NUMERICAL METHODS
Week-13
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Linear Programming (LP)

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Linear Programming (LP)

- **What** is Optimization and LP?
- **Why** do we use LP?
- **How** do we solve LP?

What is Optimization & LP?

- **Optimization** is the mathematical discipline which is concerned with finding the maxima and minima of functions, possibly subject to constraints.
- **A Linear Programming (LP) model** seeks to maximize or minimize **a linear function**, subject to a set of linear constraints.
- The linear model consists of the following components:
  - A set of decision variables.
  - An objective function.
  - A set of constraints.

Why Linear Programming?

- The Importance of Linear Programming
  - Many real world problems lend themselves to linear programming modeling.
  - Many real world problems can be approximated by linear models.
  - There are well-known successful applications in:
    - Engineering
    - Manufacturing
    - Marketing
    - Finance (investment)
    - Advertising
    - Agriculture
    - Transportation
    - …

How to represent and Solve LP?

- First Primal Form

$$
\begin{cases}
\text{maximize:} & \boldsymbol{c}^T \boldsymbol{x} \\
\text{constraints:} & \begin{cases} \boldsymbol{A}\boldsymbol{x} \leqq \boldsymbol{b} \\ \boldsymbol{x} \geqq 0 \end{cases}
\end{cases}
$$

$$
\begin{cases}
\text{maximize:} & 2x_1 + 3x_2 \\
\text{constraints:} & \begin{cases} 4x_1 + 5x_2 \leqq 6 \\ 7x_1 + 8x_2 \leqq 9 \\ 10x_1 + 11x_2 \leqq 12 \\ x_1 \geqq 0 \quad x_2 \geqq 0 \end{cases}
\end{cases}
$$

Problem Formulation

To Formulate the LP problem

(i) Pick out important information

(ii) Formulate constraints

(iii) Formulate objective function

Problem Formulation-Example-1

Suppose that a certain factory uses **two raw materials** to produce two products. Suppose also that the following are true:

→Each unit of the first product requires 5 units of the first raw material and 3 of the second.

→Each unit of the second product requires 3 units of the first raw material and 6 of the second.

→On hand are 15 units of the first raw material and 18 units of the second.

→The profits on sales of the products are 2 per unit for the first product and 3 per unit for the second product.

How should the raw materials be used to realize a maximum profit?

- To answer this question,

variables x_1 and x_2 are introduced to represent the number of units of the two products to be manufactured.

In terms of these variables, the profit is

$$2x_1 + 3x_2$$

- The process uses up $5\mathbf{x}_1 + 3\mathbf{x}_2$ units of the first raw material and $3\mathbf{x}_1 + 6\mathbf{x}_2$ units of the second.

The limitations in the fact above are expressed by these inequalities:

$$\begin{cases} 5x_1 + 3x_2 \leqq 15 \\ 3x_1 + 6x_2 \leqq 18 \end{cases}$$

The LP formulation for this problem is:

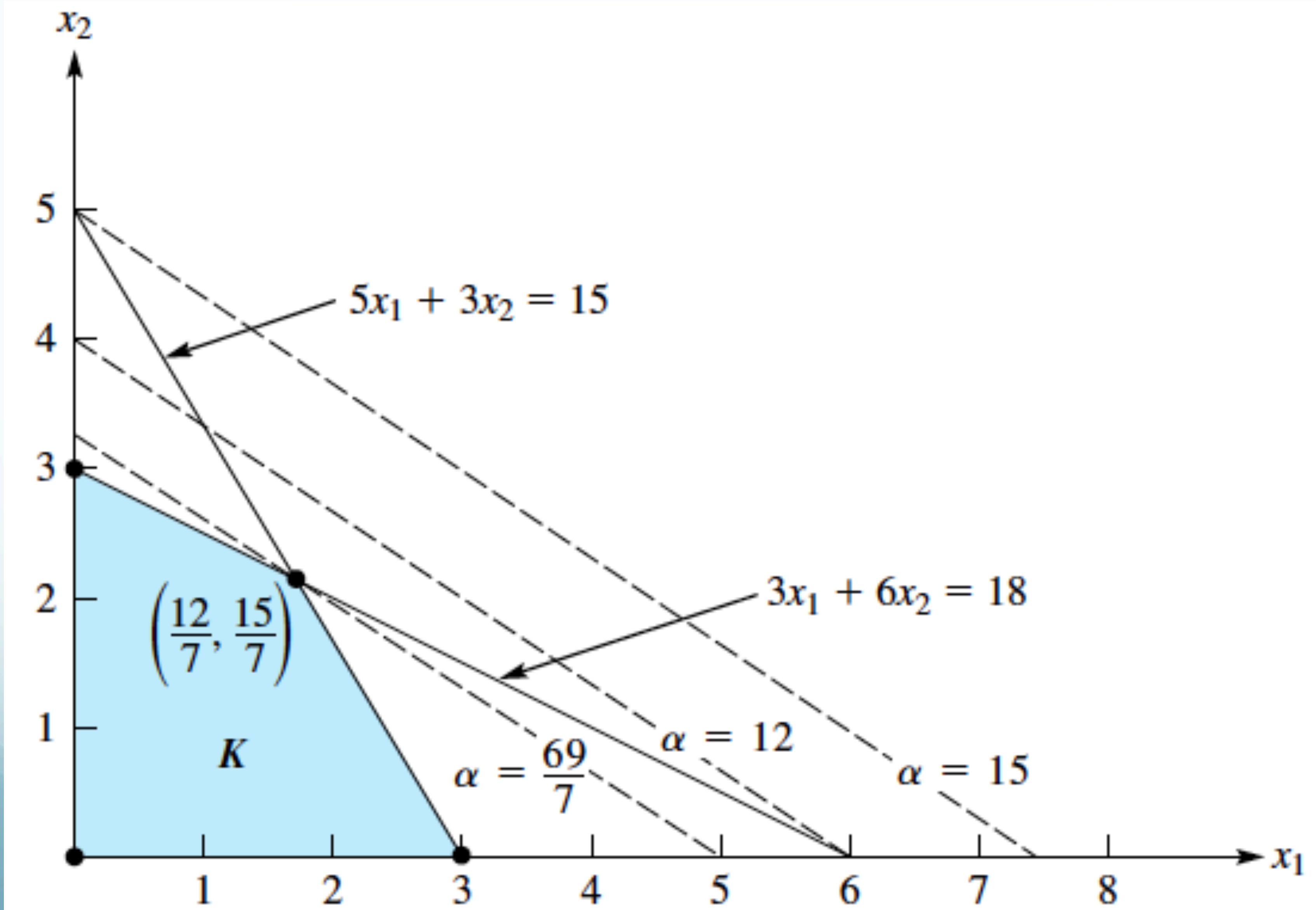
$$\begin{cases} \text{maximize:} \quad 2x_1 + 3x_2 \\ \text{constraints:} \begin{cases} 5x_1 + 3x_2 \leqq 15 \\ 3x_1 + 6x_2 \leqq 18 \\ x_1 \geqq 0 \quad x_2 \geqq 0 \end{cases} \end{cases}$$

- More precisely, among all vectors x in the set

$$\boldsymbol{K} = \{\boldsymbol{x} : \boldsymbol{x} \geqq 0,\ 5x_1 + 3x_2 \leqq 15,\ 3x_1 + 6x_2 \leqq 18\}$$

we want the one that makes $2\mathbf{x}_1 + 3\mathbf{x}_2$ as large as possible.

Graphical Solution



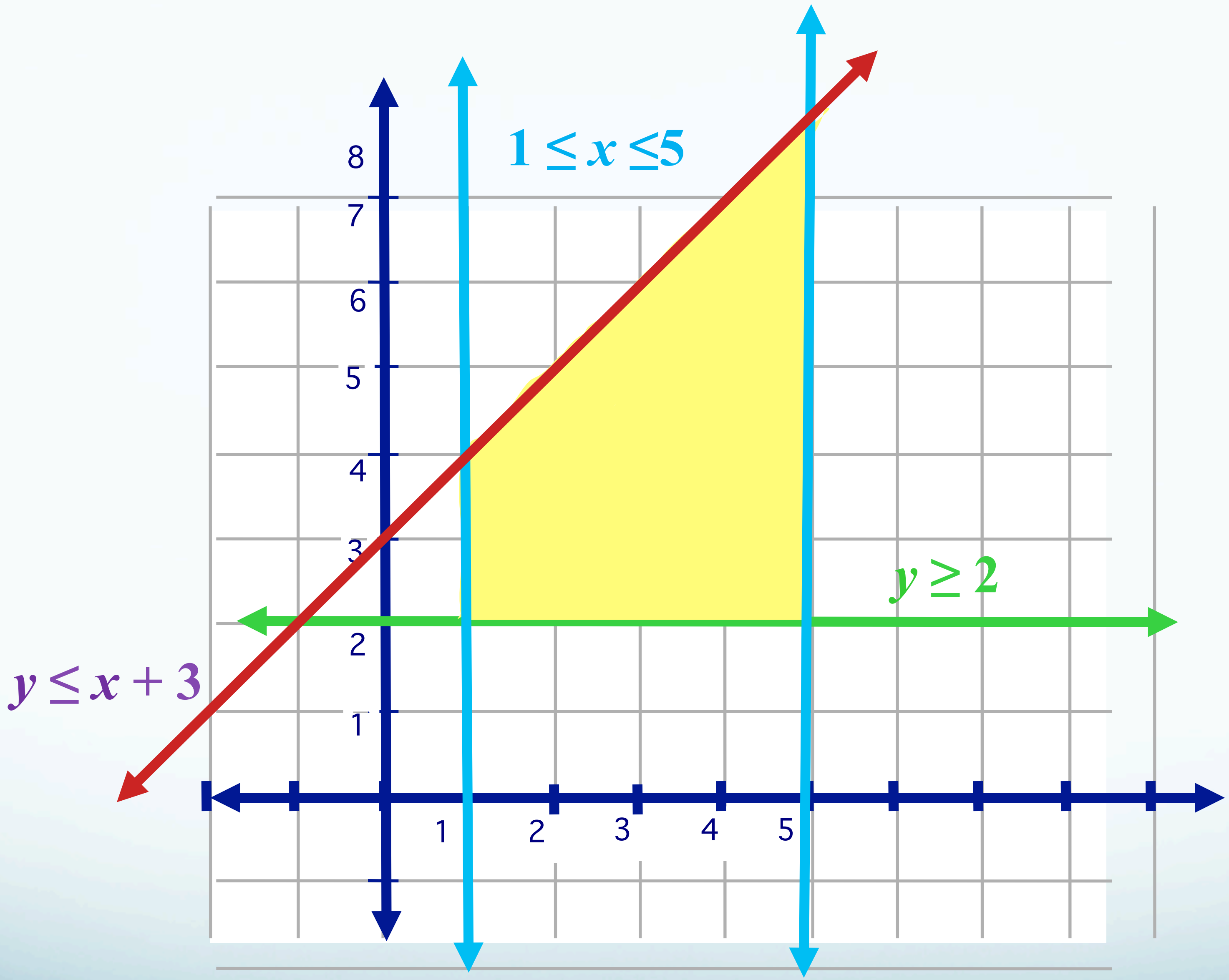
Example-2:
Find the minimum and maximum value of the function $f(x, y) = 3x - 2y$.

We are given the constraints:

- $y \geq 2$
- $1 \leq x \leq 5$
- $y \leq x + 3$

In LP Perspective..

- Find the minimum and maximum values by graphing the inequalities and finding the vertices of the polygon formed.
- Substitute the vertices into the function and find the largest and smallest values.



$1 \leq x \leq 5$
$y \geq 2$
$y \leq x + 3$
8
7
6
5
4
3
2
1
1
2
3
4
5

- The vertices of the quadrilateral formed are:

(1, 2) (1, 4) (5, 2) (5, 8)

- Plug these points into the function $f(x, y) = 3x - 2y$

$f(x, y) = 3x - 2y$

- $f(1, 2) = 3(1) - 2(2) = 3 - 4 = -1$
- $f(1, 4) = 3(1) - 2(4) = 3 - 8 = -5$
- $f(5, 2) = 3(5) - 2(2) = 15 - 4 = 11$
- $f(5, 8) = 3(5) - 2(8) = 15 - 16 = -1$

Solution

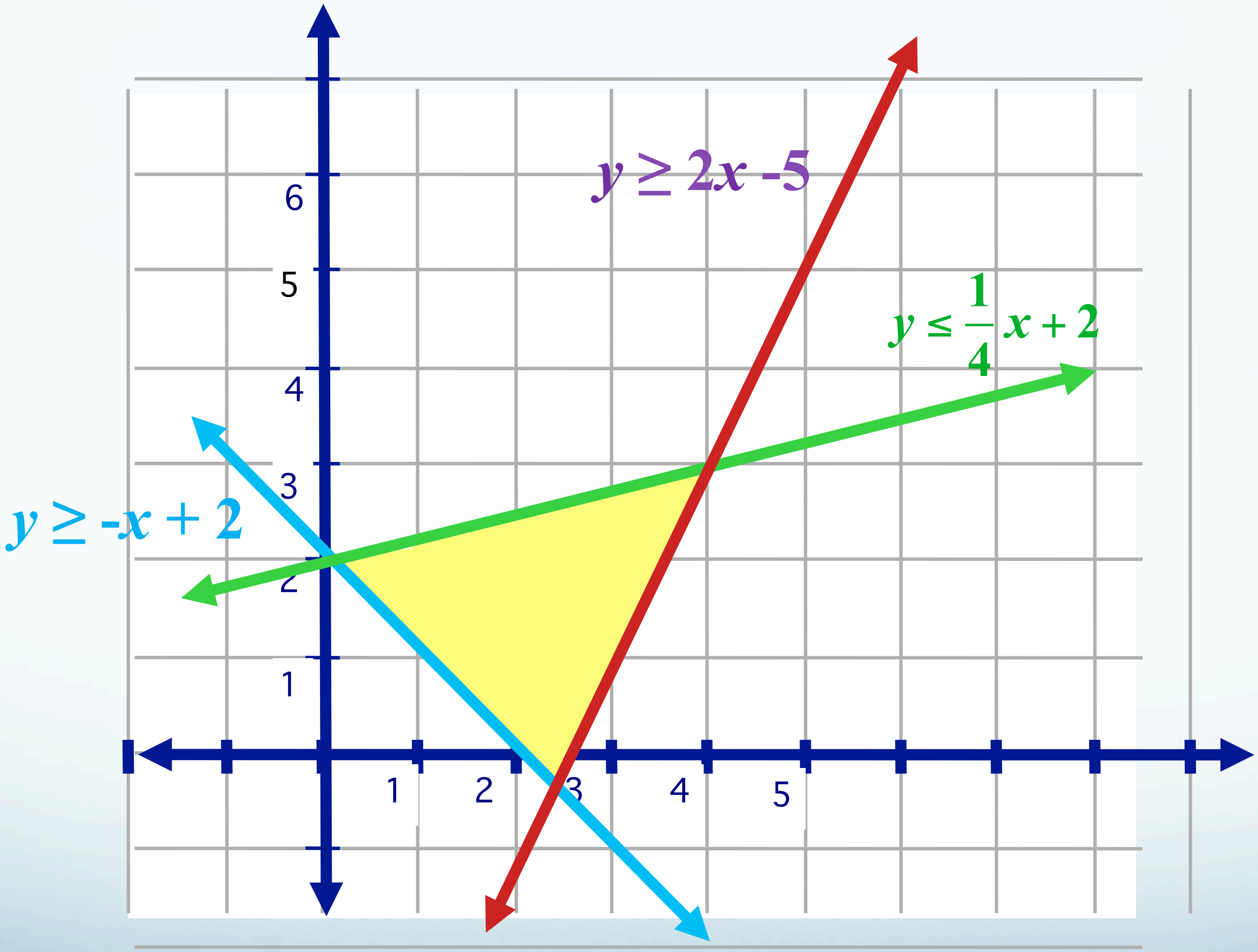
- $f(1, 4) = -5$ minimum
- $f(5, 2) = 11$ maximum

Example 3: Find the minimum and maximum value of the function

$$f(x, y) = 4x + 3y$$

We are given the constraints:

- $y \geq -x + 2$
- $y \leq \frac{1}{4}x + 2$
- $y \geq 2x - 5$


$y \geq 2x - 5$
$y \leq \frac{1}{4}x + 2$
$y \geq -x + 2$
6
5
4
3
2
1
1
2
3
4
5

$f(x, y) = 4x + 3y$

- $f(0, 2) = 4(0) + 3(2) = 6$
- $f(4, 3) = 4(4) + 3(3) = 25$
- $f\left(\frac{7}{3}, -\frac{1}{3}\right) = 4\left(\frac{7}{3}\right) + 3\left(-\frac{1}{3}\right) = \frac{28}{3} - 1 = \frac{25}{3}$

$\Downarrow$

- **$f(0, 2) = 6$ minimum**
- **$f(4, 3) = 25$ maximum**

Problem Formulation-Example 4

- A small factory produces two types of toys: cars and diggers. In the manufacturing process two machines are used: the moulder and the colouriser. A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser. The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day. Each digger gives a profit of £16 and each car gives a profit of £14. The profit needs to be maximised.

- How do we formulate this problem?

PICKING OUT IMPORTANT INFORMATION

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- A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser.

- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.

- Using the decision variables

  d = number of diggers

  c = number of cars

  make two constraints from this information.

FORMING CONSTRAINT 1

THE MOULDER

- A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser.
- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.

$$2d + c \leq 16$$

FORMING CONSTRAINT 2
THE COLOURISER

- A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser.
- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.

$$d + c \leq 9$$

FORMING THE OBJECTIVE FUNCTION

- Each digger gives a profit of £16 and each car gives a profit of £14.

$$Z = 16d + 14c$$

- MAXIMIZE $Z = 16d + 14c$

  subject to the constraints:

  (i) $2d + c \leq 16$

  (ii) $d + c \leq 9$

  (iii) $c \geq 0$, $d \geq 0$

- **VERY IMPORTANT**

DON' T FORGET YOUR NON – NEGATIVITY CONSTRAINTS !

The Prototype Production- Example 5

- A Factory manufactures two chip models:
  - Chip A.
  - Chip B.
- Resources are limited to
  - 1000 units of special material.
  - 40 hours of production time per week.

The Prototype Production- Example 5

- Marketing requirement
  - Total production cannot exceed 700 dozens.
  - Number of dozens of Chip A cannot exceed number of dozens of Chip B by more than 350.

- Technological input
  - A requires 2 units of material and 3 minutes of labor per dozen.
  - B requires 1 units of material and 4 minutes of labor per dozen.

- The current production plan calls for:
  - Producing as much as possible of the more profitable product, A ($8 profit per dozen).
  - Use resources left over to produce B ($5 profit per dozen), while remaining within the marketing guidelines.

The Linear Programming Model

- Decisions variables:
    - X_1 = Weekly production level of A (in dozens)
    - X_2 = Weekly production level of B (in dozens).
- Objective Function:
    - Weekly profit, to be maximized

The Linear Programming Model

Max $8X_1 + 5X_2$ (Weekly profit)

subject to

$2X_1 + 1X_2 \leq 1000$ (Material)

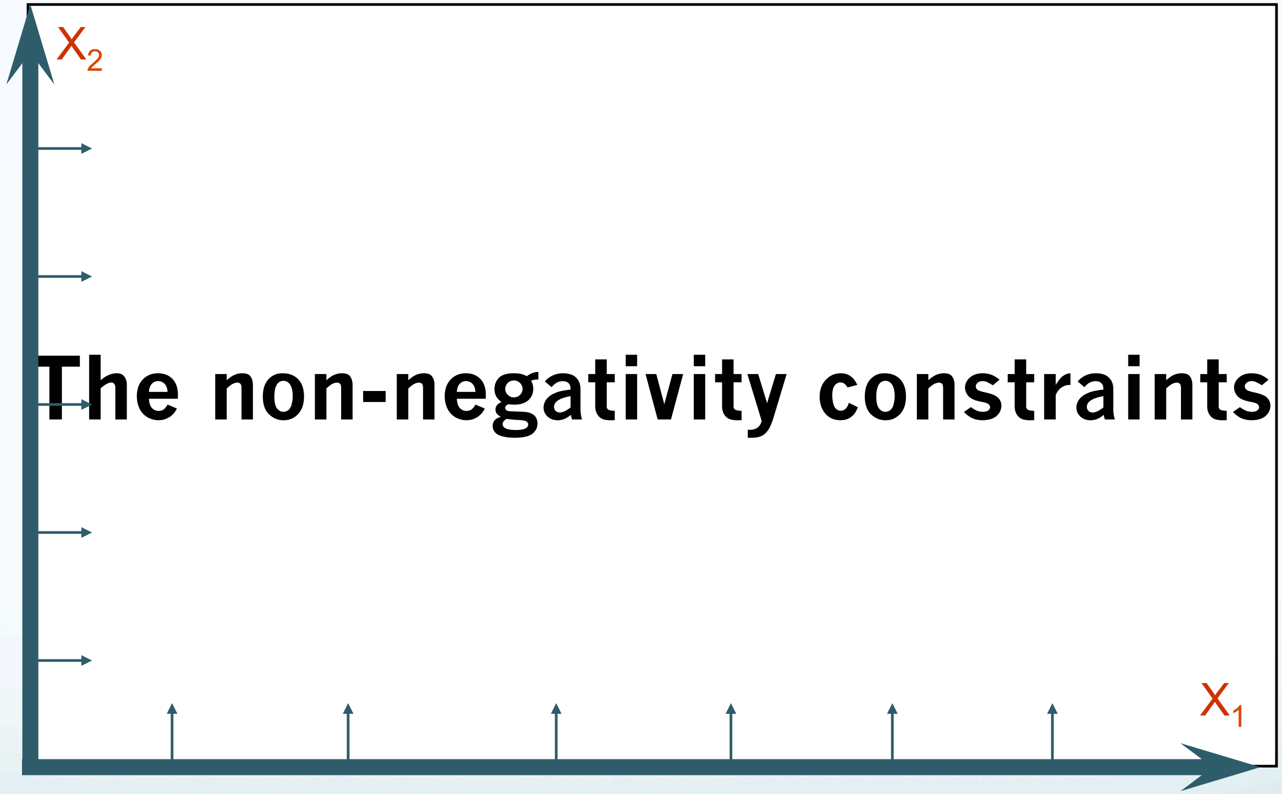
$3X_1 + 4X_2 \leq 2400$ (Production Time)

$X_1 + X_2 \leq 700$ (Total production)

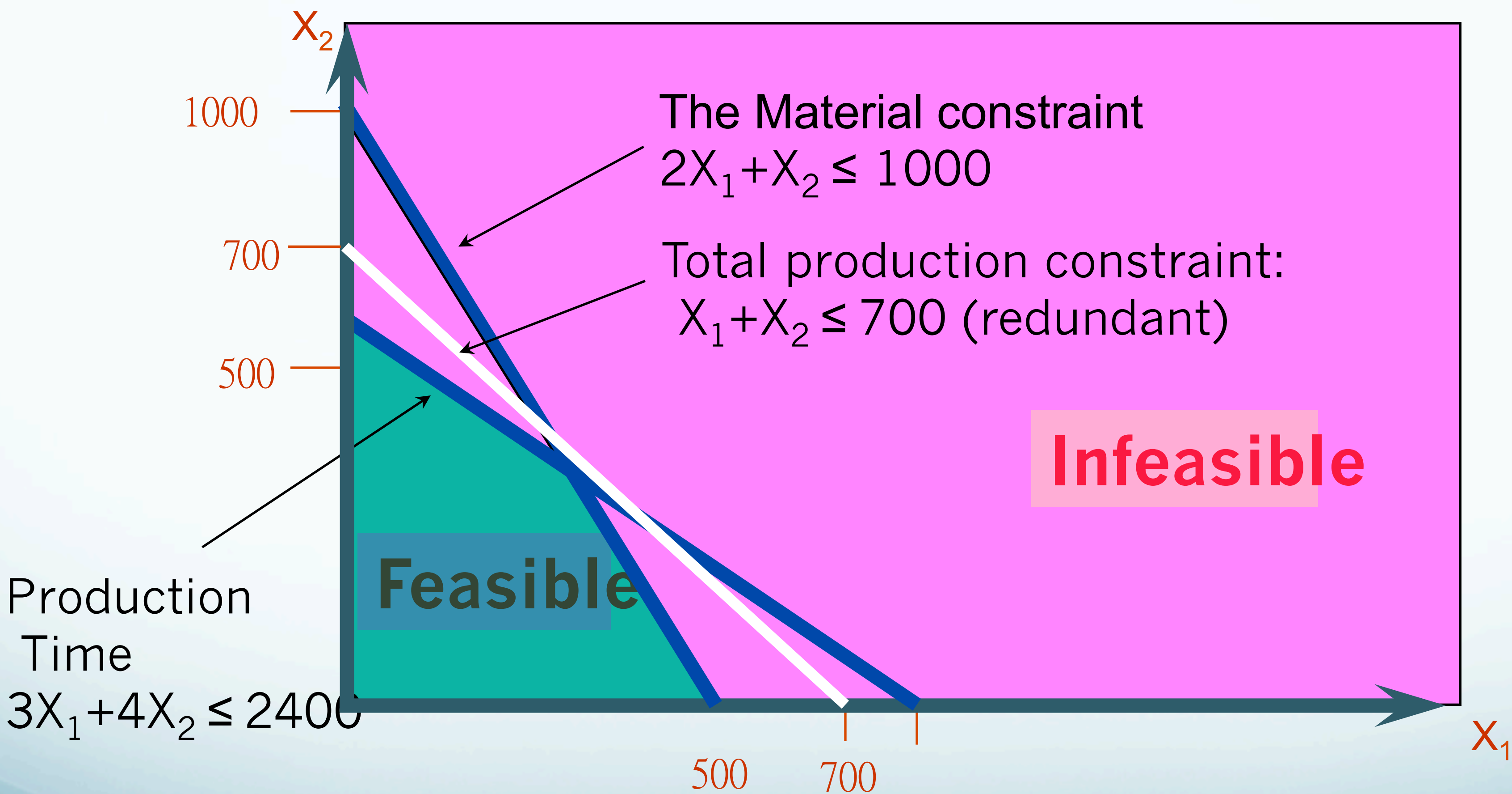
$X_1 - X_2 \leq 350$ (Mix)

$X_j >= 0, \; j = 1,2$ (Nonnegativity)

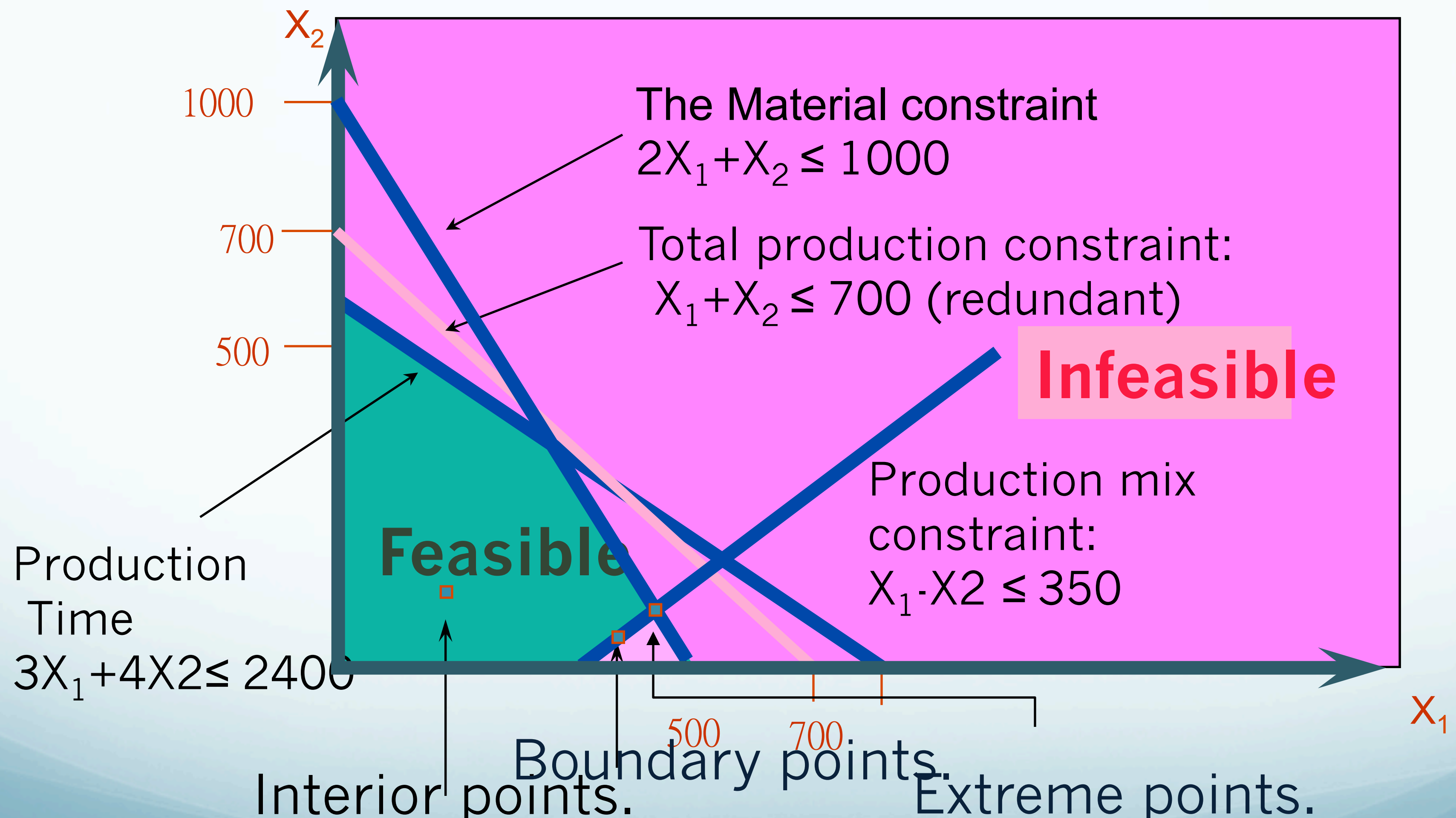
Graphical Analysis – the Feasible Region



Graphical Analysis – the Feasible Region



Graphical Analysis – the Feasible Region



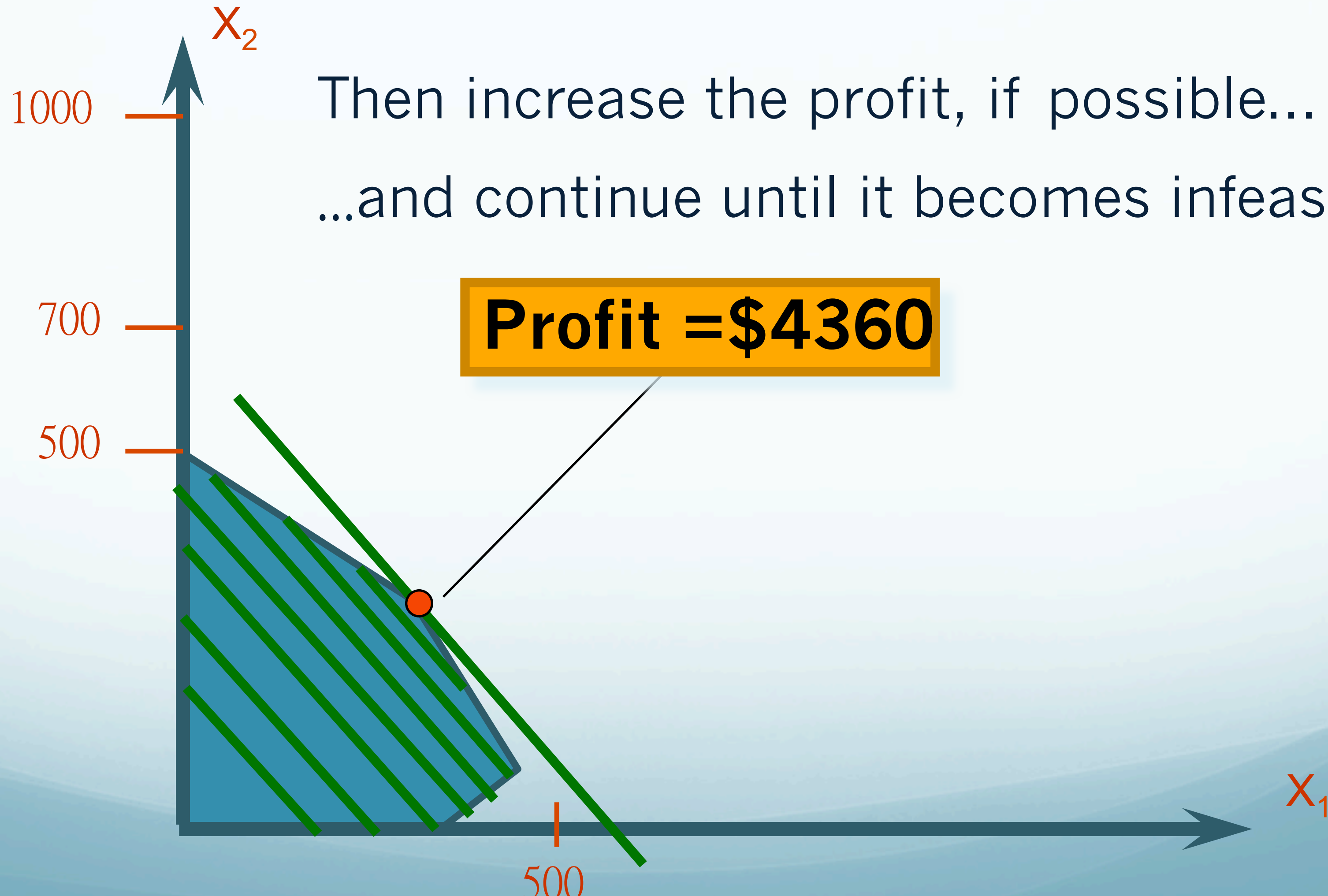
- **There are three types of feasible points**

The search for an optimal solution

Start at some arbitrary profit, say profit = $2,000...

Then increase the profit, if possible...

...and continue until it becomes infeasible



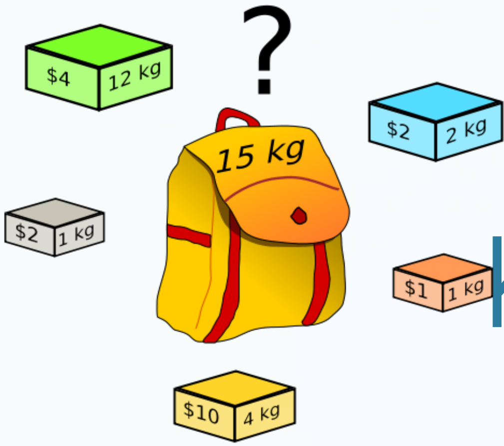
Summary of the optimal solution

Chip A = 320 dozen

Chip B = 360 dozen

Profit = $4360

- This solution utilizes all the materials and all the production hours.
- Total production is only 680 (not 700).
- Chip A production exceeds B's production by only 40 dozens.



Knapsack Problem

- You have a knapsack that has capacity (weight) C.
- You have several items $I_1,\ldots,I_n$.
- Each item I_j has a weight w_j and a benefit b_j.
- You want to place a certain number of copies of each item I_j in the knapsack so that:
  - The knapsack weight capacity is not exceeded and
  - The total benefit is maximal.

Example

Item	Weight	Benefit
A	2	60
B	3	75
C	4	90

Capacity = 5

Key question

- Suppose f(w) represents the *maximal possible benefit* of a knapsack with weight w.
- We want to find (in the example) f(5).
- Is there anything we can say about f(w) for arbitrary w?

Key observation

- To fill a knapsack with items of weight w, we must have added items into the knapsack in some order.
- Suppose the last such item was I_j with weight w_i and benefit b_i.
- Consider the knapsack with weight $(w- w_i)$. Clearly, we chose to add I_j to this knapsack because of all items with weight w_i or less, I_j had the max benefit b_i.

Key observation

- Thus, $f(w) = MAX \{ b_j + f(w\text{-}w_j) \mid I_j \text{ is an item}\}$.
- This gives rise to an immediate recursive algorithm to determine how to fill a knapsack.

f(0), f(1)

- f(0) = 0. Why? The knapsack with capacity 0 can have nothing in it.
- f(1) = 0. There is no item with weight 1.

f(2)

- f(2) = 60. There is only one item with weight 60.
- **Choose A.**

f(3)

- $f(3) = MAX \{ b_j + f(w\text{-}w_j) \mid I_j \text{ is an item}\}$.

$= MAX \{ 60+f(3\text{-}2),\ 75 + f(3\text{-}3)\}$

$= MAX \{ 60 + 0,\ 75 + 0 \}$

$= 75.$

Choose B.

f(4)

- $f(4) = MAX \{ b_j + f(w\text{-}w_j) \mid I_j \text{ is an item}\}.$

$= MAX \{ 60 + f(4\text{-}2), 75 + f(4\text{-}3), 90+f(4\text{-}4)\}$

$= MAX \{ 60 + 60, 75 + f(1), 90 + f(0)\}$

$= MAX \{ 120, 75, 90\}$

$=120.$

Choose A.

f(5)

- f(5) = MAX { b_j + f(w-w_j) | I_j is an item}.

= MAX { 60 + f(5-2), 75 + f(5-3), 90+f(5-4)}

= MAX { 60 + f(3), 75 + f(2), 90 + f(1)}

= MAX { 60 + 75, 75 + 60, 90+0}

= 135.

Choose A or B.

Result

- Optimal knapsack weight is 135.
- Two possible optimal solutions:
  - Choose A during computation of f(5). Choose B in computation of f(3).
  - Choose B during computation of f(5). Choose A in computation of f(2).
- Both solutions coincide. Take A and B.