(1)

$$3 \quad 1 \quad 3 \quad 6 \quad = \quad 8$$

$$(3+1)/3\times 6 \quad = \quad (3+1)/(3/6) \quad = \quad 8$$

- Not easy to find, but easy to verify. ("NP certificates")
- Could do exhaustive search (Arithmetic trees), but may be easier to "guess then check."
- Think outside the box: think about fractions, not only integers.

(2) Using a table form we can simulate the operations as follows:

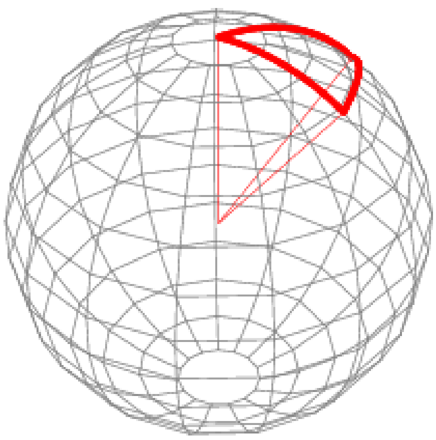
Action	A (5L)	B (3L)
	0	0
Fill A	5	0
Empty A into B	2	3
Empty B	2	0
Empty A into B	0	2
Fill A	5	2
Empty A into B	4	3

There are many other (longer) solutions too.

- Not easy to find, easy to verify. ("NP certificates")
- Could do exhaustive search (Turn into a graph), but may be easier to "guess then check."

(3) White (North pole), spherical triangle.

- Think outside the box.



(4) Python quick script to simulate the operations:

```
lockers = [False for i in range(101)]

for run in range(1,101):
    for locker in range(run,101,run):
        lockers[ locker ] = not lockers[
          locker ]

for i in range(101):
    if lockers[i]:
        print(i)
```

The lockers that remain open are: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$.

10 lockers: $1^2, 2^2, 3^3, \ldots, 10^2$ because locker ℓ remains open iff it has an odd number of divisors, which only happens if ℓ is a perfect square.

- Quick code helps find the pattern. Use Maths to justify.

(5) No.

Minimum is:

$$0 + 1 + 2 + \cdots + 9 = \frac{10 \times (0 + 9)}{2} = 45 > 44.$$

- Sometimes we have to prove a solution does not exist.
- This method is called "proof by contradiction." We assume something is possible/true and then derive a contradiction. We then conclude that the assumption must have been wrong.

(6) The answer is: 14 steps only!

Group the 100 floors into n groups, $G_1, G_2, \ldots, G_n$, and let G_k consists of g_k floors, and let the highest floor for G_k be h_k for $k = 1, 2, \ldots, n$.

If the first egg broke at floor h_1 then we need $g_1 - 1$ drops for the second egg, using linear search in the floots $1, 2, \ldots, h_k - 1$.

Similarly, if the first egg broke at floor h_2 then we need $e_2 - 1$ drops for the second egg, but we already have two drops for the first egg, so to keep the total number drops fixed, we have:

$$e_1 - 1 = e_2.$$

In general, we have

$$\begin{aligned} e_1 - 1 &= e_2. \\ &\vdots \\ e_{n-1} - 1 &= e_n. \end{aligned}$$

Therefore, in order to achieve minimum number of drops with worst case scenario, we need to have:

$$e_n = 1, e_{n-1} = 2, \ldots, e_1 = n.$$

This means minimum of 14 drops for two eggs and 100 floors: let the first egg drop at the 14th floor, if it breaks then the worst case for the second egg is to test 13 floors, 13+1 = 14. If it passes then the next test for the first egg is at the 27th floor (14+13), this time the worst case for the second egg is to test 12 floors (number: 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26). This gives 12+2 = 14.

Third drop at the 39th floor, forth drop at the 50th floor, etc. If we schedule in this way, including the worst case scenario, the minimum number of drops is 14.

Now think about how this can be generalised!

- Special cases, sub-problems, generalisations.

(7) No.

There are more black squares than white squares. (Each domino covers exactly one black and one white).

- Idea of *invariant*.