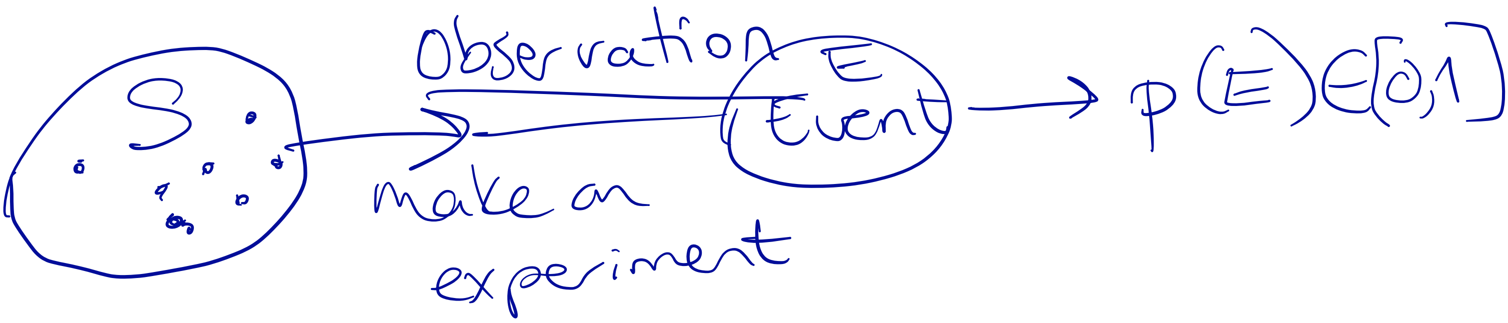
KOM 505E Lecture Notes

Sept 20, 2016

Week 1.

Probability Models have 3 components

1. Sample space: specifies everything about the experiment
2. Events / Sample points of the sample space / Outcomes
3. probabilities of events



4 Axioms of Probability:

Set theory: → Defn. of sets

Set operations. (Chapter 3)

Reading Assignment

Chapters 1, 2, 3 from the textbook.

* Two sets A & B <u>disjoint</u> when $A \cap B = \emptyset$

* Several sets $A_1, \ldots, A_k$ (could be countably ∞) are <u>mutually exclusive</u> if

$$A_i \cap A_j = \emptyset \quad \text{when } i \neq j$$

* A_i are called a <u>partition</u> of S <u>iff</u>
  1) A_i's are disjoint (mutually exclusive)
  2) $\cup A_i = S$

* $\emptyset$: empty set : impossible event

* Let A_i be a subset of S, if A_i are finitely many:

Define object: field $\boxed{\mathcal{F} = \{A_i : A_i \subset S, \ i \leq N\}}$

1) $\emptyset \in \mathcal{F}$
2) If $A_i \in \mathcal{F}$, then $A_i^c \in \mathcal{F}$
3) If $A_i \in \mathcal{F}$, $i = 1, \ldots, N$, $\bigcup_{i=1}^{N} A_i \in \mathcal{F}$

⇒ (3) Countable unions of events are events

then $\mathcal{F}$ is called a **field**.

* If A_i are ∞ly many, then it is called a Borel field.

⇒ A Borel field is closed under complement & countable unions.

Borel field:

$$\mathcal{B} = \{ A_i : A_i \subseteq S \ ; \ i \in \underbrace{\mathbb{I}}_{\text{integers}} \}$$

Any subset A of S is called an **event** iff $A \in \mathcal{B}$.

* Probability (function) assigns a unique number in $[0,1]$ interval to each event.

Axioms of Probability

(by Kolmogorov in 1933)

$$\begin{cases} 1)\ P(S) = 1 \\ 2)\ P(A) \geq 0 \quad \text{for } \forall A \in S \quad (A \in B) \\ 3)\ P\left(\bigcup_i A_i\right) = \sum_i P(A_i) \\ \quad \text{if } A_i \text{ is disjoint} \end{cases}$$

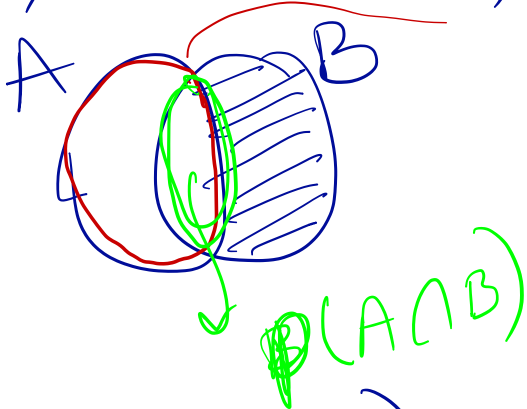
$\Rightarrow$ A few simple consequences of these axioms:

(You can derive those using the axioms)

1) $P(A^c) = 1 - P(A)$

2) $P(\emptyset) = 0$

3) $P(B \cap A^c) = P(B) - P(A \cap B)$



4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

5) If $A \subseteq B$ then $P(A) \leq P(B)$

6) $\{B_i\}$ a partition of S :

$$P(A) = \sum_i P(A \cap B_i)$$

eg. derive:

$S = \cup B_i$

$$A \cap S = A \cap (\cup B_i) \to \cup (A \cap B_i)$$

$$P(A) = P(\cup (A \cap B_i)) = \sum P(A \cap B_i)$$

<u>Ex:</u> What is the prob of getting a sum of 7 in two dice rolls? (Say a red dice, a blue dice)

Experiment: Rolling two dices $\to S = ?$

sample space for red dice

$S_1 := \{1,2,3,4,5,6\}$: 6 elements

blue die:

$S_2 = \{1,2,3,4,5,6\}$:

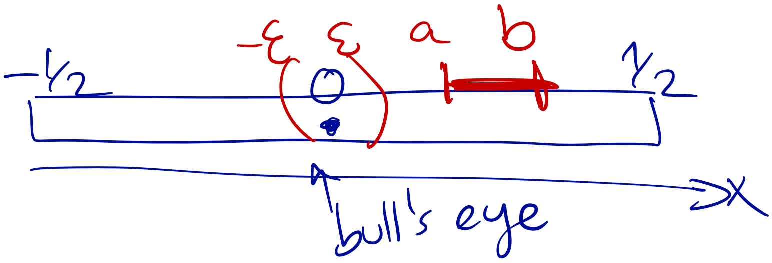
$S = S_1 \times S_2 = \{(1,1),(1,2),\ldots,(6,6)\}$

$|S| = 36$ elements

$E = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

$|E| = 6 \Rightarrow \boxed{P(E) = \frac{|E|}{|S|}} = \frac{6}{36} = \frac{1}{6}$

Ex:



$P(x=0)=?$

$S=?$ $S=\{x : x\in(-\frac{1}{2},\frac{1}{2})\}$

$|S|=?$ uncountably infinite.

$P(x=0)=0$

$P(x\in(a,b))=\frac{b-a}{1}$

$P(x\in(-\varepsilon,\varepsilon))=2\varepsilon$

$$P(E)=\frac{\text{length}(E)}{\text{length}(S)}$$

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* Prob. modeling problem: QUALITY CONTROL (Chapter 3 Real World Problem) (3.10)

Ex: Urns & ball drawing: (Sampling w/ Replacement)

3R red
2B black.

What is the prob that we get first "Red" then "Black"?

$P(\text{"}R,B\text{"})=?$

List of : R_1, R_2, R_3, B_1, B_2 : $\boxed{S=?}$

$$\left.\begin{matrix} S_1=\{R_1,R_2,R_3,B_1,B_2\} \\ S_2=\{\quad " \qquad " \quad\} \end{matrix}\right\} \Rightarrow S_1\times S_2=$$

$\Rightarrow S_1 \times S_2 = \{(R_1, R_1), (R_1, R_2), (R_1, R_3)$

$S =$ $(R_1, B_1), \ldots, (B_2, B_2)\}$

$|S| = 25$

$\boxed{E = ?}$ $E = \{(R_1, B_1), (R_1, B_2), (R_2, B_1), (R_2, B_2), (R_3, B_1), (R_3, B_2)\}$

$|E| = 6$

$$P(E) = \frac{|E|}{|S|} = \frac{6}{25}$$

Combinatorics:

ex: 1, 2, 3, 4 balls

3-tuples

$\downarrow$

inside ordering

$4 \times 3 \times 2$ $\Rightarrow$ # 3-tuples: $= 4 \times 3 \times 2$

ex:

1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

$3! = 6$

$$\binom{4}{3} = \frac{4 \times 3 \times 2}{3!} = \frac{24}{6} = 4$$

b/c ordering is not important.

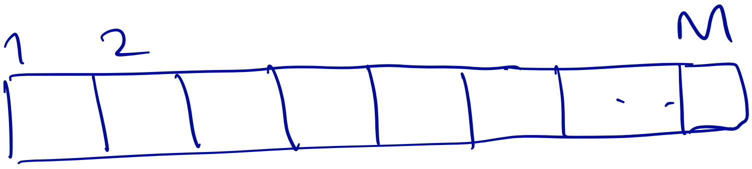
Prob. model: M coin tosses, k Heads

$\equiv$ M trials of the experiment $\Longrightarrow$ k successes.

out of

$$P(\underbrace{HHH\ldots H}_{k \text{ heads}}\ \underbrace{TTT\ldots T}_{M-k}) = p^k(1-p)^{M-k}$$

$$\left.\begin{aligned} P(\text{Heads}) &= P(\text{success}) = p \\ P(\text{Tails}) &= P(\text{fail}) = 1-p \end{aligned}\right\} a$$



$\longleftarrow$ the way k heads can be placed here.

$= \#$ k element subsets of M $= \binom{M}{k}$

$$\boxed{\Longrightarrow P(k \text{ successes (heads) in } M \text{ trials (coin tosses)}) = \binom{M}{k} p^k (1-p)^{M-k}}$$

Binomial law: prob. model for sampling w/ replacement.

Binomial law also applies to the drawing of balls from urns w/ replacement.