CMPE 352
Signal Processing & Algorithms

Spring 2019

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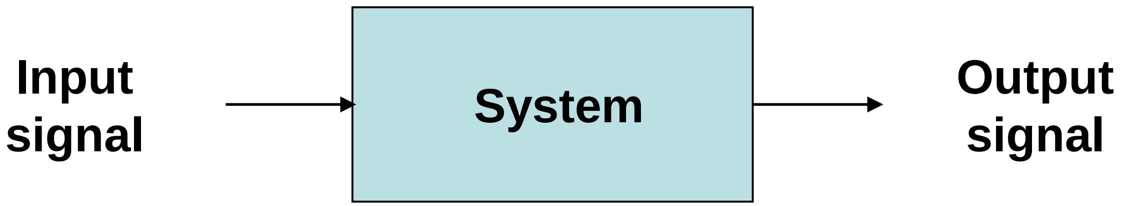
From *Signal* Spectrum to *System* Spectrum

- A (mechanical, electrical, biological, ...) system is «something» that transforms an input signal to an output signal

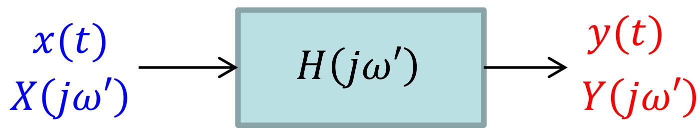


- By Fourier (Series or Transform), the input signal and the output signal can be thought of as being composed of the sum of (an infinite number of) sinusoids
- How is one such input sinusoid (say, at a frequency ω') and its corresponding output sinusoid (necessarily at same frequency ω') related?

  For example, a sinusoidal input at frequency ω' could be attenuated by 0.8 and delayed by 1.4 radians by the system

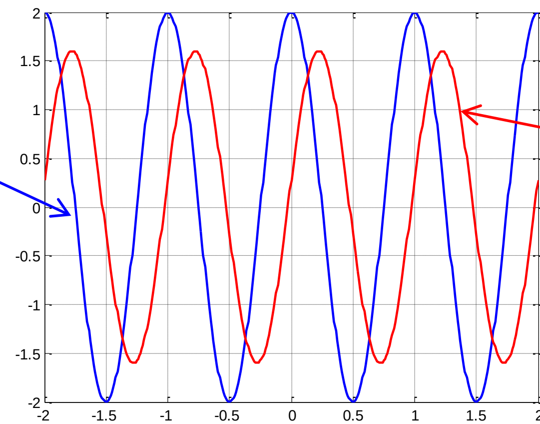
- If we plot these quantities ($|H(j\omega')|$ and $\measuredangle H(j\omega')$) not only for a particular frequency ω' but for all frequencies ω, we would have plotted the <u>magnitude and phase responses</u> (or magnitude and phase spectrum) of the system ($|H(j\omega)|$ and $\measuredangle H(j\omega)$, respectively)

From *Signal* Spectrum to *System* Spectrum

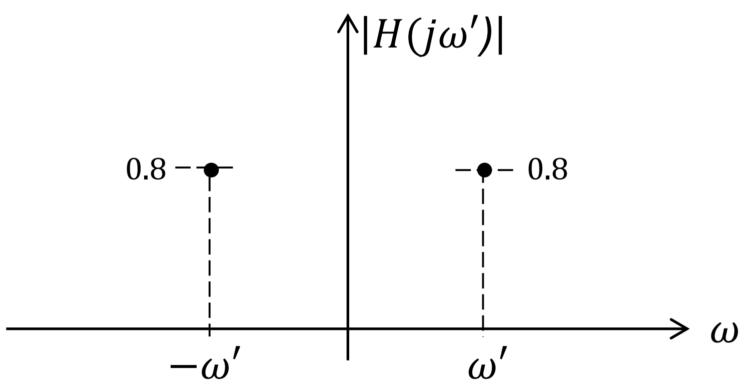


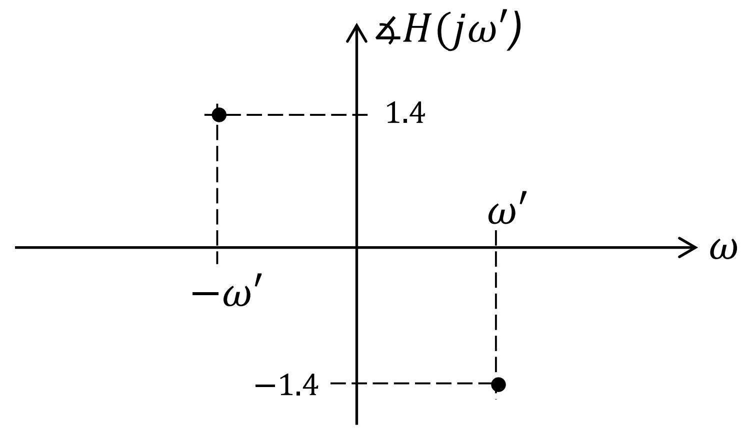
$x(t) = 2\cos(2\pi t)$

$\omega' = 2\pi$ (rad/s)



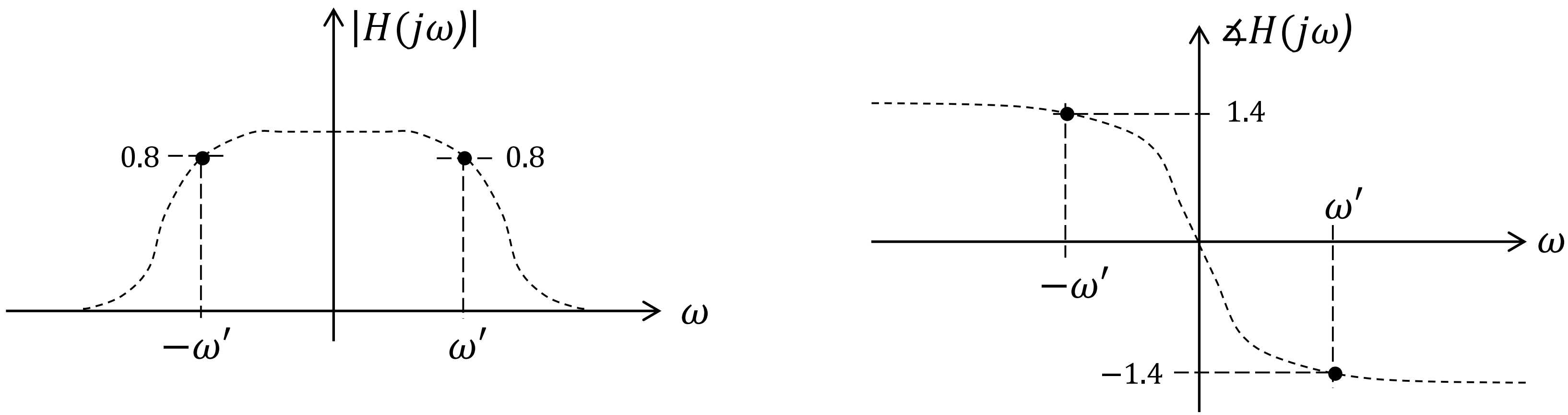
$y(t) = 1.6\cos(2\pi t - 1.4)$





From *Signal* Spectrum to *System* Spectrum

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$



Hence, $H(j\omega)$ defines the <u>system frequency response</u> (or spectrum)

$$\boxed{Y(j\omega) = H(j\omega) \cdot X(j\omega)}$$

System Frequency Response

- Hence, the system frequency response is characterized by the magnitude $|H(j\omega)|$ and the phase $\sphericalangle H(j\omega)$ of $H(j\omega)$

Frequency-domain view: $$\underbrace{X(j\omega)}_{|X(j\omega)|e^{j\sphericalangle X(j\omega)}} \longrightarrow \underset{|H(j\omega)|e^{j\sphericalangle H(j\omega)}}{\boxed{H(j\omega)}} \longrightarrow \underbrace{Y(j\omega)}_{|Y(j\omega)|e^{j\sphericalangle Y(j\omega)}} = H(j\omega)\cdot X(j\omega)$$

Note that

$$|Y(j\omega)|e^{j\sphericalangle Y(j\omega)} = |H(j\omega)|e^{j\sphericalangle H(j\omega)} \cdot |X(j\omega)|e^{j\sphericalangle X(j\omega)} = |H(j\omega)|\cdot|X(j\omega)|e^{j[\sphericalangle H(j\omega)+\sphericalangle X(j\omega)]}$$

$$\Rightarrow \boxed{\begin{aligned} |Y(j\omega)| &= |H(j\omega)| \cdot |X(j\omega)| \\ \sphericalangle Y(j\omega) &= \sphericalangle H(j\omega) + \sphericalangle X(j\omega) \end{aligned}}$$

Output magnitude is the product of the magnitudes

Output phase is the sum of the phases

Hence an input sinusoid of frequency ω will reappear at the output of the system modified in magnitude by a factor $|H(j\omega)|$ and shifted in phase by an amount $\sphericalangle H(j\omega)$.

Example 1

A sinusoidal signal $x(t) = \sin 2\pi t$ (t in seconds) is input to a system with frequency response

$$H(j\omega) = \frac{1}{1 + j\omega}$$

What signal $y(t)$ is observed at the output?

The input signal has frequency: $f = 1$ Hz.

Assuming linearity, the output signal will be a sinusoid at the same frequency:

$$y(t) = A\sin(2\pi t + \phi)$$

How do we determine A and ϕ ?

$$\left.\begin{array}{l} |H(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \Rightarrow |H(j2\pi)| = \dfrac{1}{\sqrt{1+4\pi^2}} \cong 0.16 \\[2ex] \angle H(j\omega) = -\operatorname{atan}\omega \Rightarrow \angle H(j2\pi) = -\operatorname{atan} 2\pi = 0 \end{array}\right\} \quad y(t) = 0.16\sin(2\pi t)$$

By how much (in dB) does the system attenuate this input signal? $10\log_{10}\frac{0.16^2/2}{1^2/2} \cong -15.9$ dB

(attenuation)

Example 2

A sinusoidal signal $x(t) = 5\sin(20\,\pi t + 0.5)$ (t in seconds) is input to a system with frequency response

$$H(j\omega) = \frac{2 \cdot 10^4}{1 + \omega^2}$$

What signal $y(t)$ is observed at the output?

The input signal has frequency: $2\pi f = 20\,\pi \Rightarrow f = 10$ Hz.

Assuming linearity, the output signal will be a sinusoid at the same frequency:

$$y(t) = A\sin(20\pi t + \phi)$$

How do we determine A and ϕ ?

$$\left.\begin{aligned} &|H(j\omega)| = \frac{2 \cdot 10^4}{1 + \omega^2} \Rightarrow |H(j2\pi 10)| = \frac{2 \cdot 10^4}{1 + (20\pi)^2} \cong 5.06 \\ &\angle H(j\omega) = 0 \end{aligned}\right\} \quad y(t) = 25.3\sin(20\pi t + 0.5)$$

By how much (in dB) does the system attenuate this input signal? $10\log_{10}\frac{25.3^2/2}{5^2/2} \cong 14.1$ dB

(amplification)

Filters

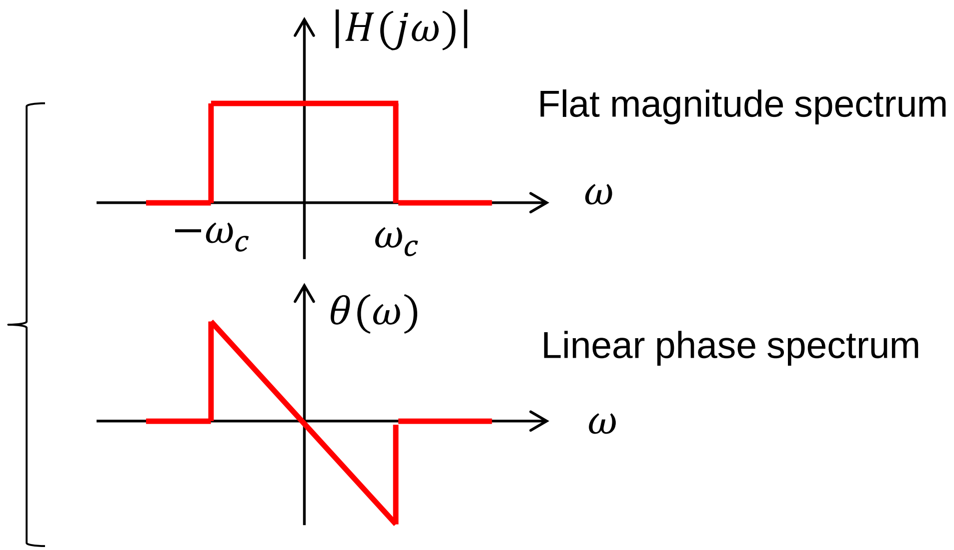
- **Filters** are systems (devices) that perform signal processing functions, specifically to remove unwanted frequency components from the signal, to enhance wanted ones, or both.

- Filters can be:
  - passive or active
  - analog or digital
  - high-pass, low-pass, band-pass, band-stop (band-rejection; notch), or all-pass
  - discrete-time (sampled) or continuous-time
  - linear or non-linear
  - etc.

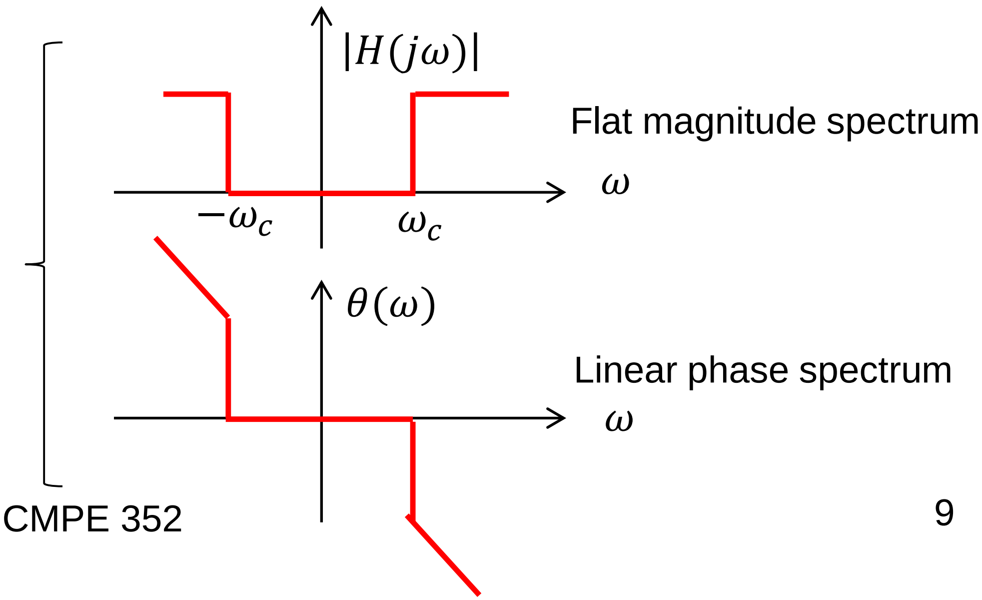
Filters with Ideal Low-Pass and High-Pass Characteristics

$$\boxed{H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}}$$

Ideal low-pass characteristic:



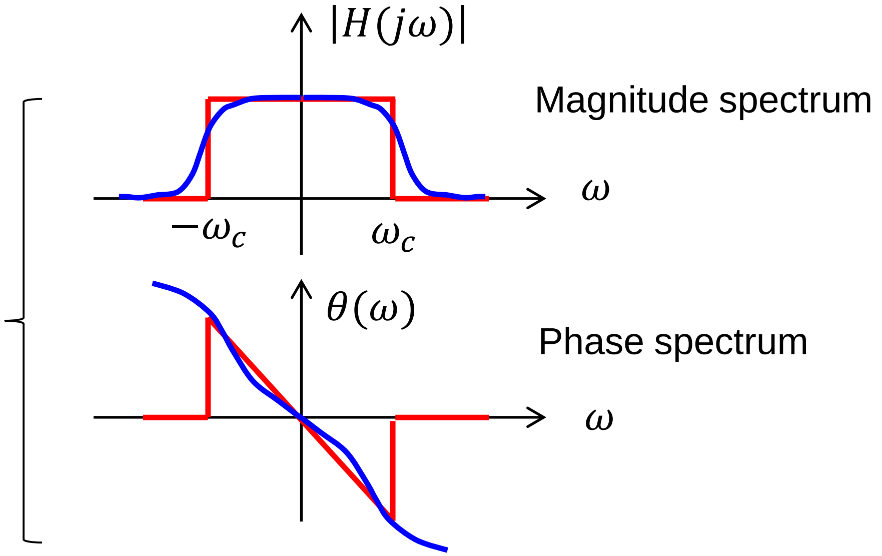
Ideal high-pass characteristic:



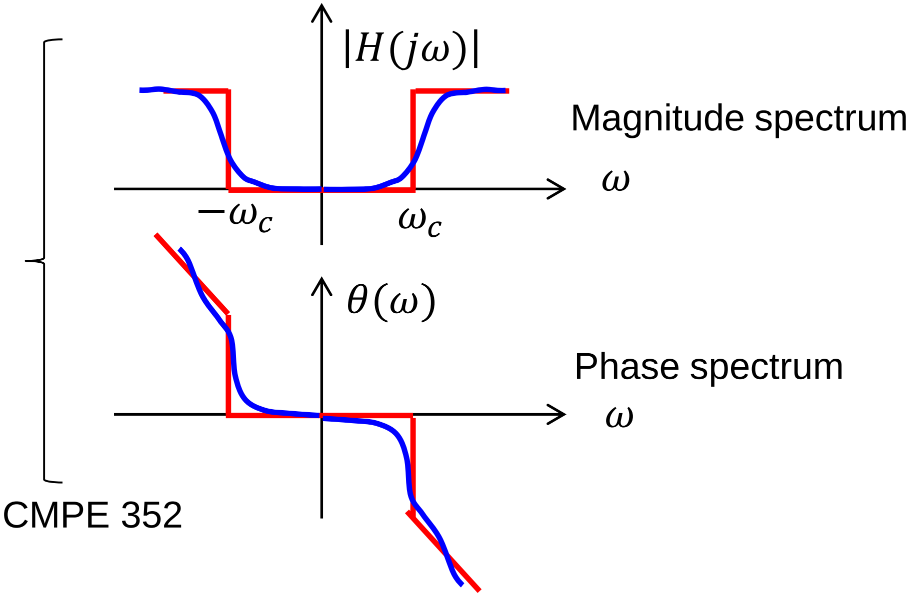
Filters with Realizable Low-Pass and High-Pass Characteristics

$$\boxed{H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}}$$

Realizable low-pass characteristic (in blue):



Realizable high-pass characteristic (in blue):

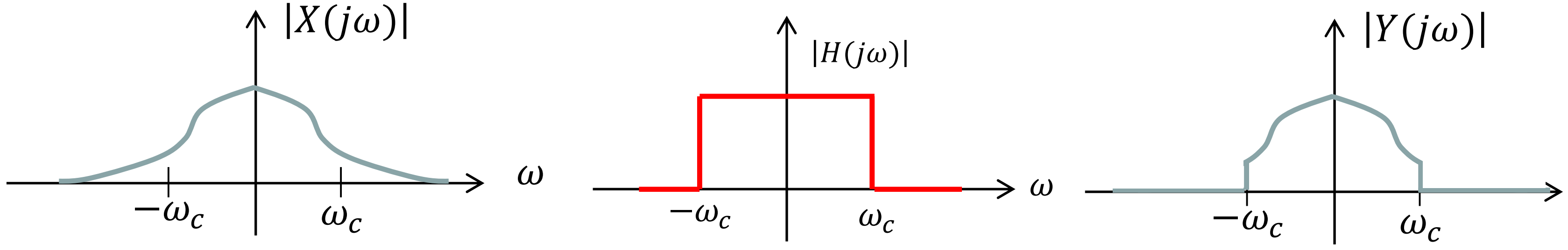


As a general rule, achieving better approximations of the ideal characteristics requires higher implementation complexity

Filters: Examples (1)

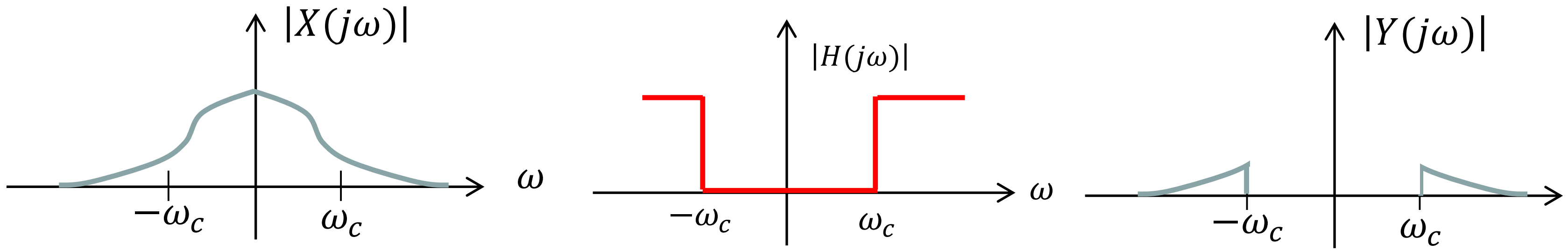
Low-pass filter

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega) \cdot X(j\omega)$$



High-pass filter

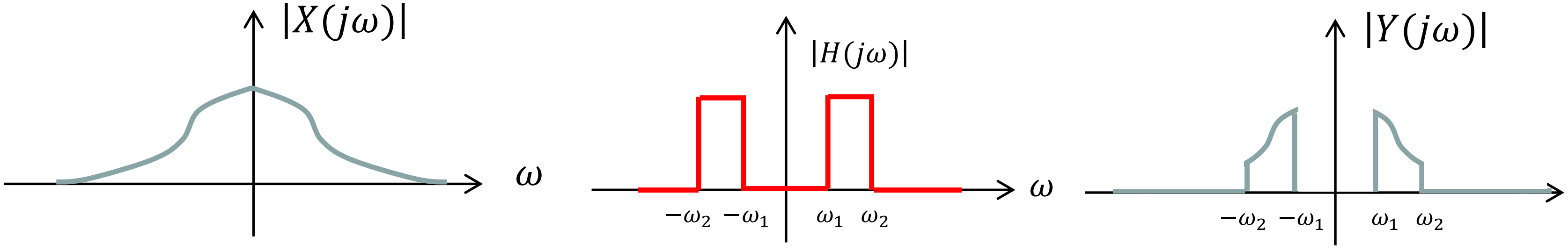
$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega) \cdot X(j\omega)$$



Filters: Examples (2)

Band-pass filter

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega) \cdot X(j\omega)$$



Filters: Examples (3)

Problem 1: A speech signal of power 10 mW extends from 0 to 4 kHz. This signal is subject to a noise (due to the environment, microphone parasitics, etc.) of 1 mW that is constant in spectrum from 0 to 10 kHz. What is the ratio of signal-power to noise-power (signal-to-noise ratio SNR) ?

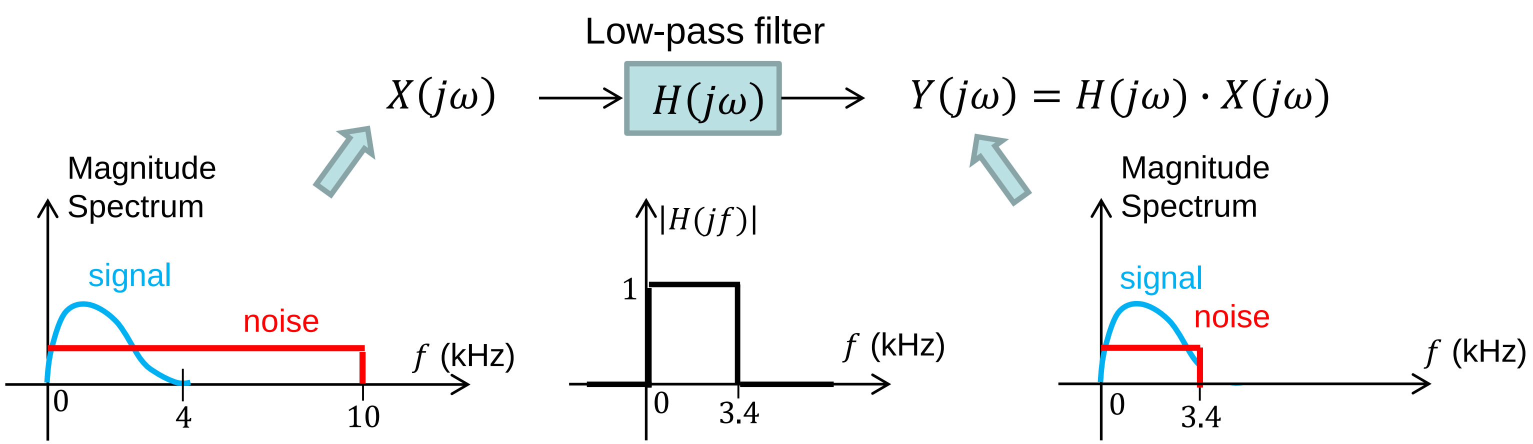
Total signal power: $10\ mW$
Total noise power: $1\ mW$

Signal to noise ratio: $\mathbf{10mW/1mW = 10 \ \rightarrow 10\ dB}$

Problem 2: What can be done to improve (increase) the SNR? Use a filter to filter out the noise.

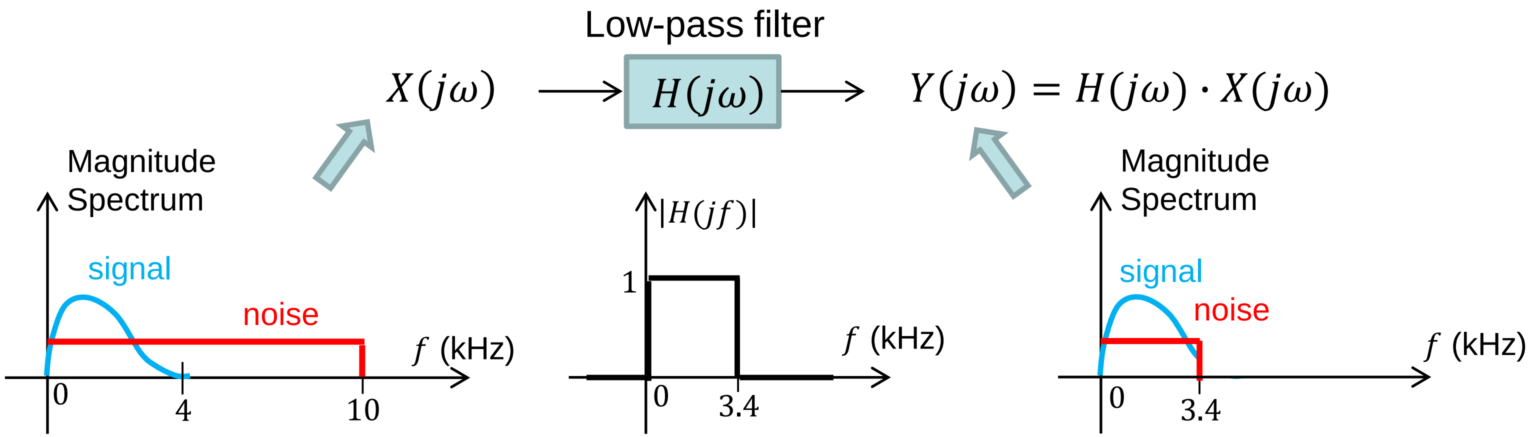
What kind of filter should be used? A low-pass filter.

What should its cutoff frequency be? For example 3.4 kHz.



Filters: Examples (4)

<u>Problem 3</u>: A speech signal of power 10 mW extends from 0 to 4 kHz. This signal is subject to a noise (due to the environment, microphone parasitics, etc.) of 1 mW that is constant in spectrum from 0 to 10 kHz. An ideal low-pass filter with a cutoff frequency of 3.4 kHz is used to filter this noisy signal. What is the ratio of signal-power to noise-power at the input and at the output of the filter, assuming that 90% of the signal power is found in the frequency band of 0 to 3.4 kHz?



At the filter input: $\left\{ \begin{array}{l} \text{Total signal power: } 10\ mW \\ \text{Total noise power: } 1\ mW \end{array} \right\}$ **Signal to noise ratio:** $\mathbf{10mW/1mW = 10 \rightarrow 10\ dB}$

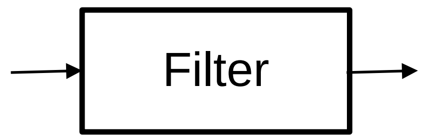
At the filter output: $\left\{ \begin{array}{l} \text{Signal power within } [0\ kHz;\ +3.4\ kHz]\text{: } 10\ mW \times 0.90 = 9\ mW \\ \text{Noise power within } [0\ kHz;\ +3.4\ kHz]\text{: } 1\ mW \times \dfrac{3.4\ kHz}{10\ kHz} = 0.34\ mW \end{array} \right.$

Signal to noise ratio: $\mathbf{9mW/0.34\ mW = 26.5 \rightarrow 14.2\ dB}$

Filters: Example (5)

Image ″denoising″:

Noisy image



Denoised image

Original

Digital Processing of Analog Signals (1)

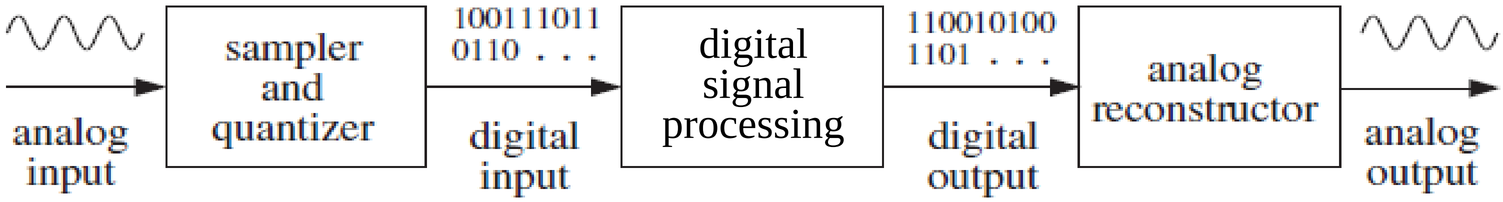
Digital processing of analog signals proceeds in three stages:

1. The analog signal is digitized, that is, it is sampled and each sample quantized to a finite number of bits. How is this process called?

   A/D conversion

2. The digitized samples are processed by digital signal processing.
3. The resulting output samples may be converted back into analog form by an analog reconstructor. How is the reconstruction process called?

   D/A conversion



Digital Processing of Analog Signals (2)

- Depending on the speed and computational requirements of the application, the digital signal processing may be realized by a
  - general purpose computer,
  - minicomputer,
  - special purpose chip (Digital Signal Processor, DSP),
  - or other digital hardware dedicated to performing a particular signal processing task.

- Digital signal processing algorithms typically require a <u>large number of mathematical operations</u> to be performed quickly and repeatedly on a series of data samples. Signals (perhaps from audio or video sensors) are constantly <u>converted from analog to digital, manipulated digitally, and then converted back to analog form</u>. <u>Sampling and quantization</u> are two key concepts which are prerequisites to every digital signal processing operation.

- Many digital signal processing applications have constraints on <u>latency</u>; that is, for the system to work, the signal processing operations must be completed within some fixed time, and deferred (or batch) processing is not viable.

Digital Processing of Analog Signals (3)

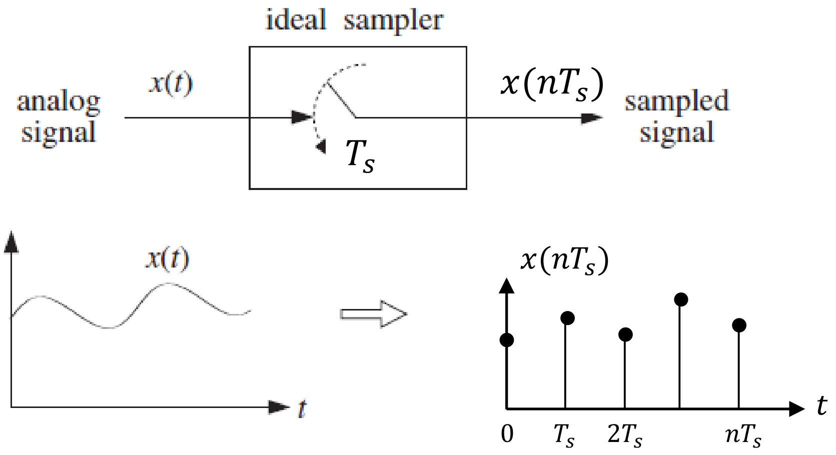
Most general-purpose microprocessors and operating systems can execute digital signal processing algorithms successfully, but are not suitable for use in portable devices such as mobile phones because of power efficiency constraints.

A specialized digital signal processor (DSP), will tend to provide a lower-cost solution, with better performance, lower latency, and no requirements for specialized cooling or large batteries.

The architecture of a DSP is optimized specifically for digital signal processing. Most also support some of the features as an applications processor or microcontroller, since signal processing is rarely the only task of a system.

Sampling Process

- During the sampling process, the analog signal $x(t)$ is periodically measured every T_s (s). Thus, time is discretized in units of the sampling interval T_s: nT_s, $n = 0,1,2,3$ ....



- For system design purposes, two questions must be answered:

1. How fast should one choose the sampling frequency $f_s = 1/T_s$, or: how small should one choose the sampling interval (sampling period) T_s ?

2. What is the effect of sampling on the frequency spectrum?

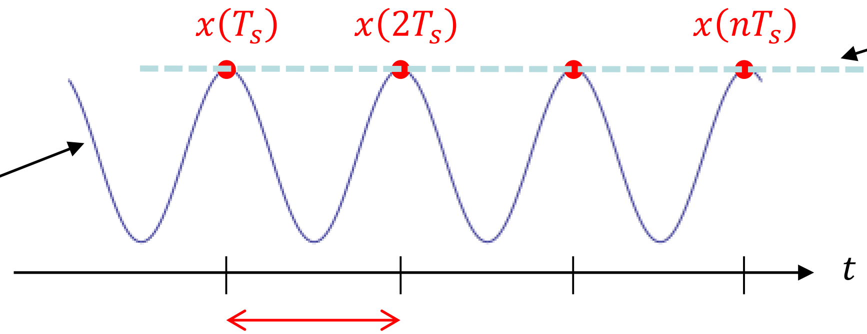
- Note: to answer these questions we will make use of what we learned from Fourier (Series or Transform): "any" signal is a linear combination of sinusoidal signals

What Sampling Frequency ?

1 Sample "very slow"

signal corresponding to samples

signal $x(t)$ of frequency f_1



sampling interval T_s

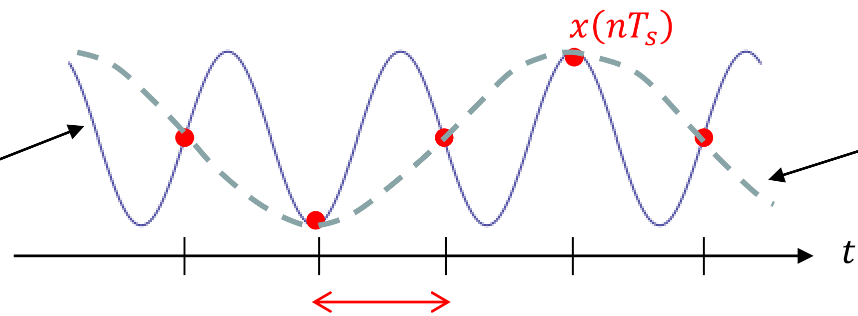
sampling frequency $f_s = 1/T_s$

$$f_s = f_1$$

What Sampling Frequency ? (2)

2 Sample ″slow″

signal frequency f_1



signal corresponding to samples

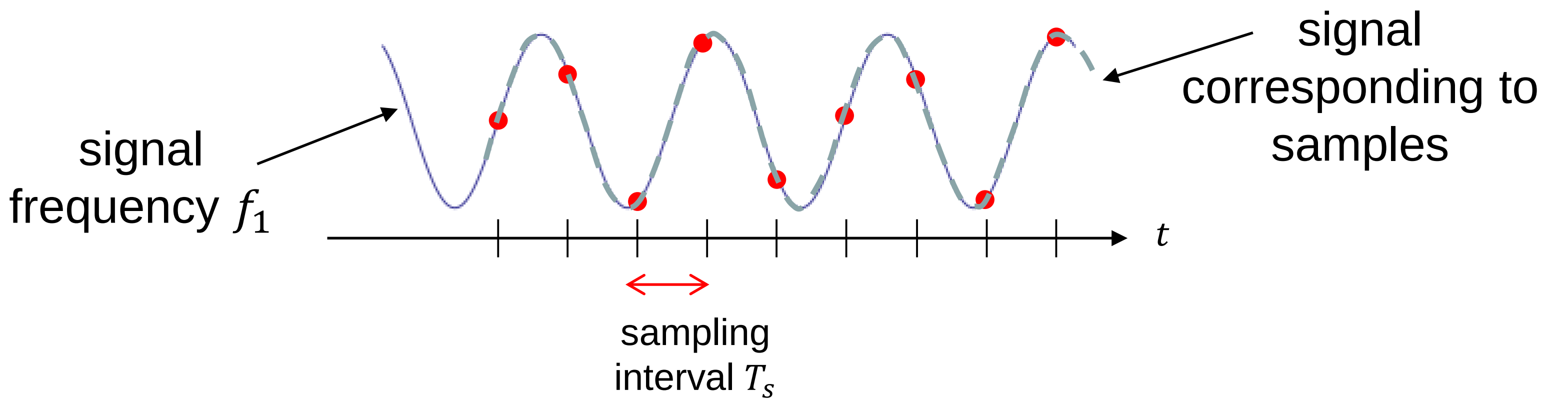
sampling interval T_s

sampling frequency $f_s = 1/T_s$

$$f_s = 1.33 f_1$$

What Sampling Frequency ? (3)

3 Sample "fast"



sampling frequency $f_s = 1/T_s$

$$f_s = 2.5 f_1$$

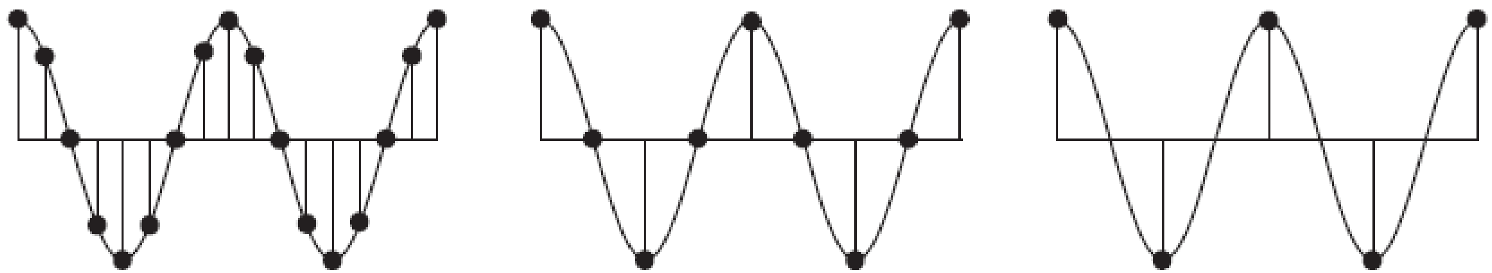
- Hence the sampling frequency should not be too low in order for the samples to constitute a good representaion of the original analog sinusoid.
- So what should the lower limit on f_s be?

$$\boxed{f_s > ?}$$

What Sampling Frequency ? (4)

Consider again the sampling of the sinusoidal signal of frequency f_1: $x(t) = \cos 2\pi f_1 t$

Sample this signal at three different sampling frequencies:



$f_s = 8f_1$ $f_s = 4f_1$ $f_s = 2f_1$

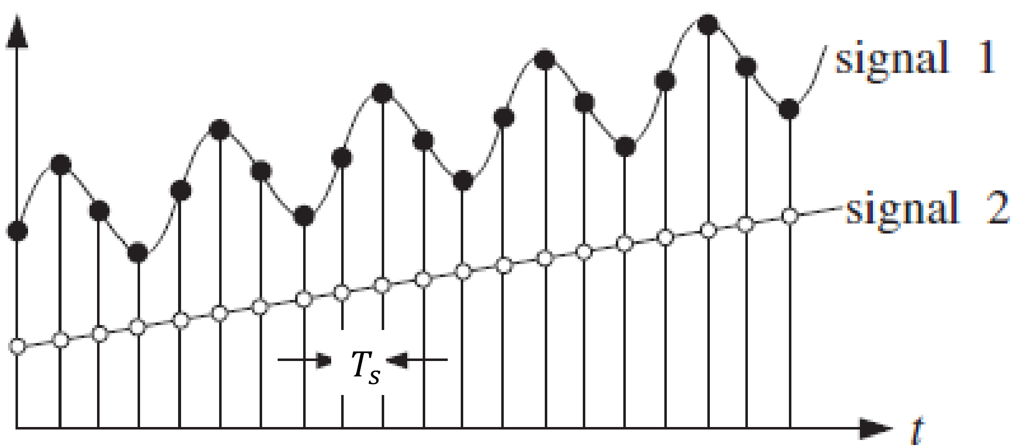
Simple inspection of these figures leads to the conclusion that the <u>minimum acceptable number of samples per cycle is two</u>. Therefore*

$$\boxed{f_s \geq 2f_1}$$

* Note that for simple sinusoids, the sampling phase also plays a role, requiring in fact that $f_s > 2f_1$.

What Sampling Frequency? (Arbitrary Signal)

- Next, consider the case of an arbitrary signal $x(t)$.
- How would you propose to choose the value of f_s (or T_s)?
- T_s must be small enough so that signal variations that occur between samples are not lost. Why not be safe and choose T_s very very small?
- It would be very impractical to choose T_s too small because then there would be too many samples to be processed.
- Is the selection of T_s adequate for the two signals below?



T_s is small enough to resolve the details of signal 1, but is unnecessarily small for signal 2.

What Sampling Frequency? (Arbitrary Signal) (2)

- Let $x(t)$ be an arbitrary signal. According to the Fourier expansion (Series or Transform), $x(t)$ can be expressed as a linear combination of sinusoids.
- Proper sampling of $x(t)$ will be achieved only if every sinusoidal component of $x(t)$ is properly sampled.
- What does this require?

  The sampling frequency must be selected so that the sinusoidal component with maximum frequency is properly sampled: hence $f_s \geq 2f_{max}$

  (This will ensure that all the other sinusoidal components are properly sampled as well!)

Note: The signal should have no spectral component above frequency f_{max} (that is, the signal must be bandlimited)

The Sampling Theorem

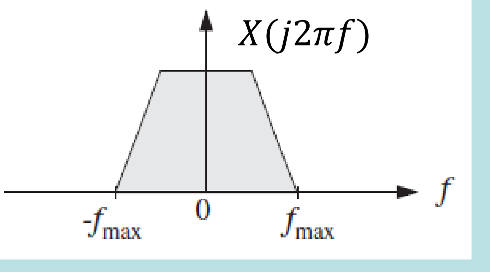
The sampling theorem states that for accurate representation of a signal $x(t)$ by its time samples $x(nT_s)$, two conditions must be met:

1. The signal $x(t)$ must be <u>bandlimited</u>, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency f_{max} and no frequencies beyond that
2. The sampling rate f_s must be chosen to be <u>at least twice the maximum frequency</u> f_{max} contained in the signal, that is,

$$f_s \geq 2f_{max}$$

or, in terms of the sampling time interval:

$$T_s \leq \frac{1}{2f_{max}}$$



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