## The Laplace transform in circuit analysis

- Laplace transform has two attractive characteristics :

  1. It transforms a set of linear constant-coefficient differential equations

     into a set of linear polynomial equations.

  2. It automatically introduces into the polynomial equations

     the initial values of the current and voltage variables.

- We will show how we can skip writing time-domain integro-differential equations

  and transforming them into the s-domain.

## Circuit elements in the s-domain

- We first write the time-domain equation

  that relates the terminal voltage to the terminal current.

- Next we take the Laplace transform of the time-domain equation.

## a resistor in the s-domain

- From Ohm's law, we have

$$\vartheta = Ri$$

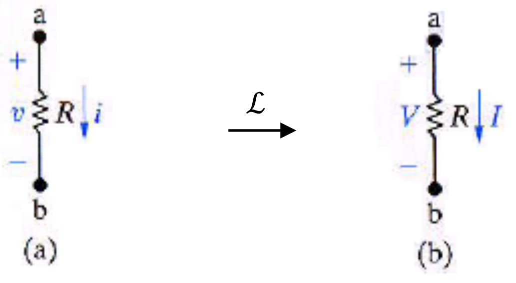
- because $R$ is constant, the Laplace transform becomes

$$V = RI$$

where

$$V = \mathcal{L}\{\vartheta\} \ , \quad I = \mathcal{L}\{i\}$$

Hence ;



## an inductor in the s-domain

- We have

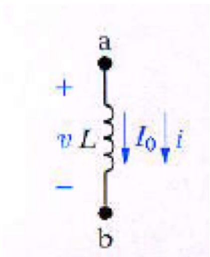
$$\vartheta = L\frac{di}{dt}$$

- and the Laplace transform of $\vartheta$ gives

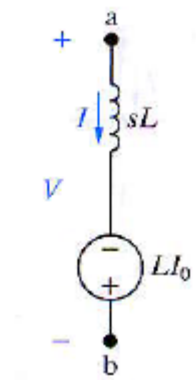
$$V = L[sI - i(0^-)]$$

$$= sLI - LI_0 \quad , \qquad I_0 \ : \ i(0^-)$$

Hence ;

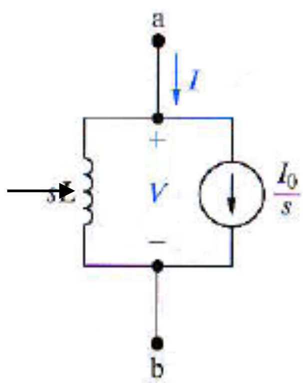



$\mathcal{L}$




“series equivalent ”

OR




“ parallel equivalent ”

## a capacitor in the s-domain

- We have

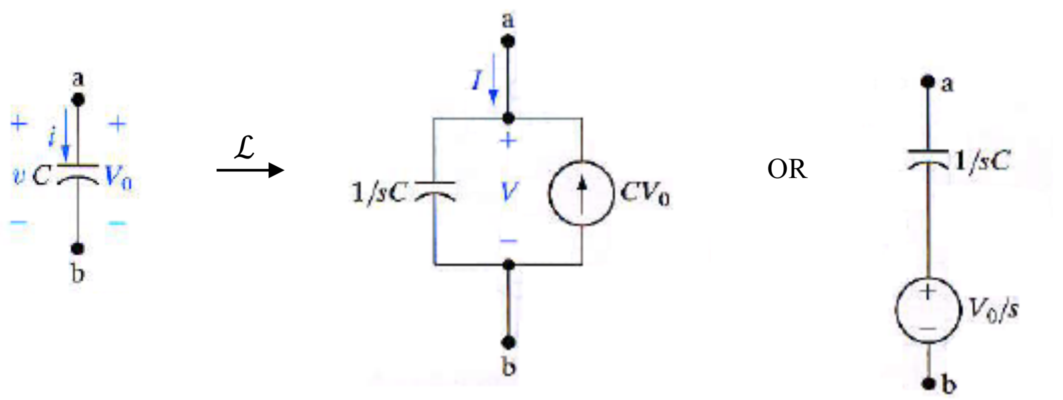
$$i = C\frac{d\vartheta}{dt}$$

- Transforming into s-domain gives

$$I = C[sV - \vartheta(0^-)]$$

$$= sCV - CV_0 \quad , \qquad V_0 \ : \ \vartheta(0^-)$$

Thus ;



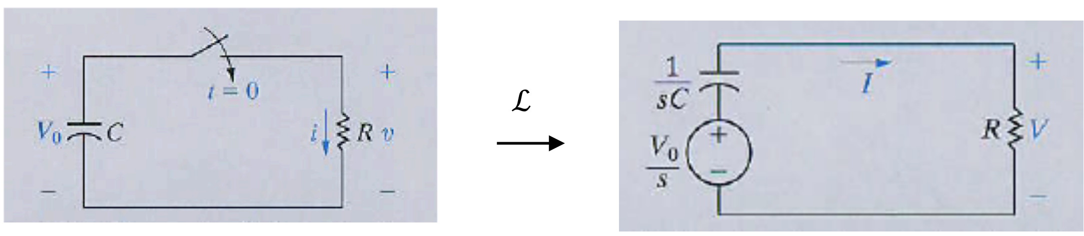

## Circuit analysis in the s-domain

- We shall now illustrate how to use the Laplace transform

  to determine the transient behavior of linear lumped-parameter circuits.

## The natural response of an RC circuit

- We consider an RC circuit




$$\frac{V_0}{s} = \frac{1}{sC}I + RI \quad \Longrightarrow \quad I = \frac{CV_0}{RCs+1}$$

$$= \frac{V_0/R}{s+(1/RC)}$$

- Using inverse Laplace transform yields

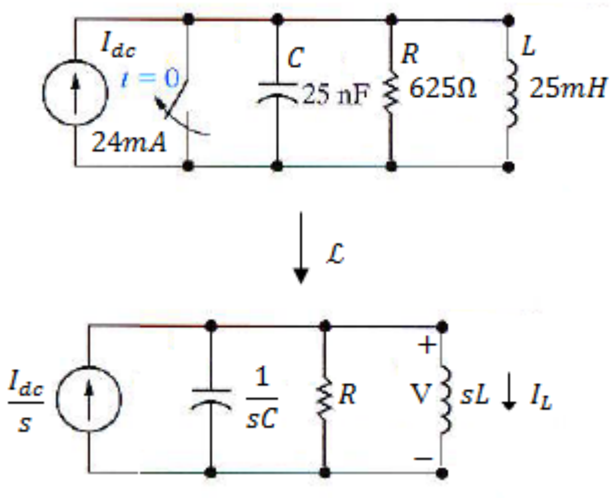
$$i(t) = \frac{V_0}{R}e^{-t/RC}\,u(t)$$

- and applying Ohm's law gives

$$\vartheta(t) = Ri(t) = V_0e^{-t/RC}\,u(t)$$

## The step response of a parallel RLC circuit

- We now analyze the parallel RLC circuit as follows




$$I_L = \frac{V}{sL}$$

$$\frac{V}{\frac{1}{sC}} + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$

$$\left(sC + \frac{1}{R} + \frac{1}{sL}\right)V = \frac{I_{dc}}{s}$$

$$V = \frac{I_{dc}/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

- and

$$I_L = \frac{I_{dc}/LC}{s\left[s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right]}$$

$$= \frac{384.10^5}{s(s^2 + 64000s + 16.10^8)} = \frac{384.10^5}{(s + 32000 - j24000)(s + 32000 + j24000)}$$

$$= \frac{C_1}{s} + \frac{C_2}{s + 32000 - j24000} + \frac{C_2^*}{s + 32000 + j24000}$$

$$C_1 = \frac{384.10^5}{16.10^8} = 24.10^{-3}$$

$$C_2 = \frac{384.10^5}{s(s + 32000 + j24000)}\bigg|_{s = -32000 + j24000}$$

$$= \frac{384.10^5}{(-32000 + j24000)j48000}$$

$= 20.10^{-3}\angle 126.87°$

Hence;

$$i_L(t) = [24.10^{-3} + 40.10^{-3}e^{-32000t}\cos(24000t + 126.87°)]\mathrm{u(t)}$$

## The transient response of a parallel RLC circuit

- We now replace the dc current source with a sinusoidal current source

$$i_g = I_m \cos \omega t \;\; A$$

- The s-domain expression for the source current is

$$I_g = \frac{sI_m}{s^2 + \omega^2}$$

- The voltage across the parallel elements is

$$V = \frac{\left(\frac{I_g}{C}\right)s}{s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)} = \frac{\left(\frac{I_m}{C}\right)s^2}{(s^2 + \omega^2)\left[s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right]}$$

from which

$$I_L = \frac{V}{sL} = \frac{\left(\frac{I_m}{C}\right)s}{(s^2 + \omega^2)\left[s^2 + \left(\frac{1}{RC}\right)s + \left(\frac{1}{LC}\right)\right]}$$

$$= \frac{384.10^5 s}{(s - j\omega)(s + j\omega)(s + \alpha - j\beta)(s + \alpha + j\beta)}$$

where

$$\omega = 40000, \qquad \alpha = 32000, \qquad and\; \beta = 24000$$

- Then we expand

$$I_L = \frac{C_1}{s - j40000} + \frac{C_1^*}{s + j40000} + \frac{C_2}{s + 32000 - j24000} + \frac{C_2^*}{s + 32000 + j24000}$$

$$C_1 = \frac{384.10^5 s}{(s + j40000)(s + 32000 - j24000)(s + 32000 + j24000)}\Bigg|_{s = j40000}$$

$$= 7.5.10^{-3}\angle -90°$$

$$C_2 = \frac{384.10^5 s}{(s - j40000)(s + j40000)(s + 32000 + j24000)}\Bigg|_{s = -32000 + j24000}$$

$$= 12.5.10^{-3}\angle 90°$$

$$i_L(t) = [15\cos(40000t - 90°) + 25e^{-32000t}\cos(24000t + 90°)]u(t)\ mA$$

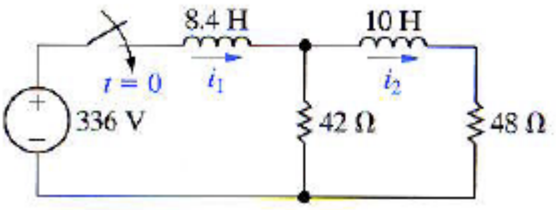
$$= [15\sin 40000t - 25e^{-32000t}\sin 24000t]u(t)\ \ mA$$

- Thus, the steady-state current is

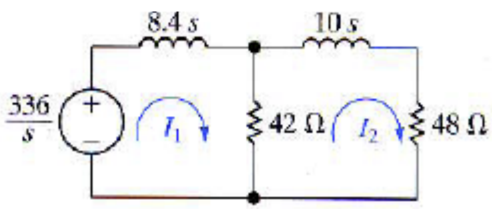
$$i_{L_{ss}} = 15\sin 40000t\ \ \ mA \qquad\qquad \left(i_{L_{ss}} = \lim_{t\to\infty} i_L(t)\right)$$

## The step response of a multiple mesh circuit

- Let us consider the following circuit example




$\downarrow \mathcal{L}$




$$\frac{336}{s} = 8.4sI_1 + 42(I_1 - I_2) \qquad \Rightarrow \qquad (8.4s + 42)I_1 - 42I_2 = \frac{336}{s}$$

$$42(I_2 - I_1) + 10sI_2 + 48I_2 = 0 \qquad \Rightarrow \qquad -42I_1 + (10s + 90)I_2 = 0$$

Hence ;

$$I_1 = \frac{10s+90}{42} I_2 \quad \Rightarrow \quad \left[\frac{(8.4s+42)(10s+90)}{42} - 42\right] I_2 = \frac{336}{s}$$

$$I_2 = \frac{168}{s(s+2)(s+12)} \quad , \quad I_1 = \frac{40(s+9)}{s(s+9)(s+12)}$$

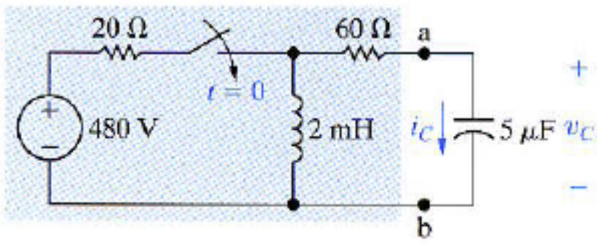
- Using partial-fraction expansion gives

$$I_1 = \frac{15}{s} - \frac{14}{s+2} - \frac{1}{s+12} \quad , \quad I_2 = \frac{7}{s} - \frac{8.4}{s+2} + \frac{1.4}{s+12}$$

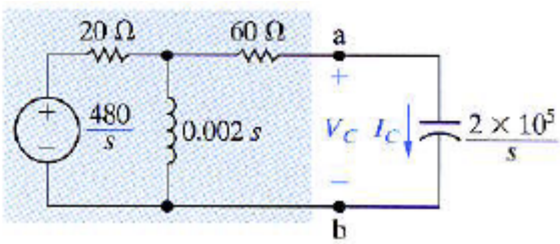
$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A \quad , \quad i_2 = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A$$

## The use of Thevenin's equivalent

- We now consider how to use Thevenin's equivalent in the s-domain for the circuit shown as




- with the s-domain equivalent circuit




- the Thevenin voltage is the open-circuit voltage across terminals a,b.

$$V_{Th} = \frac{\left(\frac{480}{s}\right)0.002s}{20 + 0.002s} = \frac{480}{s + 10^4}$$

- and the Thevenin impedance is calculated as

$$Z_{Th} = 60 + \frac{20.0.002s}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}$$

Hence ;

$$I_C = \frac{V_{Th}}{Z_{Th} + \frac{2.10^5}{s}} = \frac{\frac{480}{s} + 10^4}{\left[\frac{80(s+7500)}{s+10^4}\right] + \frac{2.10^5}{s}}$$

$$= \frac{6s}{s^2 + 10000s + 25.10^6} = \frac{6s}{(s + 5000)^2}$$

$$= -\frac{30000}{(s + 5000)^2} + \frac{6}{s + 5000}$$

$$i_C(t) = (-30000te^{-5000t} + 6e^{-5000t})u(t)A$$

Note that ;

$$i_C(0) = \lim_{t \to 0} i(t) = 6\ A$$

- which can also be found from the circuit as

$$i_C(0) = \frac{480}{20 + 60} = 6\ A$$

- and in a similar manner, we also obtain

$$V_C(t) = 2.10^5 \int_{0^-}^{t} (6 - 30000\tau)e^{-5000\tau} d\tau$$

OR

$$V_C = \frac{6s}{(s + 5000)^2} \frac{2.10^5}{s} = \frac{12.10^5}{(s + 5000)^2}$$

$$V_C(t) = \mathcal{L}^{-1}\{V_C\}$$

$$= 12.10^5 e^{-5000t} u(t)\ V$$

## The transfer function

- Defined as the s-domain ratio of the Laplace transform of the output (response)

  ↳ to the Laplace transform of the input (source)

- To compute transfer function of a circuit

  ↳ we assume ZERO initial conditions

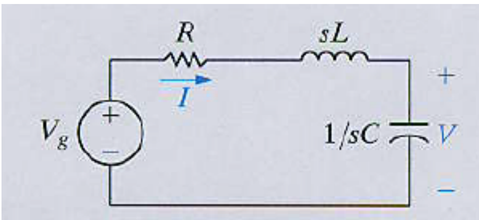
$$H(s) = \frac{Y(s)}{X(s)}$$

where

$Y(s)$ : LT of output signal ⟶ depends on how it is defined

$X(s)$ : LT of input signal

e.g. Consider




$$H(s) = \frac{I}{V_g} = \frac{1}{R + sL + \frac{1}{sC}}$$

$$= \frac{sC}{s^2LC + sRC + 1}$$

- If the voltage across the capacitor is defined as the output signal, then

$$H(s) = \frac{V}{V_g} = \frac{1/sC}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}$$

Note that ;

- For linear lumped-parameter circuits, $H(s)$ is always a rational function of $s$
- The poles of $H(s)$ must always lie in the LHP of $s$

  ↳ if the response to a bounded source is to be bounded.

## The transfer function in partial fraction expansions

- We shall write the circuit output as the product of the transfer function and the driving function

$$Y(s) = H(s)X(s)$$

- Expanding the right-hand side into a sum of partial fractions

  ↳ produces a term for each pole of $H(s)$ and $X(s)$

- The terms generated by the poles of $H(s)$

  ↳ give rise to the transient component of the total response

- And the terms generated by the poles $X(s)$

  ↳ lead to the steady-state component of the response.

## The transfer function and the steady-state sinusoidal response

- Once we have computed a circuit's transfer function

  ↳ we NO LONGER need a separate phasor analysis to determine the steady-state response.

- We use transfer function to relate the steady-state response

  ↳ to the excitation source.

- Assume that

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ &= A\cos\omega t\cos\phi - A\sin\omega t\sin\phi \end{aligned}$$

$$\Longrightarrow \quad \mathcal{L}\{x(t)\} \triangleq X(s) = \frac{(A\cos\phi)s}{s^2+\omega^2} - \frac{(A\sin\phi)\omega}{s^2+\omega^2}$$

$$= \frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}$$

$$\Longrightarrow \quad Y(s) = H(s)X(s) = H(s)\frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}$$

$$= \frac{c_1}{s-j\omega} + \frac{c_1^*}{s+j\omega} + \sum \begin{matrix} \textit{Terms generated} \\ \textit{by the poles of } H(s) \end{matrix}$$

$$\Longrightarrow \quad c_1 = \frac{H(s)A(s\cos\phi - \omega\sin\phi)}{s+j\omega}\Bigg|_{s=j\omega}$$

$$= \frac{H(j\omega)A(j\omega\cos\phi - \omega\sin\phi)}{j2\omega}$$

$$= \frac{H(j\omega)A(\cos\phi + j\sin\phi)}{2} = \frac{1}{2}H(j\omega)Ae^{j\phi}$$

Hence ;

$$c_1 = \frac{A}{2}|H(j\omega)|e^{j[\theta(\omega)+\phi]}$$

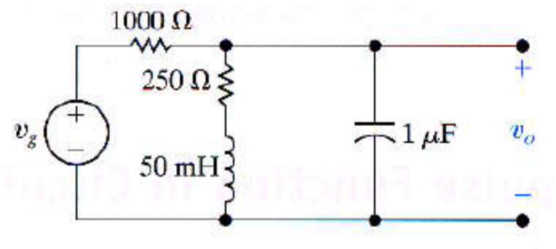
where

$$H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}$$

- and we obtain

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \theta(\omega) + \phi]$$

**e.g.** Consider the following circuit




$$\vartheta_g = 120\cos(5000t + 30^{\circ})$$

$$\vartheta_0(t) = ?$$

**Solution.**

$$(250+0.05s)\,//\,\frac{10^6}{s} \quad \Longrightarrow \quad \frac{\dfrac{(250+0.05s)10^6}{s}}{250+0.05s+\dfrac{10^6}{s}} = \frac{25.10^7 + 5.10^4 s}{0.05s^2 + 250s + 10^6}$$

$$\frac{V_0(s)}{V_g(s)}=\frac{\frac{25.10^7+5.10^4 s}{0.05s^2+250s+10^6}}{\frac{25.10^7+5.10^4 s}{0.05s^2+250s+10^6}+1000}=\frac{25.10^7+5.10^4 s}{50s^2+30.10^4 s+125.10^7}$$

$$=\frac{5.10^6+10^3 s}{s^2+6.10^3 s+25.10^6}=\frac{1000(s+5000)}{s^2+6000s+25.10^6}=H(s)$$

- and

$$H(j5000)=\frac{1000(j5000+5000)}{-25.20^6+j30.10^6+25.10^6}=\frac{10^6(j+1)5}{j30.10^6}=\frac{1-j}{6}=\frac{\sqrt{2}}{6}\angle-45°$$

$$\Longrightarrow\quad V_0(t)=120\frac{\sqrt{2}}{6}\cos(5000t+30°-45°)=20\sqrt{2}\cos(5000t-15°)\ V$$