Chapter 7 Spin Representations

7.1 Holstein-Primakoff Bosons

自旋是一个矢量算符。在对称破缺的相中，至少有一个分量的期望值不为 0。用自旋相对于其期望值的小涨落来描述有序项是很自然的，这就是自旋波理论的内容。Holstein 和 Primakoff 引入玻色算符 b 来解决这一问题，自旋的三个分量可用 b 表示为：

$$\begin{aligned} S^{+} &= \left(\sqrt{2S-n_b}\right) b \\ S^{-} &= b^{\dagger}\sqrt{2S-n_b} \\ S^{z} &= -n_b + S \end{aligned} \tag{1}$$

- $n_b = b^{\dagger} b$
- $\sqrt{2S-n_b}$ 视为其 Tayler 展开：

$$\sqrt{2S-n_b} = \sqrt{2S}\left(1 - \frac{n_b}{4S} - \frac{n_b^2}{32S^2}\cdots\right) \tag{2}$$

▼ 如何构造：

Holstein-Primakoff 变换的思想就是将 spin-S 系统的自旋算符映射为玻色子产生和湮灭算符，用玻色子的粒子数表象表示 S^z 的本征态，每个 Holstein-Primakoff 玻色子表示在 $-z$ 方向的 spin-1 moment。

按照上面的思想，玻色子态与自旋态的对应关系为：

$$|m\rangle = |S, S-m\rangle \tag{3}$$

从而

$$
\begin{aligned}
&S^z\left|m\right\rangle = S^z\left|S, S-m\right\rangle = \left(S-m\right)\left|m\right\rangle = \left(S-n_b\right)\left|m\right\rangle \\
\Rightarrow &S^z = S - n_b
\end{aligned} \tag{4}
$$

考虑降算符：

$$
S^-\left|m\right\rangle = c\left(m\right)\left|m+1\right\rangle \sim \left|m+1\right\rangle \tag{5}
$$

考虑 $S^-\left|m\right\rangle$ 的內积：

$$
\begin{aligned}
\left\langle m\right| S^+S^-\left|m\right\rangle &= \left\langle m\right|\left(S^x+iS^y\right)\left(S^x-iS^y\right)\left|m\right\rangle \\
&= \left\langle m\right|\left(S^x\right)^2+\left(S^y\right)^2-i\left[S^x,S^y\right]\left|m\right\rangle \\
&= \left\langle m\right|\left(\mathbf{S}\right)^2-\left(S^z\right)^2+S^z\left|m\right\rangle \\
&= S\left(S+1\right)-\left(S-m\right)^2+\left(S-m\right) \\
&= \left(2S-m\right)\left(m+1\right) \\
&= \left|c\left(m\right)\right|^2
\end{aligned} \tag{6}
$$

取：

$$
c\left(m\right) = \sqrt{\left(2S-m\right)\left(m+1\right)} \tag{7}
$$

此时：

$$
\begin{aligned}
S^-\left|m\right\rangle &= c\left(m\right)\left|m+1\right\rangle \\
&= \sqrt{\left(2S-m\right)\left(m+1\right)}\left|m+1\right\rangle \\
&= b^\dagger\sqrt{2S-n_b}\left|m\right\rangle
\end{aligned} \tag{8}
$$

即：

$$
S^- = b^\dagger\sqrt{2S-n_b} \tag{9}
$$

从而：

$$
S^+ = \left(S^-\right)^\dagger = \sqrt{2S-n_b}\,b \tag{10}
$$

利用 $[b,b^\dagger]=1$，可以证明上式满足自旋的对易关系：

$$
[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma \tag{11}
$$

▼ 验证

上式等价于验证：

$$
\begin{aligned}
\left[S^+,S^-\right] &= -2i\left[S^x,S^y\right] = 2S^z \\
\left[S^z,S^+\right] &= \left[S^z,S^x\right]+i\left[S^z,S^y\right] = S^+ \\
\left[S^z,S^-\right] &= \left[S^z,S^x\right]-i\left[S^z,S^y\right] = -S^-
\end{aligned} \tag{12}
$$

分别验证有：

$$
\begin{aligned}
&\left[S^{+}, S^{-}\right] \\
=&\left[\left(\sqrt{2S-n_b}\right) b, b^{\dagger} \sqrt{2S-n_b}\right] \\
=&\sqrt{2S-n_b}\left[b, b^{\dagger} \sqrt{2S-n_b}\right]+\left[\sqrt{2S-n_b}, b^{\dagger} \sqrt{2S-n_b}\right] b \\
=&\sqrt{2S-n_b}\left\{b^{\dagger}\left[b, \sqrt{2S-n_b}\right]+\sqrt{2S-n_b}\right\}+\left[\sqrt{2S-n_b}, b^{\dagger}\right] \sqrt{2S-n_b} b \\
=&2S-n_b+\sqrt{2S-n_b} b^{\dagger}\left(b \sqrt{2S-n_b}-\sqrt{2S-n_b} b\right) \\
&+\left(\sqrt{2S-n_b} b^{\dagger}-b^{\dagger} \sqrt{2S-n_b}\right) \sqrt{2S-n_b} b \\
=&2S-n_b+\sqrt{2S-n_b} n_b \sqrt{2S-n_b}-b^{\dagger}\left(2S-n_b\right) b \\
=&2S-n_b+n_b\left(2S-n_b\right)-b^{\dagger}\left[b\left(2S-n_b\right)+b\right] \\
=&2\left(S-n_b\right) \\
=&2S^{z}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\left[S^{z}, S^{+}\right] \\
=&\left[S-n_b,\left(\sqrt{2S-n_b}\right) b\right] \\
=&-\left(\sqrt{2S-n_b}\right)\left[n_b, b\right] \\
=&\left(\sqrt{2S-n_b}\right) b \\
=&S^{+}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&\left[S^{z}, S^{-}\right] \\
=&\left[S-n_b, b^{\dagger}\left(\sqrt{2S-n_b}\right)\right] \\
=&-\left[n_b, b^{\dagger}\right]\left(\sqrt{2S-n_b}\right) \\
=&-b^{\dagger}\left(\sqrt{2S-n_b}\right) \\
=&-S^{-}
\end{aligned} \tag{15}
$$

- 上面推导中用到了：$[n_b, b]=-b,\left[n_b, b^{\dagger}\right]=b^{\dagger}$

有物理意义的子空间由 b 的 $0 \sim 2S$ 粒子数态张成：

$$
\left\{\left|n_b\right\rangle\right\}_S=\{|0\rangle,|1\rangle \ldots|2S\rangle\} \tag{16}
$$

在上述子空间中：

1. $\mathbf{S}^2=S(S+1)$

▼ 验证

$$
\begin{aligned}
\mathbf{S}^2 &=\left(S^{z}\right)^2+\frac{1}{2}\left(S^{+} S^{-}+S^{-} S^{+}\right) \\
&=\left(S^{z}\right)^2+\frac{1}{2}\left(2S^{-} S^{+}+2S^{z}\right) \\
&=S^{-} S^{+}+S^{z}\left(S^{z}+1\right) \\
&=b^{\dagger}\left(2S-n_b\right) b+\left(S-n_b\right)\left(S+1-n_b\right) \\
&=b^{\dagger}\left[b\left(2S-n_b\right)+b\right]+\left(S-n_b\right)\left(S+1-n_b\right) \\
&=n_b\left(2S+1-n_b\right)+\left(S-n_b\right)\left(S+1-n_b\right) \\
&=S(S+1)
\end{aligned} \tag{17}
$$

2. S^z, S^+, S^- 在子空间内封闭：

▼ 验证

1. 对 S^z 而言是显然的，因为 $\{|n_b\rangle\}_S$ 均为其本征态，$S^z|n\rangle \sim |n\rangle$
2. 对 S^+ 而言，$S^+|n\rangle \sim |n-1\rangle$，且 $S^+|0\rangle = 0$
3. 对 S^- 而言，$S^-|n\rangle \sim |n+1\rangle$，且 $S^-|2S\rangle \sim 0$

从而 S^z, S^+, S^- 可视为 $span\{|n_b\rangle\}_S$ 上的算符

Holstein-Primakoff 表示对描述量子海森堡模型的对称破缺相很有用：

- 上述的 $\sqrt{2S-n_b}$ 展开为 z 方向上自旋涨落的半经典展开
- 将其代入海森堡模型的哈密顿量并保留至二次项，可得到无相互作用自旋波哈密顿量。
- 高阶项反应自旋波间的相互作用

上述截断将 couple 物理子空间和非物理子空间。在 Chapter 11 中，自旋波理论将会以两种方式推导：

1. 自旋相干态路径积分
2. Holstein-Primakoff 算符展开

7.2 Schwinger Bosons

Heisenberg 模型的对称相更容易用一种特殊的表示描述，这种表示能显现出哈密顿量的旋转不变性。两种 Schwinger 玻色子 a, b 将自旋算符表示如下：

$$\begin{aligned} S^x + iS^y &= a^\dagger b \\ S^x - iS^y &= b^\dagger a \\ S^z &= \frac{1}{2}(a^\dagger a - b^\dagger b) \end{aligned} \tag{18}$$

▼ 同样可以证明上述表示下的自旋算符满足其对易关系

$$\begin{aligned} \left[S^+, S^-\right] &= \left[a^\dagger b, b^\dagger a\right] \\ &= b^\dagger \left[a^\dagger, a\right] b + a^\dagger \left[b, b^\dagger\right] a \\ &= a^\dagger a - b^\dagger b \\ &= 2S^z \end{aligned} \tag{19}$$

$$\begin{aligned} \left[S^z, S^+\right] &= \left[\frac{1}{2}(a^\dagger a - b^\dagger b), a^\dagger b\right] \\ &= \frac{1}{2}\left\{\left[a^\dagger a, a^\dagger b\right] - \left[b^\dagger b, a^\dagger b\right]\right\} \\ &= \frac{1}{2}\left\{a^\dagger \left[a, a^\dagger\right] b - a^\dagger \left[b^\dagger, b\right] b\right\} \\ &= a^\dagger b \\ &= S^+ \end{aligned} \tag{20}$$

$$
\begin{aligned}
\left[S^z, S^-\right] &= \left[\frac{1}{2}(a^\dagger a - b^\dagger b), b^\dagger a\right] \\
&= \frac{1}{2}\left\{\left[a^\dagger a, b^\dagger a\right] - \left[b^\dagger b, b^\dagger a\right]\right\} \\
&= \frac{1}{2}\left\{b^\dagger\left[a^\dagger, a\right] a - b^\dagger\left[b, b^\dagger\right] a\right\} \\
&= -b^\dagger a \\
&= -S^-
\end{aligned} \tag{21}
$$

spin magnitude S 定义了物理子空间：

$$
\{|n_a, n_b\rangle | n_a + n_b = 2S\} \tag{22}
$$

子空间由投影算符给出：

$$
P_S\left(a^\dagger a + b^\dagger b - 2S\right) = 0 \tag{23}
$$

对于物理子空间的基矢，升降算符的作用为：

$$
\begin{aligned}
S^+|m, 2S-m\rangle &= |m+1, 2S-m-1\rangle \\
S^-|m, 2S-m\rangle &= |m-1, 2S-m+1\rangle
\end{aligned} \tag{24}
$$

▼ 说明

1. 三个自旋算符都保持粒子数之和 $n_a + n_b$ 不变，因此物理子空间一定满足 $n_a + n_b = N$
2. 考虑 S^z ，其最大本征值为 S，因此 $n_a + n_b = 2S$
3. 容易验证三个自旋算符在物理子空间是封闭的：$S^+|2S, 0\rangle = S^-|0, 2S\rangle = 0$

在投影子空间（物理子空间），spin magntitude 是良定的：

$$
\mathbf{S}^2 P_S = S(S+1) P_S \tag{25}
$$

▼ 验证

$$
\begin{aligned}
\mathbf{S}^2 &= (S^z)^2 + \frac{1}{2}\left(S^+S^- + S^-S^+\right) \\
&= \frac{1}{4}(n_a - n_b)^2 + \frac{1}{2}\left(a^\dagger b b^\dagger a + b^\dagger a a^\dagger b\right) \\
&= \frac{1}{4}(n_a - n_b)^2 + \frac{1}{2}\left(n_a(1-n_b) + n_b(1-n_a)\right) \\
&= \frac{1}{4}\left(n_a^2 - 2n_a n_b + n_b^2\right) + \frac{1}{2}\left(2n_a n_b + n_a + n_b\right) \\
&= \frac{1}{4}\left(n_a^2 + 2n_a n_b + n_b^2\right) + \frac{n_a + n_b}{2} \\
&= \left(\frac{n_a + n_b}{2}\right)\left(\frac{n_a + n_b}{2} + 1\right)
\end{aligned} \tag{26}
$$

对于物理子空间的任意一个基矢 $|m, 2S-m\rangle$：

$$
\begin{aligned}
&\mathbf{S}^2\left|m, 2S-m\right\rangle \\
=&\left(\frac{n_a+n_b}{2}\right)\left(\frac{n_a+n_b}{2}+1\right)\left|m, 2S-m\right\rangle \\
=&S\left(S+1\right)\left|m, 2S-m\right\rangle
\end{aligned} \tag{27}
$$

因此对于子空间的任意一个态均有：

$$
\mathbf{S}^2\left|\alpha\right\rangle = S\left(S+1\right)\left|\alpha\right\rangle \Leftrightarrow \mathbf{S}^2\big|_{\{|n_a,n_b\rangle\}} = \mathbf{I} \tag{28}
$$

Fig.7.1 画出了 Fock space 中的物理子空间

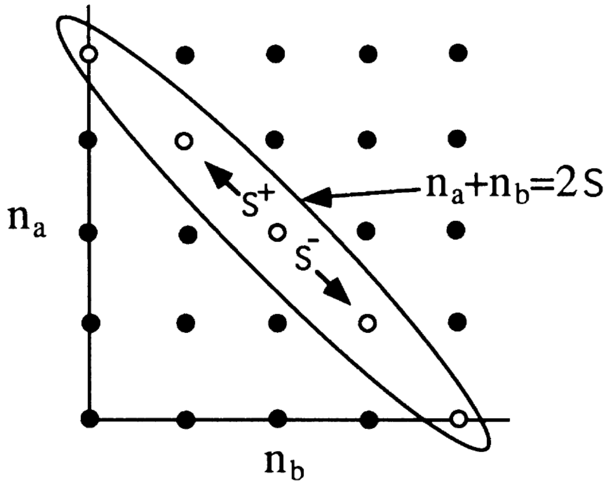


FIGURE 7.1. Projected subspace of spin S in the Schwinger bosons Fock space.

自旋态可用 Schwinger 玻色算符表示为：

$$
\left|S, m\right\rangle = \frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}}\frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}}\left|0{,}0\right\rangle \tag{29}
$$

- $\left|S, m\right\rangle$ 表示 $\mathbf{S}^2$ 本征值为 S，S^z 本征值为 m 的本征态
- $\left|0{,}0\right\rangle$ 表示 Schwinger 玻色子的真空态

特别的，对于自旋 $\frac{1}{2}$ 的粒子，有：

$$
\begin{aligned}
\left|\uparrow\right\rangle = &\quad a^\dagger\left|0\right\rangle \\
\left|\downarrow\right\rangle = &\quad b^\dagger\left|0\right\rangle
\end{aligned} \tag{30}
$$

Schwinger 玻色子对于计算自旋算符矩阵元非常有帮助，相比与之前的 Holstein-Primakoff 玻色子，Schwinger 玻色子没有平方根的项，不同 Fock 态的自旋算符矩阵元被分解为自由玻色子，然而这不一定简化非 Fock 波函数自旋关联的计算。物理子空间的限制条件引入了 a, b 玻色子占据数的关联。

这里的自旋表示可以推广 N flavor 的情况，推广的表示将 $SU(N)$ 的生成元定义为广义自旋（见 Chapter 16）。大 N 推广适用于由小参量 $\frac{1}{N}$ 控制的简单平均场理论（见 Chapter 17,18）。大 N 平均场理论由有效无相互作用玻色准粒子描述，其中不同 Flavor 间的关联被忽略。

Schwinger bosons (SB) 和 Holstein-Primakoff (HP) bosons 是密切相关的。用约束消去 a 玻色子，可以得到两者间的联系：

$$
\begin{array}{ccc}
\text{SB} & & \text{HP} \\
b & \leftrightarrow & b \\
a & \leftrightarrow & \sqrt{2S-n_b}
\end{array}
\tag{31}
$$

然而 Schwinger 玻色子在自旋空间中提供了一个对称表示， Holstein-Primakoff 玻色子选择了 S^z 方向。因此，这两种表示适用于不同的近似情况：HP 适用于对称破缺相，SB 使用与对称相。

7.2.1 Spin Rotations

Appendix A.2

自旋算符是 $SU(2)$ 群的生成元，群元 R 可用三个欧拉角 $\phi.\theta,\chi$ 参数化：

$$
\hat{\mathcal{R}} = e^{i\phi S^z} e^{i\theta S^y} e^{i\chi S^z} \tag{32}
$$

- $\mathbf{S}$ 是正规（normal）双线性算符

Schwinger 玻色产生算符旋转变换为：

$$
\begin{aligned}
\begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}' &= \hat{\mathcal{R}} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \hat{\mathcal{R}}^{-1} \\
&= e^{i\frac{1}{2}\chi\sigma^z} e^{i\frac{1}{2}\theta\sigma^y} e^{i\frac{1}{2}\phi\sigma^z} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} \\
&= \begin{pmatrix} ue^{i\frac{\chi}{2}} & ve^{i\frac{\chi}{2}} \\ -v^* e^{-i\frac{\chi}{2}} & u^* e^{-i\frac{\chi}{2}} \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}
\end{aligned}
\tag{33}
$$

💡 感觉转置后写起来更自然一些：

$$
\begin{aligned}
\left(a^\dagger, \quad b^\dagger\right)' &= \hat{\mathcal{R}} \left(a^\dagger, \quad b^\dagger\right) \hat{\mathcal{R}}^{-1} \\
&= \left(a^\dagger, \quad b^\dagger\right) e^{i\frac{1}{2}\phi\sigma^z} e^{i\frac{1}{2}\theta\sigma^y} e^{i\frac{1}{2}\chi\sigma^z} \\
&= \left(a^\dagger, \quad b^\dagger\right) \begin{pmatrix} ue^{i\frac{\chi}{2}} & -v^* e^{-i\frac{\chi}{2}} \\ ve^{i\frac{\chi}{2}} & u^* e^{-i\frac{\chi}{2}} \end{pmatrix}
\end{aligned}
\tag{34}
$$

▼ 证明

$$
\begin{aligned}
\left(a^\dagger, \quad b^\dagger\right)' &= \hat{\mathcal{R}} \left(\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \hat{\mathcal{R}}^{-1} \\
&= \left(\hat{\mathcal{R}} \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{\mathcal{R}}^{-1}, \quad \hat{\mathcal{R}} \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{\mathcal{R}}^{-1} \right) \\
&= \left(\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \mathcal{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \mathcal{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \mathcal{R} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \mathcal{R}
\end{aligned}
\tag{35}
$$

- $\mathcal{R} = e^{i\frac{1}{2}\phi\sigma^z} e^{i\frac{1}{2}\theta\sigma^y} e^{i\frac{1}{2}\chi\sigma^z}$

$$
\begin{aligned}
e^{\frac{i}{2}\phi\sigma^z} &= \exp\left(\begin{bmatrix} i\frac{\phi}{2} & 0 \\ 0 & -i\frac{\phi}{2}\end{bmatrix}\right) = \begin{bmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}}\end{bmatrix} \\
e^{\frac{i}{2}\chi\sigma^z} &= \exp\left(\begin{bmatrix} i\frac{\chi}{2} & 0 \\ 0 & -i\frac{\chi}{2}\end{bmatrix}\right) = \begin{bmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}}\end{bmatrix} \\
e^{\frac{i}{2}\theta\sigma^y} &= \exp\left(\begin{bmatrix} 0 & \frac{\theta}{2} \\ -\frac{\theta}{2} & 0\end{bmatrix}\right) = \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix} \exp\left(\begin{bmatrix} i\frac{\theta}{2} & 0 \\ 0 & -i\frac{\theta}{2}\end{bmatrix}\right) \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{bmatrix} \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}}\end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} & -ie^{i\frac{\theta}{2}} + ie^{-i\frac{\theta}{2}} \\ ie^{i\frac{\theta}{2}} - ie^{-i\frac{\theta}{2}} & e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}}\end{bmatrix} \\
&= \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}
\end{aligned} \tag{36}
$$

从而：

$$
\begin{aligned}
\mathcal{R} &= e^{i\frac{1}{2}\phi\sigma^z} e^{i\frac{1}{2}\theta\sigma^y} e^{i\frac{1}{2}\chi\sigma^z} \\
&= \begin{bmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}}\end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix} \begin{bmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}}\end{bmatrix} \\
&= \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} & \sin\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \\ -\sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} & \cos\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}}\end{bmatrix} \begin{bmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}}\end{bmatrix} \\
&= \begin{bmatrix} u & -v^* \\ v & u^*\end{bmatrix} \begin{bmatrix} e^{i\frac{\chi}{2}} & 0 \\ 0 & e^{-i\frac{\chi}{2}}\end{bmatrix} \\
&= \begin{bmatrix} ue^{i\frac{\chi}{2}} & -v^*e^{-i\frac{\chi}{2}} \\ ve^{i\frac{\chi}{2}} & u^*e^{-i\frac{\chi}{2}}\end{bmatrix}
\end{aligned} \tag{37}
$$

- 其中 $\begin{bmatrix} u(\theta,\phi) \\ v(\theta,\phi)\end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \\ -\sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}}\end{bmatrix}$,

> ❓ 书上为 $\begin{bmatrix} u(\theta,\phi) \\ v(\theta,\phi)\end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \\ \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}}\end{bmatrix}$

7.3 Spin Coherent States

7.3.1 The Defination and Properties of Spin Coherent States

在 Holstein-Primakoff 表示下，定义自旋相干态：

$$
\begin{aligned}
|\mu\rangle &\equiv e^{\mu S^-}|0\rangle \\
&= \sum_{p=0}^{\infty} \frac{(\mu S^-)^p}{p!}|0\rangle \\
&= \sum_{p=0}^{\infty} \frac{(\mu)^p}{p!}\left(b^\dagger\sqrt{2S-n_b}\right)^p|0\rangle \\
&= \sum_{p=0}^{\infty} \frac{(\mu)^p}{p!}\left(\sqrt{2S\cdot(2S-1)\cdots(2S-p+1)}\right)\cdot\sqrt{1\cdot 2\cdots p}\,|p\rangle \\
&= \sum_{p=0}^{2S}\mu^p\sqrt{\frac{(2S)!}{p!(2S-p)!}}\,|p\rangle \\
&= \sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}\,|p\rangle
\end{aligned}
\tag{38}
$$

- 类似于一般相干态的定义

相干态的性质：

1. 归一化系数：

   归一化后的相干态为：

$$
|\mu\rangle = \frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}\,|p\rangle \tag{39}
$$

▼ 证明

$$
\begin{aligned}
\langle\mu|\mu\rangle &= \sum_{p=0}^{2S}\sum_{q=0}^{2S}\mu^{*q}\mu^p\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\langle q|p\rangle \\
&= \sum_{p=0}^{2S}\binom{2S}{p}\left(|\mu|^2\right)^p \\
&= \left(1+|\mu|^2\right)^{2S}
\end{aligned}
\tag{40}
$$

2. 相干态的内积：

$$
\langle\lambda|\mu\rangle = \frac{1}{(1+|\lambda|^2)^S(1+|\mu|^2)^S}(1+\lambda^*\mu)^{2S} \tag{41}
$$

▼ 证明

$$
\begin{aligned}
&\langle\lambda|\ \mu\rangle \\
=&\frac{1}{\left(1+|\mu|^{2}\right)^{S}\cdot\left(1+|\lambda|^{2}\right)^{S}}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\left(\lambda^{*}\right)^{q}\mu^{p}\left\langle q\right|\ p\rangle \\
=&\frac{1}{\left(1+|\mu|^{2}\right)^{S}\cdot\left(1+|\lambda|^{2}\right)^{S}}\sum_{p=0}^{2S}\binom{2S}{p}\left(\lambda^{*}\mu\right)^{p} \\
=&\frac{\left(1+\lambda^{*}\mu\right)^{2S}}{\left(1+|\mu|^{2}\right)^{S}\cdot\left(1+|\lambda|^{2}\right)^{S}}
\end{aligned}
\tag{42}
$$

3. 完备性关系

$$
\int d^{2}\mu|\mu\rangle\langle\mu|m\left(|\mu|^{2}\right)=\sum_{0}^{2S}|p\rangle\langle p|=I \tag{43}
$$

- $d^{2}\mu=(dRe\mu)(dIm\mu)$
- $m\left(|\mu|^{2}\right)=\frac{2S+1}{\pi\left(1+|\mu|^{2}\right)^{2}}$
- ▼ 证明

$$
\begin{aligned}
&\int d^{2}\mu\left|\mu\right\rangle\left\langle\mu\right| \\
=&\sum_{q=0}^{2S}\sum_{p=0}^{2S}\left|p\right\rangle\left\langle q\right|\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\int d^{2}\mu\frac{1}{\left(1+|\mu|^{2}\right)^{2S}}\left(\mu^{*}\right)^{q}\mu^{p} \\
=&\sum_{q=0}^{2S}\sum_{p=0}^{2S}\left|p\right\rangle\left\langle q\right|\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\int d\rho\rho\frac{1}{\left(1+\rho^{2}\right)^{2S}}\rho^{p+q}\int d\theta e^{i(p-q)\theta} \\
=&\sum_{p=0}^{2S}\left|p\right\rangle\left\langle q\right|\binom{2S}{p}2\pi\int d\rho\frac{\rho^{2p+1}}{\left(1+\rho^{2}\right)^{2S}} \\
=&\sum_{p=0}^{2S}\left|p\right\rangle\left\langle q\right|\frac{(2S)!}{p!\,(2S-p)!}2\pi\int d\rho\frac{\rho^{2p+1}}{\left(1+\rho^{2}\right)^{2S}} \\
=&\sum_{p=0}^{2S}\left|p\right\rangle\left\langle q\right|\frac{(2S)!}{p!\,(2S-p)!}2\pi\cdot\frac{p!\,(2S-p-2)!}{2\,(2S-1)!}
\end{aligned}
\tag{44}
$$

  - 其中第二个等式用到了 $\int d\theta e^{i(p-q)\theta}=2\pi\delta_{p,q}$
  - 倒数第二个等式用到了：$(\alpha>-1,2\beta-\alpha>1)$

$$
\begin{aligned}
&\int_0^\infty dx\frac{x^\alpha}{(1+x^2)^\beta}\\
\xrightarrow{x=\tan\theta}&\int_0^{\frac{\pi}{2}} d\theta\sec^2\theta\frac{\tan^\alpha\theta}{\sec^{2\beta}\theta}\\
=&\int_0^{\frac{\pi}{2}} d\theta\sin^\alpha\theta\cos^{2\beta-\alpha-2}\\
\xrightarrow[\theta=\arcsin\sqrt{x}]{\sin^2\theta=x}&\frac{1}{2}\int_0^1 dx\,(1-x)^{-\frac{1}{2}}\,(x)^{-\frac{1}{2}}\,x^{\frac{\alpha}{2}}\,(1-x)^{\beta-\frac{\alpha}{2}-1}\\
=&\frac{1}{2}\int_0^1 dx x^{\frac{\alpha-1}{2}}\,(1-x)^{\beta-\frac{\alpha}{2}-\frac{3}{2}}\\
=&\frac{1}{2}\beta\left(\frac{\alpha+1}{2},\beta-\frac{\alpha+1}{2}\right)\\
=&\frac{1}{2}\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta-\frac{\alpha+1}{2}\right)}{\Gamma\left(\beta\right)}
\end{aligned}
\tag{45}
$$

对于上面的 $\alpha=2p+1$，$\beta=2S$，有：

$$
\begin{aligned}
&\int_0^\infty dx\frac{x^{2p+1}}{(1+x^2)^{2S}}\\
=&\frac{1}{2}\frac{\Gamma\left(p+1\right)\Gamma\left(2S-p-1\right)}{\Gamma\left(2S\right)}\\
=&\frac{p!\,(2S-p-2)!}{2\left(2S-1\right)!}
\end{aligned}
\tag{46}
$$

发现 $\int d^2\mu\,|\mu\rangle\,\langle\mu|\neq\sum_{p=0}^{2S}|p\rangle\,\langle q|$，但只需做出简单的修正，使得最后一个等式中 $\frac{(2S)!}{p!(2S-p)!}\cdot\frac{p!(2S-p-2)!}{2(2S)!}$ 含 p 的项能够消去：$2S\rightarrow 2S+2$，即在加入因子 $\frac{1}{(1+x^2)^2}$。再考虑归一化，最终选取：

$$
m\left(\rho^2\right)=\frac{2S+1}{\pi\left(1+\rho^2\right)^2}
\tag{47}
$$

这相当于在一开始引入：

$$
m\left(|\mu|^2\right)=\frac{2S+1}{\pi\left(1+|\mu|^2\right)^2}
\tag{48}
$$

7.3.2 Some Matrix Elements

这一节将计算在自旋相干态下 S^z,S^+,S^- 算符的矩阵元。

在计算 S^z 的矩阵元时相当于计算 n_b 的矩阵元，因为 $S^z=S-n_b$。定义：

$$
A\equiv S-S^z=n_b
$$

则

$$
\begin{aligned}
&\langle\lambda|\,A\,|\mu\rangle \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\left(\lambda^*\right)^q\mu^p\,\langle q|\,A\,|p\rangle \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\left(\lambda^*\right)^q\mu^p p\delta_{p,q} \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\frac{(2S)!}{(2S-p)!p!}\left(\lambda^*\mu\right)^p p \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=1}^{2S}\frac{(2S)!}{(2S-p)!\,(p-1)!}\left(\lambda^*\mu\right)^p \\
\xrightarrow{q=p-1}&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{q=0}^{2S-1}\frac{(2S)!}{(2S-1-q)!q!}\left(\lambda^*\mu\right)^{q+1} \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\cdot 2S\cdot\lambda^*\mu\sum_{q=0}^{2S-1}\frac{(2S-1)!}{(2S-1-q)!q!}\left(\lambda^*\mu\right)^q \\
=&\frac{2S\cdot\lambda^*\mu\left(1+\lambda^*\mu\right)^{2S-1}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S} \\
=&\frac{2S\cdot\lambda^*\mu}{1+\lambda^*\mu}\cdot\frac{\left(1+\lambda^*\mu\right)^{2S}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S} \\
=&\frac{2S\cdot\lambda^*\mu}{1+\lambda^*\mu}\,\langle\lambda|\,\mu\rangle
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\langle\lambda| S^{-} |\mu\rangle \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}}\sqrt{\binom{2S}{q}}\left(\lambda^*\right)^q\mu^p\langle q| S^{-} |p\rangle \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}\binom{2S}{q}}\left(\lambda^*\right)^q\mu^p\langle q| b^\dagger\sqrt{2S-n_b}\,|p\rangle \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\sum_{q=0}^{2S}\sqrt{\binom{2S}{p}\binom{2S}{q}}\left(\lambda^*\right)^q\mu^p \\
&\sqrt{(p+1)\left(2S-p\right)}\delta_{q,p+1} \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S-1}\sqrt{\binom{2S}{p}\binom{2S}{p+1}}\left(\lambda^*\right)^{p+1}\mu^p\sqrt{(p+1)\left(2S-p\right)} \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S-1}\sqrt{\frac{(2S)!}{(2S-p)!p!}}\cdot\sqrt{\frac{(2S)!}{(2S-(p+1))!\,(p+1)!}} \\
&\left(\lambda^*\right)^{p+1}\mu^p\sqrt{(p+1)\left(2S-p\right)} \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S-1}\frac{(2S)!}{(2S-(p+1))!p!}\cdot\left(\lambda^*\right)^{p+1}\mu^p \\
=&\frac{1}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\cdot 2S\cdot\lambda^*\sum_{p=0}^{2S-1}\frac{(2S-1)!}{((2S-1)-p)!p!}\cdot\left(\lambda^*\mu\right)^p \\
=&\frac{2S\cdot\lambda^*\left(1+\lambda^*\mu\right)^{2S-1}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S} \\
=&\frac{2S\cdot\lambda^*}{1+\lambda^*\mu}\cdot\frac{\left(1+\lambda^*\mu\right)^{2S}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S} \\
=&\frac{2S\cdot\lambda^*}{1+\lambda^*\mu}\langle\lambda|\,\mu\rangle
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&\langle\lambda| S^{+} |\mu\rangle \\
&=\left(\langle\mu| S^{-} |\lambda\rangle\right)^\dagger \\
&=\left[\frac{2S\cdot\mu^*}{1+\mu^*\lambda}\cdot\frac{\left(1+\mu^*\lambda\right)^{2S}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S}\right]^\dagger \\
&=\frac{2S\cdot\mu}{1+\lambda^*\mu}\cdot\frac{\left(1+\lambda^*\mu\right)^{2S}}{\left(1+|\lambda|^2\right)^S\left(1+|\mu|^2\right)^S} \\
&=\frac{2S\cdot\mu}{1+\lambda^*\mu}\langle\lambda|\,\mu\rangle
\end{aligned}
\tag{51}
$$

特别的，对于对角元有：

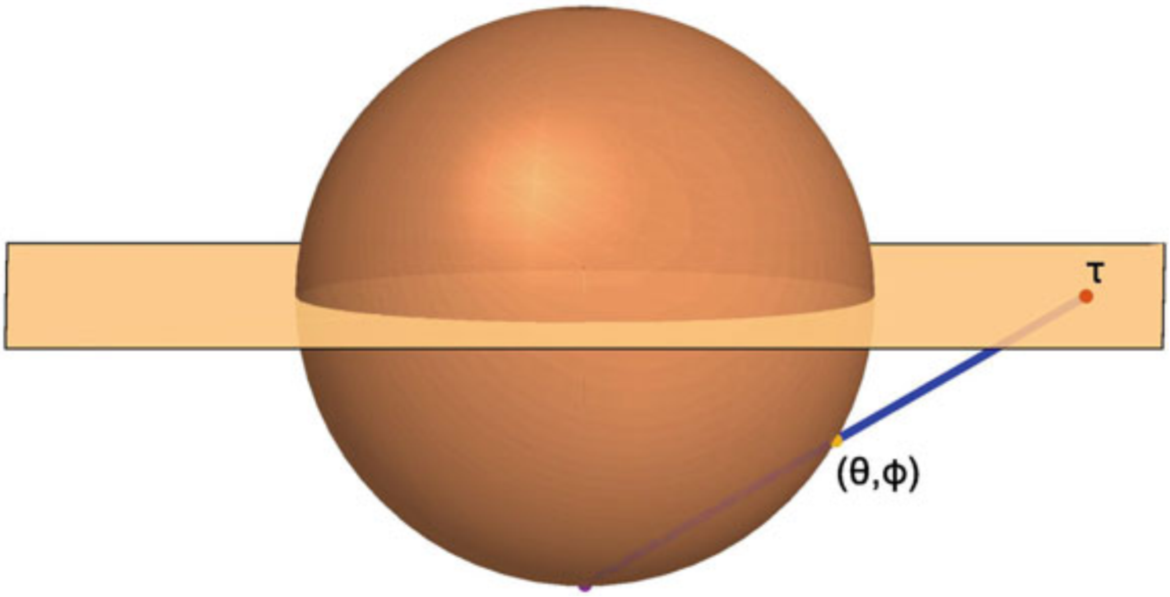
$$
\begin{aligned}
\langle\mu| A |\mu\rangle &= \frac{2S\cdot|\mu|^2\left(1+|\mu|^2\right)^{2S-1}}{\left(1+|\mu|^2\right)^S\left(1+|\mu|^2\right)^S} \\
&= \frac{2S\cdot|\mu|^2}{1+|\mu|^2}
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\langle\mu| S^- |\mu\rangle &= \frac{2S\cdot\mu^*\left(1+|\mu|^2\right)^{2S-1}}{\left(1+|\mu|^2\right)^S\left(1+|\mu|^2\right)^S} \\
&= \frac{2S\cdot\mu^*}{1+|\mu|^2}
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\langle\mu| S^+ |\mu\rangle &= \left(\langle\mu| S^- |\mu\rangle\right)^\dagger \\
&= \left(\frac{2S\cdot\mu^*}{1+|\mu|^2}\right)^\dagger \\
&= \frac{2S\cdot\mu}{1+|\mu|^2}
\end{aligned} \tag{54}
$$

7.3.3 An Alternative Parametrization

在上面的分析中，我们使用复数 μ 来参数标记自旋相干态，借助黎曼球面的球极平面投影，我们可以用单位球面上的坐标 (θ,ϕ) 来表示复数，进而参数化自旋相干态。



球面上的一点 (x,y,z)，可用球面坐标表示为：

$$
\begin{cases}
x = \sin\theta\cos\phi \\
y = \sin\theta\sin\phi \\
z = \cos\theta
\end{cases} \tag{55}
$$

黎曼球上南极与该点连线的直线方程为：

$$\frac{x'}{x}=\frac{y'}{y}=\frac{z'+1}{z+1}=t\in\mathbb{R} \tag{56}$$

令 $z'=0$，得到直线与复平面的交点为：

$$\begin{cases}x'=\frac{x}{z+1}=\frac{\sin\theta\cos\phi}{\cos\theta+1}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\cos\phi=\tan\frac{\theta}{2}\cos\phi\\ y'=\frac{y}{z+1}=\frac{\sin\theta\cos\phi}{\cos\theta+1}=\tan\frac{\theta}{2}\sin\phi\end{cases} \tag{57}$$

从而：

$$\mu=x+iy=\tan\frac{\theta}{2}\cos\phi+i\tan\frac{\theta}{2}\sin\phi=\tan\frac{\theta}{2}e^{i\phi} \tag{58}$$

相干态在新的参数下可以表示为：

$$|\Omega\rangle\equiv|\theta,\phi\rangle=|\mu\rangle=\cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^p\sqrt{\binom{2S}{p}}\,|p\rangle \tag{59}$$

- Ω 为立体角，$d\Omega=\sin\theta d\theta d\phi$

新参数下的完备性关系

考虑将新的参数视为坐标变换，计算 Jacobi 行列式：

$$\begin{aligned}d^2\mu&=d\mu_R d\mu_I\\&=\begin{vmatrix}\frac{\partial\mu_R}{\partial\theta}&\frac{\partial\mu_R}{\partial\phi}\\\frac{\partial\mu_I}{\partial\theta}&\frac{\partial\mu_I}{\partial\phi}\end{vmatrix}d\theta d\phi\\&=\begin{vmatrix}\frac{1}{2}\sec^2\frac{\theta}{2}\cos\phi&-\tan\frac{\theta}{2}\sin\phi\\\frac{1}{2}\sec^2\frac{\theta}{2}\sin\phi&\tan\frac{\theta}{2}\cos\phi\end{vmatrix}d\theta d\phi\\&=\frac{1}{2}\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}d\theta d\phi\\&=\frac{\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}}{2\sin\theta}d\Omega\end{aligned} \tag{60}$$

从而：

$$\begin{aligned}&\frac{2S+1}{\pi}\int d^2\mu\frac{1}{\left(1+|\mu|^2\right)^2}|\mu\rangle\langle\mu|\\=&\frac{2S+1}{2\pi}\int\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}d\theta d\phi\frac{1}{\left(1+\tan^2\frac{\theta}{2}\right)^2}|\theta,\phi\rangle\langle\theta,\phi|\\=&\frac{2S+1}{2\pi}\int\frac{\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}d\theta d\phi\,|\theta,\phi\rangle\langle\theta,\phi|\\=&\frac{2S+1}{4\pi}\int\sin\theta d\theta d\phi\,|\theta,\phi\rangle\langle\theta,\phi|\\=&\frac{2S+1}{4\pi}\int d\Omega\,|\Omega\rangle\langle\Omega|\end{aligned} \tag{61}$$

- $\frac{2S+1}{4\pi}d\Omega$ 即为 $SU(2)$ 李群的 Haar 测度

上面我们利用已有的结论，间接证明了新参数下的完备性关系，我们也可以采用直接证明的方式，与直接证明的过程类似，可以得到下面一个有用的结论：

$$\frac{(S+1)(2S+1)}{4\pi}\int d\Omega \hat{r}^{\alpha}\left|\Omega\right\rangle\left\langle\Omega\right| = S^{\alpha}, \alpha = x,y,z \tag{62}$$

▼ 证明

升降算符可以表示为：

$$\begin{aligned}
&S^{+}\\
=&\frac{2S+1}{4\pi}\int d\Omega S^{+}\left|\Omega\right\rangle\left\langle\Omega\right|\\
=&\frac{2S+1}{4\pi}\sum_{p,q}\sqrt{2S-n_b}b\left|p\right\rangle\left\langle p\right|\sqrt{\binom{2S}{p}\binom{2S}{q}}\\
&\int\sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\int d\phi e^{i\phi(p-q)}\\
=&\frac{2S+1}{4\pi}\sum_{p,q}\sqrt{2S-n_b}b\left|p\right\rangle\left\langle p\right|\sqrt{\binom{2S}{p}\binom{2S}{q}}\\
&\int\sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}2\pi\delta_{p,q}\\
=&\frac{2S+1}{2}\sum_{p=1}^{2S}\sqrt{(2S-p+1)\,p}\left|p-1\right\rangle\left\langle p\right|\sqrt{\binom{2S}{p}\binom{2S}{p}}\\
&\int\sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2p}\\
=&\frac{2S+1}{2}\sum_{p=1}^{2S}\left|p-1\right\rangle\left\langle p\right|\sqrt{(2S-p+1)\,p}\frac{(2S)!}{(2S-p)!p!}\\
&2\int d\theta\cos^{4S-2p+1}\frac{\theta}{2}\sin^{2p+1}\frac{\theta}{2}
\end{aligned}\tag{63}$$

其中：

$$\begin{aligned}
&\int_0^{\pi}(d\theta)\sin^{x}\frac{\theta}{2}\cos^{y}\frac{\theta}{2}\\
=&\int_0^{\pi}(d\theta)\left(\sin^2\frac{\theta}{2}\right)^{\frac{x}{2}}\left(1-\sin^2\frac{\theta}{2}\right)^{\frac{y}{2}}\\
\xrightarrow{\sin^2\frac{\theta}{2}=z}&\int_0^{1}dz\,(z)^{\frac{x-1}{2}}\,(1-z)^{\frac{y-1}{2}}\\
=&\beta\left(\frac{x+1}{2},\frac{y+1}{2}\right)\\
=&\frac{\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(\frac{y+1}{2}\right)}{\Gamma\left(\frac{x+y}{2}+1\right)}
\end{aligned}\tag{64}$$

因此

$$
\begin{aligned}
&S^{+}\\
=&\frac{2S+1}{2}\sum_{p=1}^{2S}|p-1\rangle\langle p|\sqrt{(2S-p+1)\,p}\frac{(2S)!}{(2S-p)!p!}\\
&2\int d\theta\cos^{4S-2p+1}\frac{\theta}{2}\sin^{2p+1}\frac{\theta}{2}\\
=&\frac{2S+1}{2}\sum_{p=1}^{2S}|p-1\rangle\langle p|\sqrt{(2S-p+1)\,p}\frac{(2S)!}{(2S-p)!p!}\\
&2\frac{(2S-p)!p!}{(2S+1)!}\\
=&\sum_{p=1}^{2S}|p-1\rangle\langle p|\sqrt{(2S-p+1)\,p}
\end{aligned}
\tag{65}
$$

同理：

$$
\begin{aligned}
&S^{-}\\
=&\frac{2S+1}{4\pi}\int d\Omega S^{-}|\Omega\rangle\langle\Omega|\\
=&\frac{2S+1}{4\pi}\sum_{p,q}b^{\dagger}\sqrt{2S-n_b}\,|p\rangle\langle p|\sqrt{\binom{2S}{p}\binom{2S}{q}}\\
&\int\sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\int d\phi e^{i\phi(p-q)}\\
=&\frac{2S+1}{2}\sum_{p=0}^{2S-1}\sqrt{(2S-p)\,(p+1)}\,|p+1\rangle\langle p|\binom{2S}{p}\\
&2\int d\theta\cos^{4S-2p+1}\frac{\theta}{2}\sin^{2p+1}\frac{\theta}{2}\\
=&\frac{2S+1}{2}\sum_{p=0}^{2S-1}\sqrt{(2S-p)\,(p+1)}\,|p+1\rangle\langle p|\frac{(2S)!}{(2S-p)!p!}\\
&2\frac{(2S-p)!p!}{(2S+1)!}\\
=&\sum_{p=0}^{2S-1}\sqrt{(2S-p)\,(p+1)}\,|p+1\rangle\langle p|
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&S^z \\
=&\frac{2S+1}{4\pi}\int d\Omega S^z \left|\Omega\right\rangle\left\langle\Omega\right| \\
=&\frac{2S+1}{4\pi}\sum_{p,q}(S-n_b)\left|p\right\rangle\left\langle p\right|\sqrt{\binom{2S}{p}\binom{2S}{q}} \\
&\int \sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\int d\phi e^{i\phi(p-q)} \\
=&\frac{2S+1}{2}\sum_{q}(S-q)\left|q\right\rangle\left\langle q\right|\binom{2S}{q} \\
&\int \sin\theta d\theta\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q} \\
=&\frac{2S+1}{2}\sum_{q}(S-q)\left|q\right\rangle\left\langle q\right|\binom{2S}{q} \\
&2\int d\theta\sin^{2q+1}\frac{\theta}{2}\cos^{4S-2q+1}\frac{\theta}{2} \\
=&(2S+1)\sum_{q}(S-q)\left|q\right\rangle\left\langle q\right|\frac{(2S)!}{(2S-q)!q!}\frac{q!(2S-q)!}{(2S+1)!} \\
=&\sum_{q}(S-q)\left|q\right\rangle\left\langle q\right|
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
&\int d\Omega\hat{r}^\alpha\left|\Omega\right\rangle\left\langle\Omega\right| \\
=&\sum_{q,p}\int d\Omega\hat{r}^\alpha\cos^{2S}\frac{\theta}{2}\sqrt{\binom{2S}{p}}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^p\left|p\right\rangle\cdot \\
&\cos^{2S}\frac{\theta}{2}\sqrt{\binom{2S}{q}}\left(\tan\frac{\theta}{2}e^{-i\phi}\right)^q \\
=&\sum_{q,p}\left|p\right\rangle\left\langle q\right|\sqrt{\binom{2S}{p}\binom{2S}{q}}\int d\Omega\hat{r}^\alpha\cos^4\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\left(e^{i\phi}\right)^{p-q}
\end{aligned}
\tag{68}
$$

1. $\alpha = x$

$$
\begin{aligned}
&\int d\Omega \hat{r}^x \left|\Omega\right\rangle \left\langle\Omega\right| \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta d\phi \sin\theta \cos\phi \\
&\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\left(e^{i\phi}\right)^{p-q} \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\left(e^{i\phi}\right)^{p-q} \\
&\int d\phi \cos\phi e^{i\phi(p-q)} \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q} \\
&\frac{1}{2}\int d\phi \left(e^{i\phi(p-q-1)} + e^{i\phi(p-q+1)}\right) \\
=&\sum_{q,p} \left|p\right\rangle \left\langle p\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q} \\
&\pi\left(\delta_{p,q+1} + \delta_{p,q-1}\right) \\
=&\pi\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q+1}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q+1} + \\
&\pi\sum_{q=1}^{2S} \left|q-1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q-1}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q-1} \\
=&\pi\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q+1}\binom{2S}{q}} 4\int_0^\pi (d\theta) \sin^{2q+3}\frac{\theta}{2}\cos^{4S-2q+1}\frac{\theta}{2} \\
&+\pi\sum_{q=1}^{2S} \left|q-1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q-1}\binom{2S}{q}} 4\int_0^\pi (d\theta) \sin^{2q+1}\frac{\theta}{2}\cos^{4S-2p+3}\frac{\theta}{2} \\
=&4\pi\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q+1}\binom{2S}{q}} \frac{(q+1)!\,(2S-q)!}{(2S+2)!} \\
&+4\pi\sum_{q=1}^{2S} \left|q-1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q-1}\binom{2S}{q}} \frac{q!\,(2S-q+1)!}{(2S+2)!} \\
=&4\pi\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \frac{\sqrt{(q+1)(2S-q)}}{(2S+2)(2S+1)} \\
&+4\pi\sum_{p=0}^{2S-1} \left|p\right\rangle \left\langle p+1\right| \frac{\sqrt{q(2S-q+1)}}{(2S+2)(2S+1)} \\
=&\frac{2\pi}{(1S+1)(2S+1)}\left(S^- + S^+\right) \\
=&\frac{4\pi}{(S+1)(2S+1)}S^x
\end{aligned}
\tag{69}
$$

2. $\alpha = y$

$$
\begin{aligned}
&\int d\Omega \hat{r}^y \left|\Omega\right\rangle \left\langle\Omega\right| \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \\
&\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q}\int d\phi \sin\phi \left(e^{i\phi}\right)^{p-q} \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q} \\
&\frac{1}{2i}\int d\phi \left(e^{i\phi(p-q+1)} - e^{i\phi(p-q-1)}\right) \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q} \\
&\frac{\pi}{i}\left(\delta_{p,q-1} - \delta_{p,q+1}\right) \\
=&\frac{\pi}{i}\sum_{q=1}^{2S} \left|q-1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q-1}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q-1} \\
&-\frac{\pi}{i}\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \sqrt{\binom{2S}{q+1}\binom{2S}{q}} \int \sin\theta d\theta \sin\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q+1} \\
=&\frac{4\pi}{i}\sum_{q=1}^{2S} \left|q-1\right\rangle \left\langle q\right| \frac{\sqrt{q\left(2S-q+1\right)}}{\left(2S+2\right)\left(2S+1\right)} \\
&-\frac{4\pi}{i}\sum_{q=0}^{2S-1} \left|q+1\right\rangle \left\langle q\right| \frac{\sqrt{\left(q+1\right)\left(2S-q\right)}}{\left(2S+2\right)\left(2S+1\right)} \\
=&\frac{4\pi}{\left(2S+2\right)\left(2S+1\right)}\frac{1}{i}\left(S^+ - S^-\right) \\
=&\frac{4\pi}{\left(S+1\right)\left(2S+1\right)}S^y
\end{aligned}
\tag{70}
$$

3. $\alpha = z$

$$
\begin{aligned}
&\int d\Omega \hat{r}^z \left|\Omega\right\rangle \left\langle\Omega\right| \\
=&\sum_{q,p} \left|p\right\rangle \left\langle q\right| \sqrt{\binom{2S}{p}\binom{2S}{q}} \int \sin\theta d\theta \cos\theta \\
&\cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{p+q} \int d\phi \left(e^{i\phi}\right)^{p-q} \\
=&2\pi \sum_q \left|q\right\rangle \left\langle q\right| \binom{2S}{q} \int \sin\theta d\theta \cos\theta \cos^{4S}\frac{\theta}{2}\left(\tan\frac{\theta}{2}\right)^{2q} \\
=&2\pi \sum_q \left|q\right\rangle \left\langle q\right| \binom{2S}{q} 2 \int d\theta \left(2\cos^2\frac{\theta}{2}-1\right) \cos^{4S-2q+1}\frac{\theta}{2}\sin^{2q+1}\frac{\theta}{2} \\
=&4\pi \sum_q \left|q\right\rangle \left\langle q\right| \frac{(2S)\ !}{!q!\,(2S-q)!}\left(2\frac{q!\,(2S-q+1)!}{(2S+2)!}-\frac{q!\,(2S-q)!}{(2S+1)!}\right) \\
=&4\pi \sum_q \left|q\right\rangle \left\langle q\right| \left(2\frac{(2S-q+1)}{(2S+2)(2S+1)}-\frac{1}{2S+1}\right) \\
=&\frac{4\pi}{(2S+2)(2S+1)} \sum_q \left|q\right\rangle \left\langle q\right| \left(2(2S-q+1)-(2S+2)\right) \\
=&\frac{4\pi}{(2S+2)(2S+1)} \sum_q \left|q\right\rangle \left\langle q\right| (2S-2q) \\
=&\frac{4\pi}{(S+1)(2S+1)} \sum_q \left|q\right\rangle \left\langle q\right| (S-q) \\
=&\frac{4\pi}{(S+1)(2S+1)} S^z
\end{aligned} \tag{71}
$$

新参数下态的 overlap

$$
\begin{aligned}
&\left\langle\Omega_1\middle|\,\Omega_2\right\rangle \\
=&\left\langle\mu_1\middle|\,\mu_2\right\rangle \\
=&\frac{\left(1+\mu_1^*\mu_2\right)^{2S}}{\left(1+\left|\mu_1\right|^2\right)^S\cdot\left(1+\left|\mu_2\right|^2\right)^S} \\
=&\frac{\left(1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}\right)^{2S}}{\left(1+\tan^2\frac{\theta_1}{2}\right)^S\cdot\left(1+\tan^2\frac{\theta_2}{2}\right)^S} \\
=&\frac{\left(1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}\right)^{2S}}{\sec^{2S}\frac{\theta_1}{2}\cdot\sec^{2S}\frac{\theta_2}{2}} \\
=&\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}\right)^{2S} \\
=&\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\left(\phi_2-\phi_1\right)+i\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\left(\phi_2-\phi_1\right)\right)^{2S} \\
=&\left(\rho^2e^{2i\gamma}\right)^S
\end{aligned} \tag{72}
$$

- 其中

$$
\begin{aligned}
&\rho^2 \\
=&\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\left(\phi_2-\phi_1\right)\right)^2+ \\
&\left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\left(\phi_2-\phi_1\right)\right)^2 \\
=&\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}+2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\left(\phi_2-\phi_1\right) \\
&+\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\left(\cos^2\left(\phi_2-\phi_1\right)+\sin^2\left(\phi_2-\phi_1\right)\right) \\
=&\frac{1}{4}\left[\left(1+\cos\theta_1\right)\left(1+\cos\theta_2\right)+\left(1-\cos\theta_1\right)\left(1-\cos\theta_2\right)\right] \\
&+\frac{1}{2}\sin\theta_1\sin\theta_2\cos\left(\phi_2-\phi_1\right) \\
=&\frac{1+\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\left(\phi_2-\phi_1\right)}{2} \\
=&\frac{1+\hat{r}_1\cdot\hat{r}_2}{2}
\end{aligned}
\tag{73}
$$

- $\hat{r}_1, \hat{r}_2$ 满足

$$
\begin{aligned}
\hat{r}_1 &= \left(\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\phi_1\right) \\
\hat{r}_2 &= \left(\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\phi_2\right)
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
&\hat{r}_1\cdot\hat{r}_2 \\
=&\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2+\sin\theta_1\sin\phi_1\sin\theta_2\sin\phi_2 \\
&+\cos\phi_1\cos\phi_2 \\
=&\sin\theta_1\sin\theta_2\left(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_1\right)+\cos\phi_1\cos\phi_2 \\
=&\cos\phi_1\cos\phi_2+\sin\theta_1\sin\theta_2\cos\left(\phi_2-\phi_1\right)
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
&\gamma \\
=&\arctan\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\left(\phi_2-\phi_1\right)}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\left(\phi_2-\phi_1\right)} \\
=&\arctan\frac{\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cdot 2\sin\left(\frac{\phi_2-\phi_1}{2}\right)\cos\left(\frac{\phi_2-\phi_1}{2}\right)}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(\cos^2\left(\frac{\phi_2-\phi_1}{2}\right)+\sin^2\left(\frac{\phi_2-\phi_1}{2}\right)\right)+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(\cos^2\left(\frac{\phi_2-\phi_1}{2}\right)-\sin^2\left(\frac{\phi_2-\phi_1}{2}\right)\right)} \\
=&\arctan\frac{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\tan\left(\frac{\phi_2-\phi_1}{2}\right)}{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(1+\tan^2\left(\frac{\phi_2-\phi_1}{2}\right)\right)+\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(1-\tan^2\left(\frac{\phi_2-\phi_1}{2}\right)\right)} \\
=&\arctan\frac{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\tan\left(\frac{\phi_2-\phi_1}{2}\right)}{\cos\frac{\theta_2-\theta_1}{2}+\cos\frac{\theta_2+\theta_1}{2}\tan^2\left(\frac{\phi_2-\phi_1}{2}\right)} \\
=&\arctan\frac{\left(\cos\frac{\theta_2-\theta_1}{2}-\cos\frac{\theta_2+\theta_1}{2}\right)\tan\left(\frac{\phi_2-\phi_1}{2}\right)}{\cos\frac{\theta_2-\theta_1}{2}+\cos\frac{\theta_2+\theta_1}{2}\tan^2\left(\frac{\phi_2-\phi_1}{2}\right)}
\end{aligned}
\tag{76}
$$

新参数下算符的矩阵元：

利用球面坐标与复数的关系，利用前面推导过的矩阵元的表示，可以得到：

$$
\begin{aligned}
&\langle\Omega_1|\, A\,|\Omega_2\rangle \\
=&\langle\mu_1|\, A\,|\mu_2\rangle \\
=&\frac{2S\cdot\mu_1^*\mu_2}{1+\mu_1^*\mu_2}\,\langle\mu_1|\,\mu_2\rangle \\
=&\frac{2S\cdot\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}}\,\langle\Omega_1|\,\Omega_2\rangle
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
&\langle\Omega_1|\, S^-\,|\Omega_2\rangle \\
=&\langle\mu_1|\, S^-\,|\mu_2\rangle \\
=&\frac{2S\cdot\mu_1^*}{1+\mu_1^*\mu_2}\,\langle\mu_1|\,\mu_2\rangle \\
=&\frac{2S\cdot\tan\frac{\theta_1}{2}e^{-i\phi_1}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}}\,\langle\Omega_1|\,\Omega_2\rangle
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&\langle\Omega_1|\, S^+\,|\Omega_2\rangle \\
=&\langle\mu_1|\, S^+\,|\mu_2\rangle \\
=&\frac{2S\cdot\mu}{1+\mu_1^*\mu_2}\,\langle\mu_1|\,\mu_2\rangle \\
=&\frac{2S\cdot\tan\frac{\theta_2}{2}e^{i\phi_2}}{1+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}e^{i(\phi_2-\phi_1)}}\,\langle\Omega_1|\,\Omega_2\rangle
\end{aligned}
\tag{79}
$$

特别的，对于对角元有：

$$
\begin{cases}
\langle\Omega|\, A\,|\Omega\rangle = \frac{2S\cdot|\mu|^2}{1+|\mu|^2} = \frac{2S\cdot\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = 2S\sin^2\frac{\theta}{2} = S\left(1-\cos\theta\right) \\
\langle\Omega|\, S^-\,|\Omega\rangle = \frac{2S\cdot\mu^*}{1+|\mu|^2} = \frac{2S\cdot\tan\frac{\theta}{2}e^{-i\phi}}{1+\tan^2\frac{\theta}{2}} = S\sin\theta e^{-i\phi} \\
\langle\Omega|\, S^+\,|\Omega\rangle = \frac{2S\cdot\mu}{1+|\mu|^2} = \frac{2S\cdot\tan\frac{\theta}{2}e^{i\phi}}{1+\tan^2\frac{\theta}{2}} = S\sin\theta e^{i\phi}
\end{cases}
\tag{80}
$$

从而：

$$
\begin{cases}
\langle\Omega|\, S^x\,|\Omega\rangle = \frac{S^++S^-}{2} = S\sin\theta\cos\phi \\
\langle\Omega|\, S^y\,|\Omega\rangle = \frac{S^+-S^-}{2i} = S\sin\theta\sin\phi \\
\langle\Omega|\, S^z\,|\Omega\rangle = S-A = S\cos\theta
\end{cases}
\Leftrightarrow \langle\Omega|\,\mathbf{S}\,|\Omega\rangle = S\hat{r}
\tag{81}
$$

7.3.4 The Effect of Rotating State

我们考虑将原来的 z 轴旋转到新的 z' 轴，从而新的态

$$
|\mu\rangle' = R\,|\mu\rangle
\tag{82}
$$

- R 为旋转变换

变换后的态由对应的新的算符生成：

$$
|\mu\rangle' = \frac{1}{\left(1+|\mu|^2\right)^S} e^{\mu\left(S^-\right)'}\,|0\rangle'
\tag{83}
$$

旋转变换下观测量应该保持不变，从而旋转变换前后的算符应当满足：

$$
\begin{aligned}
&{}'\langle\mu|\left(S^i\right)'|\mu\rangle' = \langle\mu|S^i|\mu\rangle \\
\Rightarrow &\langle\mu|R^\dagger\left(S^i\right)' R|\mu\rangle = \langle 0|S^i|0\rangle \\
&\Rightarrow R^\dagger\left(S^i\right)' R = S^i, \quad \forall i = x, y, z
\end{aligned}
\tag{84}
$$

一般而言，旋转变换可以表示为：

$$
R = e^{i\theta(\hat{n}\times\hat{z})\cdot\mathbf{S}} \tag{85}
$$

利用三个欧拉角，可以得到：

$$
R = e^{iS^z\phi}e^{iS^y\theta}e^{iS^z\chi} \tag{86}
$$

利用 Schwinger 表示，我们可以得到变换后的态可表示为：

$$
\begin{aligned}
&|\mu\rangle' \\
=&R|\mu\rangle \\
\xrightarrow{Holstein-Primakoff\text{表示}}&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}R|p\rangle \\
\xrightarrow{Schwinger\text{表示}}&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}R|2S-p,p\rangle \\
=&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}|0,0\rangle \\
=&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}R^{-1}R|0,0\rangle
\end{aligned}
\tag{87}
$$

- Holetein-Primakoof 表示下的态 $|p\rangle$ ，在 Schwinger 表示下为 $|2S-p,p\rangle$ （解方程组：$\begin{cases}\frac{1}{2}(a-b)=S-p\\ a+b=2S\end{cases}$ ）
- 最后一个等式中，$R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}R^{-1}$ 即为前面推导过的 Swinger 玻色子产生湮灭算符在旋转变化下的表示。

$$
R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}R^{-1} = \frac{\left(\left(a^\dagger\right)'\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(\left(b^\dagger\right)'\right)^p}{\sqrt{p!}} \tag{88}
$$

- 最后一个等式中，真空态在旋转变换下不变：

$$
\begin{aligned}
&R|0,0\rangle \\
=&e^{iS^z\phi}e^{iS^y\theta}e^{iS^z\chi}|0,0\rangle \\
=&e^{i\phi\frac{n_a-n_b}{2}}e^{i\theta b^\dagger a}e^{i\chi\frac{n_a-n_b}{2}}|0,0\rangle \\
=&|0,0\rangle
\end{aligned}
\tag{89}
$$

将上述分析代入等式，得到：

$$
\begin{aligned}
&R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}R^{-1}R\,|0,0\rangle\\
=&\frac{\left(\left(a^\dagger\right)'\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(\left(b^\dagger\right)'\right)^p}{\sqrt{p!}}\,|0,0\rangle\\
=&\frac{\left(ua^\dagger e^{i\frac{\chi}{2}}+vb^\dagger e^{i\frac{\chi}{2}}\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(-v^*a^\dagger e^{-i\frac{\chi}{2}}+u^*b^\dagger e^{-i\frac{\chi}{2}}\right)^p}{\sqrt{p!}}\,|0,0\rangle\\
=&e^{i\chi(S-p)}\frac{\left(ua^\dagger+vb^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(-v^*a^\dagger+u^*b^\dagger\right)^p}{\sqrt{p!}}\,|0,0\rangle
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
&|\mu\rangle'\\
=&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}R\frac{\left(a^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(b^\dagger\right)^p}{\sqrt{p!}}R^{-1}R\,|0,0\rangle\\
=&\frac{1}{\left(1+|\mu|^2\right)^S}\sum_{p=0}^{2S}\mu^p\sqrt{\binom{2S}{p}}e^{i\chi(S-p)}\frac{\left(ua^\dagger+vb^\dagger\right)^{2S-p}}{\sqrt{(2S-p)!}}\frac{\left(-v^*a^\dagger+u^*b^\dagger\right)^p}{\sqrt{p!}}\,|0,0\rangle
\end{aligned}
\tag{91}
$$

7.3.4 Another defination

自旋相干态还可以有最大极化态的旋转来定义：（书中采用该定义）

$$
\begin{aligned}
|\Omega\rangle &= \mathcal{R}(\chi,\theta,\phi)\,|S,S\rangle\\
&= e^{iS^z\phi}e^{iS^y\theta}e^{iS^z\chi}\,|S,S\rangle
\end{aligned}
\tag{92}
$$

与上面的推导相似，利用 Schwinger 玻色子旋转的表示可以得到：

$$
\begin{aligned}
|\Omega\rangle &= \mathcal{R}\,|S,S\rangle \\
&= \mathcal{R}\frac{\left(a^\dagger\right)^{2S}}{\sqrt{(2S)!}}\mathcal{R}^{-1}\mathcal{R}\,|0,0\rangle \\
&= \frac{\left(\left(a^\dagger\right)'\right)^{2S}}{\sqrt{(2S)!}}\,|0,0\rangle \\
&= \frac{\left(ua^\dagger+vb^\dagger\right)^{2S}}{\sqrt{(2S)!}}\,|0,0\rangle \\
&= \sum_{p=0}^{2S}\frac{(2S)!}{(2S-p)!p!}\frac{\left(ua^\dagger\right)^{2S-p}\left(vb^\dagger\right)^{p}}{\sqrt{(2S)!}}\,|0,0\rangle \\
&= \sum_{p=0}^{2S}\sqrt{\binom{2S}{p}}\left(\cos\frac{\theta}{2}e^{i\frac{\phi}{2}}\right)^{2S-p}\left(-\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}}\right)^{p}|2S-p,p\rangle \\
&= \cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\sqrt{\binom{2S}{p}}(-1)^p\tan^p\frac{\theta}{2}e^{i\phi(S-p)}\,|2S-p,p\rangle \\
&= \cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\sqrt{\binom{2S}{p}}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^{p}(-1)^p\,e^{i\phi(S-2p)}\,|2S-p,p\rangle \\
&= \cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\sqrt{\binom{2S}{p}}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^{p}\left[e^{i\phi(S-2p)+i\pi p}\,|2S-p,p\rangle\right] \\
\xrightarrow{H-P\text{表示}} &= \cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\sqrt{\binom{2S}{p}}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^{p}\left[e^{i\phi(S-2p)+i\pi p}\,|p\rangle\right]
\end{aligned}
\tag{93}
$$

❓ 该定义得到的结果似乎与之前定义得到的结果不一致：

$|\Omega\rangle\equiv|\theta,\phi\rangle=|\mu\rangle=\cos^{2S}\frac{\theta}{2}\sum_{p=0}^{2S}\left(\tan\frac{\theta}{2}e^{i\phi}\right)^p\sqrt{\binom{2S}{p}}\,|p\rangle$

7.3.5 many spins

上述对单个 spin 的结论可以很简单地推广到多个 spin。考虑一个有 N 个 格点的 spin 晶格。多体自旋相干态为单个自旋相干态的直积：

$$
|\Omega\rangle=\prod_{i=1}^{\mathcal{N}}|\Omega_i\rangle \tag{94}
$$

回顾单个 spin 相干态的 overlap，可以立刻得到多个 spin 的 overlap：

$$
\langle\Omega|\Omega'\rangle=\prod_i\left(\frac{1+\hat{r}_i\cdot\hat{r}_i'}{2}\right)^S e^{-iS\sum_i\psi[\hat{r}_i,\hat{r}_i']} \tag{95}
$$

同样的，对多体自旋相干态也有完备性关系：

$$
\int\prod_i\left(\frac{2S+1}{4\pi}d\Omega_i\right)|\Omega\rangle\langle\Omega|=I \tag{96}
$$

任意波函数 Ψ 下的两点关联函数可以表示为：

$$\frac{\langle\Psi|\mathbf{S}_i\cdot\mathbf{S}_j|\Psi\rangle}{\langle\Psi|\Psi\rangle}=\frac{(S+1-\delta_{ij})(S+1)}{Z}\int\prod_i d\Omega_i\,|\Psi[\Omega]|^2\,\hat{r}_i\cdot\hat{r}_j \tag{97}$$

- $Z=\int\prod_i d\Omega_i\,|\Psi[\Omega]|^2$
- $\Psi\,[\Omega]=\langle\Psi|\,\Omega\rangle$

▼ 证明：

前面已经证明自旋算符由自旋相干态外积的表达式，这可以推广到多体 spin 和多个自旋算符相乘的情况：

$$\begin{aligned}&\frac{(S+1)(2S+1)}{4\pi}\left(\frac{2S+1}{4\pi}\right)^{N-1}\int\prod_j d\Omega_j\hat{r}_i^\alpha\,|\Omega\rangle\,\langle\Omega|\\=&\frac{(S+1)(2S+1)}{4\pi}\int d\Omega_i\hat{r}_i^\alpha\,|\Omega_i\rangle\,\langle\Omega_i|\cdot\left(\frac{2S+1}{4\pi}\right)^{N-1}\prod_{j\neq i}\int d\Omega_j\,|\Omega_j\rangle\,\langle\Omega_j|\\=&S_i^\alpha\end{aligned} \tag{98}$$

1. $i\neq j$

$$\begin{aligned}&\left(\frac{(S+1)(2S+1)}{4\pi}\right)^2\left(\frac{2S+1}{4\pi}\right)^{N-2}\int\prod_k d\Omega_k\hat{r}_i^\alpha\hat{r}_j^\beta\,|\Omega\rangle\,\langle\Omega|\\&=\frac{(S+1)(2S+1)}{4\pi}\int d\Omega_i\hat{r}_i^\alpha\,|\Omega_i\rangle\,\langle\Omega_i|\cdot\frac{(S+1)(2S+1)}{4\pi}\\&\int d\Omega_j\hat{r}_j^\beta\,|\Omega_j\rangle\,\langle\Omega_j|\left(\frac{2S+1}{4\pi}\right)^{N-2}\prod_{k\neq i,j}\int d\Omega_k\,|\Omega_k\rangle\,\langle\Omega_k|\\&=S_i^\alpha\cdot S_j^\beta\end{aligned} \tag{99}$$

$$\begin{aligned}&\mathbf{S_i}\cdot\mathbf{S_j}\\=&S_i^xS_j^x+S_i^yS_j^y+S_i^zS_j^z\\=&\left(\frac{(S+1)\,(2S+1)}{4\pi}\right)^2\left(\frac{2S+1}{4\pi}\right)^{N-2}\int\prod_k d\Omega_k\hat{r}_i\cdot\hat{r}_j\,|\Omega\rangle\,\langle\Omega|\end{aligned} \tag{100}$$

$$\begin{aligned}&\frac{\langle\Psi|\,\mathbf{S}_i\cdot\mathbf{S}_j\,|\Psi\rangle}{\langle\Psi|\,\Psi\rangle}\\=&\frac{\left(\frac{(S+1)(2S+1)}{4\pi}\right)^2\left(\frac{2S+1}{4\pi}\right)^{N-2}}{\left(\frac{2S+1}{4\pi}\right)^N\int\prod_i d\Omega_i\,\langle\Psi|\,\Omega\rangle\,\langle\Omega|\,\Psi\rangle}\int\prod_k d\Omega_k\hat{r}_i\cdot\hat{r}_j\,\langle\Psi|\,\Omega\rangle\,\langle\Omega|\,\Psi\rangle\\=&\frac{(S+1)^2}{Z}\int\prod_k d\Omega_k\hat{r}_i\cdot\hat{r}_j\,|\Psi\,[\Omega]|^2\end{aligned} \tag{101}$$

2. $i=j$

$$
\begin{aligned}
&\mathbf{S}_i^2 \\
&= \left(\frac{2S+1}{4\pi}\right)^N \int \prod_j d\Omega_j \mathbf{S}_i^2 \left|\Omega\right\rangle \left\langle\Omega\right| \\
&= \left(\frac{2S+1}{4\pi}\right)^N S\left(S+1\right) \int \prod_j d\Omega_j \left|\Omega\right\rangle \left\langle\Omega\right|
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
&\frac{\left\langle\Psi\right| \mathbf{S}_i^2 \left|\Psi\right\rangle}{\left\langle\Psi\right|\left.\Psi\right\rangle} \\
=&\frac{\left(\frac{2S+1}{4\pi}\right)^N S\left(S+1\right)}{\left(\frac{2S+1}{4\pi}\right)^N \int \prod_i d\Omega_i \left\langle\Psi\right|\left.\Omega\right\rangle \left\langle\Omega\right|\left.\Psi\right\rangle} \int \prod_k d\Omega_k \left\langle\Psi\right|\left.\Omega\right\rangle \left\langle\Omega\right|\left.\Psi\right\rangle \\
=&\frac{S\left(S+1\right)}{Z} \int \prod_k d\Omega_k \left|\Psi\left[\Omega\right]\right|^2
\end{aligned}
\tag{103}
$$

两种情况的结果可以合并为：

$$
\begin{aligned}
&\frac{\left\langle\Psi\right| \mathbf{S}_i \cdot \mathbf{S}_j \left|\Psi\right\rangle}{\left\langle\Psi\right|\left.\Psi\right\rangle} \\
&= \frac{\left(S+1-\delta_{i,j}\right)\left(S+1\right)}{Z} \int \prod_k d\Omega_k \hat{r}_i \cdot \hat{r}_j \left|\Psi\left[\Omega\right]\right|^2
\end{aligned}
\tag{104}
$$

自旋相干态也可以用来计算算子的迹。假设 $\{|n\rangle\}$ 为一组正交完备基矢，则：

$$
\begin{aligned}
Tr\,\mathcal{O} &= \sum_n \langle n|\mathcal{O}|n\rangle \\
&= \left(\frac{2S+1}{4\pi}\right)^2 \int d\Omega \int d\Omega' \sum_n \langle n|\Omega\rangle\langle\Omega|\mathcal{O}|\Omega'\rangle\langle\Omega'|n\rangle \\
&= \left(\frac{2S+1}{4\pi}\right)^2 \int d\Omega \int d\Omega' \langle\Omega' \left(\sum_n |n\rangle\langle n|\right) \Omega\rangle\langle\Omega|\mathcal{O}|\Omega'\rangle \\
&= \frac{2S+1}{4\pi} \int d\Omega \langle\Omega|\mathcal{O}| \left(\frac{2S+1}{4\pi} \int d\Omega' |\Omega'\rangle\langle\Omega'|\right) ||\Omega\rangle \\
&= \frac{2S+1}{4\pi} \int d\Omega \langle\Omega|\mathcal{O}|\Omega\rangle
\end{aligned}
\tag{105}
$$

自旋相干态阐明了经典自旋和量子自旋的对应关系。经典极限可由 $S \to \infty$ 得到，在该极限下，不同自旋相干态的 overlap 将随 S 指数衰减 （$\frac{1+\hat{r}_1 \cdot \hat{r}_2}{2} < 1$）。自旋算符的期望值将是单位向量的函数，就像经典自旋一样。量子效应因此与自旋相干态的非正交性相联系，这也意味着 $\Psi\left[\Omega\right]$ 在 Ω 中存在一个有限宽度。

Appendix A

A.2 Normal Bilinear Operators

双线性算符定义为：

$$
\hat{A} \;= \sum_{ij} a_i^\dagger A_{ij} a_j \;\equiv \mathbf{a}^\dagger \cdot A \cdot \mathbf{a}
\tag{106}
$$

- A 为厄米矩阵

玻色子（费米子）的双线性算符与线性算符的对易（反对易）关系特别简单，线性算符 $\hat{\mathbf{v}}^\dagger$ 定义为：

$$\hat{\mathbf{v}}^\dagger = \sum_i v_i a_i^\dagger = \mathbf{v} \cdot \mathbf{a}^\dagger \tag{107}$$

则有：

$$\left[\hat{A}, \hat{\mathbf{v}}^\dagger\right] = (A\mathbf{v}) \cdot \mathbf{a}^\dagger \tag{108}$$

▼ 证明

$$\begin{aligned}\left[\hat{A}, \hat{\mathbf{v}}^\dagger\right] &= A_{ij} v_k \left[a_i^\dagger a_j, a_k^\dagger\right] \\ &= A_{ij} v_k \delta_{ij} a_i \\ &= A_{ij} v_j a_i \\ &= (A \cdot \mathbf{v}) \cdot \mathbf{a}^\dagger\end{aligned} \tag{109}$$

特别的，如果 $\mathbf{v}$ 是 A 特征值为 v 的特征向量，则 $\hat{\mathbf{v}}^\dagger$ 为 $[A, \cdot]$ 特征值为 v 的特征算符，即

$$\left[\hat{A}, \hat{\mathbf{v}}^\dagger\right] = (A \cdot \mathbf{v}) \cdot \mathbf{a}^\dagger = v\hat{\mathbf{v}}^\dagger \tag{110}$$

此时，在旋转变换下：

$$\begin{aligned}e^{i\theta\hat{A}} \hat{\mathbf{v}}^\dagger e^{-i\theta\hat{A}} &= \hat{\mathbf{v}}^\dagger + i\theta[\hat{A}, \hat{\mathbf{v}}^\dagger] + \frac{(i\theta)^2}{2}\left[\hat{A}, [\hat{A}, \hat{\mathbf{v}}^\dagger]\right] + \dots \\ &= e^{iv\theta} \hat{\mathbf{v}}^\dagger\end{aligned} \tag{111}$$

一个幺正矩阵可用一组厄米生成元 A_α 和参数 θ_α 来表示：

$$U_\theta = e^{i \sum_\alpha \theta_\alpha A_\alpha} \tag{112}$$

对应的幺正算符也有同样的表示：

$$\hat{U}_\theta = e^{i \sum_\alpha \theta_\alpha \hat{A}_\alpha} \tag{113}$$

利用对易关系有：

$$\hat{U}_\theta \hat{\mathbf{v}}^\dagger \hat{U}_\theta^{-1} = (U_\theta \mathbf{v}) \cdot \mathbf{a}^\dagger \tag{114}$$

▼ 证明

记

$$\begin{aligned}\hat{U}_\theta &= e^{\hat{A}} \\ U_\theta &= e^{A}\end{aligned} \tag{115}$$

则：

$$
\begin{aligned}
\hat{U}_\theta \hat{\mathbf{v}}^\dagger \hat{U}_\theta^{-1} &= e^{ad_{\hat{A}}} \hat{\mathbf{v}}^\dagger \\
&= \hat{\mathbf{v}}^\dagger + \left[\hat{A}, \hat{\mathbf{v}}^\dagger\right] + \frac{1}{2}\left[\hat{A}, \left[\hat{A}, \hat{\mathbf{v}}^\dagger\right]\right] + \cdots \\
&= \mathbf{v} \cdot \mathbf{a}^\dagger + (A\mathbf{v}) \cdot \mathbf{a}^\dagger + \frac{1}{2}\left(A^2\mathbf{v}\right) \cdot \mathbf{a}^\dagger + \cdots \\
&= \left(e^A \mathbf{v}\right) \cdot \mathbf{a}^\dagger \\
&= (U_\theta \mathbf{v}) \cdot \mathbf{a}^\dagger
\end{aligned} \tag{116}
$$

两个双线性算符的对易可以表示为：

$$
\left[\hat{A}, \hat{B}\right] = \mathbf{a}^\dagger [A, B] \mathbf{a} \tag{117}
$$

▼ 证明

$$
\begin{aligned}
\left[\hat{A}, \hat{B}\right] &= A_{ij} B_{lm} \left[a_i^\dagger a_j, a_l^\dagger a_m\right] \\
&= A_{ij} B_{lm} \left(a_i^\dagger \left[a_j, a_l^\dagger\right] a_m + a_l^\dagger \left[a_i^\dagger, a_m\right] a_j\right) \\
&= A_{ij} B_{lm} \left(a_i^\dagger a_m \delta_{jl} - a_l^\dagger a_j \delta_{im}\right) \\
&= A_{ij} B_{jm} a_i^\dagger a_m - B_{li} A_{ij} a_l^\dagger a_j \\
&= \mathbf{a}^\dagger [A, B] \mathbf{a}
\end{aligned} \tag{118}
$$