EEEN 460
Optimal Control

2020 Spring

Lecture II
State-Space Modeling

State-Space Modeling for Optimization

Why state-space approach ?

Typically, systems in an optimization problem may have many inputs and many outputs.

- To analyze such a system, it is essential to reduce the complexity of the mathematical model.
- It is also necessary to resort to computers for the computations necessary in soving the optimization problem.

The state-space approach to system analysis is best suited from these viewpoints.

Outline of the Lecture

1 An introduction to state-space analysis of control systems.

2 The state-space representation of transfer-function systems.
Here we present various canonical forms of state-space equations.

3 Transformation of system models (such as from transferfunction
to state-space models, and vice versa) with MATLAB.

4 Solution of time-invariant state equations.

5 Some useful results in vector-matrix analysis that are necessary in studying the state-space analysis of systems.

6 Controllability of systems.

7 Observability of systems.

State-Space Representations of Transfer-Function Systems

Consider a system defined by an nth order differential equation in which input involves no derivative terms

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

Eq.1

where u is the input and y is the output. This equation can also be written as

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Let us define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\cdots$$
$$x_n = \overset{(n-1)}{y}$$

Then Eq. $\boxed{\text{Eq.1}}$ can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\cdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{14-2}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output equation becomes

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \tag{14-3}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

Example

Consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

Where y is the output and u is the input of the system. Obtain a state-space representation of the system.

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

Let us choose the state variables as follows

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}$$

Then we obtain

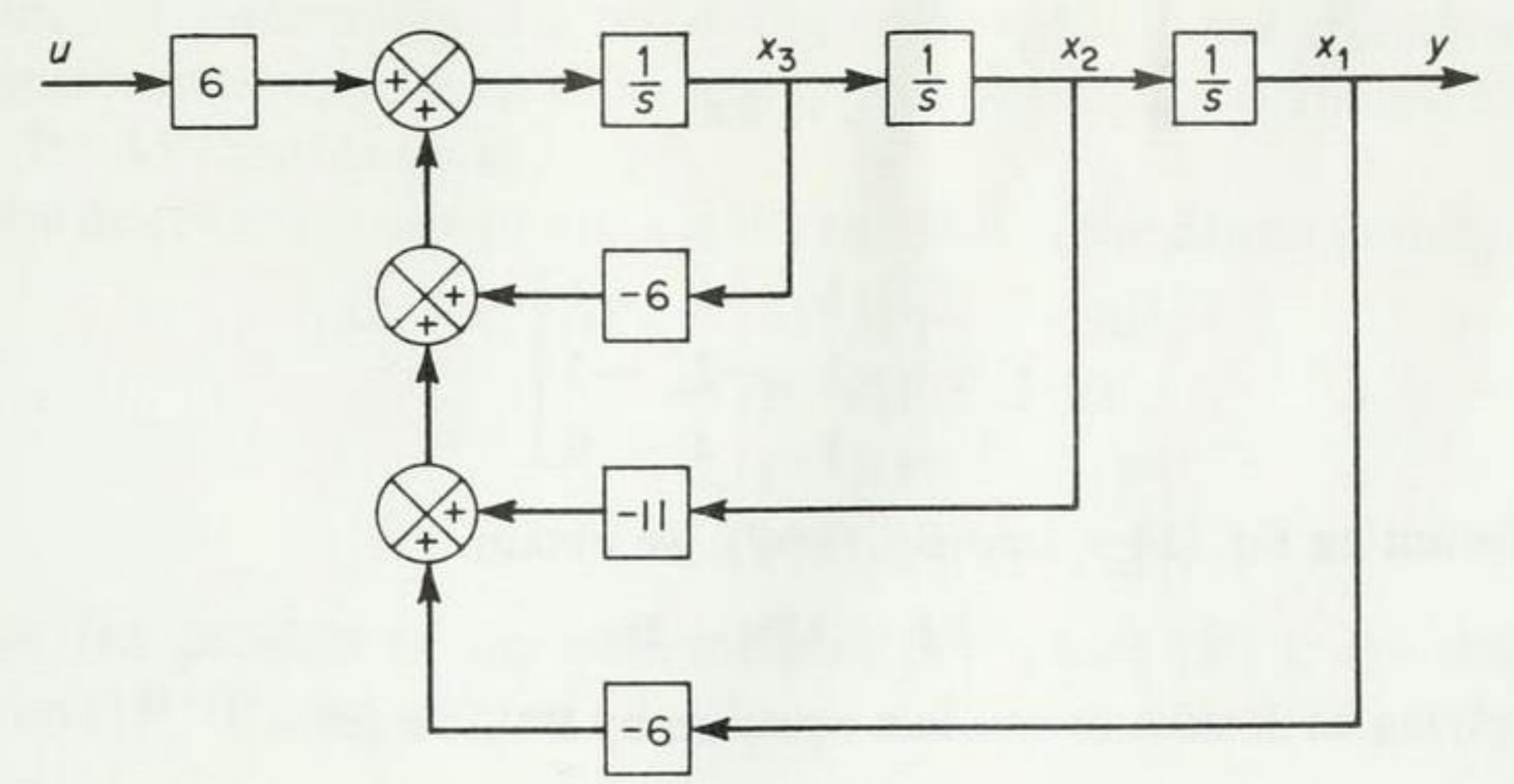
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -6x_1 - 11x_2 - 6x_3 + 6u \end{aligned}$$

By using the vector-matrix notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} [u]$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Block diagram representation



Non-uniqueness of the set of state variables

It has been stated that a set of state variables is not unique for a given system. In the following, We will see different forms of state-space modeling.

Different State-Space Representations of Transfer-Function Systems

Consider a system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

where u is the input and y is the output. Note that the derivatives of the input function appear. This equation can also be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

State-Space Representations of Transfer-Function Systems

We shall present state-space representations of the system in

a) controllable canonical form,

b)observable canonical form,

c)diagonal (or Jordan) canonical form.

Controllable Canonical Form

The following state-space representation is called a controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Observable Canonical Form

The following state-space representation is called an observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Diagonal Canonical Form

Consider the transfer-function system defined by the equation below. Here we consider the case where the denominator polynomial involves only distinct roots. For the distinct-roots case, this equation can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & & 0 \\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Diagonal (Jordan) Canonical Form

Next we shall consider the case where the denominator polynomial the transfer function equation involves multiple roots. For this case, the preceding diagonal canonical form must be modified into the Jordan canonical form. Suppose, for example, that the p_i's are different from one another, except that the first three p_i's are equal, or p_1=p$_2$=p$_3$.

Then the factored form of *Y(s)/U(s)* becomes

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)^3(s+p_4)(s+p_5)\cdots(s+p_n)}$$

The partial-fraction expansion of this equation becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \frac{c_4}{s+p_4} + \cdots + \frac{c_n}{s+p_n}$$

A state-space representation of this system in the Jordan canonical form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & & 0 \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & & \cdot \\ 0 & \cdots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Example

Consider the system given by

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}}$$

Obtain state-space representations in the

a) controllable canonical form

b) observable canonical form,

c) diagonal canonical form.

State-Space Representations of Transfer-Function Systems

Controllable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

State-Space Representations of Transfer-Function Systems

Observable Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

State-Space Representations of Transfer-Function Systems

Diagonal Canonical Form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

End of Lecture 2