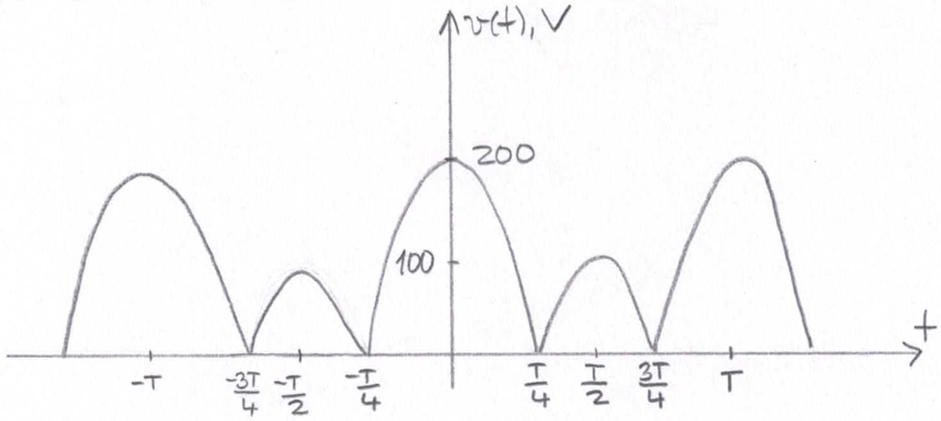
Problem 1) Derive the Fourier series for the periodic voltage shown as



given that

$$v(t) = 200 \cos \frac{2\pi}{T} t \ V, \qquad -\frac{T}{4} \leq t \leq \frac{T}{4}$$

$$v(t) = -100 \cos \frac{2\pi}{T} t \ V, \qquad \frac{T}{4} \leq t \leq \frac{3T}{4}$$

Solution. We need to calculate the Fourier series coefficients that is ;

$$v(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

where $\omega_0 = \frac{2\pi}{T}$, and

$$a_v = \frac{1}{T} \int_0^T v(t)\, dt$$

$$= \frac{1}{T} \left[\int_0^{T/4} 200 \cos \frac{2\pi}{T} t \, dt + \int_{T/4}^{3T/4} (-100) \cos \frac{2\pi}{T} t \, dt + \int_{3T/4}^{T} 200 \cos \frac{2\pi}{T} t \, dt \right]$$

$$= \frac{1}{\cancel{T}} \left[\frac{200\cancel{T}}{2\pi} \sin \frac{2\pi}{T} t \Big|_0^{T/4} - \frac{100\cancel{T}}{2\pi} \sin \frac{2\pi}{T} t \Big|_{T/4}^{3T/4} + \frac{200\cancel{T}}{2\pi} \sin \frac{2\pi}{T} t \Big|_{3T/4}^{T} \right]$$

$$= \frac{50}{\pi} \left(2 \sin \frac{2\pi}{T} t \Big|_0^{T/4} - \sin \frac{2\pi}{T} t \Big|_{T/4}^{3T/4} + 2 \sin \frac{2\pi}{T} t \Big|_{3T/4}^{T} \right)$$

$$= \frac{50}{\pi} \left(2 \sin \frac{\pi}{2} - 0 - \sin \frac{3\pi}{2} + \sin \frac{\pi}{2} + 2 \sin 2\pi - 2 \sin \frac{3\pi}{2} \right)$$

$$= \frac{50}{\pi}\left[2-(-1)+1-2(-1)\right]$$

$$= \frac{300}{\pi}$$

-and

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega_0 t \, dt$$

$$= \frac{2}{T}\left[\int_0^{T/4} 200\cos\omega_0 t \cos n\omega_0 t \, dt + \int_{T/4}^{3T/4}(-100)\cos\omega_0 t\cos n\omega_0 t\, dt + \int_{3T/4}^{T} 200\cos\omega_0 t\cos n\omega_0 t\, dt\right]$$

$$= \frac{100}{T}\left\{\int_0^{T/4}\left[\cos(n+1)\omega_0 t + \cos(n-1)\omega_0 t\right]dt - \int_{T/4}^{3T/4}\left[\cos(n+1)\omega_0 t + \cos(n-1)\omega_0 t\right]dt + \int_{3T/4}^{T}\left[\cos(n+1)\omega_0 t + \cos(n-1)\omega_0 t\right]dt\right\}$$

$$= \frac{100}{T}\left\{\frac{1}{(n+1)\omega_0}\sin(n+1)\omega_0 t\Big|_0^{T/4} + \frac{1}{(n-1)\omega_0}\sin(n-1)\omega_0 t\Big|_0^{T/4} - \frac{1}{(n+1)\omega_0}\sin(n+1)\omega_0 t\Big|_{T/4}^{3T/4} - \frac{1}{(n-1)\omega_0}\sin(n-1)\omega_0 t\Big|_{T/4}^{3T/4} + \frac{1}{(n+1)\omega_0}\sin(n+1)\omega_0 t\Big|_{3T/4}^{T} + \frac{1}{(n-1)\omega_0}\sin(n-1)\omega_0 t\Big|_{3T/4}^{T}\right\}, n \neq 1$$

$$= \frac{100}{T}\left\{\frac{1}{(n+1)\omega_0}\left[\sin\frac{(n+1)\pi}{2} - 0\right] + \frac{1}{(n-1)\omega_0}\left[\sin\frac{(n-1)\pi}{2} - 0\right]\right.$$

PS 7.2

$$= -\frac{1}{(n+1)\omega_0}\left[\sin\frac{3(n+1)\pi}{2} - \sin\frac{(n+1)\pi}{2}\right] - \frac{1}{(n-1)\omega_0}\left[\sin\frac{3(n-1)\pi}{2} - \sin\frac{(n-1)\pi}{2}\right]$$

$$+\frac{1}{(n+1)\omega_0}\left[\sin \cancelto{0}{2(n+1)\pi} - \sin\frac{3(n+1)\pi}{2}\right] + \frac{1}{(n-1)\omega_0}\left[\sin \cancelto{0}{2(n-1)\pi} - \sin\frac{3(n-1)\pi}{2}\right]$$

$$= \frac{100}{T}\left[\frac{2}{(n+1)\omega_0}\sin\frac{(n+1)\pi}{2} + \frac{2}{(n-1)\omega_0}\sin\frac{(n-1)\pi}{2} - \frac{2}{(n+1)\omega_0}\sin\frac{3(n+1)\pi}{2} - \frac{2}{(n-1)\omega_0}\sin\frac{3(n-1)\pi}{2}\right]$$

-we have

$$\sin\frac{(n+1)\pi}{2} = \sin(n\pi/2)\cancelto{0}{\cos(\pi/2)} + \cancelto{1}{\sin(\pi/2)}\cos(n\pi/2)$$

$$= \cos(n\pi/2)$$

$$\sin\frac{(n-1)\pi}{2} = \sin(n\pi/2)\cancelto{0}{\cos(\pi/2)} - \cancelto{1}{\sin(\pi/2)}\cos(n\pi/2)$$

$$= -\cos(n\pi/2)$$

$$\sin\frac{3(n+1)\pi}{2} = \sin(3n\pi/2)\cancelto{0}{\cos(3\pi/2)} + \cancelto{-1}{\sin(3\pi/2)}\cos(3n\pi/2)$$

$$= -\cos(3n\pi/2)$$

$$\sin\frac{3(n-1)\pi}{2} = \sin(3n\pi/2)\cancelto{0}{\cos(3\pi/2)} - \cancelto{-1}{\sin(3\pi/2)}\cos(3n\pi/2)$$

$$= \cos(3n\pi/2)$$

Hence;

$$a_n = \frac{200}{\omega_0 T}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos(n\pi/2) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos(3n\pi/2)\right]$$

$$= \frac{200}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left[\cos(n\pi/2) + \cos(3n\pi/2)\right]$$

$$\doteq -\frac{200}{\pi(n^2-1)}\left[\cos(n\pi/2) + \cos(n\pi/2 + n\pi)\right]$$

$$= -\frac{200}{\pi(n^2-1)}\left[\cos(n\pi/2) + \cos(n\pi/2)\underbrace{\cos(n\pi)}_{(-1)^n} - \sin(n\pi/2)\underbrace{\sin(n\pi)}_{0}\right]$$

$$= -\frac{200}{\pi(n^2-1)}\left[1 + (-1)^n\right]\cos(n\pi/2) \quad , \; n \neq 1$$

$$= \begin{cases} 0 & \text{; if } n \text{ is odd} \\ -\frac{400}{\pi(n^2-1)}\cos(n\pi/2) & \text{, if } n \text{ is even} \end{cases}$$

Moreover ;

- for $n=1$, we have

$$a_1 = \frac{2}{T}\left[\int_0^{T/4} 200\cos\omega_0 t \cos\omega_0 t \, dt + \int_{T/4}^{3T/4} (-100)\cos\omega_0 t \cos\omega_0 t \, dt + \int_{3T/4}^{T} 200 \cos\omega_0 t \cos\omega_0 t \, dt\right]$$

Note that ;

$$\cos 2A = 2\cos^2 A - 1 \quad \Rightarrow \quad \cos^2 A = \frac{1+\cos 2A}{2}$$

Hence ;

$$a_1 = \frac{2}{T}\left[\int_0^{T/4} 200 \cdot \frac{1+\cos 2\omega_0 t}{2} \, dt - \int_{T/4}^{3T/4} 100 \cdot \frac{1+\cos 2\omega_0 t}{2} \, dt + \int_{3T/4}^{T} 200 \cdot \frac{1+\cos 2\omega_0 t}{2} \, dt\right]$$

$$= \frac{2}{T}\left[100\left(t+\frac{\sin 2\omega_0 t}{2}\right)\Big|_0^{T/4} - 50\left(t+\frac{\sin 2\omega_0 t}{2}\right)\Big|_{T/4}^{3T/4}\right.$$

$$\left. + 100\left(t+\frac{\sin 2\omega_0 t}{2}\right)\Big|_{3T/4}^{T}\right]$$

$$= \frac{2}{T}\left[100\left(\frac{T}{4}+\cancel{\frac{\sin\pi}{2}}^{\,0}\right) - 50\left(\frac{3T}{4}-\frac{T}{4}+\cancel{\frac{\sin 3\pi}{2}}^{\,0}-\cancel{\frac{\sin\pi}{2}}^{\,0}\right)\right.$$

$$\left. + 100\left(T-\frac{3T}{4}+\cancel{\frac{\sin 4\pi}{2}}^{\,0}-\cancel{\frac{\sin 3\pi}{2}}^{\,0}\right)\right]$$

$$= \frac{2}{T}\left(100\frac{T}{4} - 50\frac{2T}{4} + 100\frac{T}{4}\right)$$

$$= 50$$

- and since $v(t)$ is an even function, the Fourier series is entirely a cosine series, that is

$b_n = 0, \ \forall n$

As a result;

$$v(t) = \frac{300}{\pi} + 50\cos\omega_0 - \frac{400}{\pi}\sum_{n=2,4,6}^{\infty}\frac{\cos(n\pi/2)}{(n-1)^2}\cos n\omega_0 t$$

Problem 2) It is given that

$$v(t) = 20t\cos 0.25\pi t \ \ V$$

over the interval $-6 \leq t \leq 6$ s. The function then repeats itself.

a. What is the fundamental frequency in radians per second?

b. Is the function even?

c. Is the function odd?

d. Does the function have half-wave symmetry?

Solution.

a. We first find the fundamental period, T as

$T = 6 - (-6) = 12$

$$\omega_0 = \frac{2\pi}{12} = \frac{\pi}{6} \text{ rad/sec}$$

b. We check the evenness via

$$v(t) \stackrel{?}{=} v(-t)$$

$$\Rightarrow v(t) = 20t \cos 0.25\pi t$$

$$= -20(-t)\cos(-0.25\pi t)$$

$$= -v(-t)$$

$$\neq v(-t)$$

Hence;

- the function is NOT even

c. It follows from part (b) that

$$v(t) = -v(-t)$$

implying that the function is ODD

d. We check the half-wave symmetry via

$$v(t) \stackrel{?}{=} -v\left(t - \frac{T}{2}\right)$$

$$\Rightarrow v\left(t - \frac{12}{2}\right) = 20(t-6)\cos 0.25\pi(t-6)$$

$$= 20(t-6)\cos\left(0.25\pi t - \frac{3\pi}{2}\right)$$

$$= 20(t-6)\left(\cos 0.25\pi t \underbrace{\cos\frac{3\pi}{2}}_{0} + \sin 0.25\pi t \underbrace{\sin\frac{3\pi}{2}}_{-1}\right)$$

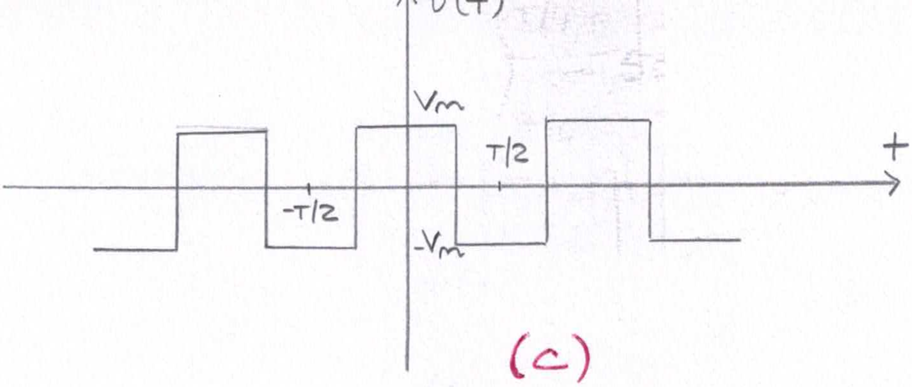
$$= -20(t-6)\sin 0.25\pi t$$

$$\neq v(t)$$

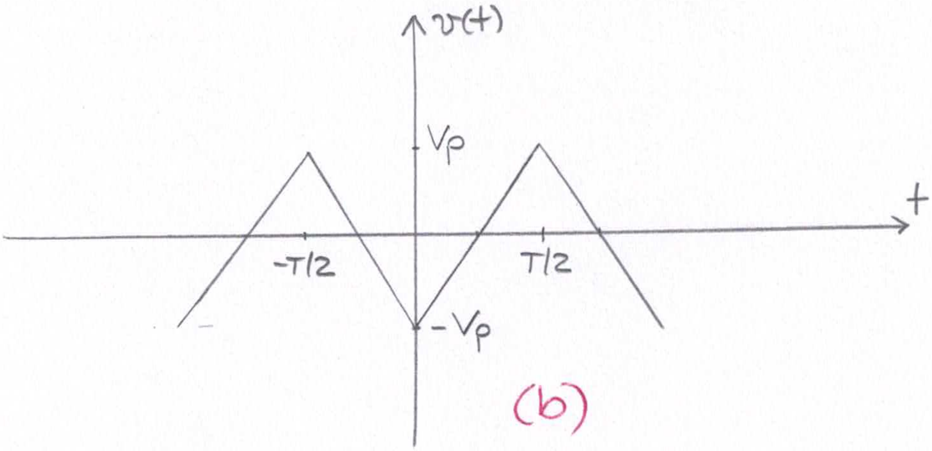
Hence;

- the function does NOT have half-wave symmetry

Problem 3) Find the Fourier series for the following periodic functions shown as



(c)



(b)

a. The function $v(t)$ in (a) is even and since

$$v(t) = -v\left(t - \frac{T}{2}\right)$$

thus having half-wave symmetry

Moreover;

- as it has even symmetry at a quarter period point

  ↳ it also has quarter-wave symmetry

Therefore;

- the corresponding Fourier series is a cosine series

  ↳ $b_n = 0, \ \forall n$

- due to half-wave symmetry, we have

$$a_v = 0$$

- and we have

$$a_n = \begin{cases} 0, & \text{for even } n \text{ (because of half-wave symmetry)} \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} v(t)\cos n\omega_0 t, & \text{for odd } n \end{cases}$$

- then we calculate

$$a_n = \frac{8}{T}\int_0^{T/4} V_m \cos n\omega_0 t \, dt \quad , n \text{ is odd}$$

$$= \frac{8V_m}{T} \left. \frac{\sin n\omega_0 t}{n\omega_0} \right|_0^{T/4}$$

$$= \frac{\cancel{8}^{4} V_m}{\cancel{T} n \left(\frac{\cancel{2}\pi}{\cancel{T}}\right)} \sin n \frac{\cancel{2}\pi}{\cancel{T}} \frac{\cancel{T}}{\cancel{4}_2}$$

$$= \frac{4V_m}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

Thus ;

$$v(t) = \frac{4V_m}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos n\omega_0 t \ \ V$$

b. The function $v(t)$ in (b) is even and since

$$v(t) = -v\left(t - \frac{T}{2}\right)$$

thus having half-wave symmetry

Moreover;

- as it has even symmetry at a quarter period point

↳ it also has quarter-wave symmetry

Thus ;

- the corresponding Fourier series is a cosine series

↳ $b_n = 0 \ , \ \forall n$

- because of half-wave symmetry, we have

$$a_v = 0$$

- and we have

$$a_n = \begin{cases} 0, \text{ for even } n \text{ (because of half-wave symmetry)} \\ \frac{8}{T}\int_0^{T/4} v(t) \cos n\omega_0 t \ , \text{ for odd } n \end{cases}$$

-we find that

$$v(t) = -V_p + \frac{4V_p}{T}t \quad , \quad 0 \leq t \leq \frac{T}{4}$$

-then we calculate

$$a_n = \frac{8}{T}\int_0^{T/4} \left(-V_p + \frac{4V_p}{T}t\right)\cos n\omega_0 t \, dt \quad , \text{ for odd } n$$

$$= -\frac{8V_p}{T}\int_0^{T/4} \cos n\omega_0 t \, dt$$

$$+ \frac{32V_p}{T^2}\int_0^{T/4} \underset{u}{t} \underbrace{\cos n\omega_0 t \, dt}_{dv}$$

$$= -\frac{8V_p}{T}\left.\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{T/4} + \frac{32V_p}{T^2}\left(\left. t\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{T/4} - \int_0^{T/4}\frac{\sin n\omega_0 t}{n\omega_0}dt\right)$$

$$\cancel{= -\frac{8V_p}{T}\frac{\sin(n\pi/2)}{n\omega_0} + \frac{\overset{8}{\cancel{32}}V_p}{T^{\cancel{2}}}\frac{\cancel{T}}{\cancel{4}}\frac{\sin(n\pi/2)}{n\omega_0}}$$

$$+ \frac{32V_p}{T^2}\left.\frac{\cos n\omega_0 t}{(n\omega_0)^2}\right|_0^{T/4}$$

$$= \frac{32V_p}{T^2}\left[\frac{1}{n\omega_0^2}\right]\left(\underset{\swarrow 0}{\cancel{\cos}}\frac{n\pi}{2} - 1\right)$$

$$= \frac{\overset{8}{\cancel{32}}V_p}{\cancel{T^2}n^2\frac{\cancel{4}\pi^2}{\cancel{T^2}}}(-1)$$

$$= -\frac{8V_p}{\pi^2 n^2}$$

As a result;

$$v(t) = -\frac{8V_p}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos n\omega_0 t$$

Problem 4) It is given that $f(t) = 0.4t^2$ over the interval $-5 < t < 5$ s.

a. Construct a periodic function that satisfies this $f(t)$ between -5 and $+5$ s, has a period of 20 s, and has half-wave symmetry.

b. Is the function even or odd?

c. Does the function have quarter-wave symmetry?

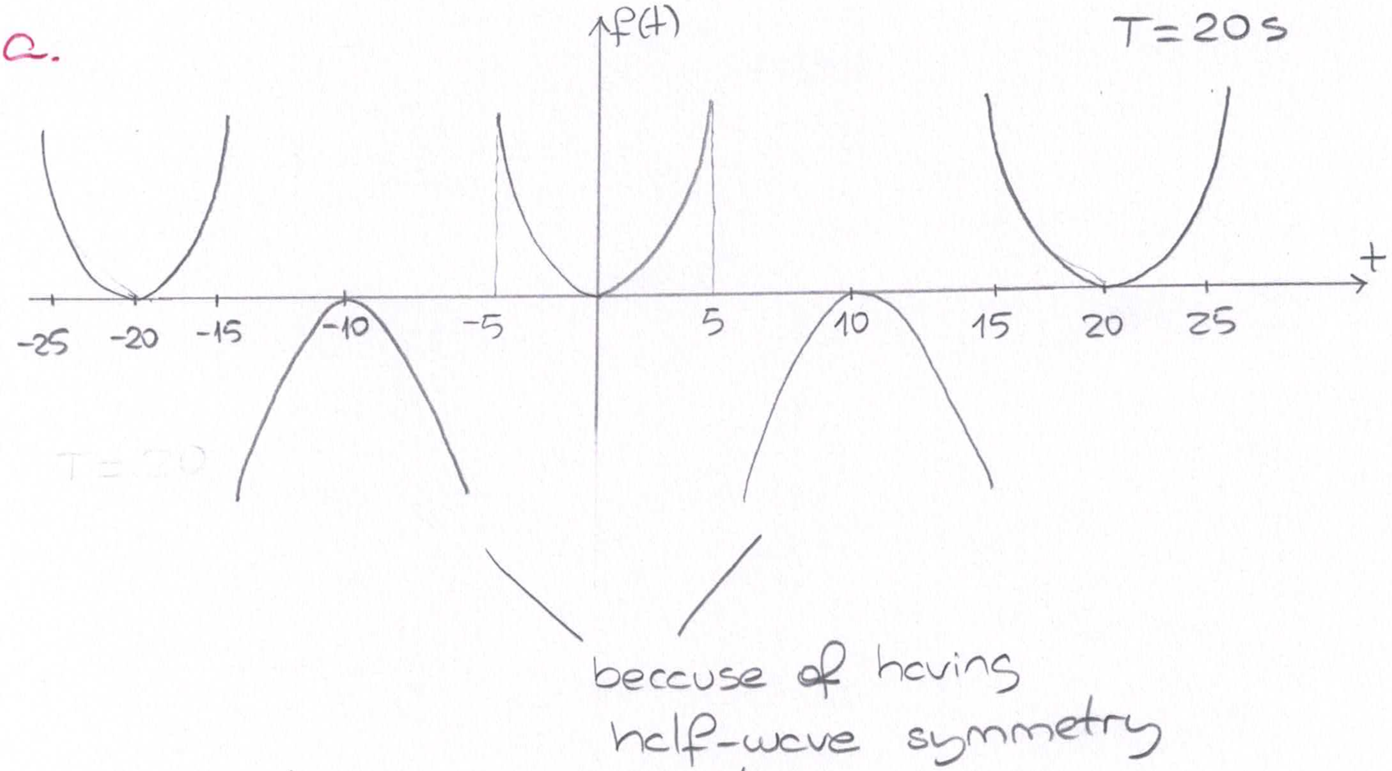
d. Derive the Fourier series for $f(t)$.

e. Write the Fourier series for $f(t)$ if $f(t)$ is shifted 5 s to the right.

Solution. We have

$$f(t) = 0.4t^2 \quad , \quad -5 < t < 5 \text{ s}$$

a.



b. The function is even, that is

$$\begin{aligned} f(t) &= 0.4t^2 \\ &= 0.4(-t)^2 \\ &= f(-t \end{aligned}$$

c. The function also has quarter-wave symmetry

↳ because at $t=5s$ which is quarter-period point, the function is even

d. As the function is even

↳ it is a cosine series, i.e. $b_n = 0, \forall n$

Because of half-wave symmetry

↳ we have $a_v = 0$

Moreover;

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{T}\displaystyle\int_0^{T/4} f(t)\cos n\omega_0 t \, dt, & \text{if } n \text{ is odd} \end{cases}$$

- we thus calculate

$$a_n = \frac{8}{T}\int_0^{T/4} 0.4t^2 \cos n\omega_0 t \, dt, \quad \omega_0 = \frac{2\pi}{T}, \quad n \text{ is odd}$$

- using integration by parts two times allows to obtain

$$a_n = \frac{3.2}{T}\int_0^{T/4} t^2 \cos n\omega_0 t \, dt$$

$$= \frac{3.2}{T}\left(\frac{2t}{n^2\omega_0^2}\cos n\omega_0 t + \frac{n^2\omega_0^2 t^2 - 2}{n^3\omega_0^3}\sin n\omega_0 t \Big|_0^{T/4}\right)$$

Note that;

- the first term is 0 at both T/4 and 0, the second term is 0 at only 0, thus

$$a_n = \frac{3.2}{n^3\omega_0^3 T}\,\frac{n^2\omega_0^2 T^2 - 32}{16}\sin\frac{n\pi}{2} = \frac{40}{\pi^3 n^3}(n^2\pi^2 - 8)\sin\frac{n\pi}{2}$$

As a result;

$$f(t) = \frac{40}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{n^2\pi^2-8}{n^3} \sin\frac{n\pi}{2} \cos n\omega_0 t$$

e. We find that

$$\cos n\omega_0\left(t-\frac{T}{4}\right) = \cos\left(n\omega_0 t - n\pi/2\right)$$

$$= \cos n\omega_0 t \underbrace{\cos(n\pi/2)}_{0} + \sin n\omega_0 t \sin(n\pi/2)$$ (since n is odd)

$$= \sin\frac{n\pi}{2} \sin n\omega_0 t$$

Hence;

$$f(t) = \frac{40}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{n^2\pi^2-8}{n^3} \underbrace{\left(\sin\frac{n\pi}{2}\right)^2}_{1} \sin n\omega_0 t$$

$$= \frac{40}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{n^2\pi^2-8}{n^3} \sin n\omega_0 t$$

(!) a sine series is obtained by shifting $f(t)$ quarter period