NUMERICAL METHODS

Week-5
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Interpolation

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Interpolation

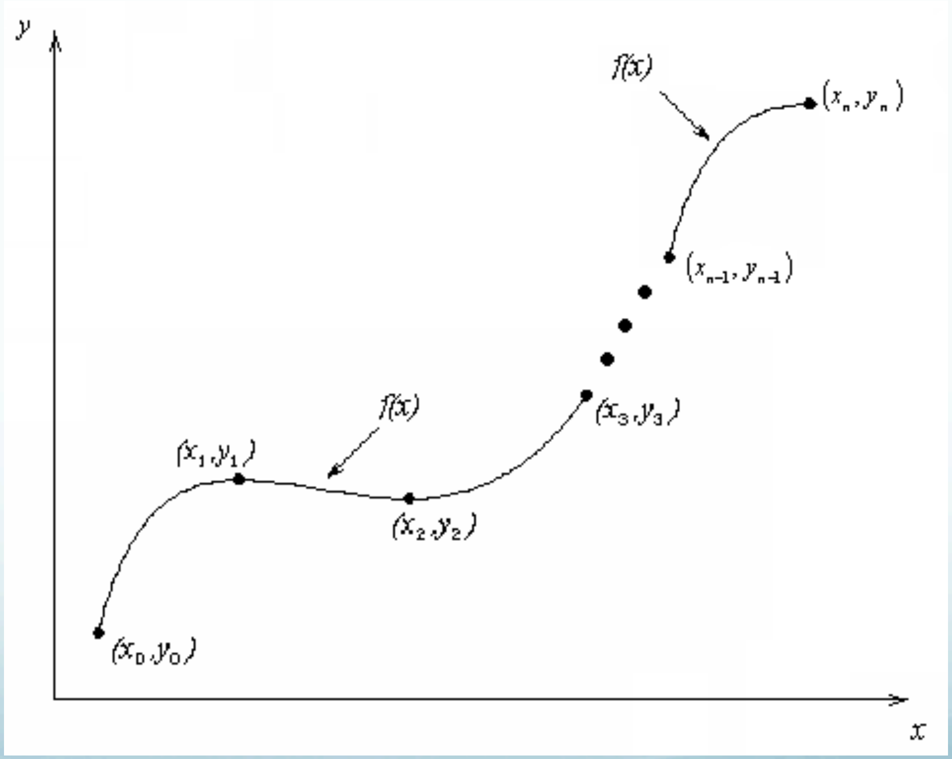
- **What** is an interpolation?
- **Why** do we use interpolation?
- **How** do we solve interpolation equations?

What is an Interpolation?

- Interpolation is the method of estimating unknown values with the help of given set of observations.
- The art of reading between the lines of the table.
- Having some number of points which are obtained by experiments or physical phenomenon forms the function of this experiment or phenomenon. The interpolation is to represent this phenomenon by a function and estimate (interpolate) the value of that function for an intermediate value within the range.
- (The process of computing the value of function outside the range of given values of the variable is called extrapolation.)

What is an interpolation ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



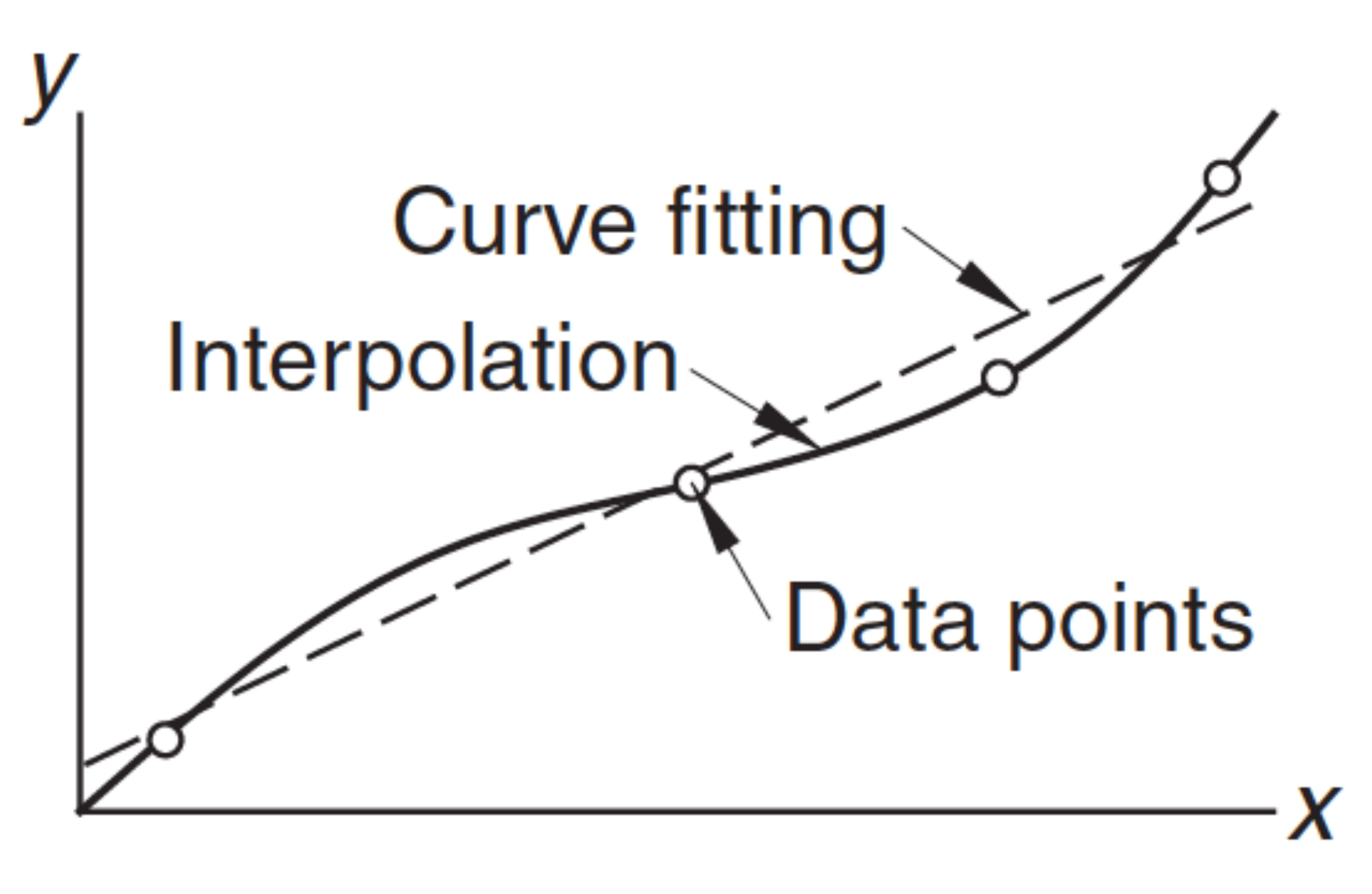
Why do we use Interpolation?

x_1	x_2	x_3	$\cdots$	x_n
y_1	y_2	y_3	$\cdots$	y_n

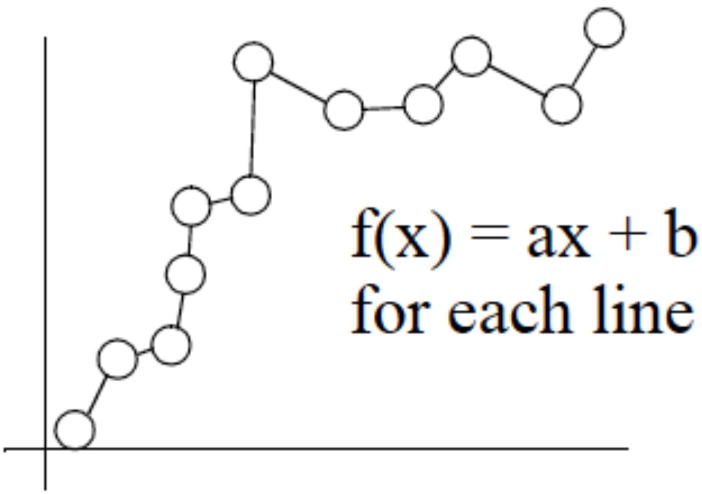
- In interpolation we **construct** a curve through **all** the data points. In doing so, we make the implicit assumption that the data points are accurate and distinct.

- Curve fitting is applied to data that contain scatter (noise), usually due to measurement errors. Here we want to find a smooth curve that **approximates the data in some sense**. Thus the curve does not have to hit the data points.

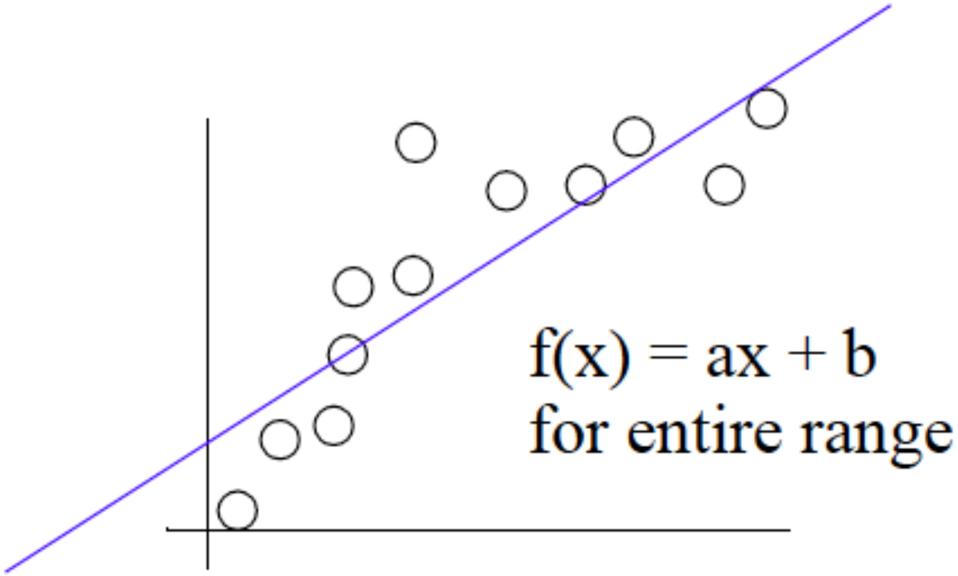
Difference between Interpolation and Curve fitting



Difference between Interpolation and Curve fitting



Interpolation



Curve Fitting

Why do we use Interpolation?

- Simply on every scientific problem
  - in which we have a set of experiments for a given range of data AND
  - IF we want to find out the function of this problem AND
  - IF we want to find a solution of this problem function for a specific value within the range.

How do we represent?

- Polynomial interpolation
  - Direct Method
  - Lagrange Method
  - Newton Method
  - …

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),………….. (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + \dots\dots\dots\dots\dots\dots + a_n x^n.$$

where a_0, a_1,………………. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example 1

The upward velocity of a rocket is given as a function of time in the table.

Find the velocity at t=16 seconds using the direct method for linear interpolation.

Table Velocity as a function of time.

$t, (\text{s})$	$v(t), (\text{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

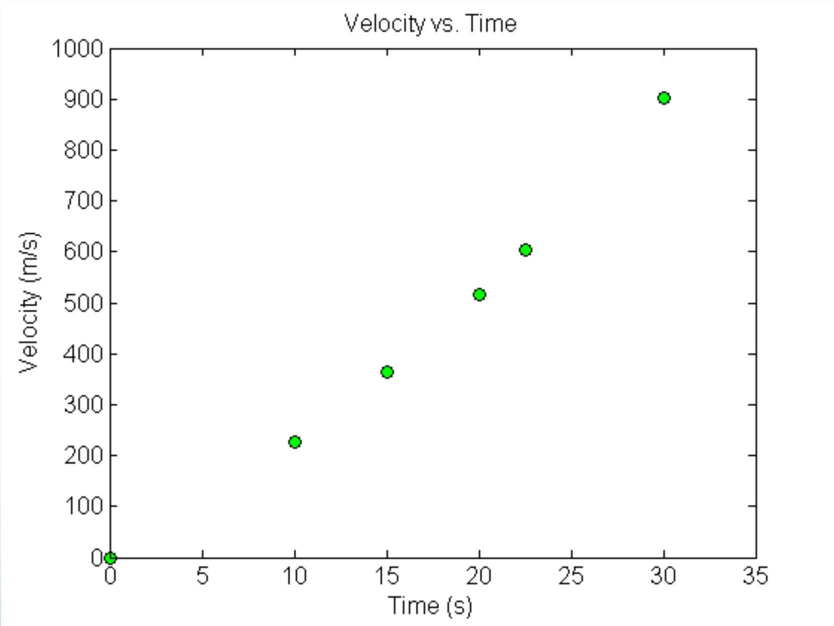


Figure: Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

Solving the above two equations gives,

$$a_0 = -100.93 \qquad a_1 = 30.914$$

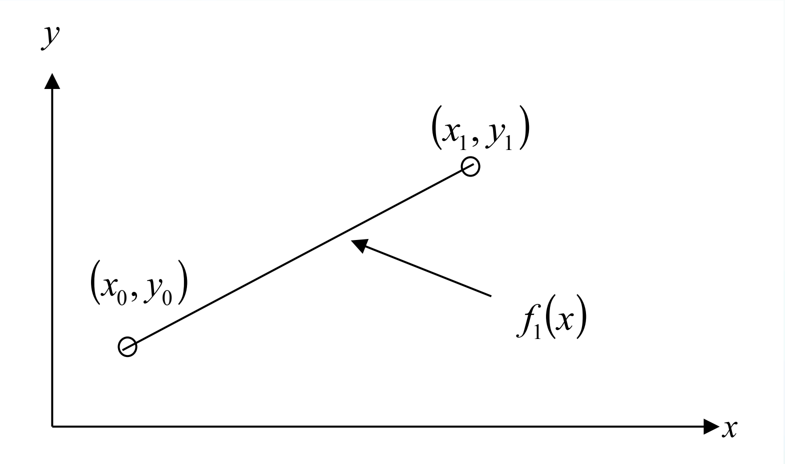


Figure Linear interpolation.

Hence

$$v(t) = -100.93 + 30.914t,\ 15 \le t \le 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

Example 2

The upward velocity of a rocket is given as a function of time in Table.

Find the velocity at t=16 seconds using the direct method for quadratic interpolation.

Table: Velocity as a function of time.

$t,(\mathrm{s})$	$v(t),(\mathrm{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

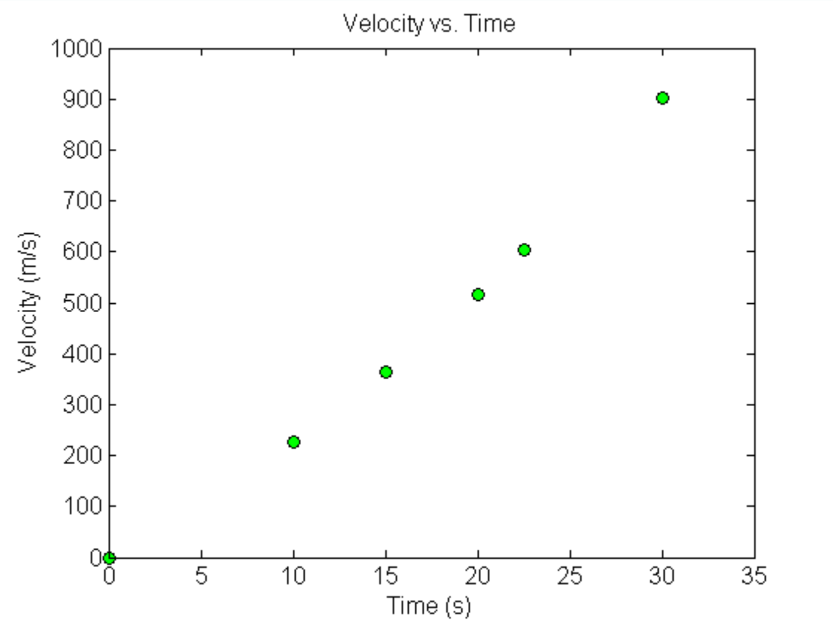


Figure Velocity vs. time data for the rocket example

Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$

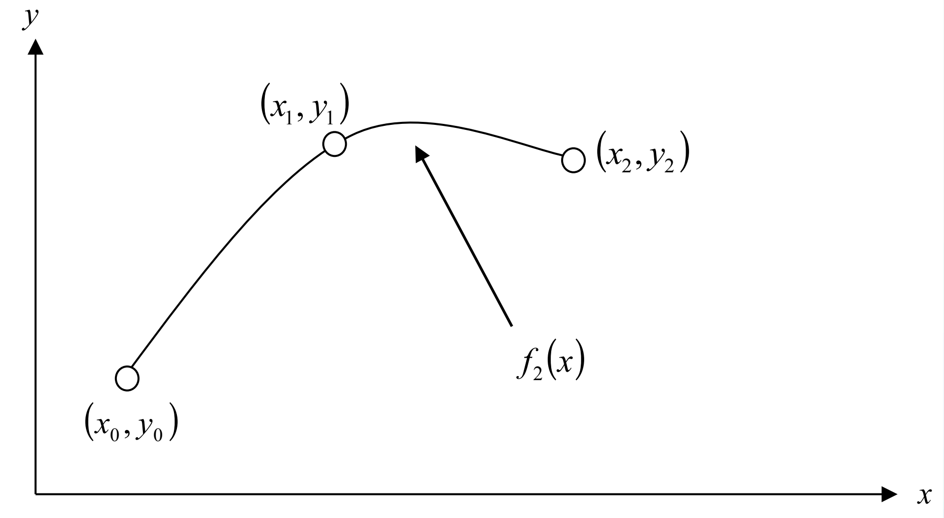


Figure Quadratic interpolation.

Solving the above three equations gives

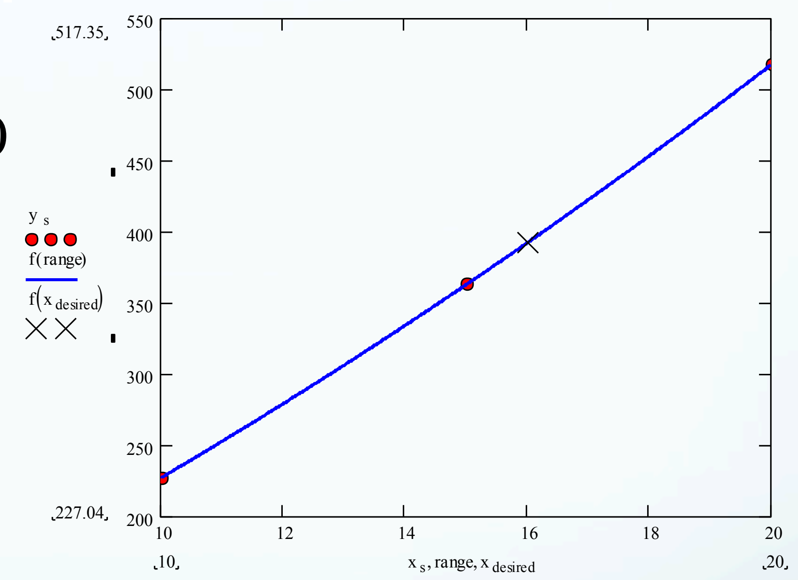
$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

Quadratic Interpolation (cont.)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \; 10 \le t \le 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$

$$= 392.19 \text{ m/s}$$



The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{392.19 - 393.70}{392.19} \right| \times 100$$

$$= 0.38410\%$$

Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

The upward velocity of a rocket is given as a function of time in the table. Find the velocity at t=16 seconds using the Lagrangian method for linear interpolation.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

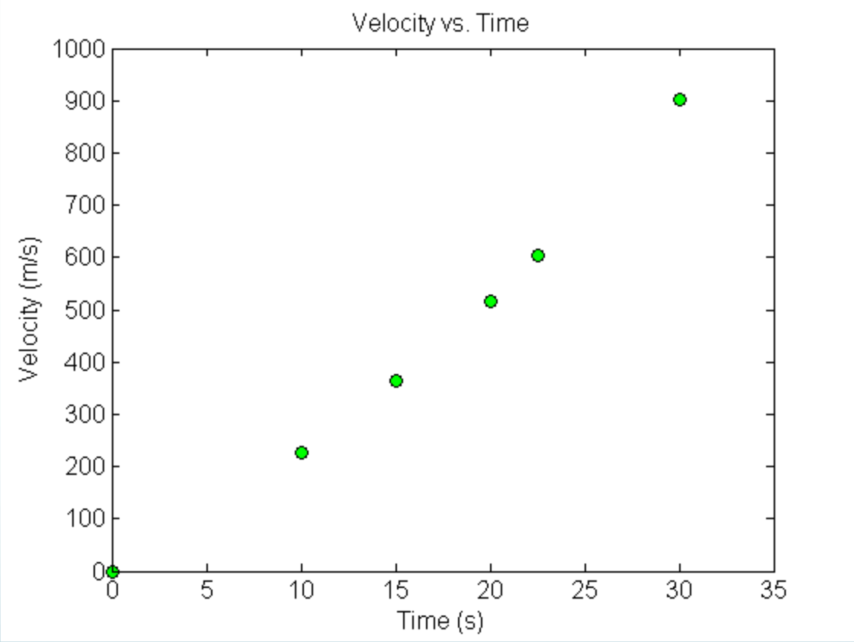


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = \sum_{i=0}^{1} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$

Linear Interpolation (contd)

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

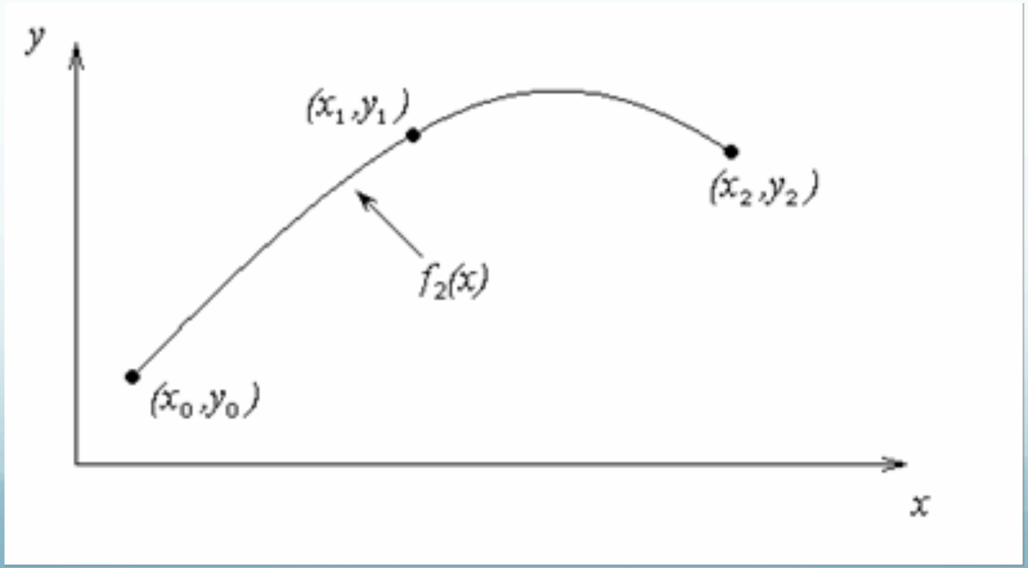
$$= 393.7 \text{ m/s.}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Quadratic Interpolation

$$t_0 = 10,\ v(t_0) = 227.04$$

$$t_1 = 15,\ v(t_1) = 362.78$$

$$t_2 = 20,\ v(t_2) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t - t_j}{t_0 - t_j} = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

Quadratic Interpolation (contd)

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2)$$

$$v(16)=\left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)(227.04)+\left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)(362.78)+\left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)(517.35)$$

$$=(-0.08)(227.04)+(0.96)(362.78)+(0.12)(527.35)$$

$$=392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

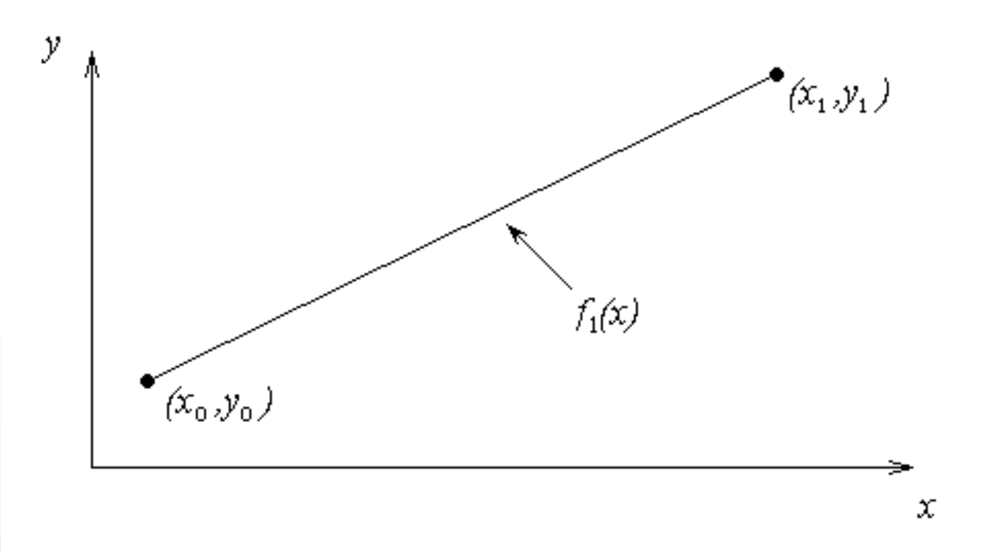
$$|\in_a|=\left|\frac{392.19-393.70}{392.19}\right|\times 100$$

$$=0.38410\%$$

Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), (x_1, y_1)$, pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where $b_0 = f(x_0)$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example

The upward velocity of a rocket is given as a function of time in Table . Find the velocity at t=16 seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
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22.5	602.97
30	901.67

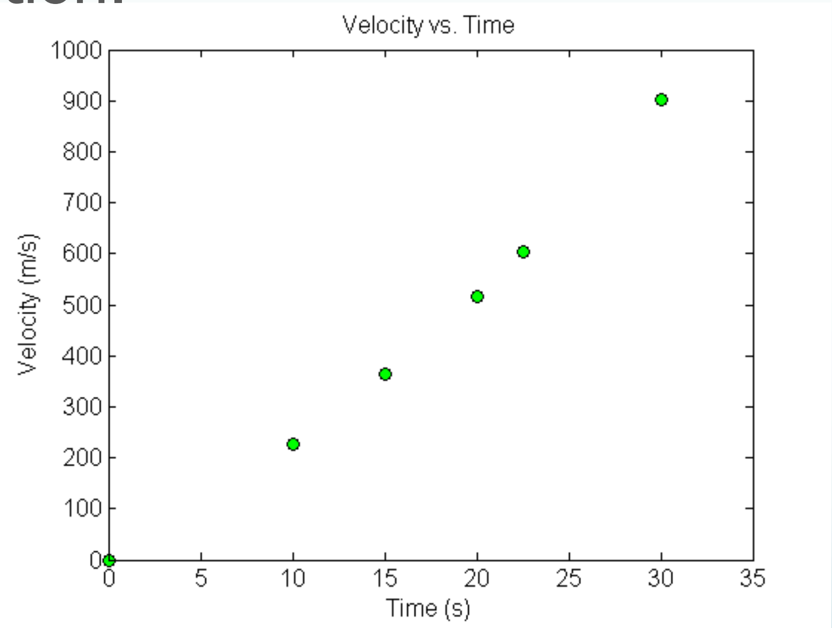


Figure. Velocity vs. time data for the rocket example

Linear Interpolation

$$v(t) = b_0 + b_1(t - t_0)$$

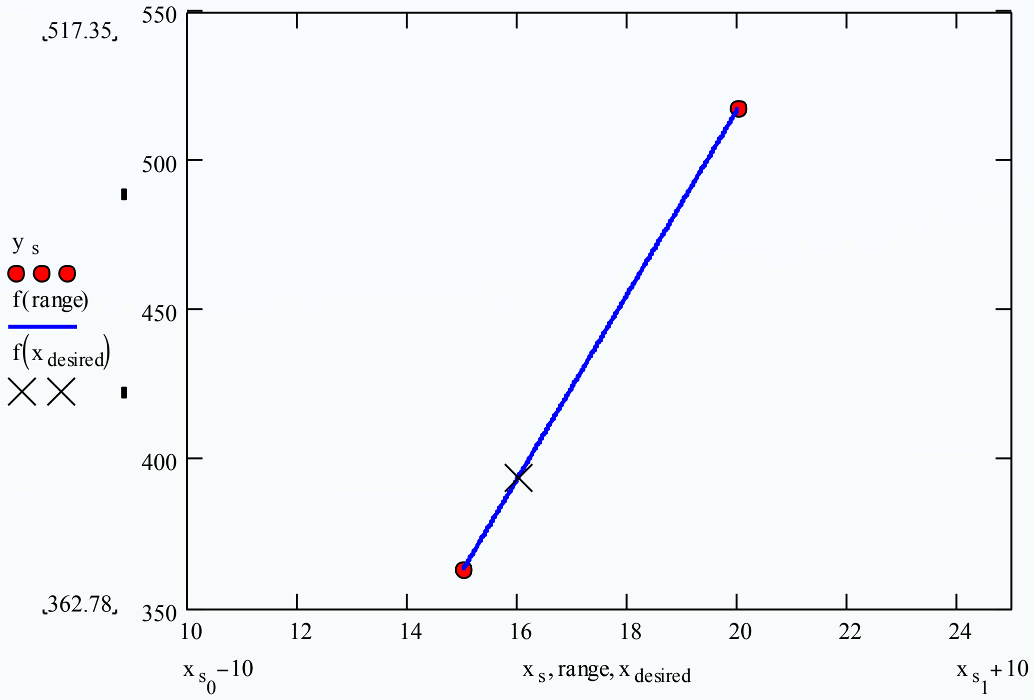
$$t_0 = 15, \ v(t_0) = 362.78$$

$$t_1 = 20, \ v(t_1) = 517.35$$

$$b_0 = v(t_0) \quad = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} \quad = 30.914$$

Linear Interpolation (contd)



$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Quadratic Interpolation

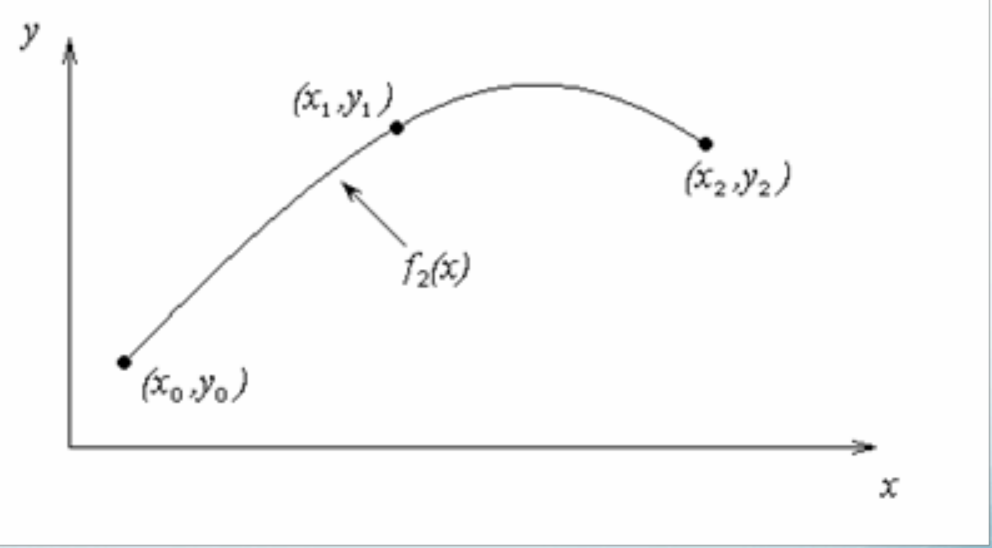
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Example

The upward velocity of a rocket is given as a function of time in the table . Find the velocity at t=16 seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

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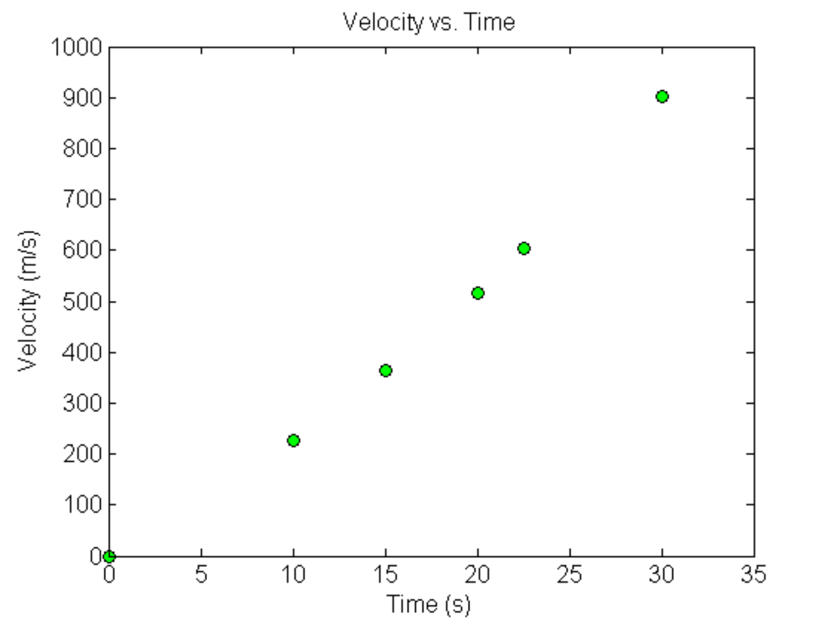
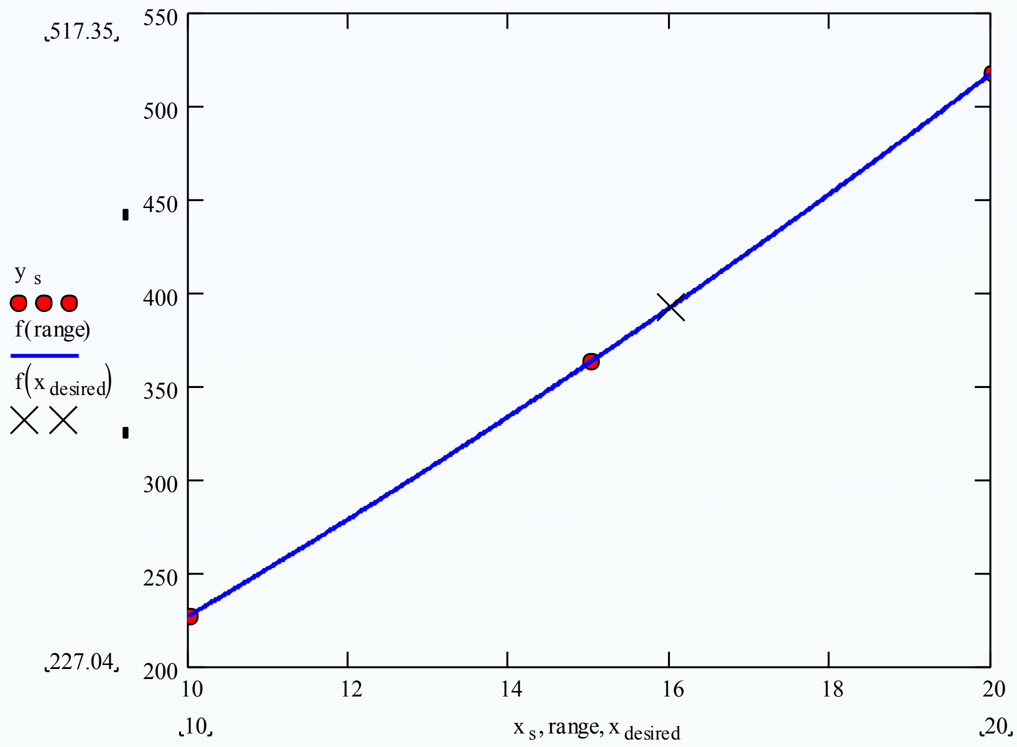


Figure. Velocity vs. time data for the rocket example

Quadratic Interpolation (contd)



$$t_0 = 10, \ v(t_0) = 227.04$$

$$t_1 = 15, \ v(t_1) = 362.78$$

$$t_2 = 20, \ v(t_2) = 517.35$$

Quadratic Interpolation (contd)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Quadratic Interpolation (contd)

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first order and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| \text{x}100$$

$$= 0.38502\ \%$$

General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

Divided Differences

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

General Form

Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\ + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

Divided Difference Table

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example to find divided differences and the Newton interpolation

- Construct a divided-difference diagram for the function f given in the following table, and write out the Newton form of the interpolating polynomial.

x	1	$\frac{3}{2}$	0	2
$f(x)$	3	$\frac{13}{4}$	3	$\frac{5}{3}$

The first entry for the table:

$$f[x_0, x_1] = \left(\tfrac{13}{4} - 3\right)/\left(\tfrac{3}{2} - 1\right) = \tfrac{1}{2}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{6} - \frac{1}{2}}{0 - 1} = \frac{1}{3}$$

The Complete Table:

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
1	3			
		$\frac{1}{2}$		
$\frac{3}{2}$	$\frac{13}{4}$		$\frac{1}{3}$	
		$\frac{1}{6}$		-2
0	3		$-\frac{5}{3}$	
		$-\frac{2}{3}$		
2	$\frac{5}{3}$			

We obtain:

$$p_3(x) = 3 + \tfrac{1}{2}(x-1) + \tfrac{1}{3}(x-1)\left(x-\tfrac{3}{2}\right) - 2(x-1)\left(x-\tfrac{3}{2}\right)x$$

Divided Differences Algorithm

integer i, j, n; **real array** $(a_{ij})_{0:n\times 0:n}, (x_i)_{0:n}$
for $i = 0$ **to** n **do**
    $a_{i0} \leftarrow f(x_i)$
end for
for $j = 1$ **to** n **do**
    for $i = 0$ **to** $n - j$ **do**
        $a_{ij} \leftarrow (a_{i+1,\, j-1} - a_{i,\, j-1})/(x_{i+j} - x_i)$
    end for
end for

Vandermonde Matrix

- **Theorem:** Every continuous function in the function space can be represented **as a linear combination of basis functions,** just as every vector in a vector space can be represented as a linear combination of basis vectors. Therefore,
- An Interpolating function f(x) can be represented by a set of basis functions φ_i for i=1,2,..,n.

$$f(x_i) = c_0\varphi_0(x_i) + c_1\varphi_1(x_i) + c_2\varphi_2(x_i) + \cdots + c_n\varphi_n(x_i) = y_i$$

Vandermonde Matrix

- For each i=1,2,...,n This is a system of linear equations→ we can represent in matrix form:

$$\mathbf{Ac} = \mathbf{y}$$

Here, A is the coefficient matrix with entries $a_{ij}=\varphi_i(x_i)$.

Monomials are the simplest and most common basis functions. The monomials:

$$\varphi_0(x) = 1, \varphi_1(x) = x, \varphi_2(x) = x^2, \ldots, \varphi_n(x) = x^n$$

Vandermonde Matrix

Consequently, a given polynomial p can be the linear combination of monomials as:

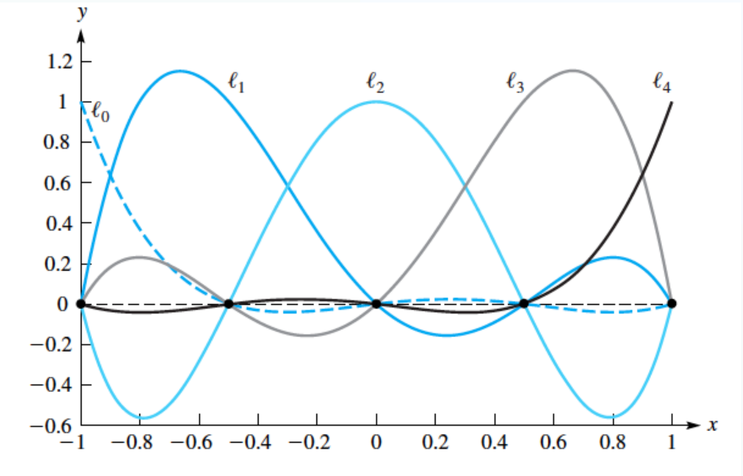
$$p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

The corresponding linear system Ac=y has the form:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

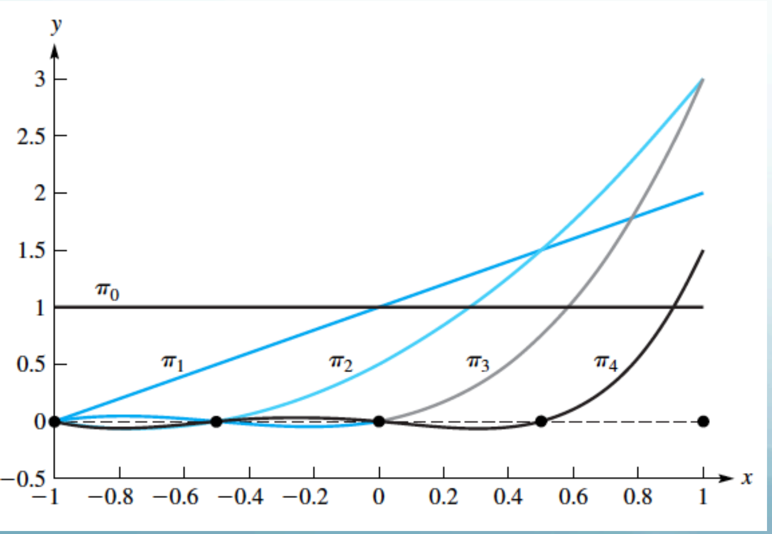
The coefficient matrix is called Vandermonde Matrix

Some basis functions we have just seen..



Lagrange Polynomials

Newton Polynomials



Monomials