RASSAL MODELLER

(14. Hafta Notları) 4 Ocak 2021

Poisson Süreci

$N(t) \sim$ parametresi $\lambda > 0$ olan Poisson Süreci olsun.

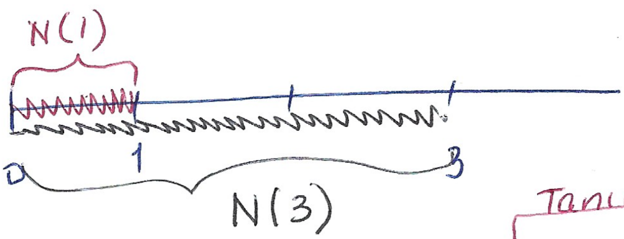
$N(t) = (0, t]$ aralığında m.g olay sayısı. gösterir ve

$N(t) \sim$ poisson (λt) dağılımına sahiptir

veya $$P[N(t) = k] = \frac{e^{-\lambda t} (\lambda t)^k}{k!} \quad k = 0, 1, 2, \ldots$$

$B(N(t)) = Var(N(t)) = \lambda t$

Daha önce gördük ki $N(1)$ ve $N(3)$ ..... değildir.



Tanım: $\boxed{Cov(X, Y) \equiv E[(X - \mu_X)(Y - \mu_Y)]}$

$Cov(N(1), N(3)) = ?$

$= E\left[[N(1) - B(N(1))][N(3) - B(N(3))]\right]$

$= B\left[(N(1) - \lambda)(N(3) - 3\lambda)\right] = B\left[N(1) . N(3) - 3\lambda N(1) - \lambda N(3) + 3\lambda^2\right]$

$= B[N(1) N(3)] - 3\lambda B(N(1)) - \lambda B(N(3)) + 3\lambda^2$

$= B[N(1) N(3)] - 3\lambda . \lambda - \lambda . 3\lambda + 3\lambda^2$

$= B[N(1) N(3)] - 3\lambda^2$

$$B[N(1)\,N(3)] = B\left[N(1) * \left(N(1) + N(3) - N(1)\right)\right]$$

$$= B\left[N^2(1) + N(1) * [N(3) - N(1)]\right]$$

$$= B[N^2(1)] + B\left[N(1) * (N(3) - N(1))\right]$$

$$= Var(N(1)) + B^2[N(1)] + B[N(1)]\, B[\underbrace{N(3) - N(1)}_{\equiv N(2)}]$$

$\equiv N(2)$ aynı dağılım neden?

$$= \lambda . 1 + \lambda^2 + \lambda . 2\lambda = 3\lambda^2 + \lambda .$$

$$= \lambda(1 + 3\lambda).$$

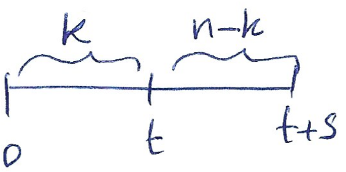
$$Cov(N(1), N(3)) = B(N(1).N(3)) - 3\lambda^2$$

$$= 3\lambda^2 + \lambda - 3\lambda^2 = \lambda . > 0 .$$

<u>SORU:</u> Korelasyon katsayısı $\rho_{N(1),N(3)} = \dfrac{Cov(N(1), N(3))}{\sqrt{Var(N(1))}\sqrt{Var(N(3))}} = ?$

<u>EX 1:</u> $N(t) \sim$ poisson süreci (λ) veya poisson (λt)

$$P[N(t) = k \mid N(t+s) = n] = \frac{P[N(t) = k, N(t+s) = n]}{P[N(t+s) = n]}$$



$$= \frac{P[N(t) = k,\ N(t+s) - N(t) = n-k]}{P[N(t+s) = n]}$$

$$= \frac{P[N(t) = k].P[N(s) = n-k]}{P[N(t+s) = n]} = \frac{\dfrac{\cancel{e^{-\lambda t}}(\lambda t)^k}{k!}\ \dfrac{\cancel{e^{-\lambda s}}(\lambda s)^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-\lambda(t+s)}}\,.\,[\lambda(t+s)]^n}{n!}} = \binom{n}{k}\left(\frac{t}{t+s}\right)^k \left(\frac{s}{t+s}\right)^{n-k}$$

Bu hangi dağılım?

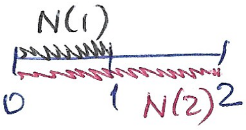
EX 2: $N(t) \sim$ poisson(λ) süreci

a) $P[N(1) \leq 2] = \sum_{k=0}^{2} P[N(1)=k] = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} + \frac{(\lambda t)^1 e^{-\lambda t}}{1!} + \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$

$= e^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2}\right)$

$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) =$

b) $P[N(1)=1, N(2)=3] = P[N(1)=1, N(2)-N(1)=2]$



$= P[N(1)=1]\; P[\underbrace{N(2)-N(1)}_{\equiv N(1)} = 2]$ Durağan Artışlar özelliği

$= \frac{e^{-\lambda} \lambda^1}{1!} \frac{e^{-\lambda} \lambda^2}{2!} = e^{-2\lambda} \frac{\lambda^3}{2}.$

c) $P[N(1) \geq 2 \mid N(1) \geq 1]$

$= \frac{P[N(1) \geq 2, N(1) \geq 1]}{P[N(1) \geq 1]} = \frac{P[N(1) \geq 2]}{P[N(1) \geq 1]} = \frac{1 - \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!}}{1 - \frac{e^{-\lambda}\lambda^0}{0!}}$

$= \frac{1 - e^{-\lambda}(1+\lambda)}{1 - e^{-\lambda}}$

EX 3:

$N(t) \sim PP(\lambda)$

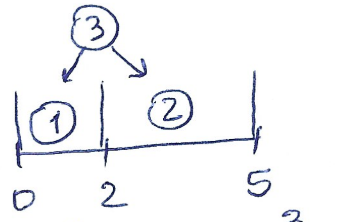
a) $B[N(2)] = 2\lambda$

b) $B[N^2(1)] = Var(N(1)) + B^2(N(1)) = \lambda + \lambda^2$

c) $B[N(1).N(2)] = ?$

d) $P[N(2)=1 \mid N(5)=3] = \binom{3}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^2$

$= 3 \cdot \frac{2}{5} \cdot \frac{9}{25} = \frac{54}{125} \cong 0.432.$

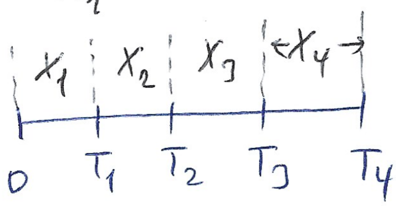


$p = \frac{2}{5}$ $\quad 1-p = \frac{3}{5}.$

EX 4:

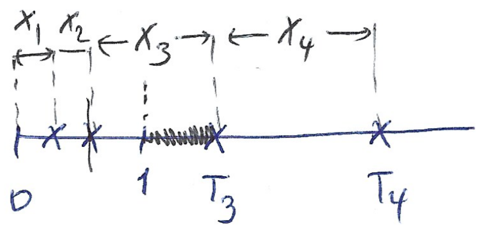
$N(t) \sim PP(\lambda)$

$X_i \sim ?$



a) $B[\text{4. müşt. geliş zamanı}] = B[T_4] = \frac{4}{\lambda}.$

$T_4 \sim Erlang(\lambda, n=4)$



b) $B[T_4 \mid N(1)=2] = 1 + B[\underbrace{T_3 - 1}_{\equiv X_3}] + B[X_4]$

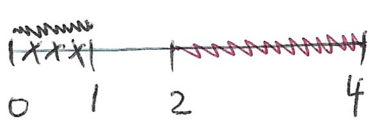
$= 1 + \frac{1}{\lambda} + \frac{1}{\lambda} = 1 + \frac{2}{\lambda}.$

$X_3 \sim exp(\lambda)$

$T_3 - 1 \mid T_3 > 1 \sim exp(\lambda)$

$P[T_3 - 1 > t \mid T_3 > 1] =$

c) $B[N(4) - N(2) \mid N(1) = 3]$



$\underset{\downarrow}{=} B[N(4) - N(2)] \underset{\downarrow}{=} B[N(2)] = 2\lambda$

neden? $\qquad$ Neden?

Üstel Hizmet Süresi

$X \sim exp(\lambda)$.

(hizmet hala devam ediyorsa)

Hizmete başladıktan s kadar sonra kalan hizmet süresinin dağılımı nedir?

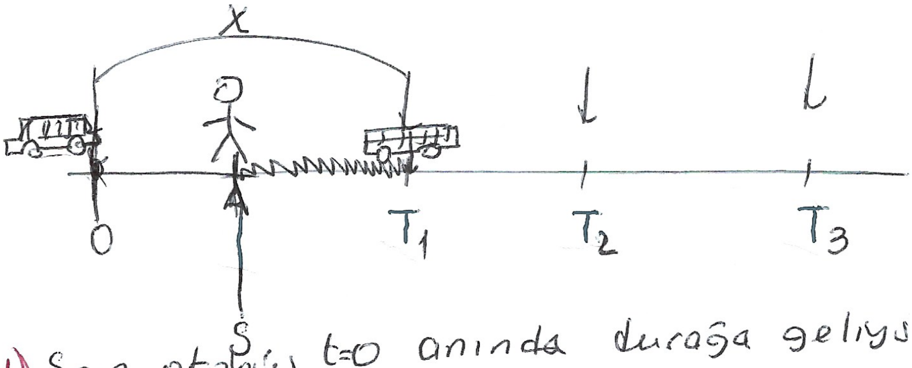
$$P(X > s+t \mid X > s) = P(X > t) = e^{-\lambda t}$$

$(X-s) > t$

$$(X-s \mid X > s) \equiv X \sim exp(\lambda)$$

üstel dağ memorilessness özelliği

Üstel Gelişlerarası Süreler



T_i : i. otobüsün durağa geliş zamanı

1) Son otobüs $t=0$ anında durağa geliyor-

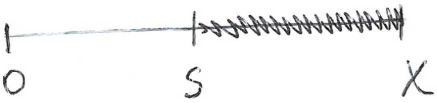
2) $t=s$ anında durağa geliyorum.

3) Sonraki otobüsün gelişine kadar geçen sürenin dağılımı nedir?

$$T_1 - s \mid T_1 > s \equiv T_1 = X \sim exp(\lambda).$$

Aynı → $X \sim exp(\lambda)$

→ $X - s \mid X > s \sim exp(\lambda)$



EX 5:

$X \sim exp(\lambda)$

$Y \sim exp(\mu)$

X, Y bağımsız r.d olsun.

SORU:

$P(X \leq Y) = ?$

Yanıt: * Y'nın değerini bilirsek cevap kolay!

$$P(X \leq y) = P(X \leq y) = 1 - e^{-\lambda y} \quad \left(\underset{y>0}{y: \text{sabit}} \right)$$

O zaman:
Y'ye göre koşullandıralım:

$$P(X \leq Y \mid Y = y) = P(X \leq y) = F_X(y) = 1 - e^{-\lambda y}$$

$$P(X \leq Y) = \int_0^\infty P(X \leq Y \mid Y = y) \underbrace{P(Y \cong y)}_{= f_Y(y)} dy$$

$$= \int_0^\infty P(X \leq y) f_Y(y) dy = \int_0^\infty (1 - e^{-\lambda y}) \mu e^{-\mu y} dy$$

$$= \int_0^\infty \mu e^{-\mu y} dy - \int_0^\infty \mu e^{-(\lambda+\mu) y} dy$$

$$= ?$$