A-2-2. Find the Laplace transform of $f(t)$ defined by

$$f(t) = 0, \qquad \text{for } t < 0$$
$$= te^{-3t}, \qquad \text{for } t \geq 0$$

Solution. Since

$$\mathcal{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2-6), we obtain

$$F(s) = \mathcal{L}[te^{-3t}] = G(s+3) = \frac{1}{(s+3)^2}$$

A-2-4. Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2-3. where $f(t) = 0$ for $t < 0$ and $2a \leq t$. Also find the limiting value of $F(s)$ as a approaches zero.

Solution. The function $f(t)$ can be written

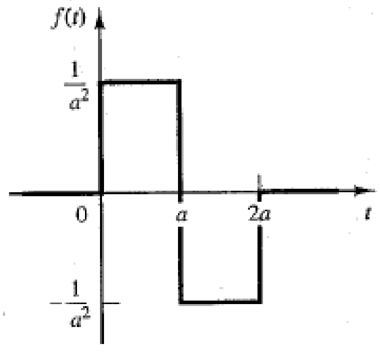
$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t-a) + \frac{1}{a^2}1(t-2a)$$

Then

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)] \\ &= \frac{1}{a^2}\mathcal{L}[1(t)] - \frac{2}{a^2}\mathcal{L}[1(t-a)] + \frac{1}{a^2}\mathcal{L}[1(t-2a)] \\ &= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as} \\ &= \frac{1}{a^2 s}(1 - 2e^{-as} + e^{-2as}) \end{aligned}$$

As a approaches zero, we have

$$\begin{aligned} \lim_{a\to 0} F(s) &= \lim_{a\to 0}\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\frac{d}{da}(1 - 2e^{-as} + e^{-2as})}{\frac{d}{da}(a^2 s)} \\ &= \lim_{a\to 0}\frac{2se^{-as} - 2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as} - e^{-2as}}{a} \\ &= \lim_{a\to 0}\frac{\frac{d}{da}(e^{-as} - e^{-2as})}{\frac{d}{da}(a)} = \lim_{a\to 0}\frac{-se^{-as} + 2se^{-2as}}{1} \\ &= -s + 2s = s \end{aligned}$$



A–2–7. Find the Laplace transform of $f(t)$ defined by

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= t^2 \sin\omega t, && \text{for } t \geq 0 \end{aligned}$$

Solution. Since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

applying the complex-differentiation theorem

$$\mathscr{L}[t^2 f(t)] = \frac{d^2}{ds^2} F(s)$$

to this problem, we have

$$\mathscr{L}[f(t)] = \mathscr{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

A–2–11. Find the inverse Laplace transform of $F(s)$, where

$$F(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Solution. Since

$$s^2 + 2s + 2 = (s + 1 + j1)(s + 1 - j1)$$

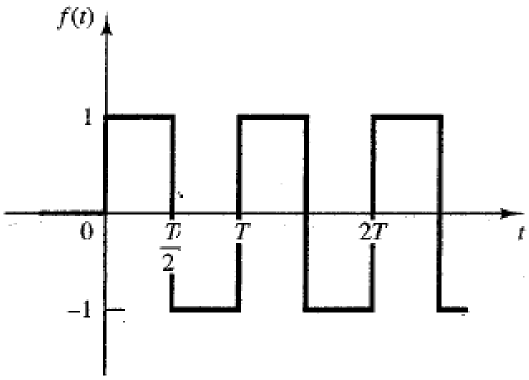


Figure 2–4
Periodic function (square wave).

we notice that $F(s)$ involves a pair of complex-conjugate poles, and so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + 2s + 2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2 + 2s + 2) + (a_2 s + a_3)s$$

By comparing coefficients of s^2, s, and s^0 terms on both sides of this last equation, respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$\begin{aligned} F(s) &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s + 2}{s^2 + 2s + 2} \\ &= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s + 1)^2 + 1^2} - \frac{1}{2}\frac{s + 1}{(s + 1)^2 + 1^2} \end{aligned}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t, \qquad \text{for } t \geq 0$$

A–2–12. Obtain the inverse Laplace transform of

$$F(s) = \frac{5(s + 2)}{s^2(s + 1)(s + 3)}$$

Solution.

$$F(s) = \frac{5(s + 2)}{s^2(s + 1)(s + 3)} = \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{a_1}{s + 1} + \frac{a_2}{s + 3}$$

where

$$\begin{aligned} a_1 &= \frac{5(s + 2)}{s^2(s + 3)}\bigg|_{s=-1} = \frac{5}{2} \\ a_2 &= \frac{5(s + 2)}{s^2(s + 1)}\bigg|_{s=-3} = \frac{5}{18} \\ b_2 &= \frac{5(s + 2)}{(s + 1)(s + 3)}\bigg|_{s=0} = \frac{10}{3} \\ b_1 &= \frac{d}{ds}\left[\frac{5(s + 2)}{(s + 1)(s + 3)}\right]_{s=0} \\ &= \frac{5(s + 1)(s + 3) - 5(s + 2)(2s + 4)}{(s + 1)^2(s + 3)^2}\bigg|_{s=0} = -\frac{25}{9} \end{aligned}$$

Thus

$$F(s) = -\frac{25}{9}\frac{1}{s} + \frac{10}{3}\frac{1}{s^2} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = -\frac{25}{9} + \frac{10}{3}t + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t}, \qquad \text{for } t \geq 0$$

A–2–17. Solve the following differential equation:

$$\ddot{x} + 2\dot{x} + 10x = t^2, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Solution. Noting that the initial conditions are zeros, the Laplace transform of the equation becomes as follows:

$$s^2X(s) + 2sX(s) + 10X(s) = \frac{2}{s^3}$$

Hence

$$X(s) = \frac{2}{s^3(s^2 + 2s + 10)}$$

We need to find the partial-fraction expansion of $X(s)$. Since the denominator involves a triple pole, it is simpler to use MATLAB to obtain the partial-fraction expansion. The following MATLAB program may be used:

```
num = [0 0 0 0 0 2];
den = [1 2 10 0 0 0];
[r,p,k] = residue(num,den)

r =

    0.0060- 0.0087i
    0.0060+ 0.0087i
   -0.0120
   -0.0400
    0.2000

p =

   -1.0000+ 3.0000i
   -1.0000- 3.0000i
    0
    0
    0

k =

    []
```

From the MATLAB output, we find

$$X(s) = \frac{0.006 - 0.0087j}{s + 1 - 3j} + \frac{0.006 + 0.0087j}{s + 1 + 3j} + \frac{-0.012}{s} + \frac{-0.04}{s^2} + \frac{0.2}{s^3}$$

Combining the first two terms on the right-hand side of the equation, we get

$$X(s) = \frac{0.012(s+1) + 0.0522}{(s+1)^2 + 3^2} - \frac{0.012}{s} - \frac{0.04}{s^2} + \frac{0.2}{s^3}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = 0.012e^{-t}\cos 3t + 0.0174e^{-t}\sin 3t - 0.012 - 0.04t + 0.1t^2, \qquad \text{for } t \geq 0$$