Kom 505E

Week 2 /27.09.2016

Lecture Notes

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Week 2. 27.09.2016

Recap Basics of Probability Theory:

Axiomatic Foundations:

Def: A collection of subsets S is called a σ-field or a Borel field $\mathcal{B}$; if it satisfies:

i. $\emptyset \in \mathcal{B}$

ii. If $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$ (closed under complementation)

iii. If $A_1, A_2, \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$.

($\mathcal{B}$ is closed under countable unions).

* If S is finite or countable, then we define $\mathcal{B}$ for a given sample space S:

$$\mathcal{B} = \{\text{all subsets of } S\text{, including itself}\}$$

ex: $S = \{1, 2, 3\}$ $\quad \mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \emptyset\}$

↳ $\mathcal{B}$ has $2^3 = 8$ elements (sets)

* Let $S = (-\infty, \infty)$; $\mathcal{B}$ field has sets of the form $[a,b]$, $(a,b]$, $[a,b)$, (a,b) $\forall a,b \in \mathbb{R}$

Here A_i are ∞ly many.

From properties of $\mathcal{B}$ field, all possible countable unions, complements of the sets are in there.

Axioms of Probability: Given a sample space S, and an associated σ-field $\mathcal{B}$, a probability function is a function P w/ domain $\mathcal{B}$ that satisfies:

1) $P(A) \geq 0$ for all $A \in \mathcal{B}$ (A is an event)
2) $P(S) = 1$
3) If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint ($A_i \cap A_j = \emptyset$, $i \neq j$)

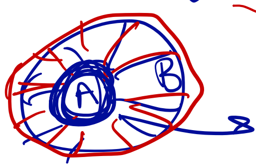
then $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Thm: If P is a prob. fn., A & B are sets in $\mathcal{B}$;

* $P(\emptyset) = 0$
* If $A \subset B$ then $P(A) \leq P(B)$

show: $B = A \cup (A^c \cap B) \Rightarrow P(B) = P(A) + P(A^c \cap B)$

disjoint

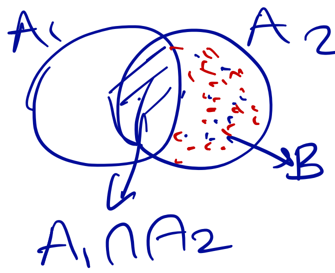


* $P(A) \leq 1 \Rightarrow A \subset S$ (from the above)
* $P(A^c) = 1 - P(A)$
* $P(B \cap A^c) = P(B) - P(A \cap B)$
* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
* $\{B_i\}$ any partition of S; $P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$

  ($\bigcup_i B_i = S$, $B_i \cap B_j = \emptyset$, $i \neq j$)

$\ast$ Boole's ineq. (Union bound): $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

A_i: any collection of sets (finite or countable) not necessarily disjoint.



$$A_1 \cup A_2 = A_1 \cup \underbrace{(A_2 \cap A_1^c)}_{} \quad \text{disjoint}$$

$$P(A_1 \cup A_2) = P(A_1) + \underbrace{P(A_2 \cap A_1^c)}_{\triangleq B}$$

$B \subset A_2$

$P(B) \leq P(A_2)$

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

(can be extended to countable sets). by induction.

COUNTING (Recap)

Ex: NY State lottery: 44 balls : pick 6 numbers for your ticket.

Recall: Sampling w/ replacement, Sampling w/o replacement.

Ordered (1), (2)

(1) Choose $\underset{=6}{r}$ balls out of $\underset{=44}{N}$ balls

$44 \times 43 \; . \; . \; . \; . \; . \; 39$

ex: $\simeq 5$ Billions

$= N(N-1) \ldots \times (N-r+1)$

$= \dfrac{N!}{(N-r)!}$

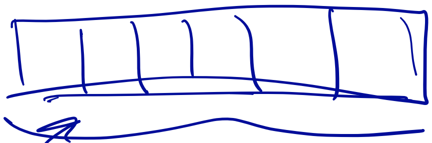
(2) W/ replacement:

ex: $\underbrace{44 \times 44 \times \cdots \times 44}_{6 \text{ times}} = (44)^6$ In general $(N)^r$

~ 7 Billion.

Unordered (when ordering is not important)

③ Sampling w/o replacement (Unordered)



r from N

$$\frac{N!}{(N-r)!} \cdot \frac{1}{r!} = \binom{N}{r}$$

inside ordering $r!$

ways choosing r objects from N objects

$r!$ ways of inside ordering r-tuples.

ex: $\binom{44}{6} \sim 7$ million.

④ Sampling w/ replacement unordered:

eg. $\{1,2,3\} \Longleftrightarrow$ sample 2 balls

$(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)$ } 6 ways.

$$= \binom{N+r-1}{r}$$

(Derivation not given).

ex: $\binom{49}{6}$

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possible arrangements of size r objects from N objects

	w/o Replacement	w/ Replacement
Ordered	$N \times (N-1) \times \cdots \times (N-r+1) = \frac{N!}{(N-r)!}$	N^r
Unordered	$\binom{N}{r}$	$\binom{N+r-1}{r}$

Ex: Choose a 5-card poker hand from a deck of 52 cards.

Prob(Having 4 aces) = ?

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*Counting Techniques are useful when S is a finite set and all outcomes in S are equally likely.

$S = \{ s_1, \ldots, s_N \}$ finite sample space
N outcomes.

$P(s_i) = \frac{1}{N}$ $\forall$ outcome s_i.

Then for any event A : $P(A) = \sum_{s_i \in A} \underbrace{P(s_i)}_{\frac{1}{N}} = \frac{|A|}{|S|}$

Back to the Ex:
$|S| = \binom{52}{5}$ ways to choose 5 cards w/o replacement from 52.
(unordered).

Event A: $= 4$ aces in a 5-card hand

1	1	1	1	X

2
$\vdots$

48 ways of specifying the 5th card.

$$P(A) = \frac{|A|}{|S|} = \frac{48}{\binom{52}{5}} = \frac{48}{2{,}598{,}960}$$ $\rightarrow$ quite a low probability!

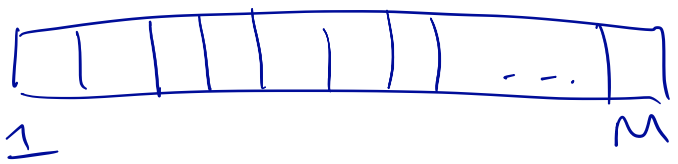
Binomial Law : probability model for Sampling w/ Replacement

(also applies to Sampling w/o replacement when M is large).

M coin tosses : H/T outcomes : $p(H) = p$.

$$P(\underbrace{HH\ldots H}_{k\text{ heads}}\ \underbrace{TT\ldots T}_{M-k\text{ tails}}) = p^k(1-p)^{M-k}$$

$$p \cdot p \cdots p \cdot (1-p) \cdots (1-p)$$

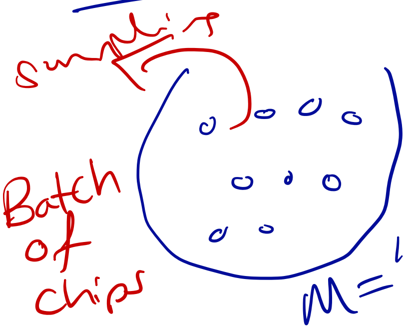


$\Rightarrow$ # ways k heads can be placed here.

$$P(k \underbrace{\text{heads}}_{\text{successes}} \text{ in } M \text{ trials}) = \binom{M}{k} p^k (1-p)^{M-k}$$

Sec 3.10 Real World Example: Quality Control

Soln: only 100 chips are tested.



$\longleftarrow$ Urn problem.

$M = 100$ chips

Success: 95 or more chips are good.

$\Rightarrow$ Prob. model : Binomial Law.

Prob. to accept a batch $\quad P(k \geq 95) =$ Prob. of 95 or more than 95 successes out of 100 draws :

p : prob. of a "good" chip $:= 0.84$

$$P(k \geq 95) = \underbrace{\binom{100}{95} p^{95}(1-p)^5}_{} + \binom{100}{96} p^{96}(1-p)^4 + \cdots + \binom{100}{100} p^{100}(1-p)^0$$

w/ $p = 0.94$

$\simeq 0.45$!. Not very good. Too high

Check $P(k \geq 98) \cong 0.05 \Rightarrow$ this is better.

Independent Events:

~~A & B~~ B are independent $\underset{\Leftrightarrow}{\text{iff}}$ $P(A \cap B) = P(A)\,P(B)$.

$\Rightarrow$ Two events are isolated, they do not affect each other.

eg. Rolling two dices: Indep. experiments.

$S_1 =$
$S_2 =$ $\Big\}$ $S_1 \times S_2 = \{1, 2, 3, \ldots, 6\} \times \{1, 2, \ldots, 6\}$

for independent prob. models.

★ These statements are equivalent:

A & B are independent
A^c & B^c "
A^c & B "
A & B^c "

Given A & B indep.

show: $P(A^c \cap B)$

$= P(B) - \underbrace{P(A \cap B)}_{P(A) \cdot P(B)}$

$= P(B)(1 - P(A))$

$P(A^c \cap B) = P(B) \cdot P(A^c)$ ✓

Conditional Probability (Chapter 4)

$P(A|B)$ = prob. of event A given event B occured.

$$P(A|B) \triangleq \frac{P(A\cap B)}{P(B)}, \quad \text{if } P(B)\neq 0.$$

Ex: Prob(obtain a 4 knowing that the outcome is an even number) = ?

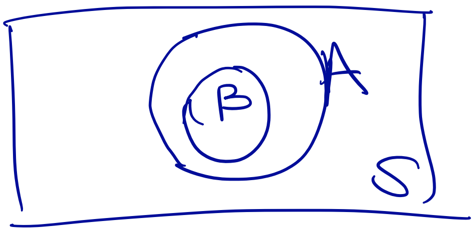
$A = \{4\}$
$B = \{2,4,6\}$ → like reduced sample space

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

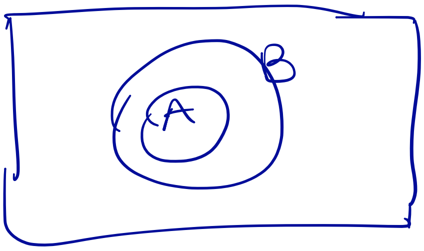
joint prob.

marginal prob.

- Study / Read prob. in Monty Hall Problem (Ex 4.4)



$P(A|B) = 1$



$$P(A|B) = \frac{P(A)}{P(B)} < 1$$

$\star$ $P(A|B)$: Axioms are satisfied for conditional probabilities :

1) $P(A|B) \geq 0$

2) $P(S|B) = 1$

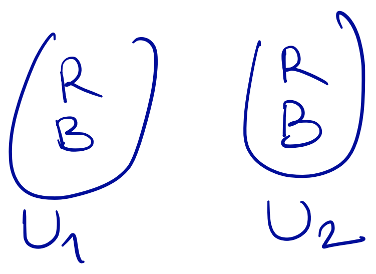
3) If A, C are mutually exclusive events (disjoint)

$$P(A \cup C|B) = P(A|B) + P(C|B)$$

Law of Total Probability: Let B_i be a partition of S ;

$$\Rightarrow \boxed{P(A) = \sum_{i=1}^{N} P(A \cap B_i)} = \sum_{i=1}^{N} P(A|B_i)P(B_i)$$

Ex: (Ex 4.2) Two urns



U_1 : p_1 : proportion of red balls in U_1.

$(1-p_1)$: " black "

U_2 : p_2 : prop. red balls

$(1-p_2)$: prob. black balls.

$P(\text{red ball is selected}) = ?$

Note: This is a Compound experiment : choose a urn at random first, then select a ball.

$A = \{\text{Red ball is selected}\}$

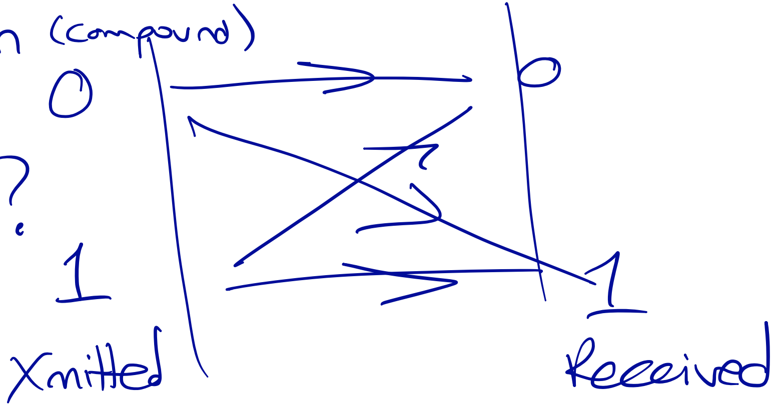
$B_1 = \{U_1 \text{ is selected}\}$

$B_2 = \{U_2 \text{ is selected}\}$

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

$$= p_1 \cdot \frac{1}{2} + p_2 \cdot \frac{1}{2}$$

Ex: 4.3: urn (compound)



$P(A|B_i) = \mathcal{E}$

$P(\text{error at receiver}) = ?$ → A.

$B_1 = \{0 \text{ xmit}\}$
$B_2 = \{1 \text{ xmit}\}$ $\Big\}$ $B_1 \cup B_2 = S$. (xmit).

$$P(A) = P(A | \underbrace{B_1}_{0\text{ xmit}}) \underbrace{P(B_1)}_{\frac{1}{2}} + P(A|B_2) \underbrace{P(B_2)}_{\frac{1}{2}}$$

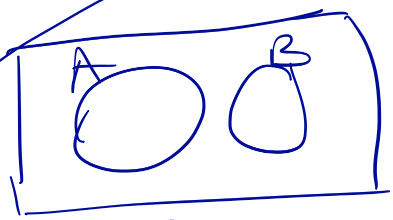
$$= \underbrace{P(1|0)}_{\mathcal{E}} \cdot \frac{1}{2} + \underbrace{P(0|1)}_{\mathcal{E}} \cdot \frac{1}{2} = \mathcal{E}.$$

Ex (4.1) Read at home.

Note: Independence

If A & B are indep. events then $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$

Q: If A & B are disjoint, are they independent?



$A \cap B = \emptyset$

$P(A \cap B) = 0 \overset{?}{\neq} P(A)P(B)$

$\Rightarrow A$ & B are not indep.

$P(A|B) = 0$
$P(B|A) = 0$.

implies only $P(A \cup B) = P(A) + P(B)$

Bayes' Thm:

$$\boxed{\underbrace{P(B|A)} = \frac{P(A|B)\,P(B)}{P(A)}}$$

Always true.

it may be difficult to calculate/evaluate $P(B|A)$, we prefer to work with the right hand side.

(Ex 4.7) $B = \{\text{person has disease}\}$
$A = \{\text{test is +ve}\}$

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

we replace $P(A)$ by $P(A|B)P(B) + P(A|B^c)P(B^c)$

$P(B) = 10^{-5}$ for the general population

$P(A|B) = 0.99$

$$= \frac{0.99 \times 10^{-5}}{0.99 \times (10^{-5}) + 0.2\,(1-10^{-5})} = \underbrace{4.95 \times 10^{-5}}_{\text{too low!}}$$

?

→ Instead, take $P(B) = 0.5$

$\Rightarrow P(B|A) = 0.83$!

Chap 4. (4.7) Real World Problem:

– Read half of Chapter 4. We'll complete it next week & start w/ Chap 5.