Selected Problems II

Problem 1) Given a set of $S = \{1, 2, 5, 10\}$, we define

$\forall a, b \in S$, $a + b \triangleq$ the smallest common factor of a and b

$a \cdot b \triangleq$ the greatest common divisor of a and b

$a' \triangleq 10/a$

a. Show that the algebraic system of $\{S, +, \cdot, '\}$ establishes a general Boolean algebra.

b. Consider the following Boolean function

$$f(x_1, x_2) = x_1'(5x_2' + x_2') + x_2(2x_1' + x_1')$$

Derive the canonical form of sum of minterms representation for $f(x_1, x_2)$.

c. Derive the canonical form of product of maxterms representation for $f(x_1, x_2)$.

d. Derive the Reed-Müller canonical form of representation for $f(x_1, x_2)$.

Solution.

a.

i. Commutative property justified as follows

$(1,2)$:	$1+2=2$ $2+1=2$	$1\cdot 2=1$ $2\cdot 1=1$
$(1,5)$:	$1+5=5$ $5+1=5$	$1\cdot 5=1$ $5\cdot 1=1$
$(1,10)$:	$1+10=10$ $10+1=10$	$1\cdot 10=1$ $10\cdot 1=1$
$(2,5)$:	$2+5=10$ $5+2=10$	$2\cdot 5=1$ $5\cdot 2=1$
$(2,10)$:	$2+10=10$ $10+2=10$	$2\cdot 10=1 \to 2$ $10\cdot 2=1 \to 2$
$(5,10)$:	$5+10=10$ $10+5=10$	$5\cdot 10=5$ $10\cdot 5=5$

ii. Associative property justified as follows

$(1,2,5)$:	$1+(2+5)=1+10=10$ $(1+2)+5=2+5=10$	$1\cdot(2\cdot 5)=1\cdot 1=1$ $(1\cdot 2)\cdot 5=1\cdot 5=1$
$(1,2,10)$:	$1+(2+10)=1+10=10$ $(1+2)+10=2+10=10$	$1\cdot(2\cdot 10)=1\cdot 2=1$ $(1\cdot 2)\cdot 10=1\cdot 10=1$
$(1,5,10)$:	$1+(5+10)=1+10=10$ $(1+5)+10=5+10=10$	$1\cdot(5\cdot 10)=1\cdot 5=1$ $(1\cdot 5)\cdot 10=1\cdot 10=1$
$(2,5,10)$:	$2+(5+10)=2+10=10$ $(2+5)+10=10+10=10$	$2\cdot(5\cdot 10)=2\cdot 5=1$ $(2\cdot 5)\cdot 10=1\cdot 10=1$

iii. Distributive property justified as follows

$(1,2,5)$: $1+2\cdot 5 = 1+1 = 1$; $1\cdot(2+5) = 1\cdot 10 = 1$

$(1+2)\cdot(1+5) = \underset{2\cdot 5}{1\cdot 1} = 1$; $1\cdot 2 + 1\cdot 5 = 1+1 = 1$

$(1,2,10)$: $1+2\cdot 10 = 1+2 = 2$; $1\cdot(2+10) = 1\cdot 10 = 1$

$(1+2)\cdot(1+10) = 2\cdot 10 = 2$; $1\cdot 2 + 1\cdot 10 = 1+1 = 1$

$(1,5,10)$: $1+5\cdot 10 = 1+5 = 5$; $1\cdot(5+10) = 1\cdot 10 = 1$

$(1+5)\cdot(1+10) = 5\cdot 10 = 5$; $1\cdot 5 + 1\cdot 10 = 1+1 = 1$

$(2,5,10)$: $2+5\cdot 10 = 2+5 = 10$; $2\cdot(5+10) = 2\cdot 10 = 2$

$(2+5)\cdot(2+10) = 10\cdot 10 = 10$; $2\cdot 5 + 2\cdot 10 = 1+2 = 2$

iv. identity element $\equiv 10$

$1+10 = 10+1 = 10$; $1\cdot 10 = 10\cdot 1 = 1$

$2+10 = 10+2 = 10$; $2\cdot 10 = 10\cdot 2 = 2$

$5+10 = 10+5 = 10$; $5\cdot 10 = 10\cdot 5 = 5$

$10+10 = 10$; $10\cdot 10 = 10$

v. zero element $\equiv 1$

$1+1 = 1$; $1\cdot 1 = 1$

$2+1 = 1+2 = 2$; $1\cdot 2 = 2\cdot 1 = 1$

$5+1 = 1+5 = 5$; $1\cdot 5 = 5\cdot 1 = 1$

$10+1 = 1+10 = 10$; $1\cdot 10 = 10\cdot 1 = 1$

vi. $1' = 10/1 = 10$, $10+1 = 1+10 = 10$, $10\cdot 1 = 1\cdot 10 = 1$

$2' = 10/2 = 5$, $5+2 = 2+5 = 10$, $5\cdot 2 = 2\cdot 5 = 1$

$5' = 10/5 = 2$, $2+5 = 5+2 = 10$, $2\cdot 5 = 5\cdot 2 = 1$

$10' = 10/10 = 1$, $1+10 = 10+1 = 10$, $1\cdot 10 = 10\cdot 1 = 1$

Hence ;

- the set $\{S, +, \cdot, '\}$ establishes a general Boolean algebra

b. $f(x_1, x_2) = x_1'(5x_2' + x_2') + x_2(2x_1' + x_1')$

$= 5x_1'x_2' + x_1'x_2' + 2x_1'x_2 + x_1'x_2$

$= f(0,0)m_0 + f(0,1)m_1 + f(1,0)m_2 + f(1,1)m_3$

$0 \equiv 1 \; , \; 1 \equiv 10$

$f(0,0) \equiv f(1,1) = 5 \cdot 1' \cdot 1' + 1' \cdot 1' + 2 \cdot 1' \cdot 1 + 1' \cdot 1$

$= 5 \cdot 10 \cdot 10 + 10 \cdot 10 + 2 \cdot 10 \cdot 1 + 10 \cdot 1$

$= 5 + 10 + 1 + 10 = 10$

$f(0,1) \equiv f(1,10) = 5 \cdot 1' \cdot 10' + 1' \cdot 10' + 2 \cdot 1' \cdot 10 + 1' \cdot 10$

$= 5 \cdot 10 \cdot 1 + 10 \cdot 1 + 2 \cdot 10 \cdot 10 + 10 \cdot 10$

$= 1 + 1 + 2 + 10 = 10$

$f(1,0) \equiv f(10,1) = 5 \cdot 10' \cdot 1' + 10' \cdot 1' + 2 \cdot 10' \cdot 1 + 10' \cdot 1$

$= 5 \cdot 1 \cdot 10 + 1 \cdot 10 + 2 \cdot 1 \cdot 1 + 1 \cdot 1$

$= 1 + 1 + 1 + 1 = 1$

$f(1,1) \equiv f(10,10) = 5 \cdot 10' \cdot 10' + 10' \cdot 10' + 2 \cdot 10' \cdot 10 + 10' \cdot 10$

$= 5 \cdot 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \cdot 10 + 1 \cdot 10$

$= 1 + 1 + 1 + 1 = 1$

$$f(x_1,x_2) = 10\cdot x_1'x_2' + 10\cdot x_1'x_2 + 1\cdot x_1x_2' + 1\cdot x_1x_2$$

$$= x_1'x_2' + x_1'x_2 \qquad \text{SOM form}$$

c.

$$f(x_1,x_2) = [10 + \overbrace{x_1 + x_2}^{10}]\cdot[10 + \overbrace{x_1 + x_2'}^{10}]$$

$$\cdot[1 + x_1' + x_2]\cdot[1 + x_1' + x_2']$$

$$= 10\cdot 10\cdot [x_1' + x_2][x_1' + x_2']$$

$$= (x_1' + x_2)(x_1' + x_2') \qquad \text{POM form}$$

d.

$$f(x_1,x_2) = a_0 \oplus a_1x_1 \oplus a_2x_2 \oplus a_3x_1x_2$$

$$f(0,0) = a_0$$

$$f(0,0) \equiv f(1,1) = 1'\cdot 1' + 1'\cdot 1 = 10\cdot 10 + 10\cdot 1$$

$$= 10 + 1 = 10$$

$$\Rightarrow \quad a_0 = 10$$

$$f(1,0) = a_0 \oplus a_1$$

$$f(1,0) \equiv f(10,1) = 10'\cdot 1' + 10'\cdot 1 = 1\cdot 10 + 1\cdot 1$$

$$= 1 + 1 = 1$$

$$a_1 = a_0 \oplus f(10,1) = 10 \oplus 1$$

$$= 10\cdot 1' + 10'\cdot 1 = 10\cdot 10 + 1\cdot 1$$

$$= 10 + 1 = 10 \qquad \Rightarrow \quad a_1 = 10$$

$f(0,1) = a_0 \oplus a_2$

$f(0,1) \equiv f(1,10) = 1' \cdot 10' + 1' \cdot 10 = 10.1 + 10.10$

$= 1 + 10 = 10$

$a_2 = a_0 \oplus f(1,10) = 10 \oplus 10$

$= 10.10' + 10'.10 = 10.1 + 1.10$

$= 1 + 1 = 1 \quad \Rightarrow \quad a_2 = 1$

$f(1,1) = a_0 \oplus a_1 \oplus a_2 \oplus a_3$

$f(1,1) \equiv f(10,10) = 10'.10' + 10'.10 = 1.1 + 1.10$

$= 1 + 1 = 1$

$\underbrace{10 \oplus 10} \oplus 1 \oplus a_3 = 1$

$\underbrace{1 \oplus 1} \oplus a_3 = 1$

$1 \oplus a_3 = 1 \quad \Rightarrow \quad a_3 = 1 \oplus 1 = 1$

Hence ;

$f(x_1, x_2) = 10 \oplus 10 \cdot x_1 \oplus 1 . x_2 \oplus 1 . x_1 . x_2$ Reed-Müller form

$= 10 \oplus x_1 \oplus 1 \oplus 1$

$= 10 \oplus x_1 \oplus 1$ → shown to be simplifiable

$= 10 \oplus x_1$

$= x_1'$

Problem 2) Express the following function as a sum of minterms and as a product of maxterms:

$$F(A,B,C,D) = B'D + A'D + BD$$

Solution. We consider

$$F(A,B,C,D) = (A+A')B'(C+C')D + A'(B+B')(C+C')D + (A+A')B(C+C')D$$

$$= AB'CD + AB'C'D + A'B'CD + A'B'C'D + A'BCD + A'BC'D + A'B'CD + A'B'C'D + ABCD + ABC'D + A'BCD + A'BC'D$$

$$= AB'CD + AB'C'D + A'B'CD + A'B'C'D + A'BCD + A'BC'D + ABCD + ABC'D$$

$$= \sum m(1,3,5,7,9,11,13,15)$$

In a similar manner, we have

$$F'(A,B,C,D) = \sum m(0,2,4,6,8,10,12,14)$$

$$= A'B'C'D' + A'B'CD' + A'BC'D' + A'BCD' + AB'C'D' + AB'CD' + ABC'D' + ABCD'$$

Hence;

$$F(A,B,C,D) = (F'(A,B,C,D))'$$

$$= (A'B'C'D' + A'B'CD' + A'BC'D' + A'BCD' + AB'C'D' + AB'CD' + ABC'D' + ABCD')'$$

$$= (A+B+C+D)\cdot(A+B+C'+D)\cdot(A+B'+C+D)\cdot(A+B'+C'+D)\cdot(A'+B+C+D)\cdot(A'+B+C'+D)\cdot(A'+B'+C+D)\cdot(A'+B'+C'+D)$$

Problem 3) Express the complement of the following functions in sum-of-minterms form:

a. $F(A,B,C,D) = \sum m(3,5,9,11,15)$

b. $F(x,y,z) = \prod M(2,4,5,7)$

Solution.

a. $F'(A,B,C,D) = \sum m(0,1,2,4,6,7,8,10,12,13,14)$

↳ that is, the remaining minterms which do NOT appear in F belong to F'

b. Note that;

$$F(x,y,z) = \prod M(2,4,5,7)$$

$$= \sum m(0,1,3,6)$$

Hence;

$$F'(x,y,z) = \sum m(2,4,5,7)$$

Problem 4) Convert each of the following to the other canonical form:

a. $F(x,y,z) = \sum m(2,5,6)$

b. $F(A,B,C,D) = \prod M(0,1,2,4,7,9,12)$

Solution.

a. $F(x,y,z) = \sum m(2,5,6)$

$$= \prod M(0,1,3,4,7)$$

b. $F(A,B,C,D) = \prod M(0,1,2,4,7,9,12)$

$$= \sum m(3,5,6,8,10,11,13,14,15)$$

Problem 5) Convert each of the following expressions into sum of products and product of sums:

a. $(AB+C)(B+C'D)$

b. $x'+x(x+y')(y+z')$

Solution.

a. $(AB+C)(B+C'D) = ABB + ABC'D + CB + \underbrace{\cancel{C}C'D}_{0}$

$= AB + ABC'D + BC$

$= AB\underbrace{(1+C'D)}_{1} + BC$

$= AB + BC$ "Sum of products"

$= (A+C)B$ "Product of sums"

b. $x'+x(x+y')(y+z') = x' + x(xy + xz' + \underbrace{\cancel{y'}y}_{0} + y'z')$

$= x' + xxy + xxz' + xy'z'$

$= x' + xy + xz' + xy'z'$

$= x' + xy + xz'\underbrace{(1+y')}_{1}$

$= x' + xy + xz'$

$= \underbrace{(x'+x)}_{1}(x'+y) + xz'$

$= x' + y + xz'$

$= \underbrace{(x'+x)}_{1}(x'+z') + y$

$= x' + y + z'$

Sum of products Product of sums

Problem 6) Write the simplest Boolean equations for the outputs which are defined by the following truth table:

a	b	c	f	g
0	0	0	1	0
0	0	1	0	0
0	1	0	0	1
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	1
1	1	1	1	0

Solution. We consider sum-of-minterms expression for f:

$$f(a,b,c) = \Sigma m(0,3,6,7)$$

$$= a'b'c' + a'bc + abc' + abc$$

$$= a'b'c' + a'bc + ab\underbrace{(c'+c)}_{1}$$

$$= a'b'c' + a'bc + ab$$

$$= a'b'c' + (a'c + a)b$$

$$= a'b'c' + \underbrace{(c'+a)}_{1}(c+a)b$$

$$= a'b'c' + ab + bc$$

- We prefer to derive product-of-maxterms expression for g:

$$g(a,b,c) = \Pi M(0,1,7)$$

$$= (a+b+c)(a+b+c')(a'+b'+c')$$

$$= [(a+b) + \underbrace{c \cdot c'}_{0}] \cdot (a'+b'+c')$$

$$= (a+b)(a'+b'+c')$$

PS 2.11