MATH 233
Fall 2018
Quiz #1

Name Lastname :
ID :

***<u>Duration:</u>* 50 minutes.**
***<u>Remark:</u>* Show your thinking/work. Do not just write a number as a result.**

1. A fair coin is flipped 10 times where each flip comes up either **head** or **tails**.

a) How many possible outcomes are there total?
b) How many of these outcomes contain a single tail or a single head?

2. Prove the following identity **<u>using induction</u>**:

$$1.2 + 2.3 + 3.4 + ... + (n-1).n = ((n-1).n.(n+1)) / 3$$

MATH 233
Fall 2018
Quiz #1 A Solutions.

<u>Duration:</u> 50 minutes.
<u>Remark:</u> Show your thinking/work. Do not just write a number as a result.

1. A fair coin is flipped 10 times where each flip comes up either **head** or **tails**.

a) How many possible outcomes are there total?
b) How many of these outcomes contain a single tail or a single head?

a) Outcomes are:
HHHHHHHHHH
HHHHHHHHHT
HHHHHHHHTH
….
….
TTTTTTTTTTTT

There are $2^{10} = 1024$ difference outcomes.

b) Outcomes containing a single head or a single tail are:

HHHHHHHHHT
HHHHHHHHTH
HHHHHHHTHH
…
THHHHHHHHH
and
TTTTTTTTTH
TTTTTTTTHT
…
HTTTTTTTTT
Thus, there are 10 + 10 = 20 such outcomes.

2. Prove the following identity **<u>using induction</u>**:

$$1.2 + 2.3 + 3.4 + \ldots + (n-1).n = ((n-1).n.(n+1)) / 3$$

a) Base case: Does the assertion hold for n=2?

left hand side of the equation is 1.2= 2 for n=2.
right hand side of the equation is (1.2.3)/3 = 2

Therefore the equation holds for n=2.

(See that it also holds for n=1)

b) **Inductive step:** Assuming that the assertion holds for k, show that it also holds for k+1.

$1.2 + 2.3 + 3.4 + \ldots + (k-1).k = ((k-1).k.(k+1)) / 3$ is given.

Show that : $1.2 + 2.3 + 3.4 + \ldots + (k-1).k + k.(k+1) = ((k).(k+1).(k+2)) / 3$.

$$\begin{aligned}
1.2 + 2.3 + 3.4 + \ldots + (k-1).k + k.(k+1) &= ((k-1).k.(k+1)) / 3 + k.(k+1) \\
&= (k+1)\ ((k^2-k)/3 + k) \\
&= (k+1)\ (k^2-k + 3k)/3 \\
&= (k+1)\ (k^2+2k)/3 \\
&= (k+1).\ k(k+2)/3
\end{aligned}$$

Base case and inductive steps show that the assertion holds for all n larger than or equal to 0.

MATH 233
Fall 2018
Quiz #1 B

Name Lastname :
ID :

***<u>Duration:</u>* 50 minutes.**
***<u>Remark:</u>* Show your thinking/work. Do not just write a number as a result.**

1. Consider all bitstrings of length 16. A bitstring is made up of bits that are either 0 or 1. For example, 00100111 is a bitstring of length 8.

a) How many possible bitstrings of length 16 are there?
b) How many of bitstrings of length 16 contain a single 1 or a single 0?

2. Prove the following identity **<u>using induction</u>**:

$$1 + 8 + 27 + \ldots + n^3 = (n^2.(n+1)^2\ ) / 4$$

MATH 233
Fall 2018
Quiz #1 B Solutions

Duration: ***50 minutes.***
Remark: ***Show your thinking/work. Do not just write a number as a result.***

1. Consider all bitstrings of length 16. A bitstring is made up of bits that are either 0 or 1. For example, 00100111 is a bitstring of length 8.

a) How many possible bitstrings of length 16 are there?
b) How many of bitstrings of length 16 contain a single 1 or a single 0?

a) Each bit can be either 0 or 1, Therefore for 16 bits there are 2^{16} = 65536 different choices.

b) Bitstrings that contain a single 0 are:

0111111111111111
1011111111111111
…
1111111111111110

There are 16 such bitstrings.

Similarly there are 16 bitstrings that contain a single 1.

Thus, in total there are **32 bitstrings** of length 16 contain a single 1 or a single 0

2. Prove the following identity **using induction**:

$$1 + 8 + 27 + \ldots + n^3 = (n^2.(n+1)^2) / 4$$

a) **Base case:** For n=1,

The left hand side of the equation is $1^3 = 1$.
The right hand side of the equation is $1.2^2 / 4 = 1$

Thus, the assertion holds for n=1.

b) **Inductive step:** Assume the assertion holds for k,
i.e., $1 + 8 + 27 + \ldots + k^3 = (k^2.(k+1)^2) / 4$

Show that it holds for k+1. That is to say, show that $1 + 8 + 27 + \ldots + k^3 + (k+1)^3 = ((k+1)^2.(k+2)^2) / 4$

$$\begin{aligned}1 + 8 + 27 + \ldots + k^3 + (k+1)^3 &= (k^2.(k+1)^2) / 4 + (k+1)^3 \\ &= (k+1)^2 (k^2 + 4k+4) / 4 \\ &= (k+1)^2 (k+2)^2 / 4\end{aligned}$$

Base case and inductive steps show that the assertion holds for all n larger than or equal to 1.

MATH 233
Fall 2018
Quiz #2 A

***<u>Duration:</u>* 50 minutes.**
***<u>Remark:</u>* Show your thinking/work. Do not just write a number as a result.**

1. A person can take <u>one stair, two stairs or three stairs</u> at a time when climbing a stairway.

- Find a **recurrence relation** for the number of ways to climb n stairs.
- What are the **initial conditions**?
- In how many ways can the person climb a 10-stair stairway?

2. A fair dice and two fair coins are tossed.

a) What is the **experiment**?
b) What is the **sample space**?
c) What is the **size** of the sample space?
d) What is the probability that a head occurs? (Describe the event E_H)
e) What is the probability that a 6 occurs? (Describe the event E_6)
f) What is the probability that the number on the dice is equal to the number of heads or tails? (Describe the event E_{same})

MATH 233
Fall 2018
Quiz #2 A Solutions

***<u>Duration:</u>* 50 minutes.**
***<u>Remark:</u>* Show your thinking/work. Do not just write a number as a result.**

1. A person can take <u>one stair, two stairs or three stairs</u> at a time when climbing a stairway.

- Find a **recurrence relation** for the number of ways to climb n stairs.

# of stairs	climbing ways	# of climbs
1	1	1
2	1-1, 2	2
3	1-1-1, 2-1, 1-2, 3	4
4	1-1-1-1, 2-1-1, 1-2-1, 3-1, 1-1-2, 2-2, 1-3	7

As can be seen from column 2, the different climbing ways for n stairs is the sum of:

1) climbing ways for n-1 stairs and a final one step
2) climbing ways for n-2 stairs and a final 2 stair-step
3) climbing ways for n-3 stairs and a final 3-stair step

If W_n is the number of ways to slimb n stairs, then the recurrence relation is:
$W_n = W_{n-1} + W_{n-2} + W_{n-3}$

- What are the **initial conditions**?

$W_1 = 1$ $W_2 = 2$ and $W_3 = 4$

- In how many ways can the person climb a 10-stair stairway?

$W_{10} = W_9 + W_8 + W_7 = (W_8 + W_7 + W_6) + W_8 + W_7 = 2W_8 + 2W_7 + W_6$
$= 4W_7 + 3W_6 + 2W_5 = 7W_6 + 6W_5 + 4W_4 = 13W_5 + 11W_4 + 7W_3 =$
$24W_4 + 20W_3 + 13W_2 = 24.7 + 20.4 + 13.2 = 274$

2. A fair dice and two fair coins are tossed.

a) What is the **experiment**?

A fair dice and two fair coins are tossed.

b) What is the **sample space**?

Sample Space = { {1,H,H}, {1,H,T}, {1,T,H}, {1,T,T},
{2,H,H}, {2,H,T}, {2,T,H}, {2,T,T},
….
{6,H,H}, {6,H,T}, {6,T,H}, {6,T,T}}.

c) What is the **size** of the sample space?

I Sample Space I = 6.2.2 = 24

d) What is the probability that a head occurs? (Describe the event E_H)

E_H = The event that a head occurs in the outcome.
It is easier to think about the complement event, the event that a head does not occur in the outcome (i.e. both coints show tails).

$\overline{E_H}$ = The event that a head does **not** occur in the outcome

$\overline{E_H}$ = {{1,T,T},{2,T,T},{3,T,T},{4,T,T},{5,T,T},{6,T,T}}

I $\overline{E_H}$ I = 6 and therefore IE_H I = 24 - 6 = 18

P(E_H) = IE_H I / I Sample Space I = 18 / 24 = 0.75

e) What is the probability that a 6 occurs? (Describe the event E_6)

E_6 = {{6,H,H}, {6,H,T}, {6,T,H}, {6,T,T}}.
I E_6 I = 4
P(E_6) = IE_6 I / I Sample Space I = 4 / 24 = 1/6 = 0.167

f) What is the probability that the number on the dice is equal to the number of heads or tails? (Describe the event E_{same})

What outcomes are in E_{same}?

E_{same}= {{1,H,T}, {1,T,H}, {2,H,H},{2,T,T}}
P(E_{same})= IE_{same}I / I Sample Space I = 4/24 = 0.167

MATH 233
Fall 2018
Quiz #2 B

***<u>Duration:</u>* 50 minutes.**
***<u>Remark:</u>* Show your thinking/work. Do not just write a number as a result.**

1. A cell divides into two in every minute. Assume we have a single cell in a laboratory tube.

   - Find a **recurrence relation** for the number of cells after n minutes.

   - What is/are the **initial condition(s)**?

   - What is the number of cells after an hour?

2. Two fair dice and a fair coin are tossed.

a) What is the **experiment**?
b) What is the **sample space**?
c) What is the **size** of the sample space?
d) What is the probability that a head occurs? (Describe the event E_H)
e) What is the probability that a 6 occurs? (Describe the event E_6)
f) What is the probability that the total number on the dice is more than the number of heads? (Describe the event E_{more})

MATH 233
Fall 2018
Quiz #2 B Solutions

Duration: ***50 minutes.***
Remark: ***Show your thinking/work. Do not just write a number as a result.***

1. A cell divides into two in every minute. Assume we have a single cell in a laboratory tube.

   - Find a **recurrence relation** for the number of cells after n minutes.

minute	cells	# of celss
0	1	1
1	1,1	2
2	1,1,1,1	4
3	1,1,1,1,1,1,1,1	8

Let C_n be the number of cells at minute n. The recurrence relation is:
$C_n = 2 \cdot C_{n-1}$

- What is/are the **initial condition(s)**?

$C_0 = 1$

- What is the number of cells after an hour?

From the recurrence relation, we see that $C_n = 2^n$

Thus, $C_{60} = 2^{60}$

2. Two fair dice and a fair coin are tossed.

a) What is the **experiment**?

Two fair dice and a fair coin are tossed.

b) What is the **sample space**?

Sample Space = {{1,1,H}, {1,1,T}, {1,2,H}, {1,2,T}, ….. {6,6,T}}.

c) What is the **size** of the sample space?

|Sample Space| = 6.6.2 = 72

d) What is the probability that a head occurs? (Describe the event E_H)

E_H= {{1,1,H}, {1,2,H}, … {6,6,H}}

$|E_H|$= 6.6 = 36

$P(E_H)= |E_H| / |\text{Sample Space}| = 36/72 = 0.5$

e) What is the probability that a 6 occurs? (Describe the event E_6)

E_6= {{1,6,H}, {1,6,T}, {2,6,H}, …. , {6,6,T}}

Consider the complementary event, i.e. a 6 does not occur at all. Let call this event E_{6c}

Size of E_{6c} = 5.5.2 = 50

$|E_6|$ = 72-50 = 22

$P(E_6)= |E_6| / |\text{Sample Space}| = 22 / 72 = 11/36 = 0.305$

f) What is the probability that the total number on the dice is more than the number of heads? (Describe the event E_{more})

E_{more}= Sample Space

The total number on the dice is any number between [2,12]. The number of heads is at most 1. Thus all outcomes are in E_{more}.

$P(E_{more})= |E_{more}| / |\text{Sample Space}| = 72 / 72 = 1$

MATH 233
Fall 2018
Quiz #3 A

<u>Duration:</u> 50 minutes.
<u>Remark:</u> Show your thinking/work. Do not just write a number or a formula as a result.

1. Prove that if $a \mid b$ and $a \mid c$ then $a \mid b+c$.

2. Prove that there is no greatest prime (There is a prime larger than any given prime).

(**Hint:** Let p be a large prime number. Consider p+1 which is not prime. Now find a prime number larger than p+1. For example, consider (p+1)! + 1.

MATH 233
Fall 2018
Quiz #3 B

<u>Duration:</u> 50 minutes.
<u>Remark:</u> Show your thinking/work. Do not just write a number or a formula as a result.

1. Fermat's Little Theorem states that:

   If p is a prime and a is an integer then, $p \mid a^p - a$

Show that Fermat's Little Theorem is invalid if we drop the assumption that p is a prime.

2. Prove that gcd(a,b) = gcd(a,r) where r is the remainder when we divide b by a.

(**Hint:** First prove that gcd(a,b) = gcd(a, b-a))

MATH 233
Fall 2018
Quiz #4 A

<u>Duration:</u> 50 minutes.
<u>Remark:</u> Show your thinking/work. Do not just write a number or a formula as a result.

1. A **graph** G=(V,E) is a set of vertices (V) and a set of edges (E) between vertices.

a) Draw all **graphs** on three vertices. Let $V = \{v_1, v_2, v_3\}$
b) What is the **number** of all possible graphs with **n** vertices?

2. Let **a** be a positive integer whose set of prime factors is $\{p_1, p_2, \ldots p_m\}$. Let **b** be a positive integer whose set of prime factors is $\{q_1, q_2, \ldots. q_n\}$. How can you form the **least common multiple** of a and b when you know the sets of their prime factors?

MATH 233
Fall 2018
Quiz #4 B

<u>Duration:</u> 50 minutes.
<u>Remark:</u> Show your thinking/work. Do not just write a number or a formula as a result.

1. A **graph** G=(V,E) is a set of vertices (V) and a set of edges (E) between vertices. A **tree** is a special graph, which is **connected** and has no **cycle**.

a) Draw all trees on **three** vertices. Let V = $\{v_1, v_2, v_3\}$
b) What is the **number** of all possible **trees** with **n** vertices?

2. Let a be **x** positive integer whose set of prime factors is $\{p_1, p_2, \ldots p_n\}$. Let **y** be a positive integer whose set of prime factors is $\{q_1, q_2, \ldots. q_m\}$. How can you form the **greatest common divisor** of **x** and **y** when you know the sets of prime factors?