

NUMERICAL METHODS

Week-3

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Linear Systems & Equations

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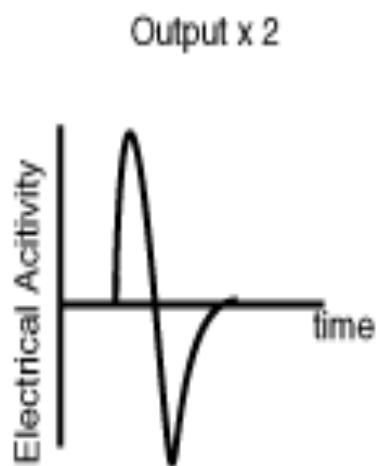
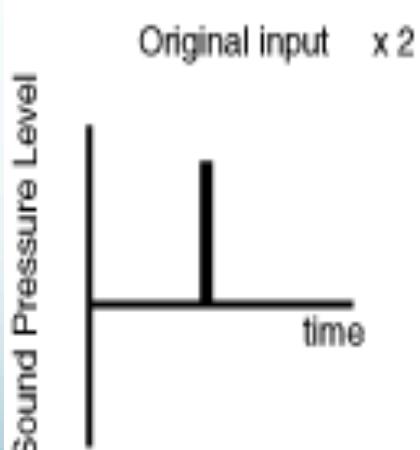
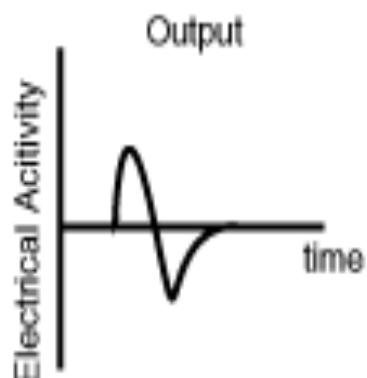
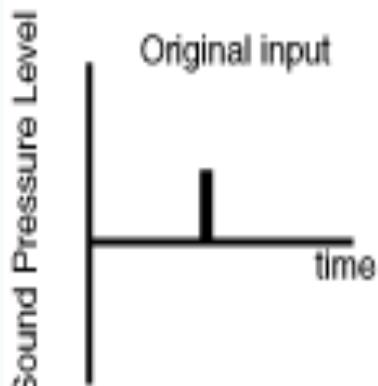
Linear System& Equation

- **What** is a Linear System/equation?
- **Why** do we use linear systems?
- **How** do we use/represent a linear system?

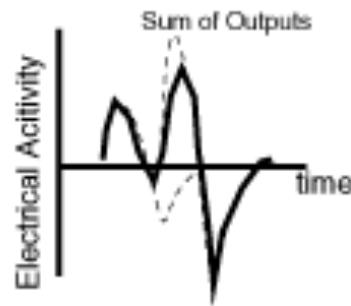
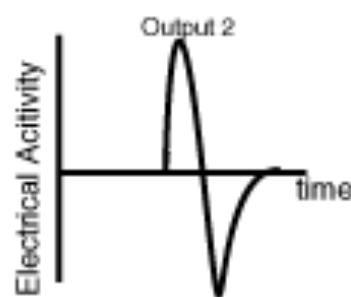
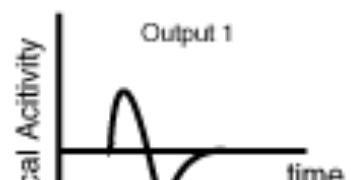
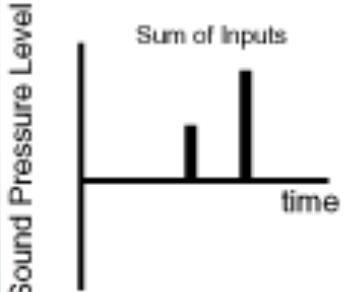
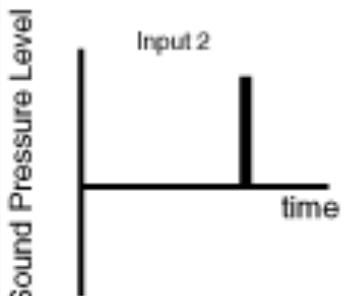
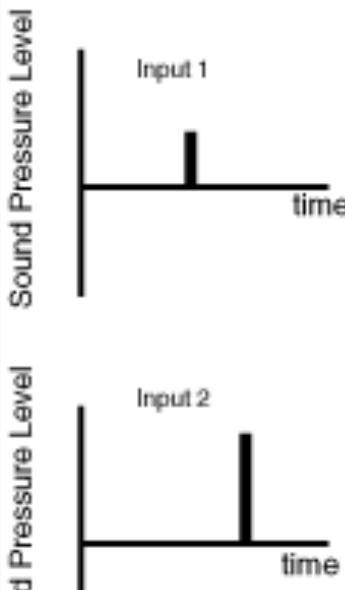
What is a Linear System?

- It is a mathematical model of a system which can be represented by a linear operator(scalar multiplication and addition)
- The basic property → SUPERPOSITION
 - Homogeneity (scalar rule)
 - Additivity

Scalar Rule



Additivity

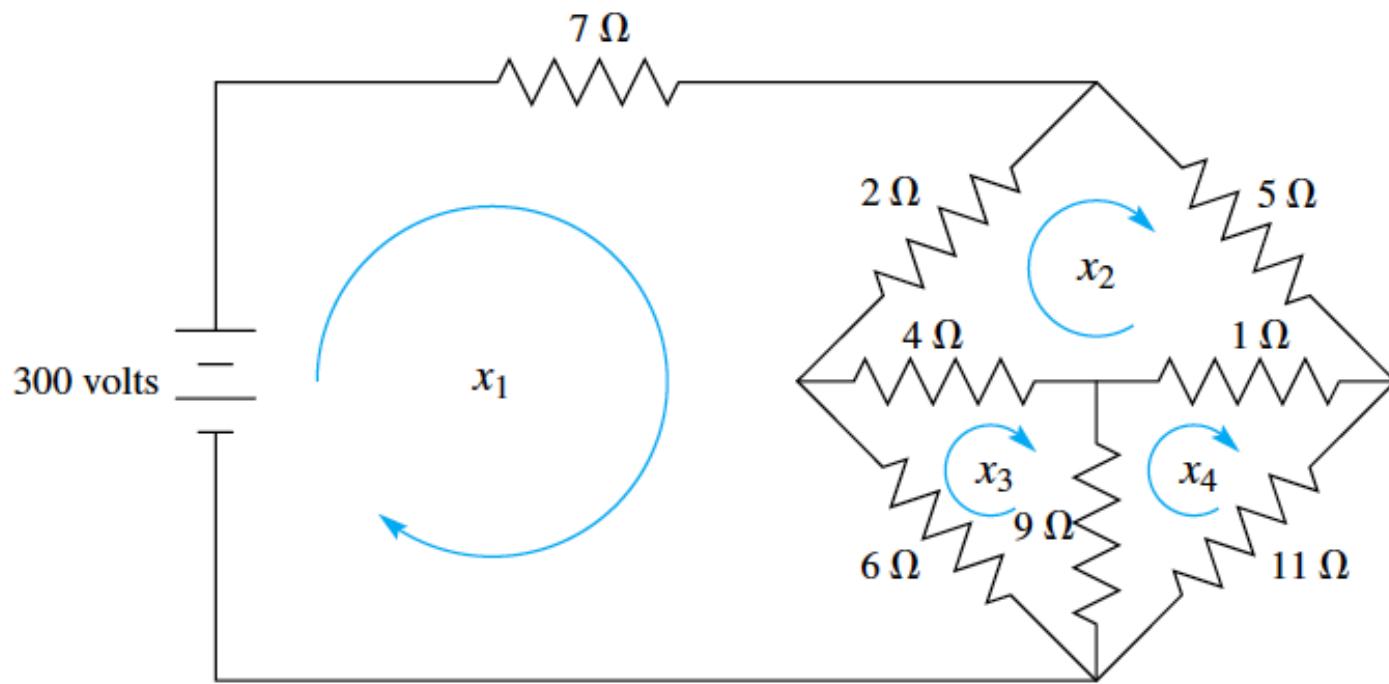


Why do we use Linear systems?

- To represent physical & chemical & electrical & electronical & mechanical phenomena..
- Wave Propagation
 - Sound and electromagnetic waves
- Electrical Circuits
 - Composed of resistors, capacitors, inductors
- Electronic Circuits
 - Amplifiers, filters
- Mechanical motions from the interaction of masses
- Recursion based models

How do we represent?

$$\begin{cases} 15x_1 - 2x_2 - 6x_3 = 300 \\ -2x_1 + 12x_2 - 4x_3 - x_4 = 0 \\ -6x_1 - 4x_2 + 19x_3 - 9x_4 = 0 \\ -x_2 - 9x_3 + 21x_4 = 0 \end{cases}$$



$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = b_i \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{array} \right.$$

Basic LS solving methods

- Naïve Gaussian Elimination
- LU Decomposition

Naïve Gaussian Elimination

A method to solve simultaneous linear equations of the form

$$[A][X]=[C]$$

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into an upper triangular matrix

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

↓

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

⋮
⋮
⋮

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

($n-1$) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21} .

$$\left[\frac{a_{21}}{a_{11}} \right] (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$\begin{array}{rcl} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n & = & b_2 \\ - \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n & = & \frac{a_{21}}{a_{11}}b_1 \\ \hline \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right)x_2 + \dots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \right)x_n & = & b_2 - \frac{a_{21}}{a_{11}}b_1 \end{array}$$

or $\overset{'}{a_{22}}x_2 + \dots + \overset{'}{a_{2n}}x_n = \overset{'}{b_2}$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$\dot{a_{22}}x_2 + \dot{a_{23}}x_3 + \dots + \dot{a_{2n}}x_n = \dot{b_2}$$

$$\dot{a_{32}}x_2 + \dot{a_{33}}x_3 + \dots + \dot{a_{3n}}x_n = \dot{b_3}$$

$$\begin{matrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{matrix}$$

$$\dot{a_{n2}}x_2 + \dot{a_{n3}}x_3 + \dots + \dot{a_{nn}}x_n = \dot{b_n}$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$\dot{a_{22}}x_2 + \dot{a_{23}}x_3 + \dots + \dot{a_{2n}}x_n = \dot{b_2}$$

$$^{''}a_{33}x_3 + \dots + ^{''}a_{3n}x_n = ^{''}b_3$$

$$\vdots$$

$$^{'''}a_{n3}x_3 + \dots + ^{'''}a_{nn}x_n = ^{'''}b_n$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$\dot{a_{22}}x_2 + \dot{a_{23}}x_3 + \dots + \dot{a_{2n}}x_n = \dot{b_2}$$

$$\ddot{a_{33}}x_3 + \dots + \ddot{a_{3n}}x_n = \ddot{b_3}$$

.

.

.

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$\dot{a_{22}}x_2 + \dot{a_{23}}x_3 + \dots + \dot{a_{2n}}x_n = \dot{b_2}$$

$$\ddot{a_{33}}x_3 + \dots + \ddot{a_n}x_n = \ddot{b_3}$$

. . .

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)}x_{i+1} - a_{i,i+2}^{(i-1)}x_{i+2} - \dots - a_{i,n}^{(i-1)}x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \dots, 1$$

Example

The upward velocity of a rocket is given at three different times

Table 1 Velocity vs. time data.

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2



The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3, \quad 5 \leq t \leq 12.$$

Find the velocity at $t=6$ seconds .

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3, \quad 5 \leq t \leq 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & : & 106.8 \\ 64 & 8 & 1 & : & 177.2 \\ 144 & 12 & 1 & : & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$$(n-1) = (3-1) = 2$$

Forward Elimination: Step 1

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 64 & 8 & 1 & : 177.2 \\ 144 & 12 & 1 & : 279.2 \end{array} \right]$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$[25 \ 5 \ 1 \ : \ 106.8] \times 2.56 = [64 \ 12.8 \ 2.56 \ : \ 273.408]$$

Subtract the result from Equation 2

$$\begin{array}{r} [64 \ 8 \ 1 \ : \ 177.2] \\ - [64 \ 12.8 \ 2.56 \ : \ 273.408] \\ \hline [0 \ -4.8 \ -1.56 \ : \ -96.208] \end{array}$$

Substitute new equation for Equation 2

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 0 & -4.8 & -1.56 & : -96.208 \\ 144 & 12 & 1 & : 279.2 \end{array} \right]$$

Forward Elimination: Step 1 (cont.)

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 0 & -4.8 & -1.56 & : -96.208 \\ 144 & 12 & 1 & : 279.2 \end{array} \right] \text{ Divide Equation 1 by 25 and multiply it by 144, } \frac{144}{25} = 5.76.$$

$$[25 \ 5 \ 1 \ : \ 106.8] \times 5.76 = [144 \ 28.8 \ 5.76 \ : \ 615.168]$$

Subtract the result from Equation 3

$$\begin{array}{r} [144 \ 12 \ 1 \ : \ 279.2] \\ - [144 \ 28.8 \ 5.76 \ : \ 615.168] \\ \hline [0 \ -16.8 \ -4.76 \ : \ -335.968] \end{array}$$

Substitute new equation for Equation 3

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 0 & -4.8 & -1.56 & : -96.208 \\ 0 & -16.8 & -4.76 & : -335.968 \end{array} \right]$$

Forward Elimination: Step 2

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 0 & -4.8 & -1.56 & : -96.208 \\ 0 & -16.8 & -4.76 & : -335.968 \end{array} \right]$$

Divide Equation 2 by -4.8
and multiply it by -16.8 ,
 $\frac{-16.8}{-4.8} = 3.5$

$$[0 \quad -4.8 \quad -1.56 \quad : \quad -96.208] \times 3.5 = [0 \quad -16.8 \quad -5.46 \quad : \quad -336.728]$$

Subtract the result from
Equation 3

$$\begin{array}{r} [0 \quad -16.8 \quad -4.76 \quad : \quad 335.968] \\ - [0 \quad -16.8 \quad -5.46 \quad : \quad -336.728] \\ \hline [0 \quad 0 \quad 0.7 \quad : \quad 0.76] \end{array}$$

Substitute new equation for
Equation 3

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & : 106.8 \\ 0 & -4.8 & -1.56 & : -96.208 \\ 0 & 0 & 0.7 & : 0.76 \end{array} \right]$$

Back Substitution

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.2 \\ 0 & 0 & 0.7 & 0.7 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 0 & -4.8 & -1.56 & -96.2 \\ 0 & 0 & 0.7 & 0.7 \end{array} \right] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned} a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25} \\ &= 0.290472 \end{aligned}$$

Naïve Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472t^2 + 19.6905t + 1.08571, \quad 5 \leq t \leq 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

Algorithm..

- Forward Elimination Phase

```
integer  $i, j, k$ ;  real array  $(a_{ij})_{1:n \times 1:n}, (b_i)_{1:n}$ 
for  $k = 1$  to  $n - 1$  do
    for  $i = k + 1$  to  $n$  do
        for  $j = k$  to  $n$  do
             $a_{ij} \leftarrow a_{ij} - (a_{ik}/a_{kk})a_{kj}$ 
        end for
         $b_i \leftarrow b_i - (a_{ik}/a_{kk})b_k$ 
    end for
end for
```

Forward Elimination Phase- Modified version

- Since the multiplier a_{ik}/a_{kk} does not depend on j , it should be moved outside the j loop.
- Notice also that the new values in column k will be 0, at least theoretically, because when $j = k$, we have

$$a_{ik} \leftarrow a_{ik} - (a_{ik}/a_{kk})a_{kk}$$

```
integer i, j, k;  real xmult;  real array (aij)1:n × 1:n, (bi)1:n
for k = 1 to n - 1 do
    for i = k + 1 to n do
        xmult ← aik/akk
        aik ← xmult
        for j = k + 1 to n do
            aij ← aij - (xmult)akj
        end for
        bi ← bi - (xmult)bk
    end for
end for
```

- Back Substitution Phase

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \vdots \\ a_{ii}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n = b_i \\ \vdots \vdots \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\ a_{nn}x_n = b_n \end{array} \right.$$

The back substitution starts by solving the n th equation for x_n :

$$x_n = \frac{b_n}{a_{nn}}$$

Then, using the $(n - 1)$ th equation, we solve for x_{n-1} :

$$x_{n-1} = \frac{1}{a_{n-1,n-1}} (b_{n-1} - a_{n-1,n}x_n)$$

We continue working upward, recovering each x_i by the formula

$$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=i+1}^n a_{ij}x_j \right) \quad (i = n-1, n-2, \dots, 1)$$

- Pseudocode of the back substitution

```
integer  $i, j, n$ ;  real  $sum$ ;  real array  $(a_{ij})_{1:n \times 1:n}, (x_i)_{1:n}$ 
 $x_n \leftarrow b_n/a_{nn}$ 
for  $i = n - 1$  to  $1$  step  $-1$  do
     $sum \leftarrow b_i$ 
    for  $j = i + 1$  to  $n$  do
         $sum \leftarrow sum - a_{ij}x_j$ 
    end for
     $x_i \leftarrow sum/a_{ii}$ 
end for
```

```

procedure Naive_Gauss( $n$ ,  $(a_{ij})$ ,  $(b_i)$ ,  $(x_i)$ )
integer  $i, j, k, n$ ;   real  $sum, xm$ 
real array  $(a_{ij})_{1:n \times 1:n}$ ,  $(b_i)_{1:n}$ ,  $(x_i)_{1:n}$ 
for  $k = 1$  to  $n - 1$  do
    for  $i = k + 1$  to  $n$  do
         $xm \leftarrow a_{ik}/a_{kk}$ 
         $a_{ik} \leftarrow xm$ 
        for  $j = k + 1$  to  $n$  do
             $a_{ij} \leftarrow a_{ij} - (xm)a_{kj}$ 
        end for
         $b_i \leftarrow b_i - (xm)b_k$ 
    end for
end for
 $x_n \leftarrow b_n/a_{nn}$ 
for  $i = n - 1$  to  $1$  step  $-1$  do
     $sum \leftarrow b_i$ 
    for  $j = i + 1$  to  $n$  do
         $sum \leftarrow sum - a_{ij}x_j$ 
    end for
     $x_i \leftarrow sum/a_{ii}$ 
end for
end procedure Naive_Gauss

```

Errors..

The **error** of a sample is the deviation of **the sample value** from **the true function value**.

The **residual** of a sample is the deviation **between the sample and the estimated function value**.

For a linear system $Ax = b$ having the true solution x and a computed solution \tilde{x} , we define

$$e = \tilde{x} - x \quad \text{error vector}$$

$$r = A\tilde{x} - b \quad \text{residual vector}$$

An important relationship between the error vector and the residual vector is

$$Ae = r$$

LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non-singular matrix $[A]$ that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear equations

If $[A] = [L][U]$ then

Multiply by

Which gives

Remember $[L]^{-1}[L] = [I]$ which leads to

Now, if $[I][U] = [U]$ then

Now, let

Which ends with

and

$$[A][X] = [C]$$

$$[L][U][X] = [C]$$

$$[L]^{-1}$$

$$[L]^{-1}[L][U][X] = [L]^{-1}[C]$$

$$[I][U][X] = [L]^{-1}[C]$$

$$[U][X] = [L]^{-1}[C]$$

$$[L]^{-1}[C] = [Z]$$

$$[L][Z] = [C] \quad (1)$$

$$[U][X] = [Z] \quad (2)$$

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

When is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Table. Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3}\right)$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination Time taken by LU Decomposition

$$= n(CT|_{FE} + CT|_{BS})$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT _{\text{inverse GE}} / CT _{\text{inverse LU}}$	3.28	25.83	250.8	2501

Method: [A] Decompose to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$[U]$ is the same as the coefficient matrix at the end of the forward elimination step.

$[L]$ is obtained using the *multipliers* that were used in the forward elimination process

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56$; $Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$\frac{144}{25} = 5.76$; $Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Finding the [U] Matrix

Matrix after Step 1:

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Step 2: $\frac{-16.8}{-4.8} = 3.5$; $Row3 - Row2(3.5) =$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step
of forward
elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second
step of forward
elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \quad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the $[L]$ and $[U]$ matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$

$$2.56z_1 + z_2 = 177.2$$

$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for [Z]

$$z_1 = 106.8$$

$$\begin{aligned} z_2 &= 177.2 - 2.56z_1 \\ &= 177.2 - 2.56(106.8) \end{aligned}$$

$$= -96.2$$

$$\begin{aligned} z_3 &= 279.2 - 5.76z_1 - 3.5z_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.21) \\ &= 0.735 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$0.7a_3 = 0.735$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using
the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned}a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\&= \frac{106.8 - 5(19.70) - 1.050}{25} \\&= 0.2900\end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Algorithm

```
integer  $i, k, n$ ;  real array  $(a_{ij})_{1:n \times 1:n}, (\ell_{ij})_{1:n \times 1:n}, (u_{ij})_{1:n \times 1:n}$ 
for  $k = 1$  to  $n$  do
     $\ell_{kk} \leftarrow 1$ 
    for  $j = k$  to  $n$  do
         $u_{kj} \leftarrow a_{kj} - \sum_{s=1}^{k-1} \ell_{ks} u_{sj}$ 
    end do
    for  $i = k + 1$  to  $n$  do
         $\ell_{ik} \leftarrow \left( a_{ik} - \sum_{s=1}^{k-1} \ell_{is} u_{sk} \right) / u_{kk}$ 
    end do
end do
```