CMPE 352
Signal Processing & Algorithms

Spring 2019

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Review Questions (1)

- What are the two main constraints that must be taken into account for the choice of the sampling frequency f_s?

  1. Constraint due to the signal spectrum (sampling theorem).
  2. Implementation also restricts the choice of f_s.

- How can the constraint due to the implementation be expressed?

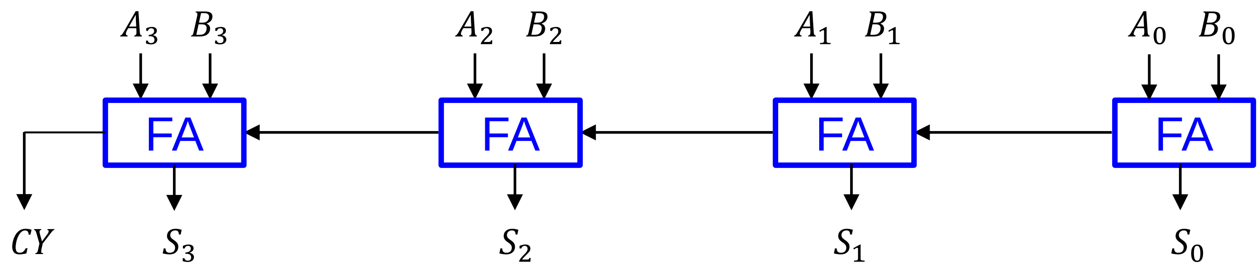
  The sampling interval T_s must be larger than total processing or computation time required for each sample T_{proc}: $T_s > T_{proc}$

  Note: *pipelining* can help to trade-off clock speed versus latency

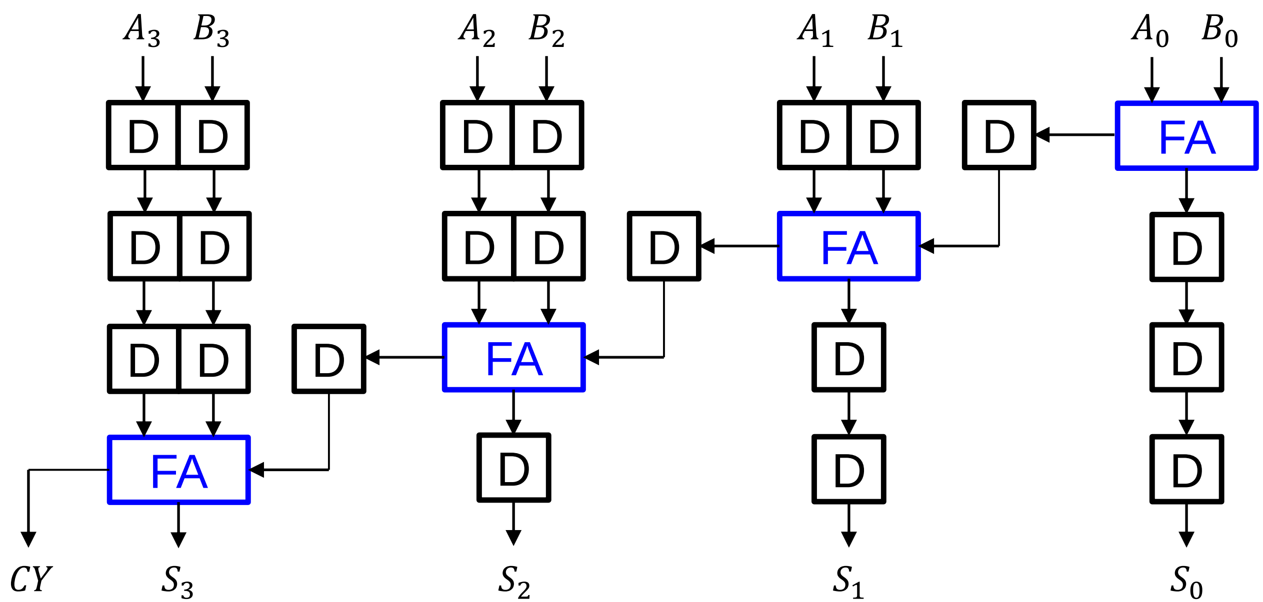
Note on Pipelining

Example: Adding two numbers.

Consider adding two 4-bit numbers: (A_3, A_2, A_1, A_0) and (B_3, B_2, B_1, B_0)



Direct Full-Adder (FA) based implementation



Implementation with <u>pipelining</u> using delay (D) elements (D flip-flops)

Review Questions (2)

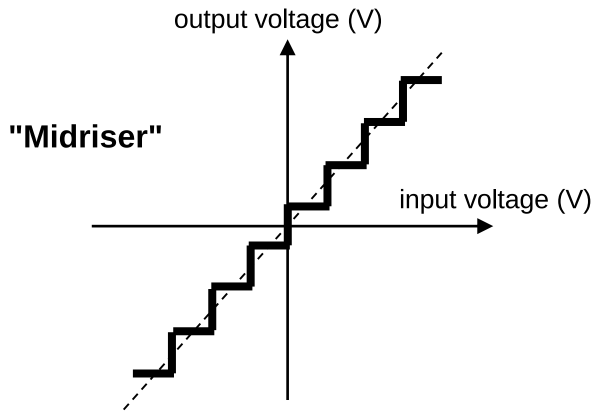
- What is the purpose of the quantization operation performed in association with sampling?

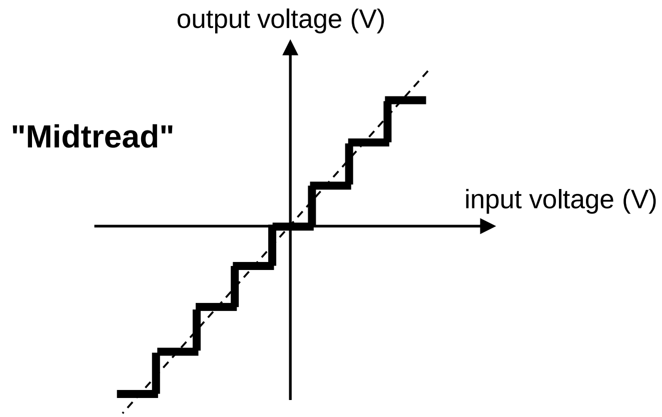
  To be able to represent a signal sample with a finite number of bits or bytes.

- How is the quantization operation realized?

  By dividing the signal variation range into a finite number of levels and rounding-off the current sample value to the nearest level.

- What is the shape of the input-output characteristic of a quantizer?

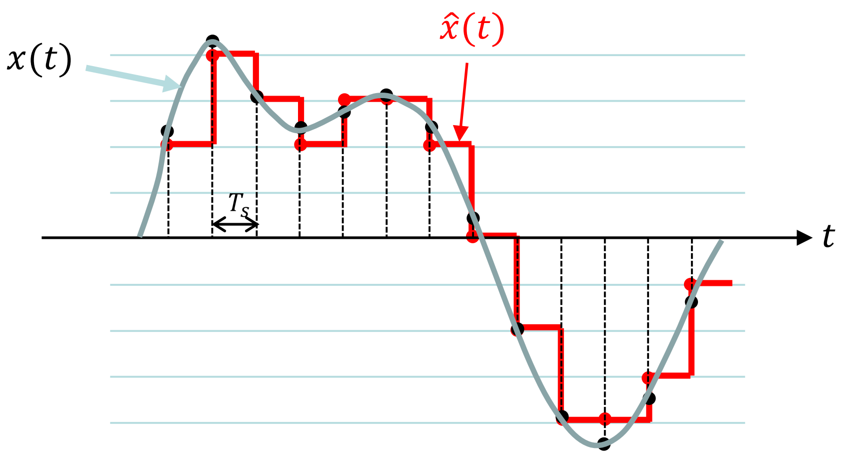




Review Questions (3)

- What is a quantization-error signal?

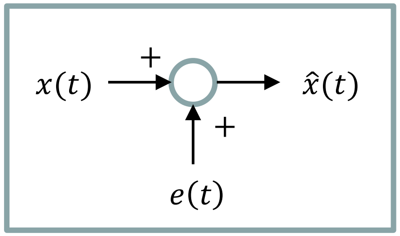
  The difference between the quantized signal and the original unquantized analog signal.



$$\boxed{e(t) = \hat{x}(t) - x(t)}$$

$$\downarrow$$

$$\hat{x}(t) = x(t) + e(t)$$



Model of the Quantizer

- How did we define a quantizer model that expresses the quantization error?

- What is the power of the quantization error (= quantization noise)?

$$\boxed{N_q = \frac{\Delta v^2}{12} = \frac{x_p^2}{3L^2}}$$

($\Delta v = \frac{2x_p}{L}$: quantization step)

- What is the effect of increasing by 1 the number of quantizer bits?

To reduce N_q by 6 dB.

Review Questions (4)

- What is a PCM?

  Pulse-code modulation (PCM) is a method used to digitally represent sampled analog signals.

- How is it realized?

  The amplitude of the analog signal is sampled regularly at uniform intervals, and each sample is quantized to the nearest value, represented by N bits, within a range of digital steps.

- In PCM, what are the two basic parameters that determine the fidelity to the original analog signal?
  - the sampling rate
  - the bit depth N

Problem 1

A sinusoidal signal is transmitted using PCM. Find the minimum number of bits required to achieve a target signal-to-quantization-error ratio (SNR) of 25 dB.

Problem 2

Consider a sinusoidal signal given by $x(t) = 3\sin(1000\pi t)$. Find the signal-to-quantization-error ratio (SNR) when the signal is quantized using a 9-bit PCM.

Problem 3

A sinusoidal TV signal with a frequency of 42 MHz is transmitted using binary PCM. The number of quantization levels is 1024. Calculate

a) The word length *N*

b) The average SNR

c) The bit rate *R*

Problem 4

At what minimum frequency can the signal

$$x(t) = e^{-2|t|}$$

be sampled if we assume that the ″essential bandwidth″ B of the signal is the frequency at which $|X(j\omega)|$ drops to 1% of its peak value ?

The Discrete Fourier Transform (DFT) - 1

- The (continuous-time) Fourier transform and the inverse Fourier transform are given by

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\ dt \qquad\qquad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\ d\omega$$

- We want to compute the (continuous-time) Fourier transform of $g(t)$ numerically.
- Hence we want to derive an **algorithm** to compute the Fourier transformation.
- This means that, practically, we have to use the samples of $g(t)$ and obtain the response (the Fourier transform) as samples of $G(\omega)$

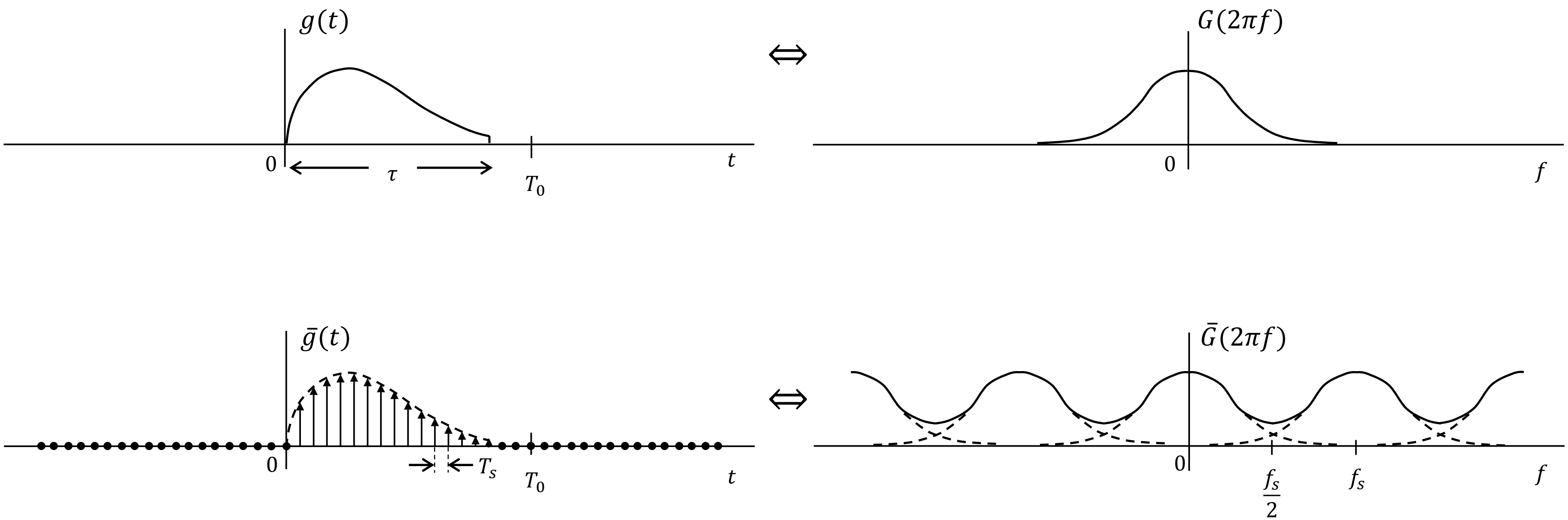
  ⇒ we need to find the relationship between the samples of $g(t)$ and the samples of $G(\omega)$.

- Furthermore, for the algorithm to be practical, the data (i.e., the number of samples of $g(t)$ and $G(\omega)$) must be finite

The Discrete Fourier Transform (DFT) -2

- Let $g(t)$ be a signal starting at $t = 0$ and having duration τ. We will consider $g(t)$ on the interval $[0, T_0]$, where $T_0 \geq \tau$ (that is: $g(t) = 0$ for $\tau \leq t \leq T_0$, hence this makes no difference in the computation of $G(\omega)$).
- We sample $g(t)$ at intervals of T_s seconds: then there are a total of N samples:

$$N = \frac{T_0}{T_s}$$



The Discrete Fourier Transform (DFT) -3

- Let us make the approximation (for T_s sufficiently small)

$$G(\omega) = \int_0^{T_0} g(t)e^{-j\omega t}dt \cong \sum_{k=0}^{N-1} g(kT_s)e^{-j\omega kT_s}\, T_s$$

- and consider the samples of $G(\omega)$ at uniform intervals of ω_0: $G_r = G(r\omega_0)$. We have therefore:

$$G_r = G(\omega)|_{\omega=r\omega_0} = \sum_{k=0}^{N-1} \underbrace{T_s\, g(kT_s)}_{g_k}\, e^{-jr\overbrace{\omega_0 T_s}^{\Omega_0}k} \quad \Rightarrow \quad \boxed{G_r = \sum_{k=0}^{N-1} g_k\, e^{-jr\Omega_0 k}}$$

Relates g_k and G_r as desired

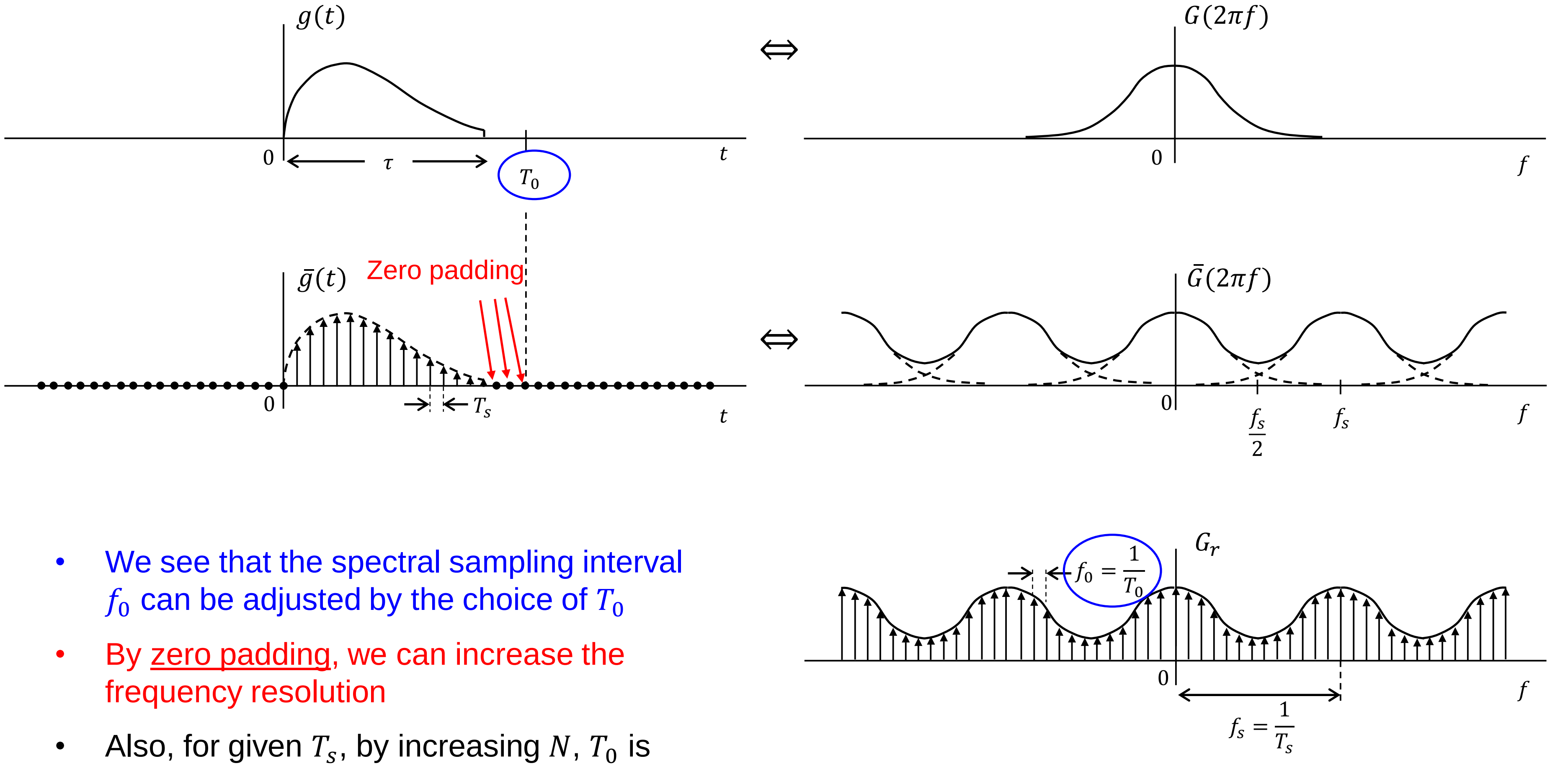
- G_r is periodic with period $\frac{2\pi}{\Omega_0}$: $G_{r+\frac{2\pi}{\Omega_0}} = G_r$

$\Rightarrow$ only $\frac{2\pi}{\Omega_0}$ samples G_r can be independent

- But G_r is also specified by N independent values of $g_k \Rightarrow$ hence $N = \frac{2\pi}{\Omega_0}$

$$N = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\omega_0 T_s} = \frac{2\pi N}{\omega_0 T_0} \Rightarrow \boxed{\omega_0 = \frac{2\pi}{T_0} \text{ or: } f_0 = \frac{1}{T_0}}$$

Relationship Between Samples of $g(t)$ and $G(\omega)$



- We see that the spectral sampling interval f_0 can be adjusted by the choice of T_0
- By zero padding, we can increase the frequency resolution
- Also, for given T_s, by increasing N, T_0 is increased ($T_0 = NT_s$)

The Discrete Fourier Transform (DFT) - 4

- We now want to find the inverse relationship, that is, express g_k in terms of G_r

$$G_r = \sum_{k=0}^{N-1} g_k\, e^{-jr\Omega_0 k} \quad \Rightarrow G_r\, e^{jm\Omega_0 r} = \sum_{k=0}^{N-1} g_k\, e^{-jr\Omega_0 k} e^{jm\Omega_0 r}$$

$$\sum_{r=0}^{N-1} G_r e^{jm\Omega_0 r} = \sum_{r=0}^{N-1} \sum_{k=0}^{N-1} g_k\, e^{-jr\Omega_0 k} e^{jm\Omega_0 r}$$

$$= \sum_{k=0}^{N-1} g_k \underbrace{\sum_{r=0}^{N-1} e^{j(m-k)\Omega_0 r}}_{} \qquad (\Omega_0 = \tfrac{2\pi}{N})$$

$$= \begin{cases} N & for\; m-k = 0, \pm N, \pm 2N, \ldots \\ 0 & otherwise \end{cases}$$

$$= g_m N$$

$$\boxed{\Rightarrow g_k = \frac{1}{N} \sum_{r=0}^{N-1} G_r e^{jk\Omega_0 r}}$$

The Discrete Fourier Transform (DFT) - 5

$$g_k = \frac{1}{N}\sum_{r=0}^{N-1} G_r e^{jk\Omega_0 r}$$

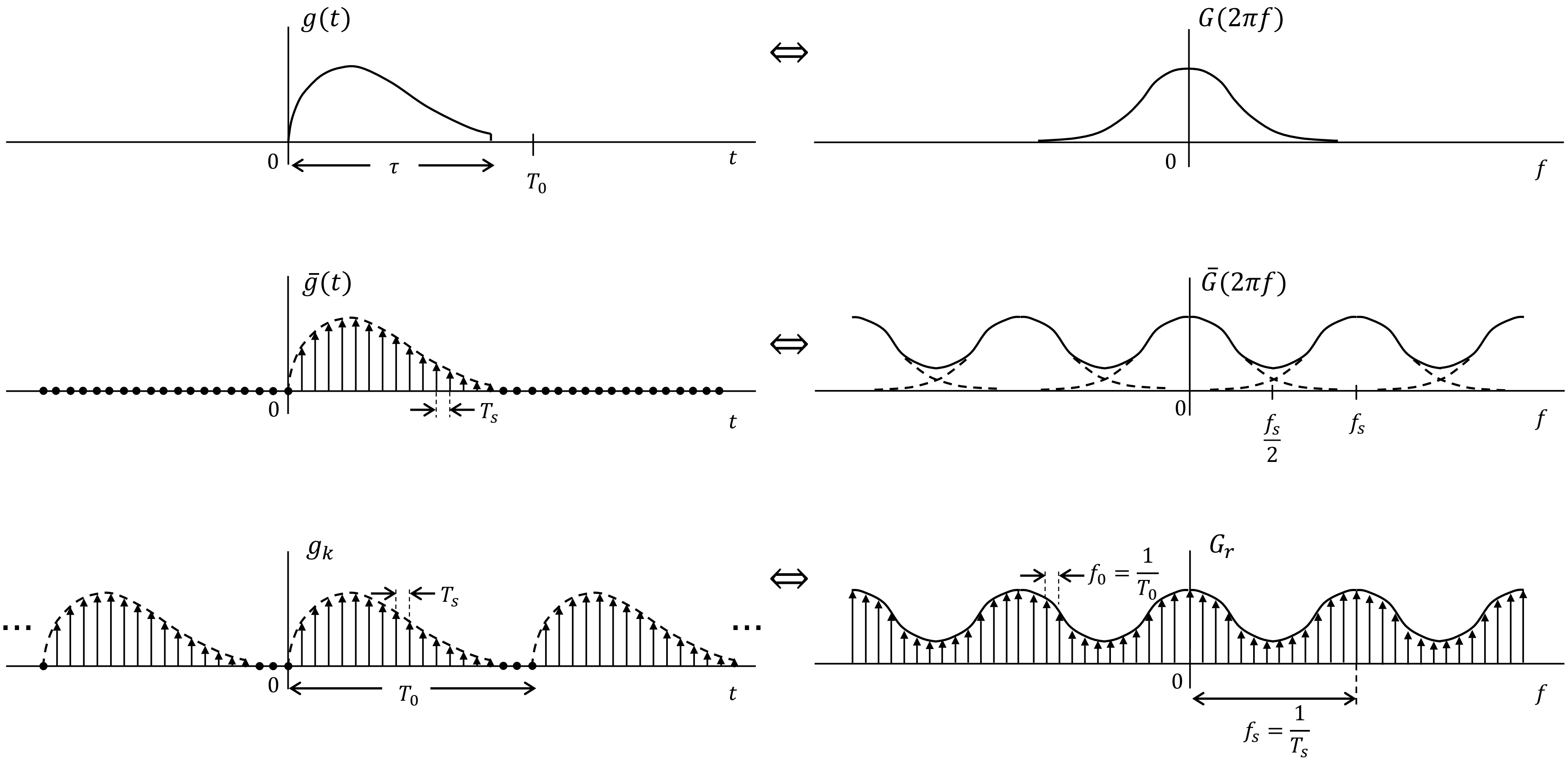
- Note that $g_{k+N} = g_k$: hence g_k is <u>periodic with period of N samples</u> (because $N\Omega_0 = 2\pi$)
- Recall that G_r is also periodic with period of N samples

g_k $(k = 0,1,\cdots,N-1)$ and G_r $(r = 0,1,\cdots,N-1)$ are both periodic with period N

- G_r's period N is equal to $\frac{1}{T_s}$:

$$N\omega_0 = \frac{T_0}{T_s}\frac{2\pi}{T_0} = \frac{2\pi}{T_s} = \omega_s \;\rightarrow \frac{1}{T_s} = f_s \quad \text{is } G_r\text{'s period}$$

Discrete Fourier Transform (DFT)



Discrete Fourier Transform (DFT) -- Summary

$$\boxed{g_k = \frac{1}{N}\sum_{r=0}^{N-1} G_r e^{jk\Omega_0 r}}$$

$$\boxed{G_r = \sum_{k=0}^{N-1} g_k e^{-jr\Omega_0 k}}$$

$$g_k = T_s g(kT_s)$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

$$N = \frac{T_0}{T_s} = \frac{\omega_s}{\omega_0} = \frac{f_s}{f_0}$$

$$G_r = G(r\omega_0)$$

$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

$$\Omega_0 = \omega_0 T_s = \frac{2\pi}{N}$$