EEEN 460
Optimal Control

2020 Spring

Lecture IX
Linear Programming
Simplex Method

Linear Programming

Linear programming (LP) is one of the widely used ways to perform optimization. It helps you solve some very complex optimization problems by making a few simplifying assumptions

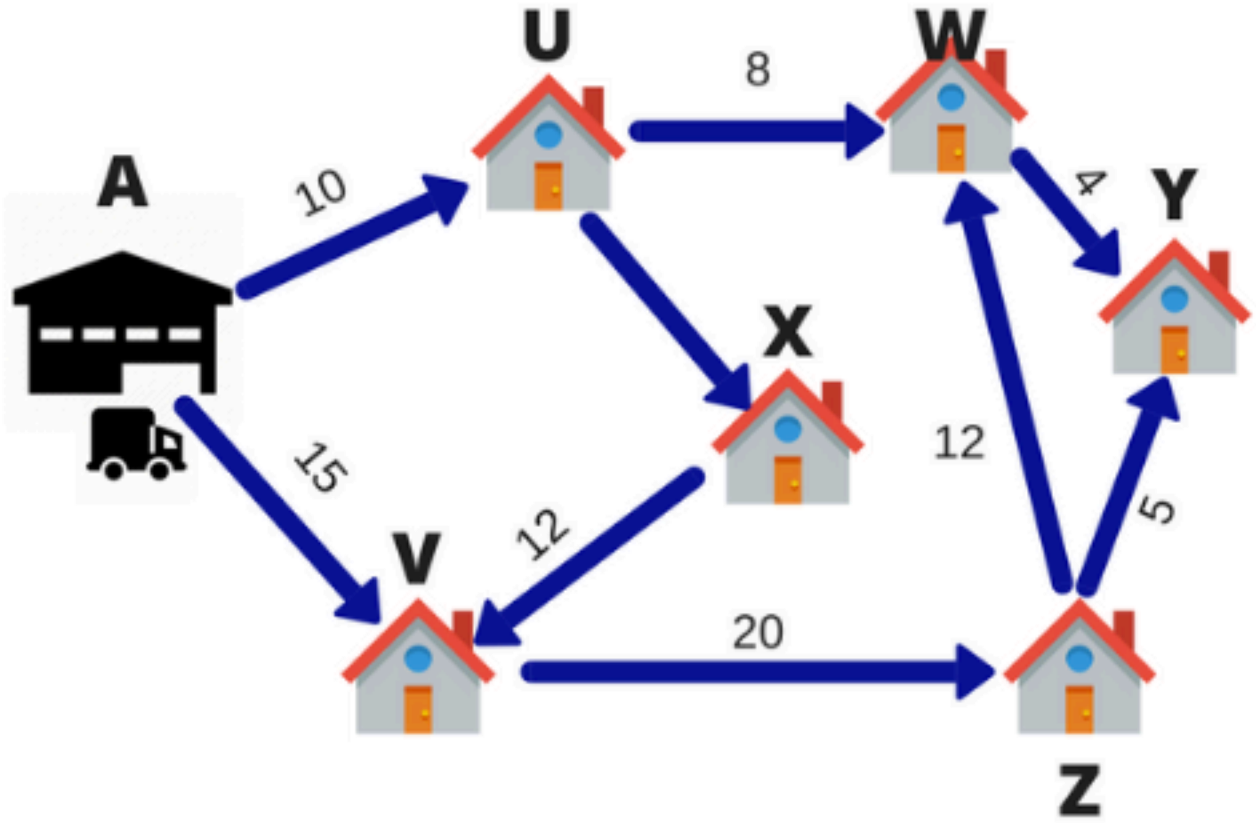
What is Linear Programming?
Basic Terminologies
The process to define an LP problem
Solve Linear Program by Graphical Method
Simplex Method

What is Linear Programming?

Linear programming is a simple technique where we **depict** complex relationships through linear functions and then find the optimum points. The important word in the previous sentence is depicted. The real relationships might be much more complex – but we can simplify them to linear relationships.

Example of a linear programming problem

Let's say a FedEx delivery man has 6 packages to deliver in a day. The warehouse is located at point A. The 6 delivery destinations are given by U, V, W, X, Y, and Z. The numbers on the lines indicate the distance between the cities. To save on fuel and time the delivery person wants to take the shortest route.



A Linear Programming Problem

Objective function

$p(x,y)=4x+3y$

Maximize it under the constraints:

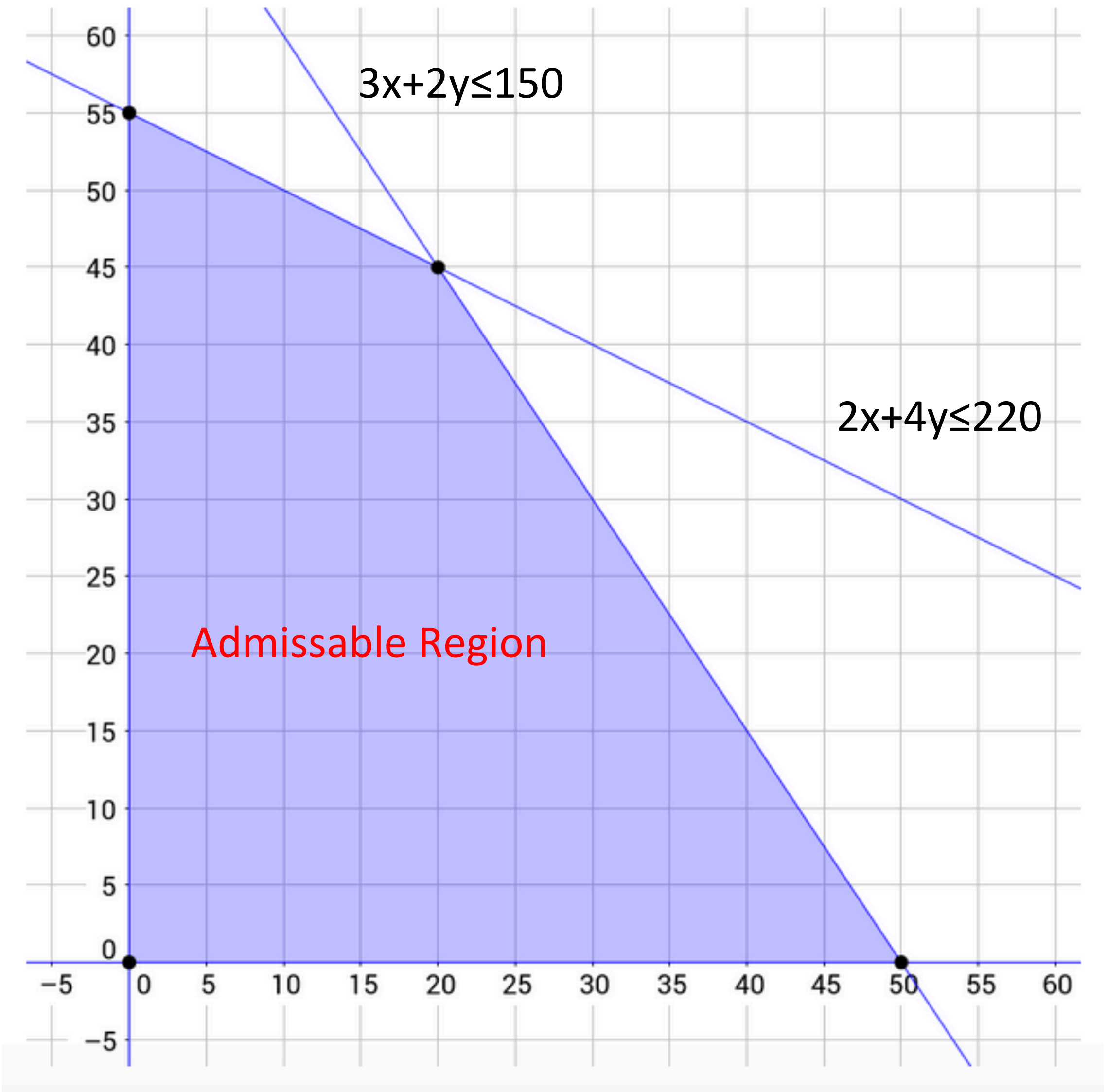
$2x+4y\leq 220$

$3x+2y\leq 150$

$x\geq 0$

$y\geq 0$

Graph the system of constraints and Admissable Region



$2x+4y \leq 220$

$3x+2y \leq 150$

$x \geq 0$

$y \geq 0$

The shaded region is the **admissable region** of this problem.
It represents the possible values of the variables that satisfy all of the constraints. In order for linear programming techniques to work.

It should be a convex polytope in 2 dimensions, a convex polygon; in 3 dimensions, a convex polyhedron and so on.

Optimization

Finding the feasible region is only sufficient to give the *possible* solutions of a problem. The goal is to maximize(minimize) the objective function.

This is the objective function for this problem:
p(x,y)=4x+3y
And our goal is to maximize it.

Let p*(x,y) be the optimal (maximum) solution in the feasible region:
p*(x,y)=4x+3y
Solve for y:

$$\boxed{y = -\frac{4}{3}x + \frac{P}{3}}$$

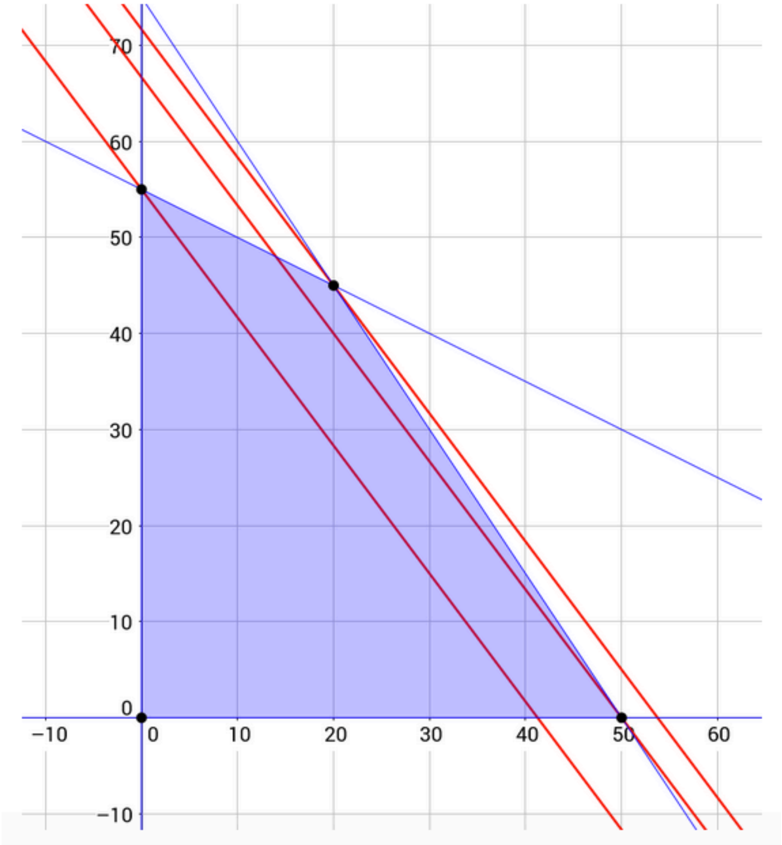
This maximum p gives an equation of a line, and whatever point in the feasible region passes through this line is the optimal solution. The y-intercept of this line is P/3. As P is maximized, this y-intercept should be maximized as well. Graph several lines with the same slope of −4?/3.

Optimization

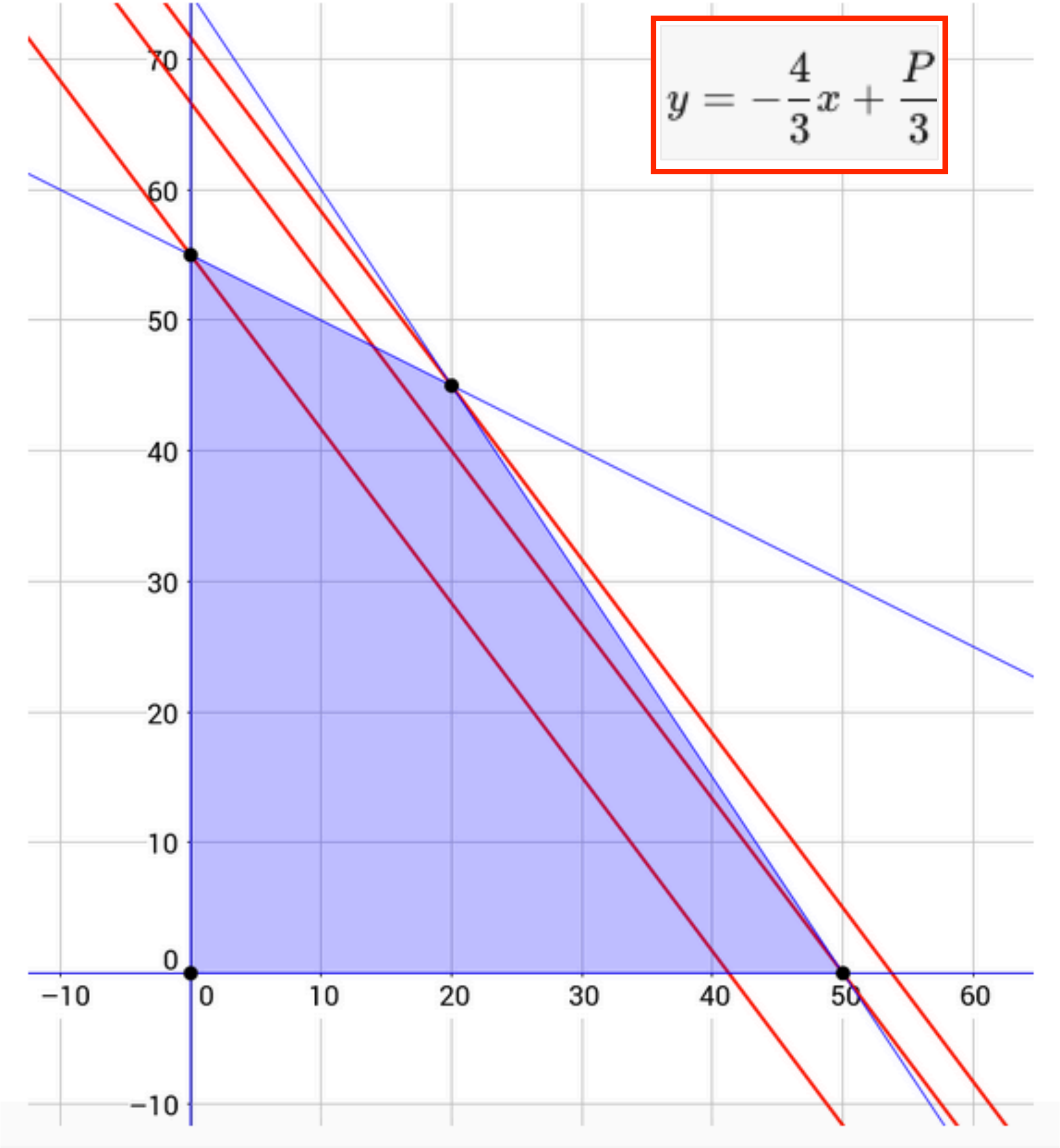
$$\boxed{y = -\frac{4}{3}x + \frac{P}{3}}$$

This is the equation of a line, The y-intercept of this line is p/3 and one of the points in the feasible region which coincides with this line is the optimal solution.

Draw several lines with the same slope of −4/3

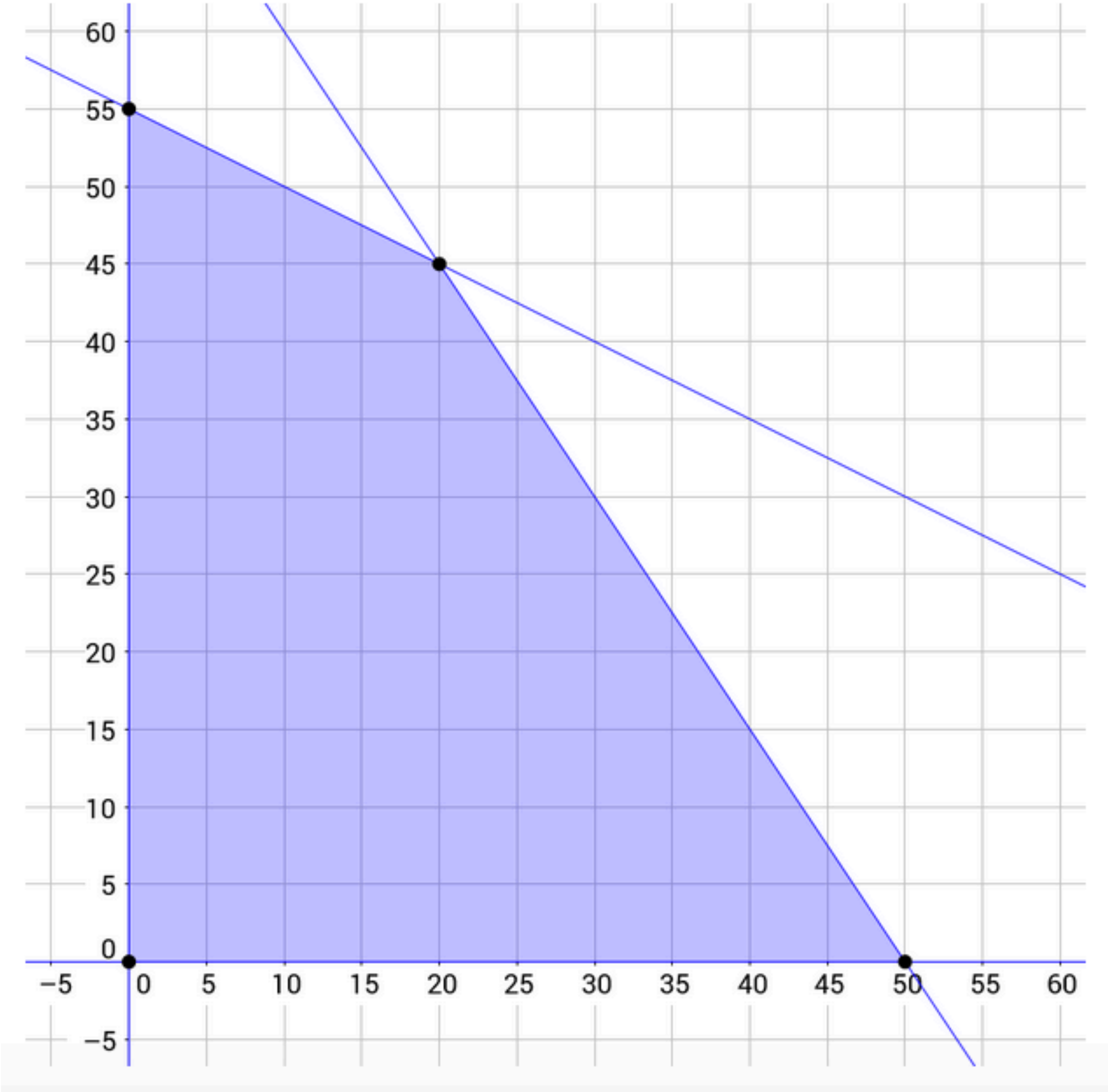


Optimization



This is the equation of a line, The y-intercept of this line is p/3 and one of the points in the feasible region which coincides with this line is the optimal solution.

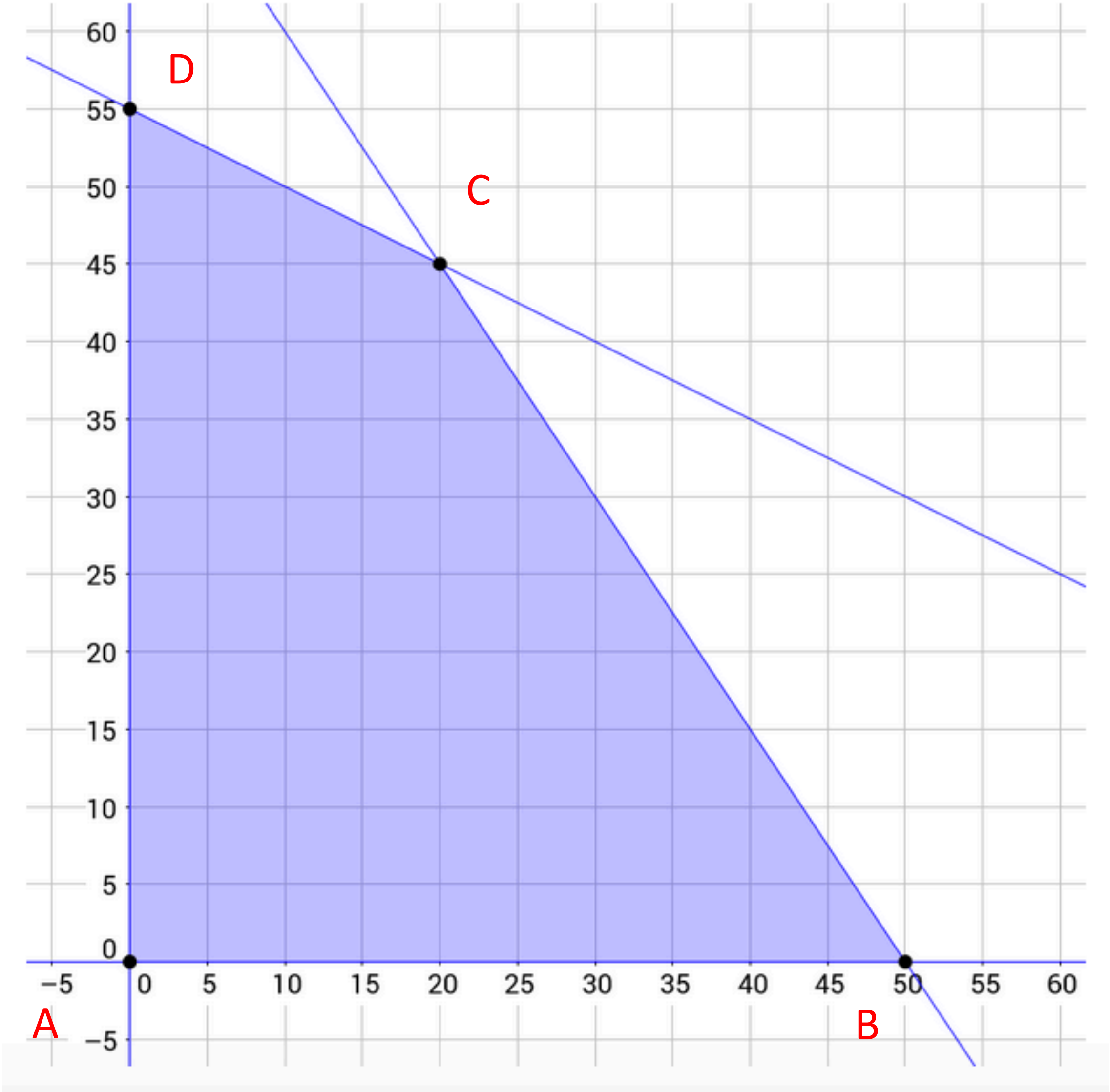
A Fundamental Theorem of Linear Programming



A Fundamental Theorem of Linear Programming: The optimal solution lies in one of the vertices of the polygon of admissable region (for 2-dimensional problems).

A

A Fundamental Theorem of Linear Programming

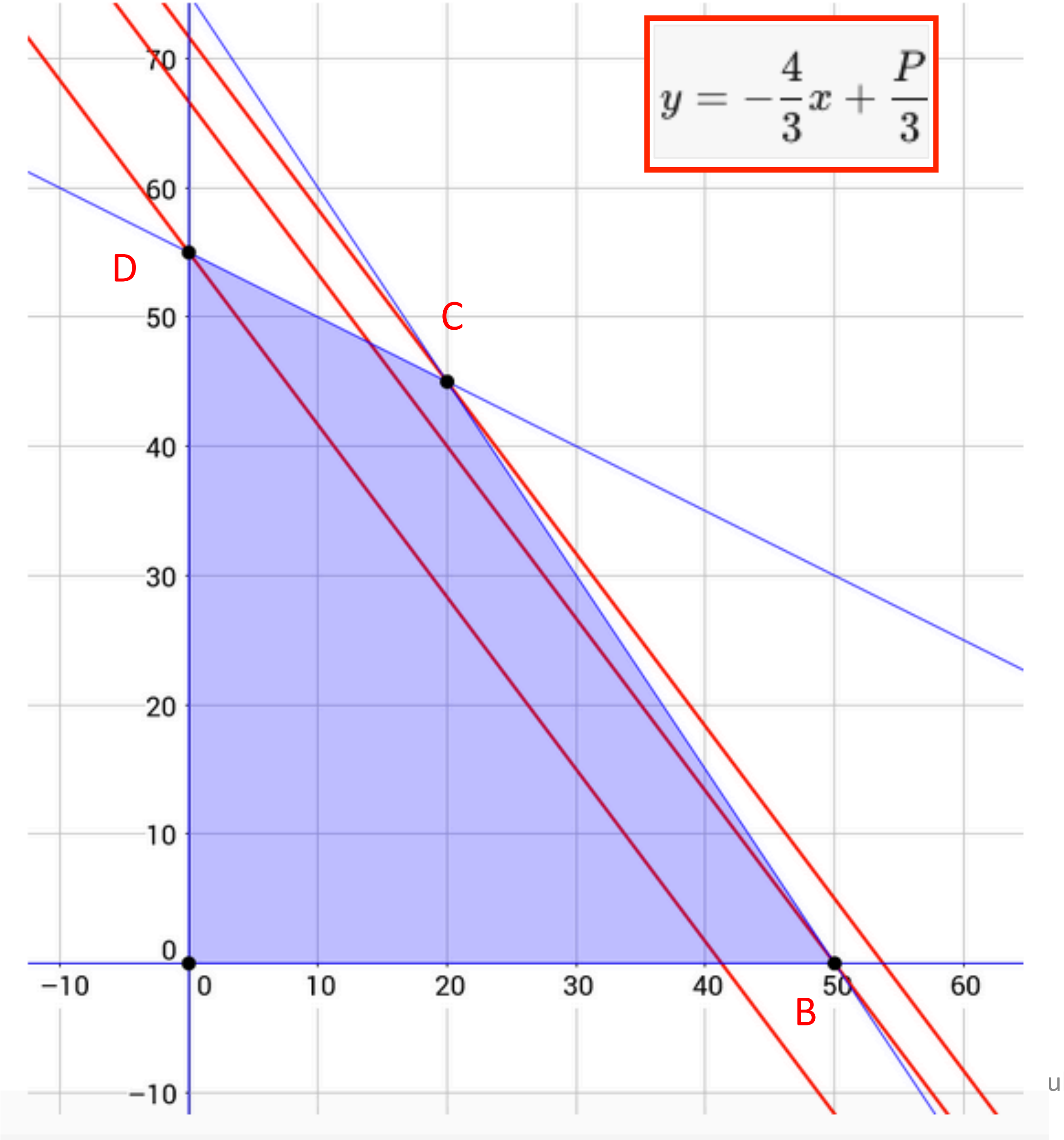


Which of these vertices corresponds to the optimal solution ?

Remember that we are trying to maximize
$p(x,y)=4x+3y$

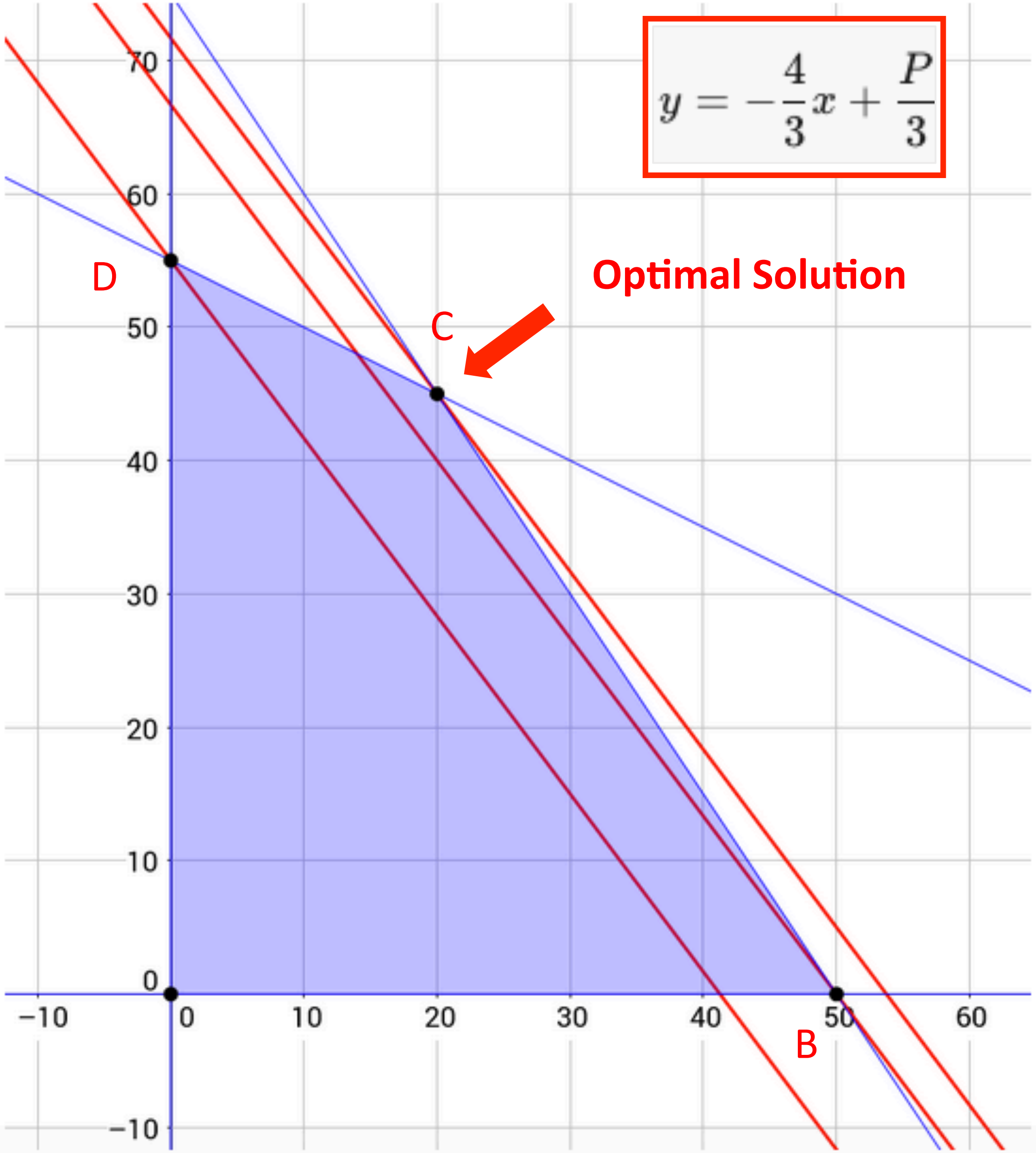
Obviously, we can eliminate A
At A, $p(x,y)= 0$ it is the mininimum solution.

Therefore, we have reduced the candidate vertices to 3, namely B, C or D.



Which vertex gives the optimal solution ?

Linear Programming

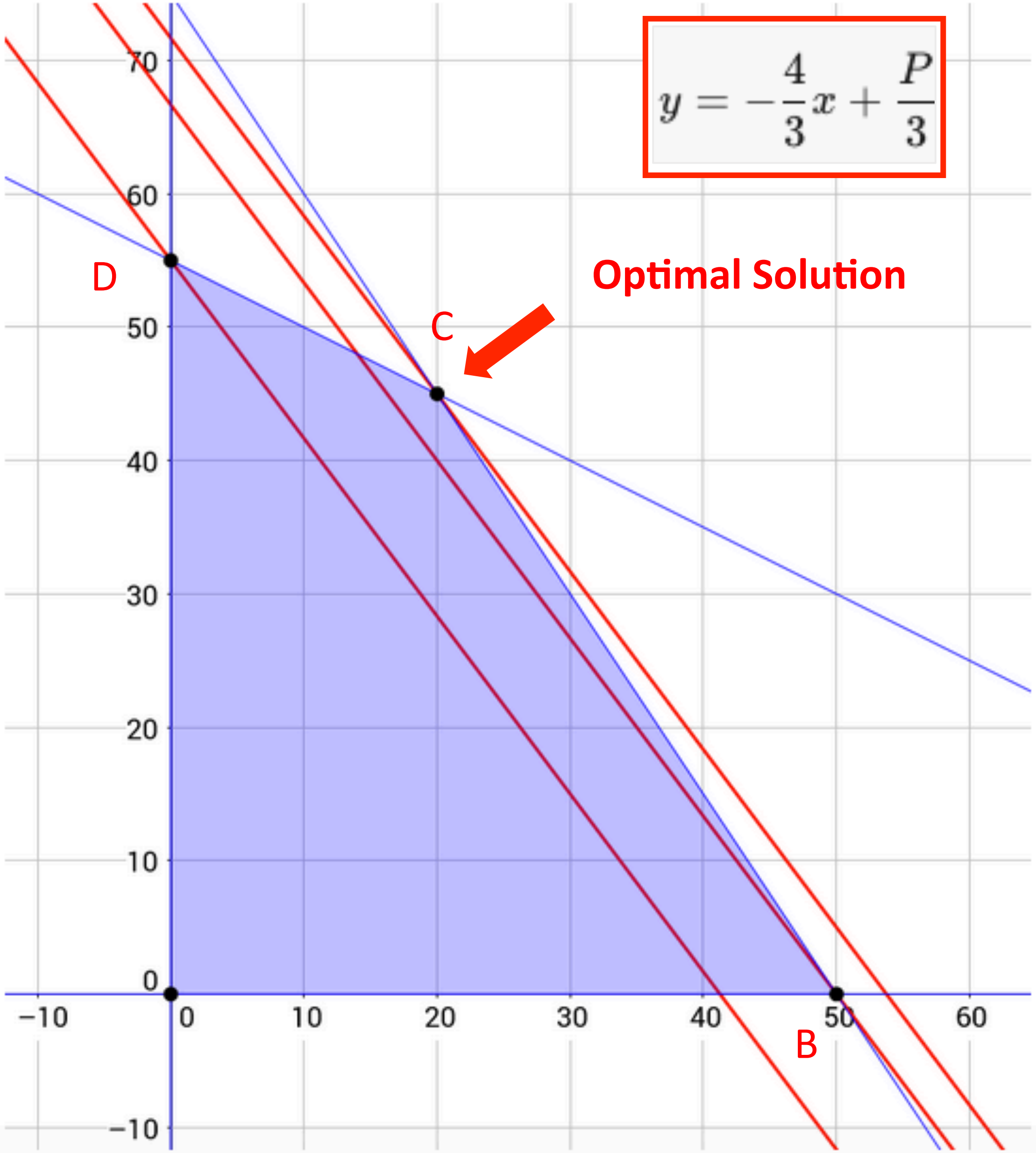


Answer:

C

Why ?

Linear Programming



Answer:

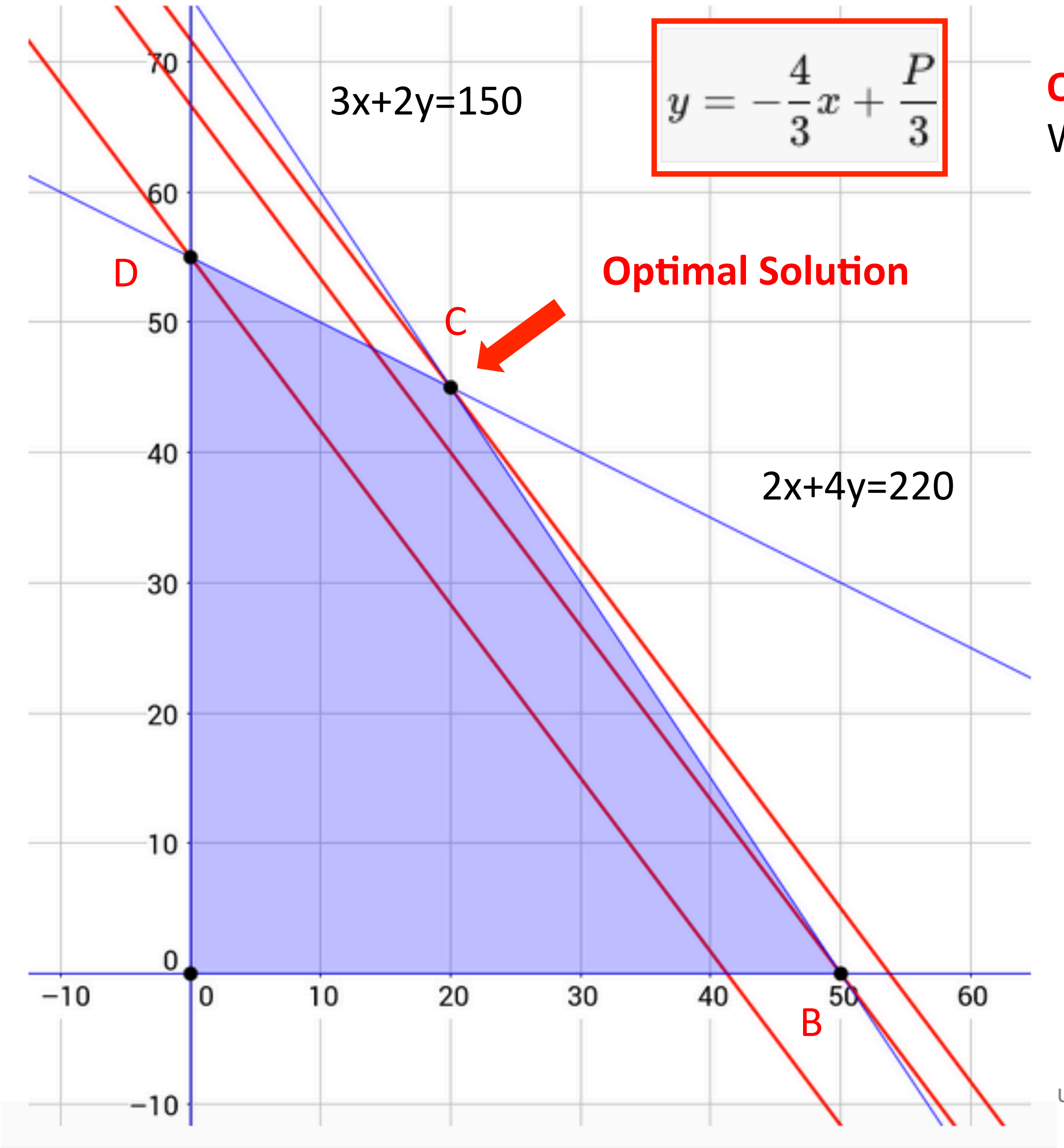
C

Why ?

The line that maximizes the y-intercept is the one that passes through the vertex at C.

Question :
What are the x and y coordinates of C ?

Linear Programming



Question :
What are the x and y coordinates of C ?

C is at the intersection of the lines

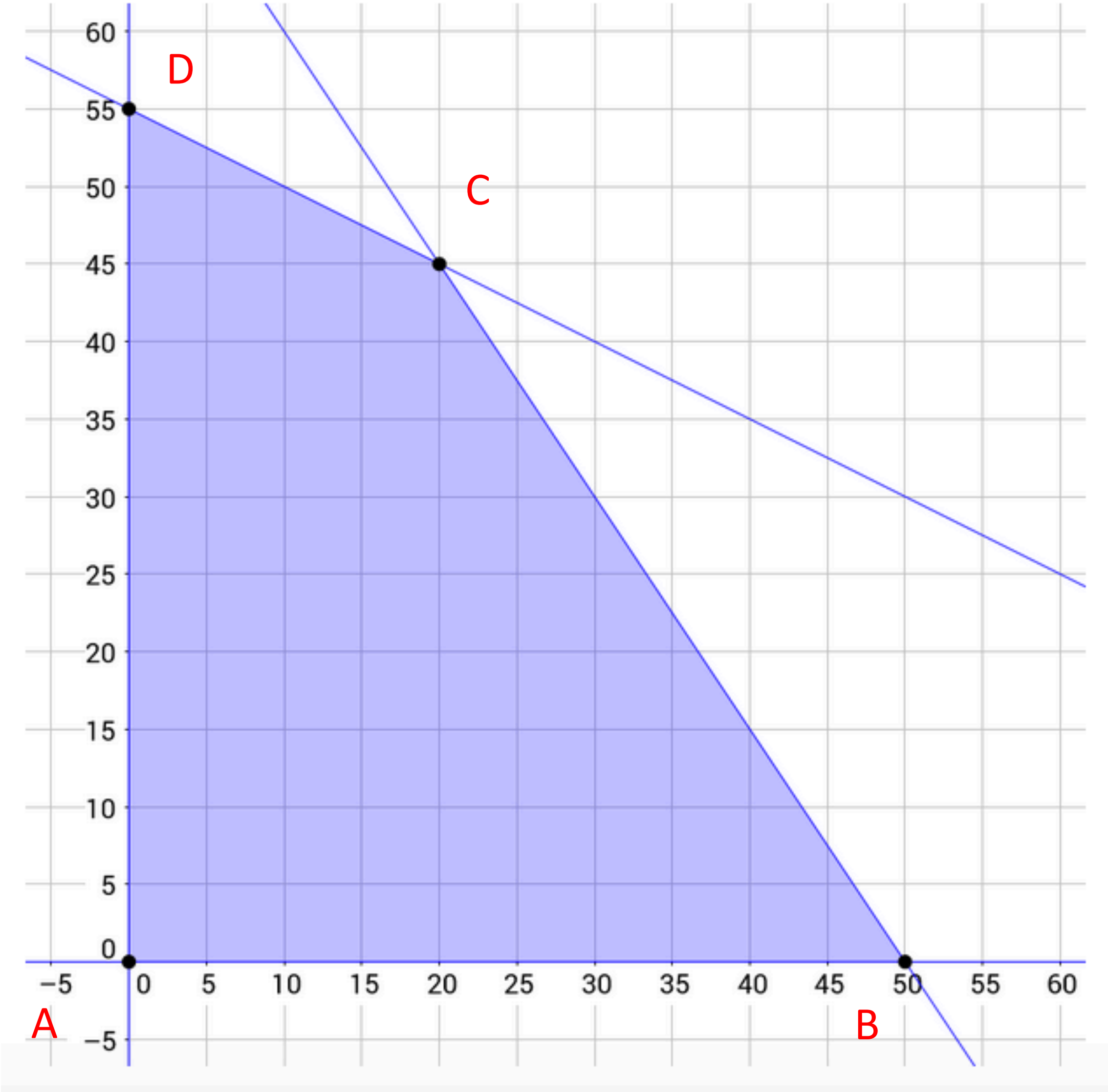
3x+2y=150

2x+4y=220

Therefore by solving these equations for x and y,
We can find
$x^*=20$
$y^*=45$

And
$p^*(x^*,y^*) = 4x^*+3y^*=215$

Linear Programming



Verification

Let us try to verify that C is actually the vertex of optimization for
p (x,y) = 4x+3y

Vertex	(x, y)	p(x,y)
A	(0,0)	0
B	(50,0)	200
C	(20,45)	215
D	(0,55)	165

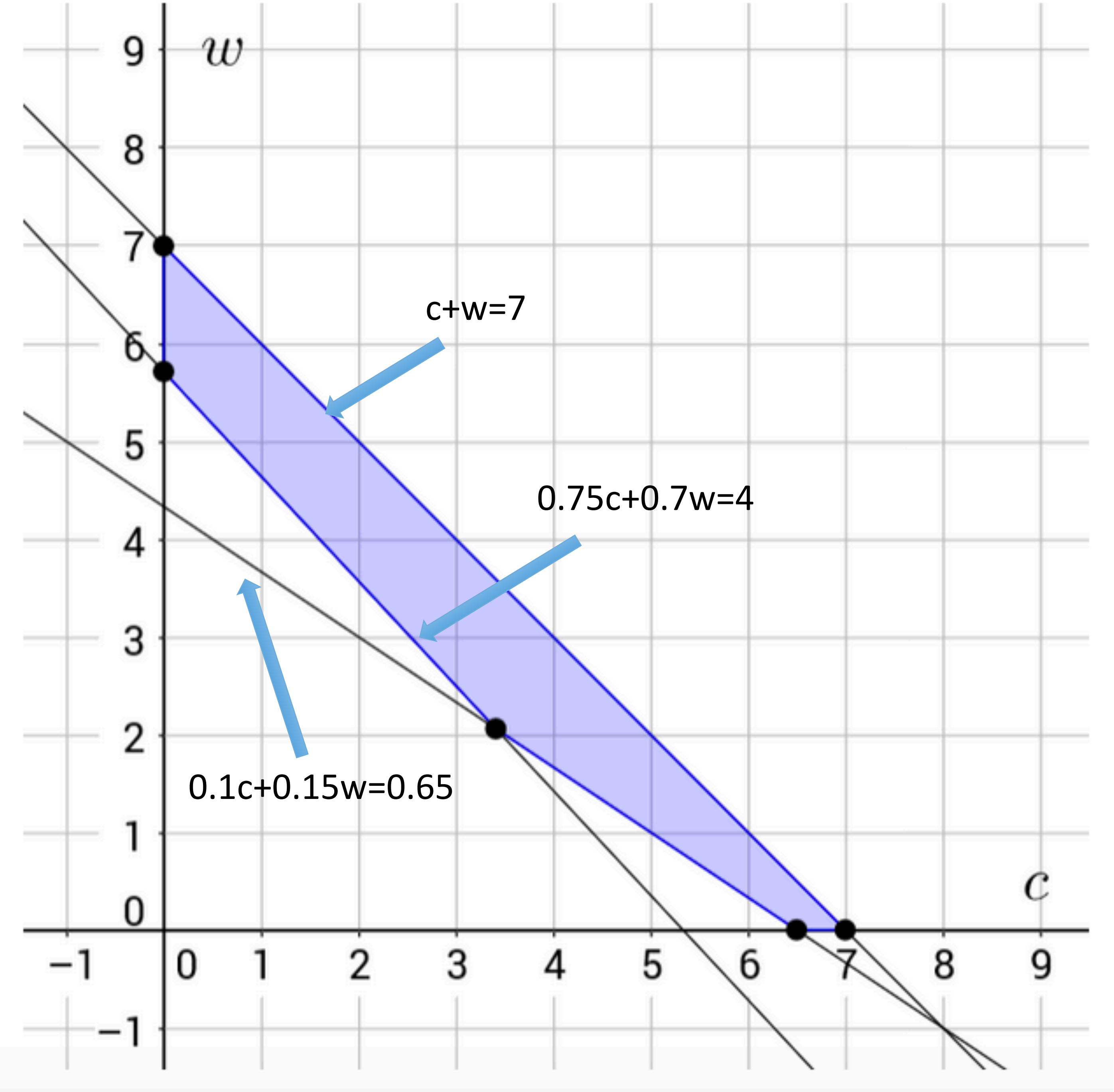
This table show at vertex C, p(x,y) is maximized

Example

Minimize: $f(c,w)=0.40c+0.45w$

Under the constraints:

$$
\begin{aligned}
0.1c+0.15w &\geq 0.65\\
0.75c+0.7w &\leq 4\\
c+w &\leq 7\\
c &\geq 0\\
w &\geq 0
\end{aligned}
$$



Solution

Drawing the system of constraints gives an idea of where the vertices of the admissible region are and which lines intersect to form them:

Sytem of constraints:

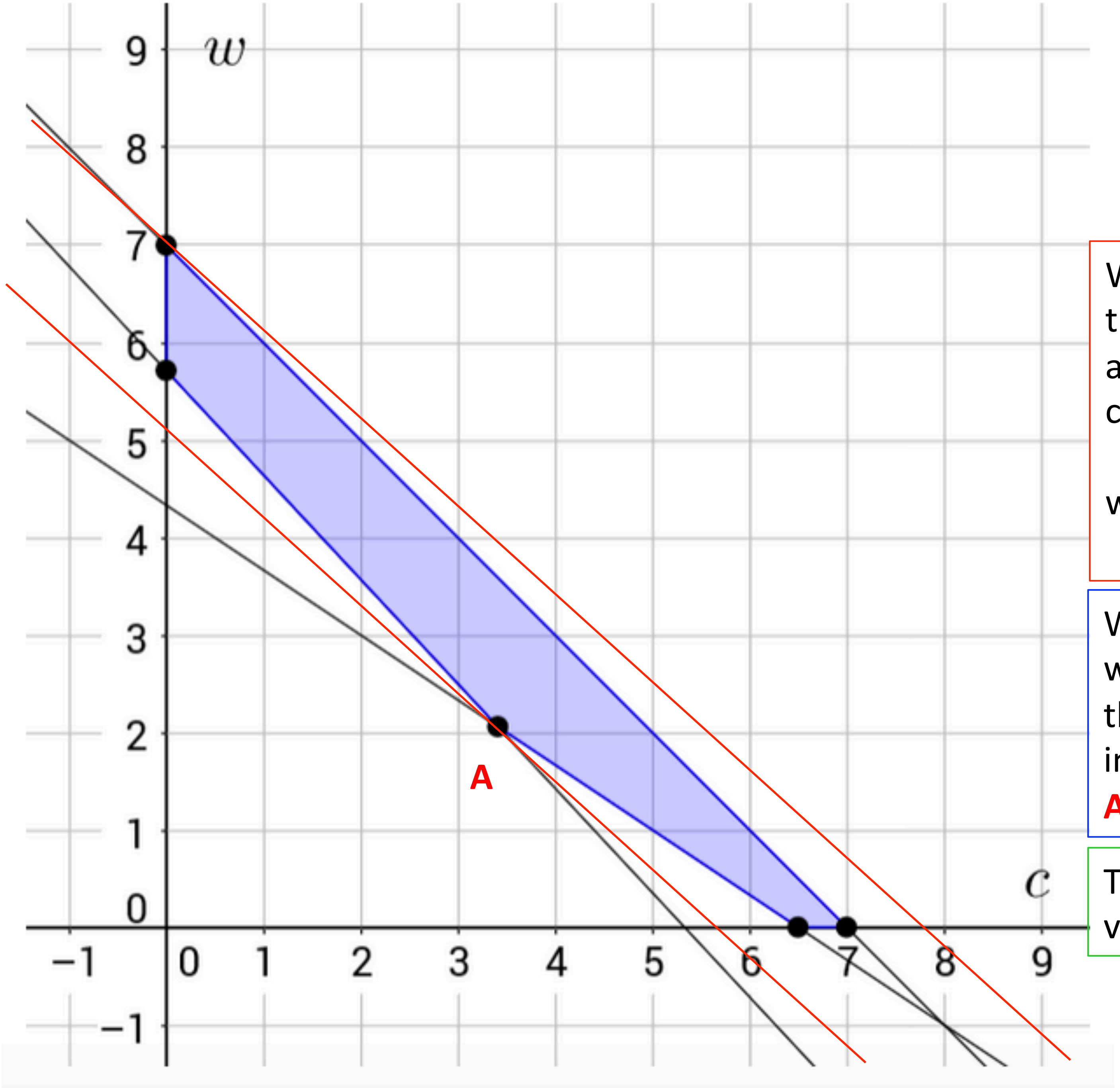
0.1c+0.15w≥0.65

0.75c+0.7w≤4

c+w≤7

c≥0

w≥0



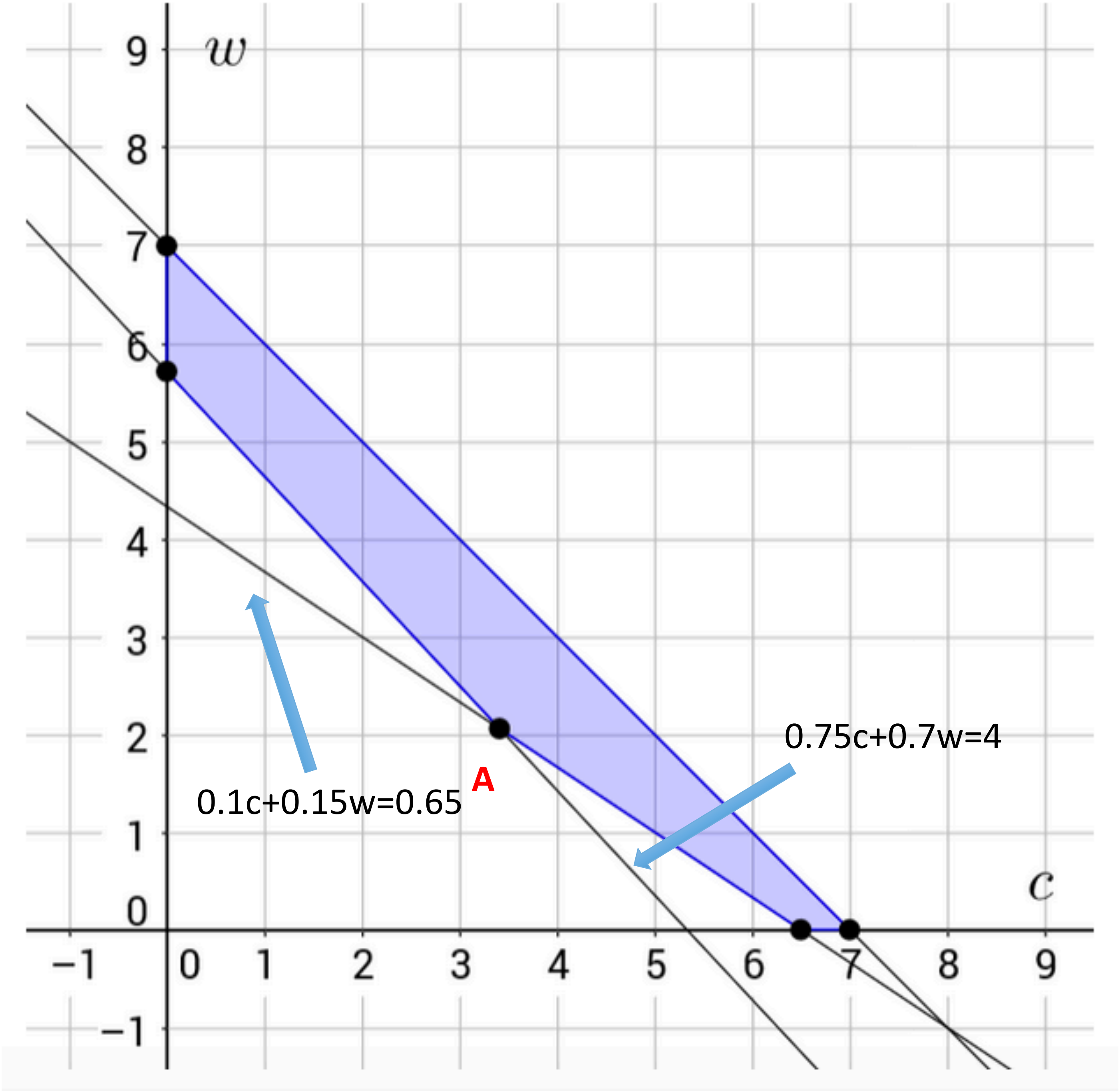
From
f(c,w)=0.40c+0.45w
We obtain:
w= -0.89c+2.22 f

We already now from a fundamental theorem of the linear programming that the solution is at one of the vertices of the admissible reigon which coincides with

w= -0.89c+2.22 f

We also know from the red lines which correspond to
w= -0.89c+2.22 f
the one with minimum f is the one with minimum y-intercept, and it is the one which goes through vertex **A**.

Therefore, if we can determine the c,w coordinates of vertex A, we solve the problem.



What are the coordinates of vertex A ?

Note that vertex A lies at the intersection of the lines

0.75c+0.7w=4

0.1c+0.15w=0.65

By solving these equations for c and w, we can find approximately:

c*= 3.411
w*=2.059

What is the minimum(optimum) value of f ?

f*(c*,w*)=0.40c*+0.45w*

f*(c*,w*)=2.29

A Fundamental Theorem of Linear Programming

Summary

In the previous examples, it was shown that the optimal solution was on a vertex of the admissible region. This is true for all linear programming problems.

Theorem:
Given a convex polygonal admissible region and a linear objective function, the solution that maximizes or minimizes the objective function will be located on one of the vertices of the admissible region.

A Fundamental Theorem of Linear Programming

Let the objective function be
 f(x,y)=ax+by..
Let the maximum value of this function be
P(x,y)
and let the minimum value of this function be
Q(x,y). There exist lines which intersect each other at the optimal solutions, (x,y) :

$$ax + by = P \qquad (1)$$
$$ax + by = Q \qquad (2)$$

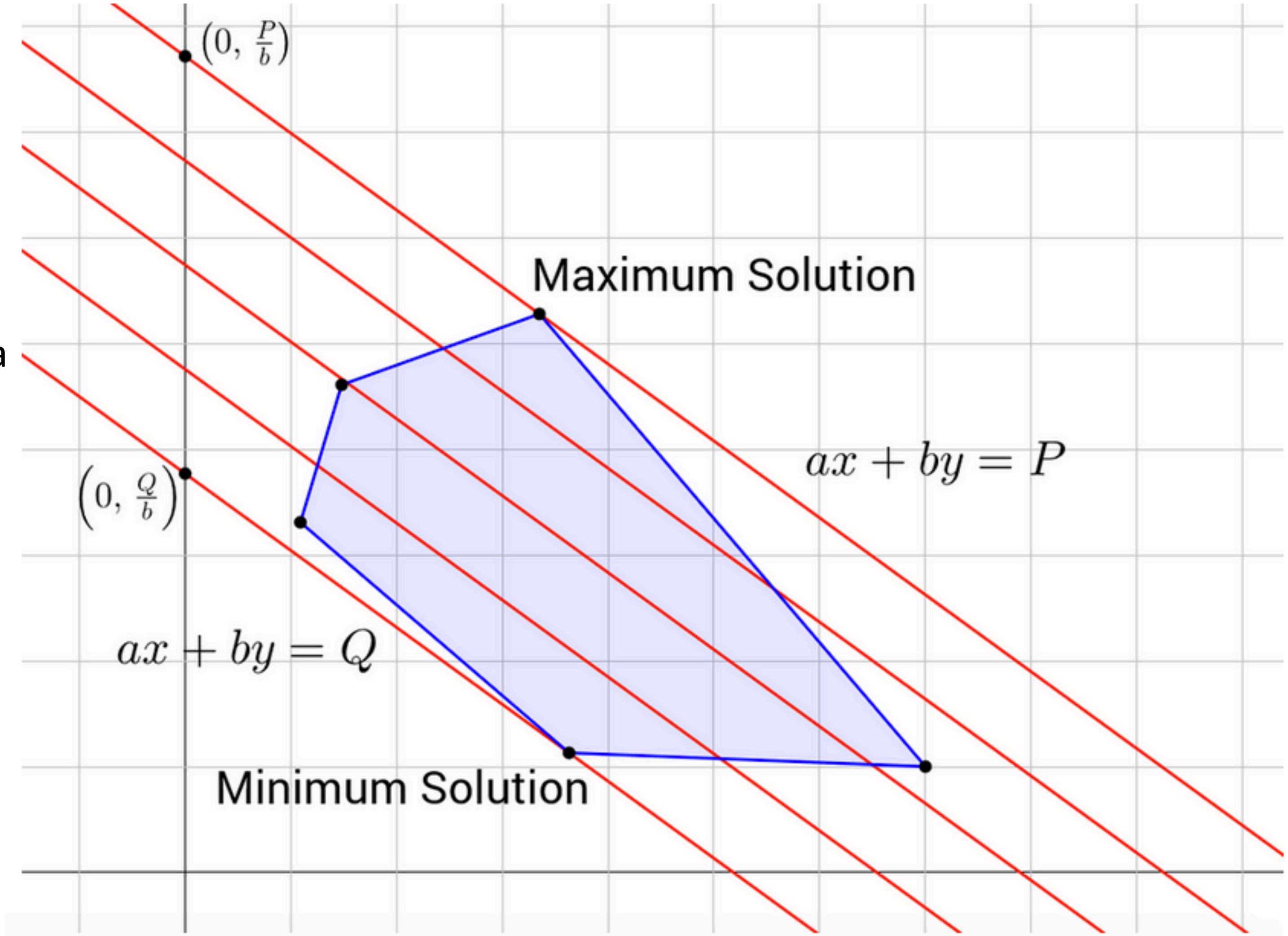
$$\Rightarrow y = -\frac{a}{b}x + \frac{P}{b} \qquad (1)$$
$$y = -\frac{a}{b}x + \frac{Q}{b}. \qquad (2)$$

A Fundamental Theorem of Linear Programming

$$\Rightarrow y = -\frac{a}{b}x + \frac{P}{b} \qquad (1)$$
$$y = -\frac{a}{b}x + \frac{Q}{b}. \qquad (2)$$

Since P is the maximum value of the objective function, (1) has the maximum y-intercept of a line with slope (−a/b) that passes through the admissible region.

Likewise, Q is the maximum value of the objective function, (2) has the minimum y-intercept of a line with slope (−a/b) that passes through the admissible region.



Introduction to The Simplex Method

Objectives

- Solve a **standard** maximization problem using the simplex method.

All the variables x, y, z, ... are nonnegative.

All constraints have the form $Ax + By + Cz + \cdots \leq N, N \geq 0$

Simplex Method

Simplex Algorithm

The simplex algorithm is a method to obtain the optimal solution of a linear system of constraints, given a linear objective function. It works by beginning at a basic vertex of the admissible region, and then iteratively moving to adjacent vertices, improving upon the solution each time until the optimal solution is found.

The simplex algorithm has many steps and rules, let us have a look at these steps and rules before proceeding with a simple example .

Simplex Method for Standard Maximization Problems

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
3. Create a simplex table or tableau and label the active or basic variables.
4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
5. Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.

Simplex Method for Standard Maximization Problems

7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry on the left side of the last row.
9. Once the left side of last row is all nonnegative, the solution can be found. The value of each row variables or active variables is equal to the right most entry in the row. All inactive variables equal zero.

Simplex Method for Standard Maximization Problems

Additional Notes

- Active or basic variables correspond to columns that contain only one and nonzero entries.
- Inactive or nonbasic variables correspond to columns that don't contain only one and nonzero entries

To help keep track of active variables, we label each row with the name of the corresponding active or basic variable.

Simplex Method

Demonstrating Simplex Method

It is helpful to understand these steps and rules with a simple example.
We will go through each step with the following example.

Simplex Method for Standard Maximization Problems

Example Problem

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

Simplex Method for Standard Maximization Problems

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

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Simplex Method for Standard Maximization Problems

Steps 1 and 2

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
3. Create a simplex table or tableau and label the active or basic variables.

$$\begin{aligned} 2x + y + s &= 5 \\ x + 2y + t &= 4 \\ -2x - 5y + P &= 0 \end{aligned}$$

Simplex Method for Standard Maximization Problems

Steps 1 and 2

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
3. Create a simplex table or tableau and label the active or basic variables.

$$2x + 1y + 1s + 0t + 0P = 5$$
$$1x + 2y + 0s + 1t + 0P = 4$$
$$-2x - 5y + 0s + 0t + 1P = 0$$

Simplex Method for Standard Maximization Problems

Step 3

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
3. Create a simplex table or tableau and label the active or basic variables.

$$2x + 1y + 1s + 0t + 0P = 5$$
$$1x + 2y + 0s + 1t + 0P = 4$$
$$-2x - 5y + 0s + 0t + 1P = 0$$

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	−2	−5	0	0	1	0

Simplex Method for Standard Maximization Problems

Maximize: $P = 2x + 5y$

$$2x + y \leq 5$$
$$x + 2y \leq 4$$
$$x \geq 0, y \geq 0$$

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
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$$2x + 1y + 1s + 0t + 0P = 5$$
$$1x + 2y + 0s + 1t + 0P = 4$$
$$-2x - 5y + 0s + 0t + 1P = 0$$

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	-2	-5	0	0	1	0

Simplex Method for Standard Maximization Problems

Maximize: $P = 2x + 5y$

$$2x + y \le 5$$
$$x + 2y \le 4$$
$$x \ge 0, y \ge 0$$

1. Convert the system of inequalities (constraints) to equations using slack variables.
2. Set the objective function equal to zero.
3. Create a simplex table or tableau and label the active or basic variables.

$$2x + 1y + 1s + 0t + 0P = 5$$
$$1x + 2y + 0s + 1t + 0P = 4$$
$$-2x - 5y + 0s + 0t + 1P = 0$$

$s = 5$
$t = 4$
$P = 0$

$x = 0$
$y = 0$

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	−2	−5	0	0	1	0

Simplex Method for Standard Maximization Problems

4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
5. Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	-2	-5	0	0	1	0

Simplex Method for Standard Maximization Problems

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	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	-2	-5	0	0	1	0

Simplex Method for Standard Maximization Problems

4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
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6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	-2	-5	0	0	1	0

Simplex Method for Standard Maximization Problems

4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
5. Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	2	0	1	0	4
P	-2	-5	0	0	1	0

$\frac{5}{1} = 5$

$\frac{4}{2} = 2$

Simplex Method for Standard Maximization Problems

4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
5. Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero.

	x	y	s	t	P		
s	2	1	1	0	0	5	$\frac{5}{1} = 5$
t	1	2	0	1	0	4	$\frac{4}{2} = 2$
P	-2	-5	0	0	1	0	

Simplex Method for Standard Maximization Problems

4. Select the pivot column. This is the column with the most negative number on the left side of the bottom row. (If all numbers on the bottom row are nonnegative, we are done.)
5. Select the pivot row. Divide each entry in the constant column by the corresponding positive entry in the pivot column. The smallest positive ratio indicates the pivot row.
6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero.

	x	y	s	t	P		
s	2	1	1	0	0	5	$\frac{5}{1} = 5$
t	1	$\boxed{2}$	0	1	0	4	$\frac{4}{2} = 2$
P	−2	−5	0	0	1	0	

6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry in the last row.

now we're on step seven

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	$\boxed{2}$	0	1	0	4
P	-2	-5	0	0	1	0

1/2

R2=1/2 * R2

	x	y	s	t	P	
s	2	1	1	0	0	5
t	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
P	-2	-5	0	0	1	0

6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry in the last row.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	1	$\boxed{2}$	0	1	0	4
P	-2	-5	0	0	1	0

We want to make these zero

	x	y	s	t	P	
s	2	$\boxed{1}$	1	0	0	5
t	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
P	-2	$\boxed{-5}$	0	0	1	0

6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry in the last row.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	$\frac{1}{2}$	$\boxed{1}$	0	$\frac{1}{2}$	0	2
P	-2	-5	0	0	1	0

$R_1 \to R_1 - R_2$

	x	y	s	t	P	
s						
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
P						

6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry in the last row.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	$\frac{1}{2}$	$\boxed{1}$	0	$\frac{1}{2}$	0	2
P	-2	-5	0	0	1	0

$R_1 \to R_1 - R_2$

$R_3 \to R_3 + 5R_2$

	x	y	s	t	P	
s						
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
P						

6. Select the pivot, which is the entry in the pivot column and pivot row. The pivot must be positive. It can't be zero. Identify the new active variable.
7. Perform row operations to make the pivot equal to 1 and the remaining elements in the pivot column equal to zero. Making the pilot equal to 1 is optional.
8. Repeat the process by identifying the most negative entry in the last row.

	x	y	s	t	P	
s	2	1	1	0	0	5
t	$\frac{1}{2}$	$\boxed{1}$	0	$\frac{1}{2}$	0	2
P	-2	-5	0	0	1	0

		x	y	s	t	P	
$R_1 \to R_1 - R_2$	s	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	3
	y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
$R_3 \to R_3 + 5R_2$	P	$\frac{1}{2}$	0	0	$\frac{5}{2}$	1	10

8. Repeat the process by identifying the most negative entry in the last row.
9. Once the left side of last row is all nonnegative, the solution can be found. The value of each row variables or active variables is equal to the right most entry in the row. All inactive variables equal zero.

Maximize: $P = 2x + 5y$

$$x + 2y \leq 4$$
$$3x + 2y \leq 3$$
$$x \geq 0, y \geq 0$$

	x	y	s	t	P	
s	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	3
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
P	$\frac{1}{2}$	0	0	$\frac{5}{2}$	1	10

All nonnegative

Step 9 says once the left side is all nonnegative the solution can be found.

That is we are done !

8. Repeat the process by identifying the most negative entry in the last row.
9. Once the left side of last row is all nonnegative, the solution can be found. The value of each row variables or active variables is equal to the right most entry in the row. All inactive variables equal zero.

Maximize: $P = 2x + 5y$

$$x + 2y \leq 4$$
$$3x + 2y \leq 3$$
$$x \geq 0, y \geq 0$$

	x	y	s	t	P	
s	$\frac{3}{2}$	0	$\boxed{1}$	$-\frac{1}{2}$	0	3
y	$\frac{1}{2}$	$\boxed{1}$	0	$\frac{1}{2}$	0	2
P	$\frac{1}{2}$	0	0	$\frac{5}{2}$	$\boxed{1}$	10

What is the solution ?

Active variables
s=3
y=2
P=10

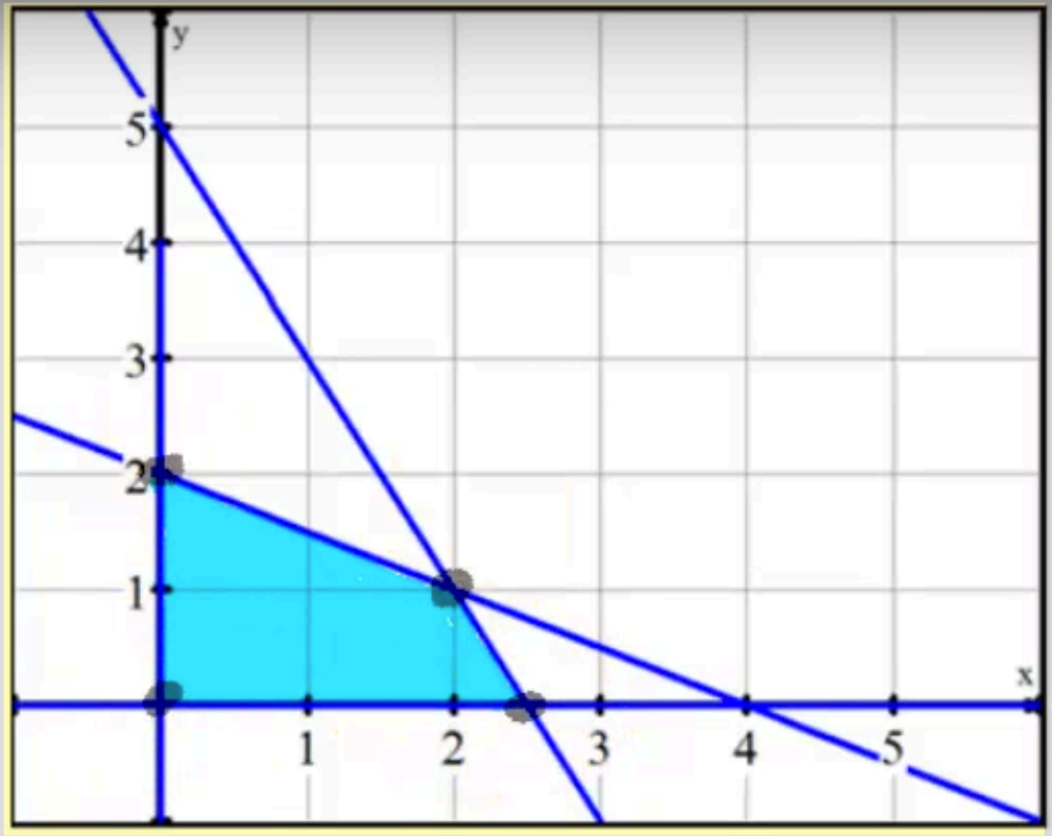
Inactive variables
x=0
t=0

Optimum Solution
x=0
y=2
P(0,2)=10

Verification with Linear Programming Method

Maximize: $P = 2x + 5y$

$$x + 2y \leq 4$$
$$3x + 2y \leq 3$$
$$x \geq 0, y \geq 0$$



$$P(0,0) = 2(0) + 5(0) = 0$$
$$P(2.5,0) = 2(2.5) + 5(0) = 5$$
$$P(2,1) = 2(2) + 5(1) = 9$$
$$\boxed{P(0,2) = 2(0) + 5(2) = 10}$$

Example Problem

EXAMPLE

Given the system of constraints

$$\begin{cases} 2x + 3y \leq 90 \\ 3x + 2y \leq 120 \\ x \geq 0 \\ y \geq 0, \end{cases}$$

maximize the objective function

$$f(x, y) = 7x + 5y.$$

Correct answer: f(36,6)=282.

Simplex Method
von Neumann Duality Principle

von Neumann duality principle

Every maximization problem has a corresponding minimization problem.

Accordingly, the simplex algorithm for minimization problems works by converting the problem to a maximization problem.

Simplex Method for minimization

EXAMPLE

Given the system of constraints

$$\begin{cases} 4x + 3y + 5z \geq 65 \\ x + 3y + 2z \geq 38 \\ 2x + 3y + 4z \geq 52 \\ x, y, z \geq 0, \end{cases}$$

minimize the function

$$f(x, y, z) = 12x + 3y + 10z.$$

This problem could be put into the form shown in the maximization examples above, but an issue would occur with finding the first basic solution: setting the x,y and z variables to 0 would give an infeasible solution with the slack variables taking on negative values. The simplex algorithm needs to start with a feasible solution, so this would not work. But there is an alternative method for this problem.

Simplex Method for minimization

$$4x + 3y + 5z \geq 65$$
$$x + 3y + 2z \geq 38$$
$$2x + 3y + 4z \geq 52$$
$$f(x, y, z) = 12x + 3y + 10z.$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 5 & 65 \\ 1 & 3 & 2 & 38 \\ 2 & 3 & 4 & 52 \\ \hline 12 & 3 & 10 & 0 \end{array}\right]$$

A "dual" of this problem can be written by transposing the coefficients. Place the coefficients of the constraints into an augmented matrix. Place the coefficients of the objective function into the bottom row, with a 0 in the right part, as shown above.

Transpose the elements of the matrix:

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 12 \\ 3 & 3 & 3 & 3 \\ 5 & 2 & 4 & 10 \\ \hline 65 & 38 & 52 & 0 \end{array}\right]$$

Note: It's tempting to divide out the 3 in the second row of this matrix, but this is not recommended since it will break the symmetry that is required to return to the original problem.

Simplex Method

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 12 \\ 3 & 3 & 3 & 3 \\ 5 & 2 & 4 & 10 \\ \hline 65 & 38 & 52 & 0 \end{array}\right]$$

This gives a new system of constraints and an objective function to be *maximized:* Given the system of constraints

$$\begin{cases} 4u + v + 2w \leq 12 \\ 3u + 3v + 3w \leq 3 \\ 5u+2v+4w \leq 10 \\ u, v, w \geq 0, \end{cases}$$

maximize the function

$$g(u, v, w) = 65u + 38v + 52w.$$

End of Lecture IX