تمرین سوم - سینا [illegible]

باسمه تعالی

1.

a.

b.

$$T=6 \rightarrow \omega_0 = \frac{2\pi}{T} = \pi/3 \quad , \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi/3\, t}$$

$$a_k = 1/6 \int_{-3}^{3} x(t) e^{-jk\omega_0 t} dt = 1/6 \int_{-2}^{-1} (t+2) e^{-jk\omega_0 t}$$

$$+ 1/6 \int_{-1}^{1} e^{-jk\omega_0 t} + 1/6 \int_{1}^{2} (2-t) e^{-jk\omega_0 t}$$

$$= 1/6\, e^{j2\omega_0 k} \left(\frac{e^{-jk\omega_0}}{-jk\omega_0} + \frac{e^{-jk\omega_0} - 1}{(jk\omega_0)^2} \right) + 1/6\, \frac{e^{jk\omega_0} - e^{-jk\omega_0}}{jk\omega_0}$$

$$+ 1/6\, e^{-2jk\omega_0} \left(\frac{e^{jk\omega_0}}{jk\omega_0} - \frac{e^{jk\omega_0} - 1}{(jk\omega_0)^2} \right)$$

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d. $$x_1(t) = \sum_{-\infty}^{\infty} \delta(t-2k) \longrightarrow x(t) = x_1(t) - 2x_1(t-1)$$

$$\left.\begin{array}{l} x_1(t) \xleftrightarrow{F_s} a_k \\ x(t) \xleftrightarrow{F_s} b_k \end{array}\right\} \longrightarrow b_k = a_k - 2e^{-jk\omega_0} a_k$$

$$\left.\begin{array}{l} T=2 \longrightarrow \omega_0 = 2\pi/2 = \pi \\ \text{است } T=2 \text{ قطار ضربه با } x_1(t) \end{array}\right\} \longrightarrow a_k = \frac{1}{T} = 1/2$$

$$\longrightarrow b_k = 1/2 - a^{-jk\pi}$$

e.

$x_1(t)$ Symmetric Square function with $T_1 = 1/2$, $T = 6$

$$\longrightarrow x(t) = x_1(t + 3/2) - x_1(t - 3/2)$$

$$\left.\begin{array}{l} x_1(t) \xleftrightarrow{Fs} a_k \\ x(t) \xleftrightarrow{Fs} b_k \end{array}\right\} \longrightarrow b_k = e^{3/2 jk\omega_0} a_k - e^{-j3/2 k\omega_0} a_k$$

$$\left.\begin{array}{l} T = 6 \longrightarrow \omega_0 = \pi/3 \longrightarrow b_k = (e^{jk\pi/2} - e^{-jk\pi/2})\, a_k \\ a_k = \begin{cases} 1/6 & k = 0 \\ \dfrac{\sin k\pi/6}{k\pi} & k \neq 0 \end{cases} \end{array}\right\} \longrightarrow$$

$$b_k = \begin{cases} \dfrac{e^{jk\pi/2} - e^{-jk\pi/2}}{6} & k = 0 \\ \dfrac{(e^{j\pi/2} - e^{-j\pi/2}) \sin k\pi/6}{k\pi} & k \neq 0 \end{cases}$$

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b.

a.

$$N = 7 \longrightarrow \omega_0 = 2\pi/7 \longrightarrow x[n] = \sum_{\langle N \rangle} a_k e^{jk 2\pi/7 n}$$

$$a_k = 1/7 \sum_{n=0}^{6} x[n] e^{-jk 2\pi/N n} = 1/7 \sum_{n=0}^{4} e^{-jk 2\pi/N n}$$

$$\longrightarrow a_k = 1/7 \, \frac{e^{-j5k 2\pi/7} - 1}{e^{-jk 2\pi/7} - 1}$$

b.

b. $N=6 \rightarrow \omega_0 = 2\pi/6 \rightarrow x[n] = \sum_{\langle 6\rangle} a_k e^{jk\frac{\pi}{3}n}$

$\rightarrow a_k = \frac{1}{6}\sum_{0}^{5} x[n]\, e^{-jk\frac{2\pi}{6}n} \rightarrow a_k = \frac{1}{6}\sum_{n=0}^{3} e^{-jk\frac{2\pi}{6}n}$

$\rightarrow \frac{1}{6}\,\frac{e^{-jk\frac{4\pi}{3}}-1}{e^{-jk\frac{\pi}{3}}-1}$

2.

a. $a_{-k}^{*} = \begin{cases} 2 & k=0 \\ -j\left(\frac{1}{2}\right)^{|k|} & \text{otherwise} \end{cases} \rightarrow a_{-k}^{*} \neq a_k$ پس x حقیقی نیست

b. $a_k = a_{-k} \rightarrow$ x زوج است

c. $x(t) = \sum a_k e^{jk\omega_0 t} \rightarrow \frac{dx(t)}{dt} = \sum jk\omega_0 a_k e^{jk\omega_0 t}$

$\rightarrow \frac{dx}{dt} \overset{\mathcal{F}}{\longleftrightarrow} jk\omega_0 a_k = c_k$

$\rightarrow c_{-k} = -jk\omega_0 a_{-k} = -jk\omega_0 a_k = -c_{-k} \rightarrow \frac{dx}{dt}$ فرد است

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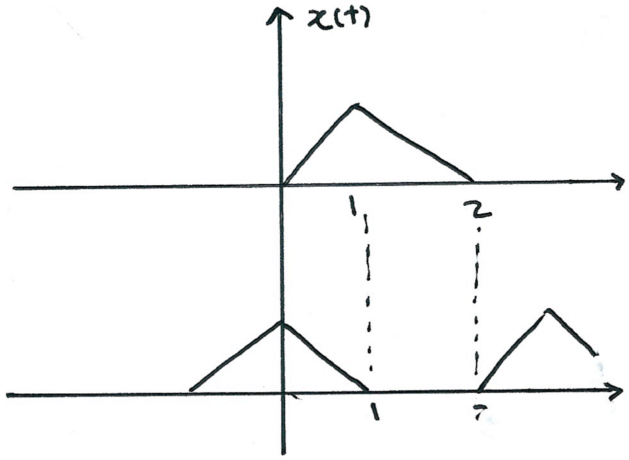
3. $a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{3}\int_0^3 x(t)\,dt = \frac{1}{3}$

$\sum_{-\infty}^{\infty} a_k e^{j2\times k\times 2\pi/3} = x(2) = 0$

$$\sum_{-\infty}^{\infty} a_K^2 e^{j4\frac{2K\pi}{3}} = T(x \circledast x)_{(t=4)} \rightarrow (x \circledast x)_4 = \int_0^3 x(\tau)\,x(4-\tau)\,d\tau$$

$$= \int_0^1 t(1-t)\,dt = t^2/2 - t^3/3 = 1/6$$



$$\rightarrow \sum_{-\infty}^{\infty} a_K^2 e^{jK2\pi/3} = 1/18$$

$$\sum_{K=-\infty}^{+\infty} a_{K-2}^2 \rightarrow (K' = K-2) \quad \sum_{-\infty}^{\infty} a_{K-2}^2 = \sum_{-\infty}^{\infty} a_{K'}^2 = 1/3 \int_0^3 |x(t)|^2\,dt$$

$$= 1/3 \int_0^1 t^2\,dt + 1/3 \int_1^2 (2-t)^2\,dt = 2/9$$

4.

a. $y(t) = x(t) * h(t) = (u(t) - u(t-1)) * e^{-t}u(t)$

$$= \int_{-\infty}^{\infty} (u(\tau) - u(\tau-1))\,e^{-(t-\tau)}\,u(t-\tau) = \int_0^{\min(1,t)} e^{-t+\tau}\,d\tau$$

$$= \begin{cases} 1 - e^{-t} & 0 < t < 1 \\ e^{1-t} - e^{-t} & t > 1 \end{cases}$$

b. $g(t) = \dfrac{dx(t)}{dt} * h(t) = \left(\delta(t) - \delta(t-1)\right) * h(t)$

$= h(t) - h(t-1) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$

c. $\frac{dx}{dt} = x(t) * u_1(t) \longrightarrow g(t) = x * u_1 * h$

$$= \underbrace{x * h}_{y} * u_1 = y(t) * u_1(t) = \frac{dy}{dt}(t)$$

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5. $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0(t-\tau)} h(\tau)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_K e^{jK\omega_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-jK\omega_0\tau} d\tau}_{H(jK\omega_0)}$$

$$\longrightarrow \left.\begin{array}{l} b_K = a_K H(jK\omega_0) \\ \omega_0 = \frac{2\pi}{T} = 14 \end{array}\right\} \longrightarrow b_K = a_K H(j14K)$$

$$\longrightarrow y(t) = x(t) \rightarrow b_K = a_K \rightarrow \forall K \quad H(j14K) \neq 1 \longrightarrow a_K = 0$$

$$\longrightarrow H(j14K) \neq 1 \rightarrow |14K| < 250 \longrightarrow |K| \leq \frac{250}{14}$$

$$\longrightarrow |K| \leq 17.85 \longrightarrow -17 \leq K \leq 17 \quad a_K = 0$$

6.

a.

$$\left.\begin{array}{l} x(t-t_0) \overset{\tilde{F.S.}}{\longleftrightarrow} a_K e^{-jK\omega_0 t_0} \\ x(t+t_0) \overset{\tilde{F.S.}}{\longleftrightarrow} a_K e^{jK\omega_0 t_0} \end{array}\right\} x(t-t_0) + x(t+t_0)$$

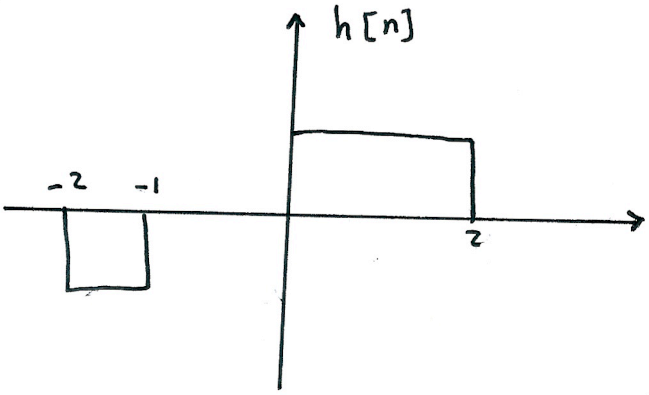
$$\overset{\tilde{F.S.}}{\longleftrightarrow} a_K\left(e^{jK\omega_0 t_0} + e^{-jK\omega_0 t_0}\right)$$

b. $ev(x(t)) = \dfrac{x(t)+x(-t)}{2} \longrightarrow ev(x(t)) \overset{F.S.}{\Longleftrightarrow} \dfrac{a_k + a_{-k}}{2}$

c. $Re\{x(t)\} = \dfrac{x(t)+x^*(t)}{2} \overset{F.S.}{\longleftrightarrow} \dfrac{a_k + a^*_{-k}}{2}$

d. $x(t) \overset{FS}{\longleftrightarrow} a_k \ , \ \dfrac{dx}{dt} \overset{Fs}{\longleftrightarrow} jk\omega_0 a_k \ , \ \dfrac{d^2x(t)}{dt^2} = -k^2\omega_0^2 a_k$

7.



$x[n] = \sum_{-\infty}^{\infty} \delta[n-4k]$

$H(e^{j\omega}) = \sum_{-\infty}^{\infty} h[n] e^{-j\omega n}$

$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$

$N=4 \rightarrow a_k = \frac{1}{4} \sum_{0}^{3} x[n] e^{-j\frac{\pi}{2}kn} = \frac{1}{4}$

$y[N] \overset{F.S.}{\longleftrightarrow} b_k \ ; \ b_k = \frac{1}{4}\left(-e^{2jk\frac{\pi}{2}} - e^{j\frac{\pi}{2}k} + 1 + e^{-j\frac{\pi}{2}k} + e^{-2j\frac{\pi}{2}k}\right)$

8.

a. $x[n-n_0] = \sum_{\langle N \rangle} a_k e^{jk(2\pi/2)(n-n_0)} = \sum \left(a_k e^{-jk\frac{2\pi}{N}n_0}\right) e^{jk\frac{2\pi}{N}n}$

$\longrightarrow b_k = a_k e^{-jk\frac{2\pi}{N}n_0}$

b. $\left.\begin{array}{l} x[n] \xrightarrow{F.s} a_k \\ x[n-1] \longrightarrow a_k e^{-jk\frac{2\pi}{N}} \end{array}\right\} \longrightarrow x[n]-x[n-1] \xrightarrow{F.S} a_k\left(1-e^{-jk\frac{2\pi}{N}}\right)$

c. $\left.\begin{array}{l} x[n] \xrightarrow{F.S} a_k \\ x[n-N/2] \xrightarrow{F.S} a_k e^{-jk\frac{2\pi}{N}\frac{N}{2}} = a_k e^{-jk\pi} \end{array}\right\} \longrightarrow$

$$\longrightarrow x[n] - x[n - N/2] \xrightarrow{F.S} a_K (1 - e^{-jK\pi}) \begin{cases} 0 & \text{even } K \\ 2a_K & \text{odd } K \end{cases}$$

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d. $$x[n] + x[n + N/2] \overset{F.S.}{\longleftrightarrow} b_K$$

$$b_K = \frac{2}{N} \sum_{0}^{N/2 - 1} \left[x[n] + x[n + N/2] \right] e^{-j \, 4\pi/N \, Kn}$$

$$= 2/N \left(\sum_{n=0}^{N/2-1} x[n] e^{-j 4\pi/N \, Kn} + \sum_{n=N/2}^{N-1} x[n] e^{-j 4\pi/N \, Kn} \right)$$

$$= 2 a_{2K}$$

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e. $$x^*[-n] \xrightarrow{FS} b_K$$

$$b_K = 1/N \sum_{n=0}^{N-1} x^*[-n] e^{-j \, 2\pi/N \, K n} = a_K^*$$

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f. $$(-1)^n = e^{jn\pi} \qquad (-1)^n x[n] \longrightarrow b_K$$

$$\longrightarrow b_K = 1/N \sum_{n=0}^{N-1} x[n] e^{jn\pi} e^{-j 2\pi/N \, Kn}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{(-j \frac{2\pi}{N} n)(\frac{-N}{2} + K)} = a_{K - N/2}$$

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g. $$y[n] = (-1)^n x[n] \longrightarrow y[n] = (-1)^n \sum_{\langle N \rangle} a_K e^{jK\frac{2\pi}{N} n}$$

$$= \sum_{\langle N \rangle} a_K e^{j(K + N/2) \, 2\pi/N \, n}$$

$$K' = 2K + N$$

$$\longrightarrow = \sum_{K' = \langle 2N \rangle} a_{\frac{K' - N}{2}} e^{jK' \frac{\pi}{N} n}$$

$$b_K = \begin{cases} a_{\frac{K-N}{2}} & K \text{ is odd} \\ 0 & K \text{ is even} \end{cases}$$

h. $$y[n] = \begin{cases} x[n] & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

$$\longrightarrow y[n] = x[n] + (-1)^n x[n]$$

از قسمت F و g داریم $\longrightarrow$

N is even $\longrightarrow b_K = a_K + a_{K-N/2}$

N is odd $\longrightarrow \begin{cases} b_K = a_K + a_{\frac{K-N}{2}} & K \text{ فرد} \\ a_K - b_K & K \text{ زوج} \end{cases}$