KOMSOSE WEEK 7

15.11.2016

(Cont'd Joint R.V.s)

$$S^{X,Y} = \underbrace{S^X}_{} \times \underbrace{S^Y}_{}$$

## Independence of r.v.s :

All joint events in $S^{X,Y}$ are independent

$\Longrightarrow X, Y$ are indep.

event $A^X \subset S^X$

$A^Y \subset S^Y$

$$P(X \in A^X, Y \in A^Y) = P(X \in A^X) \cdot P(Y \in A^Y)$$

for all $A^X \subset S^X$, $A^Y \subset S^Y$.

$X, Y$ are indep. $\underset{\text{iff}}{\Longleftrightarrow} P_{X,Y}(x,y) = P_X(x)\, P_Y(y)$

$$F_{X,Y}(x,y) = F_X(x)\, F_Y(y).$$

Ex:

| $y \backslash x$ | 0 | 1 | |
|---|---|---|---|
| 0 | 1/8 | 1/8 | 1/4 |
| 1 | 1/4 | 1/2 | 3/4 |
| | 3/8 | 5/8 | |

$y=0$

$\left.\begin{matrix}1/4\\3/4\end{matrix}\right\} p_Y(y)$

2 coin toss experiment.

Are $X$ & $Y$ indep?

Not indep.

$$P_{X,Y}(x,y) \neq P_X(x)\, P_Y(y)$$

Ex: $P_{X,Y}(x,y) = e^{-(x+y)} \underbrace{u(x)}_{\text{Heaviside (step) fn}} u(y)$

indep.

$X, Y$ are $\exp(1)$

Ex: $p_{X,Y}(x,y) = \begin{cases} 2e^{-(x+y)}, & x \geq 0,\ y \geq 0,\ \boxed{y<x} \\ 0, & o/w \end{cases}$


$$= 2e^{-(x+y)} \cdot u(x)\,u(y)\,\underbrace{u(x-y)}_{\text{not factorizable.}}$$

X & Y are not independent.

Expected Values:

$\star$ $$\boxed{E_{X,Y}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} E_X[X] \\ E_Y[Y] \end{bmatrix}}$$

$\star$ $$E_{X,Y}[g(X)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, g(x)\, dx\, dy$$

$\star$ $$\boxed{E_{X,Y}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, g(x,y)\, dx\, dy}$$
Def:

$$= \int_{-\infty}^{\infty} g(x)\, dx \underbrace{\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy}_{p_X(x)} = E_X[g(X)]$$

$\star$ $$E_{X,Y}[a g_1(X,Y) + b g_2(X,Y)] = a E_{X,Y}[g_1(X,Y)] + b E_{X,Y}[g_2(X,Y)]$$

$\star$ If X & Y are independent:

$$E_{X,Y}[g(X)\, h(Y)] = E_X[g(X)]\, E_Y[h(Y)]$$

$$E_{X,Y}[XY] = E_X[X]\, E_Y[Y]$$ for indep. r.v.s X & Y.

**Variance of the Sum:** $\mu_x = E_x[X]$, $\mu_y = E_y[Y]$

$$Var(X+Y) = E[(X-\mu_x+Y-\mu_y)^2]$$
$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

**Covariance:**

$$\boxed{Cov(X,Y) = E_{X,Y}\left[(X-\mu_x)(Y-\mu_y)\right]}$$

a measure of how r.v.s covary w.r.t. each other.

* X & Y are uncorrelated : $Cov(X,Y)=0$

$$Cov(X,Y) = E_{X,Y}[XY] - \mu_x\mu_y$$

* Correlation Coefficient : $\rho = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

* If X & Y are independent $\Rightarrow Cov(X,Y)=0$

Pf: $E_{X,Y}[XY] = \int_{-\infty}^{\infty}\int \underbrace{p_{X,Y}(x,y)}_{p_X(x)\,p_Y(y)}\, x\,y\,dx\,dy$

$$= \int_{-\infty}^{\infty} x\,p_X(x)\,dx \int_{-\infty}^{\infty} y\,p_Y(y)\,dy$$
$$= E_X[X]\,E_Y[Y]$$

$$\Rightarrow Cov(X,Y)=0$$

Independence $\Rightarrow$ Uncorrelated

Independent $\not\Leftarrow$ Uncorrelated eg. R.V. $X$ is symmetrically distrib. around zero.

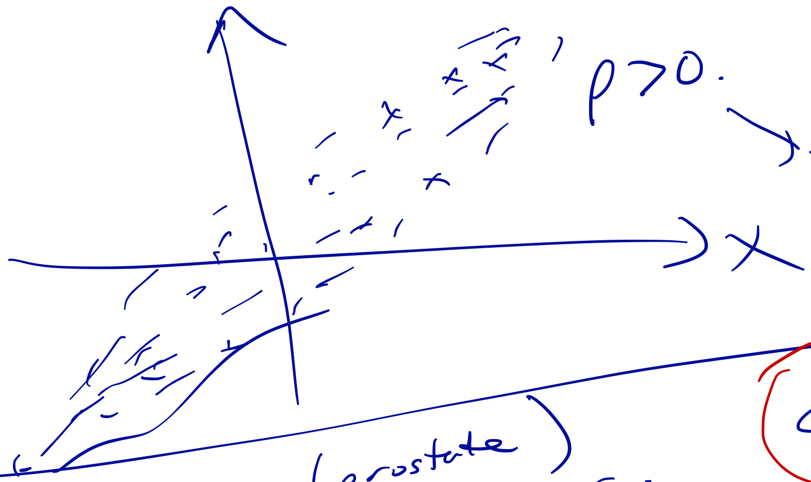
Let $Y = X^2 \Rightarrow X, Y$ are dependent.

but their correlation is zero $(\rho = 0)$.

$$\mathrm{Cov}(X,Y) = E_{x,y}[(X-\mu_x)(Y-\mu_y)]$$

with $\mu_x = 0$ and $\mu_y = 0$ marked above the two terms.

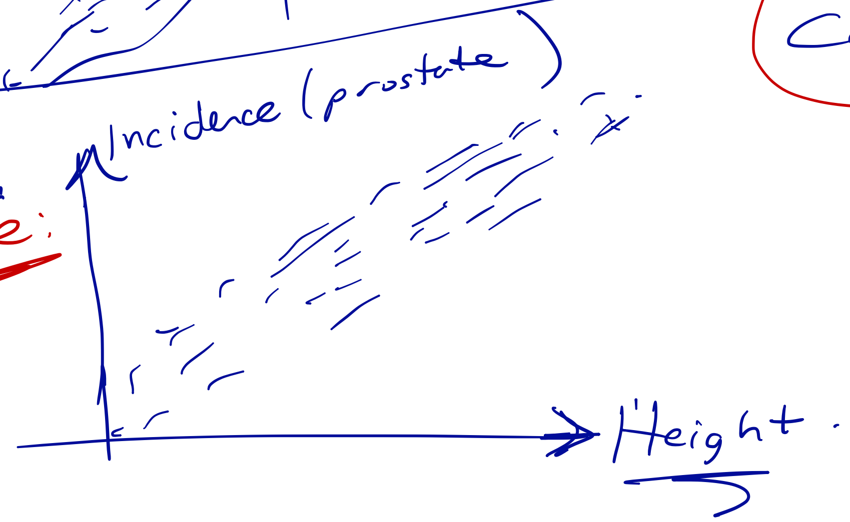
$$= E_{x,y}[X \cdot \underbrace{Y}_{X^2}] = \underbrace{E[X^3]}_{\text{for sym. distrib.}} = 0$$




$\rho > 0$. → +ve covariance.

Correlation vs. causation !!!

ex: Note:




2 r.v.s correlated does not imply one causes the other.

# Standard bivariate normal: (s.b.n.)

$$\boxed{p_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)\right\}}$$

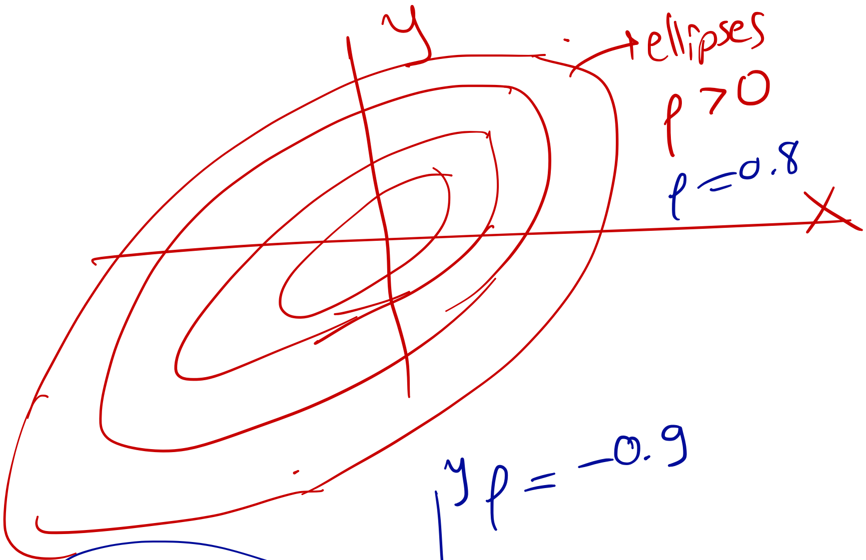
$\rho$ : correlation coeff. btw $X$ & $Y$

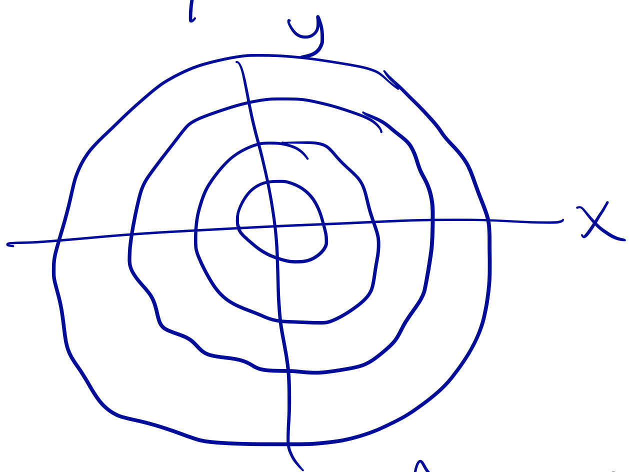
$$\rho=\frac{cov(X,Y)}{\sqrt{Var X}\sqrt{Var Y}}, \qquad -1<\rho<1$$

$$\Rightarrow \rho = Cov(X,Y) \quad \text{for s.b.n.}$$

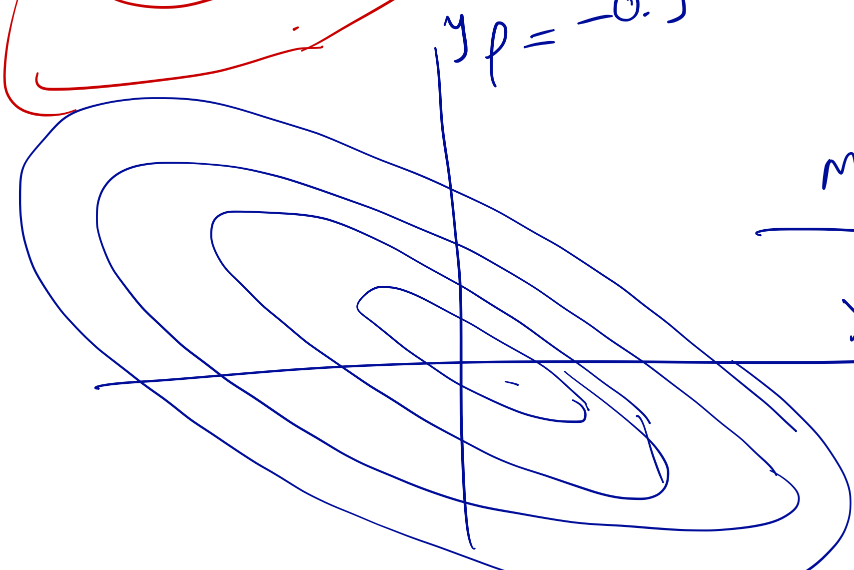
$$* \quad \boxed{\underbrace{x^2-2\rho xy+y^2}_{\text{ellipse eqn.}} = \underbrace{r^2}_{\text{const.}}} \quad \Rightarrow$$

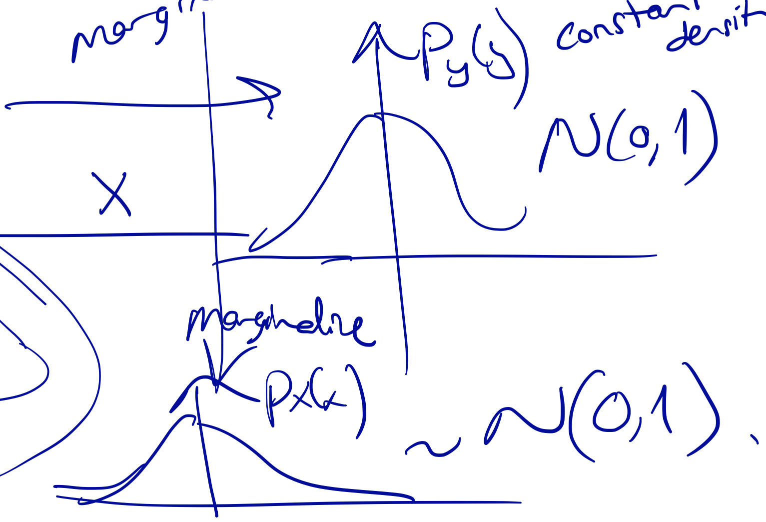
Look at 2D contours in the 2D space.




contours of constant density



$$x^2 - 2\rho xy + y^2 = \begin{bmatrix} x \\ y \end{bmatrix}^T \underbrace{\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}}_{Q} \begin{bmatrix} x \\ y \end{bmatrix}$$

$= x^T Q x$ : quadratic form.

$Q$ : symm. positive definite matrix.

— Marginals for s.b.n.

$$\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy \longrightarrow \mathcal{N}(0,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(See from the book how it is derived).

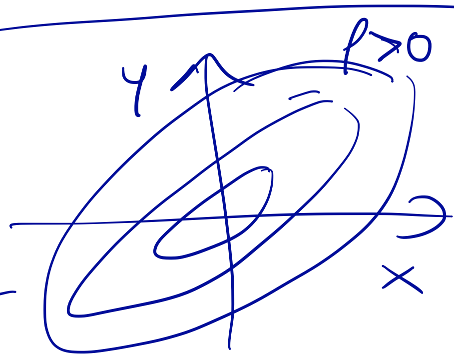
$\ast$ If $(X,Y) \sim$ s.b.n. $\Rightarrow$ $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ whatever $\rho$ is.

$\ast$ S.b.n. is independent when $\rho = 0$,

$p_{X,Y}$ becomes factorizable for //.

when $\rho \neq 0 \Rightarrow p_{X,Y}(x,y) \neq p_X(x)\, p_Y(y)$.

X & Y are dependent.

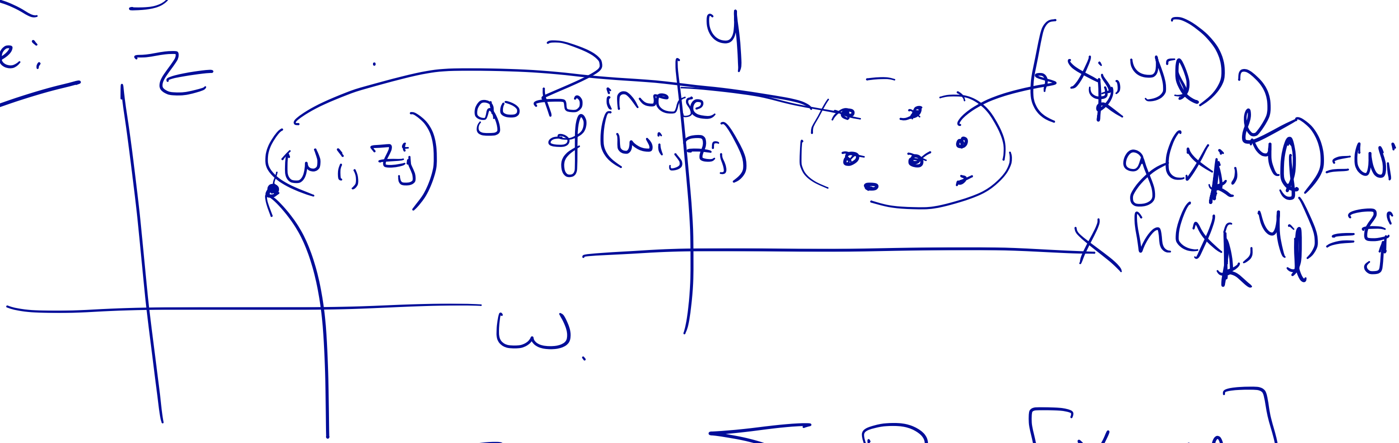


# Transformation of R.V.s :

$$\begin{bmatrix} X \\ Y \end{bmatrix} \longrightarrow \begin{bmatrix} W \\ Z \end{bmatrix}$$

$$\boxed{\begin{aligned} W &= g(X,Y) \\ Z &= h(X,Y) \end{aligned}}$$

**Discrete:**




$$P_{W,Z}[w_i, z_j] = \sum_{\substack{k,l: \\ \text{s.t.}}} P_{X,Y}[x_k, y_l]$$

$$\left\{ \begin{aligned} w_i &= g(x_k, y_l) \\ z_j &= h(x_k, y_l) \end{aligned} \right.$$

1) $\left. \begin{aligned} W &= X \\ Z &= g(X,Y) \end{aligned} \right\}$ $P_{W,Z}(w,z) \xrightarrow[\text{marginalize}]{} P_Z(z)$

**Ex:** $\left. \begin{aligned} W &= X \\ Z &= X+Y \end{aligned} \right)$ find the joint pmf $P_{W,Z}$

$$P_{W,Z}[i,j] = \sum_{k,l} P_{X,Y}[k,l] = \sum_{i,j} P_{X,Y}[i, j-i]$$

$k = i$

$\underset{(z)}{j} = k + l \Rightarrow \underline{l = j - i}$

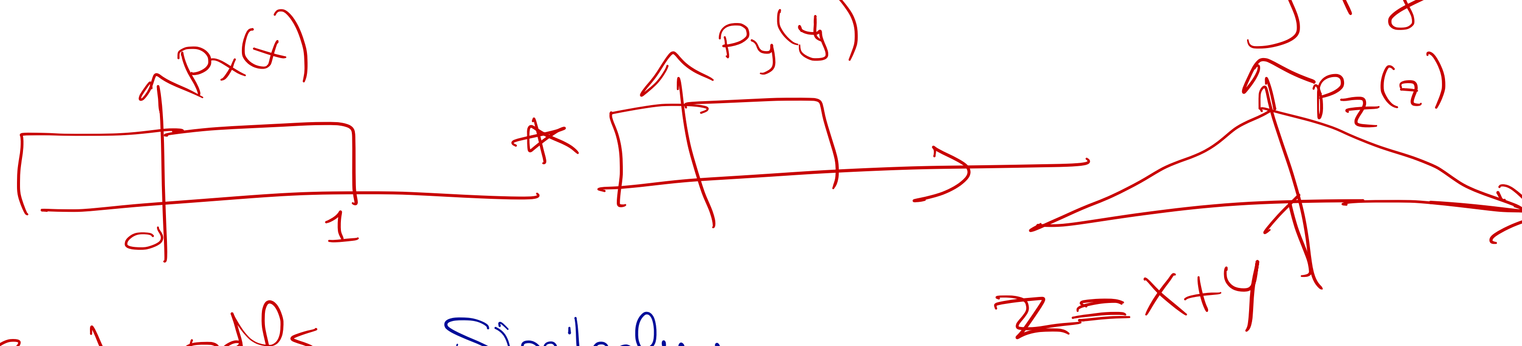
$$P_Z[j] = \sum_i P_{X,Y}[i, j-i]$$

# If $X$ & $Y$ are independent r.v.'s :

$$\boxed{Z = X + Y}$$

$$p_Z[j] = \sum_i p_X[i]\, p_Y[j-i]$$

convolution of $p_X$ & $p_Y$.




Cont. pdfs. Similarly:

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)\, p_Y(z-x)\, dx = p_X(x) * p_Y(y)$$

convolution.

Transforms:

(1) $Z = g(X,Y)$ :

$$F_Z(z) = P(Z \le z) = P(g(X,Y) \le z)$$

$$= \int_{\{(x,y):\, g(x,y) \le z\}} p_{X,Y}(x,y)\, dx\, dy$$

then $\Rightarrow p_Z(z) = \dfrac{d}{dz} F_Z(z)$ ✓

(2) Auxiliary variable method:

$W = X$
$Z = g(X,Y)$

1. Find $p_{W,Z}(w,z)$
2. Marginalize to find $p_Z(z)$.

$$p_Z(z) = \int_{-\infty}^{\infty} p_{W,Z}(w,z)\, dw$$

## (3) Generalization of the single r.v. xform:

Recall $p_Y(y) = p_X(g^{-1}(y)) \left| \dfrac{dg^{-1}(y)}{dy} \right|$

$p(y)dy = p_X(x)\,dx \Rightarrow p(y) = p_X(x) \left| \dfrac{dx}{dy} \right|$

$\boxed{w = g(x,y)\;;\; z = h(x,y)}$

$$\boxed{P_{w,z}(w,z) = P_{x,y}\left(g^{-1}(w,z), h^{-1}(w,z)\right) \left| \det(\;) \right| \qquad (\star) \quad \text{for both linear \& nonlinear transforms} \qquad \left| \det \frac{\partial(x,y)}{\partial(w,z)} \right|}$$

$$\underbrace{\frac{\partial(x,y)}{\partial(w,z)}}_{J^{-1}} = \begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial x}{\partial z} \\[2mm] \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial z} \end{bmatrix}$$

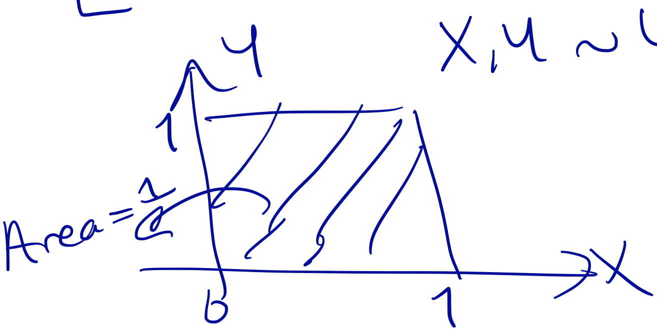
is the Jacobian matrix of the inverse transformation $[w, z] \to [x, y]$

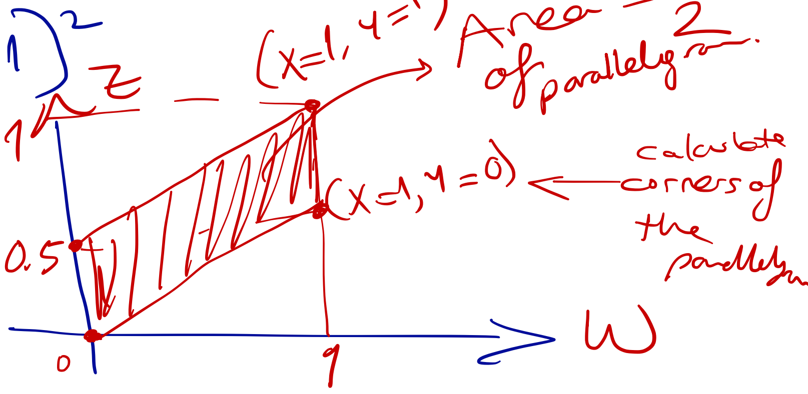
$J^{-1}$ : Jacobian of the inverse transform

**Ex:** A linear transform of the r.v.s

$$\begin{bmatrix} w \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{G} \begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{aligned} w &= X \\ z &= \tfrac{1}{2}X + \tfrac{1}{2}Y \end{aligned}$$

$X, Y \sim U[0,1]^2$




$$\frac{\text{area in } w\text{-}z \text{ plane}}{\text{area in } x\text{-}y \text{ plane}} = \frac{1}{2} = \det |G|$$

$\begin{pmatrix}\text{Support}\\ \text{Area}\end{pmatrix} = \begin{pmatrix}\text{Support region}\\ \text{where the pdf is nonzero}\end{pmatrix}$

is halved $\Rightarrow$ pdf height is doubled.

You can see it from the general transform formula ($\star$)

Ex: Linear transform for s.b.n.

$$\begin{bmatrix} w \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix}}_{G} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} 1/\sigma_w & 0 \\ 0 & 1/\sigma_z \end{bmatrix}, \quad G^{-1}\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} w/\sigma_w \\ z/\sigma_z \end{bmatrix}$$

$$\Rightarrow p_{w,z}(w,z) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{w}{\sigma_w}\right)^2 - 2\rho\left(\frac{w}{\sigma_w}\right)\left(\frac{z}{\sigma_z}\right) + \left(\frac{z}{\sigma_z}\right)^2\right)\right] \cdot \det G^{-1}$$

$$\det G^{-1} = \frac{1}{\sigma_w \sigma_z}$$

$\begin{pmatrix} -\infty < x < \infty \\ -\infty < y < \infty \end{pmatrix}$

Re-arrange the joint pdf:

$$\boxed{p_{w,z}(w,z) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2}\begin{bmatrix} w-\mu_w \\ z-\mu_z \end{bmatrix}^T C^{-1} \begin{bmatrix} w-\mu_w \\ z-\mu_z \end{bmatrix}\right)}$$

$$C = \begin{bmatrix} \sigma_w^2 & \rho\sigma_w\sigma_z \\ \rho\sigma_w\sigma_z & \sigma_z^2 \end{bmatrix} \triangleq \text{covariance matrix}$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$$

$$C = \begin{bmatrix} \sigma_w^2 & \mathrm{Cov}(W,Z) \\ \mathrm{Cov}(W,Z) & \sigma_z^2 \end{bmatrix}$$

Note: The marginals are

$W \sim N(\mu_w, \sigma_w^2)$

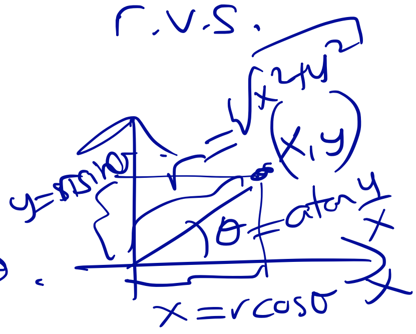
$Z \sim N(\mu_z, \sigma_z^2)$

## Ex 12.2

$$X \sim N(0,\sigma^2),\quad Y \sim N(0,\sigma^2) \;\Rightarrow\; X, Y \text{ indep. r.v.s.}$$

$$r = \sqrt{x^2+y^2},\quad r \geq 0 \longrightarrow$$

$$\theta = \operatorname{atan}\frac{y}{x},\quad 0 \leq \theta \leq 2\pi \longrightarrow$$




$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = r \geq 0.$$

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

$$p_{r,\theta}(r,\theta) = \frac{1}{2\pi\sigma^2} e^{-(r^2/2\sigma^2)} \cdot r$$

$$p_{r,\theta}(r,\theta) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \cdot \frac{1}{2\pi}$$

: $r, \theta$ are also independent

$$p_{r,\theta}(r,\theta) = p_r(r) \cdot p_\theta(\theta)$$

$r \sim$ Rayleigh distrib.

$\theta \sim$ Uniform distrib. $U[0, 2\pi]$.

**Example 12.11 :** Exercise

$$W = X,\quad \boxed{Z = Y/X}$$

$X, Y$ indep. $\sim N(0,1)$

$Z \sim$ Cauchy distributed.