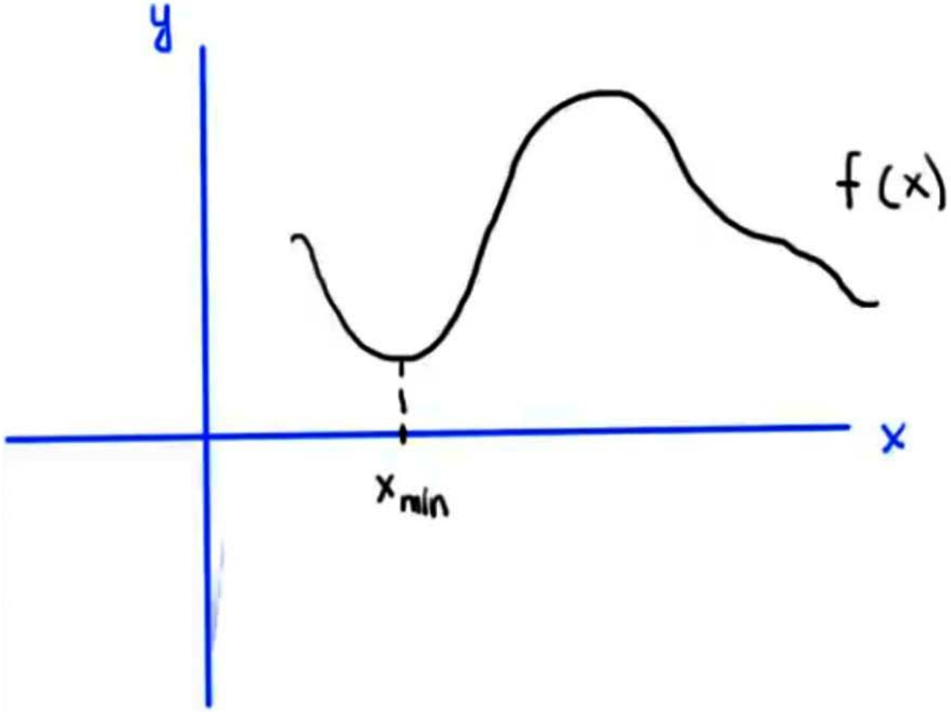
EEEN 460
Optimal Control

Spring 2020

Lecture 7

Calculus of Variations

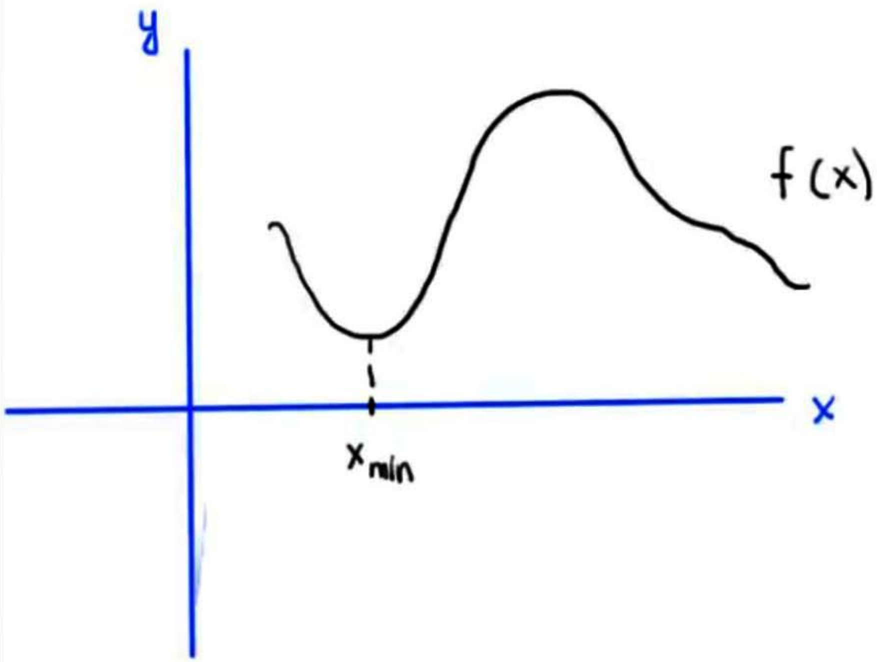
Euler-Lagrange Equations



To find x corresponding to local minimum (x_{min}):

Find $f'(x)$ or $\frac{dy}{dx}$ and set it to 0

→ values of x which satisfy this are candidates for x_{min}

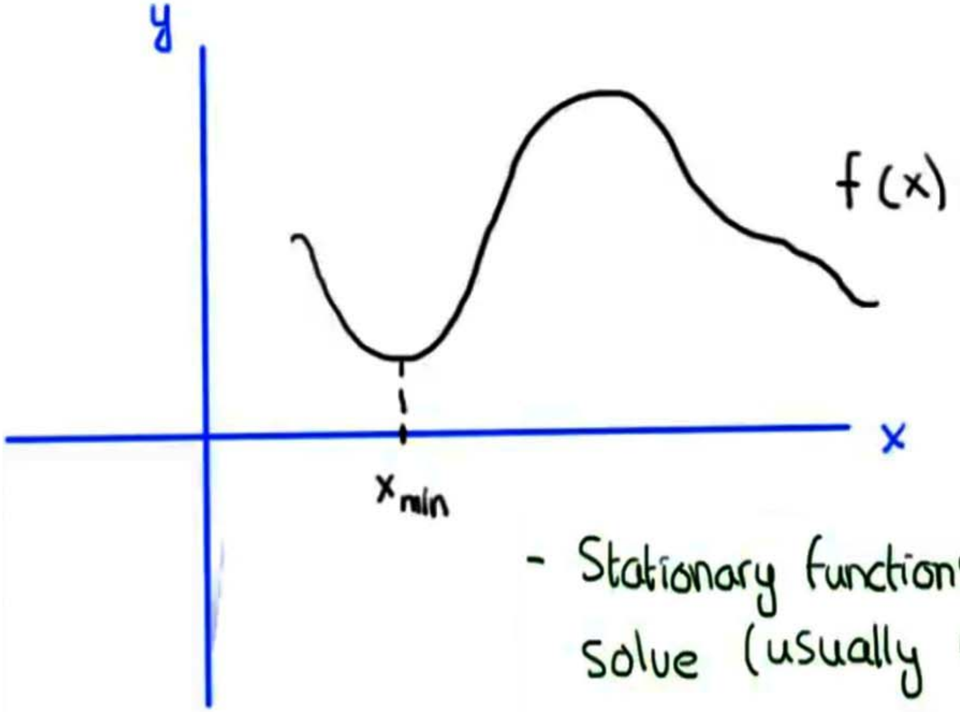


Solving $f'(x) = 0$ gives stationary points — further testing needed to determine their nature.

Thus, to find:

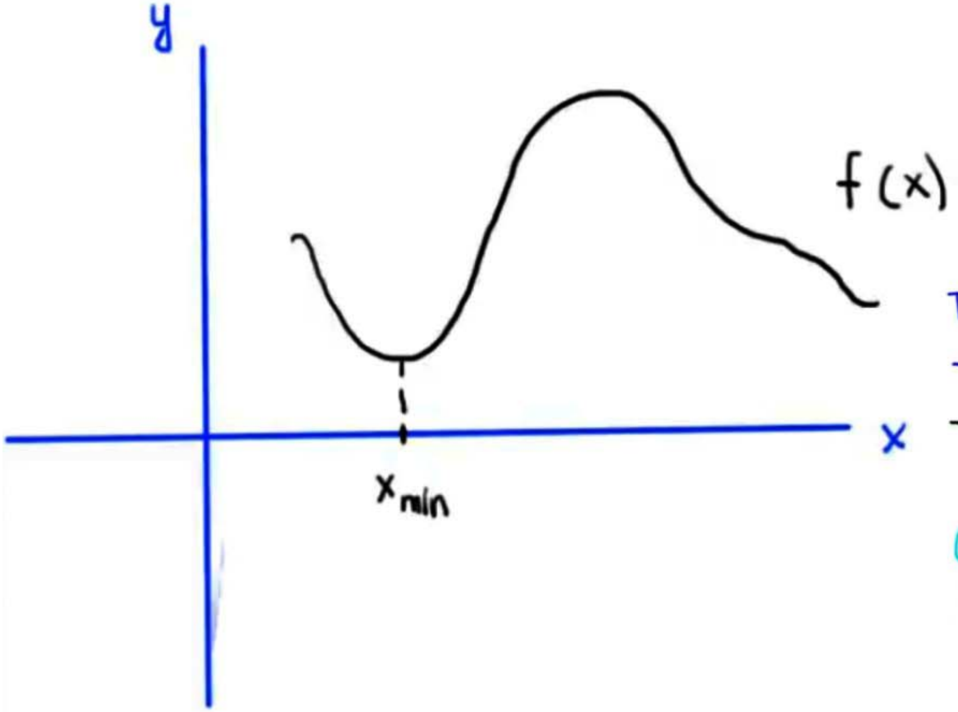
- Stationary points of $f(x)$,

solve $df/dx = 0$ for x (Regular Calculus)



- Stationary functions of a functional $I[f]$ (function of functions), solve (usually differential) equation for stationary function $f(x)$

(Calculus of Variations)

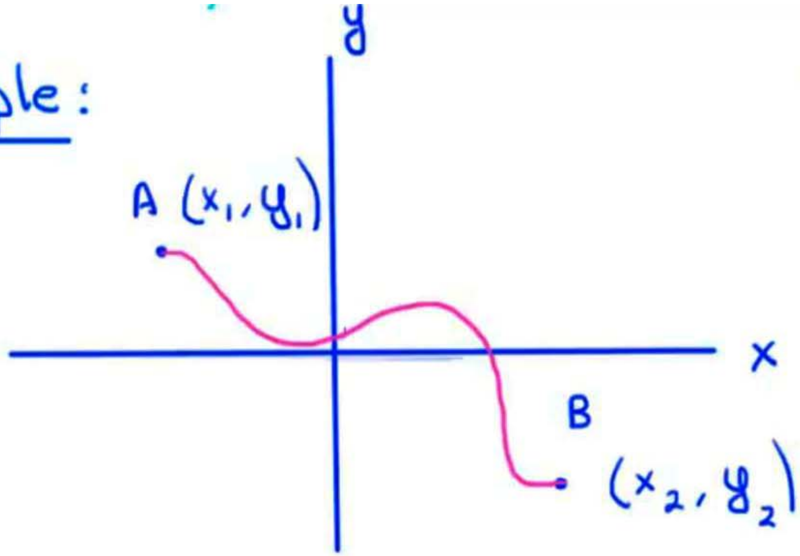


Thus, to find:

- Stationary points of $f(x)$, solve $df/dx = 0$ for x (Regular Calculus)
- Stationary functions of a functional $I[f]$ (function of functions), solve (usually differential) equation for stationary function $f(x)$

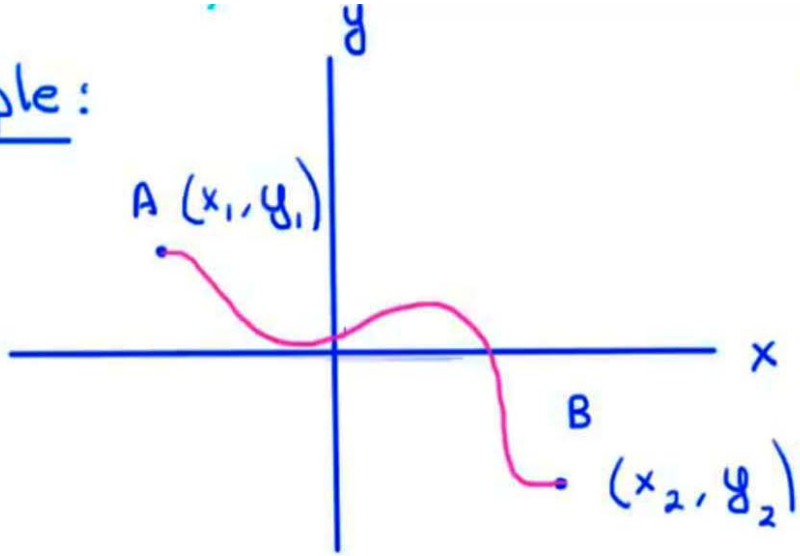
(Additional layer of functions) (Calculus of Variations)

Example:



Find path such that distance AB is minimized.

Example:



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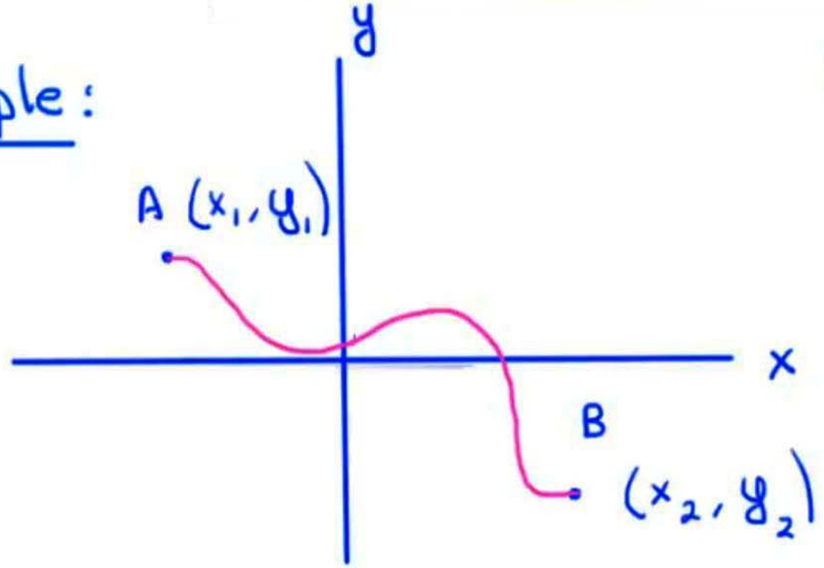


Find path such that distance AB is minimized.

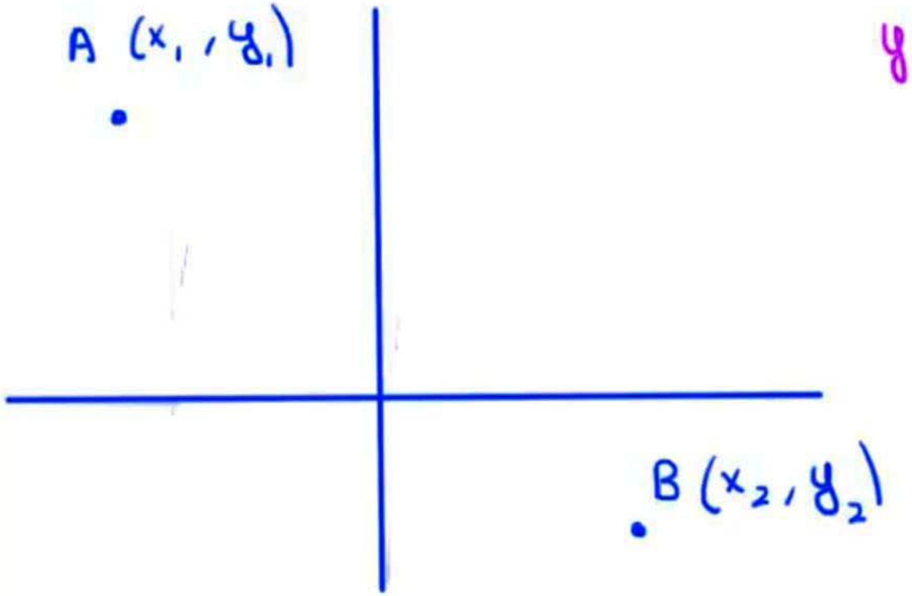
$$I = \int_A^B dS \leadsto = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Rightarrow \underbrace{I}_{\text{functional}} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Problem: Find $y = f(x)$ b/w points A and B such that the integral

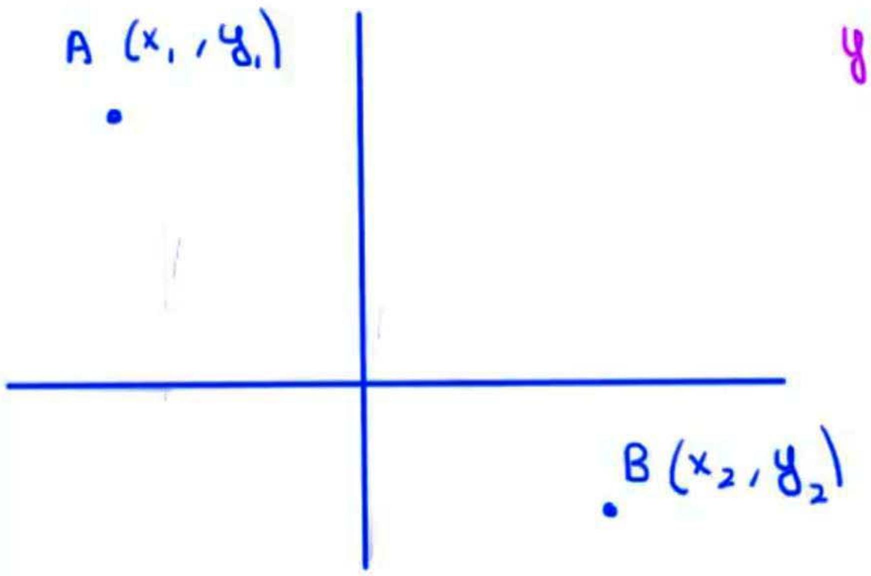
$$I = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ is MINIMIZED!

Another Example:



Given $v(x,y)$ of a particle, find the path $y = f(x)$ such that the time taken by the particle is minimized.

Another Example:



Given $v(x,y)$ of a particle, find the path $y = f(x)$ such that the time taken by the particle is minimized.

Since $dt = \dfrac{dS}{v(x,y)}$

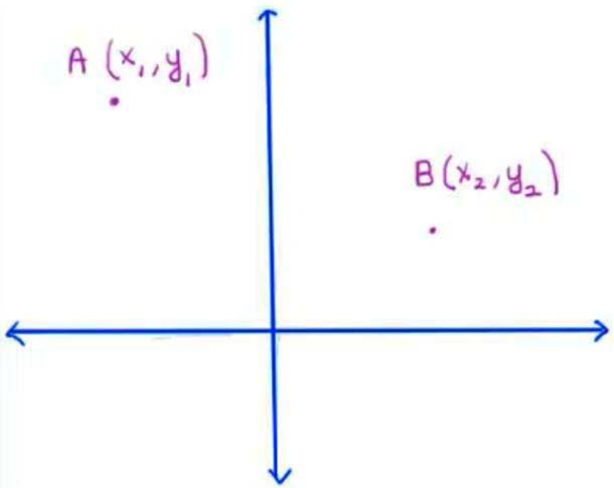
$$T = \int_{x_1}^{x_2} \frac{\sqrt{1 + (dy/dx)^2}}{v(x,y)}\, dx$$

<u>Problem</u>: Find $y = f(x)$ b/w A and B such that T is minimized.

In general, Calculus of Variations seeks to find $y = f(x)$ such that this integral :

$$I[F] = \int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}\right) dx$$ is stationary.

Deriving the Euler - Lagrange Equations

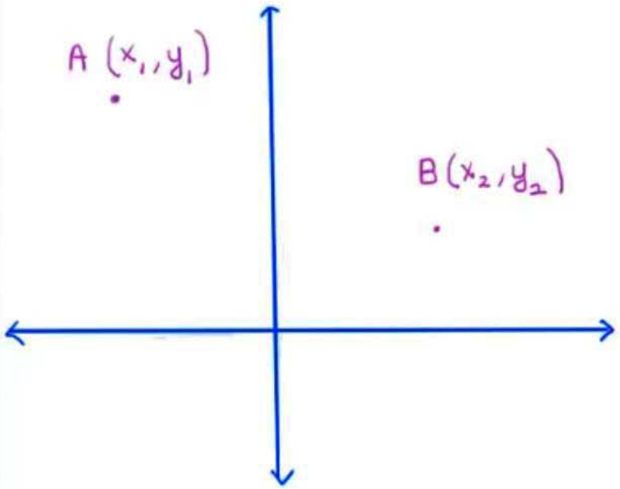


Find $y = f(x)$ such that the functional

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx \quad \text{is stationary.}$$

Boundary conditions : $y(x_1) = y_1$, $y(x_2) = y_2$

Deriving the Euler - Lagrange Equations



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$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx \quad \text{is stationary.}$$

Boundary conditions : $y(x_1) = y_1$, $y(x_2) = y_2$

Derivation / Proof : Suppose $\underbrace{y^*(x)}_{\text{Extremal}}$ makes I stationary and satisfies the above boundary conditions.

Implicit : All functions have continuous 2^{nd} derivatives.

- Introduce a function $\eta(x)$, $\eta(x_1) = \eta(x_2) = 0$
- Define : $\bar{y}(x) = y^*(x) + \varepsilon\eta(x)$

<u>Derivation / Proof</u> : Suppose $\underbrace{y^*(x)}_{\text{Extremal}}$ makes I stationary and satisfies the above boundary conditions.

Implicit : All functions have continuous 2^{nd} derivatives.

- Introduce a function $\eta(x)$, $\eta(x_1) = \eta(x_2) = 0$
- Define : $\bar{y}(x) = y^*(x) + \varepsilon\eta(x)$, satisfies same boundary conditions as y

$\bar{y}$ represents a family of curves.

x gets integrated out

<u>Problem</u> : Find the particular $\bar{y}(x)$ which makes $I(\varepsilon) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$ stationary.

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- Since I depends only on ε, to make I stationary, set :

$$\frac{dI}{d\varepsilon} = 0$$

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$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

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$$\frac{dI}{d\varepsilon}\bigg|_{\varepsilon=0} = 0 \quad , \quad \text{when} \quad \frac{dI}{d\varepsilon}\bigg|_{\varepsilon=0} = 0 :$$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx = 0 \Rightarrow \int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} F(x, \bar{y}, \bar{y}')\bigg|_{\varepsilon=0} dx = 0$$

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$$\Rightarrow \int_{x_1}^{x_2} \left.\left[\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \varepsilon} \right]\right|_{\varepsilon=0} dx = 0$$

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$$\bar{y}(x) = y^*(x) + \varepsilon\eta(x)$$

$$\bar{y}'(x) = y^{*\prime}(x) + \varepsilon\eta'(x)$$

$$\partial \bar{y}/\partial \varepsilon = \eta$$

$$\partial \bar{y}'/\partial \varepsilon = \eta'$$

Problem: Find the particular $\bar{y}(x)$ which makes $I(\varepsilon) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$ stationary.

- Since I depends only on ε, to make I stationary, set:

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0 \quad , \quad \text{when} \quad \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon} \Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\right]\Bigg|_{\varepsilon=0} dx = 0$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx = 0 \Rightarrow \int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} F(x, \bar{y}, \bar{y}')\Bigg|_{\varepsilon=0} dx = 0$$

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$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\right]\Bigg|_{\varepsilon=0} dx = 0$$

$$\boxed{\begin{aligned} \bar{y}(x) &= \overset{*}{y}(x) + \varepsilon\eta(x) \\ \bar{y}'(x) &= \overset{*}{y}'(x) + \varepsilon\eta'(x) \\ \partial\bar{y}/\partial\varepsilon &= \eta \\ \partial\bar{y}'/\partial\varepsilon &= \eta' \end{aligned}}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_{x_1}^{x_2} F(x,\bar{y},\bar{y}')\,dx = 0 \quad\Rightarrow\quad \int_{x_1}^{x_2} \left.\frac{\partial}{\partial\varepsilon} F(x,\bar{y},\bar{y}')\right|_{\varepsilon=0} dx = 0$$

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$$\partial\bar{y}/\partial\varepsilon = \eta$$
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Integrate by parts :

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'}\eta'\,dx$$
$$= \frac{\partial F}{\partial \bar{y}'}\int_{x_1}^{x_2}\eta'\,dx - \int_{x_1}^{x_2}\left(\int\eta'\right)\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]dx$$
$$= \frac{\partial F}{\partial \bar{y}'}\left[\eta\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]dx$$

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$$\bar{y}(x) = y(x) + \varepsilon\eta(x)$$

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$$\partial\bar{y}/\partial\varepsilon = \eta$$

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$$\Rightarrow \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial\bar{y}}\frac{\partial\bar{y}}{\partial\varepsilon} + \frac{\partial F}{\partial\bar{y}'}\frac{\partial\bar{y}'}{\partial\varepsilon}\right]\bigg|_{\varepsilon=0}dx = 0$$

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Integrate by parts :

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial\bar{y}'}\eta'\,dx$$

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$$= \frac{\partial F}{\partial\bar{y}'}\left[\eta\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}\left[\frac{\partial F}{\partial\bar{y}'}\right]dx$$

$$\eta(x_1) = \eta(x_2) = 0$$

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Integrate by parts :

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'}\eta' dx$$

$$= \frac{\partial F}{\partial \bar{y}'}\int_{x_1}^{x_2} \eta' dx - \int_{x_1}^{x_2} \left(\int \eta'\right) \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right] dx$$

$$= \overset{=0}{\cancel{\frac{\partial F}{\partial \bar{y}'}[\eta]_{x_1}^{x_2}}} - \int_{x_1}^{x_2} \eta \frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right] dx$$

$$\eta(x_1) = \eta(x_2) = 0$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{x_1}^{x_2} F(x,\bar{y},\bar{y}')\,dx = 0 \quad\Rightarrow\quad \int_{x_1}^{x_2}\frac{\partial}{\partial\varepsilon}F(x,\bar{y},\bar{y}')\Big|_{\varepsilon=0} dx = 0$$

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$$\Rightarrow \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial\bar{y}}\eta - \frac{d}{dx}\left(\frac{\partial F}{\partial\bar{y}'}\right)\eta\right]\Big|_{\varepsilon=0} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial\bar{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial\bar{y}'}\right)\right]\eta\,\Big|_{\varepsilon=0} dx = 0$$

Integrate by parts :

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial\bar{y}'}\eta'\,dx$$
$$= \frac{\partial F}{\partial\bar{y}'}\int_{x_1}^{x_2}\eta'\,dx - \int_{x_1}^{x_2}\left(\int\eta'\right)\frac{d}{dx}\left[\frac{\partial F}{\partial\bar{y}'}\right]dx$$
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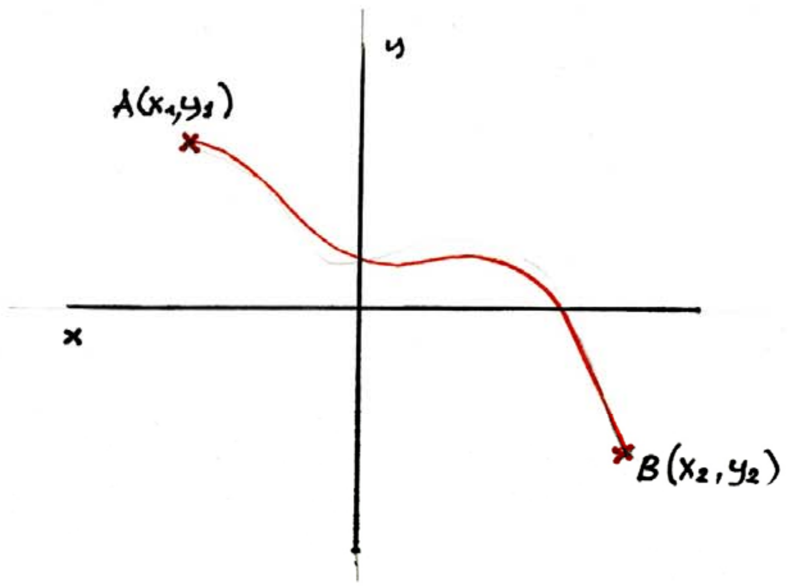
Thus, $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ (Euler-Lagrange Equation)

EXAMPLE:



Find path (A,B) such that the distance AB is minimized

Remember

$$I = \int_A^B ds$$

$$= \int_A^B \sqrt{dx^2 + dy^2}$$

$$= \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$I = \int_A^B \sqrt{1 + \dot{y}^2}\, dx \qquad \text{where} \quad \dot{y} = \frac{dy}{dx}$$

$$\downarrow$$

$$F(x, y, \dot{y})$$

Euler - Lagrange Equation

$$\frac{\partial F(x,y,\dot{y})}{\partial y} - \frac{d}{dx}\left(\frac{\partial F(x,y,\dot{y})}{\partial \dot{y}}\right) = 0$$

$$F(x,y,\dot{y}) = (1+\dot{y}^2)^{1/2}$$

Since F is only a function of $\dot{y}$

$$\frac{\partial F}{\partial y} = 0$$

$$\frac{d}{dx}\left[\frac{\dot{y}}{(1+\dot{y}^2)^{1/2}}\right] = 0$$

$$\Rightarrow \quad \ddot{y} = 0$$

$\therefore$ The optimal solution is a straight line

$$y^* = c_1 x + c_2$$

Find the solution when

$A(x_1, y_1) = (-5, 5)$

$B(x_2, y_2) = (10, -5)$

$y^* = c_1 x + c_2$

$y^*(-5) = 5$

$y^*(10) = -5$

$5 = c_1(-5) + c_2$

$-5 = c_1(10) + c_2$

$c_1 = -2/3$

$c_2 = 5/3$

$y^* = -\frac{2}{3} x + \frac{5}{3}$

End of Lecture VII