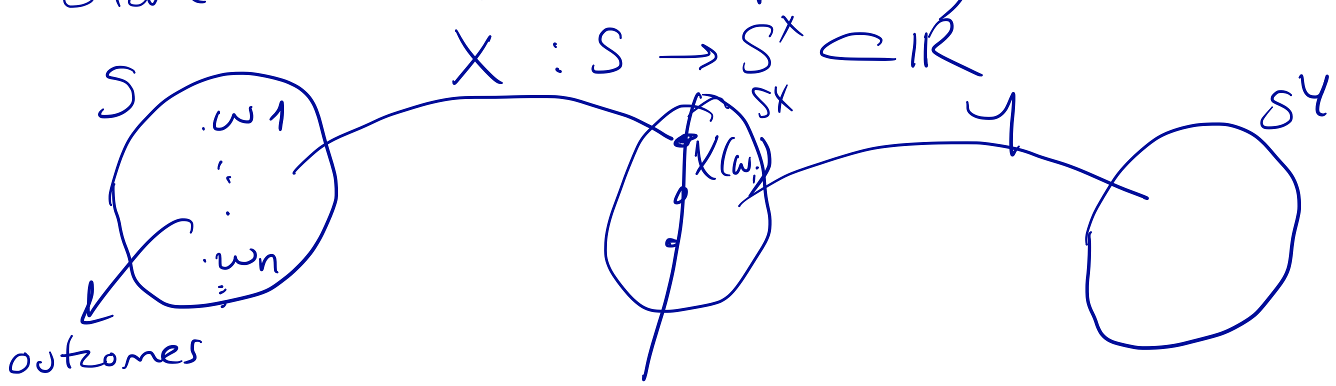
KOM505E Week 5

Lecture Notes

25.10.2016

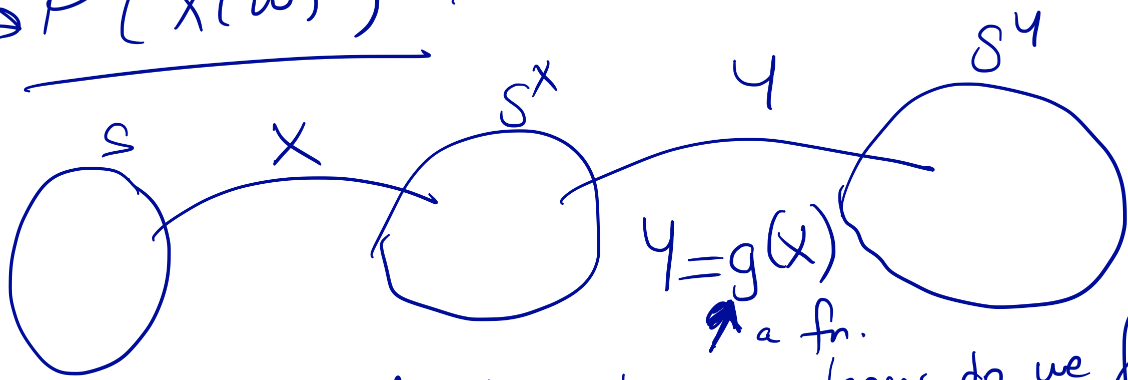
G.U.

Start with a random experiment; w/ sample space S



Probability:

$X(\omega) \to P(X(\omega))$:



Q: If the distrib. of X is known, how do we find distrib. of Y?

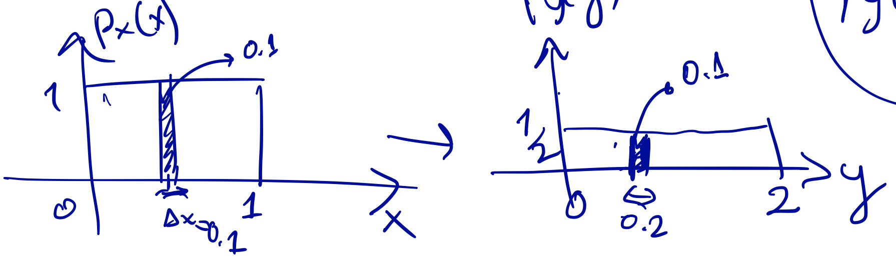
Discrete r.v.s: $P_Y[y_j] = \sum_{\forall i:\, y_j = g(x_i)} P_X[x_i]$

For cont. r.v.s; $P_X(x)\,|dx| = P_Y(y)\,|dy|$: change of variables formula

$\Rightarrow \quad P_Y(y) = P_X(x)\left|\frac{dx}{dy}\right|\Bigg|_{x=g^{-1}(y)}$

$$\boxed{P_Y(y) = P_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|}$$

$y = 2x$



Linear Xform of RV.s : $Y = aX + b$, $a\in\mathbb{R},\ a\neq 0$, $b\in\mathbb{R}$

$$p_Y(y) = p_X(g^{-1}(y))\left|\frac{d\,g^{-1}(y)}{dy}\right|$$

$$= p_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

$Y = g(X) = aX+b$

$X = \dfrac{y-b}{a}$

a: scale parameter. b: location parameter

$\star$ If the transform g is many-to-one : $Y = g(X)$

$$\boxed{p_Y(y) = \sum_{\substack{x\in g^{-1}(y)\\ i=1}}^{M} p_X(g_i^{-1}(y))\left|\frac{d g_i^{-1}(y)}{dy}\right|}$$

$Y = X^2 \Rightarrow x = \mp\sqrt{y}$

M: # elements in $g^{-1}(y)$.

Ex: $Y = X^2 = g(X)$; $\boxed{X \sim U[-1,1]} \rightarrow S^X = [-1,1]$

$\boxed{S^Y = [0,1]}$

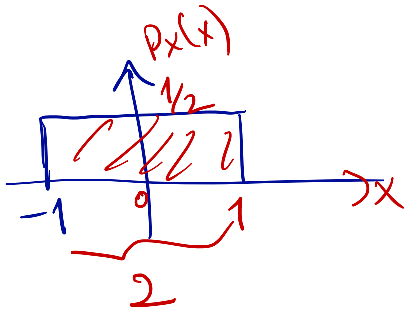
$x_1 = \sqrt{y} \Rightarrow \dfrac{dg^{-1}(y)}{dy} = \dfrac{1}{2\sqrt{y}}$

$x_2 = -\sqrt{y} \Rightarrow \dfrac{dg^{-1}(y)}{dy} = -\dfrac{1}{2\sqrt{y}}$

$$p_X(x) = \begin{cases} 1/2, & x\in[-1,1] \\ 0, & o/w \end{cases}$$

$$p_Y(y) = p_X(\sqrt{y})\left|\frac{1}{2\sqrt{y}}\right| + p_X(-\sqrt{y})\left|\frac{1}{-2\sqrt{y}}\right|$$

$$p_Y(y) = \frac{1}{2\sqrt{y}}\Big(\underbrace{p_X(\sqrt{y}) + p_X(-\sqrt{y})}_{\frac{1}{2}+\frac{1}{2}}\Big) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & y\in(0,1]\ \ \cancel{y>0} \\ 0, & y\leq 0 \end{cases}$$



As exercise: solve Ex 10.7, $Y = X^2$

$X \sim \mathcal{N}(0,1)$.

Ex 10.8 CDF Approach to determine PDF of a xformed r.v

eg. $Y = X^2$, $X \sim \mathcal{N}(0,1) \longrightarrow$ $-\infty < x < \infty$, $0 < y < \infty$

First determine the CDF of Y ito CDF of X

$$F_Y(y) = P(Y \le y) = P(\underbrace{X^2}_{=g(x)} \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Now, differentiate

$$p_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$$

$$= p_X(\sqrt{y}) \underbrace{\left(\frac{d\sqrt{y}}{dy}\right)}_{\frac{1}{2\sqrt{y}}} - p_X(-\sqrt{y}) \frac{d(-\sqrt{y})}{dy}$$

$$p_Y(y) = \begin{cases} p_X(\sqrt{y}) \frac{1}{\sqrt{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y^2/2}, & y > 0 \\ 0, & o/w. \end{cases}$$

Check w/ other nonlinear xforms ...

Ex: (2.6) Digital Communications :

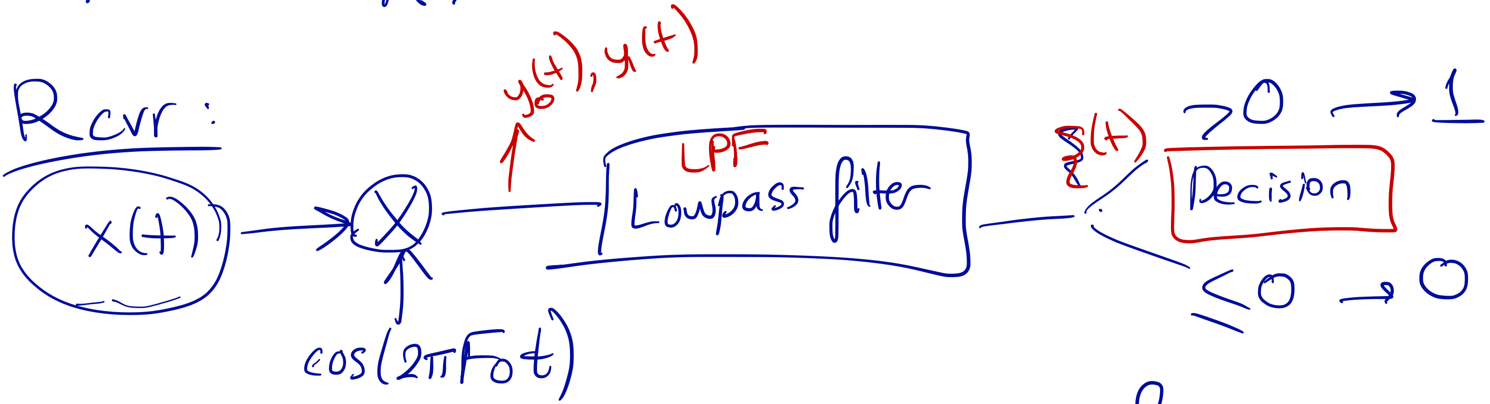
How to use Gaussian channel noise model in a communication problem to determine relation btw signal amplitude & prob. of error

Xmitter:

Xmit "0" $\Rightarrow s_0(t) = \underline{A}\cos(2\pi F_0 t + \pi)$ → represents a "0"

"1" $\Rightarrow s_1(t) = \underset{A>0}{\underline{A}}\cos(2\pi F_0 t)$ → represents a "1"

Rcvr:



$x(t) = s_i(t) + \underline{w(t)}$ ← channel noise

Ignore $w(t)$ for a moment:

$$s_0(t)\cos(2\pi F_0 t) = A\cos(2\pi F_0 t + \pi)\cos(2\pi F_0 t)$$
$$y_0(t) = -A\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi F_0 t)\right)$$

(Recall: Euler's formula: $\cos w_0 t = \dfrac{e^{jw_0 t} + e^{-jw_0 t}}{2}$)

$$s_1(t)\cos(2\pi F_0 t) = A\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi F_0 t)\right) = y_1(t)$$

LPF filters out $\cos(4\pi F_0 t)$ part of the signal (high freq.)

$$\xi(t) = \begin{cases} -A/2, & \text{for "0"} \\ A/2, & \text{for "1"} \end{cases}$$

$\Rightarrow$ Rcvr decides "1" if $\xi > 0$, "0" if $\xi < 0$.

To model channel noise:

$$\xi = \begin{cases} -\dfrac{A}{2} + w, & \text{for "0"} \\ \dfrac{A}{2} + w, & \text{for "1"} \end{cases}, \quad w \sim N(0,1)$$

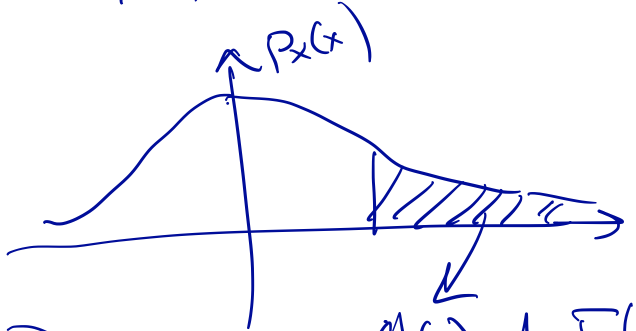
P(error) for the case "1" was xmitted.

$$P_e = P(\xi \leq 0) = P(\underbrace{\tfrac{A}{2}}_{\text{"1" xmitted}} + w \leq 0)$$

decision: "0"

$$= 1 - P\left(\frac{A}{2} + w > 0\right)$$

$$= 1 - P\left(w > -\frac{A}{2}\right)$$



$Q(x) = 1 - F_X(x)$

$\sim \text{erfc}(x)$

$$\underbrace{P_e = 1 - Q\left(-\frac{A}{2}\right) = Q\left(\frac{A}{2}\right)}$$

Can be calculated in matlab. (q.m routine in your textbook).

We can determine A to yield a given Pe.

e.g. $P_e = 10^{-7} \Rightarrow \frac{A}{2} = Q^{-1}(P_e)$

tells us how error depends on signal amplitude A.

(Note: add the error for the case "0" was xmitted).

Ex: Recall Probability Integral Xform:

If an r.v. is xformed by its cdf, the transformed r.v. $U \sim U[0,1]$ $U = F_X(x)$: xformed r.v. is a uniform r.v.

*Inverse Prob. Integral Xform: The xformation $X = F_X^{-1}(U)$

If an r.v. $X = F_X^{-1}(U)$, $U \sim U[0,1]$, then X has the pdf $p_X(x) = \frac{d}{dx} F_X(x)$.

Ex: Generate X w/ pdf:

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Generate $U[0,1]$ $\Longrightarrow$ Find CDF of X:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & o/w \end{cases}$$

$\Longrightarrow$ Apply inverse prob. integ. xform

$$u = F_X(x) = 1 - e^{-\lambda x} \Longrightarrow e^{-\lambda x} = 1 - u$$

$x = \,?\,(u)$

itö

$$x = -\frac{1}{\lambda} \ln(1-u)$$

Generate $u_1 \Longrightarrow x_1 = F_X^{-1}(u_1) = -\frac{1}{\lambda} \ln(1-u_1)$

$u_2 \Longrightarrow x_2 = F_X^{-1}(u_2) = -\frac{1}{\lambda} \ln(1-u_2)$

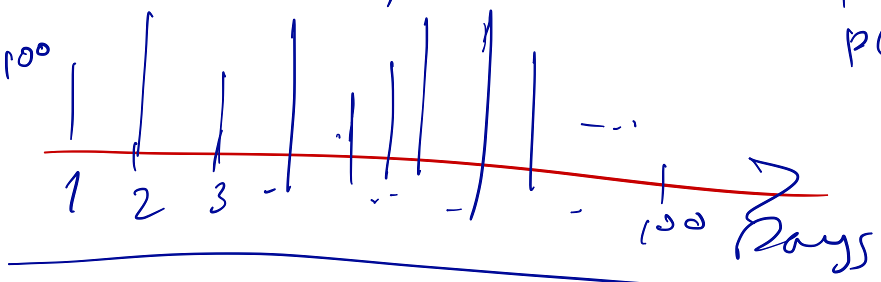
$\vdots$

Samples from the exponential distrib.

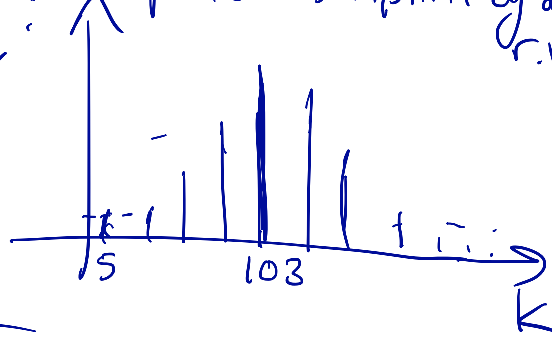
Exercise: Ex. 10.11. Laplacian r.v. generation.

Chap 6 & Chap 11 Expected Values of R.V.s.

download requests



pmf: complete description of an r.v.
pdf



For interpretation of probabilities:
you need a compact set of parameters from the pdf.

Expected Value of an R.V. (Def): Average value of outcomes of a large # experimental trials.

An r.v. w/ realizations $i = 1, \dots, N$

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \quad : \text{sample mean.}$$

Ex: (6.3)

$\left\{ \begin{array}{l} 1\$ \\ 5\$ \\ 10\$ \\ 20\$ \end{array} \right\}$ equal #.

To play you pay \$10.

Q: Will you make a profit by playing?

Expected winning per play

$$\frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 5 + \frac{1}{4}\cdot 10 + \frac{1}{4}\cdot 20 = \$9$$

No!

$$E_X[X] = \sum_i \underset{\text{bill value}}{x_i}\ \underbrace{p_X(x_i)}_{\text{prob. bill}}$$

6.4. Expected values of some r.v.s : (Discrete)

1. Bernoulli $X \sim Ber(p)$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

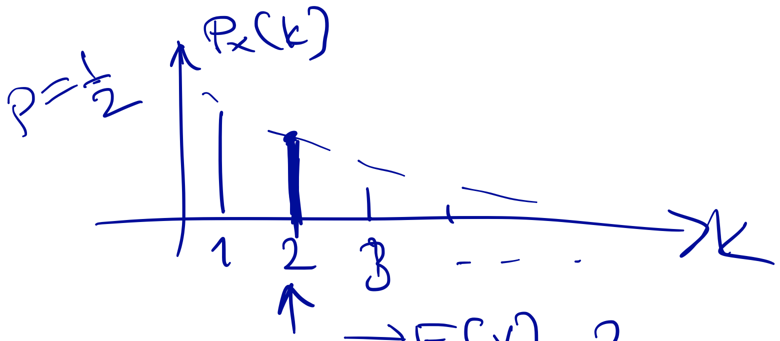
2. Binomial : $X \sim Bin(M, p)$

$$E[X] = \sum_{k=0}^{M} k \binom{M}{k} p^k (1-p)^{M-k} = M \cdot p$$

see the textbook for derivation

3. Geometric : $p_X(k) = (1-p)^{k-1} p$

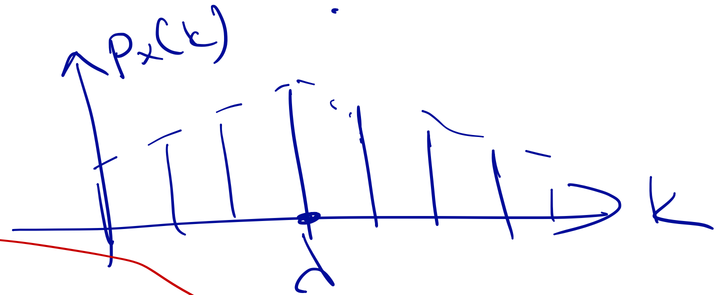
$$E[X] = \sum k (1-p)^{k-1} p = \frac{1}{p}$$



$\rightarrow E[X] = 2.$

4. Poisson : $p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, \ldots$

$$E[X] = \lambda$$

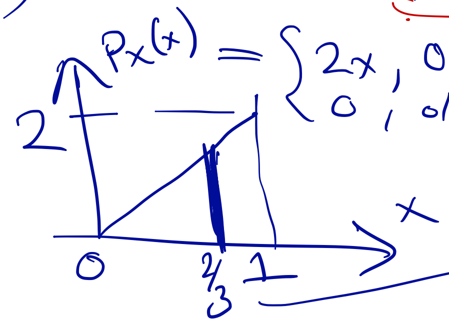


Ch. 11 Expectation (CT)

$$\boxed{E[X] = \int_{-\infty}^{\infty} x \, p_X(x) \, dx}, \quad p_X(x) : \text{pdf}.$$

ex: $p_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & o/w \end{cases}$

$$E[X] = \int_0^1 x \, 2x \, dx = \frac{2}{3}$$

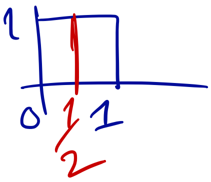


~ cm point.

$$\int_{-\infty}^{\infty} (x-\mu)\, p(x)\, dx = 0$$

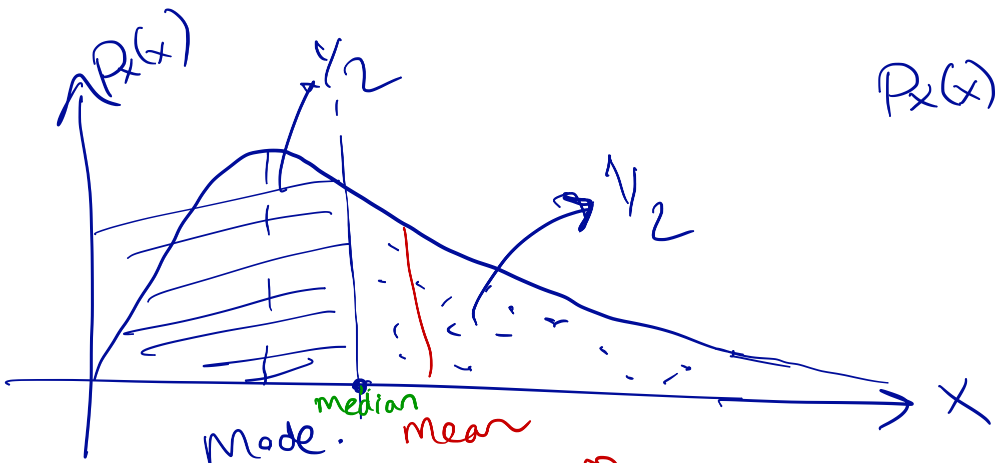
μ ↑ $E[X]$: center of mass

Ex: $X \sim U[0,1] \Rightarrow E[X] = ? = \int_0^1 x \cdot 1\, dx = \frac{1}{2}$



Note: In general: $\int_{-\infty}^{\mu} p_X(x)\, dx \neq \int_{\mu}^{\infty} p_X(x)\, dx$

$P_X(x) \propto x e^{-x^2}$



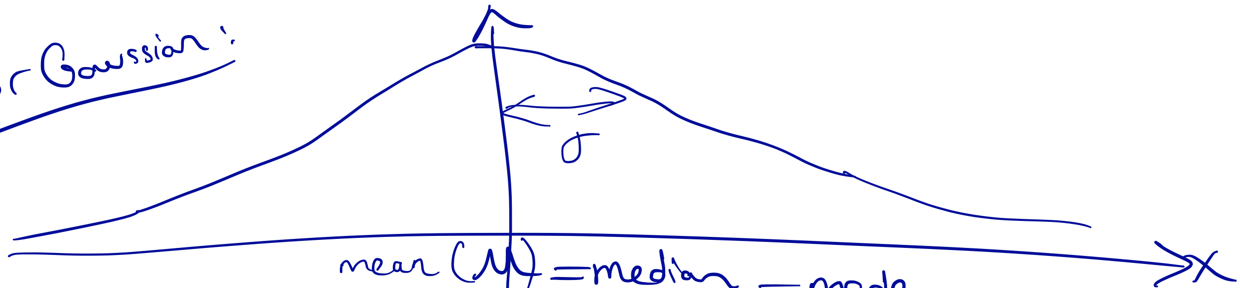
Def: mode

$$mode = \arg\max_x p_X(x)$$

$$\mu = \int_{-\infty}^{\infty} x\, p_X(x)\, dx \quad : \text{mean}$$

Median (Def): $\int_{-\infty}^{x_{med}} p_X(x)\, dx = \frac{1}{2} = F_X(x_{median})$

For Gaussian:



mean (μ) = median = mode

All 3 are equal

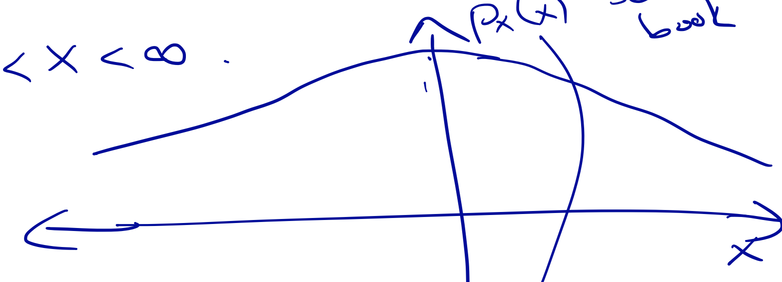
* Not all pdf's have expected values:

We should check : $\int |x| \, p_x(x) \, dx < \infty \Leftarrow$

$\Rightarrow E[|x|] < \infty$.

ex : $p_x(x) = \dfrac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$.

see the book



$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx \rightarrow \infty$$

$\int p_x(x) dx = 1$.

---

* If $Y = g(X) \Rightarrow E[Y] = ?$

$$= \int_{-\infty}^{\infty} y \, p_y(y) \, dy$$

we don't want to calculate $p_y(y)$ & this.

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \, p_x(x) \, dx}$$

rather use this.

* $Y = aX + b$: $E[\cdot]$: operator.

$$E[Y] = E[aX+b] = \int ax \, p_x(x) dx + \int b \, p_x(x) dx$$

$$= a E(X) + b$$

$\boxed{E[\cdot] \text{ is a linear operator}}$

Generally: $E[a g_1(x) + b g_2(x)] = a E[g_1(x)] + b E[g_2(x)]$

* Given the pdf, we'd like to predict the outcome of an experiment.

Q: What is the "best" guess (prediction) of the outcome of an r.v.? What is b?

$$E[(X-b)^2] = MSE = \text{Mean Squared error}$$

$$MSE = E[X^2 - 2bX + b^2]$$

$$MSE = E[X^2] - 2bE[X] + b^2$$

To find b, minimize MSE: $\frac{d}{db} MSE = -2E[X] + 2b = 0$

$$\Rightarrow b = E[X] \quad \text{: minimizer}$$

"best" predictor in the MSE sense is the expected value.

↳ What if $E[|X-c|] \Rightarrow c = ?$ median

pf: $\int_{-\infty}^{\infty} |x-c| \underbrace{p_X(x)\,dx}_{dF_X(x)} = \int_{-\infty}^{c} (c-x)\,dF_X(x) + \int_{c}^{\infty} (x-c)\,dF_X(x)$

$$\frac{d}{dc}\left(\quad \right) = \underbrace{\int_{-\infty}^{c} dF_X(x)}_{[F_X(c) - F_X(-\infty)]} - \underbrace{\int_{c}^{\infty} dF_X(x)}_{(F_X(\infty) - F_X(c))}$$

$$= 2F_X(c) - 1 = 0$$

$$\Rightarrow F_X(c) = \frac{1}{2}$$

c: median of the distribution "best" predictor in mean absolute error sense!

On board: we covered the following:

- $Var(X)$ & Properties of the variance.
- Moments of X.
- $E[X]$, $Var(X)$ for important pdf's.

Study those from your textbook.

You are responsible.

You may get the notes from your friends, who copied from the board.