KOM 505E

Week 4 Lecture Notes

11 oct 2016

G.U.

Chap 5. Poisson pmf Real world ex:

Requirement:

$$P_X[k] = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \geq 0$$

(Queueing theory: P(k arrivals in a unit time))

$\lambda = ? = 70/60$

$$P(X \leq 2) = \sum_{k=0}^{2} P_X[k] = 0.88 \quad \times \quad < 0.95$$

$\cancel{k \leq 2}$

$\Rightarrow$ Soln: open 2 lanes $\Rightarrow \lambda_{new} = \frac{7}{12}$

2 sets of arrivals $\rightarrow$ independent.

P[2 or less arrivals in both lanes]

$= P[k \leq 2 \text{ in } 1^{st} \text{ lane}]\, P[k \leq 2 \text{ in } 2^{nd} \text{ lane}]$

$= (P[k \leq 2])^2$ w/ updated λ.

$= 0.957$ ✓

Ch. 10 Continuous R.v.s

Discrete r.v.s

$S^x \rightarrow$ countable

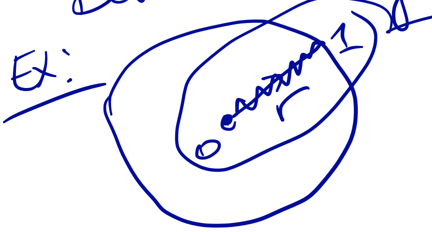
Continuous R.v.s

$S^x = (-\infty, \infty)$

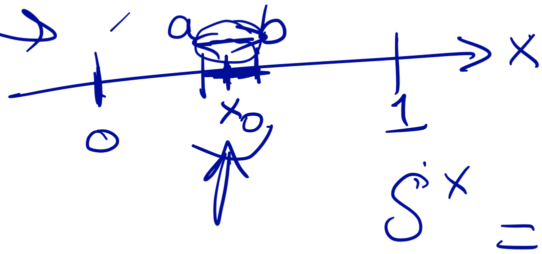
$(0,1)$

Dart throwing

Ex:



1D problem:

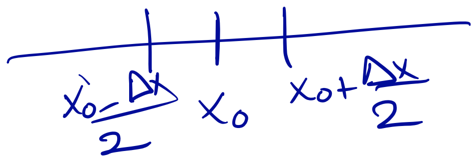


all outcomes are equally likely.

$S^x = [0,1]$

$(0,1)$

$$P(a \le X_0 \le b) = \frac{b-a}{1}$$

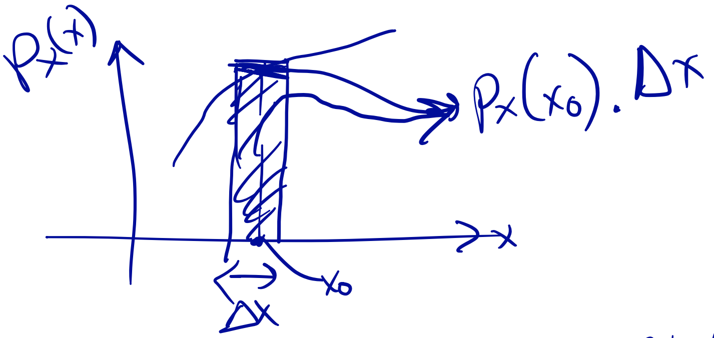


$$\boxed{P\left(x_0 - \frac{\Delta x}{2} < X_0 \le x_0 + \frac{\Delta x}{2}\right) = \int_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}} p_x(x)\, dx.}$$

for small Δx

$$\Rightarrow P_X(x_0)\Delta x = \underbrace{P\left(x_0 - \frac{\Delta x}{2} < X_0 \le x_0 + \frac{\Delta x}{2}\right)}$$

$$\Rightarrow \underline{p_X(x_0)} = \frac{P\left(x_0 - \frac{\Delta x}{2} < X_0 \le x_0 + \frac{\Delta x}{2}\right)}{\Delta x} = \frac{\text{prob.}}{\text{length}}$$



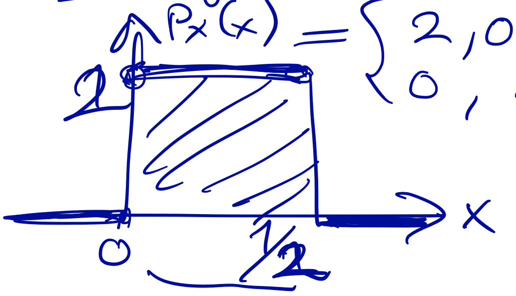
$p_X(x)$: probability per unit length.

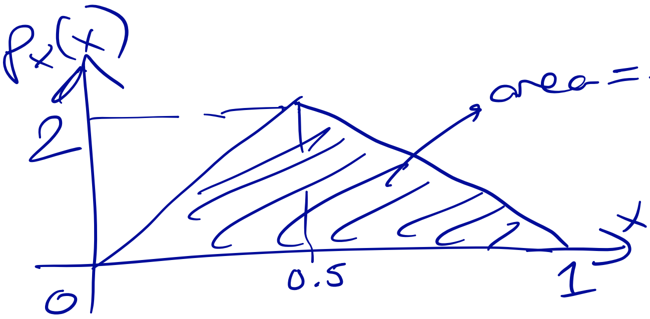
pdf: (Probability density fn.)

Analog: linear mass density $= \frac{\text{mass}}{\text{length}}$

Uniform pdf:

$$p_x(x) = \begin{cases} 2, & 0 < x \le \frac{1}{2} \\ 0, & o/w \end{cases}$$



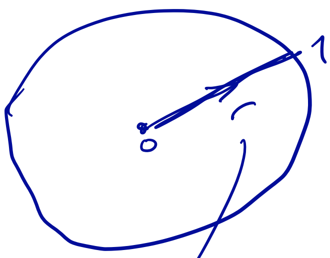


Properties of pdf:

1. $p_x(x) \ge 0$
2. $\int_{-\infty}^{\infty} p_x(x)\,dx = 1$
3. $\underbrace{p(x = x_0) = \int_{x_0}^{x_0} p_x(x)\,dx = 0}$
4. $\underbrace{p(a < x < b)} = \int_a^b p_x(x)\,dx$

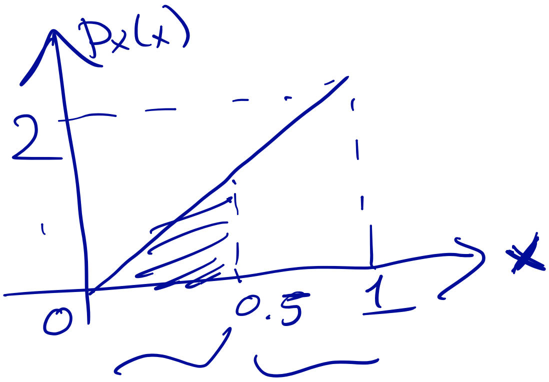
   $\left(= p(a \le x < b) = p(a < x \le b)\right)$
5. $p_x(x)$ can be larger than 1.

Ex:



x: distance to the center:

$$p_x(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & o/w \end{cases}$$



$$p(x < 0.5) = \int_0^{0.5} 2x\,dx = \underline{0.25}$$
$$x^2 \Big|_0^{0.5}$$

$$p(x > 0.5) = \int_{0.5}^{1} 2x\,dx = 0.75$$

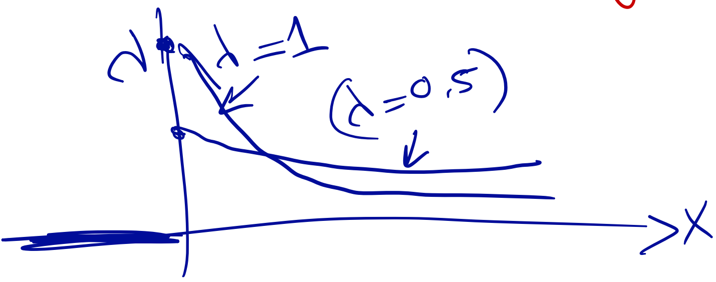
A non-uniform pdf.

Important pdfs:

1. Uniform pdf

2. Exponential pdf:

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \quad x \in (0,\infty) \\ 0, & o/w \end{cases}$$



X models lifetime of a product.

λ: # events / unit interval

Note: A pdf does not need to be continuous. Why?

eg. λ: failure rate : bulb's failure rate

Let $\boxed{\lambda = 0.01}$

Prob (bulb will last 100 days) = ?

= Prob (bulb will fail after 100 days)

$$= \int_{100}^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_{100}^{\infty} = 0.367$$

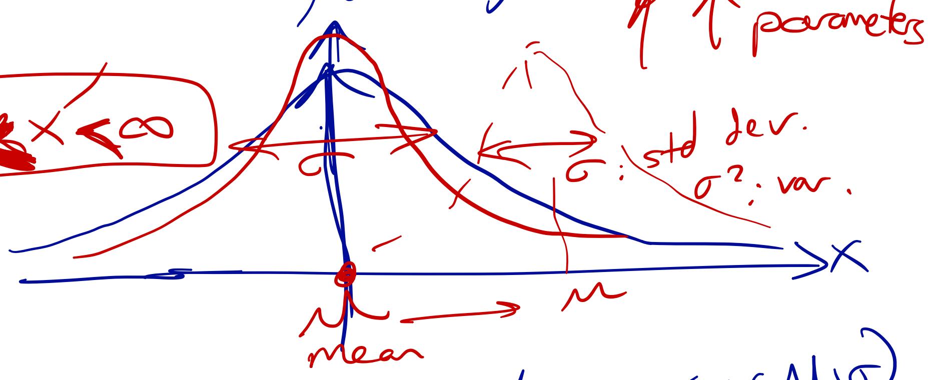
3. Gaussian pdf: (Normal pdf):

$$p_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad X \sim N(\mu, \sigma^2)$$

$\boxed{-\infty < x < \infty}$

μ, σ^2 parameters

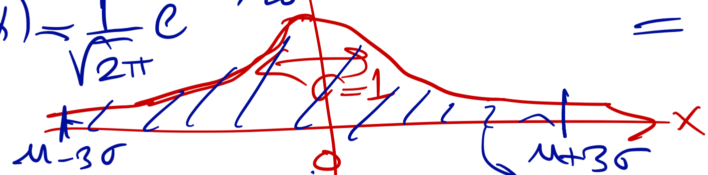
ex: $\int_{-\infty}^{\infty} p(x)dx = 1$.



Matlab: randn(.) :

Def: Std Normal : $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$X \sim N(0,1)$

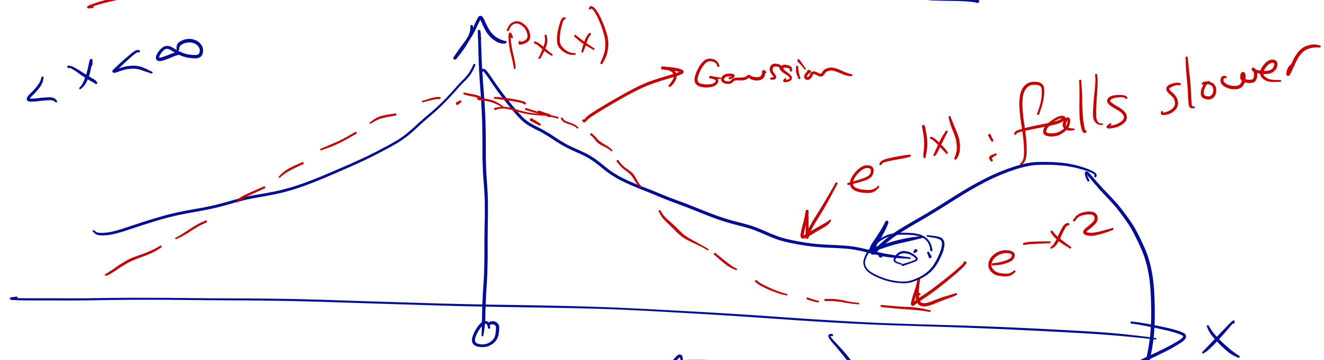


$p(\mu - \sigma < x < \mu + \sigma) = \underline{0.66}$

$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.997$ (?)

(4) Laplacian pdf: $p_X(x) \propto e^{-|x|}$ $\left(p_X(x) \propto e^{-x^2}\right)$

$-\infty < x < \infty$



$$p_X(x) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|x|\right), \quad \sigma^2 > 0.$$

mean $= 0$.

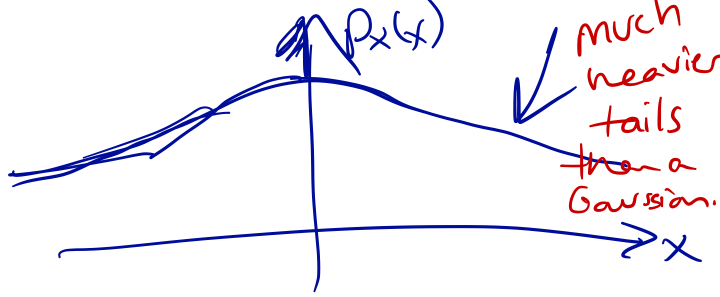
* Laplacian "heavier tails" than a Gaussian

"tails": region of pdf for which $|x|$ is large

eg. used as a model for speech amplitudes.

(5) Cauchy pdf:

$$p_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$



eg. arises as a pdf of the ratio of two independent $\mathcal{N}(0,1)$ r.v.s. (Ch. 12).

(6) Gamma pdf: $p_X(x) \propto x^{\alpha-1} e^{-\lambda x}$, $x \geq 0$

$X \sim \Gamma(\alpha, \lambda)$. $\quad \alpha, \lambda > 0$ are parameters

reduces to some well-known pdfs

(i) $\alpha = 1 \Rightarrow$ exponential pdf

(ii) $\alpha = \frac{N}{2}, \lambda = \frac{1}{2} \Rightarrow$ chi-squared pdf $\quad X \sim \Gamma\left(\frac{N}{2}, \frac{1}{2}\right)$

pdf of sum of squares of N indep. r.v.s w/ pdf $\mathcal{N}(0,1)$. $\Rightarrow$

$$Y = \sum_{i=1}^{N} X_i^2 \,, \quad X_i \sim N(0,1) \text{ i.i.d.}$$

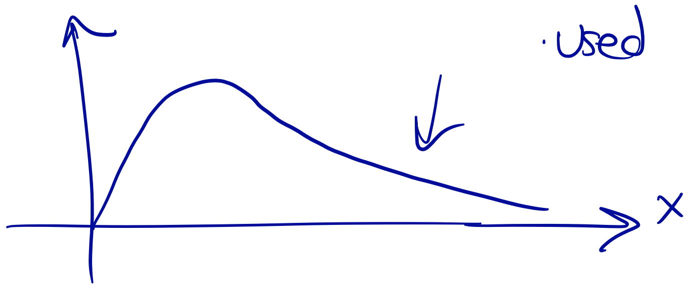
Chi-squared pdf.

(iii) $\alpha = N \Rightarrow$ Erlang distrib. = sum of N i.i.d. exponential r.v.s.

7. Rayleigh pdf:

$$p_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0 \\ 0 & \text{o/w} \end{cases}$$

$$p_X(x) \propto x e^{-x^2}$$



· used a lot in communications.

used in modeling magnitude of the signal in ultrasound imaging.

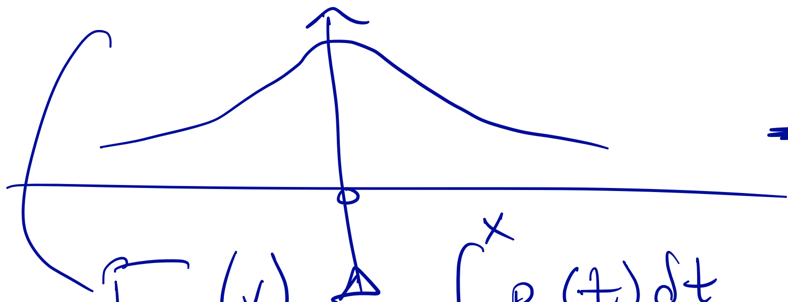
* arises as the pdf square root of sum of squares of 2 indep. normal r.v.s.

$$\boxed{X = \sqrt{X_1^2 + X_2^2}}$$

indep. $N(0,1)$

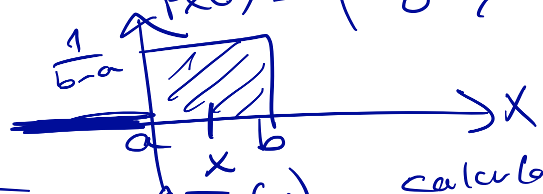
CDF (Cumulative Distribution Fn):

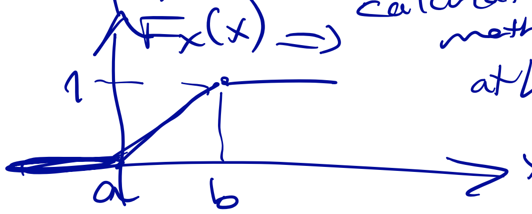
$$F_X(x) = \underline{P(X \leq x)} \,, \quad -\infty < x < \infty$$



$$F_X(x) \triangleq \int_{-\infty}^{x} p_X(t)\, dt \,, \quad -\infty < x < \infty$$

$$p_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{o/w} \end{cases}$$

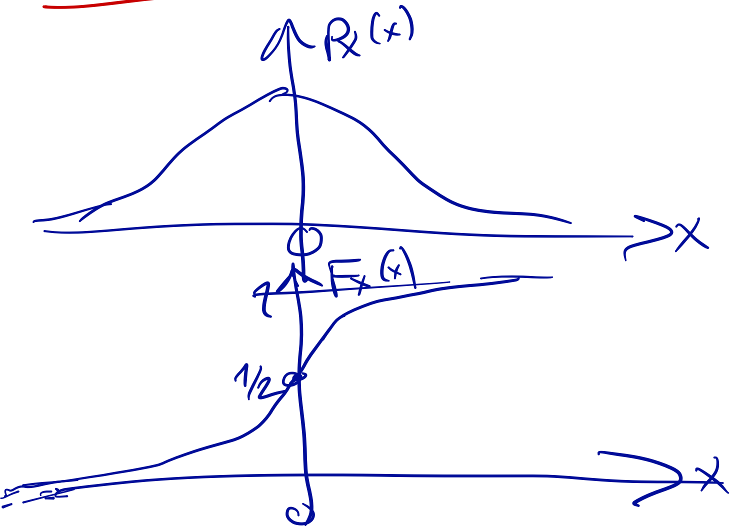




$F_X(x) \Rightarrow$ calculate mathematically at home.

Uniform cdf

Gaussian CDF: $X \sim \mathcal{N}(0,1)$ std Gaussian



$$F_X(x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

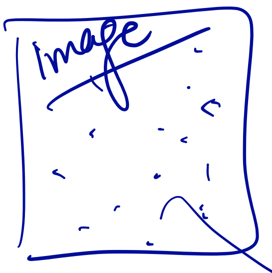
$$Q(x) = 1 - \Phi(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

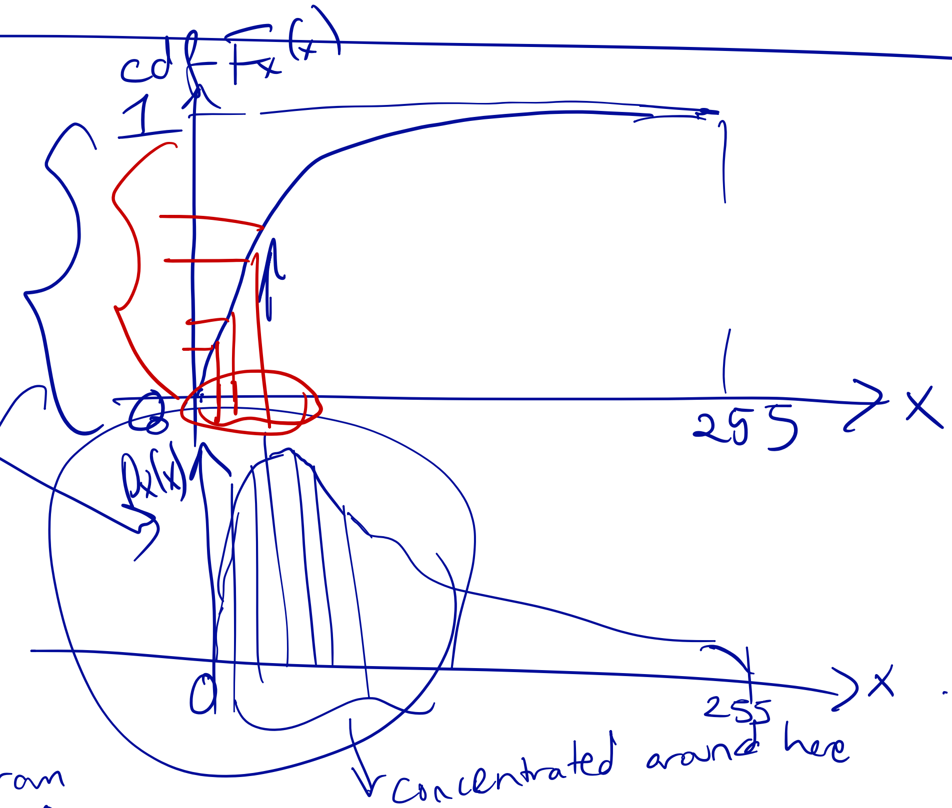
tail prob

Use look-up tables to get the value of $\Phi(x)$ at any x. → (Recall the error fn.

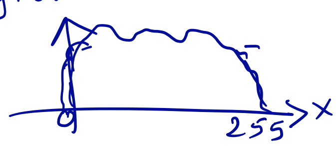
$$\mathrm{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2/2} dt \qquad Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

ex: cdf of the exponential pdf ?





Histogram Equalization = Uniformization of the histogram



Thm (1) If a cont r.v. X is given as $X = F_X^{-1}(U)$, $U \sim U[0,1]$

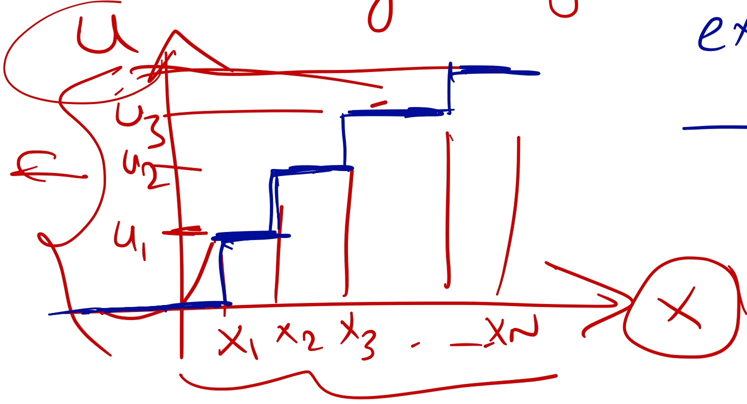
then X has the cdf $F_X(x)$. (Inverse Prob. Integral Transform)

pdf $p_X(x) = \frac{d}{dx} F_X(x)$. (from Fund. Thm of calculus)

$F_X(x) = \int_{-\infty}^{x} p_X(t)\,dt$

Thm: If an r.v. is transformed according to its CDF, then the xformed r.v. is $U \sim U[0,1]$.

(Probability Integral Transform).



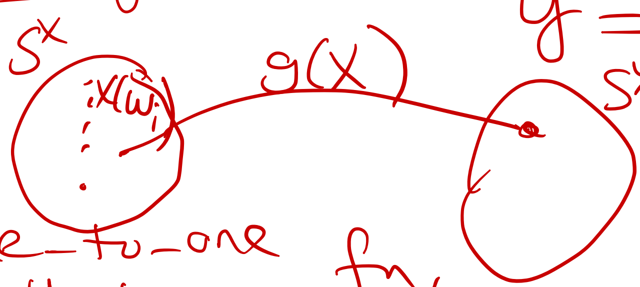
ex: Given pmf:

$$\rightarrow p_X(x) = \begin{cases} 0.2, & x=1 \\ 0.5, & x=2 \\ 0.3, & x=3 \end{cases}$$

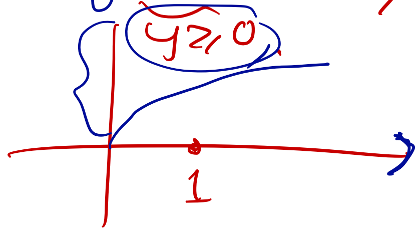
☆ USE the above THEOREM (1) to simulate r.v.s.

Transformation of Continuous R.V.s.

$y = g(x)$: eg. $Y = X^2$ ok.



eg. $Y = \sqrt{X}$, $x \geq 0$



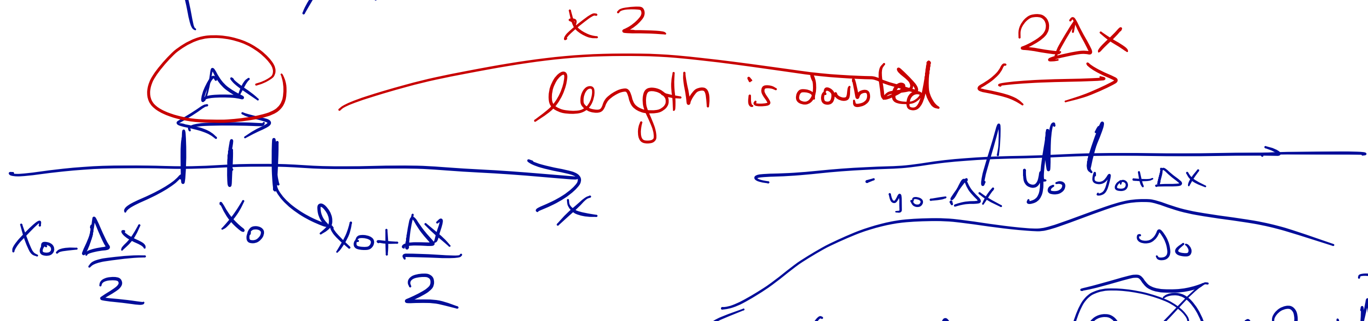
g: one-to-one fn, many-to-one.

$X \rightarrow Y = g(X)$: g a fn.

Ex: $X \sim U[0,1]$, $Y = 2X$

$$P_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & o/w \end{cases} \Rightarrow 0 \leq y \leq 2.$$

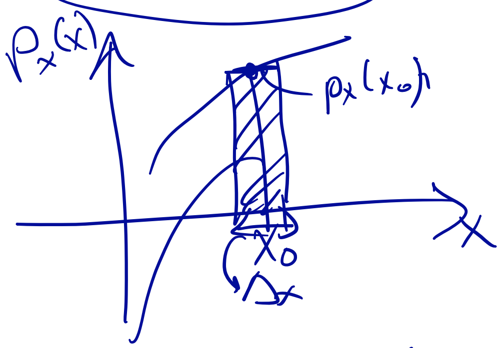
(determine domain of y)

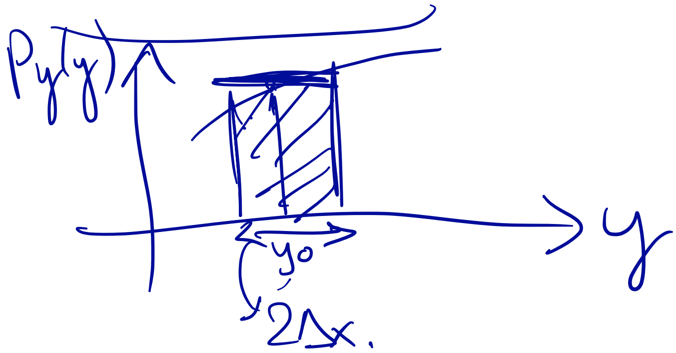


$$P\left(x_0 - \frac{\Delta x}{2} \leq X_0 \leq x_0 + \frac{\Delta x}{2}\right) = P\Big(2x_0 - \Delta x \leq \underbrace{\overset{y_0}{2x_0}}_{y} \leq 2x_0 + \Delta x\Big)$$

$$\Rightarrow \boxed{P_Y(y) = \frac{P_X(x)}{2}}$$

$$\int_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}} P_X(x)\,dx = \int_{2x_0 - \Delta x}^{2x_0 + \Delta x} P_Y(y)\,dy$$





Δx small $\approx P_X(x_0) \cdot \Delta x = p_Y(y_0)\, 2\Delta x$

$$\Rightarrow P_Y(y_0) = \frac{P_X(x_0)}{2}$$

In general: If $g(\cdot)$ is 1-1: $Y = g(X)$

$$|dy| = \left|\frac{d\,g(x)}{dx}\right|$$

Jacobian of the transform

$$\boxed{p_Y(y) = \frac{p_X(x)}{\left|\frac{dg(x)}{dx}\right|}}$$

$x = g^{-1}(y)$

$$\boxed{p_Y(y) = p_X(g^{-1}(y))\left|\frac{d\,g^{-1}(y)}{dy}\right|}$$

exercise show using the formula.

ex: $y = g(x) = 2x$ $\quad p_Y(y) = ?$

Ex: $Y = aX + b$ (affine transform) $a > 0$, a, b scalars

$X \sim \mathcal{N}(0,1) \rightarrow S^x = (-\infty, \infty)$

$Y \sim ?$ $\quad S^y = (-\infty, \infty)$ ← check this first

$$p_Y(y) = p_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}, \quad \left|\frac{dg(x)}{dx}\right| = |a| \quad a > 0.$$

$$= \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(y-b)^2}{2a^2}}$$

$X \sim \mathcal{N}(0,1)$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

insert $\left(\frac{y-b}{a}\right)$

$$= \mathcal{N}(b, a^2)$$

⇒ A linear (affine) xform of a Gaussian r.v. results in another Gaussian r.v. w/ different parameters.

∴ To generate $Y \sim \mathcal{N}(\mu, \sigma^2)$: use randn$(\cdot) \Rightarrow x$ ⇒ $y = \sigma x + \mu$ ✓

Thm: (More general) (We'll do this next time)

When $g(\cdot)$ is many-to-one; use the same formula &
add it for all roots

$y = g(x)$

$x_i = g_i^{-1}(y)$

for $i = 1, \dots, M$

\# element in $g^{-1}(y)$

$$P_y(y) = \sum_{i=1}^{M} P_x(g_i^{-1}(y)) \left| \frac{d g_i^{-1}(y)}{dy} \right|$$

M ← no. of roots.