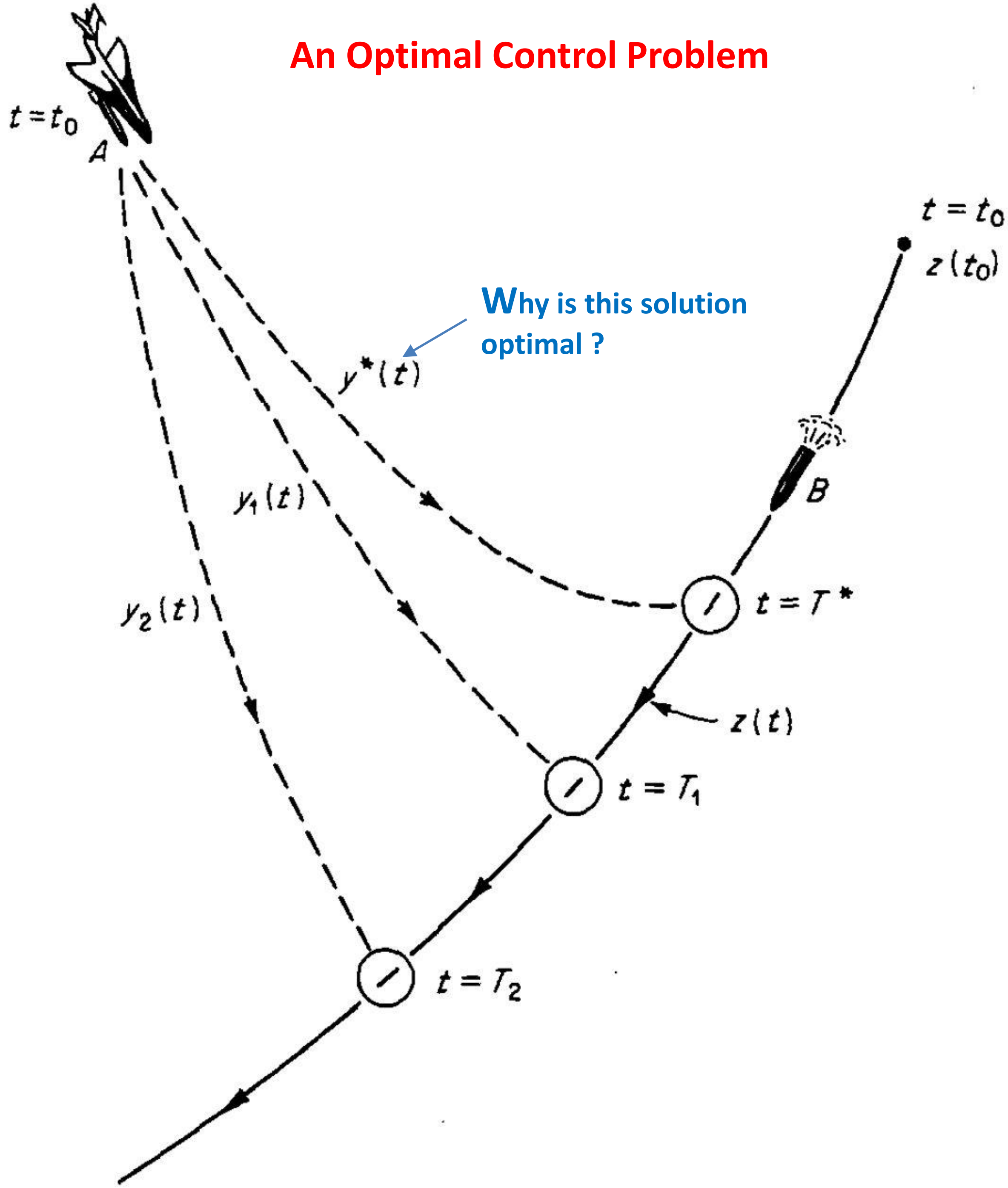
EEEN 460
Optimal Control

2020 Spring

Lecture I

Introduction
Describing the System and Evaluating its Performance
Conditions of Optimality

An Optimal Control Problem



The Objective of Optimal Control

The objective of optimal control theory is *to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion.*

The Formulation of an Optimal Control Problem

The formulation of an optimal control problem requires:

1. A mathematical description (or model) of the process to be controlled.
2. A statement of the physical constraints.
3. Specification of a performance criterion.

The Mathematical Model

The Mathematical Model

A nontrivial part of any control problem is modeling the process. The objective is to obtain the simplest mathematical description that adequately predicts the response of the physical system to all anticipated inputs. Our discussion will be restricted to systems described by ordinary differential equations (in state variable form).† Thus, if

$$x_1(t), x_2(t), \dots, x_n(t)$$

are the *state variables* (or simply the *states*) of the process at time t, and

$$u_1(t), u_2(t), \dots, u_m(t)$$

are *control inputs* to the process at time t, then the system may be described by n first-order differential equations

$$\begin{aligned}
\dot{x}_1(t) &= a_1(x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_m(t), t) \\
\dot{x}_2(t) &= a_2(x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_m(t), t) \\
&\vdots \\
\dot{x}_n(t) &= a_n(x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots, u_m(t), t).\ddagger
\end{aligned} \tag{1.1-1}$$

We shall define

$$\mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

as the *state vector* of the system, and

$$\mathbf{u}(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_m(t) \end{bmatrix}$$

as the *control vector*. The state equations can then be written

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{1.1-1a}$$

Example

Figure 1-1 A simplified control problem

Example 1.1-1. The car shown parked in Fig. 1-1 is to be driven in a straight line away from point O. The distance of the car from O at time t is denoted by $d(t)$. To simplify the model, let us approximate the car by a unit point mass that can be accelerated by using the throttle or decelerated by using the brake.

The differential equation is

$$\ddot{d}(t) = \alpha(t) + \beta(t), \tag{1.1-2}$$

where the control α is throttle acceleration and β is braking deceleration. Selecting position and velocity as state variables, that is,

$$x_1(t) \triangleq d(t) \quad \text{and} \quad x_2(t) \triangleq \dot{d}(t),$$

and letting

$$u_1(t) \triangleq \alpha(t) \quad \text{and} \quad u_2(t) \triangleq \beta(t),$$

we find that the state equations become

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) + u_2(t), \end{aligned} \tag{1.1-3}$$

or, using matrix notation,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t). \tag{1.1-3a}$$

This is the mathematical model of the process in state form.

Physical Constraints

After we have selected a mathematical model, the next step is to define the physical constraints on the state and control values. To illustrate some typical constraints, let us return to the automobile whose model was determined in Example 1.1-1.

Example 1.1-2. Consider the problem of driving the car in Fig. 1-1 between the points O and e. Assume that the car starts from rest and stops upon reaching point e.

First let us define the state constraints. If t_0 is the time of leaving O, and t_f is the time of arrival at e, then, clearly,

$$\begin{aligned} x_1(t_0) &= 0 \\ x_1(t_f) &= e. \end{aligned} \tag{1.1-4}$$

In addition, since the automobile starts from rest and *stops* at e,

$$\begin{aligned} x_2(t_0) &= 0 \\ x_2(t_f) &= 0. \end{aligned} \tag{1.1-5}$$

In matrix notation these *boundary conditions* are

$$\mathbf{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad \text{and} \quad \mathbf{x}(t_f) = \begin{bmatrix} e \\ 0 \end{bmatrix}. \tag{1.1-6}$$

If we assume that the car does not back up, then the additional constraints

$$\begin{aligned} 0 \le x_1(t) &\le e \\ 0 \le x_2(t) & \end{aligned} \tag{1.1-7}$$

are also imposed.

Constraints on the Control Inputs

What are the constraints on the control inputs (acceleration)? We know that the acceleration is bounded by some upper limit which depends on the capability of the engine, and that the maximum deceleration is limited by the braking system parameters. If the maximum acceleration is $M_1 > 0$, and the maximum deceleration is $M_2 > 0$, then the controls must satisfy

$$\begin{aligned} 0 \le u_1(t) &\le M_1 \\ -M_2 \le u_2(t) &\le 0. \end{aligned} \tag{1.1-8}$$

Performance Constraint

In addition, if the car starts with G gallons of gas and there are no service stations on the way, another constraint is

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]\, dt \leq G \tag{1.1-9}$$

which assumes that the rate of gas consumption is proportional to both acceleration and speed with constants of proportionality k_1 and k_2.

Some Definitions

DEFINITION 1-1

A history of control input values during the interval $[t_0, t_f]$ is denoted by $\mathbf{u}$ and is called a *control history*, or simply a *control*.

DEFINITION 1-2

A history of state values in the interval $[t_0, t_f]$ is called a *state trajectory* and is denoted by $\mathbf{x}$.

DEFINITION 1-3

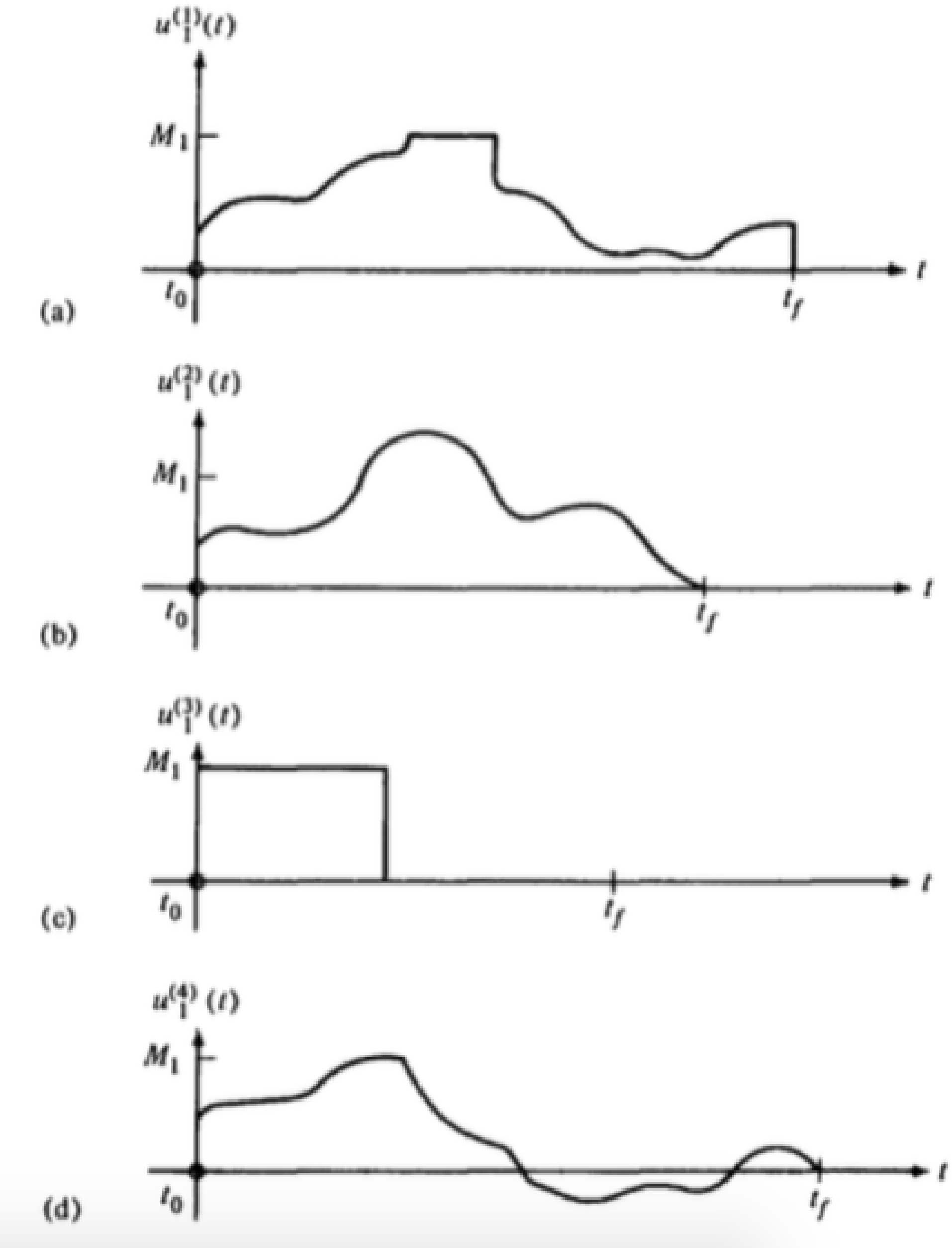
A control history which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an *admissible control*.

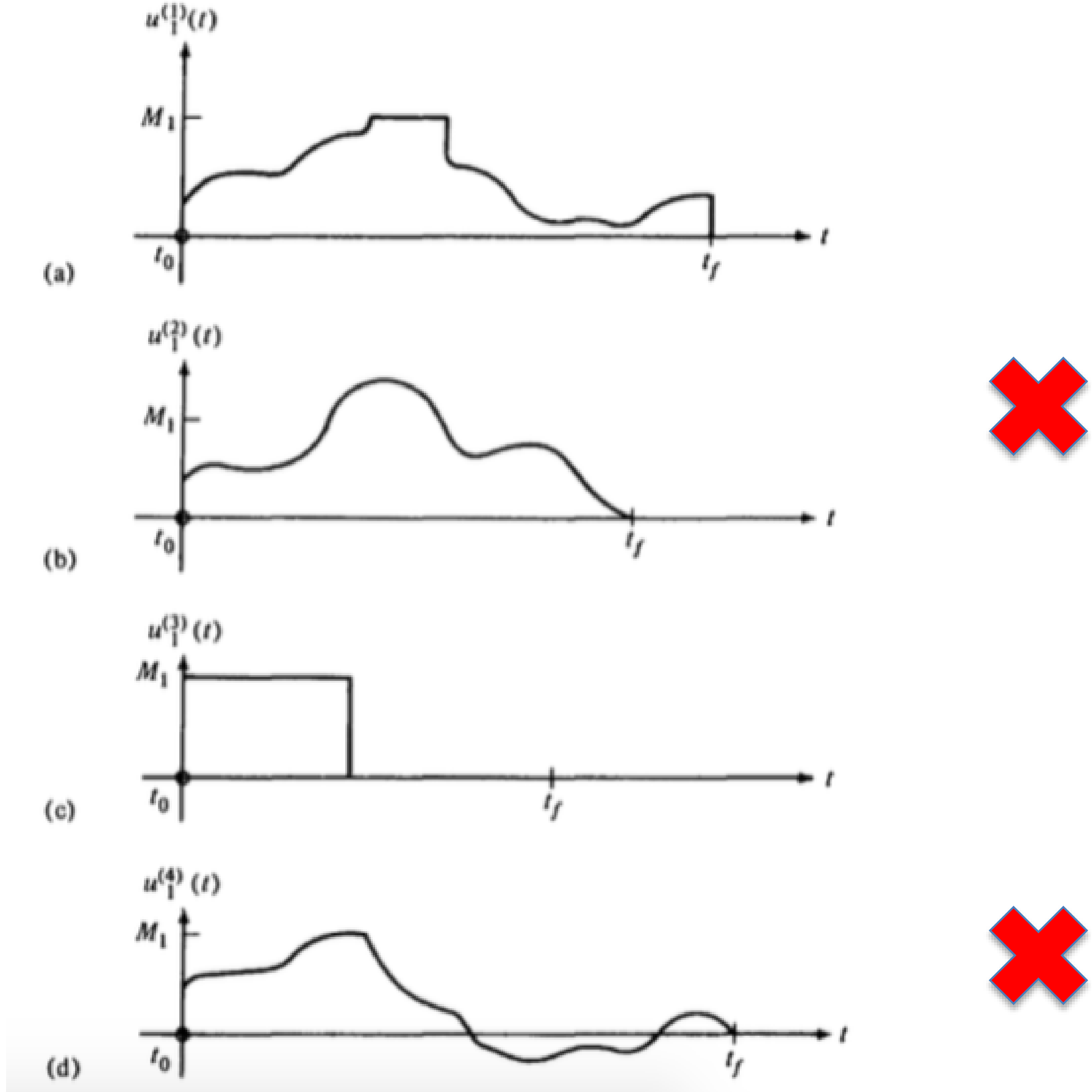
DEFINITION 1-4

A state trajectory which satisfies the state variable constraints during the entire time interval $[t_0, t_f]$ is called an *admissible trajectory*.

Example

Which of these Controls are not admissible ?




$u_1^{(1)}(t)$
M_1
t_0
t_f
t
(a)
$u_1^{(2)}(t)$
M_1
t_0
t_f
t
(b)
$u_1^{(3)}(t)$
M_1
t_0
t_f
t
(c)
$u_1^{(4)}(t)$
M_1
t_0
t_f
t
(d)

Performance Measure

The Performance Measure

In order to evaluate the performance of a system quantitatively, the designer selects a performance measure. An *optimal control* is defined as one that *minimizes* (or maximizes) the performance measure.

Example

For example, the statement, "Transfer the system from point A to point B as quickly as possible," clearly indicates that elapsed time is the performance measure to be minimized.

Performance Measure for the Automobile Problem

Example 1.1-3. Let us return to the automobile problem begun in Example 1.1-1. The state equations and physical constraints have been defined; now we turn to the selection of a performance measure. Suppose the objective is to make the car reach point e as quickly as possible; then the performance measure J is given by

$$J = t_f - t_0. \tag{1.1-10}$$

Performance Measure

In all that follows it will be assumed that the performance of a system is evaluated by a measure of the form

$$\boxed{J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt,} \qquad (1.1\text{-}11)$$

where t_0 and t_f are the initial and final time; h and g are scalar functions. t_f may be specified or "free," depending on the problem statement.

Starting from the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and applying a control signal $\mathbf{u}(t)$, for $t \in [t_0, t_f]$, causes a system to follow some state trajectory; the performance measure assigns a unique real number to each trajectory of the system.

Optimal Control Problem Statement

Find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.1-12}$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that *minimizes* the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt. \tag{1.1-13}$$

$\mathbf{u}^*$ is called an *optimal control* and $\mathbf{x}^*$ an *optimal trajectory*.

Minimizing the Performance Measure

when we say that $\mathbf{u}^*$ causes the performance measure to be minimized, we mean that

$$J^* \triangleq h(\mathbf{x}^*(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\, dt$$
$$\leq h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt \qquad (1.1\text{-}14)$$

Example

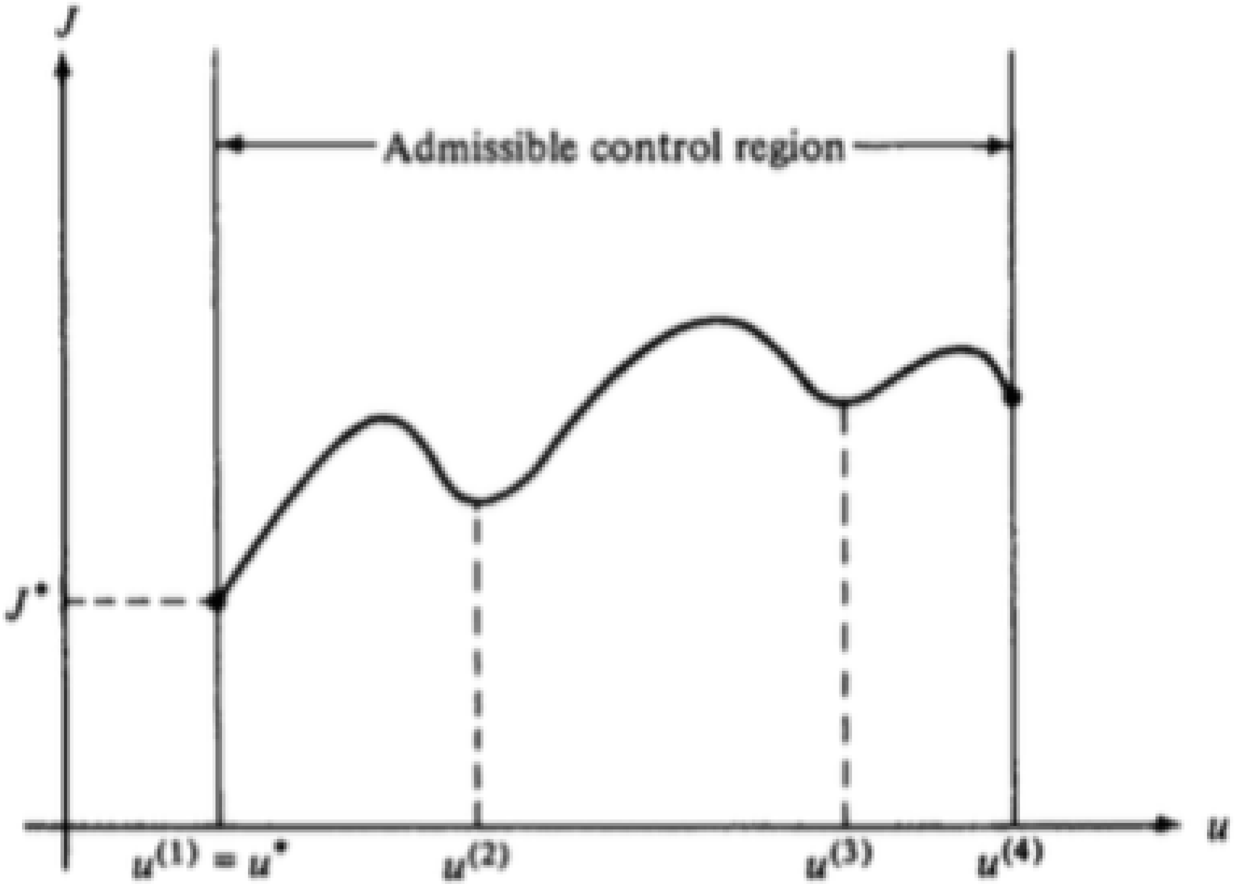


Figure 1-5 A representation of the optimization problem

It may be helpful to visualize the optimization as shown in Fig. 1-5. $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, and $u^{(4)}$ are "points" at which J has local, or relative, minima; $u^{(1)}$ is the "point" where J has its global, or absolute, minimum.

A Comment

Finally, observe that if the objective is to maximize some measure of system performance, the theory we shall develop still applies because this is the same as minimizing the negative of this performance measure. Henceforth, we shall speak, with no lack of generality, of minimizing the performance measure.

Back to Automobile Problem

Example 1.1-4. To illustrate a complete problem formulation, let us now summarize the results of Example 1.1-1, using the notation and definitions which have been developed.

The state equations are

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) + u_2(t). \end{aligned} \tag{1.1-3}$$

The set of admissible states X is partially specified by the boundary conditions

$$\mathbf{x}(t_0) = \mathbf{0}, \qquad \mathbf{x}(t_f) = \begin{bmatrix} e \\ 0 \end{bmatrix}$$

and the inequalities

$$\begin{aligned} 0 \leq x_1(t) &\leq e \\ 0 \leq x_2(t)&. \end{aligned} \tag{1.1-7}$$

The set of admissible controls U is partially defined by the constraints

$$\begin{aligned} 0 \leq u_1(t) &\leq M_1 \\ -M_2 \leq u_2(t) &\leq 0. \end{aligned} \tag{1.1-8}$$

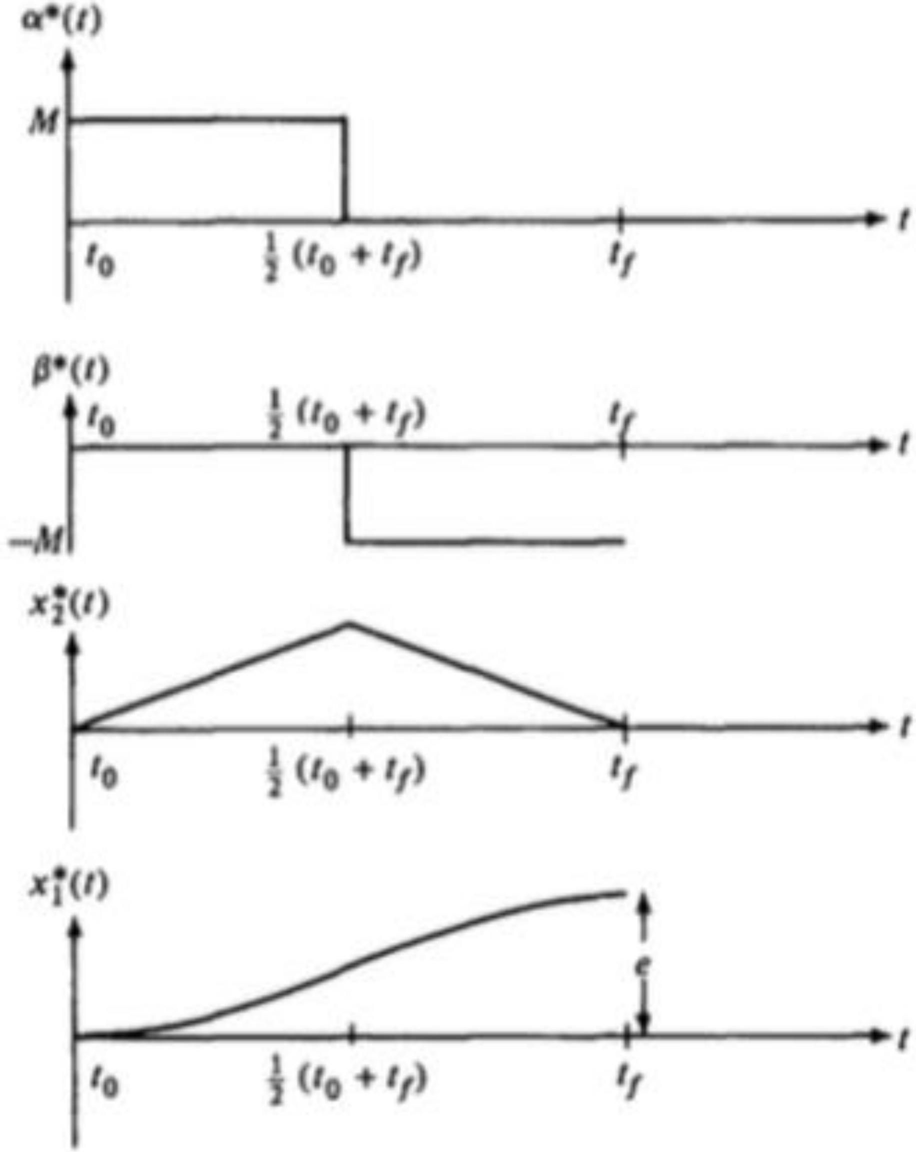
Performance Measure

The inequality constraint

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)]\, dt \le G \qquad (1.1\text{-}9)$$

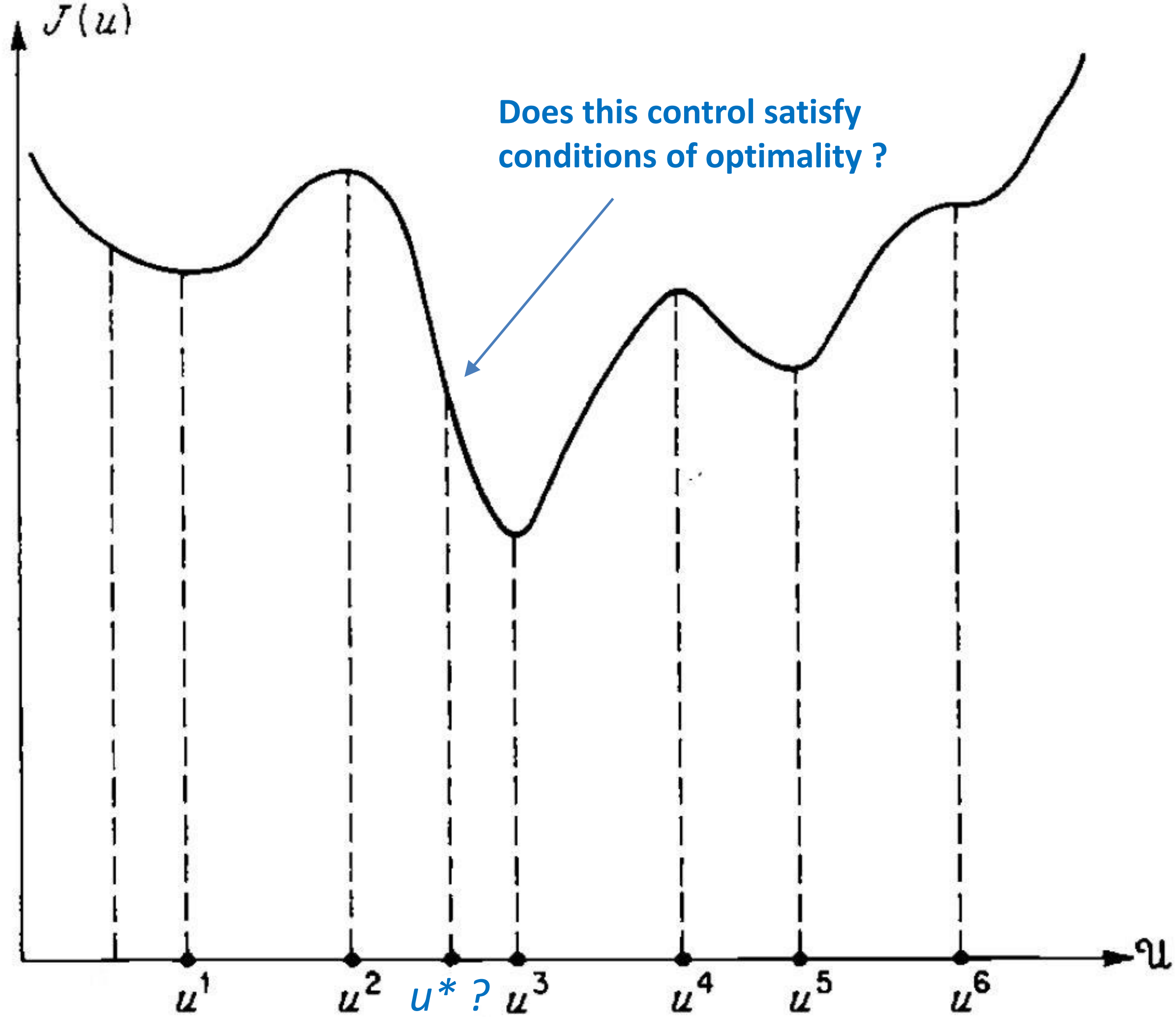
completes the description of the admissible states and controls.

Solution

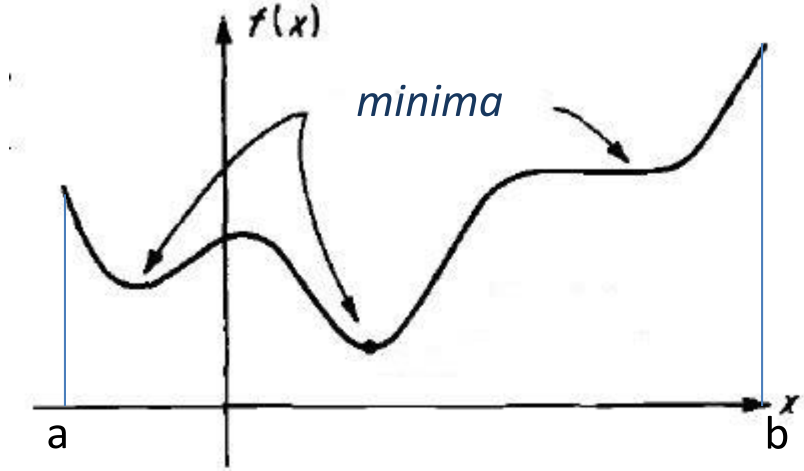


How do we find this solution ?

Conditions of Optimality



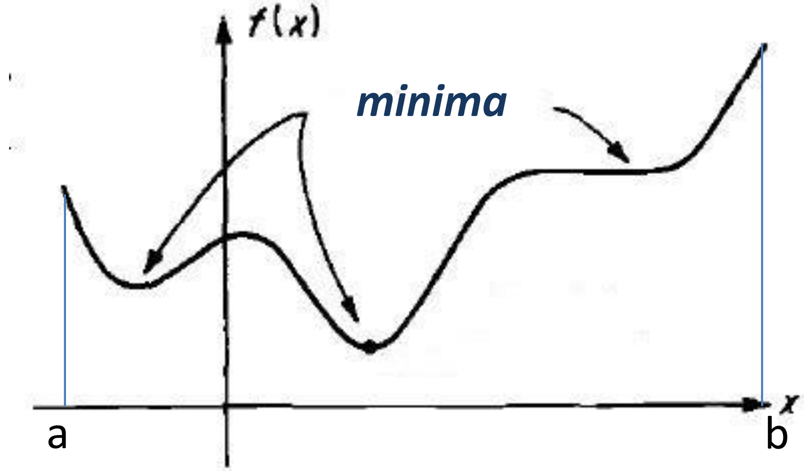
Conditions of Optimality



f is a real-valued function which is defined and *continuous* on the *closed* interval [a, b].

We observe that *f* has *minima, on* [a, b].
How can we find them?

Conditions of Optimality



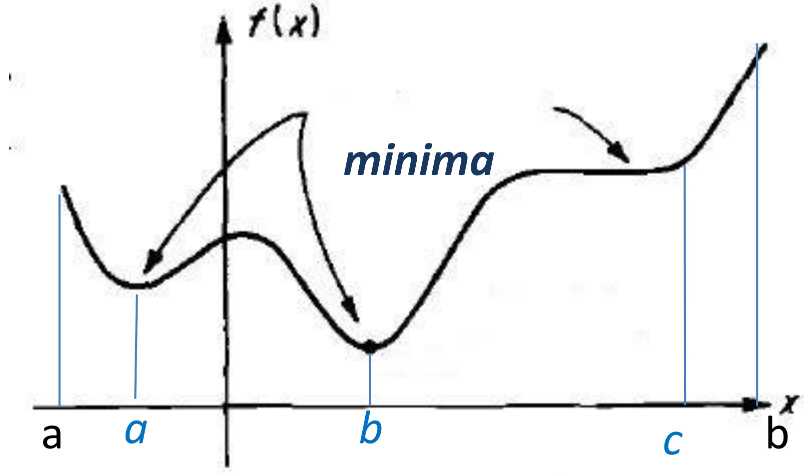
First note that if *x* is an interior point of [a, b] and if the derivative *f* '(x) exists and is *not* zero, then *x cannot* be a minimum of *f*. This leads us to the necessary condition:

NCI *If x* is an interior point of* [a, b], *if f'(x*) exists, and if x* is a minimum of f, then*

$$f'(x^*) = \frac{df}{dx}(x^*) = 0$$

A point *x* at which *f* '(x) is zero is called an *extremum* of *f*.

Conditions of Optimality



NC2 *If x* is an interior point of* [a, b] *which is a minimum of f, then*

either

$$f'(x^*) = 0$$

or

$$\lim_{x \to x^*+} f'(x) \geq 0 \quad \textit{and} \quad \lim_{x \to x^*-} f'(x) \leq 0‡$$

If a is a minimum of f, then

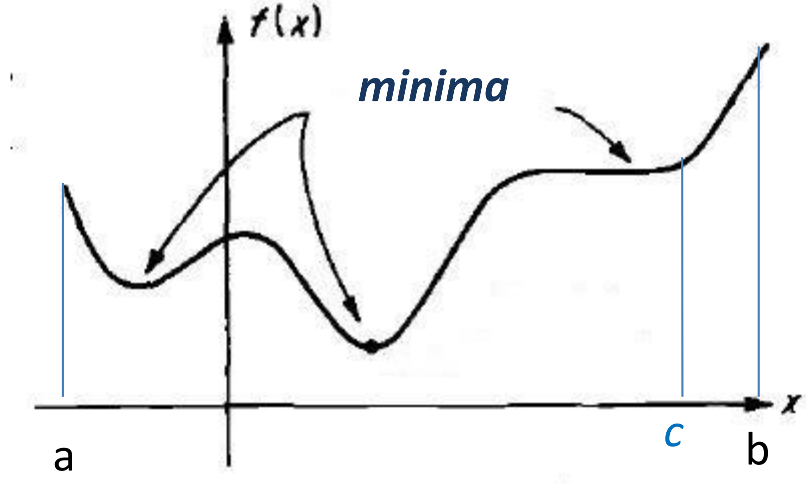
$$\lim_{x \to a+} f'(x) \geq 0$$

and if b is a minimum of f, then

$$\lim_{x \to b-} f'(x) \leq 0$$

Conditions of Optimality

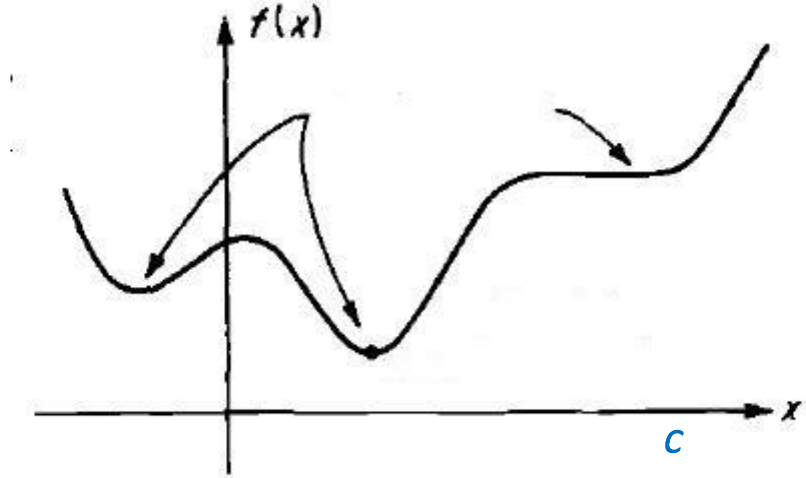
Exercise



If x=c is an interior point of* [a, b] *which is a minimum of f, then what is the necessary condition for c ?*

Conditions of Optimality

Solution



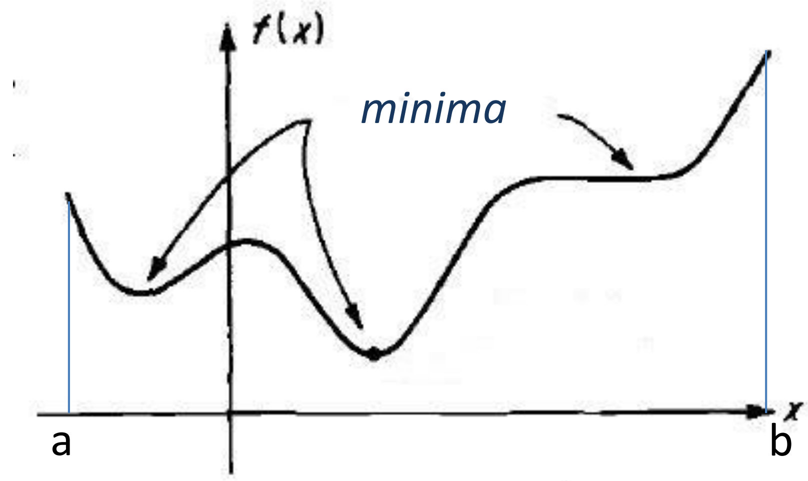
If $x^*=c$ is an interior point of [a, b] *which is a minimum of f, then what is the necessary condition for c ?*

$$f'(x^*) = 0$$

$$\lim_{x \to c^+} f'(x) \geq 0$$

$$\lim_{x \to c^-} f'(x) \leq 0$$

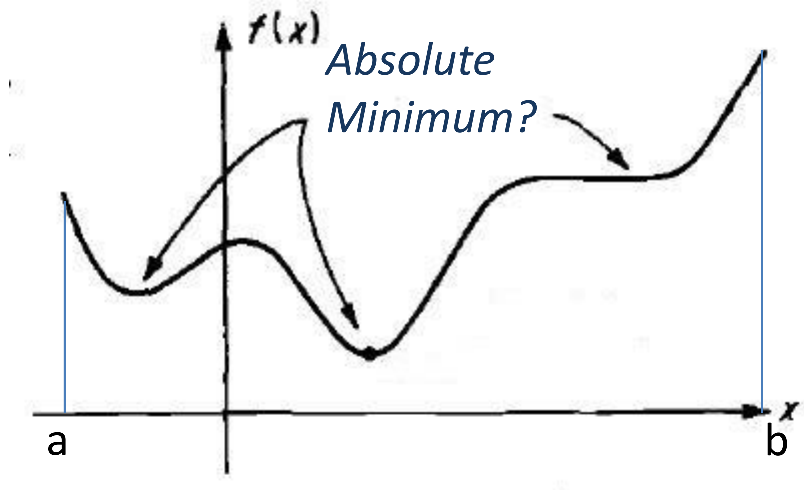
Conditions of Optimality



We can deduce from the necessary conditions NCI and NC2 that if x^* is a point at which $f'(x^*)$ is zero and if the derivative of f changes from negative to positive when passing through x^*, then x^* is a minimum of f. This sufficiency condition for a minimum may be phrased as a sufficient condition as follows:

SCI *If x^* is an interior point of* [a, b] *at which f is zero [that is, $f'(x^*) = 0$] and if $f''(x^*) > 0$, then x^* is a (local) minimum of f.*

Conditions of Optimality



Definition *A point* x* *in* R_n *is said to be an absolute minimum of g if* f(x*) ≤ *f(x) for all x in* R_n.

Conditions of Optimality

So we shall now consider a multivariable function *g(x)* in a given domain D_n, such that $g(x) = g(x_1, x_2, \ldots, x_n)$

NC3 *If a point x* in D is a minimum of g, then all the partial derivatives of g vanish at x*, that is,*

$$\frac{\partial g}{\partial x_1}(\mathbf{x}^*) = \frac{\partial g}{\partial x_2}(\mathbf{x}^*) = \cdots = \frac{\partial g}{\partial x_n}(\mathbf{x}^*) = 0$$

SC2 *If the gradient of g is 0 at x* and if the matrix Q is positive definite then x* is a minimum of g.*

$$\mathbf{Q} = \begin{bmatrix} \frac{\partial^2 g}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 g}{\partial x_1\,\partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 g}{\partial x_1\,\partial x_n}(\mathbf{x}^*) \\ \frac{\partial^2 g}{\partial x_2\,\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 g}{\partial x_2^2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 g}{\partial x_2\,\partial x_n}(\mathbf{x}^*) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 g}{\partial x_n\,\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 g}{\partial x_n\,\partial x_2}(\mathbf{x}^*) & \cdots & \frac{\partial^2 g}{\partial x_n^2}(\mathbf{x}^*) \end{bmatrix}$$

Conditions of Optimality

Special Case

if g is a function on D_2

$$\mathbf{Q} = \begin{bmatrix} \boxed{\begin{matrix} \dfrac{\partial^2 g}{\partial x_1^2}(\mathbf{x}^*) & \dfrac{\partial^2 g}{\partial x_1\,\partial x_2}(\mathbf{x}^*) \\ \dfrac{\partial^2 g}{\partial x_2\,\partial x_1}(\mathbf{x}^*) & \dfrac{\partial^2 g}{\partial x_2^2}(\mathbf{x}^*) \end{matrix}} & \cdots & \dfrac{\partial^2 g}{\partial x_1\,\partial x_n}(\mathbf{x}^*) \\ & \cdots & \dfrac{\partial^2 g}{\partial x_2\,\partial x_n}(\mathbf{x}^*) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{\partial^2 g}{\partial x_n\,\partial x_1}(\mathbf{x}^*) \quad \dfrac{\partial^2 g}{\partial x_n\,\partial x_2}(\mathbf{x}^*) & \cdots & \dfrac{\partial^2 g}{\partial x_n^2}(\mathbf{x}^*) \end{bmatrix}$$

$$\frac{\partial g}{\partial x_1}(\mathbf{x}^*) = \frac{\partial g}{\partial x_2}(\mathbf{x}^*) = 0$$

$$\frac{\partial^2 g}{\partial x_1^2}(\mathbf{x}^*) > 0$$

$$\frac{\partial^2 g}{\partial x_1^2}(\mathbf{x}^*)\,\frac{\partial^2 g}{\partial x_2^2}(\mathbf{x}^*) - \left[\frac{\partial^2 g}{\partial x_1\,\partial x_2}(\mathbf{x}^*)\right]^2 > 0$$

guarantee that x* is a minimum of *g.*

Conditions of Optimality

Example

Consider the function *g* defined on D_2 by

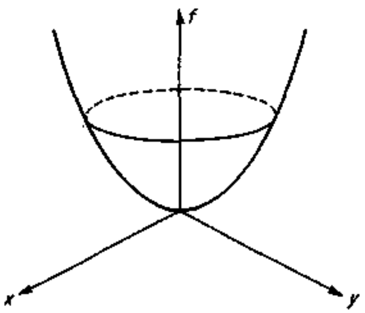
$g(x, y) = (x^2 + y^2)/2$

Show that the origin (0,0) is the minimum of *g*.

Conditions of Optimality

Solution

When we plot $g(x, y) = (x^2 + y^2)/2$



we see the function *g,* viewed as a surface in R_2, is a paraboloid of revolution as illustrated above. Its minimum is (0,0).

$$\partial g/\partial x = x$$
$$\partial g/\partial y = y$$
$$\partial^2 g/\partial x\,\partial y = 0$$
$$\partial^2 g/\partial x^2 = 1$$
$$\partial^2 g/\partial y^2 = 1$$

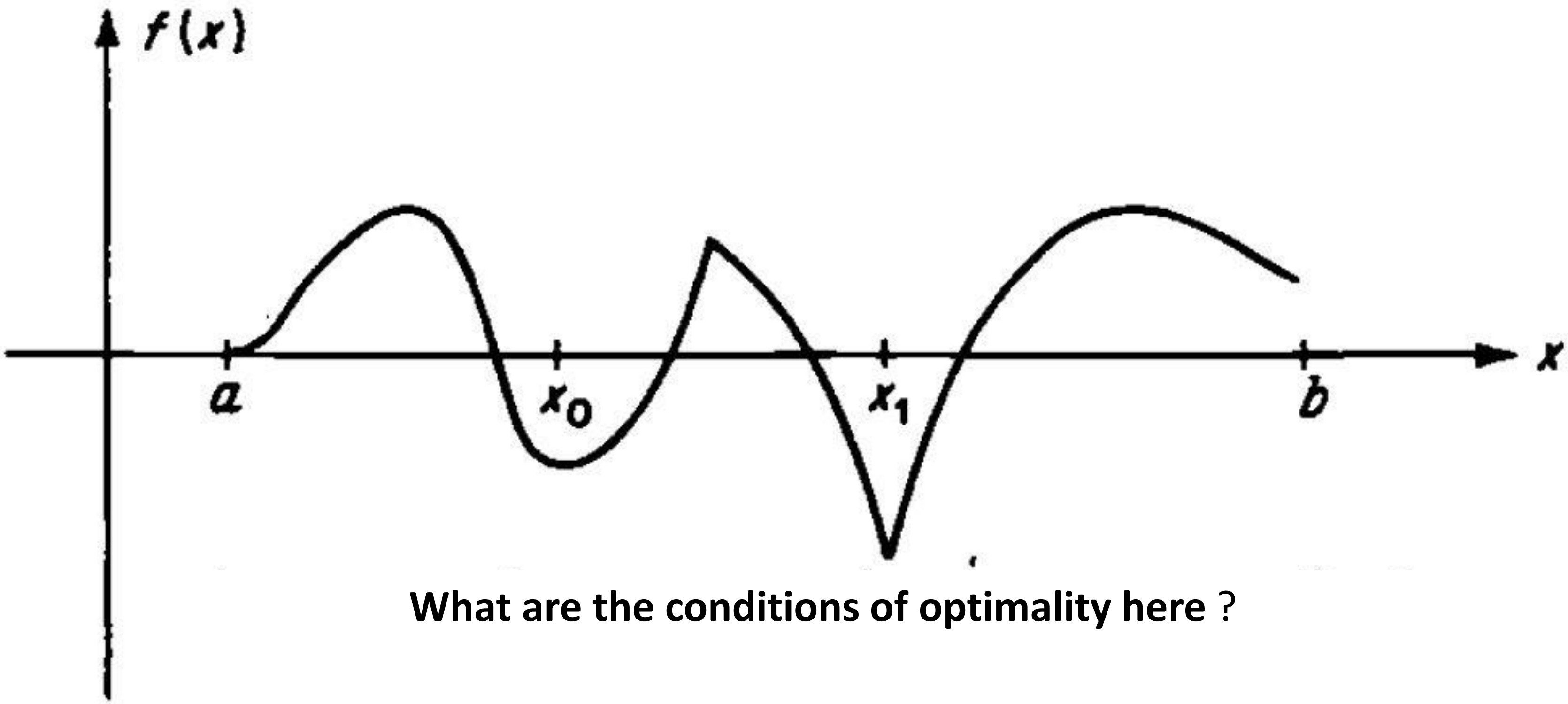
We note that the matrix Q in this case is :

$$Q = \begin{bmatrix} \dfrac{\partial^2 g}{\partial x^2}(\mathbf{x}^*) & \dfrac{\partial^2 g}{\partial x\,\partial y}(\mathbf{x}^*) \\ \dfrac{\partial^2 g}{\partial y\,\partial x}(\mathbf{x}^*) & \dfrac{\partial^2 g}{\partial y^2}(\mathbf{x}^*) \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

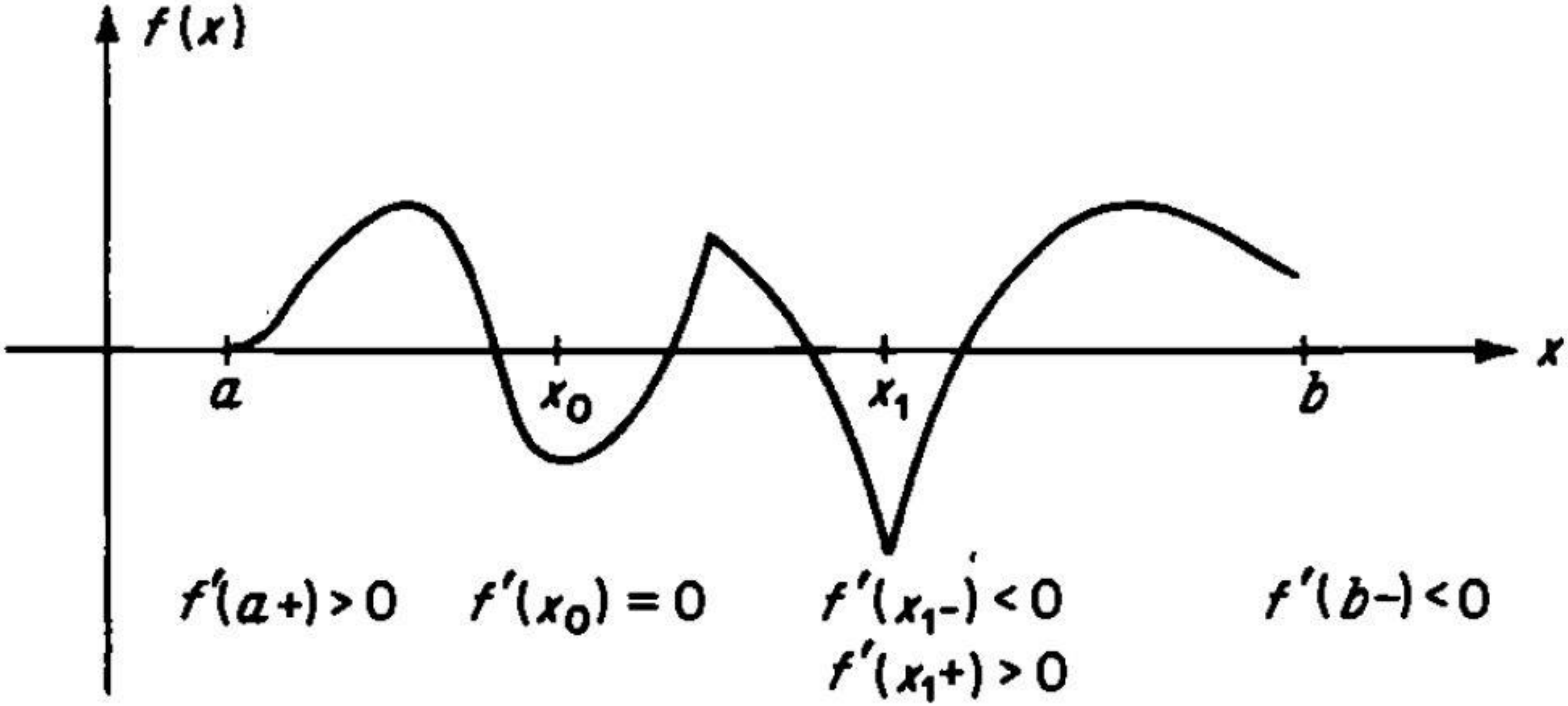
Conditions of Optimality

Summary



What are the conditions of optimality here ?

Conditions of Optimality



End of Lecture 1