NUMERICAL METHODS
Week-7
26.03.2013

Approximation by Spline functions & Smoothing of Data

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Approximation by Spline functions

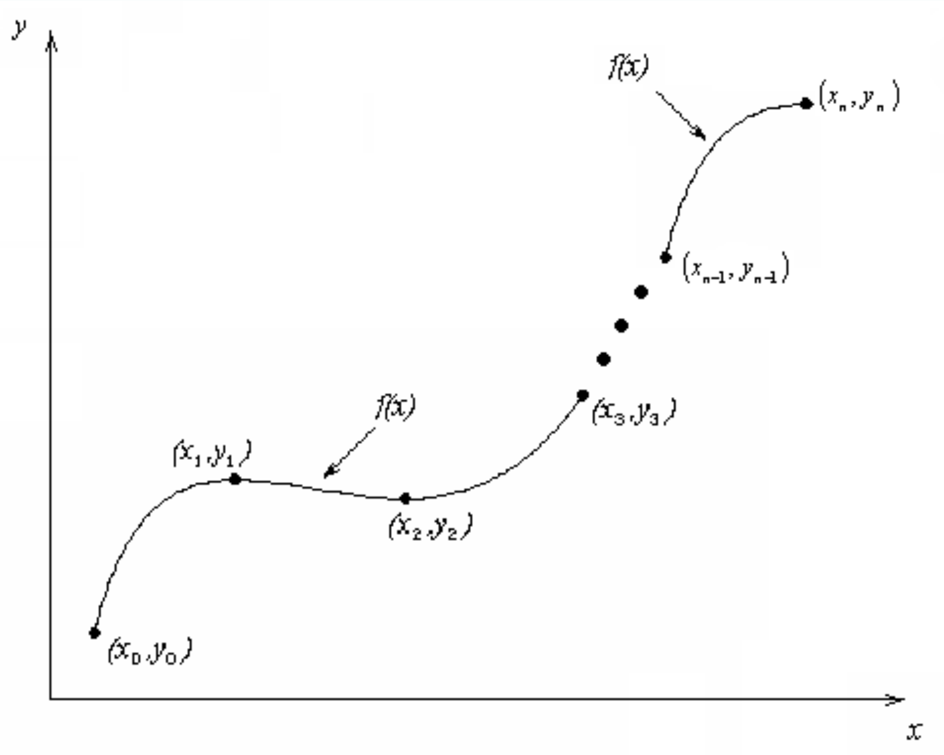
- **What** is a spline function?
- **Why** do we use approximation by splines?
- **How** do we solve spline equations?

What is a Spline?

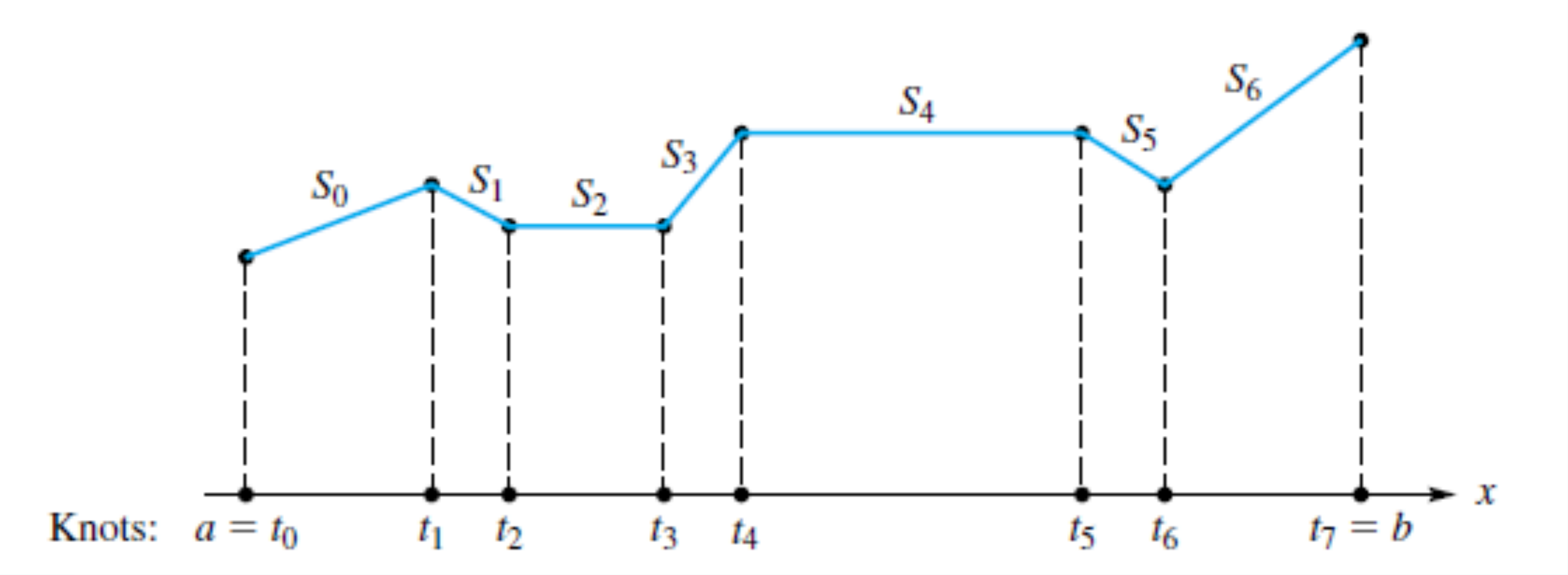
- A spline is a function that consisting of simple functions joined together.
- As with polynomial functions, splines are used to interpolate tabulated data as well as functions.
- A spline is different from a polynomial interpolation, which consists of a single well defined function that approximates a given shape; splines are normally piecewise polynomial.

What is a spline function ?

Given (x_0,y_0), (x_1,y_1), …… (x_n,y_n), find the value of 'y' at a value of 'x' that is not given.



What is a spline function ?



- A simple example is the **polygonal** function (or spline of degree 1), whose pieces are linear polynomials joined together to achieve **continuity**, as in figure. The points t_0, $t_1, \ldots, t_n$ at which the function changes its character are termed **knots** in the theory of splines.

What is a spline function ?

$$S(x)=\begin{cases} S_0(x) & x\in[t_0,t_1] \\ S_1(x) & x\in[t_1,t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x\in[t_{n-1},t_n] \end{cases}$$

where;

$$S_i(x)=a_i x+b_i$$

- If the knots $t_0, t_1, \ldots, t_n$ were given and if the coefficients $a_0, b_0, a_1, b_1, \ldots, a_{n-1}, b_{n-1}$ were all known, then the evaluation of $S(x)$ at a **specific** x would proceed by first determining the interval that contains x and then using the appropriate linear function for that interval.

Why Splines ?

- Splines are used to approximate complex functions and shapes.
- Drawbacks of higher order polynomials in interpolating functions.
- Splines are normally piecewise polynomials so provides better approximation then polynomial interpolatings.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

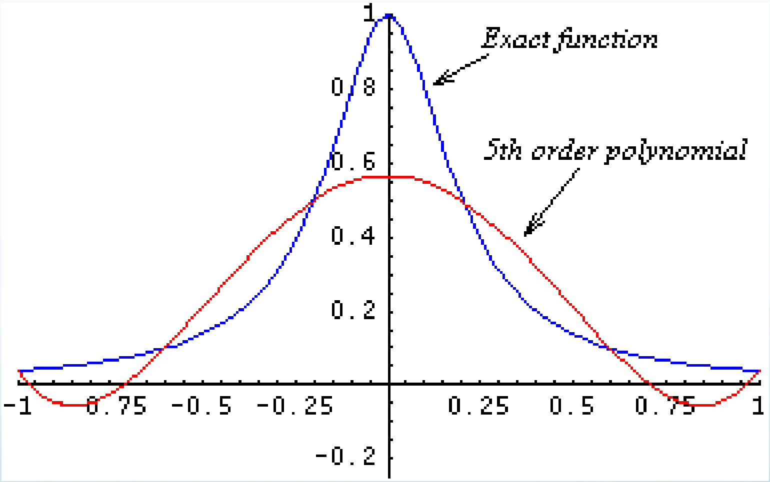


Figure : 5th order polynomial vs. exact function

Why Splines ?

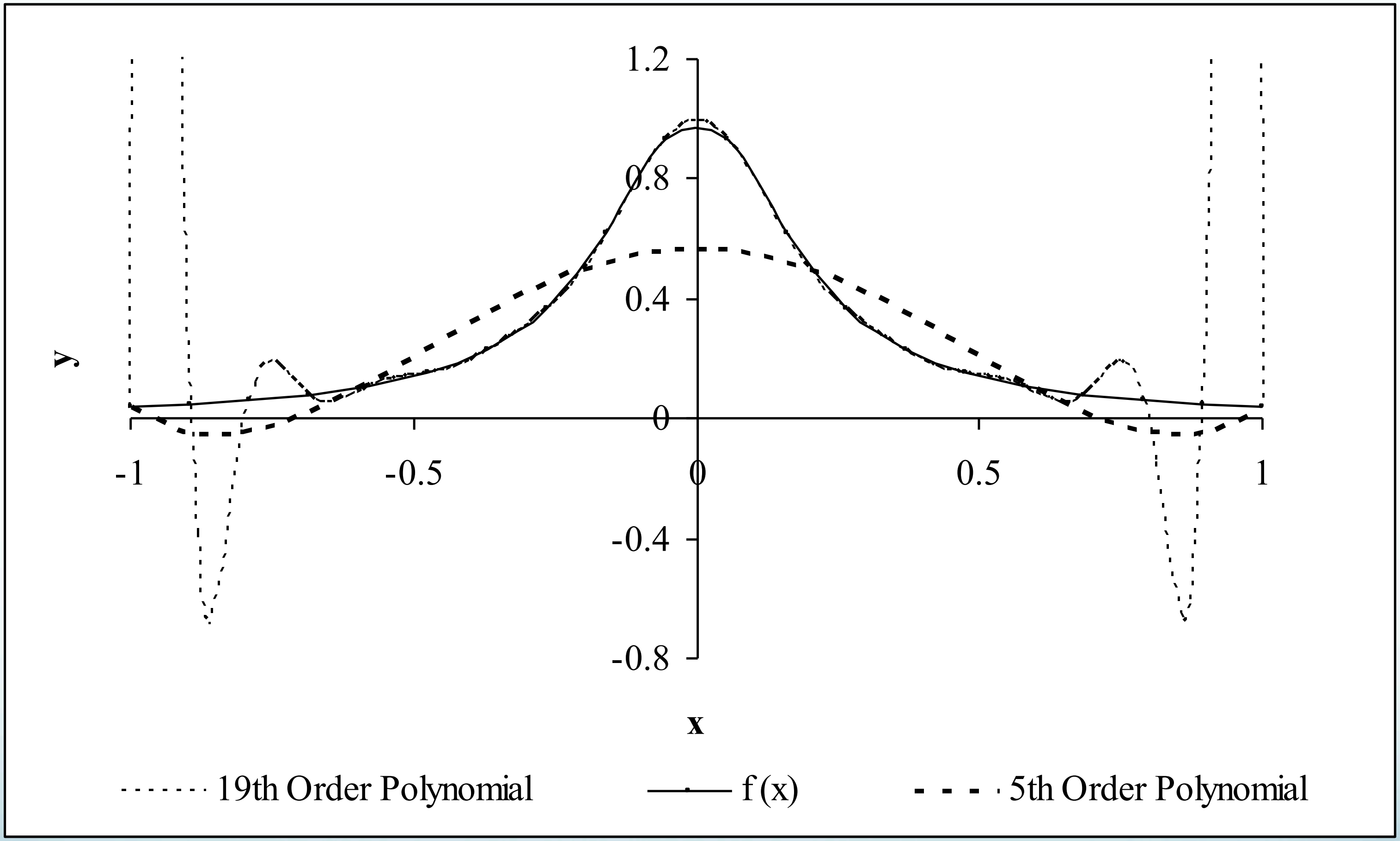
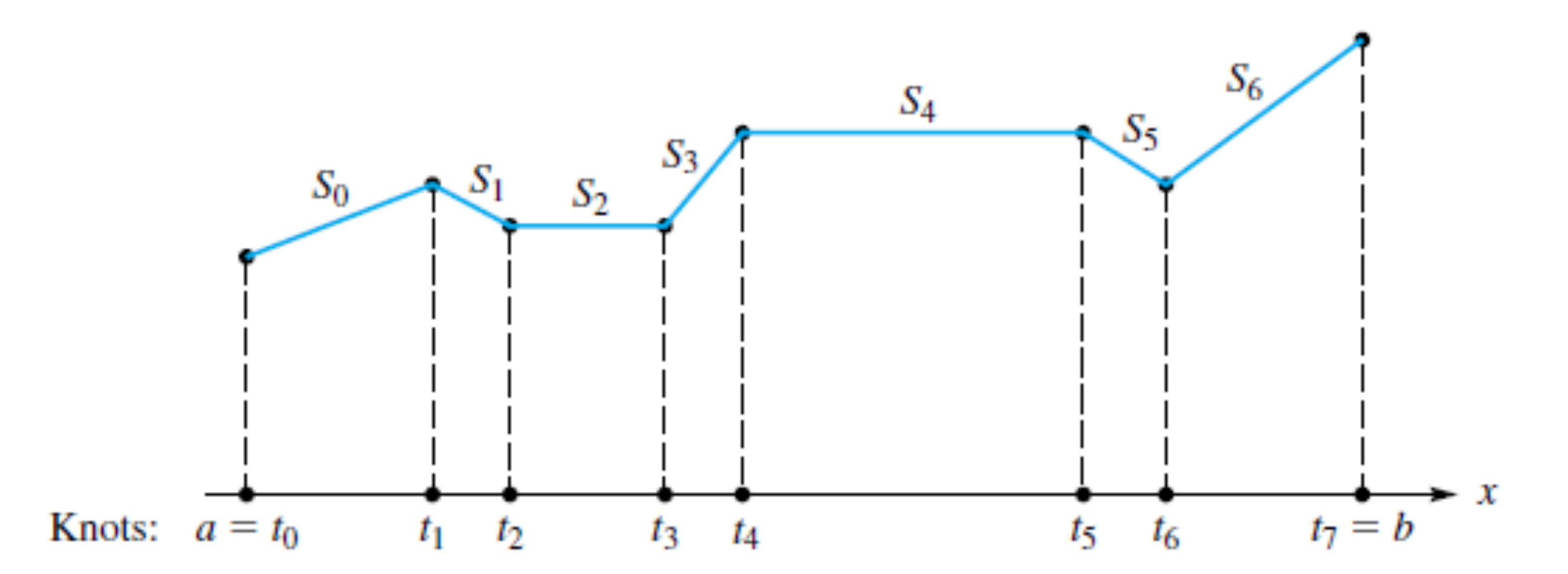


Figure : Higher order polynomial interpolation is a bad idea

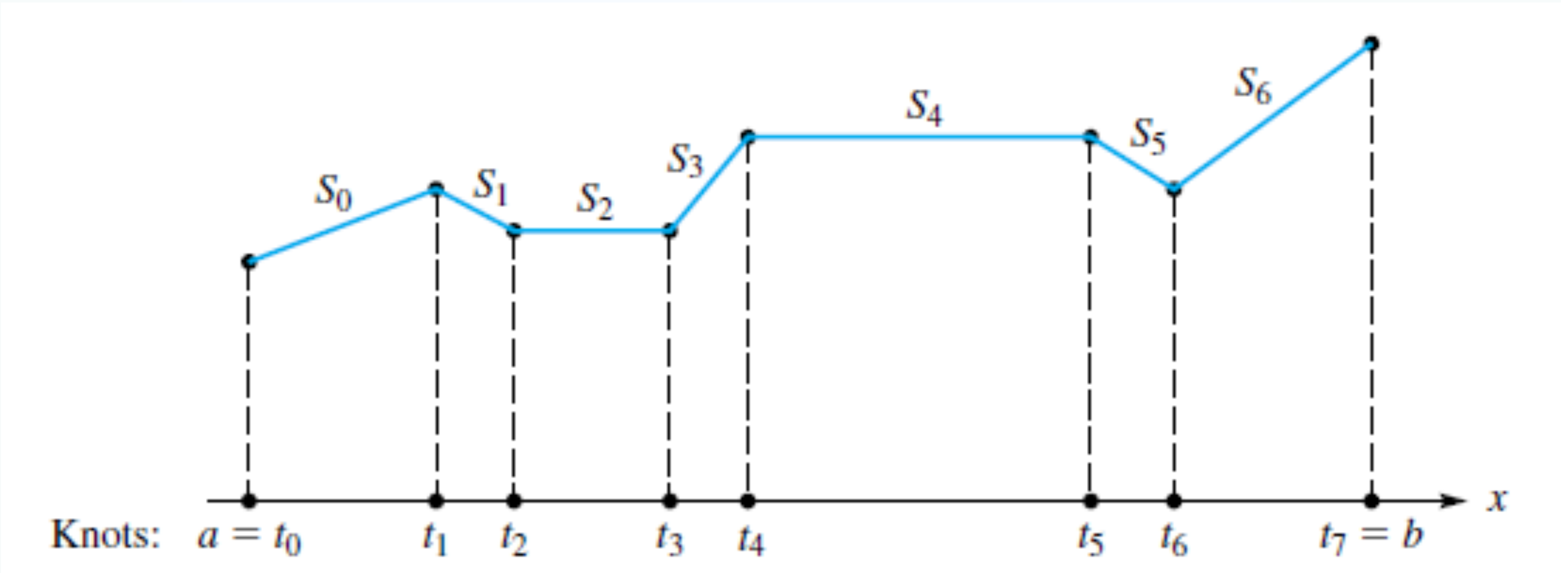
First Degree Splines

- Splines make use of partitions, which are a way of cutting an interval into a number of subintervals.

- The spline functions of degree 1 can be used for interpolation



First Degree Splines

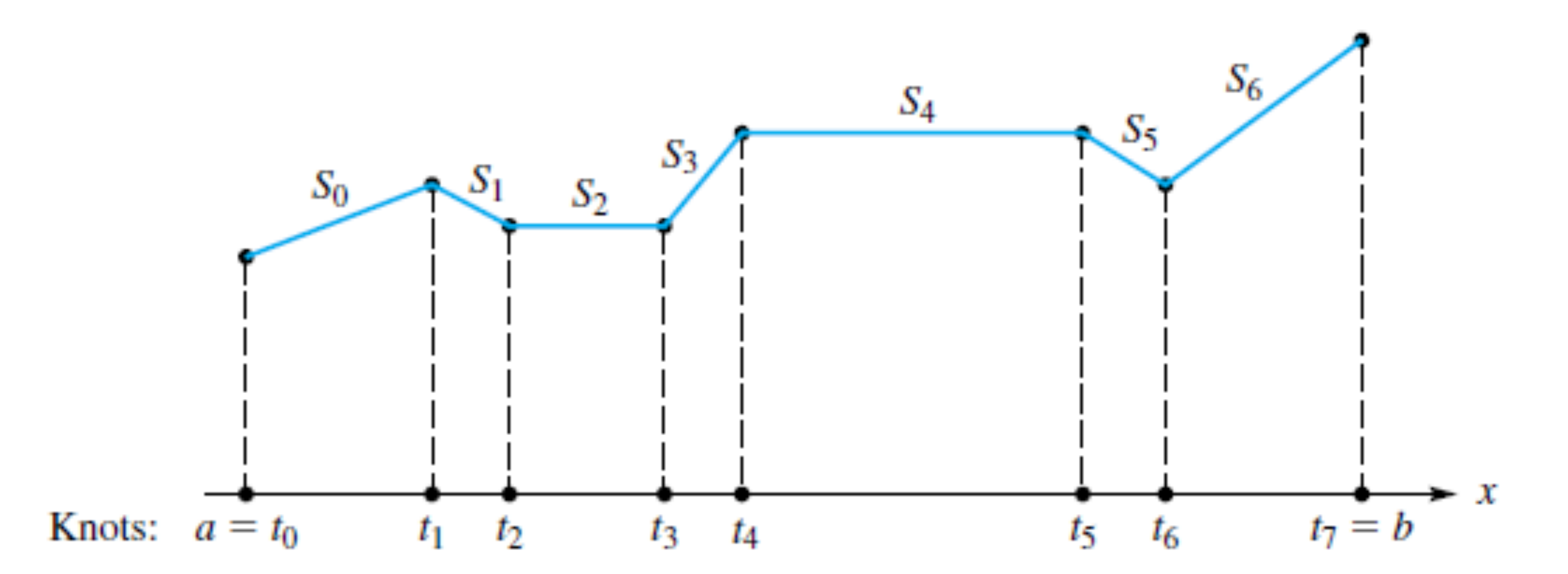


$$S(x) = \begin{cases} S_0(x) & x \in [t_0, t_1] \\ S_1(x) & x \in [t_1, t_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [t_{n-1}, t_n] \end{cases}$$

where;

$$S_i(x) = a_i x + b_i$$

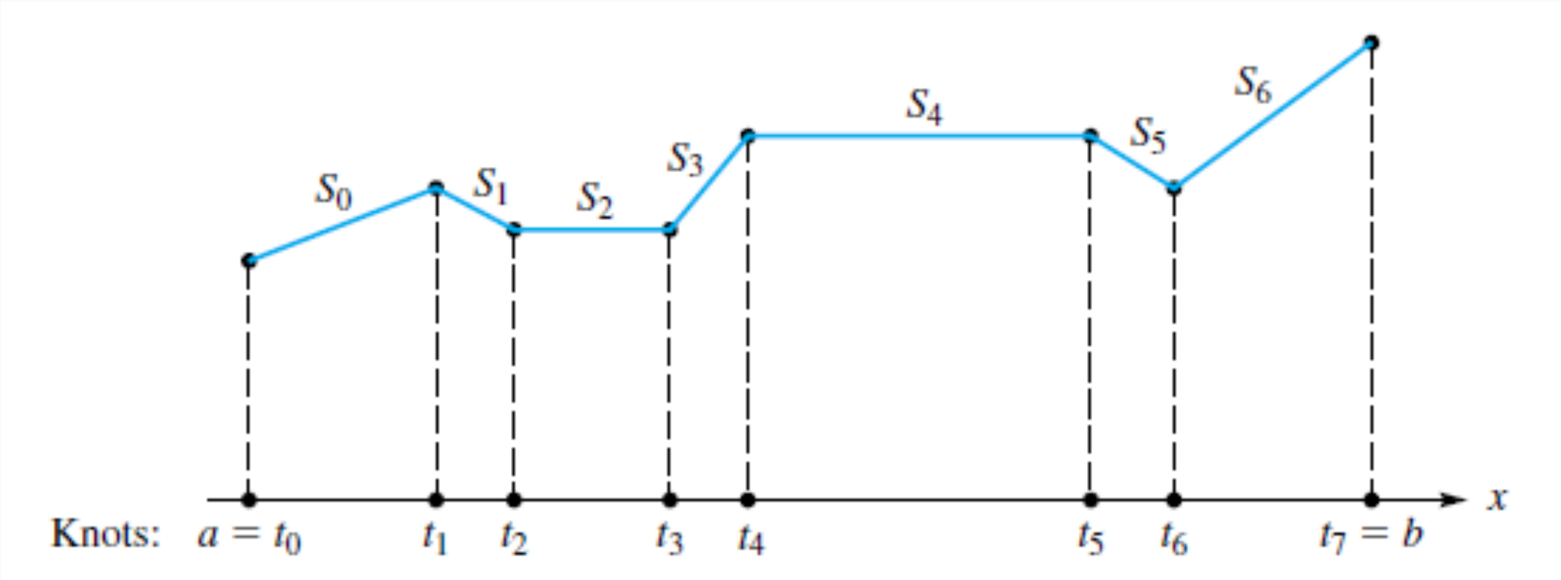
First Degree Splines



A function S is called a **spline of degree 1** if:

1. The domain of S is an interval $[a, b]$.
2. S is **continuous** on $[a, b]$.
3. There is a partitioning of the interval $a = t_0 < t_1 < \cdots < t_n = b$ such that S is a linear polynomial on each subinterval $[t_i, t_{i+1}]$.

First Degree Splines



Continuity of a function *f* at a point *s* can be defined by the condition

$$\lim_{x \to s^+} f(x) = \lim_{x \to s^-} f(x) = f(s)$$

Here, $\lim_{x \to s^+}$ means that the limit is taken over *x* values that converge to *s* from above *s*; that is, *(x − s)* is positive for all *x* values. Similarly, $\lim_{x \to s^-}$ means that the *x* values converge to *s* from below.

First Degree Splines

Continuity of a function *f* at a point *s* can be defined by the condition

$$\lim_{x \to s^+} f(x) = \lim_{x \to s^-} f(x) = f(s)$$

$$S(x) = \begin{cases} x & x \in [-1,0] \\ 1-x & x \in [0,1] \\ 2x-2 & x \in [1,2] \end{cases}$$

The function is obviously piecewise linear but is not a spline of degree 1 because it is discontinuous at $x = 0$. Notice that

$\lim_{x \to 0^+} S(x) = \lim_{x \to 0} (1-x) = 1$, whereas $\lim_{x \to 0^-} S(x) = \lim_{x \to 0} (x) = 0$.

First Degree Splines

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

$$\vdots$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

The upward velocity of a rocket is given as a function of time in table.

Find the velocity at t=16 seconds using linear splines.

Table : Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

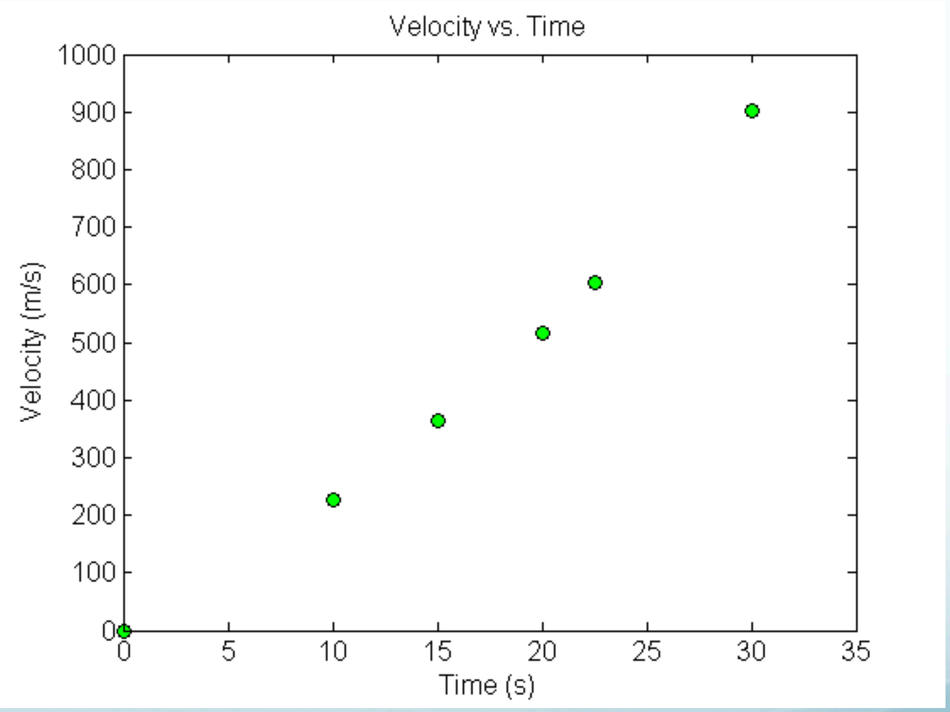


Figure : Velocity vs. time data for the rocket example

Linear Splines

$t_0 = 15, \qquad v(t_0) = 362.78$

$t_1 = 20, \qquad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

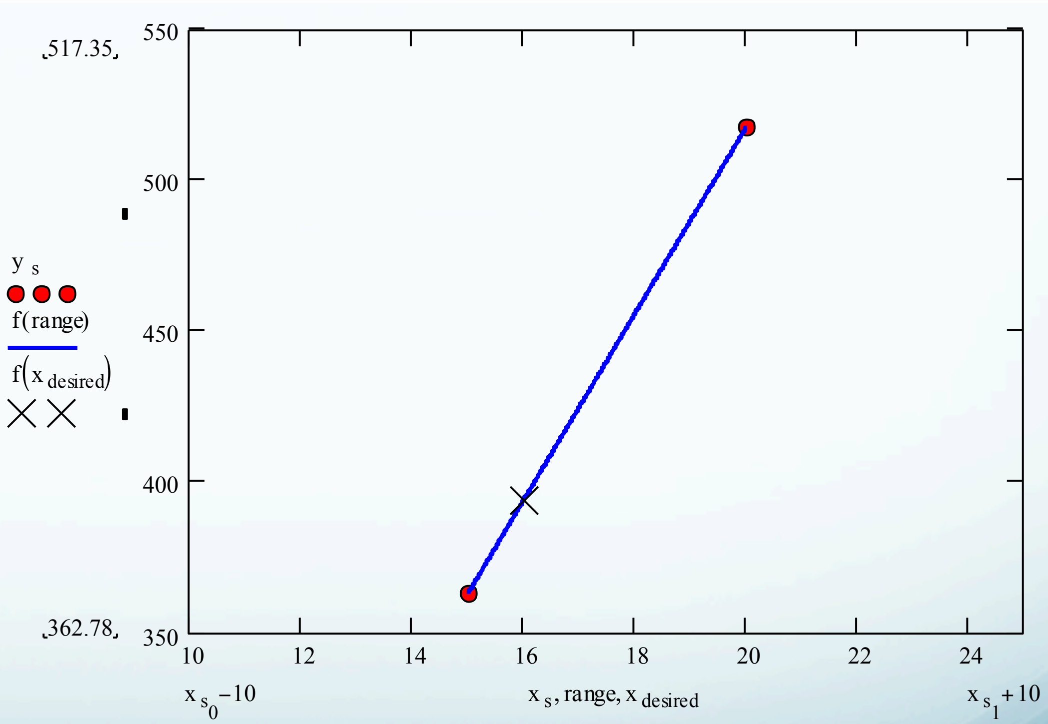
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

$$= 393.7 \text{ m/s}$$



Linear Spline Algorithm

real function *Spline1*$(n, (t_i), (y_i), x)$
integer i, n; **real** x; **real array** $(t_i)_{0:n}, (y_i)_{0:n}$
for $i = n - 1$ **to** 0 **step** -1 **do**
  if $x - t_i \geqq 0$ **then** exit loop
end for
Spline1 $\leftarrow y_i + (x - t_i)[(y_{i+1} - y_i)/(t_{i+1} - t_i)]$
end function *Spline1*

Quadratic Splines

A function Q is a **second-degree spline** if it has the following properties

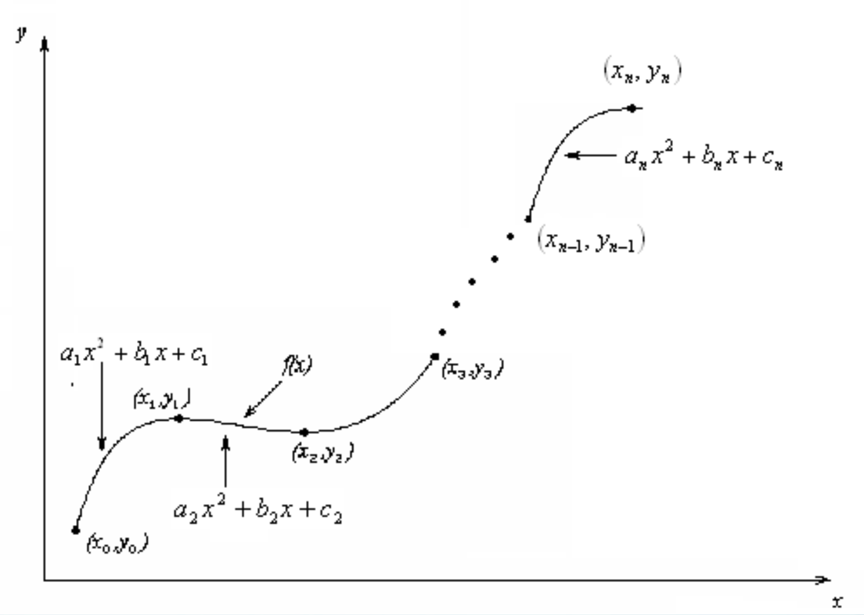
A function Q is called a **spline of degree 2** if:

1. The domain of Q is an interval **[**a, b**]**.
2. Q and Q are continuous on **[**a, b**]**.
3. There are points t_i (called **knots**) such that $a = t_0 < t_1 < \cdots < t_n = b$ and Q is a polynomial of degree at most 2 on each subinterval $[t_i , t_{i+1}]$.

Quadratic Splines

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, && x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, && x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_nx^2 + b_nx + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots, n$

Quadratic Splines

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1) \quad .$$

$$.$$

$$.$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

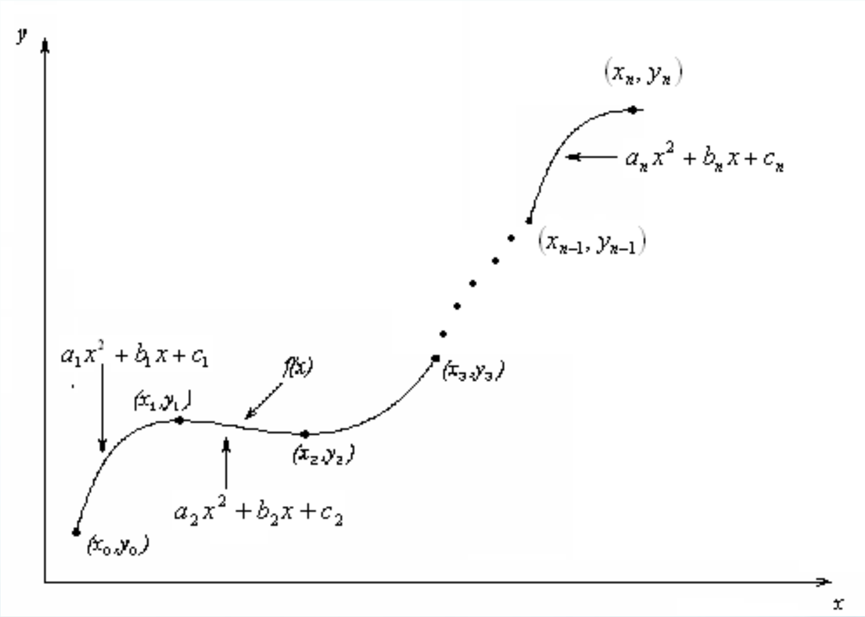
$$a_i x_i^2 + b_i x_i + c_i = f(x_i) \quad .$$

$$.$$

$$.$$

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

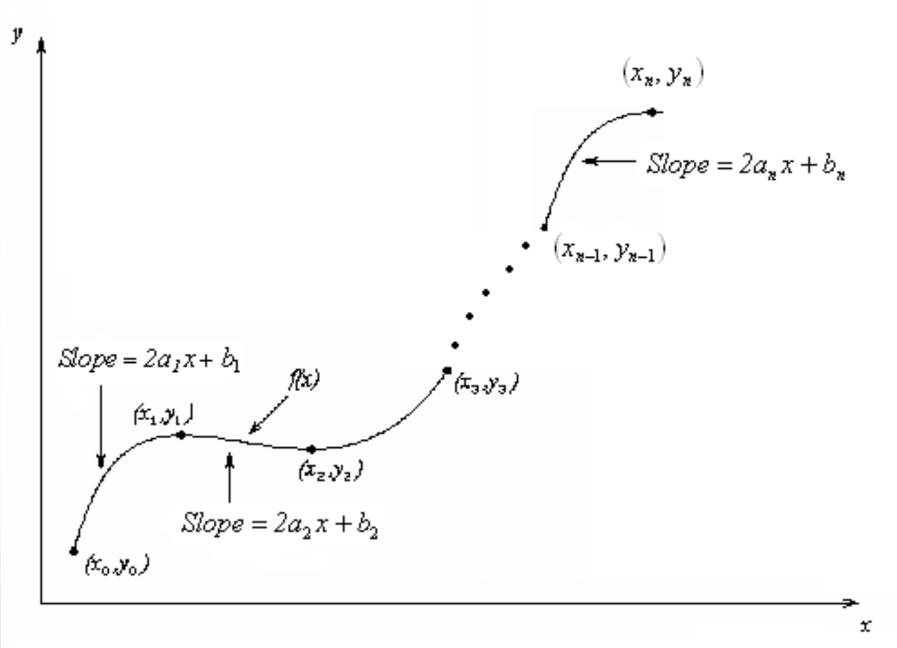
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

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$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

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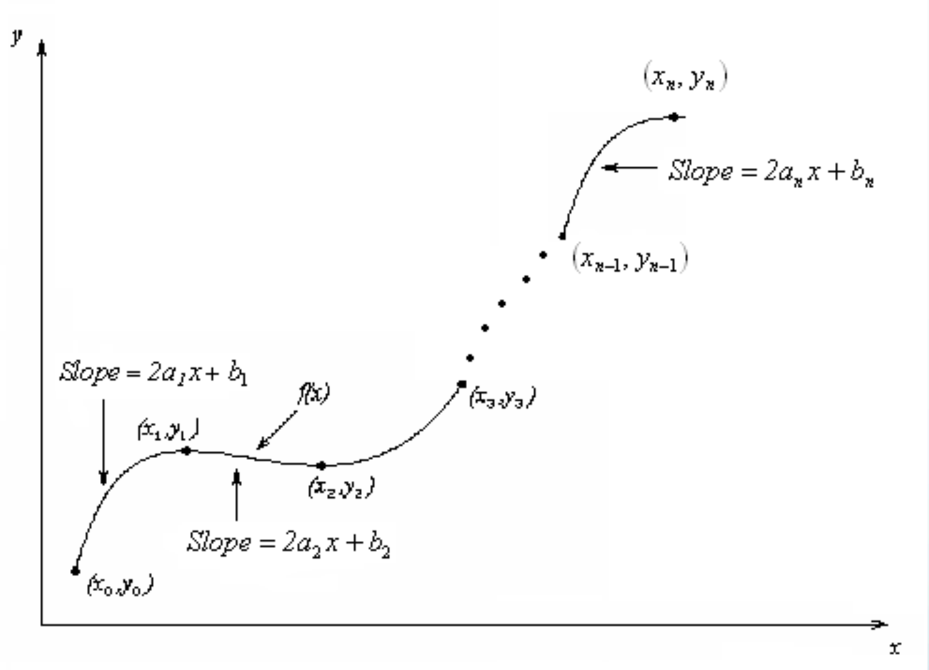
$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



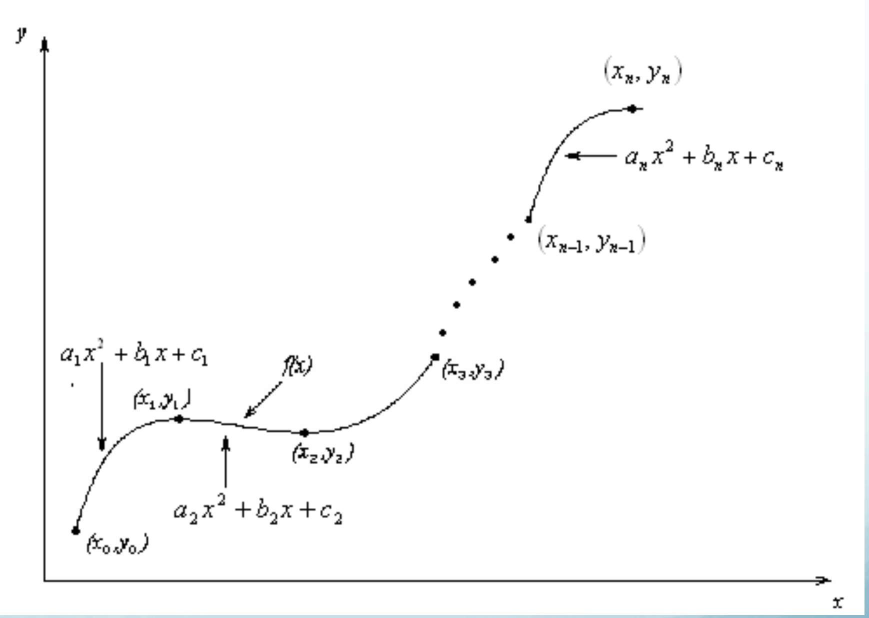
We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, && x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, && x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_nx^2 + b_nx + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table : Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

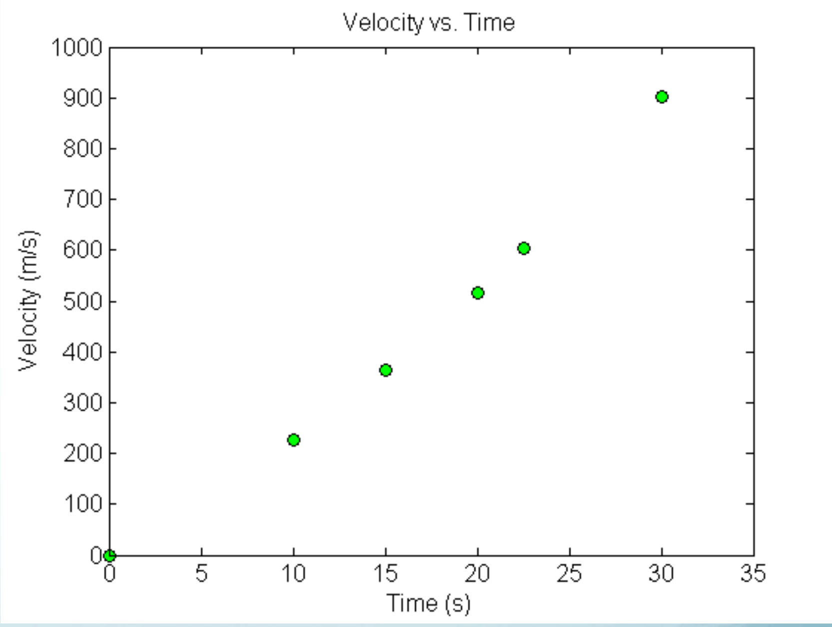


Figure : Velocity vs. time data for the rocket example

Solution

$$
\begin{aligned}
v(t) &= a_1 t^2 + b_1 t + c_1, & 0 \le t \le 10 \\
&= a_2 t^2 + b_2 t + c_2, & 10 \le t \le 15 \\
&= a_3 t^2 + b_3 t + c_3, & 15 \le t \le 20 \\
&= a_4 t^2 + b_4 t + c_4, & 20 \le t \le 22.5 \\
&= a_5 t^2 + b_5 t + c_5, & 22.5 \le t \le 30
\end{aligned}
$$

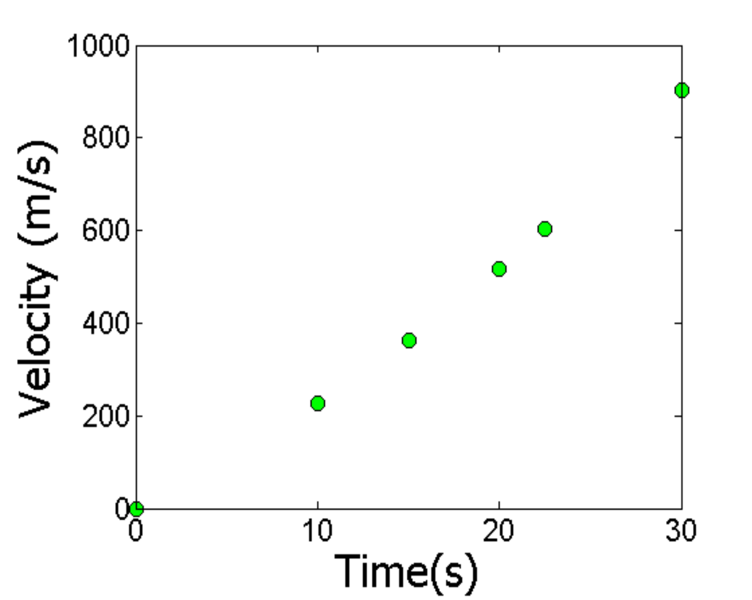
Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \qquad 0 \le t \le 10$$

$$a_1 (0)^2 + b_1 (0) + c_1 = 0$$

$$a_1 (10)^2 + b_1 (10) + c_1 = 227.04$$



Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	**m/s**
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$
$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$
$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$
$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$
$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$
$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$
$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$
$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1,\ 0 \le t \le 10$$
$$= a_2 t^2 + b_2 t + c_2, 10 \le t \le 15$$

$$\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\Big|_{t=10} = \frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\Big|_{t=10}$$

$$\left(2a_1 t + b_1\right)\Big|_{t=10} = \left(2a_2 t + b_2\right)\Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are Continuous at Interior Data Points

At t=10

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

At t=15

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

At t=20

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

At t=22.5

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

Last Equation $a_1 = 0$

Final Set of Equations

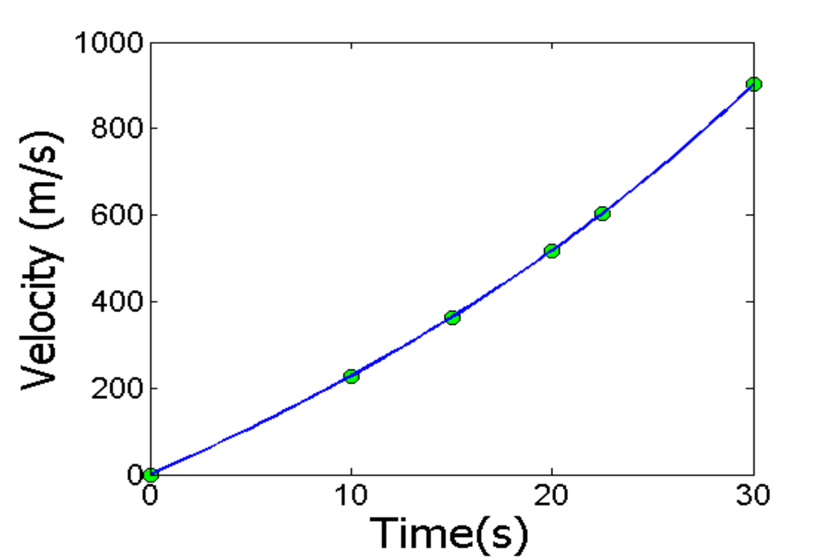
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\ 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	−0.1356	35.66	−141.61
4	1.6048	−33.956	554.55
5	0.20889	28.86	−152.13

Final Solution

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$



Velocity at a Particular Point

a) Velocity at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

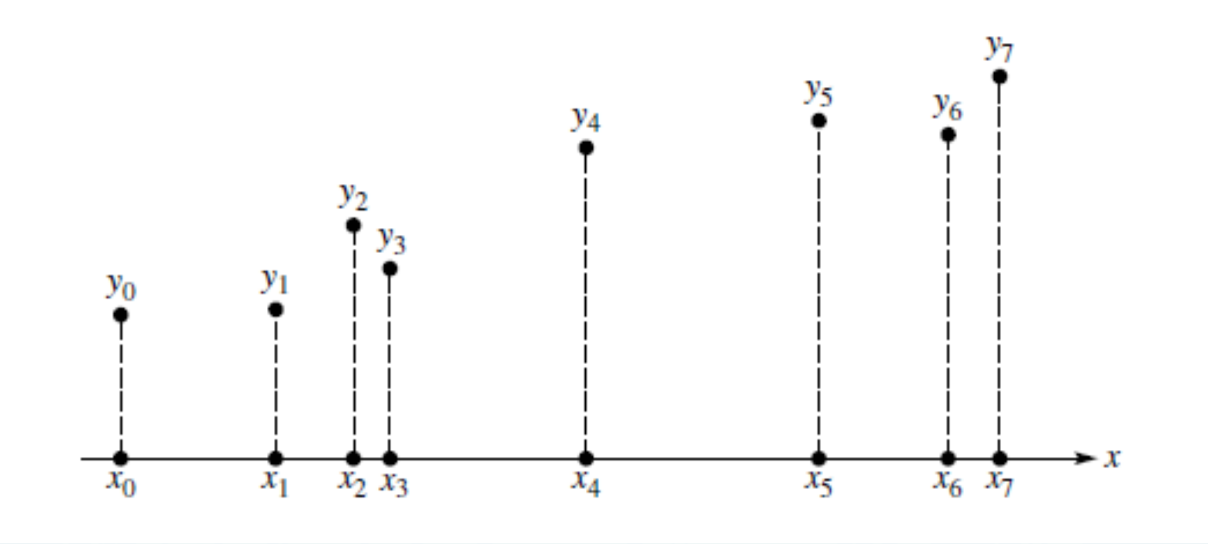
$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

Smoothing of Data & Least Squares Method

- **What** is data smoothing?
- **Why** do we use data smoothing & least squares?
- **How** do we solve method of least squares?

data smoothing

- Fitting a smooth curve to tabulated data by experiments.
- Draw a curve that defines the characteristic of data best.
- Obta



- Dispersed data points

Least Squares Method

- Using error function fits a curve. Gist of method is minimizing the error of function.

Linear Form

$f(x) = a + bx$

In this case, the function to be minimized is

$$S(a, b) = \sum_{i=0}^{n} [y_i - f(x_i)]^2 = \sum_{i=0}^{n} (y_i - a - bx_i)^2$$

Linear Least Squares

$$S(a, b) = \sum_{i=0}^{n} [y_i - f(x_i)]^2 = \sum_{i=0}^{n} (y_i - a - bx_i)^2$$

$$\frac{\partial S}{\partial a} = \sum_{i=0}^{n} -2(y_i - a - bx_i) = 2\left[a\,(n+1) + b\sum_{i=0}^{n} x_i - \sum_{i=0}^{n} y_i\right] = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=0}^{n} -2(y_i - a - bx_i)x_i = 2\left(a\sum_{i=0}^{n} x_i + b\sum_{i=0}^{n} x_i^2 - \sum_{i=0}^{n} x_i y_i\right) = 0$$

Dividing both equations by 2(n+1), rearrange terms;

$$a + \bar{x}b = \bar{y} \qquad \bar{x}a + \left(\frac{1}{n+1}\sum_{i=0}^{n} x_i^2\right) b = \frac{1}{n+1}\sum_{i=0}^{n} x_i y_i$$

Linear Least Squares

$$a + \bar{x}b = \bar{y} \qquad \bar{x}a + \left(\frac{1}{n+1}\sum_{i=0}^{n} x_i^2\right) b = \frac{1}{n+1}\sum_{i=0}^{n} x_i y_i$$

Where;

$$\bar{x} = \frac{1}{n+1}\sum_{i=0}^{n} x_i \qquad \bar{y} = \frac{1}{n+1}\sum_{i=0}^{n} y_i$$

are the mean values of x and y data. The solution for parameters is;

$$a = \frac{\bar{y}\sum x_i^2 - \bar{x}\sum x_i y_i}{\sum x_i^2 - n\bar{x}^2} \qquad b = \frac{\sum x_i y_i - \bar{x}\sum y_i}{\sum x_i^2 - n\bar{x}^2}$$

Linear Least Squares

Data obtained by an experiment;

x	4	7	11	13	17
y	2	0	2	6	7

System of two equations;

$$\begin{cases} 644a + 52b = 227 \\ 52a + 5b = 17 \end{cases}$$

Corresponding values;

$$a = 0.4864$$

$$b = -1.6589$$