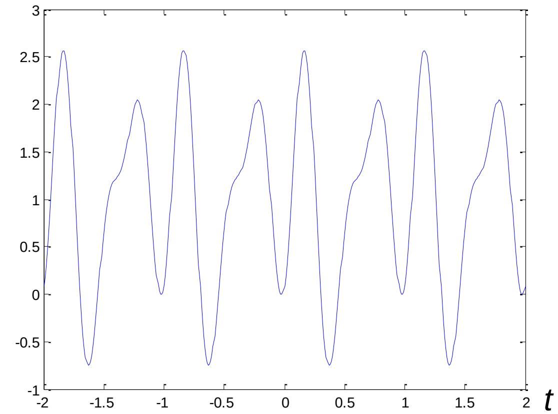
We can also add a constant:

$$x_0(t) = a_0 = 1.0$$

$$x_1(t) = \frac{1}{2}\cos(2\pi t + 0.7)$$

$$x_2(t) = \cos(4\pi t - 2.25)$$

$$x_3(t) = \frac{2}{3}\cos(6\pi t - \pi)$$



$$x_0(t) + x_1(t) + x_2(t) + x_3(t)$$

Sum of sinusoidal functions (cntd)

In summary, we have obtained a fairly general expression, such as:

$$\begin{cases} x_0(t) = a_0 = 1.0 \\ x_1(t) = \frac{1}{2}\cos(2\pi t + 0.7) = \frac{1}{2}\cos(\omega_1 t + \theta_1) = \frac{1}{2}\cos(1 \cdot \omega_0 t + \theta_1) \\ x_2(t) = \cos(4\pi t - 2.25) = \cos(\omega_2 t + \theta_2) = \cos(2\omega_0 t + \theta_2) \\ x_3(t) = \frac{2}{3}\cos(6\pi t - \pi) = \frac{2}{3}\cos(\omega_3 t + \theta_3) = \frac{2}{3}\cos(3\omega_0 t + \theta_3) \end{cases} \qquad (\omega_0 = 2\pi \text{ rad/s})$$

$$\Rightarrow x(t) = x_0(t) + x_1(t) + x_2(t) + x_3(t)$$

$$= 1.0 + \frac{1}{2}\cos(1 \cdot \omega_0 t + \theta_1) + \cos(2\omega_0 t + \theta_2) + \frac{2}{3}\cos(3\omega_0 t + \theta_3)$$

In this example we have added 3 sinusoids (harmonics), but we can in general add as many (harmonic) sinusoids as we want.

By adding an infinite number of sinusoids (each properly scaled (A) and shifted (θ)) with frequencies that are multiples of a reference (fundamental) frequency (harmonic frequencies) we can essentially generate (synthesize) any periodic signal shape !

The Fourier Series

Hence we can write in general

Fourier Series
(trigonometric form)

$$\boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)}$$

By adding an infinite number of sinusoids (each properly scaled A_k and shifted θ_k) with frequencies $k\omega_0$ that are harmonics of a fundamental frequency ω_0, we can synthesize any periodic signal !

We can also do the converse: given any periodic signal $x(t)$, we can determine what "cos" terms it contains, that is: determine A_k and θ_k for each frequency $k\omega_0$. This way, we can analyze any given signal. How? More later...

The Fourier Series

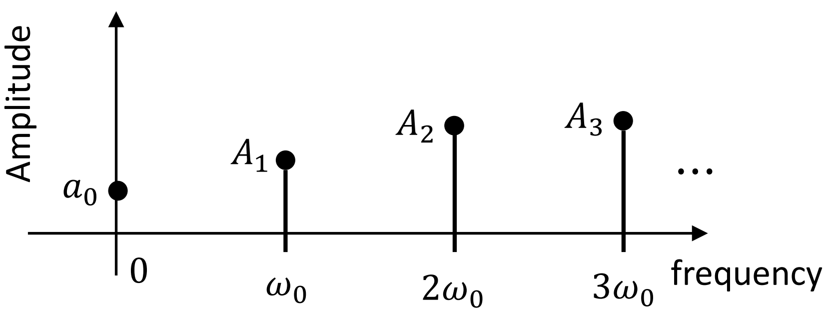
- The Fourier series is a method of expressing (most) periodic time-domain signals in the frequency domain (that is, with A_k's and $\phi_k{}'$s for all k).
- Then how can we represent this signal in the frequency domain graphically?

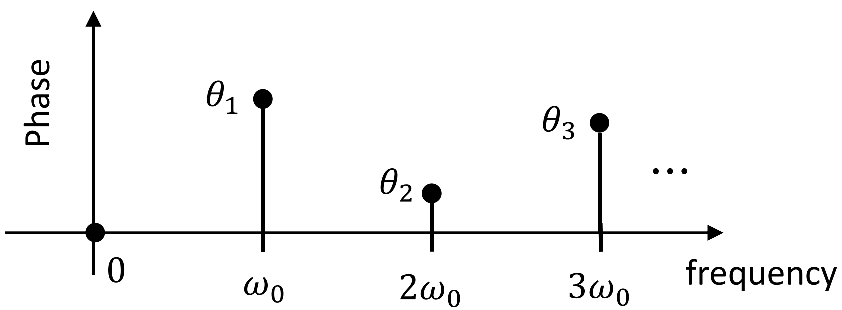
$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Amplitude at frequency $k\omega_0$

frequency: $k\omega_0$

Phase at frequency $k\omega_0$





The Fourier Series

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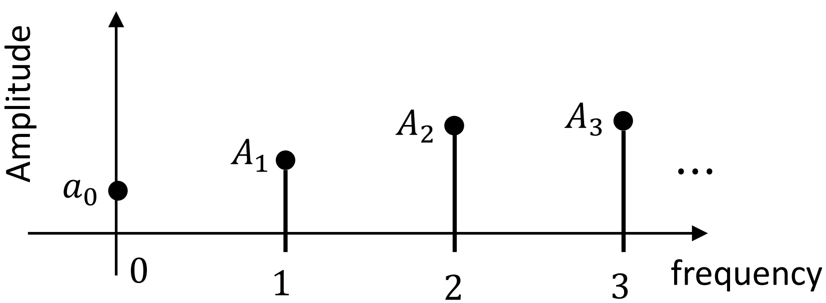
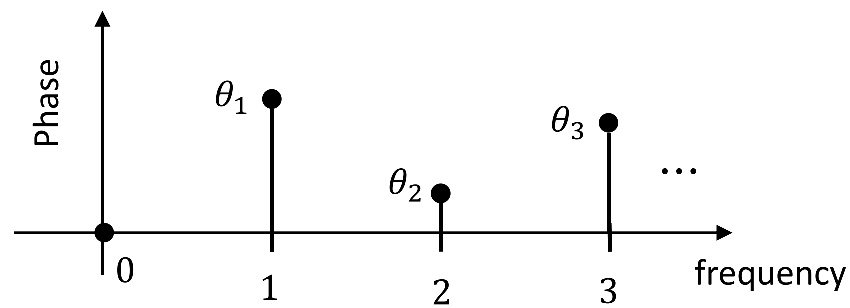




The Fourier Series

The frequency-domain representation appears graphically as a series of <u>lines</u> occurring at the fundamental frequency (determined by the period of the original signal) and its harmonics.

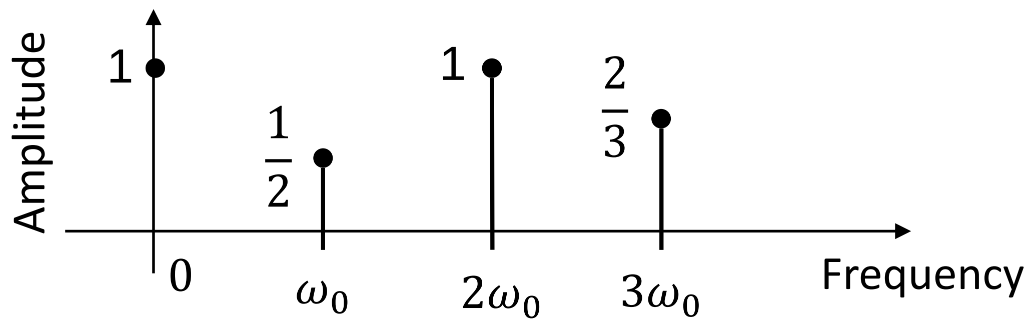
The magnitudes A_k of these lines are the **<u>Fourier coefficients</u>**.

This series of components are called the **<u>signal spectrum</u>**.

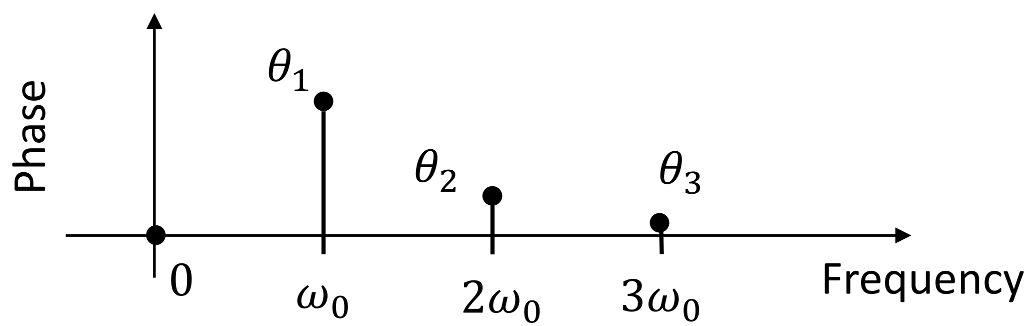
Magnitude and Phase Spectra

Example:

$$x(t) = 1.0 + \frac{1}{2}\cos(1 \cdot \omega_0 t + \theta_1) + \cos(2\omega_0 t + \theta_2) + \frac{2}{3}\cos(3\omega_0 t + \theta_3)$$



Amplitude spectrum



Phase spectrum

The Fourier Series

Fourier Series: $$\boxed{x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)}$$ $(\omega_0 = \frac{2\pi}{T_0})$

Fundamental question: if I know the signal $x(t)$, can I compute a_0, A_k, and θ_k?

That is: if I know the time-domain signal, can I compute its frequency spectrum?

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\, dt$$

$$b_k = \frac{1}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\, dt$$

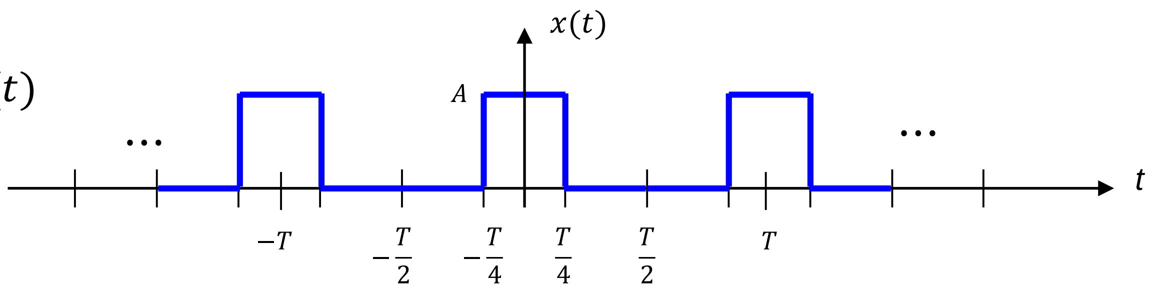
$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \operatorname{atan}\frac{-b_k}{a_k}$$

Can you show this?

The Fourier Series: Example

The signal $x(t)$ is given:



$a_0 = A/2$ (= average value of $x(t)$ over 1 period)

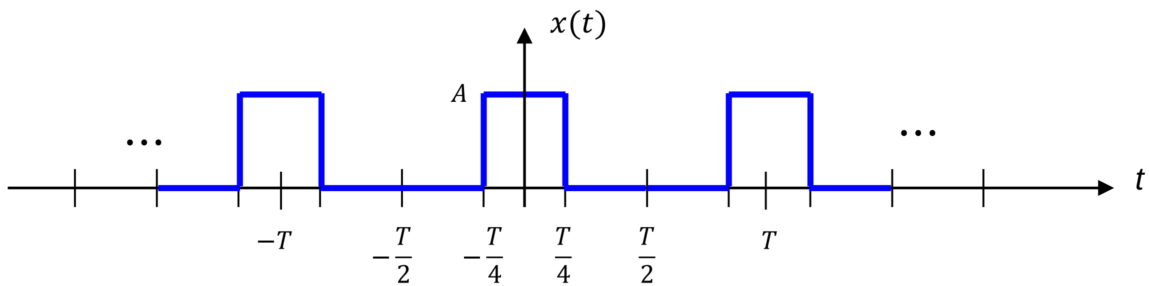
$$\alpha_k = \frac{1}{T}\int_T x(t)\cos\left(k\frac{2\pi}{T}t\right)dt = \frac{A}{T}\int_{-T/4}^{T/4}\cos\left(k\frac{2\pi}{T}t\right)\,dt = \cdots = \frac{A}{\pi k}\sin(k\frac{\pi}{2})$$

$$\beta_k = \frac{1}{T}\int_T x(t)\sin\left(k\frac{2\pi}{T}t\right)dt = \frac{A}{T}\int_{-T/4}^{T/4}\sin\left(k\frac{2\pi}{T}t\right)\,dt = 0$$

$$A_k = \sqrt{\alpha_k^2+\beta_k^2} = \alpha_k = \frac{A}{\pi k}\sin\left(k\frac{\pi}{2}\right) = \begin{cases} \frac{A}{\pi k}(-1)^{(k-1)/2} & k \text{ odd} \\ 0 & k \text{ even } (\neq 0) \end{cases}$$

$$\theta_k = \operatorname{atan}\left(\frac{-\beta_k}{\alpha_k}\right) = 0$$

The Fourier Series: Example (cntd)



$$a_0 = A/2 \qquad A_k = \begin{cases} \dfrac{A}{\pi k}(-1)^{(k-1)/2} & k \text{ odd} \\ 0 & k \text{ even } (\neq 0) \end{cases} \qquad \theta_k = 0$$

$$\Rightarrow x(t) = \frac{A}{2} + \frac{2A}{\pi}\cos(\omega_0 t) - \frac{2A}{3\pi}\cos(3\omega_0 t) + \frac{2A}{5\pi}\cos(5\omega_0 t) - \frac{2A}{7\pi}\cos(7\omega_0 t) + \cdots$$