MATH 233 **Fall 2018** **Midterm Exam Solutions**

Remarks: Show your work. Do not just write a number or a formula as the result.
Duration is 90 minutes and all questions are worth 20 pts.

1. In a sport shop, there are T-shirts of 5 different colors, shorts of 4 different colors, and socks of 3 different colors and shoes of 2 different colors. How many different uniforms can you compose from these items?

A uniform consists of a T-shirt, short, socks and shoes. There are 5 different ways to select a T-shirt, 4 different ways to choose a short, 3 different ways to choose a sock and 2 different ways to choose a shoe. Thus there are $5 * 4 * 3 * 2 = 120$ different uniforms (composed of different color combinations).

2. On a ticket for a **soccer sweepstake**, you have to guess 1, 2, or X for each of 13 games. How many different ways can you fill out the ticket?

There are three different guesses for each game: 1,2,X. Thus, there are $3 * 3 \ldots . * 3 = 3^{13}$ different guesses for 13 games. The ticket can be filled in 3^{13} different ways.
$3^5 = 3 * 3 * 3 * 3 * 3 = 243$. Thus, $3^{13} = (3^5)^2 * (3^3) = 243 * 243 * 27 = 1{,}594{,}323$

3. From a class of 24 students, a comittee of **five students** are going to be choosen randomly to represent the class.

   A. In how many ways can this five-student committee be formed?

This is selecting a 5-element set from a 24-element set. Therefore $C(24,5) = 24! / (19! * 5!)$ $= 24 * 23 * 22 * 21 * 20 / (5 * 4 * 3 * 2) = 23 * 22 * 21 * 20/5 = 23 * 22 * 21 * 4 = 42{,}504$ ways.

   B. One of the students in the class is Haydar Pekgül. What is the **probability** that Haydar Pekgül is in that committee?

Experiment is selecting randomly 5-students from a student body of 24 students.
Sample space is all possible 5-student sets from the 24-student class. Size of the sample space is C(24,5).
E_{HP} = The event that Haydar Pekgül (HP) is among the 5 students,
What is the size of E_{HP}?
What is the number of all 5-student sets that contain HP?
C(23,4) is the number of all 5-student committees that contain HP, because the event reduces to choosing 4 students from a set of 23 students (HP is already choosen).
$C(23,4) = 23! / (19! * 4!) = 23 * 22 * 21 * 20 / (4 * 3 * 2) = 23 * 22 * 21 * 5 / (3 * 2) = 23 * 11 * 7 * 5 = 8855$
$|E_{HP}| = 8855$
$P(E_{HP}) = |E_{HP}| / |\text{Sample Space}| = 8855 / 42504 = 0.2083$

C. Assume another student in the class is Ela Sel. What is the **probability** that both Haydar Pekgül and Ela Sel **are both** in the chosen five-student committee?

In a similar spirit to B,

$|E_{HP,ES}| = C(22,3) = 22 * 21 * 20 / (3 * 2) = 11 * 7 * 20 = 1540$
$P(E_{HP,ES}) = |E_{HP,ES}| / |\text{Sample Space}| = 1540 / 42504 = 0.0362$

A much smaller probability than E_{HP} which makes sense.

4. A string that contain only 0s, 1s and 2s is called a **ternary string**. For example, 10220101 is a ternary string and so are 0102, 221, 0101 and 11.

A. Find a **recurrence relation** for the number of ternary strings of length n that do NOT contain consecutive 0s, 1s or 2s.

Let T_n be the number of ternary strings of length n that do not contain consecutive 0's, 1's or 2's. See what values T_n take starting from n=1:

n	ternary strings w/o consecutive 0's, 1's or 2's	T_n
1	0,1,2	3
2	01, 02, 10, 12, 20, 21	6
3	010, 012, 020, 021, 101, 102, 121, 120, 201, 202, 210, 212	12

Notice, looking at the table that for every string in T_{n-1} there are two new strings in T_n. This is because, the last digit can be followed by 2 different digits (to avoid consecutive digits). Thus, the recurrence relation is:

$T_n = 2 * T_{n-1}$

B. What are the **initial conditions**?

$T_1 = 3$

C. How many ternary strings of length 16 do NOT contain consecutive 0s, 1s or 2s?

$T_{16} = 2 * T_{15} = 2 * 2 * T_{14} = 2^{15} * T_1 = 32{,}768 * 3 = 98{,}304$

5. We roll a dice twice.

a) What is the **experiment?**

We roll a dice twice.

(Note that this experiment is equivalent to rolling two dice, because both result in the same sample space (same set of equiprobable outcomes)).

b) What is the **sample space**?

Sample Space = {1-1, 1-2, 1-3, … 1-6,
2-1, 2-2, 2-3, … 2-6,
…
6-1, 6-2, 6-3, …. 6-6}

c) What is the **size** of the sample space?

|Sample Space| = 6*6 = 36

d) Consider the event E_{even_sum} that corresponds to getting an even number as the sum of two dice. What is the **probability** of this event?

E_{even_sum} = {1-1, 1-3, 1-5, 2-2, 2-4, 2-6, …. 6-6}

$|E_{even\_sum}| = 18$

$P(E_{even\_sum}) = |E_{even\_sum}| / |\text{Sample Space}| = 18 / 36 = 0.5$

e) Assume along with the two dice, we also toss a coin. What is the **probability** of E_{even_sum} if we know that the coin showed a tail?

The Event that the coin showed a tail (E_{tail}) and E_{even_sum} are **independent events**. Thus,

$P(E_{even\_sum} | E_{tail}) = P(E_{even\_sum}) = 0.5$

f) Assume along with the two dice, we also toss a coin. What is the **probability** of E_{even_sum} if we know that one of the dice shows a 2?

Sample Space = {1-1-H, 1-1-T, 1-2-H, 1-2-T, …. 6-6-H, 6-6-T}.
|Sample Space| = 6*6*2 = 72

Let E_2 be the event that one of the dice shows a 2.

E_2 = {1-2-H, 1-2-T,
2-1-H, 2-1-T, 2-2-H, 2-2-T, 2-3-H, 2-3-T, … 2-6-H, 2-6-T,
3-2-H, 3-2-T,
4-2-H, 4-2-T,
5-2-H, 5-2-T,
6-2-H, 6-2-T}

E_{even_sum} = {1-1-H, 1-1-T, 1-3-H, 1-3-T, 1-5-H, 1-5-T, … 6-6-H, 6-6-T}

$P(E_{even\_sum} \mid E_2) = P(E_{even\_sum} \cap E_2) / P(E_2)$

$E_{even\_sum} \cap E_2$ = {2-2-H, 2-2-T, 2-4-H, 2-4-T, 2-6-H, 2-6,T,
4-2-H, 4-2-T,
6-2-H, 6-2-T}

$P(E_{even\_sum} \cap E_2) = |E_{even\_sum} \cap E_2| / |\text{Sample Space}| = 10 / 72$

$P(E_2) = |E_2| / |\text{Sample Space}| = 22 / 72$

Therefore, $P(E_{even\_sum} \mid E_2) = 10/22 = 5 / 11 = 0.4545$

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