Practice Questions for TEL252E

(1)

LTI system

if $x[n] = e^{j\omega n}$

then show that $y[n] = Ce^{j\omega n}$, where C is a complex number. Compute C.

Answer:

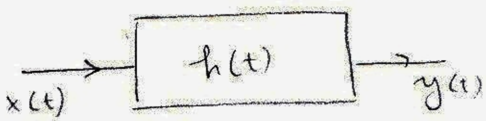
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega k}$$

and

$$C = \sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega k}$$

(2)



LTI system

Assume that

$$h(t) = \sin\left(\frac{2\pi}{T}t\right)u(t) \quad \text{and} \quad x(t) = u(t)$$

then compute $y(t)$

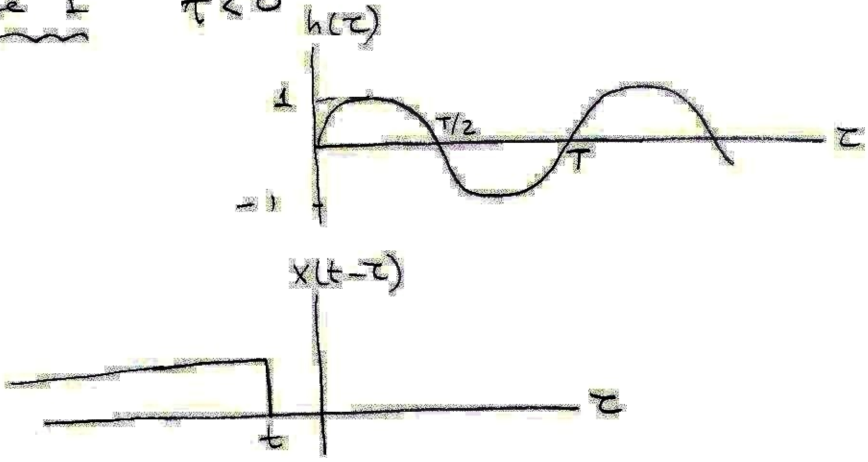
Answer:

$$y(t) = h(t) * x(t)$$

$h(t)$ is periodic in time

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Case 1 $\quad t<0$



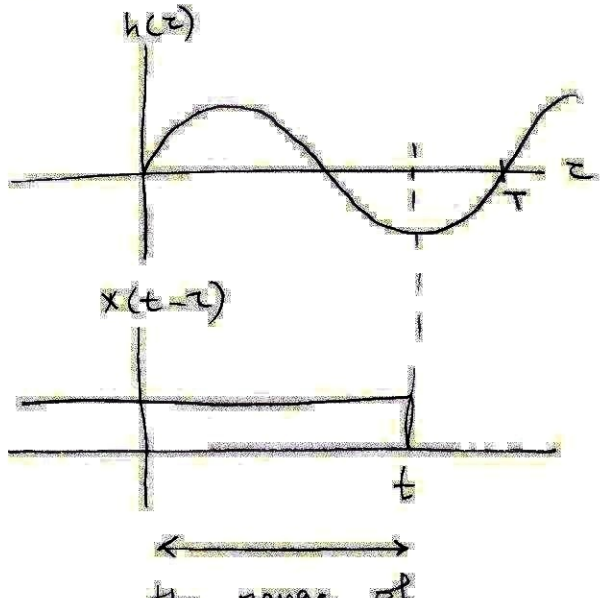
no overlap

$h(\tau)\, x(t-\tau) = 0 \quad \forall \tau$

therefore

$y(t) = 0 \quad t<0$

Case 2 $\quad 0<t<T$



$$y(t) = \int_0^t \underbrace{h(\tau)}_{\sin\frac{2\pi}{T}\tau}\, \underbrace{x(t-\tau)}_{1}\, d\tau = \int_0^t \sin\left(\frac{2\pi}{T}\tau\right) d\tau$$

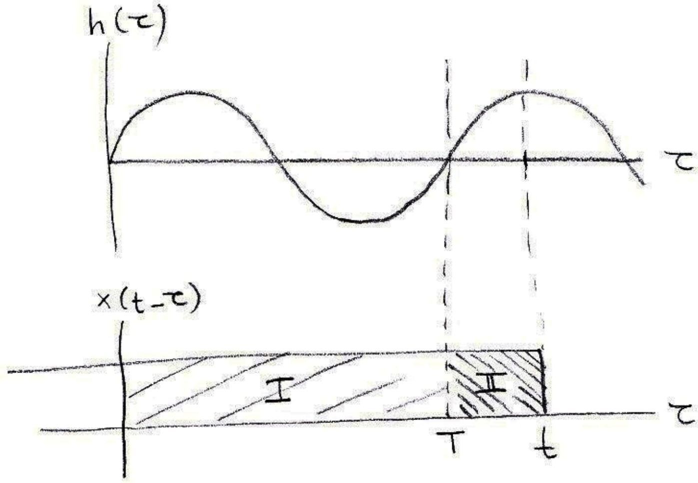
$$= \frac{-T}{2\pi} \cos \frac{2\pi}{T}\tau \Big|_0^t$$

$$= \frac{T}{2\pi} \left\{ 1 - \cos \frac{2\pi}{T} t \right\}$$

Note that at $t=T$

$$y(t) = \frac{T}{2\pi} \left\{ 1 - \overbrace{\cos \frac{2\pi}{T} T}^{1} \right\} = 0$$

Case 3 $\quad T < t < 2T$



$$y(t) = \underbrace{\cancel{\int_0^T \sin \frac{2\pi}{T}\tau \, d\tau}}_{\mathrm{I}} + \underbrace{\int_T^t \sin \frac{2\pi}{T}\tau \, d\tau}_{\mathrm{II}}$$

0

integral over a period of sin

$$\int_T^t \sin\left(\frac{2\pi}{T}\tau\right) d\tau = \frac{-T}{2\pi} \cos \frac{2\pi}{T}\tau \Big|_T^t$$

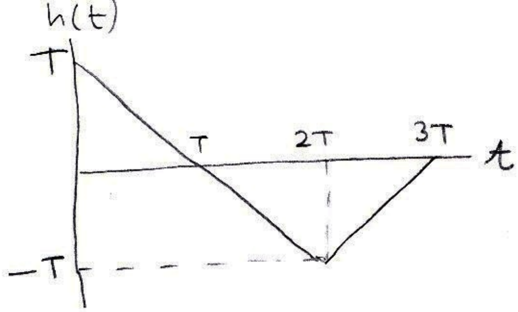
$$= \frac{T}{2\pi}\left\{1 - \cos \frac{2\pi t}{T}\right\}$$

Note that is equal to what we compute for

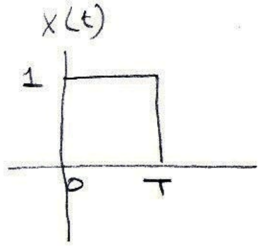
Therefore $y(t)$ is also periodic with period T.

$$y(t) = \frac{T}{2\pi}\left\{1 - \cos \frac{2\pi t}{T}\right\} u(t)$$

3



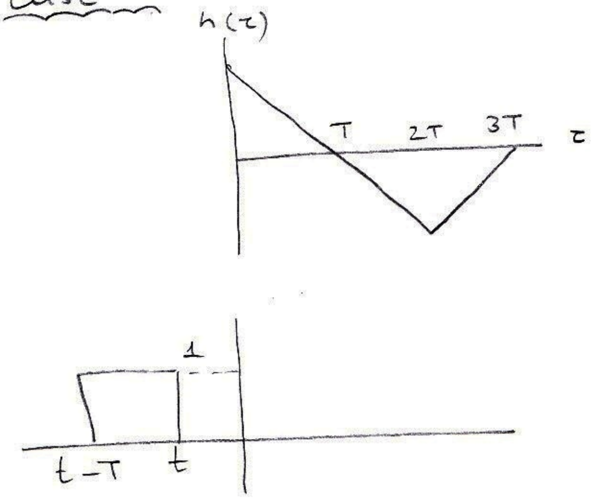
and



Compute $h(t) * x(t)$

Answer: $$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

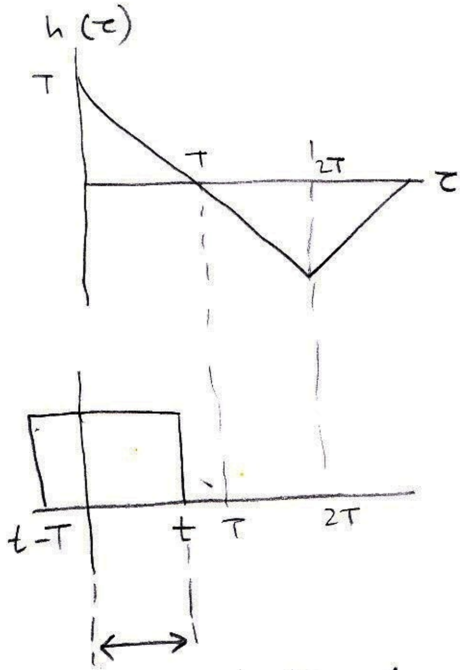
Case 1: $t<0$



no overlap

$y(t) = 0$ for $t<0$

Case 2: $0<t<T$

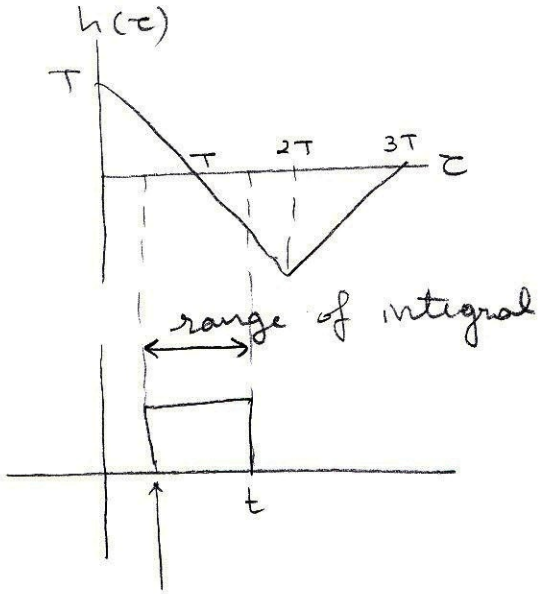


range of integral

$$y(t) = \int_0^t (T-\tau)\, d\tau$$

$$= Tt - \frac{t^2}{2}$$

Case 3 $\quad T < t < 2T$



$0 < t - T < T$

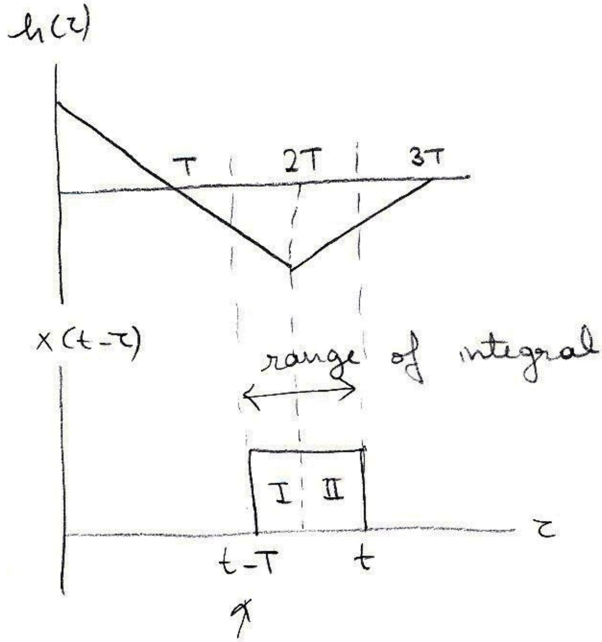
$$y(t) = \int_{t-T}^{t} (T - \tau)\, d\tau$$

$$= T^2 - \frac{t^2 - (t-T)^2}{2}$$

$$= T^2 - \frac{\cancel{t^2} - \cancel{t^2} + 2Tt - T^2}{2}$$

$$= \frac{T^2 + 2Tt}{2}$$

Case 4 $\quad T < t < 3T$



note that

$T < t - T < 2T$

$$y(t) = \underbrace{\int_{t-T}^{2T} (T - \tau)\, d\tau}_{\mathrm{I}} + \underbrace{\int_{2T}^{t} (\tau -}_{\mathrm{II}}$$

Part I

$$3T^2 - tT - \frac{4T^2 - (t-T)^2}{2}$$

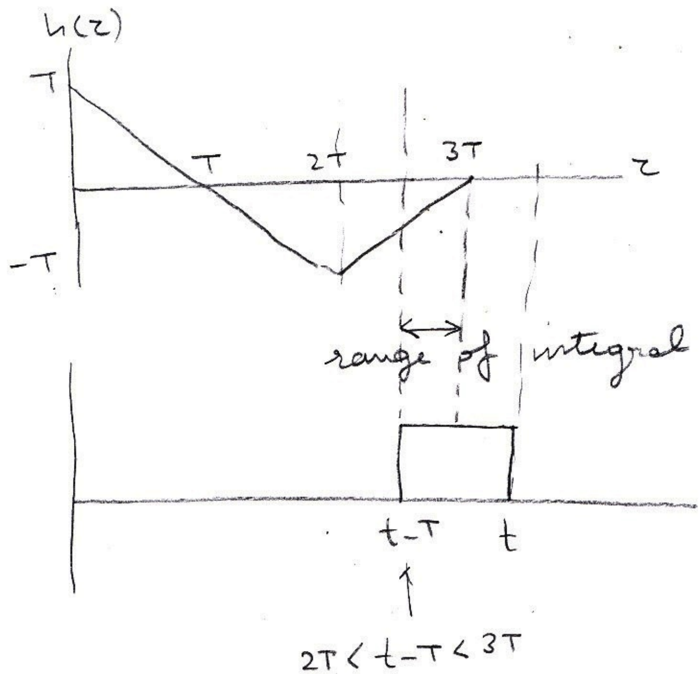
$$= \frac{3T^2 + t^2 - 4tT}{2}$$

Part II

$$= \frac{t^2 - 6Tt + 8T^2}{2}$$

$$y(t) = t^2 - 5Tt + \frac{11}{2} T^2$$

Case 5: $3T < t < 4T$



$$y(t) = \int_{t-T}^{3T} (\tau - 3T)\, d\tau$$

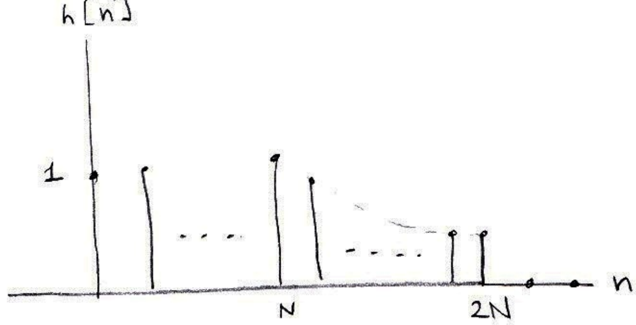
$$= \frac{-t^2 + 10Tt - 16T^2}{2}$$

Case 6: $4T < t$, no overlap $y(t) = 0$

Finally

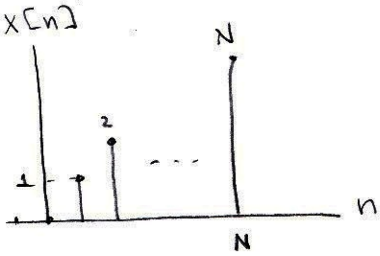
$$y(t) = \begin{cases} 0 & t < 0 \\ Tt - \frac{t^2}{2} & 0 < t < T \\ \frac{T^2 + 2Tt}{2} & T < t < 2T \\ t^2 - 5Tt + \frac{11}{2}t^2 & 2T < t < 3T \\ \frac{-t^2 + 10Tt - 16T^2}{2} & 3T < t < 4T \\ 0 & 4T < t \end{cases}$$

4



$$h[n] = \begin{cases} 1 & 0 \leq n \leq N \\ a^{n-N} & N < n \leq 2N \end{cases}$$

$|a| < 1$

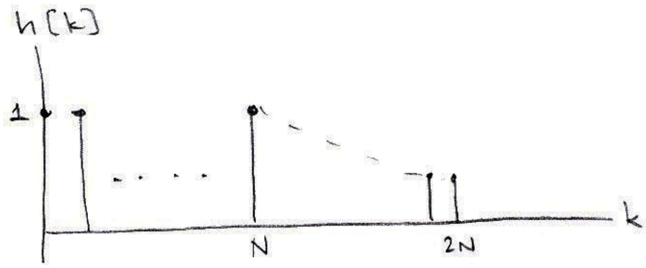


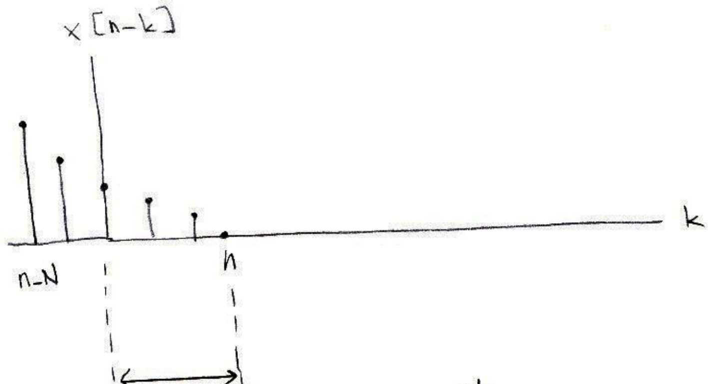
$$x[n] = \begin{cases} n & 0 \leq n \leq N \\ 0 & \text{else} \end{cases}$$

Then compute $x[n] * h[n]$

Case 1: $n < 0$, no overlap $y[n] = 0$

Case 2: $0 \leq n \leq N$

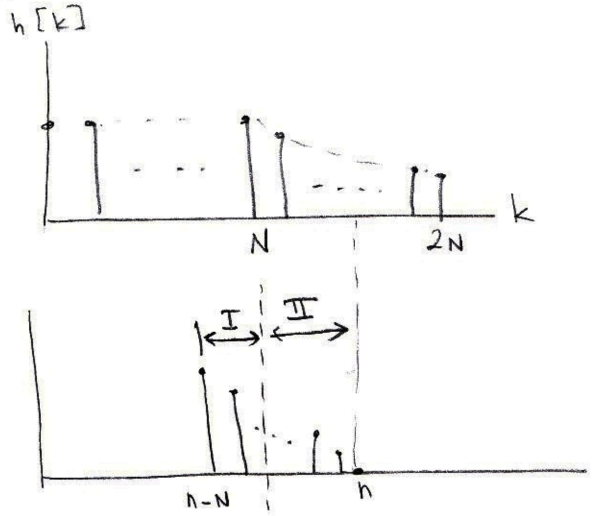




range of summation

$$y[n] = \sum_{k=0}^{n} \overset{1}{\cancel{h[k]}}\, \underbrace{x[n-k]}_{n-k} = n(n+1) - \underbrace{\sum_{k=0}^{n} k}_{\frac{n(n+1)}{2}} = \frac{(n+1)n}{2}$$

Case 3 $\quad N < n \leq 2N$



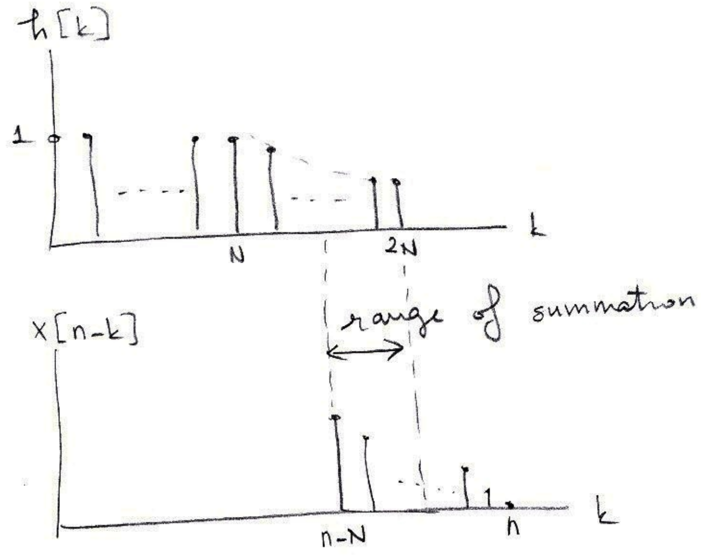
note that

$0 < n-N < N$

$$y[n] = \sum_{k=n-N}^{N} \cancelto{1}{h[k]}\ \cancelto{n-k}{x[n-k]} + \sum_{k=N+1}^{n} \cancelto{a^k}{h[k]}\ \cancelto{(n-k)}{x[n-k]}$$

Any closed-form expression ?

Case 4: $\quad 2N < n \leq 3N$



$$y[n] = \sum_{k=n-N}^{2N} a^k \, (n-k)$$

Case 5: $\quad 3N < n$, no overlap , $y[n] = 0$

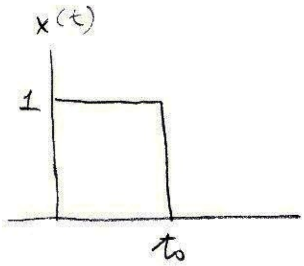
11-04-07

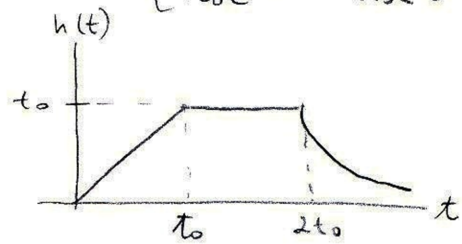
(1)

LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = ?$$

$$x(t) = \begin{cases} 1 & 0 \leq t \leq t_0 \\ 0 & \text{else} \end{cases} \qquad h(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t \leq t_0 \\ t_0 & t_0 < t \leq 2t_0 \\ t_0 e^{-\alpha t} & 2t_0 \leq t \end{cases}$$

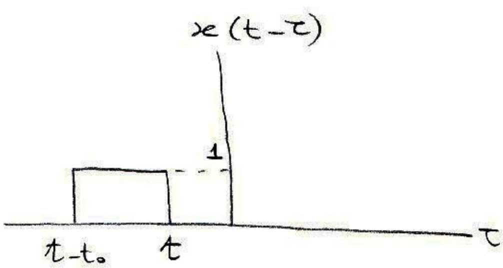


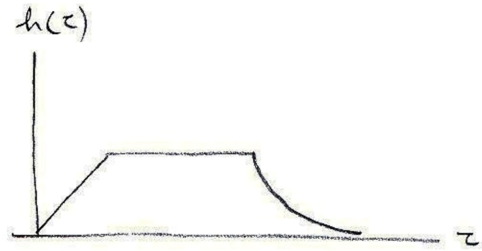


Find $y(t)$

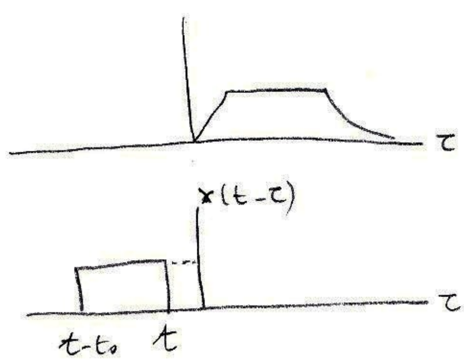
Answer: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

Convolution with graphic method





Case 1: $t < 0$

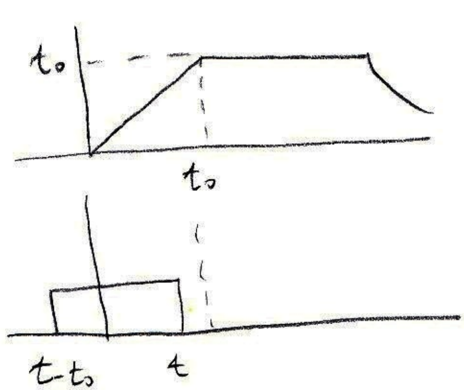


No overlap

$y(t) = 0 \qquad t < 0$

Case 2:

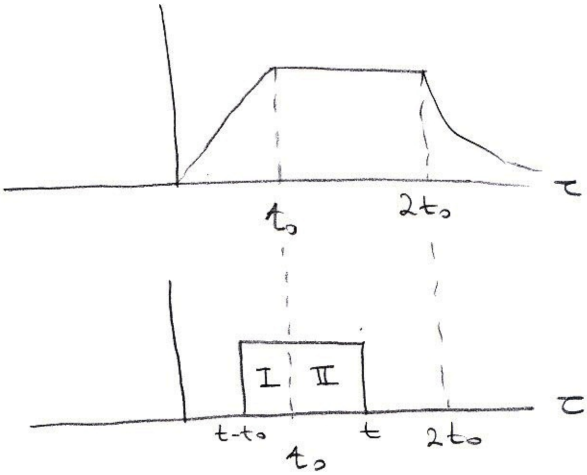
$0 < t \leq t_0$



$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^t \tau \, d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$

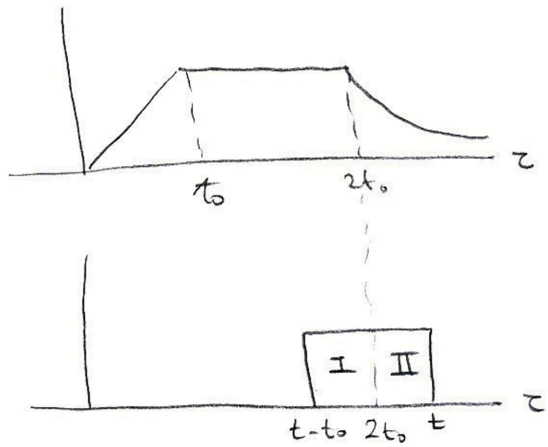
Case 3: $t_0 < t \leq 2t_0$



$$y(t) = \int_{t-t_0}^{t_0} \tau \, d\tau + \int_{t_0}^{t} t_0 \, d\tau = \frac{t_0^2 - (t-t_0)^2}{2} + t_0(t-t_0)$$

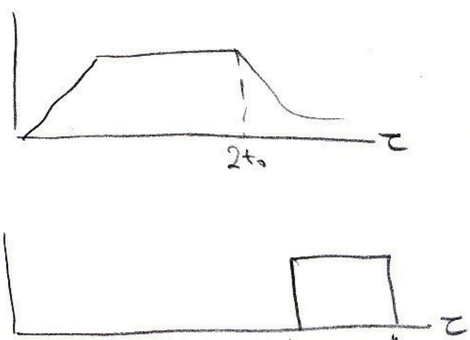
$$= \frac{-t^2 + 4tt_0 - 4t_0^2}{2}$$

Case 4: $2t_0 \leq t < 3t_0$



$$y(t) = \int_{t-t_0}^{2t_0} t_0 \, d\tau + \int_{2t_0}^{t} t_0 e^{-\alpha\tau} \, d\tau = (3t_0 - t) + \frac{-t_0}{\alpha}\left\{ e^{-\alpha t} - e^{-2\alpha t_0} \right\}$$

Case 5: $3t_0 < t$

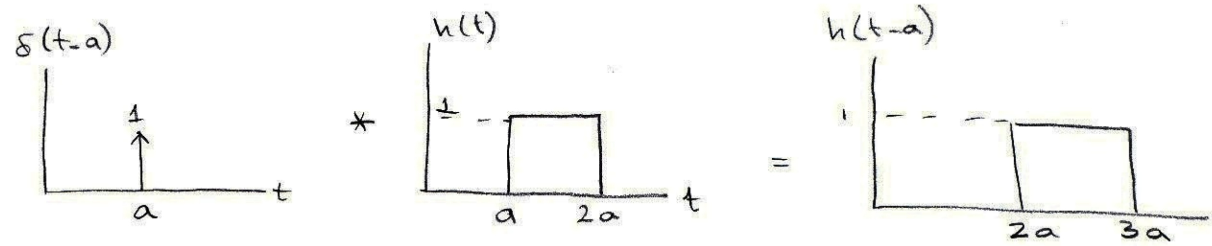


$$y(t) = \int_{t-t_0}^{t} t_0 e^{-\alpha\tau} \, d\tau$$

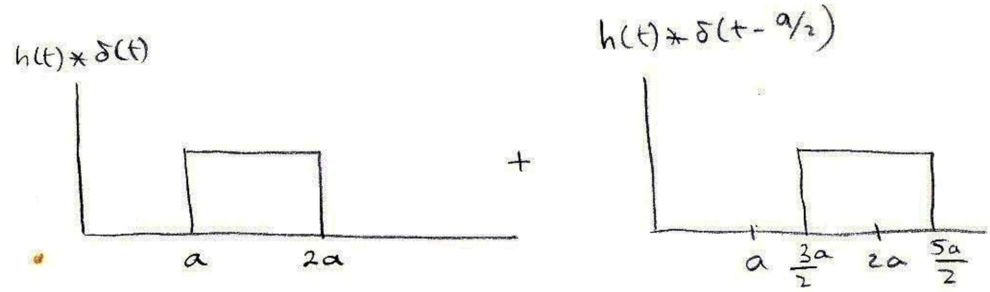
$$= \frac{t_0 e^{-\alpha t}}{\alpha}\left\{ e^{+\alpha t_0} - 1 \right\}$$

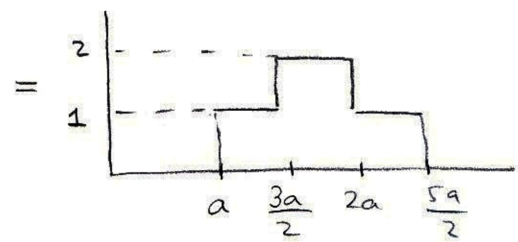
2) a) $\delta(t) * h(t) = h(t)$

$\delta(t-a) * h(t) = h(t-a)$

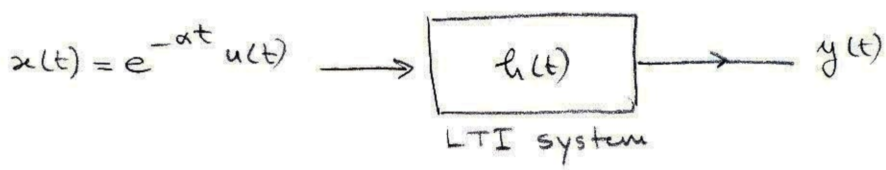


b) $\{\delta(t) + \delta(t - \frac{a}{2})\} * h(t)$





3)



a) If $h(t) = e^{-\beta t} u(t)$ $\alpha \neq \beta$, find $y(t) = ?$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau} \underbrace{u(\tau)} \, e^{-\beta(t-\tau)} \underbrace{u(t-\tau)} \, d\tau$$

$$u(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

$$u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & t < \tau \end{cases}$$

$$= \int_{0}^{t} e^{-\beta t} \cdot e^{-(\alpha-\beta)\tau} \, d\tau$$

$$= \frac{-e^{-\beta t}}{\alpha-\beta} \, e^{-(\alpha-\beta)\tau} \Big|_{0}^{t}$$

$$= \frac{e^{-\beta t}}{\alpha-\beta} \left\{ 1 - e^{-(\alpha-\beta)t} \right\} u(t)$$

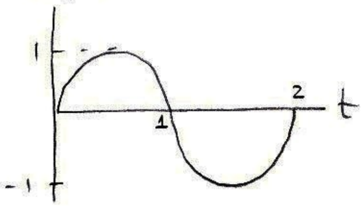
b) Find $y(t)$ if $h(t) = e^{-\alpha t} u(t)$

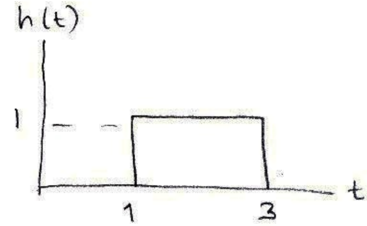
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau)\, e^{-\alpha(t-\tau)} u(t-\tau)\, d\tau$$

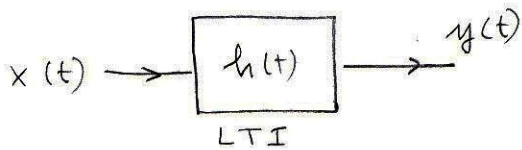
$$= e^{-\alpha t} \int_0^t d\tau = t e^{-\alpha t} u(t)$$

4)

$x(t) = \sin \pi t$





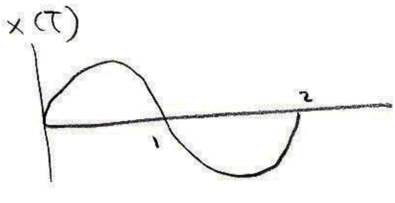


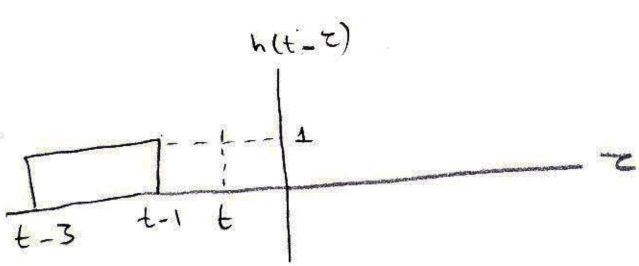
Find $y(t) = ?$

Answer: It's easier to flip and shift $h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Case 1: $t < 1$

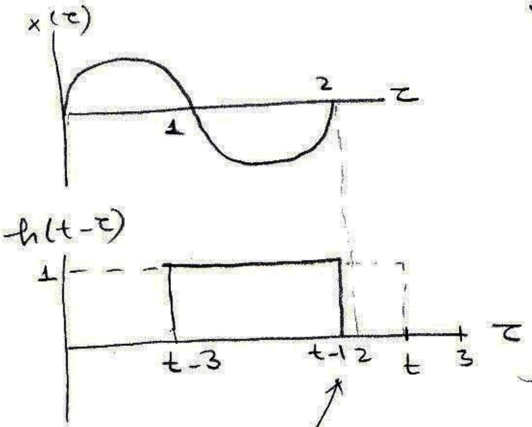




No overlap

$y(t) = 0$

Case 2: $1 < t \leq 3$



$$y(t) = \int_0^{t-1} \sin(\pi\tau)\,d\tau$$

$$= \frac{-1}{\pi} \cos(\pi\tau)\Big|_0^{t-1}$$

notice that

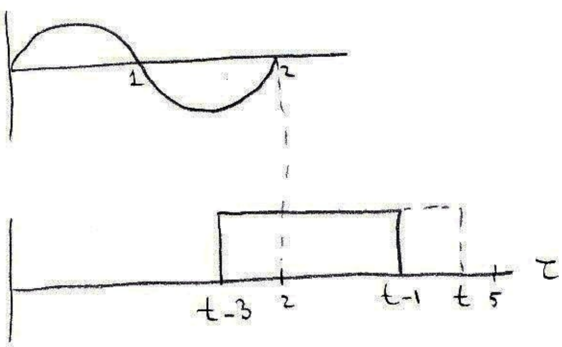
$t-1 \leq 2$

$t-3 < 0$

$$= \frac{-1}{\pi}\left\{\cos \pi(t-1) - 1\right\}$$

$$= \frac{1}{\pi}\left\{1 - \cos \pi(t-1)\right\}$$

Case 3: $3 < t < 5$



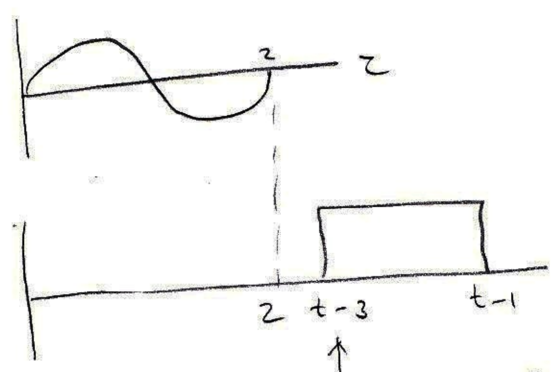
now $3 < t < 5$

$t-3 < 2$

$t-1 > 2$

$$y(t) = \int_{t-3}^{2} \sin(\pi\tau)\,d\tau = \frac{-1}{\pi} \cos(\pi\tau)\Big|_{t-3}^{2} = \frac{1}{\pi}\left\{\cos \pi(t-3) - \overset{1}{\cancel{\cos 2\pi}}\right\}$$

Case 4: $5 < t$



Note that $t-3 > 2$

No overlap

$y(t) = 0$

Therefore

$$y(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{\pi}\left\{1 - \cos \pi(t-1)\right\} & 1 < t < 3 \\ \frac{1}{\pi}\left\{\cos \pi(t-3) - 1\right\} & 3 < t < 5 \\ 0 & 5 < t \end{cases}$$

5

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

LTI system

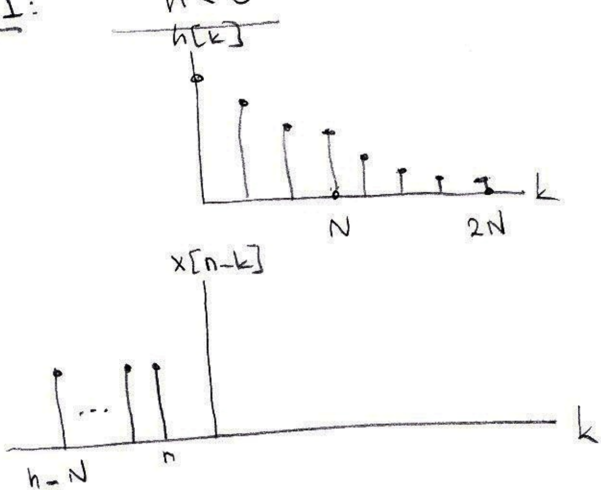
$$h[n] = \begin{cases} 0 & n < 0 \\ a^n & 0 \leq n < N \\ (ab)^n & N \leq n \leq 2N \\ 0 & 2N < n \end{cases} \qquad |a| + |b| < 1$$

a) if $x[n] = u[n] - u[n-N-1]$

then compute $y[n] = ?$

Answer: $$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$$

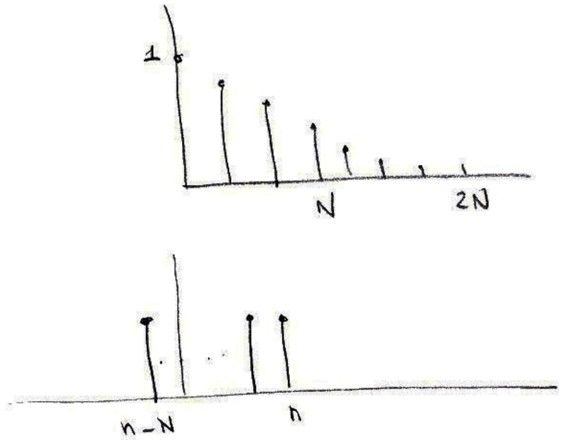
Case 1: $n < 0$



No overlap

$y[n] = 0$

Case 2 $\quad 0 \leqslant n < N$

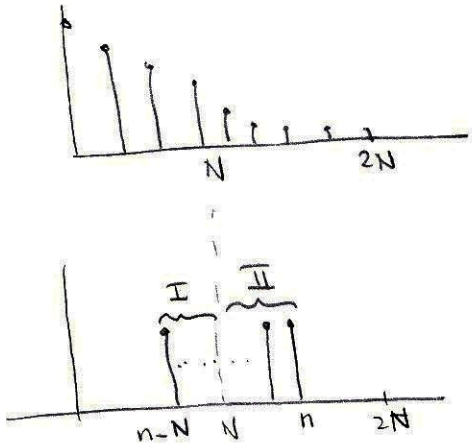


$$y[n] = \sum_{k=0}^{n} a^k$$

$$= \frac{1-a^{n+1}}{1-a}$$

note that $\quad n-N<0$

Case 3 $\quad N \leqslant n \leqslant 2N$



$h-N<N$

$$y[n] = \underbrace{\sum_{k=n-N}^{N-1} a^k}_{\mathrm{I}} + \underbrace{\sum_{k=N}^{n} (a.b)^k}_{\mathrm{II}}$$

define $\quad k' = k-n+N$

$n-N \longrightarrow 0$

$N-1 \longrightarrow 2N-n-1$

part I

$$\sum_{k'=0}^{2N-n-1} a^{k'+n-N} = a^{n-N} \frac{1-a^{2N-n}}{1-a}$$

part II

define $\quad k' = k-N$

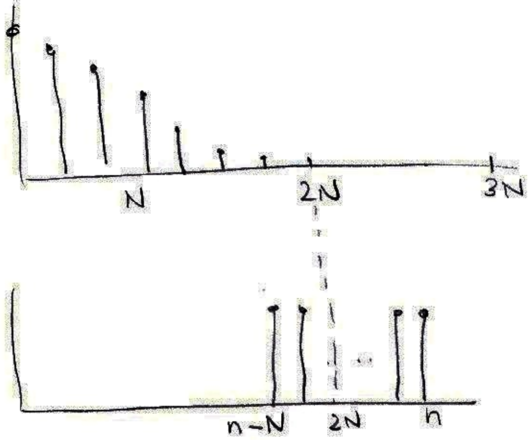
$N \rightarrow 0$

$n \rightarrow n-N$

$$\sum_{k'=0}^{n-N} (a.b)^{k'+N} = (a \cdot b)^N \cdot \frac{(ab)^{n-N+1}}{1-(ab)}$$

$$y[n] = a^{n-N} \frac{1-a^{2N+1-n}}{1-a} + (ab)^N \cdot \frac{(ab)^{n-N+1}}{1-(ab)}$$

Case 4: $2N < n \leq 3N$



note that

$n - N < 2N$

$$y[n] = \sum_{k=n-N}^{2N} (a \cdot b)^k$$

define $k' = k - n + N$

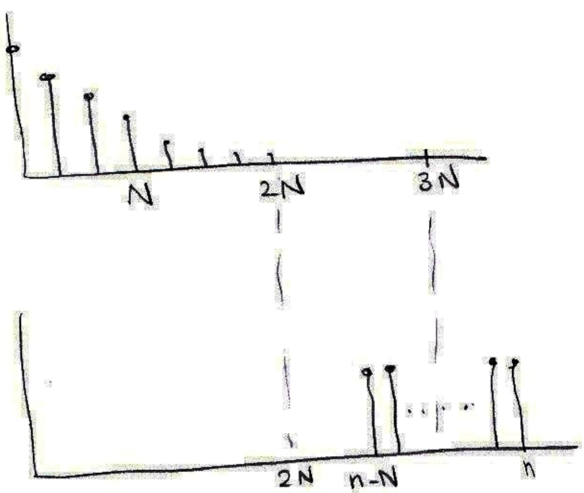
$n - N \rightarrow 0$

$2N \rightarrow 3N - n$

$$y[n] = \sum_{k'=0}^{3N-n} (a \cdot b)^{k'+n-N}$$

$$= (a \cdot b)^{n-N} \frac{1 - (a \cdot b)^{3N-n+1}}{1 - (a \cdot b)}$$

case 5: $3N < n$



Note that

$n - N > 2N$

No overlap

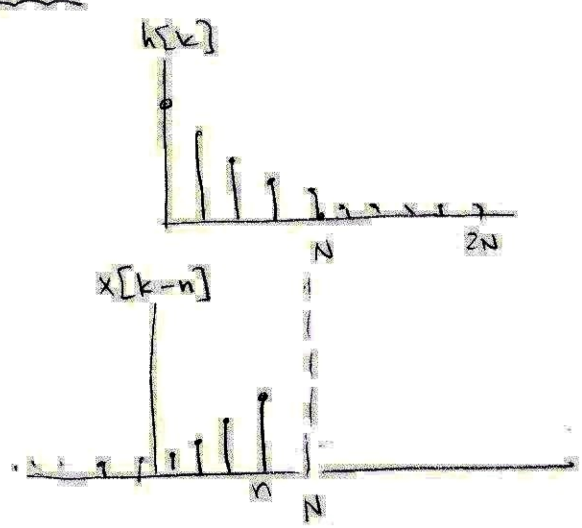
$y[n] = 0$

b) if $x[n] = b^n u[n]$, compute $y[n] = ?$

Case 1: $n < 0$ (similar to part a) no overlap

$$y[n] = 0$$

Case 2: $0 \leq n < N$

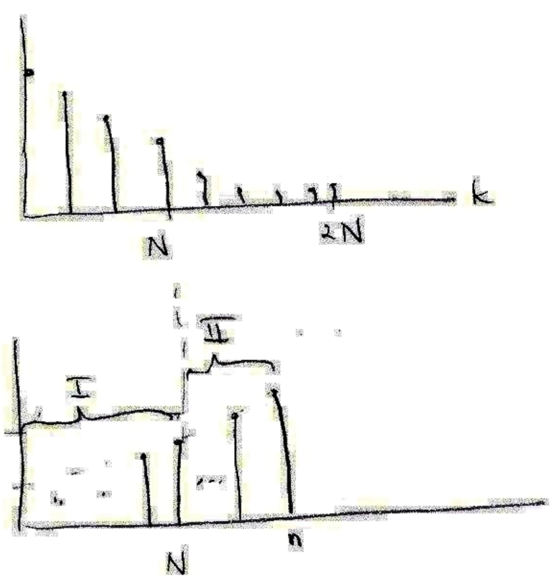


$$y[n] = \sum_{k=0}^{n} a^k b^{n-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n \frac{1-(a/b)^{n+1}}{1-a/b}$$

Case 3: $N \leq n \leq 2N$



$$y[n] = \sum_{k=0}^{N-1} a^k b^{n-k} + \underbrace{\sum_{k=N}^{n} (a\cdot b)^k b^{n-k}}$$

$$b^n \sum_{k=N}^{n} a^k = b^n \frac{1-a^{n-N+1}}{1-a}$$

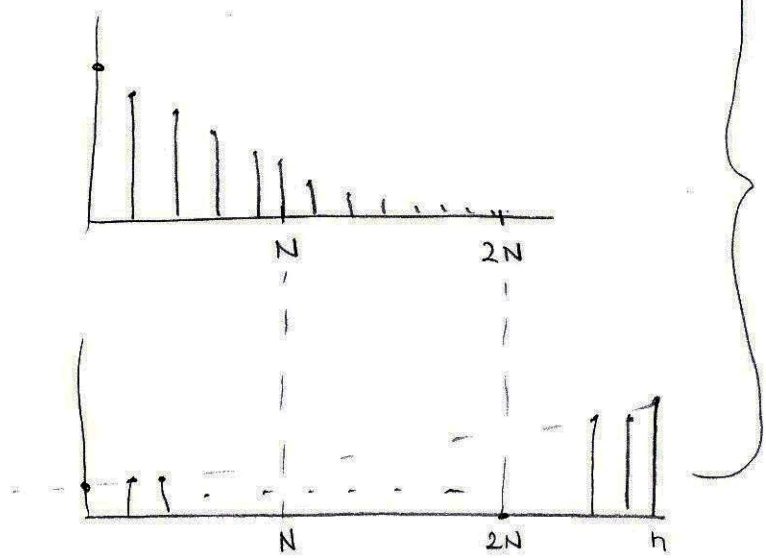
define $k' = k - N$

$N \to 0$

$n \to n - N$

$$y[n] = b^n \left\{ \frac{1-(a/b)^N}{1-(a/b)} + \frac{1-a^{n-N+1}}{1-a} \right\}$$

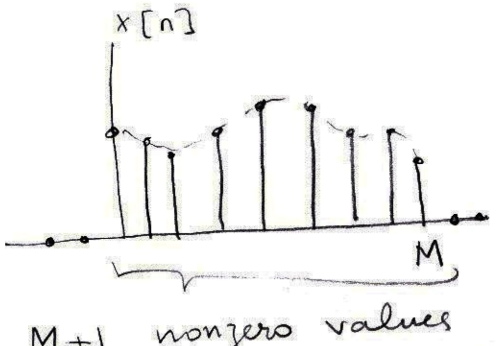
Case 4: $n > 2N$



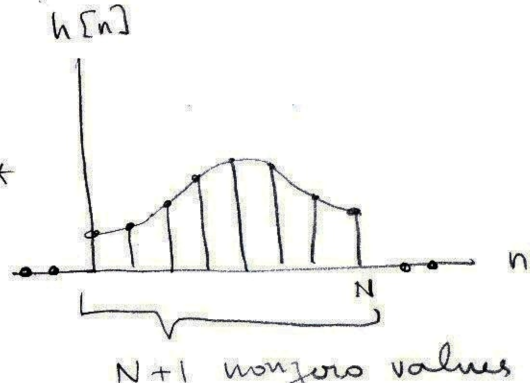
$$y[n] = \sum_{k=0}^{N-1} a^k\, b^{n-k} + \sum_{k=N}^{2N} (ab)^k\, b^{n-k}$$

$$y[n] = b^n \left\{ \frac{1-(a/b)^N}{1-(a/b)} + \frac{1-a^{N+1}}{1-a} \right\}$$

(6)



$*$



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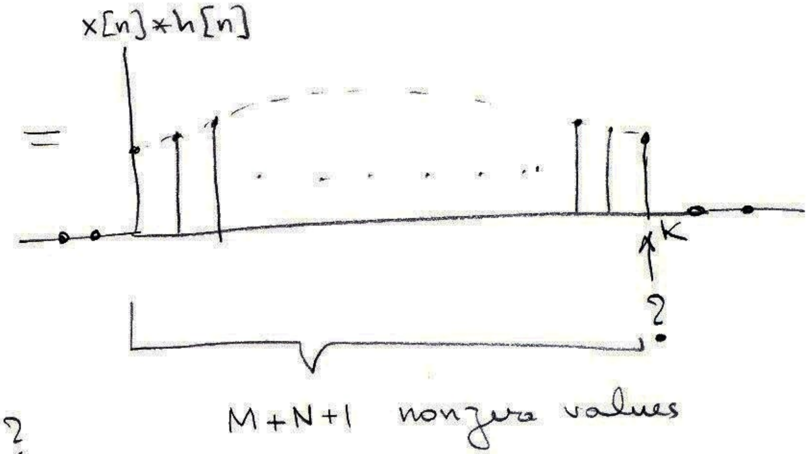
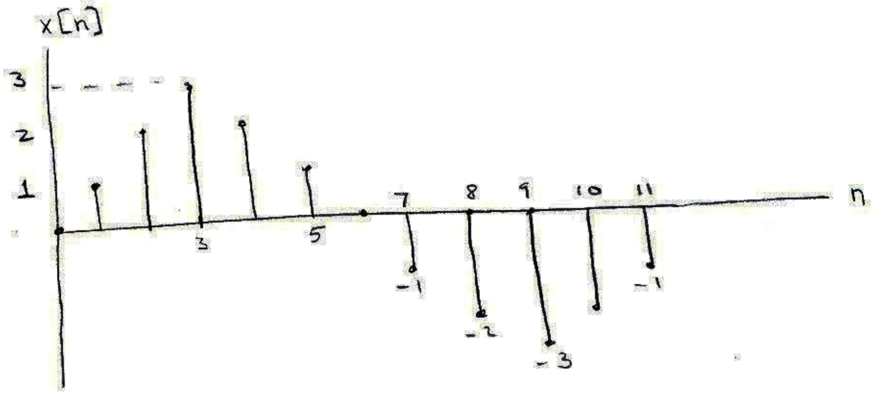


$K = ?$

$K = M + N$

7



a) Compute $y[n]$ if $h[n] = u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k]$$

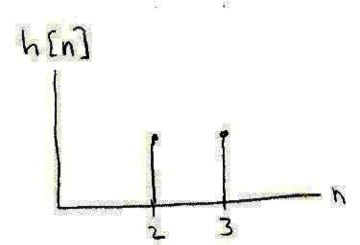
$$u[n-k] = \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{else } k > n \end{cases}$$

and $x[k] = 0$ if $k < 0$

then

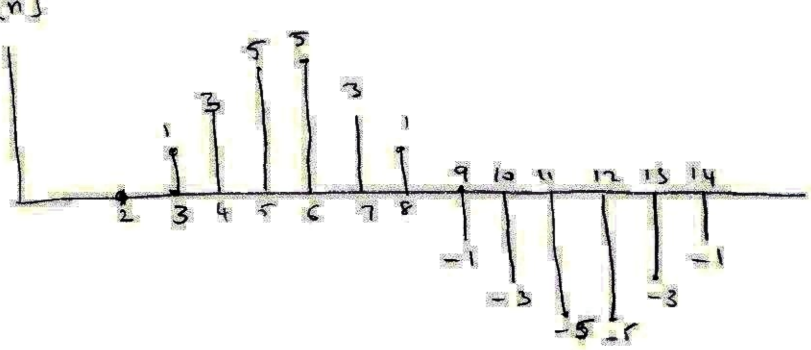
$$y[n] = \sum_{k=0}^{n} x[n] = \{1, 3, 6, 8, 9, 9, 8, 6, 3, 1, 0, 0, \ldots\}$$

b) compute $y[n]$ if $h[n] = \begin{cases} 1 & n = 2, 3 \\ 0 & \text{else} \end{cases}$

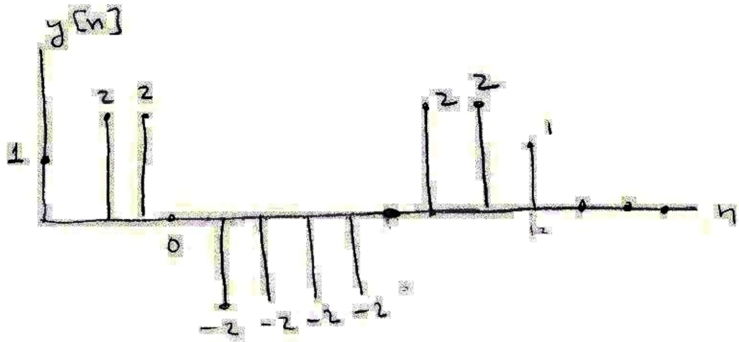


$$h[n] = \delta[n-2] + \delta[n-3]$$

$$y[n] = x[n] * h[n] = x[n-2] + x[n-3]$$



c) $h[n] = \begin{cases} -1 & n=-1 \\ 0 & n=0 \\ 1 & n=1 \\ 0 & \text{else} \end{cases}$ $\Rightarrow$ compute $y[n] = ?$



)

$$x[n] = u[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

LTI system

If $h[n] = \frac{a^n}{n!} u[n]$ then compute $\lim_{n\to\infty} y[n]$

nswer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} \frac{a^k}{k!} \underbrace{u[k]\, u[n-k]}_{= \begin{cases} 1 & 0 \le k \le n \\ 0 & \text{else} \end{cases}}$$

Then

$$y[n] = \sum_{k=0}^{n} \frac{a^k}{k!}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a^k}{k!} = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$