Prepared for submission to JHEP

Weekly/Monthly: 2021.3.7.-

Yuan Yao[a,b]

[a] *Institute for Solid State Physics, University of Tokyo, Kashiwa, Chiba 277-8581, Japan*

[b] *Condensed Matter Theory Laboratory, RIKEN CPR, Wako, Saitama 351-0198, Japan*

Contents

1 Spinorial WEEKLY: 2021. 3. 7.-13.

1.1 Lorentz Group

Let us consider a set of element which transforms basis vector $\{e_\mu : \mu = 0, \cdots, D-1\}$ in D-dimensional spacetime:

$$\hat{\Lambda}(e_\mu) = e_\nu \Lambda^\nu{}_\mu, \tag{1.1}$$

where the dumb indices are summed implicitly and the matrix Λ satisfies:

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}; \tag{1.2}$$

$$\eta_{\mu\nu} \equiv e_\mu \cdot e_\nu = \mathrm{diag}[-1, +1, \cdots, +1]_{D\times D}, \tag{1.3}$$

which exactly means that the Λ transformation preserves the inner dot products: $\hat{\Lambda}(e_\mu) \cdot \hat{\Lambda}(e_\nu) = e_\mu \cdot e_\nu = \eta_{\mu\nu}$ so that we can let such inner products to be a fixed matrix η.

For an arbitrary vector, the definition of action by $\hat{\Lambda}$ is to be the linear extension:

$$\begin{aligned}\hat{\Lambda}(x^\mu e_\mu) &= x^\mu \hat{\Lambda}(e_\mu) \\ &= (\Lambda^\mu{}_\nu x^\nu) e_\mu,\end{aligned} \tag{1.4}$$

so that we can consider the transformation is on the coordinate $\{x^\mu\}$ effectively. Obviously, $\hat{\Lambda}$ has an inverse represented by

$$\left(\Lambda^{-1}\right)^\mu{}_\nu \equiv \Lambda_\nu{}^\mu, \tag{1.5}$$

where we have used η to raise or lower the indices [1].

Proof: $\Lambda^\alpha{}_\mu \left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda^\alpha{}_\mu \eta_{\nu\beta}\eta^{\mu\varphi}\Lambda^\beta{}_\varphi = \eta_{\nu\beta}\eta^{\beta\mu}$ by Eq. (1.2).

Then we consider the successive actions:

$$\begin{aligned}\hat{\Lambda}_2\hat{\Lambda}_1(x^\mu e_\mu) &= \hat{\Lambda}_2[(\Lambda_1)^\mu{}_\nu x^\nu e_\mu] \\ &= (\Lambda_1)^\mu{}_\nu x^\nu e_\alpha (\Lambda_2)^\alpha{}_\mu \\ &= [(\Lambda_2\Lambda_1)^\mu{}_\nu x^\nu] e_\mu,\end{aligned} \tag{1.6}$$

where we have defined the matrix $\Lambda_{1,2}$ to have $(\Lambda_{1,2})^\mu{}_\nu$ as the element at the (μ, ν) position. In the above sense, the action can be seen as a mapping from the Minkowski 4-manifold to itself: $\Lambda : x^\mu \to \Lambda^\mu{}_\nu x^\nu$ in a fixed coordinate system and we will always use this *active* picture instead of coordinate transformation etc. The product of the mappings is simply defined as the composition of mapping. Therefore, we have the (one-directed) correspondence between the element products:

$$\hat{\Lambda}_2\hat{\Lambda}_1 \mapsto \Lambda_2\Lambda_1. \tag{1.7}$$

Furthermore, we have an identity element $\hat{\mathbb{I}}$ which does nothing $\hat{\mathbb{I}}(e_\mu) = e_\mu$ corresponding to the identity matrix. In a short summary, we have obtained a group homomorphism from

[1] Here raising or lowering orientation is irrelevant because of the symmetry of matrix η, which will bother us later in the spinor case.

$\{\hat{\Lambda}\}$, so-called Lorentz group, satisfying Eq. (1.2) with the multiplication as the successive action to the group of invertible $D \times D$ matrices by

$$\hat{\Lambda} \mapsto \Lambda^{\mu}{}_{\nu}. \tag{1.8}$$

In addition, the D-plet $(x^0, x_1, \cdots, x^{D-1})$ gives D-dimensional (real) linear representation of the group $\{\hat{\Lambda}\}$.

Finally, the connected component to the identity contains the infinitesimal transformation represented by

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}, \tag{1.9}$$

with $\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$. Proof: $-\delta\omega^{\mu}{}_{\nu} = \delta\omega_{\nu}{}^{\mu}$ by Eq. (1.5).

1.2 Quantum generators

In quantum mechanics, the group element is represented by the transformation operator acting on the state vectors:

$$\hat{\Lambda}|\Phi\rangle \equiv \hat{U}(\Lambda)|\Phi\rangle. \tag{1.10}$$

We consider again the infinitesimal transformations and assume the operator correspondence takes the form as

$$\hat{U}(\Lambda) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}\hat{M}^{\mu\nu}, \tag{1.11}$$

where the "1/2" takes into account the over-counting and $\hat{M}^{\mu\nu} = -\hat{M}^{\nu\mu}$ are hermitian operators so that the transformation $\hat{U}$ is unitary.

To be consistent in Eq. (1.10), we have the justification of the form of the infinitesimal transformation above

$$\hat{U}(\Lambda_2)\hat{U}(\Lambda_1) = \hat{U}(\Lambda_2\Lambda_1), \tag{1.12}$$

i.e., the operator correspondence is a group homomorphism and the state vector space is a representation (unnecessarily linear and potentially projective in half-integral spin cases).

We know that

$$\begin{aligned} \Lambda_2^{-1}\Lambda_1^{-1}\Lambda_2\Lambda_1 &\mapsto \delta_{\nu} + (\delta\omega_2\delta\omega_1)^{\mu}{}_{\nu} - (\delta\omega_1\delta\omega_2)^{\mu}{}_{\nu}, \\ \text{and} \quad \hat{U}(\Lambda_2^{-1}\Lambda_1^{-1}\Lambda_2\Lambda_1) &= \hat{U}(\Lambda_2^{-1})\hat{U}(\Lambda_1^{-1})\hat{U}(\Lambda_2)\hat{U}(\Lambda_1) \\ &= 1 - \frac{1}{4}\delta\omega_2{}^{\mu\nu}\delta\omega_1{}^{\alpha\beta}[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}]. \end{aligned} \tag{1.13}$$

Combining them together, we obtain at the order $\delta\omega_2\delta\omega_1$ coefficient matching (actually the lowest nonzero, why? It is because when $\Lambda_{1 \text{ or } 2} = \mathbb{I}$, everything should be vanishing.)

$$-\frac{1}{4}\delta\omega_2{}^{\mu\nu}\delta\omega_1{}^{\alpha\beta}[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}] = \frac{i}{2}\left[(\delta\omega_2\delta\omega_1)^{\mu}{}_{\nu} - (\delta\omega_1\delta\omega_2)^{\mu}{}_{\nu}\right]\hat{M}_{\mu}{}^{\nu}, \tag{1.14}$$

which implies a *realization-independent* commutator relation

$$[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}] = \frac{1}{i}\left\{\eta^{\nu\alpha}\hat{M}^{\mu\beta} - \eta^{\beta\mu}\hat{M}^{\alpha\nu} - (\mu \leftrightarrow \nu)\right\}, \tag{1.15}$$

which is determined by Lorentz group (or algebra) completely.

Proof hint: notice that the equation $\delta\omega^{\mu\nu}\hat{M}_{\mu\nu} = \delta\omega^{\mu\nu}A_{\mu\nu}$ true for any antisymmetric $\delta\omega$ only tells that $A_{\mu\nu} - (\mu \leftrightarrow \nu) = 2\hat{M}_{\mu\nu}$.

Then we can derive down the transformation rule of $\hat{M}_{\mu\nu}$:

$$\begin{aligned} \hat{U}(\Lambda^{-1})\hat{M}^{\mu\nu}\hat{U}(\Lambda) &= \hat{M}^{\mu\nu} - \frac{i}{2}\delta\omega_{\alpha\beta}[\hat{M}^{\mu\nu}, \hat{M}^{\alpha\beta}] \\ &= \hat{M}^{\mu\nu} + \delta\omega^{\mu}{}_{\alpha}\hat{M}^{\alpha\nu} + \delta\omega^{\nu}{}_{\alpha}\hat{M}^{\mu\alpha} \\ &= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\hat{M}^{\alpha\beta}. \end{aligned} \tag{1.16}$$

Let us go one step further to include the translation

$$\hat{T}_a[x^\mu e_\mu] = (x^\mu + a^\mu)e_\mu, \tag{1.17}$$

which is clearly nonlinear and, effectively, $\hat{T}_a : x^\mu \to x^\mu + a^\mu$, i.e., it maps every spacetime point with coordinate value x^μ to the coordinate value $x^\mu + a^\mu$ in our quantum system[2].

Thus, we can consider the composition as the successive action:

$$a\Lambda \equiv \hat{T}_a\Lambda, \tag{1.18}$$

which also gives

$$\hat{U}_a(\Lambda) \equiv \hat{U}_a\hat{U}(\Lambda), \tag{1.19}$$

where $\hat{U}_a$ is the unitary transformation associated with $\hat{T}_a$

$$\hat{U}_a(\Lambda)^{-1}\varphi(x)\hat{U}_a(\Lambda) = \varphi\left[\Lambda^{-1}(x-a)\right] \tag{1.20}$$

and, for infinitesimal transformations,

$$\hat{U}_a = 1 - iP_\mu a^\mu. \tag{1.21}$$

Here $P^\mu = (H_{\text{Hamiltonian}}, \vec{P}_{\text{momentum}})$. Think about the minus sign here, e.g., the time evolution by Hamiltonian actually translates the time-series of states back to *past*!

Repeating the previous steps including the translations,

$$[P^\mu, M^{\alpha\beta}] = \frac{1}{i}\eta^{\mu\alpha}P^\beta - (\alpha \leftrightarrow \beta) \tag{1.22}$$

$$[P^\mu, P^\nu] = 0, \tag{1.23}$$

and [Proof hint: prove first that only the translation of the long composition "-1-1+1+1" has the lowest order as $(\delta\omega_2 a_1 - \delta\omega_1 a_2)$.]

$$U(\Lambda^{-1})P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu. \tag{1.24}$$

[2]Note that we are using active picture thereby the background universe is still there and we are pushing our quantum system by vector a^μ.

To re-interpret this equation, we label the state vector by its 4-momentum eigenvalue as $|P^\mu = p^\mu\rangle$, i.e., $P^\mu|p^\mu\rangle = p^\mu|p^\mu\rangle$, then

$$U(\Lambda)|p^\mu\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle, \tag{1.25}$$

up to some phase factor which can be absorbed into the definition of state vector. (Could we play the same game on $M^{\mu\nu}$? What should be noticed? $M^{\mu\nu}$'s do not necessarily commute with each other, so we need to extract out the abelian subalgebra, e.g., Cartan subalgebra.)

1.3 Field operator representation of Poincaré group

Observing Eq. (1.24), we can see that operators $\{P^0, P^1, \cdots, P^{D-1}\}$ themselves might potentially turn out to be a representation. It is true because

$$\begin{aligned}
U[(\Lambda_2\Lambda_1)^{-1}]P^\mu U[(\Lambda_2\Lambda_1)] &= U({\Lambda_1}^{-1})U({\Lambda_2}^{-1})P^\mu U(\Lambda_2)U(\Lambda_1) \\
&= U({\Lambda_1}^{-1}){\Lambda_2}^\mu{}_\nu P^\nu U(\Lambda_1) \\
&= {\Lambda_2}^\mu{}_\nu U({\Lambda_1}^{-1})P^\nu U(\Lambda_1) \\
&= {\Lambda_2}^\mu{}_\nu {\Lambda_1}^\nu{}_\alpha P^\alpha \\
&= (\Lambda_2\Lambda_1)^\mu{}_\nu P^\nu.
\end{aligned} \tag{1.26}$$

This can be also directly seen by the action of Λ's on the P^μ-eigenstate $|p^\mu\rangle$.

Therefore, we could generalize this idea to arbitrary set of local operators that are denoted by $\{\psi_a(x)\}$: (Justify the change on the spacetime coordinate below.)

$$U(\Lambda^{-1})\psi_a(x)U(\Lambda) = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x), \tag{1.27}$$

where $L_a{}^b(\Lambda)$ proves to satisfy similarly to Eq. (1.12) as:

$$L_a{}^b(\Lambda_2)L_b{}^c(\Lambda_1) = L_a{}^c(\Lambda_2\Lambda_1). \tag{1.28}$$

An interesting thing can be noticed once one defines the infinitesimal transformation coefficient as

$$L_a{}^b(\Lambda) = \delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}S_L^{\mu\nu}{}_a{}^b, \tag{1.29}$$

which is exactly the same form as Eq. (1.11)! However, a slight difference that should be also paid adequate attention to is that $M^{\mu\nu}$ is operator on Hilbert space while $S_L^{\mu\nu}{}_a{}^b$ is purely a number. Nevertheless, we can define a matrix $S_L^{\mu\nu}$ with the element at the position (a, b) as $S_L^{\mu\nu}{}_a{}^b$. Hence, $S_L^{\mu\nu}$ is also an operator on the d-dimensional "vector" $(\psi_1, \psi_2, \cdots, \psi_d)$ though, while it makes everything derived before for $M^{\mu\nu}$ applicable here for $S_L^{\mu\nu}$:

$$[S_L^{\mu\nu}, S_L^{\alpha\beta}]_{\text{matrix}} = \frac{1}{i}\left\{\eta^{\nu\alpha}S_L^{\mu\beta} - \eta^{\beta\mu}S_L^{\alpha\nu} - (\mu \leftrightarrow \nu)\right\}, \tag{1.30}$$

and, to be re-emphasized, here the commutator $[\cdots]_{\text{matrix}}$ is a matrix commutator.

Then we can also take into account the spacetime coordinate mapping in Eq. (1.27) as

$$\psi_a(\Lambda^{-1}x) = \left(1 + \frac{i}{2}\delta\omega_{\mu\nu}\mathcal{L}^{\mu\nu}\right)\psi_a(x), \tag{1.31}$$

where, it can be calculated that,

$$\mathcal{L}^{\mu\nu} = \frac{1}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{1.32}$$

By the same logic as above,

$$[\mathcal{L}^{\mu\nu}, \mathcal{L}^{\alpha\beta}]_{\text{diff}} = \frac{1}{i}\left\{\eta^{\nu\alpha}\mathcal{L}^{\mu\beta} - \eta^{\beta\mu}\mathcal{L}^{\alpha\nu} - (\mu \leftrightarrow \nu)\right\}, \tag{1.33}$$

where the commutator is understood as that for the differential (diff) operators. In addition, we can also immediately derive that

$$\begin{aligned} U(\Lambda^{-1})\partial_\mu \psi(x) U(\Lambda) &= \frac{\partial}{\partial x^\mu} U(\Lambda^{-1})\psi(x)U(\Lambda) \\ &= \frac{\partial}{\partial x^\mu}\psi(\Lambda^{-1}x) \\ &= \frac{\partial}{\partial x^\mu}\psi(x^\mu {\Lambda_\mu}^\nu) \\ &= {\Lambda_\mu}^\nu \partial_\nu \psi(\Lambda^{-1}x), \end{aligned} \tag{1.34}$$

which transforms like a vector if ψ does not carry any other index, i.e., a scalar.

Wrapping everything up by Eq. (1.11), we obtain

$$[\psi_a(x), \hat{M}^{\mu\nu}] = ({S_L^{\mu\nu}}_a{}^b + \mathcal{L}^{\mu\nu})\psi_b(x). \tag{1.35}$$

For the sake of later usage, let us transform Eq. (1.24) into the form above:

$$\begin{aligned} U(\Lambda^{-1})P_\mu U(\Lambda) &= ({\delta_\mu}^\nu + {\delta\omega_\mu}^\nu)\, P_\nu \\ &= \left({\delta_\mu}^\nu + \frac{i}{2}\delta\omega_{\alpha\beta} {S_V^{\alpha\beta}}_\mu{}^\nu\right) P_\nu, \end{aligned} \tag{1.36}$$

which means

$${S_V^{\alpha\beta}}_\mu{}^\nu = \frac{1}{i}(\eta^{\beta\nu}\delta^\alpha_\mu - \eta^{\alpha\nu}\delta^\beta_\mu), \tag{1.37}$$

where the subscript "V" denotes "vector".

1.4 Spinor representation

We have now prepared everything that is needed to approach spinors. The basic logic is to find the *matrices* $\{S_L^{\mu\nu}\}$ that satisfy Eq. (1.30). Then the operator transformation can be characterized by Eq. (1.27) before $U(\Lambda)$ is concretely realized or constructed. Of course, there are infinitely many construction, but we will concentrate on something called spinor. Before that, we make a small digression to the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\}_{\text{matrix}} = 2\eta^{\mu\nu}\mathbb{I}_{d\times d}, \tag{1.38}$$

where $\{\}_{\text{matrix}}$ denotes matrix anticommutator and γ^μ is a $d \times d$ matrix. To simplify the notation, we will suppress the subscript "matrix", which wouldn't induce much confusion hopefully. We will also denote the matrix element at (a, b) as ${\gamma^\mu}_a{}^b$.

Let us consider (or try)

$$S^{\mu\nu} = \frac{1}{4i}[\gamma^\mu, \gamma^\nu], \tag{1.39}$$

which is indeed already antisymmetrized by $\mu \leftrightarrow \nu$. Then

$$\begin{aligned}
[S^{\mu\nu}, S^{\alpha\beta}] &= \frac{1}{4i}[\gamma^\mu\gamma^\nu, S^{\alpha\beta}] - (\mu \leftrightarrow \nu) \\
&= \left(\frac{1}{(4i)^2}[\gamma^\mu\gamma^\nu, \gamma^\alpha\gamma^\beta] - (\alpha \leftrightarrow \beta)\right) - (\mu \leftrightarrow \nu) \\
&= \frac{1}{(4i)^2}4\left(\gamma^\mu\gamma^\beta\eta^{\nu\alpha} - \gamma^\mu\gamma^\alpha\eta^{\nu\beta} + \gamma^\beta\gamma^\nu\eta^{\alpha\mu} - \gamma^\alpha\gamma^\nu\eta^{\mu\beta}\right) - (\mu \leftrightarrow \nu) \\
&= \frac{1}{i}(\eta^{\nu\alpha}S^{\mu\beta} - \eta^{\mu\beta}S^{\alpha\nu} + \eta^{\nu\beta}S^{\alpha\mu} - \eta^{\alpha\mu}S^{\nu\beta}) \\
&= \frac{1}{i}\left\{\eta^{\nu\alpha}S^{\mu\beta} - \eta^{\mu\beta}S^{\alpha\nu} - (\mu \leftrightarrow \nu)\right\}.
\end{aligned} \tag{1.40}$$

Hint: The identity $[A, BC] = \{A, B\}C - B\{A, C\}$ is suitable for the calculation above, which enables us to obtain a frequently to-be-used identity:

$$[\gamma^\mu, \gamma^\alpha\gamma^\beta] = 2\eta^{\mu\alpha}\gamma^\beta - 2\eta^{\mu\beta}\gamma^\alpha. \tag{1.41}$$

One more useful identity could be

$$\begin{aligned}
\gamma^{\mu\nu\rho} &\equiv \frac{1}{6}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu) \\
&= \frac{1}{6}(2\gamma^\mu\gamma^{\nu\rho} + [\gamma^\nu\gamma^\rho, \gamma^\mu] + 2\gamma^\mu\gamma^\nu\gamma^\rho - 2\eta^{\mu\nu}\gamma^\rho \\
&\quad 2\eta^{\mu\rho}\nu - 2\gamma^\mu\gamma^\rho\gamma^\nu + [\gamma^\mu, \gamma^\rho\gamma^\nu]) \\
&= \gamma^\mu\gamma^{\nu\rho} - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\nu,
\end{aligned} \tag{1.42}$$

where

$$\gamma_{\mu_1\cdots\mu_p}(= \Gamma_{\mu_1\cdots\mu_p}) \equiv \frac{1}{p!}A_{\{i_j\}}\gamma_{\mu_{i_1}}\cdots\gamma_{\mu_{i_p}}, \tag{1.43}$$

where A denotes the antisymmetrization and we will simply write a general form $\Gamma_{(p)}$ or $\Gamma^{(p)}$ depending on the (contra-)covariance.

Invertibly, there is a power equation to merge the products of γ matrices into a sum of entities, each of which contains only one Γ matrix:

$$\gamma^{\mu_1\cdots\mu_p}\gamma_{\nu_1\cdots\nu_m} = \gamma^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_m} + \cdots \frac{p!}{(p-r)!}\frac{m!}{(m-r)!}\gamma^{[\mu_1\cdots\mu_{p-r}}{}_{[\nu_{r+1}\cdots\nu_m}\delta^{\mu_{p-r+1}\cdots\mu_p]}_{\nu_r\cdots\nu_1]} + \cdots.$$

Therefore, $\{S_{\mu\nu a}{}^b\}$ is an appropriate choice for the generators here. The matrix γ^μ carries a spacetime index, which implies that it can be used to create a vector partially. It is natural to consider the infinitesimal transformation of $[\gamma^\mu\psi(x)]_a$:

$$
\begin{aligned}
& U(\Lambda^{-1})\gamma^\mu{}_a{}^b\psi_b(x)U(\Lambda) \\
&= \gamma^\mu{}_a{}^b U(\Lambda^{-1})\psi_b(x)U(\Lambda) \\
&= \gamma^\mu{}_a{}^b \left\{\delta_b{}^c + \frac{i}{2}\delta\omega^{\alpha\beta}(\delta_b{}^c\mathcal{L}_{\alpha\beta} + S_{\alpha\beta b}{}^c)\right\}\psi_c(x) \\
&= \left\{1 + \frac{i}{2}\delta\omega^{\alpha\beta}(\mathcal{L}_{\alpha\beta} + S_{\alpha\beta})\right\}\gamma^\mu{}_a{}^b\psi_b(x) + \frac{i}{2}\delta\omega^{\alpha\beta}\frac{1}{i}\left(\eta^{\mu\alpha}\delta^\beta_\varphi - \eta^{\mu\beta}\delta^\alpha_\varphi\right)\gamma^\varphi\psi(x) \\
&= \left\{1 + \frac{i}{2}\delta\omega^{\alpha\beta}(\mathcal{L}_{\alpha\beta} + S_{\alpha\beta})\right\}\gamma^\mu{}_a{}^b\psi_b(x) + \frac{i}{2}\delta\omega_{\alpha\beta}S_V^{\alpha\beta\mu}{}_\nu\gamma^\nu\psi(x) \\
&= \exp\left(\frac{i}{2}\delta\omega_{\alpha\beta}S^{\alpha\beta}\right)_a{}^b\Lambda^\mu{}_\nu\gamma^\nu{}_b{}^c\psi_c(\Lambda^{-1}x),
\end{aligned}
\tag{1.44}
$$

where one should make clear about the indices of various matrices that the field carries. From now on for simplicity, we will omit the scalar transformation part in the spacetime coordinate.

1.5 Heavily used section

It is attempting to raise the spinor index

$$
\psi^a \equiv \mathcal{C}^{ab}\psi_b, \tag{1.45}
$$

so that

$$
U(\Lambda^{-1})\psi^a U(\Lambda) = \psi^b \exp\left(-\frac{i}{2}\delta\omega_{\alpha\beta}S^{\alpha\beta}\right)_b{}^a, \tag{1.46}
$$

which exactly means $\chi^a\psi_a$ is a scalar.

We can also define a lowering procedure

$$
\psi_a = \psi^b\mathcal{C}_{ba}, \tag{1.47}
$$

where we always use the "northwest-southeast & nearest" contraction convention. For one thing, the raising and lowering should be consistent, namely,

$$
\mathcal{C}^{ac}\mathcal{C}_{ab} = \delta^c_b, \tag{1.48}
$$

which gives an auxiliary (not quite necessary in general) convenience [3]

$$
\mathcal{C}^{ab}\mathcal{C}_{ac}\mathcal{C}_{bd} = \mathcal{C}_{cd} \tag{1.49}
$$

so that we can also feel free to raise or lower indices of $\mathcal{C}$ by itself. Then, the consistent condition in Eq. (1.48) can be re-written as

$$
\mathcal{C}_b{}^c = \delta^c_b. \tag{1.50}
$$

[3] It strongly depends on the contraction convention and not true in general conventions when $\mathcal{C}$ is asymmetric.

For notational convenience, we will denote ψ^a as $\bar{\psi}^a$ to remind ourselves that the index has been raised up, which is called "Majorana adjoint". We define a matrix $(C)_{ab} \equiv \mathcal{C}^{ba}$ thereby $(C^{-1})_{ab} = \mathcal{C}_{ab}$. The issue in Eq. (1.46) can be turned into the following form

$$C^\intercal \exp(iS_{\mu\nu}) = \exp(-iS_{\mu\nu}^\intercal)C^\intercal, \tag{1.51}$$

which can be sufficiently satisfied by

$$(CS_{\mu\nu})^\intercal = -C^\intercal S_{\mu\nu}, \tag{1.52}$$

or equivalently, [c.f. the definition Eq. (1.43)]

$$(C\Gamma_{\mu_1\mu_2})^\intercal = -C^\intercal \Gamma_{\mu_1\mu_2}. \tag{1.53}$$

It somehow motivates us to impose a not-much stronger condition with $t_p \in \{\pm 1\}$:

$$(C\Gamma_{(p)})^\intercal = -t_p C\Gamma_{(p)}, \tag{1.54}$$

to be called C-parity of $\Gamma_{(p)}$. It means

$$\begin{aligned} -t_p C\Gamma_{(p)} &= \Gamma_{\mu_p}^\intercal \cdots \Gamma_{\mu_1}^\intercal C^\intercal \\ &= \Gamma_{\mu_p}^\intercal \cdots \Gamma_{\mu_2}^\intercal C^\intercal (-t_0 t_1)\Gamma_{\mu_1} \\ \cdots &= -t_0(t_0 t_1)^p C\Gamma_{(p)^\intercal} \quad (\text{Here } (p)^\intercal \equiv \mu_p \cdots \mu_1.) \\ &= -t_o(t_0 t_1)^p (-1)^{p(p-1)/2} C\Gamma_{(p)}. \end{aligned} \tag{1.55}$$

Therefore,

$$t_p = t_0(t_0 t_1)^p (-1)^{p(p-1)/2}, \tag{1.56}$$

which implies the cyclicity $t_r = t_{r+4}$, and Eq. (1.54) is not so strong as it seems to be because the remaining symmetries are held automatically after the C-parities of $p = 0, 1$.

Then the condition Eq. (1.53) is translated as $t_0 t_2 = -1$ which is always correct! The explicit form of C depends on representations that is not important at this point and we will give the information of t_p in Fig. 2, and, additionally,

$$C^\dagger = C^{-1}. \tag{1.57}$$

Remark: In the above discussion and motivation of C, we didn't even need to specify the signature since we only focus on the symmetric properties of matrices without doing any complex congjugation. Thus, everything holds in **arbitrary signature** $(-,-,\cdots,+,+,\cdots)$.

1.6 Reality: extremely useful tools

[Starting from this part, we will partially fix the representation such that γ^0 is antihermitian and γ^i's are hermitian for $i = 1, \cdots, D-1$. This can be justified by the definition of Clifford algebra.]

Table 3.1 Symmetries of γ-matrices. The entries contain the numbers r mod 4 for which $t_r = \pm 1$. For even dimensions, in bold face are the choices that are most convenient for supersymmetry.

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0, 3	2, 1
	0, 1	**2, 3**
1	0, 1	2, 3
2	0, 1	2, 3
	1, 2	**0, 3**
3	1, 2	0, 3
4	**1, 2**	**0, 3**
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	**1, 2**
7	0, 3	1, 2

Figure 1. The bolded ones are the conventional choice in SUSY. This table is copied from "Supergravity" by D. Z. Freedman, & A. Van Proeyen.

Therefore, we can write down many scalars like $\bar\psi\lambda$, $\bar\psi\gamma^\mu\partial_\mu\lambda$. However, we need hermitian forms and, to do so, we seek for a convenient way to tell hermiticity. It is known that spinors are Grassmannian fields, i.e., $\psi_\alpha\chi_\beta = -\chi_\beta\psi_\alpha$ in the Lagrangian formalism.

A basic idea is to first construct a spinor $B_a{}^b(\psi_b)^\dagger$ constructed from $(\psi_a)^\dagger$. We know that its transformation rule as

$$U(\Lambda^{-1})B_a{}^b(\psi_b)^\dagger U(\Lambda) = B_a{}^b\left\{\exp\left(\frac{i}{2}\delta\omega_{\alpha\beta}S^{\alpha\beta}\right)_b{}^c\psi_c\right\}^\dagger, \tag{1.58}$$

which is wanted to behave like a spinor and it can be satisfied if

$$B\Gamma^*_{\mu\nu} = \Gamma_{\mu\nu}B. \tag{1.59}$$

Then we can find a *unitary* B matrix with some conventional phase factor

$$B \equiv it_0\gamma^0 C^{-1}, \tag{1.60}$$

since it satisfies

$$B^{-1}\gamma_\mu B = (-t_0t_1)\gamma^*_\mu; \tag{1.61}$$

$$B^*B = -t_1\mathbb{I}_{d\times d}. \tag{1.62}$$

Hint: $\gamma^\dagger_\mu = \gamma_0\gamma_\mu\gamma_0$ in our basis fixing.

Thus, we have the following new spinor, called charge conjugation, constructed from the complex conjugate of ψ_a:

$${}^c\psi_a \equiv B_a{}^b\psi^\dagger_b. \tag{1.63}$$

This gives us a powerful tool to write down the complex conjugate of any *bosonic* entity easily once we know their charge conjugation because we can prove:

$$(\bar\chi M\lambda)^\dagger = {}^c(\bar\chi M\lambda) \equiv (-t_0t_1)\,{}^c\bar\chi\,{}^cM\,{}^c\lambda, \tag{1.64}$$

where any matrix including differential operator or spinor matrix M with

$$^{c}M \equiv BM^{*}B^{-1}. \tag{1.65}$$

Specifically,

$$^{c}\gamma^{\mu} = (-t_0 t_1)\gamma^{\mu}. \tag{1.66}$$

Sometimes, we need a "double" hermicity check, namely checking hermicity in two distinct ways and matching them together to get something unaccessible by each single way alone.

Let us define another adjoint called Dirac adjoint:

$$\tilde{\psi}^{a} \equiv \sum_{b} \psi_{b}^{\dagger} i\gamma^{0}{}_{b}{}^{a}, \tag{1.67}$$

where one should be cautious since the Dirac adjoint does not have well-behaving index contraction. Since we rarely use Dirac adjoint later except as a hermiticity checker, we use "tilde" to denote it temporarily even though it is hard to distinguish from the "bar" in Majorana adjoint before. It is useful to prove that

$$^{c}\bar{\psi} = (-t_0 t_1)\tilde{\psi}. \tag{1.68}$$

Let us manifest how to use them to exclude non-hermitian terms in Lagrangian for Majorana spinors defined below.

1.7 Majorana spinors: Useful results

Majorana spinors satisfy the following Majorana condition:

$$^{c}\psi (= B\psi^{*}) = \psi, \tag{1.69}$$

which implies that $\psi = B\psi^{*} = BB^{*}\psi$ thereby $BB^{*} = -t_1 = +1$. Therefore, we can find a basis such that $B = 1$ and this step contains a unitary transformation of d-dimensional basis and a field redefinition by a $U(1)$ phase. Such a new basis is called Majorana basis where γ matrices are either all purely real ($t_0 = -1$) or all purely imaginary ($t_0 = +1$), c.f. Eq. (1.61). We will call them Majorana and pseudo-Majorana and by Eqs. (1.66,1.68):

$$t_1 = -1 : \begin{cases} t_0 = +1 : \ {}^{c}\gamma = \gamma; \quad {}^{c}\bar{\gamma} = \tilde{\gamma}; \quad (\gamma^{*} = \gamma \text{ basis-dependent}), \\ t_0 = -1 : {}^{c}\gamma = -\gamma; \ {}^{c}\bar{\gamma} = -\tilde{\gamma}; \ (\gamma^{*} = -\gamma \text{ basis-dependent}). \end{cases} \tag{1.70}$$

Then let us doubly check several Lagrangian terms.

1.7.1 $t_0 = +1$: Majorana spinors

We first check mass term: $m\bar{\psi}\chi + m\bar{\chi}\psi$

$$\begin{aligned} \text{Maj-checker: } & {}^{c}(m\bar{\psi}\chi) = (-t_0 t_1) m {}^{c}\bar{\psi} {}^{c}\chi = m\bar{\psi}\chi; \\ \text{Dirac-checker: } & (m\bar{\psi}\chi)^{\dagger} = (m\psi^{\dagger} i\gamma^{0}\chi)^{\dagger} = m\bar{\chi}\psi. \end{aligned} \tag{1.71}$$

This implies that the mass term is non-vanishing and permitted.

Then the kinetic term: $\bar{\psi}\gamma^\mu\partial_\mu\chi + \bar{\chi}\gamma^\mu\partial_\mu\psi$

$$\begin{aligned} \text{Maj-checker: } {}^c(\bar{\psi}\gamma^\mu\partial_\mu\chi) &= (-t_0t_1)^2\bar{\psi}\gamma^\mu\partial_\mu\chi = \bar{\psi}\gamma^\mu\partial_\mu\chi; \\ \text{Dirac-checker: } (\bar{\psi}\gamma^\mu\partial_\mu\chi)^\dagger &= -\chi^\dagger\gamma^0\gamma^\mu\gamma^0 i\gamma^0\partial_\mu\psi = \bar{\chi}\gamma^\mu\partial_\mu\psi. \end{aligned} \tag{1.72}$$

Thus the kinetic term is also permitted.

1.7.2 $t_0 = -1$: pseudo-Majorana spinors

We again first check the mass term $m\bar{\psi}\chi + m\bar{\chi}\psi$:

$$\begin{aligned} \text{Maj-checker: } {}^c(m\bar{\psi}\chi) &= (-t_0t_1)m\,{}^c\bar{\psi}{}^c\chi = -m\bar{\psi}\chi; \\ \text{Dirac-checker: } (m\bar{\psi}\chi)^\dagger &= (-m\psi^\dagger i\gamma^0\chi)^\dagger = m\bar{\chi}\psi. \end{aligned} \tag{1.73}$$

Therefore, the mass term $m\bar{\psi}\chi+m\bar{\chi}\psi$ must vanish, which indicates a gravitational anomaly!

Next, the kinetic term: $\bar{\psi}\gamma^\mu\partial_\mu\chi + \bar{\chi}\gamma^\mu\partial_\mu\psi$

$$\begin{aligned} \text{Maj-checker: } {}^c(\bar{\psi}\gamma^\mu\partial_\mu\chi) &= (-t_0t_1)^2\bar{\psi}\gamma^\mu\partial_\mu\chi = \bar{\psi}\gamma^\mu\partial_\mu\chi; \\ \text{Dirac-checker: } (\bar{\psi}\gamma^\mu\partial_\mu\chi)^\dagger &= \chi^\dagger\gamma^0\gamma^\mu\gamma^0 i\gamma^0\partial_\mu\psi = \bar{\chi}\gamma^\mu\partial_\mu\psi. \end{aligned} \tag{1.74}$$

Thus the kinetic term is still permitted.

1.8 Unrelated points

1.8.1 Gauged internal symmetry

Gauge symmetry is actually required by CPT theorem and Lorentz invariance of massless particles so that on-shell degrees of freedom can be matched (Think about the helicity!).

1.8.2 Helicity

Helicity should be learnt in more details, e.g., why helicity takes integer/half-integer values? Does the topological argument given by Weinberg Vol. 1 really make sense? In my understanding, the massless particle cannot "see" the full Lorentz group, so the topological argument is subtle.

1.8.3 C matrices

Existence of C matrix in various dimensions should be learnt in more detail.

1.8.4 Additional Notes

So far, we haven't touched the chiral transformation and Weyl fermion. They will be included when I have more time, but Dirac's dotted/undotted notation is important and worthy learning!

1.9 Formula summarized and including chiral matrices properties

We will summarize all the important results below.

Table 3.1 Symmetries of γ-matrices. The entries contain the numbers r mod 4 for which $t_r = \pm 1$. For even dimensions, in bold face are the choices that are most convenient for supersymmetry.

D (mod 8)	$t_r = -1$	$t_r = +1$
0	0, 3	2, 1
	0, 1	**2, 3**
1	0, 1	2, 3
2	0, 1	2, 3
	1, 2	**0, 3**
3	1, 2	0, 3
4	**1, 2**	**0, 3**
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	**1, 2**
7	0, 3	1, 2

Figure 2. The bolded ones are the conventional choice in SUSY. This table is copied from "Supergravity" by D. Z. Freedman, & A. Van Proeyen.

$$\gamma^{\mu_1\cdots\mu_p}\gamma_{\nu_1\cdots\nu_m} = \gamma^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_m} + \cdots \frac{p!}{(p-r)!}\frac{m!}{(m-r)!}\gamma^{[\mu_1\cdots\mu_{p-r}}{}_{[\nu_{r+1}\cdots\nu_m}\delta^{\mu_{p-r+1}\cdots\mu_p]}_{\nu_r\cdots\nu_1]} + \cdots .$$

$$(C\Gamma_{(p)})^\intercal \equiv -t_p C\Gamma_{(p)}; \quad t_p = t_0(t_0t_1)^p(-1)^{p(p-1)/2}. \tag{1.75}$$

$$B \equiv it_0\gamma^0 C^{-1}; \quad B^* B = -t_1 \mathbb{I}_{d\times d}; \tag{1.76}$$

$$B^{-1}\gamma_\mu B = (-t_0t_1)\gamma_\mu^*; \quad {}^c\psi = B(\psi^\dagger)^\intercal; \tag{1.77}$$

$$\bar\lambda\Gamma^{(r_1)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1}t_{r_1}\cdots t_{r_p}\bar\chi\Gamma^{(r_p)}\cdots\Gamma^{(r_1)}\lambda. \tag{1.78}$$

$$(\bar\chi M\lambda)^\dagger = {}^c(\bar\chi M\lambda) \equiv (-t_0t_1)\overline{{}^c\chi}\,{}^cM\,{}^c\lambda, \tag{1.79}$$

$$ {}^cM \equiv BM^*B^{-1}; \quad {}^c\gamma^\mu = (-t_0t_1)\gamma^\mu. \tag{1.80}$$

$$\overline{{}^c\psi} = (-t_0t_1)\tilde{\bar\psi}. \tag{1.81}$$

$$\text{Maj condition: } {}^c\psi = \psi \;\Rightarrow t_1 = -1. \tag{1.82}$$

$$t_1 = -1 : \begin{cases} t_0 = +1 \text{ (gappable)}: \; {}^c\gamma = \gamma; \quad \overline{{}^c\gamma} = \tilde\gamma; \quad (\gamma^* = \gamma \text{ basis-dependent}), \\ t_0 = -1 \text{ (massless)}: \; {}^c\gamma = -\gamma; \; \overline{{}^c\gamma} = -\tilde\gamma; \; (\gamma^* = -\gamma \text{ basis-dependent}). \end{cases}$$

2 Gravitational WEEKLY: 2021.3.14.-20.

In this week, let us try to understand the general gauge theory in more detail. As a nontrivial example, we will consider gauged algebra which includes translation symmetry. The subtle and "hidden" points in this viewpoint will be manifested.

Einstein's general covariance principle and Yang-Mill's local invariance are the main motivation to gauge every global symmetry in the theory. We will first consider how to gauge a general algebra.

2.1 General gauge theory

First, we have a global symmetry acting on the matter fields as:

$$\begin{aligned} \delta\varphi &\equiv U(\epsilon)^{-1}\varphi U(\epsilon) - \varphi = \epsilon^A T_A(\varphi); \quad (2.1) \\ U(\epsilon) &\equiv 1 - \epsilon^A \hat{T}_A; \quad (2.2) \\ \left[\hat{T}_A, \hat{T}_B\right] &\equiv f_{AB}{}^C \hat{T}_C. \quad (2.3) \end{aligned}$$

The definition above deserves one additional sentence to be explained: $\hat{T}_A$ is a generator (anti-hermitian for unitary symmetry) while $T_A(\varphi)$ or $T_A\varphi$ for further simplicity later is

$$T_A(\varphi) \equiv \mathrm{adj}_{\hat{T}_A}(\varphi) = [\hat{T}_A, \varphi]. \quad (2.4)$$

It should be noted that, if $\hat{T}_A$ admits a differential/matrix generator $\mathcal{T}_A$, then $\mathcal{T}_A\varphi = -T_A\varphi$ by observing the earlier result like Eq. (1.35) or by definition:

$$U(\epsilon)^{-1}\varphi U(\epsilon) = (1 - \epsilon^A \mathcal{T}_A)\varphi. \quad (2.5)$$

Additionally, all the commutators [...] is defined as

$$[a, b] = \begin{cases} ab + ba, \text{ if both are fermionic;} \\ ab - ba, \text{ otherwise.} \end{cases} \quad (2.6)$$

Then let us see the commutator of two actions:

$$\begin{aligned} [\epsilon_1^A T_A, \epsilon_2^B T_B]\varphi &\equiv \epsilon_1^A T_A(\epsilon_2^B T_B(\varphi)) - \epsilon_2^B T_B(\epsilon_1^A T_A(\varphi)) \\ &= [\epsilon_1^A \hat{T}_A, [\epsilon_2^B \hat{T}_B, \varphi]] - (A\&1 \leftrightarrow B\&2) \\ &= [[\epsilon_1^A \hat{T}_A, \epsilon_2^B \hat{T}_B], \varphi] \\ &= [\epsilon_2^B \epsilon_1^A [\hat{T}_A, \hat{T}_B], \varphi] \\ &= \epsilon_2^B \epsilon_1^A f_{AB}{}^C T_C(\varphi), \quad (2.7) \end{aligned}$$

which will make it clear why we have a minus sign in the convention of $U(\epsilon)$ above and the ordering of $\epsilon^B\epsilon^A$ is essential in the case of fermionic cases.

2.1.1 Covariant derivative and covariant general coordinate transformation

Next, we promote the parameters ϵ to be space-time dependent $\epsilon(x)$. The first thing to do is to define a well-behaved or covariant derivative. For simplicity, we restrict to space-time scalars φ. We can see how "ill-behaved" the original derivative is:

$$\begin{aligned}\delta\partial_\mu\varphi &= \partial_\mu(\epsilon^A T_A\varphi)\\ &= \partial_\mu(\epsilon^A)T_A\varphi + \epsilon^A\partial_\mu(T_A\varphi).\end{aligned} \tag{2.8}$$

To cancel the first term, we introduce a series of gauge field h_μ^A in

$$D_\mu^C\varphi \equiv (\partial_\mu - h_\mu^A T_A)\varphi, \tag{2.9}$$

so that

$$\delta D_\mu^C\varphi = \epsilon^A D_\mu^C(T_A\varphi). \tag{2.10}$$

It implies that

$$\delta h_\mu^A = \partial_\mu\epsilon^A + \epsilon^C h_\mu^B f_{BC}{}^A, \tag{2.11}$$

where the second term means that the gauge field transform in the adjoint representation of the algebra up to the first derivative term. Indeed,

$$\begin{aligned}\delta D_\mu^C\varphi &= \epsilon^A\partial_\mu(T_A\varphi) - \epsilon^C h_\mu^B f_{BC}{}^A T_A\varphi - h_\mu^A\epsilon^B T_B T_A\varphi\\ &= \epsilon^A\partial_\mu(T_A\varphi) - [h_\mu^A T_A, \epsilon^B T_B]\varphi - h_\mu^A\epsilon^B T_B T_A\varphi\\ &= \epsilon^A\partial_\mu(T_A\varphi) - \epsilon^A h_\mu^B T_B T_A\varphi\\ &= \epsilon^A D_\mu^C(T_A\varphi).\end{aligned} \tag{2.12}$$

In the above derivation, we treat h_μ^A as a gauge parameter, e.g., it commuting with T_A's. In this sense, we can compactly rewrite the covariant derivative as:

$$D_\mu^C = D_\mu - \delta(h_\mu), \tag{2.13}$$

where D_μ is the space-time covariant derivative for general space-time tensors. However, it is tricky when we encounter gauged Poincaré case in the next section, where, except for the gauge field, other fields are space-time scalars, or to be made as so by vierbein.

In order that the covariant derivatives can transform properly under general coordinate transformation, we need the gauge field to transform as

$$\delta_\xi^{\text{gct}} h_\mu^A = \mathcal{L}_\xi h_\mu^A \equiv \xi^\rho\partial_\rho h_\mu^A + (\partial_\mu\xi^\rho)h_\rho^A, \tag{2.14}$$

where $\mathcal{L}_V$ denotes the Lie derivative along the vector field V^μ. This can be seen as the local version (in the absence of other local symmetries) of the global translation with the gauge-parameter reduction as $\xi^\mu(x) = -a^\mu$ in Eq. (1.21). This additional minus sign is simply a convention and it could be attributed by a switch of passive and active pictures. Similarly, gct acts on the matter field by Lie derivative $\delta_\xi^{\text{gct}}\varphi = \mathcal{L}_\xi\varphi$ for general spacetime tensor field φ not restricted to spacetime scalars.

2.1.2 Curvatures

With the algebraic index "A" in gauge fields h_μ^A, we also need a well-controlled term that can transform in the adjoint representation even in the presence of the local (space-time dependent) transformation $\epsilon(x)$:

$$\delta F_{\mu\nu}^A = \epsilon^C F_{\mu\nu}^B f_{BC}{}^A. \tag{2.15}$$

Since h_μ^A also carries one algebraic index, it is natural to guess that $F \propto h$. To cancel the derivatives, it is even more natural (try it!) to guess that

$$F_{\mu\nu}^A \equiv \partial_\mu h_\nu^A - \partial_\nu h_\mu^A - h_\mu^C h_\nu^B f_{BC}{}^A, \tag{2.16}$$

which can be also re-written into

$$F_{\mu\nu}^A = D_\mu^C(g) h_\nu^A, \tag{2.17}$$

where $D_\mu(g)$ here is torsion-free uniquely determined by metric $g_{\mu\nu}$. The proof that $F_{\mu\nu}^A$ needs the graded Jacobi identity:

$$\epsilon_2^B \epsilon_1^A \epsilon_3^C f_{AB}{}^D f_{CD}{}^E + (\text{cyclic } 1 \to 2 \to 3) = 0, \tag{2.18}$$

where whether ϵ_i^A is fermionic or not matches the corresponding properties of h^A.

2.2 Gauging (super-)Poincaré algebra

In this part, we will try to figure out the gauging procedure for Poincaré algebra which is easily extended to super-Poincaré algebra, i.e., including supersymmetry algebra.

The global symmetries (namely in the flat Minkowski spacetime) include

$$\delta_{\xi,\lambda}\varphi = \left(\xi^a \partial_a - \frac{1}{2}\lambda_{ab} S_{\text{Lor}}^{ab} + \cdots\right)\varphi, \tag{2.19}$$

where $\lambda_{ab} = -\omega_{ab}$ in Eq. (1.11) before and S_{Lor} is the (numerical) matrix operator for the Lorentz transformation, e.g., $S_{\text{Lor}} = iS_V + iS$ for a spinor carrying a vector index ψ_a in the Rarita-Schwinger model, which serves as the gauge field for local supersymmetry. Moreover, as mentioned before, we will concentrate on space-time scalar, namely φ does not carry the space-time index "μ".

It should be noted that, although "a" indices above coincide with "μ" in the case of flat spacetime, they denotes different transformation properties. "a"s means that the quantity transforms under local Lorentz transformation while inert under general coordinate transformation. On the other hand, "μ"s are active only under general coordinate transformation. Let us motivate this subtle or seemingly artificial rule by the following quantity called vierbein:

$$e_\mu^a e_\nu^b \eta_{ab} \equiv g_{\mu\nu}, \tag{2.20}$$

where the matrix-valued field is invertible and we denotes its inverse as e_a^μ. Clearly, $g_{\mu\nu}$ is invariant if we transform e_μ^a by a local Lorentz transformation $\delta_{\text{local-Lor}} e_\mu^a = \lambda^a{}_b(x) e_\mu^b$.

Additionally, the general coordinate transformation of $g_{\mu\nu}$ can be realized by that of e^a_μ once we treat the "μ" and "a" indices of e^a_μ as above. As a gift, we can attach e^a_μ to arbitrary space-time vector V^μ to get a space-time scalar but a Lorentz vector $V^a \equiv e^a_\mu V^\mu$.

Furthermore, since the local Lorentz transformation rule of e^a_μ matches that of the gauge transformation of the gauge field for translation symmetry, we can do such an identification. In the same spirit, we can also identify the spin connection ω^{ab}_μ as the gauge field for the local Lorentz symmetry.

Clearly, there is a loop hole in the above discussion because ∂_a is only defined for locally flat spacetime. In other words, we haven't defined the local transformation for translation on curved spacetime even for the matter field! One natural choice could be the abuse of notation by defining $\partial_a \equiv e^\mu_a \partial_\mu$ and $\xi^a \equiv \xi^\mu e^a_\mu$. Although these definitions are innocent, the corresponding transformation $\xi^a \partial_a \varphi$ may not be an appropriate one since it might not be gauge covariant for onsite gauge symmetries. This requirement is reasonable because $\delta\varphi$ can be seen as the difference between two matter fields both of which are covariant. Therefore, we define

$$\delta_\xi \varphi \equiv \xi^\mu D^C_\mu \varphi = \xi^a D^C_a \varphi, \tag{2.21}$$

where $D^C_a \equiv e^\mu_a D^C_\mu$ and D^C_μ only contains the gauge symmetries other than translations [4], which we will call "standard" gauge symmetries with label "S":

$$D^C_\mu \equiv D_\mu - \delta(h^S_\mu). \tag{2.22}$$

Equivalently, at least for scalars, the local translation is the following covariant general coordinate transformation (cgct)

$$\delta^{\text{cgct}}_\xi \equiv \delta^{\text{gct}}_\xi - \delta(h^S_\mu). \tag{2.23}$$

At this point, one (including me) might be wondering why we only concentrate on space-time scalars? Actually, a question in advance could be what kinds of extension of δ_ξ on vectors should be? It is natural to guess that

$$\delta^{(1)}_\xi V_\mu = \xi^\rho D^C_\rho V_\mu, \tag{2.24}$$

but it is obviously incorrect! It can be seen by taking $V_\mu \equiv \partial_\mu \varphi$:

$$\delta^{(1)}_\xi V_\mu = \partial_\mu \delta^{(1)}_\xi \varphi \neq \xi^\rho D^C_\rho (\partial_\mu \varphi), \tag{2.25}$$

where we have used the fact that the symmetry transformation commutes with the differential operator consisting of only pure numbers and derivatives. The remaining choice is just cgct defined above. Then it is a good point to return our main question, of which the answer is rather clear now: δ^{gct} in δ^{cgct} will introduce the ξ-derivative terms so that it is impossible to have $\delta^{\text{cgct}} V_\mu = \xi^a (T_a V_\mu)$ without derivatives of gauge parameter in $(T_a V_\mu)$ [5].

[4] If the translation gauge field were also included, the derivative term would have been cancelled.

[5] Of course, gauge fields, as spacetime vectors, are not restricted by this regulation, since they are not matter fields. Their local translation can be still given by δ^{cgct}_ξ.

Therefore, from the former half of the above discussion, using D^C_μ in Eq. (2.21) is misleading, which would better be rewritten as $\delta^{\rm cgct}$, although the space-time scalars couldn't realize the difference. Anyway, the discussion here just means that we should restrict ourselves to space-time scalars as our matter-field content.

In a short summary, we have

$$\delta_{\xi,\lambda}\varphi = \left(\delta^{\rm cgct}_\xi + \frac{1}{2}\lambda_{ab}S^{ab}_{\rm Lor} + \cdots\right)\varphi, \tag{2.26}$$

on a curved manifold. The prize to be paid is that the commutator of two local translations probably is non-vanishing due to the gauge transformation in D^C_μ. We need another assumption on this modified algebra that the commutator of two local supersymmetric transformation still leads to a local translation, or more generally, any two standard symmetry transformation whose commutator includes translations [6]. *Under such an assumption*, you can prove, although not obviously, that $D_a\varphi$ transforms covariantly under local standard transformations:

$$\delta_\epsilon D_a\varphi = \epsilon^S D_a(T_S\varphi) - \epsilon^S f_{aS}{}^b D_b\varphi. \tag{2.27}$$

A final remark could be that, for the gauge field, we should use the $\delta^{\rm cgct}$ above as the re-definition of the local translation because the gauge field has been coupled with matter field by D^C_a so they should be transformed in the same manner under a single transformation. Of course, in the absence of any matter field, this re-definition is completely unnecessary since all the problem we met above is due to the matter field (c.f. footnote 4.) Eventually, in either situation, the required symmetries can be concluded by 1) general coordinate transformation and 2) standard gauge symmetries, since cgct can be split into gct and standard ones. However, to be more careful, we could find that the pure-gauge field action has more symmetry since it is also invariant under the original local translation gauge symmetry. I'd admit that I haven't understood this aspect thoroughly, but it seems that the closure of the gauge symmetry algebra on the pure-gauge side, which has never been modified of course, already ensures the existence of the original local translation gauge symmetry no matter whether we impose or not. Therefore, there is actually no logic gap, at least for the super-Poincaré case.

2.3 Additional remarks

1) I haven't understood quite thoroughly why we need a curvature vanishing condition $F^a_{\mu\nu} = 0$ for the translation. At least, it is not invariant under super-Poincaré and it needs a manufactural modification on the spin connection to *keep it variant*, which seems very artificial. Perhaps, instead, we should write down the whole action first and find an equation of motion of the algebraic constraint relating spin connection and other gauge fields.

2) I wouldn't go further into the conformal gravity, but it can be checked quite fluently by Freedman's supergravity.

[6] Although I cannot prove it by the first principle, we can take it for granted as an axiom, which is justified by covariance and understood by parallel transportation.

3) I also find a small trick to easily derive or to recall the Lie derivative of a vector. First, we could construct a vector $V_\mu \equiv \partial_\mu \varphi$, and then evaluate its gct:

$$\begin{aligned}
\delta_\xi^{\mathrm{gct}} V_\mu &= \partial_\mu \delta_\xi^{\mathrm{gct}} \varphi \\
&= \partial_\mu (\xi^\rho \partial_\rho \varphi) \\
&= \xi^\rho \partial_\rho V_\mu + (\partial_\mu \xi^\rho) V_\rho,
\end{aligned} \tag{2.28}$$

where in the last line we have made of use that $\partial_\mu \partial_\rho \varphi = \partial_\rho \partial_\mu \varphi$.

3 Dotted-Undotted WEEKLY: 2021.3.21.-27.

In this week, I would review the convention on Weyl fermion in ($D = 2m$)-dimensional Minkowski spacetime ($\eta = \text{diag}[-1, +1, \cdots, +1]$), i.e., dotted-undotted Van der Waerden notation. As follows, we will always assume that γ^0 is anti-hermitian while all the other space-like components γ^i's are hermitian.

3.1 Weyl basis

The Weyl basis is characterized by the following chirality matrix:

$$\gamma_* \equiv (-i)^{m+1}\gamma^{0\cdots D-1}, \tag{3.1}$$

which satisfies

$$\gamma_*^\dagger = \gamma_*;\ \gamma_*^2 = 1. \tag{3.2}$$

Interestingly, the proofs of the above two relations follow exactly the same mathematical stream. Moreover, by Clifford algebra, it is easy to see that

$$\{\gamma_*, \gamma^\mu\} = 0;\ \text{Tr}\gamma_* = 0. \tag{3.3}$$

As a byproduct, we have

$$\text{Tr}\Gamma^{\{r_p\}} = 0, \tag{3.4}$$

which is clear if $p \in 2\mathbb{Z}$ and we can use the fact that $\{\Gamma^{\{l_{m_{\text{odd}}}\}}, \gamma_*\} = 0$ to prove it for $p \in 2\mathbb{Z}+1$ by letting $\{l_m\} = \{\bar{r_p}\}$.

Anyway, there exists a basis after a unitary transformation, by which the (anti-)hermiticity of γ^μ is invariant, so that

$$\gamma_* = \begin{pmatrix} \mathbb{I}_{m\times m} & \\ & -\mathbb{I}_{m\times m} \end{pmatrix}, \tag{3.5}$$

and this basis is called Weyl basis.

Additionally, the anticommutator $\{\gamma_*, \gamma^\mu\}$ and the (anti-)hermiticities of γ^μ's tell us that the general form of γ^μ has the following matrix relation

$$(\gamma^\mu)_\alpha{}^\beta = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}_{\alpha,\beta}, \tag{3.6}$$

where the submatrices

$$(\sigma^0)^\dagger = -\bar{\sigma}^0 \text{ and } (\sigma^j)^\dagger = \bar{\sigma}^j, \tag{3.7}$$

thereby *numerically* $(\sigma^\mu)^\dagger = \bar{\sigma}_\mu$ and satisfy

$$\sigma^\mu\bar{\sigma}^\nu + (\mu \leftrightarrow \nu) = \bar{\sigma}^\mu\sigma^\nu + (\mu \leftrightarrow \nu) = 2\eta^{\mu\nu}\mathbb{I}_{m\times m}. \tag{3.8}$$

Therefore, the spinor-representation matrix generator defined before is:

$$
\begin{aligned}
(S^{\mu\nu})_\alpha{}^\beta &= \frac{1}{4i}[\gamma^\mu, \gamma^\nu] \\
&= \frac{1}{4i}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - (\mu \leftrightarrow \nu) & \\ & \bar{\sigma}^\mu\sigma^\nu - (\mu \leftrightarrow \nu) \end{pmatrix}_{\alpha,\beta} \\
&\equiv \begin{pmatrix} S_L & \\ & S_R \end{pmatrix}_{\alpha,\beta}
\end{aligned}
\tag{3.9}
$$

which exactly means the irreducible spinor representation of Clifford algebra is actually reducible under Lorentz algebra so$(D-1,1)$. It implies that the two chirality subspaces do not mix together, so we can further polish the spinor indices as:

$$
\Psi_\alpha \equiv \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix}_\alpha , \tag{3.10}
$$

where the dots and undots indicate the chirality, namely how they transform under Lorentz transformations. Here we also add a bar over dotted terms to indicate the chirality when we suppress the indices in the future, but if already clear by the context, e.g., dotted indices already explicitly present, we would omit the bars.

Therefore, the γ matrices should carry the index as follows:

$$
(\gamma^\mu)_\alpha{}^\beta = \begin{pmatrix} & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & \end{pmatrix}, \tag{3.11}
$$

and clearly the undotted indices are contracted northwest-southeast while the dotted are in the other way, because of the previous convention of general spinor contraction. Partially due to that we haven't introduce the explicit forms of C matrices, one might be wondering why we should artificially raise up the dotted indices at the beginning, which bestows the distinction of the contraction. It turns out to be a convenience to be back to later.

3.2 A digression: a convenient basis choice

There is a basis choice that is consistent with all the requirements so far:

$$
\gamma^0 \equiv i\sigma_2 \otimes \mathbb{I}_{m\times m} = \begin{pmatrix} & \mathbb{I}_{m\times m} \\ -\mathbb{I}_{m\times m} & \end{pmatrix}, \tag{3.12}
$$

together with $\gamma^j \equiv \sigma_1 \otimes \gamma_{\mathrm{E}}^j$ for $m \geq 2$:

$$\gamma^1 = \sigma_1 \otimes \sigma_1 \otimes \mathbb{I} \otimes \cdots \tag{3.13}$$

$$\gamma^2 = \sigma_1 \otimes \sigma_2 \otimes \mathbb{I} \otimes \cdots \tag{3.14}$$

$$\gamma^3 = \sigma_1 \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbb{I} \otimes \cdots \tag{3.15}$$

$$\gamma^4 = \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbb{I} \otimes \cdots \tag{3.16}$$

$$\cdots$$

$$\gamma^{D-1} = \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \tag{3.17}$$

where γ_{E}^j is the Euclidean γ matrix obtained by Jordan-Wigner transformations. Then the corresponding C matrices can be summarized as follows up to a phase (we make them real):

3.2.1 $m \in 2\mathbb{Z}$, i.e., $D = 0 \mod 4$:

$$C = \mathbb{I} \otimes i\sigma_2 \otimes (\sigma_1 \otimes i\sigma_2)^{\otimes(m-2)/2} \text{ with } t_0 = -(-1)^{m/2} \text{ and } t_0 t_1 = -1; \tag{3.18}$$
$$C = \sigma_3 \otimes i\sigma_2 \otimes (\sigma_1 \otimes i\sigma_2)^{\otimes(m-2)/2} \text{ with } t_0 = -(-1)^{m/2} \text{ and } t_0 t_1 = +1. \tag{3.19}$$

3.2.2 $m \in 2\mathbb{Z}+1$, i.e., $D = 2 \mod 4$:

$$C = i\sigma_2 \otimes (\sigma_1 \otimes i\sigma_2)^{\otimes(m-1)/2} \text{ with } t_0 = (-1)^{(m-1)/2} \text{ and } t_0 t_1 = -1; \tag{3.20}$$
$$C = \sigma_1 \otimes (\sigma_1 \otimes i\sigma_2)^{\otimes(m-1)/2} \text{ with } t_0 = -(-1)^{(m-1)/2} \text{ and } t_0 t_1 = +1. \tag{3.21}$$

Hint: Prove them by force and the local movement is sufficient. Find our result in FIG. 2!

Two cases of the same dimension is related by a chiral transformation by γ_* (up to a phase) in that γ_* has a well-defined C-parity.

3.3 Restrictions to $D = 0 \mod 4$ and real C's

These results imply that when $D = 4\mathbb{Z}$, the C-matrix is diagonalized in blocks as:

$$C^{\alpha\beta} = \begin{pmatrix} c^{ab} & \\ & \bar{c}_{\dot{a}\dot{b}} \end{pmatrix}, \tag{3.22}$$

so that $\psi^a \chi_a$ and $\bar{\psi}_{\dot{a}} \bar{\chi}^{\dot{a}}$ are Lorentz scalars, where c and $\bar{c}$ are used to lower or raise the indices according to dotted-undotted contraction rule before and $c^{ab} = -t_0 t_1 \bar{c}_{\dot{a}\dot{b}}$ by Eq. (3.19). The consistency condition of original C matrices becomes

$$c^{ab} c_{ac} = \delta^b_c; \ \bar{c}^{\dot{a}\dot{b}} \bar{c}_{\dot{a}\dot{c}} = \delta^{\dot{b}}_{\dot{c}}. \tag{3.23}$$

By the explicit real C's above, we have $c_{ab} = c^{ab}$ for both Eqs. (3.18 and 3.19) [7]. Then we also have $c_{ab} = (-t_0 t_1) \bar{c}_{\dot{a}\dot{b}}$ thereby

$$\bar{c}_{\dot{a}\dot{b}} (\bar{\sigma}^\mu)^{\dot{b}c} c_{cd} = -t_0 t_1 (\sigma^\mu)_{d\dot{a}}. \tag{3.24}$$

However, if we only focus on two chirality separately, we are free to relax the phase of lowering or raising procedures. Since only the relative sign matters, we fix the undotted ones and introduce a freedom ($\rho = \pm 1$) for the dotted ones:

$$\chi^a \equiv \chi^a c_{ba}; \ \bar{\chi}_{\dot{a}} = \rho \bar{c}_{\dot{a}\dot{b}} \bar{\chi}^{\dot{b}} = -\rho t_0 t_1 c_{ab} \bar{\chi}^{\dot{b}}. \tag{3.25}$$

It is because what is required is that $\chi\psi \equiv \chi^a \psi_a$ and $\bar{\chi}\bar{\psi} \equiv \bar{\chi}_{\dot{a}} \bar{\psi}^{\dot{a}}$ are Lorentz scalars. To avoid to be buried over by notations, let us redefine a "new" $\bar{c}_{\dot{a}\dot{b}} \equiv \rho \bar{c}_{\dot{a}\dot{b}}$ thereby now

$$\bar{\chi}_{\dot{a}} = \bar{c}_{\dot{a}\dot{b}} \bar{\chi}^{\dot{b}} = -\rho t_0 t_1 c_{ab} \bar{\chi}^{\dot{b}}; \tag{3.26}$$
$$\bar{c}_{\dot{a}\dot{b}} (\bar{\sigma}^\mu)^{\dot{b}c} c_{cd} = -\rho t_0 t_1 (\sigma^\mu)_{d\dot{a}}. \tag{3.27}$$

[7] Nevertheless, it should be still noted that there is no canonical relation between c_{ab} and c^{ab} because we can multiply a general phase to C.

one can prove that under infinitesimal Lorentz transformation, *whatever* values that $t_{0,1}$ take (technically, we should derive this equation right after Eq. (3.24)):

$$
\begin{aligned}
(\psi^a)^* = c^{ab}\psi_b^* \mapsto c^{ab}\psi_b^* \; &+ c^{ab}(i^* S_L^* \psi^*)_b \\
&= c^{ab}(iS_L)_b^{*c}\psi_c^* \\
&= \frac{1}{4}c^{ab}(\sigma^{\mu *}\sigma_\nu^\intercal - \mu \leftrightarrow \nu)_b{}^c c_{dc}c^{de}\psi_e^* \\
&= i(S_R)^{\dot a}{}_{\dot b}(\psi^b)^*(-t_1 \cdot -t_0 \cdot -t_0 t_1)^2 \\
&= i(S_R)^{\dot a}{}_{\dot b}(\psi^b)^*.
\end{aligned}
\tag{3.28}
$$

Therefore, $(\psi^a)^*$ is transforming exactly as a (-1)-chiral spinor and it is tempting to introduce the notation

$$
(\bar{\psi}^\dagger)^{\dot a} \equiv (\psi^a)^*. \tag{3.29}
$$

However, there remains a notational consistency check:

$$
\psi^\dagger{}_{\dot b}\bar{c}^{\dot b \dot a} = \psi^{\dagger \dot a} = (c^{ab}\psi_b)^\dagger = c^{ab}\psi_b{}^\dagger, \tag{3.30}
$$

where we have used the prescription of real c^{ab}. In order to take a convenient, but strictly unnecessary, identification

$$
\psi^\dagger{}_{\dot b} = \psi_b{}^\dagger \tag{3.31}
$$

as well, the new (just as a reminder) $\bar{c}^{\dot a \dot b} = -t_0 c^{ab}$, i.e., $\rho = t_1$ being preferred and Eqs. (3.26, 3.27) being

$$
\begin{cases}
\bar\chi_{\dot a} = \bar{c}_{\dot a \dot b}\bar\chi^{\dot b} = -t_0 c_{ab}\bar\chi^{\dot b}; \\
\bar{c}_{\dot a \dot b}(\bar\sigma^\mu)^{\dot b c}c_{cd} = -t_0(\sigma^\mu)_{d\dot a}.
\end{cases}
\tag{3.32}
$$

In other words, when $t_1 = -1$, we need to modify, by $\rho \neq 1$, the raising/lowering matrix which is *different* from the general-spinor raising/lowering rule by the large C matrix just for the sake of the convenient notational identification $\psi^\dagger{}_{\dot b} = (\psi_b)^*$ to be consistent with the definition Eq. (3.29). As a consequence, the reconstruction of Majorana condition (existing only for $t_1 = -1$) needs a little bit more care, which is out of our scope although quite straightforward to be done. Another useful identity could be $\chi\psi = -t_0\chi_a\psi^a$ and $\bar\chi\bar\psi = -t_0\chi^{\dot a}\psi_{\dot a}$ once the the contractions are not done properly.

Unfortunately, when $D = 2 \mod 4$, we don't have dotted/undotted convenience to construct Lorentz scalars, which is directly seen in the explicit forms of C's, due to the absence of its block diagonalization.

3.4 Hermiticity check *etc.*

Let us exercise our notation developed so far by checking hermiticity of several Lagrangian terms. The first one could be the kinetic term:

$$
\begin{aligned}
\left(i\bar{\psi}^\dagger\bar{\sigma}^\mu\partial_\mu\psi\right)^\dagger &= \left[i\psi^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\psi_b\right]^\dagger \\
&= -i\partial_\mu\psi^\dagger_{\dot{b}}(\bar{\sigma}^\mu)^{a\dot{b}*}\psi_a \\
&= i\psi^\dagger_{\dot{b}}(\bar{\sigma}^\mu)^{\dot{b}a}\partial_\mu\psi_a
\end{aligned} \tag{3.33}
$$

where we have used the basis where σ^μ are hermitian and done an integration by part.

The second term could be a mass term $m\chi\xi + m\bar{\xi}^\dagger\bar{\chi}^\dagger$, but it is already automatically hermitian due to our convention of the contractions:

$$
\bar{\xi}^\dagger\bar{\chi}^\dagger = \xi^\dagger{}_{\dot{a}}\chi^{\dagger\dot{a}}, \tag{3.34}
$$

since the hermitian conjugate of a positive-chirality spinor transforms as a dotted spinor. It should be noted that, if we hadn't applied the convenient identification Eq. (3.31), we would have a relative minus sign between $\chi\xi$ and $\bar{\xi}^\dagger\bar{\chi}^\dagger$ in the mass term, which would turn out to be quite annoying. This reflects the convenience of our current notation.

The third term to be considered is not directly relevant to hermiticity and it is a generalized kinetic term which would be helpful to reconstruct Dirac Lagrangian:

$$
\begin{aligned}
i\bar{\psi}^\dagger\bar{\sigma}^\mu\partial_\mu\chi &= i\psi^\dagger_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\chi_b \\
&= i\chi_b(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu\psi^\dagger_{\dot{a}} \\
&= (-t_0)i\chi\sigma^\mu\partial_\mu\bar{\psi}^\dagger.
\end{aligned} \tag{3.35}
$$

3.5 Relation to standard textbooks

In most textbooks, a different raising or lowering rule is applied and we will sketch how to move from our convention to theirs. First, we define that

$$
\begin{aligned}
&\mathcal{E}_{ab} \equiv \bar{c}_{ab}; \\
&\mathcal{E}^{ab} \equiv -t_0\bar{c}^{ab},
\end{aligned} \tag{3.36}
$$

which imply the following rules

$$
\mathcal{E}^{ab}\mathcal{E}_{bc} = \delta^a_c; \tag{3.37}
$$

$$
\bar{\psi}^{\dot{a}} = \mathcal{E}^{\dot{a}\dot{b}}\bar{\psi}_{\dot{b}};\ \ \bar{\psi}_{\dot{a}} = \mathcal{E}_{\dot{a}\dot{b}}\bar{\psi}^{\dot{b}}; \tag{3.38}
$$

$$
\psi^a = \mathcal{E}^{ab}\psi_b;\ \ \psi_a = \mathcal{E}_{ab}\psi^b; \tag{3.39}
$$

$$
\mathcal{E}_{\dot{a}\dot{b}}\mathcal{E}_{dc}(\bar{\sigma}^\mu)^{\dot{b}c} = -(\sigma^\mu)_{d\dot{a}}, \tag{3.40}
$$

where the additional minus sign is originated from $\{\gamma^0, \gamma^0\} = -2\mathbb{I}$ and it disappears when we choose the opposite Minkowski signature in Clifford algebra at the beginning.

The advantage of this convention is that the formula are t_0-free and dotted/undotted are raised or lowered by the same two tensors $\mathcal{E}$'s and the same rule. Such t_0 dependences are all packed into the raising/lowering of $\mathcal{E}$ by itself and the relation between $\mathcal{E}^{ab}$ and $\mathcal{E}_{ab}$:

$$
\mathcal{E}_{ca}\mathcal{E}_{db}\mathcal{E}^{ab} = -t_0\mathcal{E}_{cd};\ \ \mathcal{E}^{ca}\mathcal{E}^{db}\mathcal{E}_{ab} = -t_0\mathcal{E}^{cd}; \mathcal{E}^{ab} = -t_0\mathcal{E}_{ab}. \tag{3.41}
$$

which need more care when we raise or lower the indices of $\mathcal{E}$ tensors by $\mathcal{E}$ themselves.

3.6 Draft in trash here in the source Tex (neglect it)

4 Covariant WEEKLY: 2021.3.28.-4.3.

In this week, I would, briefly, explain a confusion associated with covariant derivatives.

4.1 A conventional one on vectors and co-vectors

Conventionally, we define the following covariant derivative

$$D_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \tag{4.1}$$

for a vector field V^μ. This definition has a clear physical meaning from the parallel transportation. Namely, we first transport $V^\mu(x)e_\mu(x)$ to $x+\Delta x$ which is an action on the basis vectors:

$$e_\mu(x) \to e_\mu(x+\Delta x) - \Gamma^\nu_{\rho\mu}\Delta x^\rho e_\nu(x+\Delta x) \tag{4.2}$$

thereby

$$V^\mu(x)e_\mu(x) \mapsto V^\mu(x)[e_\mu(x+\Delta x) - \Gamma^\nu_{\rho\mu}\Delta x^\rho e_\nu(x+\Delta x)]. \tag{4.3}$$

Then we can compare the vector field at $(x+\Delta x)$ with this transported one:

$$\Delta V = V^\mu(x+\Delta x)e_\mu(x+\Delta x) - V^\mu(x)[e_\mu(x+\Delta x) - \Gamma^\nu_{\rho\mu}\Delta x^\rho e_\nu(x+\Delta x)] \tag{4.4}$$

which turns out to be

$$\lim_{\Delta x\to 0} \Delta V/\Delta x^\mu = e_\nu D_\mu V^\nu. \tag{4.5}$$

Similarly, we can do the same thing on the co-vectors since, by the constraint that $\langle e^\mu(x), e_\nu(x)\rangle = \delta^\mu_\nu$ is kept during the parallel transportation, we obtain the parallel transportation for the co-vector basis vectors as

$$e^\mu(x) \to e^\mu(x+\Delta x) + \Gamma^\mu_{\rho\nu}\Delta x^\rho e^\nu(x+\Delta x) \tag{4.6}$$

which gives

$$D_\mu V_\nu \equiv \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho. \tag{4.7}$$

It is also tempting to understand the formula above by means of tensor entity, such as:

$$D_\mu V_\nu = (\nabla V)_{\mu\nu}. \tag{4.8}$$

This notation reminds us that the covariant derivative just seems to be acted on the components, but actually induced by the action on the (basis) vectors.

4.2 Algebraic covariant derivative

If we see the indices μ as algebraic indices transforming by general coordinate transformations, we can also define the covariant derivatives concentrating on the indices. However, there is something strange, because this time the components are covariantly differentiated due to being carrying an index rather than being associated to the vector basis! Such a strangeness can be explicitly seen by

$$D(dx^\nu V_\nu) = D(dx^\nu)V_\nu + dx^\nu D(V_\nu) = dx^\mu \wedge dx^\nu \partial_\nu V_\mu, \tag{4.9}$$

where the key point is that now $D(V_\nu)$ also has a connection part, which will cancel that of $D(dx^\nu)$. This covariant derivative differs from $dx^\nu dx^\mu(\nabla V)_{\mu\nu}$ by a torsion form.

4.3 Additional but Essential Remarks

- It should be noted that the second derivative is acted on differential forms rather than simply vectors or co-vectors as in the first type. However, one might attempt to identify the 1-forms in the second kind of derivative as a co-vector, namely, $dx^\mu \equiv e^\mu$, and then use the first derivative. But, as we have seen, it is different from the result obtained by the second definition. The correct viewpoint may be the object to be differentiated in the second type is the *graded algebra*, e.g., dx^μ also being viewed as the component of $e_\mu dx^\mu$. Then we can use the first type to define the derivative for it and the result matches the second type! Additionally, we should view $dx^\mu V_\mu$ as the "inner product" $\langle dx^\mu e_\mu, e^\nu V_\nu \rangle \equiv dx^\mu \wedge V_\nu \langle e_\mu, e^\nu \rangle$ rather than simply a co-vector.

- The second type acting on zero-forms but possibly carry indices of course gives the same result as in the first type if we treat them as corresponding tensor fields. Therefore, the first type of tensors of any rank is actually the **zero-form special case** of the second type!

- The second type is useful to define covariant vectors (with antisymmetric indices of course) and one of the most well-known example is $(F)_{ij} = D(A)_{ij}$ while the only difference is that here D also has a nontrivial effect on the U(N) indices "i, j", and F does not contain connections Γ's *not* depending on being torsion-free or not. In constrast, $D(A_\mu)_{ij}$ is a 1-form (since $(A_\mu)_{ij}$ is a zero-form, i.e., a matrix function) and it contains connections Γ's.

- It is interesting to understand why the anticommuator of differential 1-form is important. It is because $ddx^\mu = 0$ and $0 = ddy^\mu = \partial^2_{\alpha,\beta} y dx^\alpha dx^\beta$ need to be consistent with each other. Or to be more complicated, it is required to be consistent

$$d(dy^\mu \tilde{V}_\mu) = d(dx^\alpha V_\alpha) \tag{4.10}$$

  if $\tilde{V}_\alpha = V_\mu \partial_\alpha y^\mu$.

  We will see that the second kind of differential covariant derivative could be quite useful in supergravity formalism in future.

5 Gravitational MONTHLY: 2021.4.4-2021.5.3

In this month, I completed the reading of Wald's General Relativity halfly, i.e., Chapters 1-6. I attach the Solutions to Exercises and a reduction to Euler's formula which seems not clear in the textbook, as in the following three pages.

$= \frac{dr^2}{\cos^2\psi} = \frac{dr^2}{1-r^2} \Rightarrow k=1$

$\frac{dr^2}{1} \Rightarrow k=0$

$\Rightarrow d\psi^2 = \frac{dr^2}{\cosh^2\psi} = \frac{dr^2}{1+r^2} \Rightarrow k=-1$

assumed to be nonnegative.

$, \frac{\pi}{2})$ is covered.

$+ r^2(d\theta^2 + \sin^2\theta d\phi^2)]$

$\sigma - \partial_\sigma g_{\mu\nu})$

$\Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\alpha\mu}$

$(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$

$g_{\nu\rho} + \partial_\nu g_{\mu\rho})$

$\frac{1}{2}g^{\mu\mu}(\partial_\nu g_{\mu\mu})$

$= \frac{1}{2}g^{\rho\rho}(\partial_\rho g_{\mu\mu})$

$= \frac{1}{2}g^{\mu\mu}\partial_\mu g_{\mu\mu}$

$= \frac{1}{r}$; $\Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$

kr^2} ; $\Gamma^0_{\theta\theta} = \dot{a}a \cdot r^2$; $\Gamma^0_{\phi\phi} = \dot{a}ar^2\sin^2\theta$

$\cdot r$; $\Gamma^r_{\phi\phi} = -(1-kr^2)r\sin^2\theta$

$\cos\theta$; $\Gamma^r_{rr} = \frac{kr}{1-kr^2}$

$+ \Gamma^\alpha_{00}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu 0}\Gamma^\nu_{\alpha 0}$

$-3\left(\frac{\dot{a}^2}{a^2}\right)$

$\Gamma^\alpha_{ii}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu i}\Gamma^\nu_{\alpha i}$

$+ \frac{2}{r^2} + \frac{2k}{1-kr^2} - \frac{2}{r^2} - \frac{k^2r^2}{(1-kr^2)^2}$

$+ \frac{3\dot{a}^2}{1-kr^2} - \frac{2\dot{a}^2}{1-kr^2}$

$\frac{a\ddot{a}+2k}{a^2}) = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$

$= \frac{3}{2}\left(-\frac{\ddot{a}}{a} + \frac{2\dot{a}^2 + a\ddot{a} + 2k}{a^2}\right) = \frac{3\dot{a}}{a^2} + \frac{3k}{a^2}$

$\cdot R)g^{rr} = -\frac{1}{2}\left(\frac{3\ddot{a}}{a} + \frac{2\dot{a}^2 + a\ddot{a} + 2k}{a^2}\right)$

$= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}$

$\Rightarrow \begin{cases} \frac{3\dot{a}}{a^2} = 8\pi\rho - \frac{3k}{a^2} \\ \frac{3\ddot{a}}{a} = -4\pi(\rho + 3P) \end{cases}$ ※

$\begin{cases} \partial^a T_{ab} u^b = 0 \quad (u^a u_a = -1) \\ \partial^a T_{ab} + \partial^a T_{ac} u^c u_b = 0 \end{cases}$

if $\partial^a T_{ab} \cdot V^b = 0$ for $\forall V^b$. ⇑

Since $V^b = \delta^b_a V^a = (\delta^b_a + u^b u_a - u^b u_a)V^a$

$\therefore \begin{cases} \partial^a((\rho + P)u_a)(-1) + \partial_b P \cdot u^b = -\partial^a(\rho u_a) - P\partial_a u^a = 0 \\ \quad (*) \Rightarrow u^a \partial_a \rho + (P+\rho)\partial_a u^a = 0 \\ \partial^a((\rho+P) u_a u_b) + \partial_b P + \partial^a((\rho+P)u_a u_c)u^c u_b + \partial_c P \cdot u^c u_b = 0 \\ \Rightarrow (\rho+P)u_a \partial^a u_b + (\rho+P)u_a \partial^a u_c \cdot u^c u_b \\ \quad + \partial_b P + \partial_c P \cdot u^c u_b = 0 \\ \xrightarrow{(**)} (P+\rho)u^a \partial_a u_b + (g_{ab} + u_a u_b)\partial^a P = 0 \end{cases}$

I) Non-relativistic limit $\rho \gg P$; $u^\mu = (1, \vec{v})$

NOTE: $\vec{v} \ll 1$ is not able to be inserted since $u^0\partial_0 \sim u^i\partial_i$ & ∂_0 has c.

$\therefore (*) \Rightarrow \partial_t \rho + \partial_i(\rho \cdot v^i) = 0$

0-th component of $(**)$: $(P+\rho)u^\mu\partial_\mu(-1) + \partial_0 P - u_\mu \partial^\mu P = 0$

$\Rightarrow u^\mu \partial_\mu P = \partial_0 P$

i-th of $(**)$: $(P+\rho)u^\mu \partial_\mu u^i + \partial^i P + v^i u^\mu \partial_\mu P = 0$

$\rho(\partial_t V^i + V \cdot \nabla(v^i)) = -\partial^i P$

(Since $\frac{1}{c^2}v^i \partial_t P \ll \vec{\nabla}_i P$) if $v^i \partial_t P \ll \vec{\nabla}_i P$.

It's strange that $u^i\partial_i P \approx 0$. To see it, we restore the c's : $\rho c^2 u^\mu \partial_\mu(-u^0) - u^i\partial_i P = 0$

$\partial_\mu(-u^0) = \partial_\mu \frac{-1}{\sqrt{1 - v^2/c^2}} \simeq -\frac{1}{2c^2}\partial_\mu \vec{v}^2$

$\therefore -\frac{\rho}{2}u^i\partial_i \vec{v}^2 - \frac{\rho}{2}\partial_0 \vec{v}^2 = \vec{v} \cdot \vec{\nabla} P$

$\therefore$ if $\vec{v}$ is steady i.e. $\partial_0 \vec{v} = 0$ and ρ is const.

then $\frac{d}{dt}\left(\frac{\rho v^2}{2} + P\right) = 0$

$\therefore \frac{\rho v^2}{2} + P = \text{const.}$

Bernoulli's principle!

Key : Restoring c's & checking terms as an entity rather than partially.

Problem 3.1.

a) Pf. Let $\tilde{\nabla}_a$ be torsion-free e.g. $\tilde{\nabla}_a = \partial_a$

Since $\nabla_a - \tilde{\nabla}_a$ is a local operator,

$\therefore (\nabla_a - \tilde{\nabla}_a)V_b = \cancel{V_d \Gamma_{ab}{}^d} - C_{ab}{}^d V_d$

$\therefore [\nabla_a, \nabla_b]f$

$= \tilde{\nabla}_a \tilde{\nabla}_b f - C_{ab}{}^d \tilde{\nabla}_d f$
$-(a \leftrightarrow b)$

$= -2C_{[ab]}{}^d \nabla_d f$

$\therefore T_{ab}{}^d = 2C_{[ab]}{}^d$ ※

b) Pf. Choose a torsion-free basis $\tilde{\nabla}_a$

$[X, Y]^c = X^a \tilde{\nabla}_a Y^c - Y^a \tilde{\nabla}_a X^c$

$= X^a \nabla_a Y^c - Y^a \nabla_a X^c$
$+ (-1)\underbrace{[X^a C_{ab}{}^c Y^b - Y^a C_{ab}{}^c X^b]}_{T_{ab}{}^c X^a Y^b}$

c) Pf. $C_{(ab)}{}^d \equiv \Gamma_{ab}{}^d = \frac{1}{2}(C_{ab}{}^d + C_{ba}{}^d)$ ※

$(*)$ $0 = \nabla_c g_{ab} = \partial_c g_{ab} - C_{ca}{}^d g_{db} - C_{cb}{}^d g_{ad}$

$(**)$ $0 = \partial_a g_{bc} - C_{ab}{}^d g_{dc} - C_{ac}{}^d g_{bd}$

$(***)$ $0 = \partial_b g_{ca} - C_{bc}{}^d g_{da} - C_{ba}{}^d g_{cd}$

$0 = (*) + (***) - (**)$

$= \partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} - 2\Gamma_{bc}{}^d g_{ad}$
$+ T_{ab}{}^d g_{dc} + T_{ac}{}^d g_{db} \Rightarrow$

$\Gamma_{bc}{}^d = \frac{1}{2}[\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}]g^{ad}$
$+ (T^d{}_b{}^e g_{ec} + T^d{}_c{}^e g_{eb})\frac{1}{2}$

$\therefore C_{ab}{}^c = \Gamma_{ab}{}^c + \frac{1}{2}T_{ab}{}^c$

$= \underbrace{\Gamma_{ab}{}^c}_{\text{Christoffel connection.}} + \frac{1}{2}\{T^c{}_{ab} + T^c{}_{ba} + T_{ab}{}^c\}$

Problem 3.2.

a) Pf. if $\nabla_a\beta = \epsilon_{ab}\nabla^b\alpha$ has solution $\Leftrightarrow \nabla_c \epsilon_{ab}\nabla^b\alpha = \nabla_a \epsilon_{cb}\nabla^b\alpha$

$\Leftrightarrow \nabla_c(\epsilon_{ab}\nabla^b\alpha) \cdot \epsilon^{ac} = 0$

since α harmonic $\Leftrightarrow (\nabla_c \epsilon_{ab}) \cdot \epsilon^{ac} \cdot \nabla^b\alpha = 0$

$0 = \nabla_c(\epsilon_{ab}\epsilon^{ac}) = \nabla_c\epsilon_{ab} \cdot \epsilon^{ac} + \epsilon_{ab}\nabla_c\epsilon^{ac}$

$-\epsilon_{ab}\nabla_c\epsilon^{ac} = -\epsilon_{ab}(\partial_c\epsilon^{ac} + \Gamma_c{}^a{}_\sigma\epsilon^{\sigma c} + \Gamma_c{}^c{}_\sigma\epsilon^{a\sigma})$

what about $\cancel{\partial_a\beta = \epsilon_{ab}g^{bd}\partial_d\alpha}$ $\epsilon_{ab} = \epsilon_{ab} \cdot \sqrt{g}$

By Equivalence Principle $\nabla_c\epsilon_{ab} = 0$ ※

$\nabla_a\nabla^a\beta = \nabla_a(\epsilon^{ab}\nabla_b\alpha) = \epsilon^{ab}\nabla_a\nabla_b\alpha = 0$ if torsion-free.

Problem 3.3

a) Pf. $0 R_{abcd} - R_{bacd} + 2R_{bcad} + 2R_{cabd} = R_{abcd} + R_{bcad} + R_{cabd}$

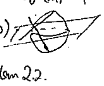
cyclicity $a \to b \to c \to d$ 4 times by $(+,-,-,+)$ ※

Problem 2.1.

a) : $f_2 \circ f_1^{-1}$ with $f_1 : (x_1 x_2 x_3) \mapsto (x_2 x_3)$, $f_2 : (x_1 x_2 x_3) \mapsto (x_1 x_3)$

if $\text{sgn}(x_2) = S_2$ $\therefore f_1^{-1}(x_2, x_3) = (\sqrt{1 - x_2^2 - x_3^2}\, S_1, x_2, x_3)$

$\text{sgn}(x_1) = S_1$ $\therefore f_2^{S_2} \circ f_1^{S_1^{-1}}(x_2, x_3) = (\sqrt{1 - x_2^2 - x_3^2}\, S_1, x_3)$ is smooth.

b)  S^n is compact ; $\mathbb{R}^n$ is noncompact.

$\therefore$ $\mathbb{R}^n$ cannot singly chart-covers.

Problem 2.2.

Pf. $F(x_n, \vec{x}_{(n-1)}) = F(x_n, \vec{a}_{(n-1)}) + (x^i - a^i)H_i(x_n, \vec{x}_{(n-1)})$

$\overset{\text{by H}}{=} F(\vec{a}) + (x_n - a_n) \cdot H_n(x_n, \vec{x}_{(n-1)}) + \{i\}$

$= F(\vec{a}) + (x^\mu - a^\mu)H_\mu(\vec{x})$ ※

Problem 2.3. Pf:

(a) linearity: $[v,w](af + bg) = v(w(af + bg)) - w(v(af + bg))$

$= a\begin{pmatrix} v(w(f)) \\ -wv(f) \end{pmatrix} + b\begin{pmatrix} v(w(g)) \\ -wv(g) \end{pmatrix}$

$= a[v,w](f) + b[v,w](g)$

N-L: $[v,w](fg) = vwfg - wvfg$

$= v(fwg + wf \cdot g) - w(fvg + vf \cdot g)$

$= f \cdot vwg + vf\,wg + vwf \cdot g + wf \cdot vg$
$- fwvg - wf \cdot vg - vf \cdot wg - wvf \cdot g$

$= [v,w]f \cdot g + f \cdot [v,w]g$ ※

(b) LHS $f = [X,Y]Zf - Z \cdot [X,Y]f + \cdots$

$= X(Y(Z(f))) - Y(X(Z(f))) - ZXYf + ZYXf + \cdots$

$\begin{matrix} + YZX & -ZYX & -XYZ & +XZY \\ + ZXY & -XZY & -YZX & +YXZ \end{matrix}$

$= 0$ ※

(c) $C^\gamma{}_{\alpha\beta}C^\lambda{}_{\gamma\theta} + C^\gamma_{\beta\theta}C^\lambda{}_{\gamma\alpha} + C^\gamma_{\theta\alpha}C^\lambda{}_{\gamma\beta} = 0$ if C is const.

$-Y_\theta(C^\gamma_{\alpha\beta})\delta^\lambda_\gamma - Y_\alpha(C^\gamma_{\beta\theta})\delta^\lambda_\gamma - Y_\beta(C_{\theta\alpha}{}^\gamma)\delta^\lambda_\gamma$

Problem 2.4.

(a) $[v,w]f = (v^\mu \partial_\mu w^\nu \partial_\nu - w^\mu \partial_\mu v^\nu \partial_\nu)f$

$\therefore [v,w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$

(b) In any coordinate basis,

$[Y_\alpha, Y_\beta] = (Y_\alpha)^\mu \partial_\mu (Y_\beta)^\nu \cdot \partial_\nu - (\alpha \leftrightarrow \beta)$

$(Y^{\alpha *})_\mu (Y_\beta)^\mu = \delta^\alpha_\beta \Rightarrow \boxed{(Y^{\alpha})^*_\mu (Y_\alpha)^\nu = \delta_\mu^\nu}$

(Cheat-work) $(Y_\alpha)^\mu \partial_\mu (Y_\beta)^\nu = (Y_\lambda)^\mu \cdot (Y^{\lambda *})_\kappa (Y_\alpha)^\kappa \partial_\mu [(Y_\omega)^\nu \cdot (Y^{\omega *})_\sigma (Y_\beta)^\sigma]$

$(Y^{\sigma *})_\nu \cancel{(Y^{\beta *})_\mu} [(Y_\alpha)^\mu \partial_\mu (Y_\beta)^\nu] - \alpha \leftrightarrow \beta$

$= -(Y_\alpha)^\mu (Y_\beta)^\nu \partial_\mu (Y^{\sigma *})_\nu - \alpha \leftrightarrow \beta = C^\sigma_{\alpha\beta}$ ※

$\therefore -(\partial_\mu (Y^{\sigma *})_\nu - \mu \leftrightarrow \nu) = C^\sigma_{\alpha\beta}(Y^{\alpha *})_\mu (Y^{\beta *})_\nu$ ※

correct!)

$-\frac{3\Lambda}{8\pi})$

if $k=1$ & $\Lambda>0$

$\frac{\Lambda}{\pi} \Rightarrow a_c^2=\Lambda^{-1}$

$(\Lambda-\frac{3k}{a^2})a-\frac{3\dot{a}^2}{a}$

4 perturbation $\dot{a}\approx 0$; if $a>a_c, \ddot{a}>0$

$\ddot{a}=-W_a a$, $W_a<0$

$\Leftarrow$ if $a<a_c$, $\ddot{a}<0$, $W_a>0$

$-\frac{3k}{a^2} \Rightarrow \dot{a}^2=-k \Rightarrow a^2(t)=-k(t+d)^2$

$\therefore$ if $k=0 \Rightarrow a=$ const.

$$\begin{cases} 3\frac{\dot{a}^2}{a^2}=8\pi(\rho+\frac{\Lambda}{8\pi})-\frac{3k}{a^2} \\ 3\frac{\ddot{a}}{a}=-4\pi(\rho+3P-\frac{\Lambda}{4\pi}) \end{cases}$$

$\frac{3\Lambda}{8\pi}$; $\rho+\frac{\Lambda}{8\pi}=\frac{3k}{8\pi a^2}$

>0, $k>0$

$a=a_c=\sqrt{\frac{k}{\Lambda}}$

$3\frac{\dot{a}^2}{a^2}=3(\Lambda-\frac{k}{a^2})$

$\ddot{a}=\frac{1}{2}(\Lambda-\frac{k}{a^2})a-\frac{1}{2}\frac{\dot{a}^2}{a} \Rightarrow$ Unstable

$=\Lambda-\frac{3k}{a^2}$

$=\Lambda$

$=a_0\exp(\pm\sqrt{\frac{\Lambda}{3}}t)$

...tion $\Rightarrow a(t)=a_0\exp(\sqrt{\frac{\Lambda}{3}}t)$ ※

the negative pressure and positive energy ... same ratios, so $\rho+3P$ with $3>1$ is key ... (accelerating).

$\therefore U_b=-\delta^0_b$

$\Gamma^0_{ab}=\frac{\dot{a}}{a}h_{ab}$ with $\dot{a}=\dot{a}(t)$

...ordinates in Riemann normal coordinate

$)=D_{\xi^0}(-k^a U_a)=k^a\nabla_a(-k^b U_b)$

since $k^a\equiv\frac{d}{d\xi^0}x^a$

ω^2 massless $=-k^a k^b\nabla_a U_b$

$+\frac{dt}{d\xi^0}$

$\cdot(-1) \Rightarrow \frac{\omega(t_1)}{\omega(t_2)}=\frac{a(t_2)}{a(t_1)}$ ※

b) $\Delta\psi=\int\frac{(1-\cos\eta)d\eta}{(1-\cos\eta)}=\int d\eta=2\pi$ ※ ($\eta=\pi$ maximum)

$\Rightarrow$ halfway at the "big crunch" ※

$\Rightarrow \alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$

$\cancel{\partial_b(\frac{\partial_d A_e}{\partial e A_b})(\partial^d A^e-\partial^e A^d)}$

$\cancel{=\partial_b[\partial_d A_e\partial^d A^e\cdot 2-\partial_e A_d\partial^d A^e\cdot 2]}$

$\cancel{=2\partial_b\{\partial_d A_e[\partial^d A^e-\partial^e A^d]\}}$

~~furthermore~~ $\cancel{\partial^a(\partial_a A_b-\partial_b A_a)}=0$

$\therefore \cancel{\partial_b\partial_d A_e=}$

$\cancel{\propto\partial_b\partial_d A_e F^{de}}$

$\cancel{+\partial_d A_e\partial_b F^{de}}$

$F_{ac}\partial^a F_b{}^c-\frac{1}{4}\partial_b(F^{de}F_{de})$

$=F_{ac}\partial^a F_b{}^c-\frac{1}{2}F^{de}\partial_b F_{de}$

$=F^{de}(\partial_d F_{be}-\frac{1}{2}\partial_b F_{de})$

$=\frac{1}{2}F^{de}(\partial_d F_{be}-\partial_d F_{eb}-\partial_b F_{de})$

$=\frac{1}{4}F^{de}(-\partial_e F_{bd}+\partial_e F_{db}+\partial_b F_{ed}+\partial_d F_{be}-\partial_d F_{eb}-\partial_b F_{de})$

$=\frac{6}{4}F^{de}\partial_{[d}F_{be]}$

$=0$ by Bianchi identity ※

$\partial_a\partial^a A_b=0$

$\partial_a\partial^a e^{iS}=0$

$\partial_a(\partial^a S e^{iS})$

$\sigma=\square S+\partial_a S\partial^a S$

$\nabla^a\nabla^b F_{ab}$

$\propto[\nabla^a,\nabla^b]F_{ab}$

$=R_{ab}{}^{bc}{}_{a}F_{cb}$

$+R_{ab}{}^{b}{}_{c}F_{ac}$

$=R^{ba}{}_{b}{}^{c}F_{ca}$

$+R^{ab}{}_{b}{}^{c}F_{ac}$

$=2R^{ac}F_{ca}$

$=0$ since $R^{ac}=R^{ca}$ ※

$m^2\phi\nabla_b\phi$

$+\nabla_a\phi\nabla^a\nabla_b\phi$

$-\frac{1}{2}\nabla_b\nabla_c\phi\cdot\nabla^c\phi\, 2$

$-m^2\phi\nabla_b\phi$

$\propto T_{bc}{}^{d}\nabla_d\phi\nabla\phi$

$=0$ if ...

---

5. RIKEN will not retain personal data for purposes beyond those described above.
6. You may file a complaint with the supervisory authority of EEA member states in case there are complaints to be resolved regarding RIKEN's handling of your personal data. However, we would highly appreciate if we could be notified in advance to seek a solution prior to your communication to a supervisory authority.

<Contact> Public Relations Office

E-mail: lab-hp @riken.jp

3. If you ask RIKEN to delete your personal data on the RIKEN Website, we will delete it promptly.

The following provisions (4 to 6) apply specifically to research personnel from the European Economic Area (EEA).

4. The collected personal data will be stored on a server in Japan, where the personal data of registered individuals will be managed in an appropriate manner. In addition, RIKEN will not transfer the collected personal data outside of EEA member nations. Japan has obtained an adequacy decision from the European Commission as of January 23, 2019; however, the adequacy decision does not apply to administrative organizations and independent administrative institutions, including RIKEN.

Problem 3.5. a) Pf. $\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=\alpha\cdot\frac{dx^\mu}{d\tau}$ $\therefore \frac{d^2\tilde{x}^\mu}{ds^2}=\frac{d^2x^\mu}{d\tau^2}\left(\frac{d\tau}{ds}\right)^2+\frac{dx^\mu}{d\tau}\left(\frac{d^2\tau}{ds^2}\right)$

Let $\tau=\tau(s)$, $\tilde{x}(s)\equiv x^\mu(\tau(s))$

$\tilde{\alpha}(s)\equiv\alpha(\tau(s))$: $\frac{dx^\mu(\tau)}{d\tau}=\frac{dx^\mu(\tau(s))}{ds}\cdot\frac{ds}{d\tau}$

or $\frac{d\tilde{x}^\mu(s)}{ds}=\frac{dx^\mu}{d\tau}\frac{d\tau}{ds}$ Let $\left(\frac{d\tau}{ds}\right)^{-1}\frac{d^2\tau}{ds^2}$

b) if $\tilde{\alpha}$ has been zero $\Rightarrow \tau''(s)=0$ since $\frac{d\tau}{ds}\neq 0$ ※ $=\tilde{\alpha}(s)$ ※

Problem 2.5.

Pf. $(Y^\mu_\alpha(x))\frac{\partial}{\partial x^\mu}\overset{?}{=}\frac{\partial}{\partial y^\alpha}$

$\left\{\text{if so} \Rightarrow \left[\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}=\frac{\partial}{\partial y^\beta}\frac{\partial}{\partial y^\alpha} \Leftrightarrow [Y_\alpha,Y_\beta]=0\right]\right\}$

It has sol iff $(Y^{\alpha*}_\mu)dx^\mu=dy^\alpha$ has sol. by matrix inverse.

(by hint) iff $\partial_\nu Y^{\alpha*}{}_\mu-\partial_\nu Y^{\alpha*}{}_\mu=0$

(by 4(b)) iff $C^\gamma{}_{\alpha\beta}=0$ since Y & Y^* are invertible ※

Problem 2.6. Pf.

(a) $\forall\,\Omega\in V^* \Rightarrow \Omega=v^{\mu*}\Omega(v_\mu)$ & $\Sigma a_\mu v^{\mu*}=0 \Leftrightarrow a_\mu=0$ ※

(c) $e^{a*}(e_b)=\delta^a_b$ if $e'_b=\beta_b{}^a e_a$, $e'^{b*}=\tilde{\beta}_a{}^b e^{a*}$

$\therefore \delta^b_a=\tilde{\beta}_\alpha{}^b e^{\alpha*}(\beta_a{}^\gamma e_\gamma)=\beta_a{}^\gamma\tilde{\beta}_\gamma{}^b$

$\Rightarrow \tilde{\beta}=\beta^{-1}$

Since $T'^{a\cdots}_{a\cdots}=\tilde{\beta}\beta T^{a\cdots}_{a\cdots}\tilde{\beta}\tilde{\beta}$ ※

(or formally $T'^{a}{}_{c}{}^{b} e_b\otimes e^{c*}=T(e'^{a*},e'_a)$

$=\tilde{\beta}_b{}^a\beta_a{}^\gamma T(e^{b*},e_\gamma)$

$=T(e^{b*},e_b)$ Q.E.D.)

Problem 2.7. Pf

(a) if $n=1$, since g invertible $\therefore g\neq 0 \Rightarrow$ statement holds.

$g=O^T DO$ with $v_i=(O^T)_i{}^j\partial_j$

(b) Since $\mathrm{Spec}(g)$ is invariant $\Rightarrow$ # positive/negative eigenvalue invariant

$g(v_i, v_i)\cdot\|v_i\|$ positive definite $\Rightarrow$ signature is invariant.

Problem 2.8.

b). $t=t'$; $x'^2+y'^2=x^2+y^2$, $z'=z$, $\frac{y'}{x'}=\tan(\phi-\omega t)$

$dx^2+dy^2=r^2d\theta^2+dr^2$

$dx'^2+dy'^2$ — too complicated ~

$\therefore$ ~~Let $r = f(\tilde{r})$~~ $\quad \frac{d\tilde{r}}{\tilde{r}} = \sqrt{h}\,\frac{dr}{r}$

~~$d\tilde{r}^2 = f'^2 dr^2$~~

$$\therefore \tilde{r} = C\exp\left(\int^r \sqrt{h(r)}\,\frac{dr}{r}\right)$$

$$\& \quad H(\tilde{r}) = \left(\frac{r}{\tilde{r}}\right)^2$$

b) Pf. if $h(r) = \left(1 - \frac{2}{r}\right)^{-1}$

$$\Rightarrow \tilde{r} = C\cdot\left[(r-1) + \sqrt{r^2 - 2r}\right]$$

Let $C \equiv 1$

then $r^2 - 2r = r^2 - 2r(1+\tilde{r}) + (1+\tilde{r})^2$

$$\therefore \tilde{r}^2 + 2\tilde{r} + 1 - 2r\cdot\tilde{r} = 0$$

$$\therefore r = \frac{(\tilde{r}+1)^2}{2\tilde{r}}$$

$$\therefore H(\tilde{r}) = \frac{(\tilde{r}+1)^4}{4\tilde{r}^4}$$

Take $\tilde{r} = 2\tilde{r}'$

$$H(\tilde{r}') = 4\cdot\frac{(\tilde{r}'2+1)^4}{4\cdot 2^4 \tilde{r}'} = \left(\frac{1+2\tilde{r}'}{2\tilde{r}'}\right)^4$$

Renaming $\tilde{r}' \mapsto \tilde{r}$ $\therefore$
$$\begin{cases} r \equiv \dfrac{(2\tilde{r}+1)^2}{4\tilde{r}} & \text{does the job.} \\ H(\tilde{r}) = \left(1 + \dfrac{1}{2\tilde{r}}\right)^4 \end{cases}$$

$$\therefore\ 1 - \frac{2M}{r} = 1 - \frac{8\tilde{r}}{(2\tilde{r}+1)^2} = \left(\frac{1-2\tilde{r}}{1+2\tilde{r}}\right)^2 = \left(\frac{1-\frac{1}{2\tilde{r}}}{1+\frac{1}{2\tilde{r}}}\right)^2$$

Problem 6.3.

a) Pf. Due to $U(1) = SO(2) \subset SO(3)$ little group $\divideontimes$

$e_{0,1} \wedge e_{2,3}$ cannot appear

but $e_2 \wedge e_3$ is $SO(2)$ volume form

so it's invariant

Due to $SO(3)/SO(2) \Rightarrow e_2 \wedge e_3 \cdot F(r)$ and stationary

Due to stationary & $SO(3)$ $e_0 \wedge e_1 \cdot \tilde{F}(r)$

Other terms are all forbidden.

$$\therefore F = A(r)\cdot e_0 \wedge e_1 + B(r)\cdot e_2 \wedge e_3$$

b) Pf. $\because \nabla^\mu F_{\mu\nu} = 0$

$$\therefore \nabla^\mu (e_\mu^a e_\nu^b F_{ab}) \overset{\text{vielbein postulate}}{=} (\nabla^\mu F_{ab}) e_\mu^a e_\nu^b$$
$$= \nabla_\mu F_{ab}\cdot e^{a\mu} e_\nu^b$$
$$= (\partial_\mu F_{ab} + \omega_{\mu a}{}^c F_{cb} + \omega_{\mu b}{}^c F_{ac})\cdot e^{a\mu} e_\nu^b$$

Taking $\nu = t \Rightarrow \nabla^\mu F_{\mu t} = -\sqrt{\frac{f}{h}}\,\frac{1}{r^2}\partial_r (F_{01} r^2) = 0$

$A(r) = F_{01}(r) \Rightarrow A(r) = \frac{-q}{r^2}$ $\divideontimes$

Problem 6.4.

a) Pf. $u = \frac{1}{\sqrt{-g_{tt}}}\partial_t$, $\xi = \frac{\partial}{\partial t}$

$$\Rightarrow u^a \nabla_a u^b = u^t \nabla_t u^b = u^t \Gamma^b_{tt} u^t$$
$$= \frac{1}{2} u^t u^t \cdot g^{b\sigma}\cdot(-1)\cdot\partial_\sigma g_{tt}$$
$$= \frac{1}{2}\partial^b \ln(-g_{tt}) = \partial^b \ln V \ \divideontimes$$

by a force ... with ...

$$m\sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a^\mu a_\mu}\cdot m = mV^{-1}(\nabla_a V \cdot \nabla^a V)^{1/2}$$

otherwise, the object would follow geodesic rather than stationary.

Since being stationary, the distance δs in the freely falling frame is proper

$\Rightarrow W \equiv m\sqrt{\vec{a}\cdot\vec{a}}\cdot\delta s$ is assumed to be released and becomes radiation energy W in the MCRF.

Then its ~~energy~~ four-momentum satisfies $\langle p, \frac{1}{\sqrt{-g_{tt}}}\partial_t\rangle = W$

$$\therefore p_0 \cdot \frac{1}{\sqrt{-g_{tt}}} = W$$

$$\therefore p_0 = W\sqrt{-g_{tt}}$$

which is the energy in the asymptotic observer

$$\Rightarrow W\sqrt{-g_{tt}} = F_\infty \cdot \delta s$$

$$\therefore F_\infty = \sqrt{-g_{tt}}\,F = V\cdot F \ \divideontimes$$

Problem 6.6.

Pf. Let's continue to use original coordinates

Since $d\tau^2 = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left[\left(\frac{2M}{r} - 1\right) dt^2 + r^2 d\Omega^2\right]$

$$\therefore \left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2M}{r} - 1} \quad \Leftarrow r < 2M$$

$$\therefore \tau_m = \left|\int_{2M}^{0} \frac{dr}{\sqrt{\frac{2M}{r} - 1}}\right| = \pi M$$

If a particle is falling freely from $r_M = 2M$ with $P_t = E = 0$, then $\frac{dt}{d\tau}\left(\frac{2M}{r} - 1\right) = 0$ later

$$\Rightarrow \frac{dt}{d\tau} = 0 \Rightarrow 1 = \left(\frac{dr}{d\tau}\right)^2 \Big/ \left(\frac{2M}{r} - 1\right)$$

$$\Rightarrow \Delta\tau = \left|\int_{2M}^{0} \frac{dr}{\sqrt{\frac{2M}{r} - 1}}\right| = \tau_m$$

$\divideontimes$

6 Conformal Weekly: 2021.5.4.-2021.5.19.

Let us first focus on the Ward identities, which keeps confusing me for quite a long time.

To do so, we define the energy-momentum tensor as:

$$\langle T^{\mu\nu}(x)O^{\alpha}[\varphi(x_i)]_g\rangle_g \equiv \frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}}\langle O^{\alpha}[\varphi(x_i)]_g\rangle_g, \tag{6.1}$$

where $\langle\cdots\rangle_g$ is the path integral average but *without* normalization by $1/Z_g$. Here $O^{\alpha}[\varphi(x_i)]_g$ denotes a product of, perhaps spinful, local operators at x_i's and they are normal-ordered by the metric g.

We re-write it as

$$\frac{\sqrt{g}}{2}\int \delta g_{\mu\nu}\langle T^{\mu\nu}(x)O^{\alpha}[\varphi(x_i)]_g\rangle_g = \delta\langle O^{\alpha}[\varphi(x_i)]_g\rangle_g. \tag{6.2}$$

6.1 The first Ward identity

We consider the Ward identity related with translations, so we define

$$\tilde{x} \equiv x + a(x) \tag{6.3}$$

with $a(x)$ being nonzero only around x_i's while *bumped* to zero elsewhere. This transformation induces the transformation at $g=\delta$,

$$\delta g \equiv \tilde{g} - g = \mathcal{L}_{a(x)}g = -\partial_\mu a_\nu(x) - \partial_\nu a_\mu(x), \tag{6.4}$$

similar for the fundamental field $\varphi \mapsto \tilde{\varphi}$.

Thus, around the flatness $g=\delta$ implying no Weyl anomaly,

$$\begin{aligned}
&\delta\langle O^{\alpha}[\varphi(x_i)]_g\rangle_{g=\delta} \\
&= \int [\mathcal{D}\varphi]_{\tilde{g}} e^{-S[\varphi,\tilde{g}]}O^{\alpha}[\varphi(x_i)]_{\tilde{g}} - \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]}O^{\alpha}[\varphi(x_i)]_g \\
&= \int [\mathcal{D}\tilde{\varphi}]_{\tilde{g}} e^{-S[\tilde{\varphi},\tilde{g}]}O^{\alpha}[\tilde{\varphi}(x_i)]_{\tilde{g}} - \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]}O^{\alpha}[\varphi(x_i)]_g \quad [\varphi\mapsto\tilde{\varphi}] \\
&= \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]}\left\{O^{\alpha}(x_i-a)_g + \partial_\beta a^{\alpha}O^{\beta}(x_i)\right\} \quad [\text{General covariance of vector}] \\
&\quad - \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]}O^{\alpha}[\varphi(x_i)]_g \\
&= \sum_i -a_\nu(x_i)\partial_i^\nu\langle O^{\alpha}[\varphi(x_i)]_\delta\rangle_\delta + \partial_\beta a^{\alpha}(x_i)\langle O^{\beta}[\varphi(x_i)]_\delta\rangle_\delta.
\end{aligned} \tag{6.5}$$

On the other hand of LHS of Eq. (6.2), due to the bumping of $a(x)$, we can safely integrating it by parts to obtain

$$-a_\nu(x_i)\partial_i^\nu\langle O^{\alpha}[\varphi(x_i)]_\delta\rangle_\delta + \partial_\beta a^{\alpha}(x_i)\langle O^{\beta}[\varphi(x_i)]_\delta\rangle_\delta = \int a_\nu(x)\partial_\mu\langle T^{\mu\nu}(x)O^{\alpha}[\varphi(x_i)]_\delta\rangle_\delta,$$

where LHS depends only on $a(x)$ near x_i's and $\bar{a}_i$'s are arbitrary. Therefore, at the flatness,

$$\partial_\mu\langle T^{\mu\nu}(x)O^{\alpha}[\varphi(x_i)]_\delta\rangle_\delta = -\sum_i \delta(x-x_i)\left[\delta^{\alpha}_{\beta}\partial_i^\nu + \delta^{\nu\alpha}\overleftarrow{\partial}_\beta\right]\langle O^{\alpha}[\varphi(x_i)]_\delta\rangle_\delta, \tag{6.6}$$

where $f(x)\overleftarrow{\partial} \equiv \partial f(x)$ and the second term in $[\cdots]$ results from the vector (in general tensor) general coordinate transformation. By defining

$$P^\nu(\Sigma) \equiv -\oint_\Sigma dS_\mu T^{\mu\nu}, \tag{6.7}$$

we obtain

$$[P^\nu, O^\alpha(x_i)] = \partial_i^\nu O^\alpha(x_i). \tag{6.8}$$

One should be cautious that the derivative of δ-function does not come here while it will play an essential role later in Eq. (6.19).

6.2 Conformal Ward identities

We can proceed to more Ward identities if the action is classically Weyl invariant:

$$S[\varphi, \Omega^2(x)g] = S[\varphi, g], \tag{6.9}$$

specifically conformal invariant:

$$S[\varphi, g + \mathcal{L}_{\epsilon^{\mathrm{CK}}}] = S[\varphi, g], \tag{6.10}$$

where ϵ^{CK} is the conformal Killing vector satisfying at the flatness

$$\mathcal{L}_{\epsilon^{\mathrm{CK}}} = -\partial_\mu \epsilon_\nu^{\mathrm{CK}} - \partial_\nu \epsilon_\mu^{\mathrm{CK}} = -\frac{2}{d}\partial \cdot \epsilon^{\mathrm{CK}} \delta_{\mu\nu} \equiv -2\omega \delta_{\mu\nu}. \tag{6.11}$$

First, we can readily obtain from Eqs. (6.2,16.10) around flatness (no Weyl anomaly)

$$\langle O[\varphi(x_i)]_{\delta - 2\omega\delta} - O[\varphi(x_i)]_\delta \rangle_\delta = \int -\omega(x) \langle T_\mu^\mu(x) O^\alpha[\varphi(x_i)]_\delta \rangle_\delta, \tag{6.12}$$

and the more general Weyl invariant Eq. (6.9) further gives a point-wise precision:

$$\sum_i \delta(x - x_i) \langle O[\varphi(x_i)]_{\delta - 2\omega\delta} - O[\varphi(x_i)]_\delta \rangle_\delta = -\omega(x) \langle T_\mu^\mu(x) O^\alpha[\varphi(x_i)]_\delta \rangle_\delta, \tag{6.13}$$

Then by the trick as before, we consider an almost CK field:

$$\tilde{x} = x + \epsilon(x) = x + a(x)\epsilon^{\mathrm{CK}}, \tag{6.14}$$

where $a(x)$ is constantly a unity around x_i while bumped to zero elsewhere.

Since if $a(x)$ were globally a constant function, the conformal invariance (16.10) would imply the $S[\varphi, g + \mathcal{L}_\epsilon g] = S[\varphi, g]$, which means that, at the lowest order, there exists some J^μ:

$$S[\varphi, \delta + \mathcal{L}_\epsilon \delta] = S[\varphi, g] - \int J^\mu \partial_\mu a(x). \tag{6.15}$$

Let us evaluate RHS of Eq. (6.2) alternatively putting $\tilde{g} = \delta + \mathcal{L}_\epsilon \delta$ into LHS of Eq. (6.2):

$$\begin{aligned}
&\delta\langle O[\varphi(x_i)]_\delta\rangle_\delta \\
&= \int [\mathcal{D}\varphi]_{\tilde{g}} e^{-S[\varphi,\tilde{g}]} O^\alpha[\varphi(x_i)]_{\tilde{g}} - \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]} O^\alpha[\varphi(x_i)]_g \\
&= \int [\mathcal{D}\varphi]_{\tilde{g}} e^{-S[\varphi,\delta]} (1 + \int J^\mu \partial_\mu a) \left\{O^\alpha + \delta_g O^\alpha\right\} [\varphi(x_i)]_\delta - \int [\mathcal{D}\varphi]_g e^{-S[\varphi,g]} O^\alpha[\varphi(x_i)]_g \\
&= \langle \delta_g O^\alpha[\varphi(x_i)]_\delta\rangle_\delta + \int \partial_\mu a \langle J^\mu(x) O[\varphi(x_i)]_\delta\rangle_\delta. \qquad (6.16)
\end{aligned}$$

Back to Eq. (6.2) and using Eq. (6.13), we obtain, by integration by parts,

$$\begin{aligned}
&\int a(x) \partial_\mu \langle J^\mu(x) O^\alpha[\varphi(x_i)]_\delta\rangle_\delta \\
&= \langle \delta_g O^\alpha[\varphi(x_i)]_\delta\rangle_\delta + \int \langle T^{\mu\nu} \partial_\mu [a(x) \epsilon^{\rm CK}_\nu] O^\alpha[\varphi(x_i)]_\delta\rangle \\
&= \int -\frac{1}{d}\delta(x - x_i) \partial\cdot\epsilon^{\rm CK} a(x) \langle T^\mu_\mu O^\alpha[\varphi(x_i)]_\delta\rangle_\delta - \int a(x) \epsilon^{\rm CK}_\nu \partial_\mu \langle T^{\mu\nu}(x) O^\alpha[\varphi(x_i)]_\delta\rangle_\delta \\
&= -\int \delta(x - x_i) a(x) \partial_\mu \langle (T^{\mu\nu}\epsilon^{\rm CK}_\nu) O^\alpha[\varphi(x_i)]_\delta\rangle_\delta, \qquad (6.17)
\end{aligned}$$

where, in the last line, we have used the Killing condition of $\epsilon^{\rm CK}$. Thus,

$$\begin{aligned}
J^\mu &= -T^{\mu\nu}\epsilon^{\rm CK}_\nu; \\
Q(\Sigma) &= \oint_\Sigma dS_\mu J^\mu = -\oint_\Sigma dS_\mu T^{\mu\nu}\epsilon^{\rm CK}_\nu. \qquad (6.18)
\end{aligned}$$

Let us obtain the following algebraic result by Eq. (6.6) into the second or third line above:

$$\partial_\mu \langle J^\mu(x) O^\alpha(x_i)\rangle = \delta(x - x_i) \left[\langle \delta_{\mathcal{L}_{\epsilon^{\rm CK}}\delta} O^\alpha[\varphi(x_i)]_\delta\rangle - \mathcal{L}_{\epsilon^{\rm CK}} \langle O^\alpha(x_i)\rangle\right], \qquad (6.19)$$

where, for instance of a vector,

$$\mathcal{L}_\xi O^\alpha \equiv -\xi^\nu \partial_\nu O^\alpha + \partial_\nu \xi^\alpha O^\mu. \qquad (6.20)$$

Our final result Eq. (6.19) has a clear physical meaning: the first term on its RHS represents a Weyl transformation while the second term the tensor transformation due to the general covariance.

Eventually, for a field with a definite Weyl weight Δ_O:

$$\delta_{-2\omega\delta} O = \omega \Delta_O O, \qquad (6.21)$$

which also means

$$\Omega^{\Delta_O} O_{\Omega^2 g} = O_g. \qquad (6.22)$$

Then we obtain, e.g., for a scalar,

$$[Q, O(x)] = \left[\epsilon^{\rm CK}_\mu \partial^\mu + \frac{1}{d}\partial\cdot\epsilon^{\rm CK}\Delta_O\right] O(x), \qquad (6.23)$$

thereby, if $\partial_\nu \tilde{x}^\mu = \Omega(\tilde{x}) R^\mu_\nu(\tilde{x})$,

$$O(x)_\delta = \Omega(\tilde{x})^{\Delta_O} \tilde{O}(\tilde{x})_\delta = [\det(\partial_\nu \tilde{x}^\mu)]^{\frac{\Delta_O}{d}} \tilde{O}(\tilde{x})_\delta. \qquad (6.24)$$

7 Topology MONTHLY: 2021.5.20.-2021.6.15.

In this month, I reviewed the book by James Munkres about point-set topology. Since I will systematically self-study the complex analysis in the coming month, I find that such a review will be helpful. Here, I will discuss several tricky points worthy much more attention than I expected before.

We always the real line $\mathbb{R}$ is equipped with its standard topology.

7.1 Completeness

Def: Cauchy sequence A series of elements labelled by positive integers $\{x_n\}_{n\in\mathbb{Z}^+}$ in a metric $|.|$ topological space satisfying that

$\forall\epsilon > 0$, $\exists N_\epsilon$, $\forall N, M \geq N_\epsilon \Rightarrow |x_N - x_M| \leq \epsilon$.

Theorem: Convergent sequences in a metric topological space are Cauchy sequences.

Proof: straightforward by triangle inequality.

Theorem: If a Cauchy sequence has a converging subsequence, it is convergent.

Proof: Again by triangle inequality.

Theorem: A Cauchy sequence is bounded in the sense that there exists an open ball containing the whole sequence.

Theorem: If the metric space is $\mathbb{R}^n$ with its standard Euclidean metric, any Cauchy sequence is convergent.

Proof: First of all, a Cauchy sequence is bounded, so since we are in the standard metric, it is contained in a cube $[-L, +L]^n$ which is compact since $[-L, +L]$ is so. Thus, this Cauchy sequence has a convergent subsequence and then it is convergent.

Remark: The property that every Cauchy sequence is convergent is called "completeness". From the above proof, we can see that completeness is not only a topological property but intrinsically metric depending on the metric. It is because "being bounded" is NOT a topological property, but depending on metric! To reflect its topological issue, consider $\mathbb{Q}$ instead of $\mathbb{R}$. To reflect its metric aspect, consider the metric as $d(x, y) = |\arctan x - \arctan y|$ instead of the standard one on $\mathbb{R}$ and this metric effectively *homeomorphically* maps or squeezes the real line to the open interval $(-\pi/2, +\pi/2)$ which is clearly incomplete.

7.2 Weierstrass M test

In real analysis, we always assume that the real line is dressed with the standard Euclidean metric. However, the completeness can be still somewhere even we do not rely on any metric. To do so, let us consider the useful Weierstrass M test.

Theorem 6.1: If a series of real numbers $\{s_n\}_{n\in\mathbb{Z}^+}$ is upper bounded by the standard ordering and $s_{n+1} > s_n$, it is convergent.

Proof: The limit is the supreme of the series which exists by the axiom of real line.

Theorem 6.2: If $|a_n| \leq b_n$ and $\sum_{n\in\mathbb{Z}^+} b_n$ exists, then $\sum_{n\in\mathbb{Z}^+} a_n$ exists. Here $|a| \equiv a$ if $a \geq 0$ and $|a| \equiv -1 \cdot a$ if $a < 0$.

Proof: Let us define $S_N \equiv \sum_{n=1}^{N} |a_n|$, which clearly converges as $N \to \infty$. $L_N \equiv \sum_{n=1}^{N} |a_n| + a_n$. Clearly $L_{n+1} \geq L_n$ and it is upper bounded (of ordering) by $2\sum_{n\in\mathbb{Z}^+} b_n$.

Remark: The above theorem can be easily proven by arguing $S_{N\in\mathbb{Z}^+}$ as a Cauchy sequence by introducing the standard metric.

Weierstrass M test: Given a summation of continuous function $\sum_n f_n(x)$ from a topological space X to $\mathbb{R}$. If $|f_n(x)| \leq M_n$ where M_n is in a convergent summation $\sum_n M_n$. Then $\sum_n f_n(x)$ converges uniformly in X. (Then the limit function is itself continuous.)

Proof: By the preceding Theorem, the function converges pointwisely, i.e., the limit $N \to \infty$ of $F_N(x) = \sum_{n=1}^{N} f_n(x)$ exists for each fixed x, and we denote it as $F(x)$. Since

$$|F_N(x) - F_{N+k}(x)| \leq \sum_{n=N}^{N+k} M_n, \tag{7.1}$$

then $\forall \epsilon > 0$, $\exists N_\epsilon$, $\forall N \geq N_\epsilon$ and $\forall k \geq 0 \Rightarrow \sum_{n=N}^{N+k} M_n \leq \epsilon/2$ because $\sum_{n=1}^{N} M_n$ is a Cauchy sequence indexed by $N \in \mathbb{Z}^+$. Thus

$$|F_N(x) - F_{N+k}(x)| \leq \epsilon/2, \tag{7.2}$$

We know that $F_N(x)$ converges to $F(x)$, so $\exists k_{\epsilon,x} > 0$, $\forall k \geq k_{\epsilon,x} \Rightarrow |F_{N+k}(x) - F(x)| \leq \epsilon/2$. Since Eq. (7.2) is true all integer k, we have $|F_N(x) - F(x)| \leq \epsilon$ for $\forall N \geq N_\epsilon$.

Remark: Throughout the proof of the M test, we can see the completeness of the real numbers is actually reflected in the standard ordering property by which we define the absolute value $|.|$ although we do not rely on any metric. So why? The key is the fact that the absolute value also obeys the triangle inequality purely by the axioms of real numbers!

8 Algebraic topology MONTHLY: 2021.6.16-2021.8.31.

In these months, I reviewed the famous book by Bott and Tu on differential forms in algebraic topology, especially the spectral sequence. However, it is hard to type them into the current Tex file, while I encouraged everybody to be familiar with that powerful tool. Later, I also tried to solve Allen Hatcher's problem sets of Chapters 2&3. I put my solutions to part of problems that I know how to do in the following pages.

9 Superconducting MONTHLY: 2021.9.1-9.30

In this month, I have (re)viewed the superconductivity, including the deeply in-sighted BCS theory in a superficial sense and the topological superconductors.

9.1 An extremely brief review of 2nd quantizations

A common misunderstanding that quantum field theory is a different theory from our textbook quantum mechanics mainly results from the 2nd quantization. In this short paragraph, I want to emphasize that quantum field theory or many-body quantum system is even a SUBSET of our textbook quantum mechanics! In one word, when we consider indistinguishable particles in quantum mechanics, quantum field theory is simply a convenient TOOL to formulate such quantum mechanical systems, rather than any new fundamental theory!

In quantum mechanics, we know that the indistinguishable particles in three dimensions have the property that

$$\begin{aligned}&\Psi(x_1\sigma_1, x_2\sigma_2, \cdots, x_i\sigma_i, \cdots, x_j\sigma_j, \cdots)\\ =&\begin{cases} +\Psi(x_1\sigma_1, x_2\sigma_2, \cdots, x_j\sigma_j, \cdots, x_i\sigma_i, \cdots), & \text{called boson;}\\ -\Psi(x_1\sigma_1, x_2\sigma_2, \cdots, x_j\sigma_j, \cdots, x_i\sigma_i, \cdots), & \text{called fermion,}\end{cases}\end{aligned} \tag{9.1}$$

while in two dimensions other phase factors are possible, so-called anyonic systems.

We should see the above equation as a constraint on the wave function we need to consider. Mathematically speaking, indistinguishable-particle systems are a subset of the distinguishable-particle systems, which satisfy Eq. (9.1). Therefore, the basis of Hilbert space we need to consider is also shrunk or restricted:

$$\text{Boson: } S[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots]; \tag{9.2}$$

$$\text{Fermion: } A[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots], \tag{9.3}$$

which obviously satisfy Eq. (9.1), and in which S and A are symmetrization and antisymmetrization taking into account the normalization, respectively.

2nd quantization is just a rewriting of the above basis elements as:

$$\begin{aligned}\text{Boson: } S[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots] &\equiv b^\dagger_{x_1\sigma_1} b^\dagger_{x_2\sigma_2} \cdots |\text{vac}\rangle;\\ \text{Fermion: } A[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots] &\equiv c^\dagger_{x_1\sigma_1} c^\dagger_{x_2\sigma_2} \cdots |\text{vac}\rangle,\end{aligned} \tag{9.4}$$

where $|\text{vac}\rangle$ represents the unique vacuum state without any particle.

From Eq. (9.1), or by the definitions of S and A, it is clear that

$$[b^\dagger_{..}, b^\dagger_{...}] = 0; \ \ \{c^\dagger_{..}, c^\dagger_{...}\} = 0. \tag{9.5}$$

In addition, due to the appearance of A or the anticommutator above, $(c^\dagger_{..})^2 = 0$ as an operator equation on any quantum state. Moreover, $b(c)_{x_i\sigma_i}$, when acted on a state without such a state of particle, vanishes by the definition (9.4). Then the following commutator can be derived from all the definitions:

$$[b_{x_1\sigma_1}, b^\dagger_{x_2\sigma_2}] = \delta_{x_1,x_2}\delta_{\sigma_1,\sigma_2}; \ \ \{c_{x_1\sigma_1}, c^\dagger_{x_2\sigma_2}\} = \delta_{x_1,x_2}\delta_{\sigma_1,\sigma_2}. \tag{9.6}$$

In many textbooks, they use the state-occupation number to label states like $|n_1, n_2, \cdots\rangle$, but I don't think it is a good idea in the case of fermions since we need to know how to order different states, which is actually inconvenient!

Then let us study how the operators are REWRITTEN (not defined) in the new language.

9.1.1 Single particle operators

Single particle operators imply the operator can be even defined when there is only one particle, so $\hat{T} = \sum_{i,j} T_{ij}|i\rangle\langle j|$ if there is only one particle. When there are N indistinguishable particles, the operator is N-copy (since they are all the same particles):

$$\hat{T} = \sum_{n=1}^{N} T_{ij}|i\rangle_n\langle j|_n, \tag{9.7}$$

where the summation over i, j is done implicitly, and the index "n" indicates which bra or ket is acted upon, e.g.,

$$\begin{aligned} T_{ij}|i\rangle_n\langle j|_n(|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots) &= T_{ij}(|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots |i\rangle\langle j|x_n\sigma_n\rangle \otimes |x_{n+1}\sigma_{n+1}\rangle \cdots) \\ &= T_{ij}\langle j|x_n\sigma_n\rangle(|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots |i\rangle \otimes |x_{n+1}\sigma_{n+1}\rangle \cdots), \end{aligned}$$

while keeping other kets untouched.

Then the representation of $\hat{T}$ on the basis Eq. (9.1) by the equation:

$$\hat{T}b^\dagger_{x_1\sigma_1}b^\dagger_{x_2\sigma_2}\cdots|\text{vac}\rangle \equiv \hat{T}\left\{S[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots]\right\}; \tag{9.8}$$
$$\hat{T}c^\dagger_{x_1\sigma_1}c^\dagger_{x_2\sigma_2}\cdots|\text{vac}\rangle \equiv \hat{T}\left\{A[|x_1\sigma_1\rangle \otimes |x_2\sigma_2\rangle \cdots]\right\}, \tag{9.9}$$

which implies that

$$\hat{T} = \sum_{i,j} T_{ij}a^\dagger_i a_j, \tag{9.10}$$

the same expression for both bosonic ($a^\dagger = b^\dagger$) and fermionic ($a^\dagger = c^\dagger$) cases.

9.1.2 Two-particle operators

Likewise, we can also 2nd-quantize the two-particle operators:

$$\hat{V} = \sum_{ijkm} V_{ijkm}(|i\rangle_1\langle k|_1)(|j\rangle_2\langle m|_2) \tag{9.11}$$

for two particles labelled by "1" and "2", and clearly $V_{ijkm} = V_{kmij}$ can be set without loss of generality since only this symmetric part contributes to the above formula of $\hat{V}$. Equivalently, we can do the over counting and then kill by "1/2":

$$\hat{V} = \sum_{ijkm}\sum_{p\neq q:1,2}\frac{1}{2}V_{ijkm}(|i\rangle_p\langle k|_p)(|j\rangle_q\langle m|_q), \tag{9.12}$$

where i, j, k, m are also implicitly summed, because $|i\rangle_p\langle k|_p$ commutes with $|j\rangle_q\langle m|_q$ if $p \neq q$.

Then for N-particle case:

$$\hat{V} = \sum_{ijkm} \sum_{p\neq q:1,2,\cdots,N} \frac{1}{2} V_{ijkm} (|i\rangle_p \langle k|_p)(|j\rangle_q \langle m|_q), \tag{9.13}$$

We can quickly derive its form in the creation/annihilation operators by Eq. (9.10) as follows:

$$\hat{V} = \sum_{ijkm} \left[\sum_{p,q:1,\cdots,N} - \sum_{p=q:1,\cdots,N} \right] \frac{1}{2} V_{ijkm} (|i\rangle_p \langle k|_p)(|j\rangle_q \langle m|_q) \tag{9.14}$$

$$= \sum_{ijkm} \frac{1}{2} V_{ijkm} \left[(a_i^\dagger a_k)(a_j^\dagger a_m) - |i\rangle_p \langle m|_p \delta_{k,j} \right] \quad (\text{since } \langle k|j\rangle = \delta_{k,j}) \tag{9.15}$$

$$= \sum_{i,j,k,m} \frac{1}{2} V_{ijkm} a_i^\dagger a_j^\dagger a_m a_k, \tag{9.16}$$

again for both bosonic and fermionic cases, but notice the order of i, j, k, m in the last line.

Remark:

1) The 2nd quantization is simply a basis rewriting when we restrict our attention to the indistinguishable particles;

2) The new notation system is advantageous in that it helps us avoid writing out a complicated form of interaction terms, especially, we don't need to specify how many particles we are considering [e.g., comparing Eq. (9.7) and Eq. (9.10)]. Thus it is a suitable language in studying many-body (indistinguishable-particle) physics.

3) Logically and ironically, many-body distinguishable systems are more difficult to formulate, where 2nd quantization doesn't work by definition, and the Hilbert space is literally too large.

9.2 BCS theory from BdG Hamiltonian

I am not going to make the BCS theory in details since it requires a long-term course including phonon-mediated electron-electron interaction etc., but I would sketch the idea roughly. First of all, electron and electron interacts by repulsive Coulomb interaction which instantaneous in the condensed-matter time scale (rather than the high-energy fundamental particle level). The Coulomb interaction can be screened due to the positively charged background (no matter mobile or immobile). Moreover, electron also interacts with the ions at the lattice sites (let us assume they are fixed around each site) by Coulomb interactions. Interestingly, such interactions will induce an attractive interaction between electrons: Electron A attracts a lattice ion when it passes by the ion, and then the ion attracts Electron B even after Electron A has been away. Thus, effectively, Electron A attracts B indirectly. Obviously, unlike the (repulsive) Coulomb interaction between A and B, such lattice-induced attractive force is not instantaneous and it is retarded. Thus it has a characteristic frequency ω_{D} which is a typical frequency scale of the lattice vibration, called Debye frequency. In addition, back to the Coulomb interactions, due to the screening, they will decay in a long range such that they behavior like free fermion or Landau Fermi liquid more precisely.

From the above argument, we expect that the ground state under the condition above should be almost the same deep in the Fermi sea and only different from the conventional Fermi sea around the Fermi energy by the order of $\pm\omega_{\mathrm{D}}$.

9.3 BdG Hamiltonian and its ground state

Let us directly consider the following BdG Hamiltonian:

$$\mathcal{H}_{\mathrm{BdG}} = \sum_{\sigma,\alpha,\beta,p} \xi_p c^\dagger_{p,\sigma} c_{p,\sigma} + \left[\Delta_{\alpha,\beta}(\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta} + \text{h.c.}\right], \ \ (\xi_p \equiv \frac{\vec{p}^2}{2m} - \mu) \tag{9.17}$$

which is obviously translation invariant due to its conservation of momentum, while it breaks the charge conservation. Thus the ground state does not have a definite charge number. The complex number $\Delta_{\alpha,\beta}(\vec{p})$ is called the BCS gap function, and it is restricted by the fermionic anticommutativity as shown as follows:

$$\begin{aligned}
&\sum_{\alpha,\beta,p} \Delta_{\alpha,\beta}(\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta} \\
=& \frac{1}{2} \sum_{\alpha,\beta,p} \Delta_{\alpha,\beta}(\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta} + \Delta_{\alpha,\beta}(-\vec{p}) c^\dagger_{-p,\alpha} c^\dagger_{p,\beta} \\
=& \frac{1}{2} \sum_{\alpha,\beta,p} \Delta_{\alpha,\beta}(\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta} - \Delta_{\beta,\alpha}(-\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta},
\end{aligned} \tag{9.18}$$

which is nonzero only when

$$\Delta_{\alpha,\beta}(\vec{p}) = -\Delta_{\beta,\alpha}(-\vec{p}). \tag{9.19}$$

Specifically for the spinless fermion (i.e., spin-polarized electrons like all spin-up), α or β can only take a single value, so the only choice is $\Delta(\vec{p}) = -\Delta(-\vec{p})$.

There is a terminology called X-wave BdG Hamiltonian:

$$X = \begin{cases} s, d, f, \cdots, & \text{if } \Delta_{\alpha,\beta}(\vec{p}) = \Delta_{\alpha,\beta}(-\vec{p}); \\ p, e, g, \cdots, & \text{if } \Delta_{\alpha,\beta}(\vec{p}) = -\Delta_{\alpha,\beta}(-\vec{p}). \end{cases} \tag{9.20}$$

Therefore, spinless fermion can only form $p, e, g, \cdots$-wave superconductor. We will see why it is named in this way after we derive the wave function of the ground state.

9.4 Nambu spinor and charge conjugation

Let us define a useful vector called Nambu spinor:

$$\Psi_{\vec{p}} \equiv \begin{pmatrix} c_{\vec{p}\uparrow} \\ c_{\vec{p}\downarrow} \\ c^\dagger_{-\vec{p}\downarrow} \\ -c^\dagger_{-\vec{p}\uparrow} \end{pmatrix}, \tag{9.21}$$

which is consistent with its notation that $\Psi_{\vec{p}}$ possesses a definite momentum quantum number $\vec{p}$, namely $\hat{P}\Psi^\dagger_p \hat{P}^{-1} = p\Psi^\dagger_p$ with the momentum operator $\hat{P} \equiv \sum_{p,\sigma} p c^\dagger_{p\sigma} c_{p\sigma}$, whereas it does not have a definite spin number. The additional minus sign in Eq. (9.21) is simply a notational convenience for s-wave Hamiltonian, but we will use it universally.

Then we can rewrite the BdG Hamiltonian in terms of Nambu spinors as:

$$\mathcal{H}_{\text{BdG}} = \frac{1}{2}\sum_p \Psi_p^\dagger H_{\text{BdG}}(p)\Psi_p + \text{const.}, \tag{9.22}$$

where $H_{\text{BdG}}(p)$ is a 4-by-4 matrix for each $\vec{p}$, formed by pure numbers rather than an operator, different from the curly $\mathcal{H}_{\text{BdG}}$, so we will call $H_{\text{BdG}}(p)$ by **Hamiltonian matrix**. We will also omit the constant term below since it just changes the (many-body) energy zero point.

The Nambu spinor introduces additional degrees of freedom, e.g., originally each momentum has only two bands spanned by linear combination of $\uparrow, \downarrow$, while there are four bands now. Thus there is a description redundancy:

$$\Psi_p = (\Psi_{-p}^\dagger C)^T, \tag{9.23}$$

which is called charge-conjugation redundancy because we use particle and hole simultaneously, and in which

$$C \equiv \tau_y \otimes \sigma_y = C^T = C^\dagger = C^* = C^{-1}. \tag{9.24}$$

It means that

$$H_{\text{BdG}}(p) = -C^{-1}H_{\text{BdG}}^*(-p)C, \tag{9.25}$$

namely given an eigenvector Φ of $H_{\text{BdG}}(p)$ with eigenvalue E, we can immediately obtain $C\Phi^*$ as an eigenvector of $H_{\text{BdG}}(-p)$ but with eigenvalue $-E$. Its physical meaning is concise that creating a hole with energy $-E$ and momentum $-p$ is equivalent to annihilating a particle with opposite energy and momentum. Then we can expand the Hamiltonian by the eigenmodes:

$$\mathcal{H}_{\text{BdG}} = \frac{1}{2}\sum_{p;n=1,2,\bar{1},\bar{2}} E_{p,n}\Gamma_{p,n}^\dagger\Gamma_{p,n}, \tag{9.26}$$

$$\Gamma_{p,n} \equiv \Phi_{p,n}^\dagger \Psi_p, \tag{9.27}$$

where $\Phi_{p,n}$ is one of the four eigenvectors of $H_{\text{BdG}}(p)$ with eigenvalue $E_{p,n}$. By the anti-commutator relation of fermionic operator $c_{p\sigma}$,

$$\{\Gamma_{p,\alpha}, \Gamma_{p',\beta}^\dagger\} = \delta_{p,p'}\delta_{\alpha,\beta}. \tag{9.28}$$

However, by an appropriate re-ordering such that $E_{p,1}, E_{p,2} \geq 0$ while $E_{p,\bar{1}}, E_{p,\bar{2}} \leq 0$, and

$$\Phi_{-p,\bar{n}} = C\Phi_{p,n}^*, \tag{9.29}$$

which implies that

$$\Gamma_{p,n}^\dagger = \Psi_p^\dagger \Phi_{p,n} = \Psi_{-p}^T C C^{-1}\Phi_{-p,\bar{n}}^* = \Gamma_{-p,\bar{n}}, \tag{9.30}$$

exactly the particle-hole redundancy above. Then

$$\mathcal{H}_{\mathrm{BdG}} = \sum_{p;n=1,2} E_{p,n}\Gamma^\dagger_{p,n}\Gamma_{p,n}, \tag{9.31}$$

which indeed has the correct numbers of degree of freedom, i.e., two bands per momentum. Thus the ground state is simply:

$$\Gamma_{p,n}|\mathrm{G.S.}\rangle = 0, \tag{9.32}$$

for each momentum $\vec{p}$ and only for $n = 1, 2$ **not** including $\bar{1}, \bar{2}$.

9.5 s-wave superconductor: BCS ground state

For s-wave superconductors,

$$\sum_{\alpha,\beta,p} \Delta_{\alpha,\beta}(\vec{p}) c^\dagger_{p,\alpha} c^\dagger_{-p,\beta} = \sum_p \Delta c^\dagger_{p\uparrow} c^\dagger_{-p\downarrow} = \sum_p \frac{1}{2}\Delta(c^\dagger_{p\uparrow} c^\dagger_{-p\downarrow} - c^\dagger_{p\downarrow} c^\dagger_{-p\uparrow}), \tag{9.33}$$

where Δ is a constant, which can be made real by a gauge transformation, so we set $\Delta \in \mathbb{R}$.

The matrix $H_{\mathrm{BdG}}(p)$ (not the curly $\mathcal{H}_{\mathrm{BdG}}$) is

$$H_{\mathrm{BdG}}(p) = \begin{pmatrix} \xi_p & & \Delta & \\ & \xi_p & & \Delta \\ \Delta & & -\xi_p & \\ & \Delta & & -\xi_p \end{pmatrix} = \xi_p \tau_z \otimes \sigma_0 + \Delta \tau_x \otimes \sigma_0, \tag{9.34}$$

where σ_0 denotes the identity matrix.

It is straightforward to solve its eigenvectors with positive eigenvalues:

$$\Phi_{p,1} = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} \Delta \\ 0 \\ \sqrt{\xi_p^2 + \Delta^2} - \xi_p \\ 0 \end{pmatrix}; \ \Phi_{p,2} = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} 0 \\ \Delta \\ 0 \\ \sqrt{\xi_p^2 + \Delta^2} - \xi_p \end{pmatrix}, \tag{9.35}$$

with some normalizer $\mathcal{N}$ (we will denote all normalizer by $\mathcal{N}$) and

$$E_{p,1} = E_{p,2} = \sqrt{\xi_p^2 + \Delta^2}. \tag{9.36}$$

It should be noted that this degeneracy is not fundamental since a spin-orbit coupling can break it. Then

$$\Gamma_{p,1} = \frac{\Delta}{\sqrt{\mathcal{N}}} \left(c_{p\uparrow} + \frac{\sqrt{\xi_p^2 + \Delta^2} - \xi_p}{\Delta} c^\dagger_{-p\downarrow} \right) \equiv \frac{\Delta}{\sqrt{\mathcal{N}}} \left(c_{p\uparrow} + A_p c^\dagger_{-p\downarrow} \right); \tag{9.37}$$

$$\Gamma_{p,2} = \frac{\Delta}{\sqrt{\mathcal{N}}} \left(c_{p\downarrow} - \frac{\sqrt{\xi_p^2 + \Delta^2} - \xi_p}{\Delta} c^\dagger_{-p\uparrow} \right) = \frac{\Delta}{\sqrt{\mathcal{N}}} \left(c_{p\downarrow} - A_p c^\dagger_{-p\uparrow} \right). \tag{9.38}$$

Therefore, we can solve out the ground state as:

$$|\text{G.S.}\rangle_{\text{BCS}} \propto \prod_p (1 + A_p c^\dagger_{p\downarrow} c^\dagger_{-p\uparrow})|\text{vac}\rangle, \tag{9.39}$$

up to some normalizer. Let us take a close look at this ground state: 1) when $|p| \ll p_{\text{F}}$ with $p_{\text{F}}^2/2m = \mu \gg \Delta$, then $A_p \gg 1$ which means that the deep Fermi sea is conventionally filled; 2) when $|p| \gg p_{\text{F}}$, $A_p \ll 1$ which is also natural and means that large-momentum single-particle states are empty. Therefore, the ground state looks like the conventional Fermi sea far away from p_{F}, while different only near the Fermi surface like $[p_{\text{F}} - m\Delta/p_{\text{F}}, p_{\text{F}} + m\Delta/p_{\text{F}}]$ by a simple dimension analysis. Then, it is reasonable to expect that $\Delta \propto \omega_{\text{D}}$ up to a number (which is quite large though).

To see why it is called s-wave, we extract out the two-particle component from the above ground state:

$$|\text{G.S.}\rangle_{\text{BCS}} = |\text{vac}\rangle + \sum_p A_p c^\dagger_{p\downarrow} c^\dagger_{-p\uparrow}|\text{vac}\rangle + (\text{4- or more-particle states}) \tag{9.40}$$

as

$$\begin{aligned} |\text{pair}\rangle &\propto \sum_p A_p c^\dagger_{p\downarrow} c^\dagger_{-p\uparrow}|\text{vac}\rangle \\ &= \sum_{p,x,y} A_p c^\dagger_{x\downarrow} c^\dagger_{y\uparrow} \exp[ip \cdot (x-y)]|\text{vac}\rangle \\ &= \sum_{x,y} A(x-y) c^\dagger_{x\downarrow} c^\dagger_{y\uparrow}|\text{vac}\rangle \\ &= \sum_{x,y} A(x-y)[|x\downarrow\rangle|y\uparrow\rangle - |y\uparrow\rangle|x\downarrow\rangle], \end{aligned} \tag{9.41}$$

where in the last line we write the state back to 1st quantization, and

$$A(\vec{x} - \vec{y}) \equiv \sum_{\vec{p}} A_{\vec{p}} \exp[i\vec{p} \cdot (\vec{x} - \vec{y})]. \tag{9.42}$$

Then, since $A_{\vec{p}} = \frac{\sqrt{\xi_p^2 + \Delta^2} - \xi_p}{\Delta}$ is only function of $|\vec{p}|$, we obtain

$$A(\vec{x} - \vec{y}) = A(|\vec{x} - \vec{y}|) \sim \begin{cases} \exp(-|x-y|/\xi) & \text{strong pairing if } \mu < 0 \\ \text{const.} & \text{weak pairing if } \mu > 0 \end{cases}, \tag{9.43}$$

where ξ is a correlation length. Therefore, BCS ground state corresponds to weak pairing where Cooper pair is weakly paired in the **real** space, while the strong pairing is less interesting in that it is simply filled from an empty vacuum like an insulator. It implies that the orbital part and the spin part disentangled:

$$|\text{pair}\rangle \propto \sum_{x,y} A(|x-y|)|x\rangle|y\rangle \otimes \frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\downarrow\rangle), \tag{9.44}$$

and the relative orbital angular momentum of the pair is zero because the orbital wave function of the pair

$$(\langle x\downarrow|\langle y\uparrow|)|\text{pair}\rangle = A(|x-y|) \tag{9.45}$$

is isotropic, which is the reason it is called s-wave.

9.6 $p+ip$ superconductor in two dimensions

Let us consider p-wave spinless superconductor and the Nambu spinor is simply:

$$\Psi_p = [c_p, c^\dagger_{-p}], \tag{9.46}$$

and the BdG Hamiltonian to be considered is

$$\mathcal{H}_{\mathrm{BdG}} = \frac{1}{2}\sum_p \Psi_p^\dagger \begin{pmatrix} \xi_p & i\Delta(p_x+ip_y) \\ -i\Delta(p_x-ip_y) & -\xi_p \end{pmatrix} \Psi_p, \tag{9.47}$$

with the charge conjugation redundancy

$$\Psi_p = (\Psi^\dagger_{-p}\tau_x)^T. \tag{9.48}$$

Similarly, we can solve out the eigenmodes:

$$\begin{aligned}\Gamma_p &= \frac{1}{\sqrt{\mathcal{N}}}(-i\Delta(p_x-ip_y), \xi_p)\Psi_p \\ &= \frac{1}{\sqrt{\mathcal{N}}}\left[-i\Delta(p_x-ip_y)c_p + \xi_p c^\dagger_{-p}\right],\end{aligned} \tag{9.49}$$

with single-particle energy eigenvalue as

$$E_p = \sqrt{\xi_p^2 + \Delta^2 p^2}. \tag{9.50}$$

It should be noted that we have only one band per momentum rather than two for the same reason of particle-hole redundancy. Then the Hamiltonian is

$$\mathcal{H}_{\mathrm{BdG}} = \sum_p E_p \Gamma_p^\dagger \Gamma_p, \tag{9.51}$$

and $\{\Gamma_p, \Gamma^\dagger_{p'}\} = \delta_{p,p'}$, so the ground state is

$$\Gamma_p|\mathrm{G.S.}\rangle = 0, \tag{9.52}$$

which can be solved out to be

$$|\mathrm{G.S.}\rangle \propto {\prod_p}' \left[1 + \frac{\xi_p}{i\Delta(p_x-ip_y)} c_p^\dagger c^\dagger_{-p}\right]|\mathrm{vac}\rangle, \tag{9.53}$$

where ${\prod_p}'$ means that we only sum up p in half of the momentum space to avoid over counting (because we don't have additional index like spin). To see why it is called a p-wave superconductor, we extract out its pair state:

$$\begin{aligned}|\mathrm{pair}\rangle &\propto \sum_p \frac{\xi_p}{\Delta p^2}(p_x+ip_y)|p\rangle|-p\rangle \\ &= \sum_{p,x,y} \frac{\xi_p}{\Delta p^2}(p_x+ip_y)\exp[ip(x-y)]|x\rangle|y\rangle \\ &= \sum_{x,y} B(\vec{x}-\vec{y})|x\rangle|y\rangle,\end{aligned} \tag{9.54}$$

with

$$B(\vec{x}-\vec{y}) \equiv \sum_p \frac{\xi_p}{\Delta p^2}(p_x + ip_y)\exp[ip(x-y)]. \tag{9.55}$$

Therefore, the orbital wave function is $B(\vec{x}-\vec{y})$. Then we can study their relative angular momentum and find that $B(\vec{r})$ is eigen-function of the differential operator $l_z \equiv \vec{r}\times(-i\partial_{\vec{r}})_z$ with eigenvalue 1 and eigen-function of the operator $\vec{l}^2$ with eigenvalue $\sqrt{1(1+1)}$, i.e., spin-one. Here comes its name p-wave. The reason that it is called $p+ip$ is due to the appearance of $p_x + ip_y$, which also directly means that $l_z = 1$.

9.7 $p+ip$ superconductor is topological!

Let us study a fantastic aspect of $p+ip$ superconductor, its boundary chiral mode(s).

Since $\mu < 0$ should corresponds to the vacuum phase and $\mu > 0$ the superconducting phase, we can create an interface or boundary for the material by a space-dependent chemical potential

$$\mu(x,y) = \begin{cases} -\mu_0, & \text{if } x \ll 0; \\ +\mu_0, & \text{if } x \gg 0, \end{cases} \tag{9.56}$$

with $\mu_0 > 0$. In the y direction, we still have a periodic boundary condition.

Then we can solve out the boundary mode from (let's gauge $\Delta > 0$)

$$H_{\text{BdG}}(-i\partial_x, p_y) = \begin{pmatrix} -\mu(x) & i\Delta(-i\partial_x + ip_y) \\ -i\Delta(-i\partial_x - ip_y) & \mu(x) \end{pmatrix}, \tag{9.57}$$

which is just a modification omitting p^2-terms of Eq. (9.47) and obtain

$$\begin{aligned} \gamma_{p_y}(x) &= \exp\left[-\int_0^x \frac{\mu(s)}{\Delta} ds\right] \frac{i}{\sqrt{2}}(1,-1)\begin{pmatrix} c_{x,p_y} \\ c^\dagger_{x,-p_y} \end{pmatrix}, \\ E_{p_y} &= \Delta p_y, \end{aligned} \tag{9.58}$$

which is a chiral mode. Furthermore, by the charge conjugation redundancy:

$$\gamma_{p_y} = \gamma^\dagger_{-p_y}, \tag{9.59}$$

which means the edge BdG Hamiltonian is

$$\mathcal{H}_{\text{BdG;edge}} = \sum_{p_y \geq 0} \Delta p_y \gamma^\dagger_{p_y}(x \approx 0)\gamma_{p_y}(x \approx 0). \tag{9.60}$$

This edge mode seems rather strange at the first sight:

1. It is chiral, namely the velocity is always of the same sign, so it can never be realized in a pure 1D lattice system;

2. When we have a periodic boundary condition along y direction, then (the unique) $p_y = 0$ mode exists. However, it doesn't have any charge conjugation partner because $\gamma_0^\dagger = \gamma_0$! It means that the band number at $p_y = 0$ is odd due to this zero mode, which contradicts with the charge-conjugation redundancy that requires band number per momentum must be even since the Hamiltonian matrix is even-dimensional. A simple understanding on this issue is that $(p_y = 0, E = 0)$ is a charge-conjugation fixed point. Additionally, the same argument works for the relativistic Majorana fermions since the Dirac-Γ matrices or Clifford-algebra representations (which can be reducible) are always even-dimensional (except for (0+1) dimension).

3. This ill-defined edge Hamiltonian is called Majorana-Weyl fermion.

So where is the problem? The answer is that we have another edge with modes $\tilde{\gamma}_{p_y}$:

$$\tilde{\mathcal{H}}_{\text{BdG;edge}} = \sum_{p_y \le 0} -\Delta p_y \tilde{\gamma}^\dagger_{p_y}(x \approx L_x)\tilde{\gamma}_{p_y}(x \approx L_x); \tag{9.61}$$

$$\gamma_{p_y}(x) = \exp\left[+\int_{L_x}^{x} \frac{\mu(s)}{\Delta} ds\right] \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix} c_{x,p_y} \\ c^\dagger_{x,-p_y}\end{pmatrix},$$

$$E_{p_y} = -\Delta p_y, \tag{9.62}$$

just like a cylinder has two boundary circles while we only consider one. Then the **exactly solved** edge modes, when both edges are taken into account in the function $\mu(x)$, are linear combinations of γ and $\tilde{\gamma}$. However, when $p_y \neq 0$, the solution should almost be γ and $\tilde{\gamma}$ by simply considering the charge-conjugation redundancy. However, the story is different for the zero modes, the exact solution must be

$$f_0 = \frac{1}{\sqrt{2}}(\tilde{\gamma}_0 - i\gamma_0); \tilde{f}_0 = \frac{1}{\sqrt{2}}(\tilde{\gamma}_0 + i\gamma_0), \tag{9.63}$$

$$\lim_{L_x \to 0^+} f_0 \approx c_{x=0,p_y=0}, \tag{9.64}$$

which are charge-conjugation partner to each other $f_0^\dagger = \tilde{f}_0$ — we can say that $f_0^\dagger|\text{vac}\rangle$ is a well-defined quantum state at the lattice scale just like $\gamma^\dagger_{p_y>0}|\text{vac}\rangle$ now! Nevertheless, when $L_x \gg 0$, the modes on two far-separated edges don't talk to each other since we only consider local interactions. Then the edges cannot be gapped out.

On the other hand, when $L_x \to 0$, two edges will meet and it is meaningful to consider the sum of them as a one-dimensional Hamiltonian with γ and $\tilde{\gamma}$: ($\Delta \equiv 2$)

$$\begin{aligned}\mathcal{H}_{\text{BdG;edge}} + \tilde{\mathcal{H}}_{\text{BdG;edge}} &= \sum_{p_y>0} \Delta p_y \left[\gamma^\dagger_{p_y}\gamma_{p_y} + \tilde{\gamma}^\dagger_{-p_y}\tilde{\gamma}_{-p_y}\right] + 0 \cdot f_0^\dagger f_0 \\ &= \int dy (\psi_L, \psi_R) i\partial_y \tau_z \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},\end{aligned} \tag{9.65}$$

where we have formally included the zero mode, and define a two-component Majorana spinor as (no restriction on the summations over p_y below)

$$\psi_R(y) \equiv \sum_{p_y} \gamma_{p_y} \exp(ip_y y); \tag{9.66}$$

$$\psi_L(y) \equiv \sum_{p_y} \tilde{\gamma}_{p_y} \exp(ip_y y), \tag{9.67}$$

which both satisfy the Majorana reality condition $\psi^\dagger_{L(R)} = \psi_{L(R)}$ also because $C = \mathbb{I}_{2\times 2}$ in the current basis called Majorana(-Weyl) basis. The composite edge Hamiltonian is exactly a massless Majorana fermion which is well-defined and realizable in pure 1D, e.g., the famous Kitaev chain at criticality, with Lagrangian density:

$$\mathcal{L}_{\text{Maj}} = (\psi_L, \psi_R) \begin{pmatrix} i\partial_{t-y} & \\ & i\partial_{t+y} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{9.68}$$

I won't consider Kitaev chain, but one can do it without any difficulty, like solve out its bulk ground state and the edge mode(s). Furthermore, $p + ip$ superconductor can also host Majorana mode in the bulk when there is a π-flux inside. This is also an easy exercise, but what is nontrivial is that such Majorana mode have non-abelian statistics! Knowledgeable readers must have realized that the mathematical source of such zero modes is exactly the zero mode γ_0 or $\tilde{\gamma}_0$ in the above calculation. The reason is in that the disk geometry only allows the antiperiodic boundary condition at its boundary (spin structure extendiblity argument), so insertion of π-flux is equivalent to apply periodic boundary condition around the small circle around the insertion point.

Further remark:

1) When we take a look on $\mathcal{L}_{\text{Maj}}$ above, we find that in $(1+1)$-dimensional spacetime we need BOTH chiralities. However, in $(3+1)$ dimensions, we know that, without considering U(1) symmetry transformations, one massless Majorana is equivalent with a single Weyl fermion rather than both chiralities. Actually, the crucial reason is that in $(8k+2)$-dimensional spacetime, charge conjugation and chiral operator are consistent so that we construct Majorana fermion and Weyl fermion from the parent Dirac fermion by killing degrees of freedom in two distinct ways. Thus in $(8k+2)$-dimensional spacetime, one Majorana and one Weyl fermion are strictly inequivalent, which can be also seen by checking their **many-body** spectra, like the single-particle excitation (above the **many-body** vacuum) spectra.

2) It is also interesting to see the mode expansion of Dirac fermion in the current Majorana basis: (note that $C = \mathbb{I}_{2\times 2}$ instead of $C = \tau_x$ in earlier Nambu basis)

$$\Psi_{\text{Dirac}} = \sum_p b_p \begin{pmatrix} \Theta(-p) \\ \Theta(p) \end{pmatrix} \exp(ipy - i|p|t) + d^\dagger_p \begin{pmatrix} \Theta(-p) \\ \Theta(p) \end{pmatrix} \exp(-ipy + i|p|t), \tag{9.69}$$

which reduces to Majorana fermion $[\psi_L, \psi_R]$ by requiring

$$b_p = d_p = \begin{cases} \tilde{\gamma}_p & \text{if } p < 0; \\ \gamma_p, & \text{if } p > 0, \end{cases} \tag{9.70}$$

which effectively kills one band per momentum.

Caveat: The $p+ip$ or s-wave superconductor is actually a small-momentum part of some parent lattice superconductor, which has far richer structure. However, I don't have time to write them here, but one can check the book by Bernevig and Hughes.

9.8 Three-dimensional time-reversal topological superconductor

After I learnt topological superconductor protected by time-reversal symmetry, I found that it is interesting and technically impressive. Let us switch to a little-bit different notation that

$$\Psi_{\vec{p}} \equiv (c_{p\uparrow}, c_{p\downarrow}, -c^{\dagger}_{-p\downarrow}, c^{\dagger}_{-p\uparrow})^T, \tag{9.71}$$

with BdG Hamiltonian:

$$\mathcal{H}_{\text{BdG}} = \frac{1}{2}\sum_p \Psi^{\dagger}_p \begin{pmatrix} \xi_p & 0 & \Delta p_z & \Delta p_- \\ 0 & \xi_p & \Delta p_+ & -\Delta p_z \\ \Delta p_z & \Delta p_- & -\xi_p & 0 \\ \Delta p_+ & -\Delta p_z & 0 & -\xi_p \end{pmatrix} \Psi_p, \tag{9.72}$$

where $\Delta \equiv 1$ and $p_\pm \equiv p_x + i \pm p_y$.

We are interesting in its topology, so we can extract out the boundary excitation by

$$\mu(x,y,z) = \begin{cases} -\mu_0, & \text{if } z \ll 0; \\ +\mu_0, & \text{if } z \gg 0, \end{cases} \tag{9.73}$$

with $\mu_0 > 0$. We omit the p^2 terms and thus $\xi_p \approx -\mu(x,y,z)$. It is clear that the normalizable boundary modes have the eigenfunctions in the following subspace as:

$$\Phi_p(a,z) = \exp\left(-\int_0^z \frac{\mu(s)}{\Delta} ds\right) \frac{1}{2} \begin{pmatrix} i \\ \exp(ia) \\ 1 \\ i\exp(ia) \end{pmatrix} \quad \text{with } a \in \mathbb{R}. \tag{9.74}$$

Well, we could directly solve out a so that the vector above is the eigenfunction of BdG Hamiltonian matrix. However, our goal is to understand its gappability constrained from the symmetry transformation. If we solve out the eigenfunction problem, such symmetry information would be vague in the sense that the transformation will be not globally well-defined in the whole momentum space (you can try it) due to nontrivial Berry curvatures. Thus, we would keep ignorant and work within the whole space before diagonalization. Then, the boundary theory subspace above is two-dimensional and can be spanned by $a = 0$ and $a = \pi$. Those two modes are

$$\psi_p \equiv \begin{pmatrix} \Phi^{\dagger}_p(0,0)\Psi_p \\ \Phi^{\dagger}_p(\pi,0)\Psi_p \end{pmatrix}. \tag{9.75}$$

Then the Hamiltonian is

$$\begin{aligned}\mathcal{H}_{\text{BdG:bdry}} &= \frac{1}{2}\sum_p \Psi_p^\dagger[\Phi_p\Phi_p(0,0)^\dagger + \Phi_p\Phi_p(\pi,0)^\dagger] H_{\text{BdG}}[\Phi_p\Phi_p(0,0)^\dagger + \Phi_p\Phi_p(\pi,0)^\dagger]\Psi_p \\ &= \frac{1}{2}\sum_p \psi_p^\dagger(p_x\sigma_z + p_y\sigma_y)\psi_p \left(\equiv \frac{1}{2}\sum_p \psi_p^\dagger H_{\text{BdG:bdry}}(p)\psi_p\right). \end{aligned} \tag{9.76}$$

The charge conjugation redundancy is

$$\psi_p = \psi_p^c \equiv -\sigma_z(\psi_{-p}^\dagger)^T, \tag{9.77}$$

and the time-reversal symmetry is

$$T:\ \psi_p \mapsto -\sigma_y\psi_{-p}. \tag{9.78}$$

Here we use the convention that:

$$T^{-1}c_{p\uparrow}T = -c_{-p\downarrow};\ T^{-1}c_{p\downarrow}T = c_{-p\uparrow};\ T^{-1}iT = -i. \tag{9.79}$$

Charge conjugation allows a mass term $m\sigma_x$ which is prohibited by time reversal. Indeed the symmetries restrict the Hamiltonian **matrix** to satisfy:

$$C:\ \sigma_z H^*_{\text{BdG:bdry}}(-p)\sigma_z = -H_{\text{BdG:bdry}}(p); \tag{9.80}$$

$$T:\ \sigma_y H^*_{\text{BdG:bdry}}(-p)\sigma_y = H_{\text{BdG:bdry}}(p). \tag{9.81}$$

What is tricky is that the time reversal alone has already implied the ingappability even without the charge conjugation. This is natural since the Hamiltonian matrix above is the same as that of the boundary theory on the surface of time-reversal invariant topological insulator where there is no charge conjugation. So it seems that everything is similar to topological insulator? The difference is in that the topological insulator is $\mathbb{Z}_2$ classified which means its boundary could be gapped once we have two copies. However, the situation is different here in that even though we have two copies, the charge conjugation prohibits the gapping mass term! Here is the charge conjugation playing its role, and the classification of topological superconductor in 3D is enhanced to $\mathbb{Z}$ rather than $\mathbb{Z}_2$ of its TI cousin. Furthermore, when the interaction terms are allowed, the classification will be reduced: $\mathbb{Z} \to \mathbb{Z}_{16}$, while TI is still $\mathbb{Z}_2$ classified.

10 Fermionic MONTHLY: 2021.10.1-10.31

Recently, I am trying to have a systematic treatment on relativistic fermions and their path integral in both Minkowskian and Euclidean signatures. I will follow M. Stone's notation in his nice Notes.

The object here is to clarify the symmetry correspondence between two signatures, but I don't know how to make it precise any further to fixing the phase factors due to U(1) and $\mathbb{Z}_2^{\mathrm{F}}$ for Dirac fermion and Majorana fermion, respectively.

In this note, let us restrict to the γ-matrices so that mostly-plus (East-coast) signature:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu};\ \eta = \mathrm{diag}[-1, +1, +1, \cdots], \tag{10.1}$$

in Minkowski. And the Euclidean signature gives:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij},\ \ \left(\gamma^i\right)^\dagger = \gamma^i. \tag{10.2}$$

We won't distinguish both superscripts which ranges from 0 to $D-1$, and $\gamma^{\mu=0} = i\gamma^{i=0}$ can be taken. This convention is good since γ-matrices have the same symmetric property in both signatures. Moreover, in even dimensions, we use the convention on the chiral matrix $\bar{\gamma}$ to be Hermitian and $\bar{\gamma}^2 = 1$.

The most important quantities are $\mathcal{C}$ and $\mathcal{T}$ whose existences depend on the spacetime dimension D, and we gauge them to be **unitary**:

$$\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -\left(\gamma^\mu\right)^T; \tag{10.3}$$

$$\mathcal{T}\gamma^\mu\mathcal{T}^{-1} = \left(\gamma^\mu\right)^T. \tag{10.4}$$

Both $\mathcal{C}$ and $\mathcal{T}$ have definite basis-independent symmetry properties.

In Minkowski, Dirac equation reads:

$$(\gamma\nabla + m)\psi = 0, \tag{10.5}$$

which is an **operator equation** where $\nabla \equiv \partial + iA +$ (gravitational terms). Thus, even without discussing Lagrangian, we can find that the symmetry of the Dirac (operator) equation is equivalent to the symmetry of Hamiltonian (Exercise).

However, when we are Euclidean signature, the symmetry must be defined on the Lagrangian, i.e., through Grassmann integrated variables:

$$\mathcal{L}_E = \bar{\psi}(\gamma\nabla + m)\psi. \text{ (up to some prefactor)} \tag{10.6}$$

When $\mathcal{C}$ is antisymmetric, we can define Majorana fermions with Dirac mass allowed by requiring

$$\psi = \mathcal{C}^{-1}\bar{\psi}^T;\ (\Rightarrow \bar{\psi} = -\psi^T\mathcal{C}). \tag{10.7}$$

When $\mathcal{T}$ is symmetric, we can define Majorana fermions with Dirac mass forbidden by requiring

$$\psi = \mathcal{T}^{-1}\bar{\psi}^T;\ (\bar{\psi} = \psi^T\mathcal{T}). \tag{10.8}$$

Here $^-$ in Minkowski is defined as an antilinear **spinor-mapping** $\bar{\psi} \equiv \psi^\dagger\gamma^0$ while in Euclid $\bar{\psi}$ is just another Grassmann not related to ψ by any operation since there is no conjugation on Grassmann.

10.1 Majorana in $D = 1 \mod 8$: $\mathcal{T}$ sym

There is only one way to impose the Majorana condition: $\psi = \mathcal{T}^{-1}\bar{\psi}^T$ and the Dirac mass is forbidden.

I. Majorana fermion in mostly-plus Minkowski:

1) Time-reversal T is: $(T^{-1}iT = -i)$

$$T^{-1}\psi(t,x)T = i\mathcal{T}\psi(-t,x), \tag{10.9}$$

with $T^2 = 1$ and, up to a sign, the inclusion of the i-prefactor is compulsory to ensure that the kinetic Hamiltonian $i\int d^{D-1}x\psi^T\mathcal{T}\vec{\gamma}\vec{\nabla}\psi$ is invariant.

2) The second time reversal will be CT, but C is identity on Majorana fermions thereby the same as 1).

3) Reflection symmetry is $R_1 : (t, x_1, \cdots) \to (t, -x_1, \cdots)$ for $D > 1$:

$$(R_1)^{-1}\psi(t,x)(R_1) = i\gamma^1\psi(t, -x_1, .), \tag{10.10}$$

and ${R_1}^2 = (-1)^F$. However, we still have ± 1 unknown prefactor here.

II. Majorana fermion in Euclid:

1,2)' Up to $\pm$, the only possibility that time-reversal T is continued to is:

$$T: \ \psi(\tau,x) \to \pm i\gamma^0\psi(-\tau,x), \tag{10.11}$$

so that the Euclidean Lagrangian can be a scalar:

$$T : \psi^T\mathcal{T}\gamma\nabla\psi(\tau,x) \to \psi^T\mathcal{T}\gamma\nabla\psi(-\tau,x). \tag{10.12}$$

Now $T^2 = (-1)^F$ opposite to the Minkowski, which is consistent since continuation does not respect the product of operators due to the antilinearity of T.

3)' The form of R_1 is naturally continued and kept again up to a sign:

$$R_1 : \psi(\tau,x) \to \pm i\gamma^1\psi(\tau, -x_1, .), \tag{10.13}$$

which gives ${R_1}^2 = (-1)^F$, the same as the Minkowskian case.

Additionally, we uniformly use $\pm$ but we don't mean that they are the same value. I don't known how to fix these $\pm$ prefactors which cannot be done in the current framework.

10.2 Dirac in $D = 1 \mod 8$: $\mathcal{T}$ sym

Let us first decompose a Dirac fermion ψ into two Majorana fermions:

$$\psi = \frac{1}{2}(\psi + \mathcal{T}^{-1}\bar{\psi}^T) + i\frac{1}{2i}(\psi - \mathcal{T}^{-1}\bar{\psi}^T) \equiv \psi_1 + i\psi_2; \tag{10.14}$$

$$\Rightarrow \bar{\psi} = \bar{\psi}_1 - i\bar{\psi}_2. \tag{10.15}$$

The charge conjugation $C^{-1}\psi C = \mathcal{T}^{-1}\gamma^{0T}\psi^*$ with $C^{-1}iC = i$ and $\psi^* \equiv \psi^{\dagger T}$ gives $C^{-1}\psi_1 C = \psi_1$ and $C^{-1}\psi_2 C = -\psi_2$, but $(\psi_i)^c \equiv \mathcal{T}^{-1}\gamma^{0T}\psi_i^* = \psi_i$ because $(.)^c$ is just antilinear **spinor mapping** rather than a transformation adjoined by operators. The decomposition above will be useful when we do the analytic continuation to Euclidean signature since we know how to do it for Majorana fermions as before.

I. Dirac fermion in mostly-plus Minkowski:

1) The first time-reversal symmetry is ($T^{-1}iT = -i$)

$$\begin{gathered} T^{-1}\psi(t,x)T = i\mathcal{T}\psi(-t,x), \\ \Rightarrow \begin{cases} T^{-1}\psi_1 T = i\mathcal{T}\psi_1(-\tau); \\ T^{-1}\psi_2 T = -i\mathcal{T}\psi_2(-\tau); \end{cases} \end{gathered} \tag{10.16}$$

$$T^{-1}A_0(t,x)T = A_0(-t,x); \tag{10.17}$$

$$T^{-1}\vec{A}(t,x)T = -\vec{A}(-t,x); \tag{10.18}$$

which means $T^2 = 1$ and Dirac equation transforms is invariant so:

$$m \to m. \tag{10.19}$$

2) The second time-reversal symmetry is CT with

$$(CT)^{-1}\psi(t,x)(CT) = i\gamma^{0*}\psi^*(-t,x) = -i\bar{\psi}^T, \tag{10.20}$$

$$(CT)^{-1}\psi_i(t,x)(CT) = i\gamma^{0*}\psi_i^*(-t,x) = -i\mathcal{T}\psi_i, \tag{10.21}$$

$$(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); \tag{10.22}$$

$$(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); \tag{10.23}$$

and then $(CT)^2 = +1$ and Dirac mass is flipped:

$$m \to -m. \tag{10.24}$$

3) Reflection symmetry is $R_1 : (t, x_1, \cdots) \to (t, -x_1, \cdots)$ for $D > 1$:

$${R_1}^{-1}\psi(t,x)R_1 = i\gamma^1\psi(t,-x_1,.), \tag{10.25}$$

$${R_1}^{-1}\psi_i(t,x)R_1 = i\gamma^1\psi_i(t,-x_1,.), \tag{10.26}$$

$${R_1}^{-1}A_\mu(t,x)R_1 = (-1)^{\delta_{\mu,1}}A_\mu(t,-x_1,.), \tag{10.27}$$

with ${R_1}^2 = (-1)^F$ and the Dirac mass is also flipped: $m \to -m$.

II. Dirac fermion in Euclid:

1)' We assume that the continuation does respect the decomposition (10.14), i.e., C is continued to be still C. By combining Eq. (10.16) and our result in Majorana cases:

$$T : \begin{cases} \psi(\tau,x) \equiv \psi_1 + i\psi_2 \to i\gamma^0\psi_1 - i\gamma^0\psi_2 = i\gamma^0\mathcal{T}^{-1}\bar{\psi}_1^T - i\gamma^0\mathcal{T}^{-1}\bar{\psi}_2^T = i\gamma^0\mathcal{T}^{-1}\bar{\psi}^T(-\tau,x); \\ \bar{\psi}(\tau,x) \equiv \bar{\psi}_1 - i\bar{\psi}_2 = (\psi_1 - i\psi_2)^T\mathcal{T}^T \to i\psi^T(-\tau,x)\mathcal{T}\gamma^0. \end{cases} \tag{10.28}$$

We should note that T is linear now thereby unaffecting i's. By defining R_τ as

$$R_\tau : \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}(\tau,x) \to \begin{pmatrix} i\gamma^0\psi \\ i\bar{\psi}\gamma^0 \end{pmatrix}(-\tau,x), \tag{10.29}$$

we find T is continued to CR_τ. Indeed,

$$CR_\tau : \psi(\tau,x) \xrightarrow{C} \mathcal{T}^{-1}\bar{\psi}^T(\tau,x) \xrightarrow{R_\tau} i\mathcal{T}^{-1}\gamma^{0T}\bar{\psi}^T(-\tau,x); \tag{10.30}$$
$$CR_\tau : \bar{\psi}(\tau,x) \xrightarrow{C} \psi^T(\tau,x)\mathcal{T} \xrightarrow{R_\tau} i\psi^T(-\tau,x)\gamma^{0T}\mathcal{T}, \tag{10.31}$$

which implies that $(CR_\tau)^2 = (-1)^F$ and Dirac mass term is respected:

$$T \mapsto CR_\tau : \bar{\psi}m\psi \to \bar{\psi}m\psi. \tag{10.32}$$

Here $T^2 = 1$ is continued to $(-1)^F$ as expected from our earlier result on Majorana fermions. It is also natural once noting that gauge fields transform covariantly as 1-form under R_τ.

2)' CT: Equation (10.21) and the earlier result on Majorana fermions give

$$CT : \begin{cases} \psi(\tau,x) \equiv \psi_1 + i\psi_2 \to -i\gamma^0\psi(-\tau,x); \\ \bar{\psi}(\tau,x) \equiv \bar{\psi}_1 - i\bar{\psi}_2 = (\psi_1 - i\psi_2)^T\mathcal{T}^T \to -i\bar{\psi}(-\tau,x)\gamma^0, \end{cases} \tag{10.33}$$

i.e., CT should be continued to R_τ and indeed it has the expected mass flipping:

$$CT \mapsto R_\tau : \bar{\psi}m\psi \to -\bar{\psi}m\psi, \tag{10.34}$$

and $(CT)^2 = (-1)^F$ has an opposite sign before continuation.

3)' R_1 is naturally continued to R_1 with the same form as in Minkowski.

In the current dimension, we are lucky to fix the prefactor up to $\pm$-sign due to the existence of Majorana fermion. However, in some other dimensions like $D = 5,6,7 \mod 8$ not Majorana permitted, I don't know how to do it properly.

10.3 Majorana in $D = 0 \mod 8$: $\mathcal{C}, \mathcal{T}$ sym

Since $\mathcal{C}$ is symmetric, there is also only one way to impose the Majorana condition: $\psi = \mathcal{T}^{-1}\bar{\psi}^T$ and the Dirac mass is forbidden. The situation is totally the same as above $D = 1 \mod 8$ cases, except that we have one more possibility of time reversal.

I. Majorana fermion in mostly-plus Minkowski:

4) Time-reversal T_5: $(T_5^{-1}iT_5 = -i)$

$$T_5^{-1}\psi(t,x)T_5 = \mathcal{T}\bar{\gamma}\psi(-t,x), \tag{10.35}$$

which means ${T_5}^2 = +1$ since $\bar{\gamma}^2 = 1$ and $\mathcal{T}\bar{\gamma}\mathcal{T}^{-1} = \bar{\gamma}^T$. Here the subscript "5" is used since in $D = 4$, $\bar{\gamma}$ is denoted by γ^5.

II. Majorana fermion in Euclid:

4)' T_5 is continued to:

$$T_5 : \ \psi(\tau,x) \to \gamma^0\bar{\gamma}\psi(-\tau,x), \tag{10.36}$$

where we have suppressed the $\pm$ ambiguity above. In addition, ${T_5}^2 = (-1)^F$ in the Euclidean signature as expected.

Similarly, we can also combine $\bar{\gamma}$ with spatial reflections in a direct way.

10.4 Dirac in $D = 0 \mod 8$: $\mathcal{C}, \mathcal{T}$ sym

For Dirac fermions, we can say something more, but we just present the simplest case:

I. Dirac fermion in mostly-plus Minkowski:

4)+1) It is straightforward to take

$$T_5^{-1}\psi(t,x)T_5 = \mathcal{T}\bar{\gamma}\psi(-t,x),$$
$$\Rightarrow \begin{cases} T_5^{-1}\psi_1 T_5 = \mathcal{T}\bar{\gamma}\psi_1(-\tau); \\ T_5^{-1}\psi_2 T_5 = -\mathcal{T}\bar{\gamma}\psi_2(-\tau); \end{cases} \tag{10.37}$$

$$T_5^{-1}A_0(t,x)T_5 = A_0(-t,x); \tag{10.38}$$

$$T_5^{-1}\vec{A}(t,x)T_5 = -\vec{A}(-t,x); \tag{10.39}$$

which means $T_5^2 = 1$, but the Dirac mass is flipped

$$T_5^{-1}i\bar{\psi}m\psi T_5 = -i\bar{\psi}m\psi. \tag{10.40}$$

Moreover, we can consider Weyl mass term:

$$\begin{cases} T_5^{-1}m_w\bar{\psi}\bar{\gamma}\psi T_5 = m_w\bar{\psi}\bar{\gamma}\psi(-t,x); \\ T^{-1}m_w\bar{\psi}\bar{\gamma}\psi T = -m_w\bar{\psi}\bar{\gamma}\psi(-t,x). \end{cases} \tag{10.41}$$

It is generally true that the inclusion of $\bar{\gamma}$ changes both the transformation rules of Dirac and Weyl masses, (and the rule of Weyl mass flippling is always opposite to Dirac mass — I haven't checked this point), so we omit other cases. Furthermore, the invariance of the Weyl mass under T_5 is dual to the invariance of the Dirac mass under T.

I'. Dirac fermion in Euclid:

4)' By the result of Majorana, T_5 is continued to

$$T_5 : \begin{cases} \psi(\tau,x) \to \bar{\gamma}\gamma^0\mathcal{T}^{-1}\bar{\psi}^T(-\tau,x); \\ \bar{\psi}(\tau,x) \to \psi^T(-\tau,x)\mathcal{T}\gamma^0\bar{\gamma}, \end{cases} \tag{10.42}$$

with ${T_5}^2 = (-1)^F$. Actually, after noting that the product of $\mathcal{T}$ and $\mathcal{C}$ is proportional to $\bar{\gamma}$, we find that T and T_5 are dual to each other simply by replacing $\mathcal{T}$ in C by $\mathcal{C}$.

10.5 Majorana in $D = 2 \mod 8$: $\mathcal{C}$ antisym, $\mathcal{T}$ sym

This dimension is special in that it supports Majorana-Weyl fermions, but we don't want to discuss this topic here. Moreover, the result here can be seen as a hybridization of $D = 0$ and $D = 1$, except that now

$$\mathcal{C}\bar{\gamma}\mathcal{C}^{-1} = \mathcal{T}\bar{\gamma}\mathcal{T}^{-1} = -\bar{\gamma}^T. \tag{10.43}$$

Therefore, the only difference happens in T_5 as below.

I. Majorana fermion in mostly-plus Minkowski:

4) Time-reversal T_5: ($T_5^{-1}iT_5 = -i$)

$$T_5^{-1}\psi(t,x)T_5 = i\mathcal{T}\bar{\gamma}\psi(-t,x), \tag{10.44}$$

which means ${T_5}^2 = (-1)^F$ due to that now $\mathcal{T}\bar{\gamma}\mathcal{T}^{-1} = -\bar{\gamma}^T$ rather than the added i.

This change has a significant implication that the theory and T_5 can describe spinful electrons and their time-reversal symmetry in condensed matter, and indeed it can describe the boundary theory of time-reversal invariant topological superconductor. On the other side, T can only describe the time reversal of spin-polarized electrons.

II. Majorana fermion in Euclid:

4)' T_5 is continued to:

$$T_5: \ \psi(\tau, x) \to i\gamma^0\bar{\gamma}\psi(-\tau, x), \tag{10.45}$$

and ${T_5}^2 = +1$ in the Euclidean signature as expected.

Remark: There is one more potential confusion that, if we were using $\mathcal{C}$ to define Majorana condition, we would be permitted to have a Dirac mass, which seems unfair since here we are forbidden to add Dirac mass. Actually, the point is that Majorana by $\mathcal{T}$ allows Weyl mass, while Majorana by $\mathcal{C}$ does not! Thus you can see that Dirac and Weyl masses are dual to each other when we swap the definitions of Majorana between $\mathcal{T}$ and $\mathcal{C}$. Therefore, it is imprecise to state something like "Majorana by $\mathcal{T}$ is enforced to be massless".

10.6 Dirac in $D = 0 \mod 8$: $\mathcal{C}$ antisym, $\mathcal{T}$ sym

The Dirac fermion also has the same change on ${T_5}^2$.

I. Dirac fermion in mostly-plus Minkowski:

4) It is again direct to take

$$\begin{aligned}
&T_5^{-1}\psi(t, x)T_5 = i\mathcal{T}\bar{\gamma}\psi(-t, x), \\
\Rightarrow &\begin{cases} T_5^{-1}\psi_1 T_5 = i\mathcal{T}\bar{\gamma}\psi_1(-\tau); \\ T_5^{-1}\psi_2 T_5 = -i\mathcal{T}\bar{\gamma}\psi_2(-\tau); \end{cases}
\end{aligned} \tag{10.46}$$

$$T_5^{-1}A_0(t, x)T_5 = A_0(-t, x); \tag{10.47}$$

$$T_5^{-1}\vec{A}(t, x)T_5 = -\vec{A}(-t, x); \tag{10.48}$$

which means ${T_5}^2 = (-1)^F$, and both the Dirac and Weyl masses are flipped

$$T_5^{-1} i\bar{\psi}m\psi T_5 = -i\bar{\psi}m\psi(-t, x). \tag{10.49}$$

$$T_5^{-1} m_w\bar{\psi}\bar{\gamma}\psi T_5 = -m_w\bar{\psi}\bar{\gamma}\psi(-t, x). \tag{10.50}$$

Indeed, the theory describes the boundary theory of quantum spin Hall which is known as time-reversal invariant topological insulator. (It should be noted that although its name "spin", the conserved U(1) charge is still electron charge rather than spin-z.)

I'. Dirac fermion in Euclid:

4)' By the result of Majorana, T_5 is continued to

$$T_5: \begin{cases} \psi(\tau, x) \to i\bar{\gamma}\gamma^0\mathcal{T}^{-1}\bar{\psi}^T(-\tau, x); \\ \bar{\psi}(\tau, x) \to i\psi^T(-\tau, x)\mathcal{T}^{-1}\gamma^0\bar{\gamma}. \end{cases} \tag{10.51}$$

with ${T_5}^2 = 1$, and T and T_5 are again dual to each other simply by replacing $\mathcal{T}$ in C by $\mathcal{C}$.

10.7 Majorana in $D = 3, 4 \mod 8$: $\mathcal{C}, \mathcal{T}(4)$ antisym

We consider together $D = 3$ and $D = 4 \mod 8$ and the result related to $\bar{\gamma}$ is irrelevant to $D = 3 \mod 8$ implicitly.

I. Majorana in mostly-plus Minkowski:

Now we go back to our reality $D = 4(\text{mod } 8)$. Here the Majorana condition can only imposed by $\mathcal{C}$ rather than $\mathcal{T}$ since $\mathcal{T}$ is antisymmetric:

$$\psi = \mathcal{C}^{-1}\bar{\psi}^T; \ \Rightarrow \bar{\psi} = -\psi^T\mathcal{C}. \tag{10.52}$$

Differently from the Majorana defined by $\mathcal{T}$ before, we can add a Dirac mass now. Actually, we can also add a Weyl mass as well by checking the Table in M. Stone's Notes. Thus the full Dirac (operator) equation turns out to be

$$(\gamma\nabla + m + im_w\bar{\gamma})\,\psi(t,x) = 0. \tag{10.53}$$

1) Time-reversal T_5: $(T_5^{-1}iT_5 = -1)$

$$T_5^{-1}\psi(t,x)T_5 = \mathcal{C}\psi(-t,x), \tag{10.54}$$

with ${T_5}^2 = (-1)^F$. It is straightforward to read from the Dirac equation above that

$$T_5 : m \to -m; \ m_w \to m_w. \tag{10.55}$$

Here, no additional i factor is needed since the kinetic Hamiltonian is already invariant.

2) Time-reversal T: $(T^{-1}iT = -1)$

$$T^{-1}\psi(t,x)T = i\mathcal{C}\bar{\gamma}\psi(-t,x), \tag{10.56}$$

with $T^2 = (-1)^F$ due to that $\mathcal{C}\bar{\gamma}\mathcal{C}^{-1} = \bar{\gamma}^T$ rather than the additional i factor which is just to ensure that the kinetic Hamiltonian to commute with T. The flipping rule for masses is:

$$T : m \to m; \ m_w \to -m_w. \tag{10.57}$$

3) Reflection R_1:

$${R_1}^{-1}\psi(t,x)R_1 = i\bar{\gamma}\gamma^1\psi(t,-x_1,.), \tag{10.58}$$

which means that ${R_1}^2 = 1$. The mass terms change as:

$$R_1 : m \to m; \ m_w \to -m_w. \tag{10.59}$$

Of course, we could also include $\bar{\gamma}$ again which changes both the flipping rules of the Dirac mass and Weyl mass, but the square property does not change.

II. Majorana in Euclid:

1)' T_5 should be continued to:

$$T_5 : \psi(\tau,x) \to \pm\gamma^0\psi(-\tau,x), \tag{10.60}$$

so that the masses can transform in the same way as the Minkowskian correspondence. Then ${T_5}^2 = 1$.

2)' The continuation of T is then obtained as:

$$T : \psi(\tau, x) \to \pm i\bar{\gamma}\gamma^0\psi(-\tau, x), \tag{10.61}$$

with $T^2 = 1$.

3)' R_1 is still of the same form after continuation:

$$R_1 : \psi(\tau, x) \to \pm i\bar{\gamma}\gamma^1\psi(\tau, -x_1, .), \tag{10.62}$$

with a sign ambiguity, and ${R_1}^2 = 1$.

10.8 Dirac in $D = 3, 4 \mod 8$: $\mathcal{C}, \mathcal{T}$ antisym

Let us first decompose a Dirac fermion ψ into two Majorana fermions:

$$\psi = \frac{1}{2}(\psi + \mathcal{C}^{-1}\bar{\psi}^T) + i\frac{1}{2i}(\psi - \mathcal{C}^{-1}\bar{\psi}^T) \equiv \psi_1 + i\psi_2; \tag{10.63}$$

$$\Rightarrow \bar{\psi} = \bar{\psi}_1 - i\bar{\psi}_2 = (\psi_1 - i\psi_2)^T\mathcal{C}^T. \tag{10.64}$$

The charge conjugation $C^{-1}\psi C = \mathcal{C}^{-1}\gamma^{0T}\psi^*$ with $C^{-1}iC = i$ and $\psi^* \equiv \psi^{\dagger T}$ gives $C^{-1}\psi_1 C = \psi_1$ and $C^{-1}\psi_2 C = -\psi_2$, but $(\psi_i)^c \equiv \mathcal{C}^{-1}\gamma^{0T}\psi_i^* = \psi_i$ since again $(.)^c$ is only a spinor mapping rather than a transformation adjoined by operators.

I. Dirac fermion in mostly-plus Minkowski:

1) The first time-reversal symmetry is ($T_5^{-1}iT_5 = -i$)

$$T_5^{-1}\psi(t,x)T_5 = \mathcal{C}\psi(-t,x),$$
$$\Rightarrow \begin{cases} T_5^{-1}\psi_1 T_5 = \mathcal{C}\psi_1(-\tau); \\ T_5^{-1}\psi_2 T_5 = -\mathcal{C}\psi_2(-\tau); \end{cases} \tag{10.65}$$

$$T_5^{-1}A_0(t,x)T_5 = A_0(-t,x); \tag{10.66}$$

$$T_5^{-1}\vec{A}(t,x)T_5 = -\vec{A}(-t,x); \tag{10.67}$$

which, by combining our earlier results on Majorana and Eq. (10.70), gives:

$$T_5 : m \to -m;\ m_w \to m_w; \tag{10.68}$$

$${T_5}^2 = (-1)^F. \tag{10.69}$$

2) The second time-reversal symmetry is ($T^{-1}iT = -i$)

$$T^{-1}\psi(t,x)T = i\mathcal{C}\bar{\gamma}\psi(-t,x),$$
$$\Rightarrow \begin{cases} T^{-1}\psi_1 T = i\mathcal{C}\bar{\gamma}\psi_1(-\tau); \\ T^{-1}\psi_2 T = -i\mathcal{C}\bar{\gamma}\psi_2(-\tau); \end{cases} \tag{10.70}$$

$$T^{-1}A_0(t,x)T = A_0(-t,x); \tag{10.71}$$

$$T^{-1}\vec{A}(t,x)T = -\vec{A}(-t,x); \tag{10.72}$$

which, by combining our earlier results on Majorana and Eq. (10.70), gives:

$$T : m \to m;\ m_w \to -m_w; \tag{10.73}$$
$$T^2 = (-1)^F. \tag{10.74}$$

3) The third time-reversal symmetry is $[(CT)^{-1} iCT = -i]$:

$$(CT)^{-1}\psi(t,x)(CT) = i\gamma^{0*}\bar{\gamma}^*\psi^*(-t,x), \tag{10.75}$$
$$(CT)^{-1}\psi_i(t,x)(CT) = i\gamma^{0*}\bar{\gamma}^*\psi_i^*(-t,x), \tag{10.76}$$
$$(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); \tag{10.77}$$
$$(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); \tag{10.78}$$

which gives

$$CT : m \to m;\ m_w \to -m_w; \tag{10.79}$$
$$(CT)^2 = (-1)^F. \tag{10.80}$$

4) Reflection symmetry R_1:

$$R_1{}^{-1}\psi(t,x)R_1 = i\bar{\gamma}\gamma^1\psi(t,-x_1,.), \tag{10.81}$$
$$R_1{}^{-1}\psi_i(t,x)R_1 = i\bar{\gamma}\gamma^1\psi_i(t,-x_1,.), \tag{10.82}$$
$$R_1{}^{-1}A_\mu(t,x)R_1 = (-1)^{\delta_{\mu,1}}A_\mu(t,-x_1,.), \tag{10.83}$$

with $R_1{}^2 = 1$ and

$$R_1 : m \to m;\ m_w \to -m_w. \tag{10.84}$$

II. Dirac fermion in Euclid:

1,2)' We assume that the continuation does respect the decomposition (10.63), i.e., C is continued to be still C. By combining Eq. (10.70) and our result in Majorana cases:

$$T : \begin{cases} \psi(\tau,x) = \psi_1 + i\psi_2 \to i\bar{\gamma}\gamma^0\psi_1 + \bar{\gamma}\gamma^0\psi_2 = i\bar{\gamma}\gamma^0\mathcal{C}^{-1}\bar{\psi}^T(-\tau,x); \\ \bar{\psi}(\tau,x) \equiv \bar{\psi}_1 - i\bar{\psi}_2 = (\psi_1 - i\psi_2)^T\mathcal{C}^T \to i\psi^T(-\tau,x)\mathcal{C}\gamma^0\bar{\gamma}. \end{cases} \tag{10.85}$$

It implies that $T^2 = 1$ and yields the correct mass transformation rule:

$$T : m \to m;\ m_w \to -m_w. \tag{10.86}$$

The continuation of T_5 can be also done similarly by comparing the continuation of Majorana fermion and the decomposition of Dirac fermion into two Majorana fermions:

$$T_5 : \begin{cases} \psi \to \gamma^0\mathcal{C}^{-1}\bar{\psi}^T(-\tau,x); \\ \bar{\psi} \to \psi^T\mathcal{C}\gamma^0, \end{cases} \tag{10.87}$$

with $T_5{}^2 = 1$ and it has the **opposite** rule of mass flipping to that of T's above.

3)' The continuation of CT symmetry can be also similarly done:

$$CT:\begin{cases}\psi(\tau,x)=\psi_1+i\psi_2\to i\bar{\gamma}\gamma^0\psi_1-\bar{\gamma}\gamma^0\psi_2=i\bar{\gamma}\gamma^0\psi(-\tau,x);\\ \bar{\psi}(\tau,x)\equiv\bar{\psi}_1-i\bar{\psi}_2\to i\bar{\psi}(-\tau,x)\bar{\gamma}\gamma^0,\end{cases} \tag{10.88}$$

with $(CT)^2=1$ and the correct mass transformation rules.

4)' The reflection R_1 is continued naturally:

$$R_1:\begin{cases}\psi=\psi_1+i\psi_2\mapsto i\bar{\gamma}\gamma^1\psi(\tau,-x_1,.)\\ \bar{\psi}\equiv\bar{\psi}_1-i\bar{\psi}_2\mapsto i\bar{\psi}\bar{\gamma}\gamma^1\end{cases} \tag{10.89}$$

with ${R_1}^2=1$ and as expected

$$R_1:m\to m;\ m_w\to -m_w. \tag{10.90}$$

10.9 Dirac fermion in $D=5,6 \mod 8$, $\mathcal{T}$ antisym, $\mathcal{C}(6)$ sym

In the present dimensions, no Majorana fermion exists, but I have found a more fundamental method to deal with it. The nature of the Majorana approach before is that we actually, implicitly, allow U(1)-breaking terms. Thus, here we simply break U(1) symmetry by the following Hamiltonian density:

$$\mathcal{H}=\mathcal{H}_{\text{Dirac}}+\Delta\mathcal{H}, \tag{10.91}$$

where $\mathcal{H}_{\text{Dirac}}=i\bar{\psi}(\gamma^j\partial_j+m)\psi$ is the massive Dirac Hamiltonian for two identical Dirac fermions, and their **Hermitian** interaction:

$$\Delta\mathcal{H}=\psi^T\mathcal{T}\psi-\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T. \tag{10.92}$$

Due to the existence of $\mathcal{T}$, we can still have the charge conjugation:

$$C^{-1}\psi C=\mathcal{T}^{-1}\bar{\psi}^T\Rightarrow C^{-1}\bar{\psi}^TC=-\mathcal{T}\psi, \tag{10.93}$$

which flips the Dirac mass $C:m\to -m$, but keeps $\Delta\mathcal{H}$ invariant.

I. Dirac fermions in mostly-plus Minkowski

1) The first time-reversal symmetry is $(T^{-1}iT=-i)$:

$$\begin{aligned}T^{-1}\psi(t,x)T&=\eta_T\mathcal{T}\psi(-t,x),\\ \Rightarrow T^{-1}\bar{\psi}^TT&=\eta_T^*\mathcal{T}^{-1}\bar{\psi}^T; \qquad (10.94)\\ T^{-1}A_0(t,x)T&=A_0(-t,x); \qquad (10.95)\\ T^{-1}\vec{A}(t,x)T&=-\vec{A}(-t,x); \qquad (10.96)\end{aligned}$$

which means $T^2=(-1)^F$ and Dirac mass is invariant:

$$m\to m. \tag{10.97}$$

Furthermore,

$$T^{-1}\Delta\mathcal{H}T=\eta_T^2\psi^T\mathcal{T}\psi-\eta_T^{*2}\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T. \tag{10.98}$$

2) The second time-reversal symmetry is CT with

$$
\begin{aligned}
&(CT)^{-1}\psi(t,x)(CT) = -\eta_T^*\bar{\psi}^T, && (10.99)\\
&\Rightarrow (CT)^{-1}\bar{\psi}^T(CT) = \eta_T\psi, && (10.100)\\
&(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); && (10.101)\\
&(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); && (10.102)
\end{aligned}
$$

and then $(CT)^2 = -\eta_T^2$ on ψ and Dirac mass is flipped $m \to -m$, and

$$
(CT)^{-1}\Delta\mathcal{H}(CT) = \eta_T^2\psi^T\mathcal{T}\psi - \eta_T^{*2}\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T. \tag{10.103}
$$

3) Reflection symmetry R_1:

$$
\begin{aligned}
{R_1}^{-1}\psi(t,x)R_1 &= \eta_R\gamma^1\psi(t,-x_1,.), && (10.104)\\
{R_1}^{-1}A_\mu(t,x)R_1 &= (-1)^{\delta_{\mu,1}}A_\mu(t,-x_1,.), && (10.105)
\end{aligned}
$$

with ${R_1}^2 = \eta_R^2$ and

$$
{R_1}^{-1}\Delta\mathcal{H}R_1 = \eta_R^2\psi^T\mathcal{T}\psi - \eta_R^{*2}\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T. \tag{10.106}
$$

I'. Dirac fermions in Euclid:

We need to assume that the form of ΔH is unchanged when continued to Euclid.

1') Time reversal T is naturally continued to

$$
T: \begin{cases} \psi(\tau,x) \to \lambda_T\gamma^0\mathcal{T}^{-1}\bar{\psi}^T \\ \bar{\psi}(\tau,x) \to -\lambda_T^*\psi^T\mathcal{T}\gamma^0, \end{cases} \tag{10.107}
$$

which gives $T^2 = 1$ independing on λ_T, while λ_T is fixed by the transformation rule of $\Delta\mathcal{H}$:

$$
T: \Delta\mathcal{H} \to \lambda_T^{*2}\psi^T\mathcal{T}\psi - \lambda_T^2\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T, \tag{10.108}
$$

which can be the same as its Minkowski counterpart only if

$$
\lambda_T^2 = \eta_T^{*2}. \tag{10.109}
$$

2') The second time reversal CT is naturally continued to

$$
CT: \begin{cases} \psi(\tau,x) \to \lambda_{CT}\gamma^0\psi; \\ \bar{\psi}(\tau,x) \to -\lambda_{CT}^*\bar{\psi}\gamma^0, \end{cases} \tag{10.110}
$$

and $(CT)^2 = \exp(2F\ln\lambda_{CT})$ or simply its effect on ψ as $(CT)^2 = \lambda_{CT}^2$, and

$$
CT: \Delta\mathcal{H} \to \lambda_{CT}^2\psi^T\mathcal{T}\psi - \lambda_{CT}^{*2}\bar{\psi}\mathcal{T}^{-1}\bar{\psi}^T, \tag{10.111}
$$

which is the same as the Minkowski case only if

$$
\lambda_{CT}^2 = \eta_T^2, \tag{10.112}
$$

which means $(CT)^2$ must be opposite after continuation, independing on η_T!

3)' Reflection R_1 is continued to

$$R_1 : \begin{cases} \psi \mapsto \lambda_R \gamma^1 \psi(\tau, -x_1, .) \\ \bar{\psi} \mapsto -\lambda_R^* \bar{\psi}(\tau, -x_1, .)\gamma^1 \end{cases} \tag{10.113}$$

with ${R_1}^2 = \lambda_R^2$ and

$$R_1 : \Delta\mathcal{H} \mapsto \lambda_R^2 \psi^T \mathcal{T} \psi - \lambda_R^{*2} \bar{\psi}^T \mathcal{T}^{-1} \bar{\psi}^T, \tag{10.114}$$

which compared with the Minkowski case gives:

$$\eta_R^2 = \lambda_R^2, \tag{10.115}$$

thereby ${R_1}^2$ is invariant under continuation.

In $D = 6 \mod 8$, we could have also mixed $\bar{\gamma}$ into various symmetries above to create something like T_5 etc., but as we have seen before, such mixtures are equivalent to replacing $\mathcal{T}$ by $\mathcal{C}$ (sym) everywhere, of which the situation we will consider in the next part $D = 7 \mod 8$ where, though, no concept of $\bar{\gamma}$ exists (only the formal expression is useful to us).

10.10 Dirac fermion in $D = 7 \mod 8$, $\mathcal{C}$ sym

Here we will omit R_1 which is basically the same as the last part. We define the charge conjugation as $C^{-1}\psi C = \mathcal{C}^{-1}\bar{\psi}^T \Rightarrow C^{-1}\bar{\psi}^T C = -\mathcal{C}\psi$. Let us consider two identical species:

$$\mathcal{H} = \mathcal{H}_{\text{Dirac}} + \Delta\mathcal{H}, \tag{10.116}$$

where $\mathcal{H}_{\text{Dirac}} \equiv \sum_{n=1,2} i\bar{\psi}_n(\gamma^j\partial_j + m)\psi_n$. Furthermore, the interaction between two species, consistently with Pauli principle, becomes

$$\begin{aligned} \Delta\mathcal{H} &= \psi_1^T \mathcal{C}\psi_2 + \bar{\psi}_2 \mathcal{C}^{-1}\bar{\psi}_1^T \\ &= -\psi_2^T \mathcal{C}\psi_1 - \bar{\psi}_1 \mathcal{C}^{-1}\bar{\psi}_2^T. \end{aligned} \tag{10.117}$$

I. Dirac fermion in Minkowski:

1) The first time-reversal symmetry is $(T^{-1}iT = -i)$ and identically for each species:

$$\begin{aligned} T^{-1}\psi(t,x)T &= \eta_T \mathcal{C}\psi(-t,x), \\ \Rightarrow T^{-1}\bar{\psi}^T T &= \eta_T^* \mathcal{C}^{-1}\bar{\psi}^T; \end{aligned} \tag{10.118}$$

$$T^{-1}A_0(t,x)T = A_0(-t,x); \tag{10.119}$$

$$T^{-1}\vec{A}(t,x)T = -\vec{A}(-t,x); \tag{10.120}$$

which means $T^2 = 1$ and Dirac mass is flipped:

$$m \to -m. \tag{10.121}$$

Furthermore,

$$T^{-1}\Delta\mathcal{H}T = \eta_T^2 \psi_1^T \mathcal{C}\psi_2 + \eta_T^{*2}\bar{\psi}_2 \mathcal{C}^{-1}\bar{\psi}_1^T. \tag{10.122}$$

2) The second time-reversal symmetry is CT with

$$(CT)^{-1}\psi(t,x)(CT) = \eta_T^*\bar{\psi}^T, \tag{10.123}$$

$$\Rightarrow (CT)^{-1}\bar{\psi}^T(CT) = -\eta_T\psi, \tag{10.124}$$

$$(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); \tag{10.125}$$

$$(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); \tag{10.126}$$

and then $(CT)^2 = -\eta_T^2$ on ψ and Dirac mass is flipped $m \to -m$, and

$$(CT)^{-1}\Delta\mathcal{H}(CT) = -\eta_T^2\psi_1^T\mathcal{C}\psi_2 - \eta_T^{*2}\bar{\psi}_2\mathcal{C}^{-1}\bar{\psi}_1^T. \tag{10.127}$$

I'. Dirac fermions in Euclid:

We need to assume that the form of ΔH is unchanged when continued to Euclid.

1') Time reversal T is naturally continued to

$$T: \begin{cases} \psi(\tau,x) \to \lambda_T\gamma^0\mathcal{C}^{-1}\bar{\psi}^T \\ \bar{\psi}(\tau,x) \to \lambda_T^*\psi^T\mathcal{C}\gamma^0, \end{cases} \tag{10.128}$$

which gives $T^2 = (-1)^F$ independing on λ_T. λ_T is fixed by the transformation rule of $\Delta\mathcal{H}$:

$$T: \Delta\mathcal{H} \to \lambda_T^{*2}\psi_1^T\mathcal{C}\psi_2 + \lambda_T^2\bar{\psi}_2^T\mathcal{C}^{-1}\bar{\psi}_1^T, \tag{10.129}$$

which can be the same as its Minkowski counterpart only if

$$\lambda_T^2 = \eta_T^{*2}. \tag{10.130}$$

2') The second time reversal CT is naturally continued to

$$CT: \begin{cases} \psi(\tau,x) \to \lambda_{CT}\gamma^0\psi; \\ \bar{\psi}(\tau,x) \to -\lambda_{CT}^*\bar{\psi}\gamma^0, \end{cases} \tag{10.131}$$

and $(CT)^2 = \exp(2F\ln\lambda_{CT})$ or simply its effect on ψ as $(CT)^2 = \lambda_{CT}^2$, and

$$CT: \Delta\mathcal{H} \to -\lambda_{CT}^2\psi_1^T\mathcal{C}\psi_2 - \lambda_{CT}^{*2}\bar{\psi}_2\mathcal{C}^{-1}\bar{\psi}_1^T, \tag{10.132}$$

which is the same as the Minkowski case only if

$$\lambda_{CT}^2 = \eta_T^2, \tag{10.133}$$

which means $(CT)^2$ must be opposite after continuation, independing on η_T as expected.

10.11 Supplemental: Dirac fermion with $\mathcal{T}$ sym

To make the previous U(1)-symmetry breaking approach complete, we will apply to the case with $\mathcal{T}$ symmetric and

$$\begin{aligned}\Delta\mathcal{H} &= \psi_1^T\mathcal{T}\psi_2 - \bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1{}^T \\ &= -\psi_2^T\mathcal{T}\psi_1 + \bar{\psi}_1\mathcal{T}^{-1}\bar{\psi}_2^T.\end{aligned} \tag{10.134}$$

Due to the existence of $\mathcal{T}$, we can still have the charge conjugation:

$$C^{-1}\psi C = \mathcal{T}^{-1}\bar{\psi}^T \Rightarrow C^{-1}\bar{\psi}^T C = \mathcal{T}\psi, \tag{10.135}$$

which flips the Dirac mass $C : m \to -m$, and keeps $\Delta\mathcal{H}$ invariant.

I. Dirac fermions in mostly-plus Minkowski

1) The first time-reversal symmetry is ($T^{-1}iT = -i$) and identically for each species:

$$\begin{aligned} T^{-1}\psi(t,x)T &= \eta_T \mathcal{T}\psi(-t,x), \\ \Rightarrow T^{-1}\bar{\psi}^T T &= -\eta_T^* \mathcal{T}^{-1}\bar{\psi}^T; \end{aligned} \tag{10.136}$$

$$T^{-1}A_0(t,x)T = A_0(-t,x); \tag{10.137}$$

$$T^{-1}\vec{A}(t,x)T = -\vec{A}(-t,x); \tag{10.138}$$

which means $T^2 = 1$ and Dirac mass is invariant:

$$m \to m. \tag{10.139}$$

Furthermore,

$$T^{-1}\Delta\mathcal{H}T = \eta_T^2 \psi_1^T \mathcal{T}\psi_2 - \eta_T^{*2}\bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1^T. \tag{10.140}$$

2) The second time-reversal symmetry is CT with

$$(CT)^{-1}\psi(t,x)(CT) = -\eta_T^*\bar{\psi}^T, \tag{10.141}$$

$$\Rightarrow (CT)^{-1}\bar{\psi}^T(CT) = \eta_T\psi, \tag{10.142}$$

$$(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); \tag{10.143}$$

$$(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); \tag{10.144}$$

and then $(CT)^2 = -\eta_T^2$ on ψ and Dirac mass is flipped $m \to -m$, and

$$(CT)^{-1}\Delta\mathcal{H}(CT) = \eta_T^2\psi_1^T\mathcal{T}\psi_2 - \eta_T^{*2}\bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1^T. \tag{10.145}$$

3) Reflection symmetry R_1:

$${R_1}^{-1}\psi(t,x)R_1 = \eta_R\gamma^1\psi(t,-x_1,.), \tag{10.146}$$

$${R_1}^{-1}A_\mu(t,x)R_1 = (-1)^{\delta_{\mu,1}}A_\mu(t,-x_1,.), \tag{10.147}$$

with ${R_1}^2 = \eta_R^2$ and

$${R_1}^{-1}\Delta\mathcal{H}R_1 = \eta_R^2\psi_1^T\mathcal{T}\psi_2 - \eta_R^{*2}\bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1^T. \tag{10.148}$$

I'. Dirac fermions in Euclid:

We need to assume that the form of ΔH is unchanged when continued to Euclid.

1') Time reversal T is naturally continued to

$$T : \begin{cases} \psi(\tau,x) \to \lambda_T\gamma^0\mathcal{T}^{-1}\bar{\psi}^T \\ \bar{\psi}(\tau,x) \to -\lambda_T^*\psi^T\mathcal{T}\gamma^0, \end{cases} \tag{10.149}$$

which gives $T^2 = (-1)^F$ independing on λ_T that is fixed by the transformation rule of $\Delta\mathcal{H}$:

$$T : \Delta\mathcal{H} \to \lambda_T^{*2}\psi_1^T\mathcal{T}\psi_2 - \lambda_T^2\bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1^T, \tag{10.150}$$

which can be the same as its Minkowski counterpart only if

$$\lambda_T^2 = \eta_T^{*2}. \tag{10.151}$$

2') The second time reversal CT is naturally continued to

$$CT : \begin{cases} \psi(\tau, x) \to \lambda_{CT}\gamma^0\psi; \\ \bar{\psi}(\tau, x) \to -\lambda_{CT}^*\bar{\psi}\gamma^0, \end{cases} \tag{10.152}$$

and $(CT)^2 = \exp(2F\ln\lambda_{CT})$ or simply its effect on ψ as $(CT)^2 = \lambda_{CT}^2$, and

$$CT : \Delta\mathcal{H} \to \lambda_{CT}^2\psi_1^T\mathcal{T}\psi_2 - \lambda_{CT}^{*2}\bar{\psi}_2\mathcal{T}^{-1}\bar{\psi}_1^T, \tag{10.153}$$

which is the same as the Minkowski case only if

$$\lambda_{CT}^2 = \eta_T^2, \tag{10.154}$$

which means $(CT)^2$ must be opposite after continuation, independing on η_T!

3)' Reflection R_1 is continued to

$$R_1 : \begin{cases} \psi \mapsto \lambda_R\gamma^1\psi(\tau, -x_1, .) \\ \bar{\psi} \mapsto -\lambda_R^*\bar{\psi}(\tau, -x_1, .)\gamma^1 \end{cases} \tag{10.155}$$

with ${R_1}^2 = \lambda_R^2$ and

$$R_1 : \Delta\mathcal{H} \mapsto \lambda_R^2\psi_1^T\mathcal{T}\psi_2 - \lambda_R^{*2}\bar{\psi}_2^T\mathcal{T}^{-1}\bar{\psi}_1^T, \tag{10.156}$$

which compared with the Minkowski case gives:

$$\eta_R^2 = \lambda_R^2, \tag{10.157}$$

thereby ${R_1}^2$ is invariant under continuation.

10.12 Supplemental: Dirac fermion in $\mathcal{C}$ antisym

Here we will omit R_1 which is basically the same as the last part. We define the charge conjugation as $C^{-1}\psi C = \mathcal{C}^{-1}\bar{\psi}^T \Rightarrow C^{-1}\bar{\psi}^T C = \mathcal{C}\psi$. Furthermore, we add a mass term, consistently with Pauli principle, becomes

$$\Delta\mathcal{H} = \psi^T\mathcal{C}\psi + \bar{\psi}\mathcal{C}^{-1}\bar{\psi}^T, \tag{10.158}$$

which is flipped by C, while the Dirac mass is invariant under $C : m \to m$.

I. Dirac fermion in Minkowski:

1) The first time-reversal symmetry is ($T^{-1}iT = -i$):

$$\begin{aligned} T^{-1}\psi(t,x)T &= \eta_T \mathcal{C}\psi(-t,x), \\ \Rightarrow T^{-1}\bar{\psi}^T T &= -\eta_T^* \mathcal{C}^{-1}\bar{\psi}^T; \end{aligned} \tag{10.159}$$

$$T^{-1}A_0(t,x)T = A_0(-t,x); \tag{10.160}$$

$$T^{-1}\vec{A}(t,x)T = -\vec{A}(-t,x); \tag{10.161}$$

which means $T^2 = (-1)^F$ and Dirac mass is flipped:

$$m \to -m. \tag{10.162}$$

Furthermore,

$$T^{-1}\Delta\mathcal{H}T = \eta_T^2 \psi^T \mathcal{C}\psi + \eta_T^{*2}\bar{\psi}\mathcal{C}^{-1}\bar{\psi}^T. \tag{10.163}$$

2) The second time-reversal symmetry is CT with

$$(CT)^{-1}\psi(t,x)(CT) = \eta_T^* \bar{\psi}^T, \tag{10.164}$$

$$\Rightarrow (CT)^{-1}\bar{\psi}^T(CT) = -\eta_T \psi, \tag{10.165}$$

$$(CT)^{-1}A_0(t,x)(CT) = -A_0(-t,x); \tag{10.166}$$

$$(CT)^{-1}\vec{A}(t,x)(CT) = \vec{A}(-t,x); \tag{10.167}$$

and then $(CT)^2 = -\eta_T^2$ on ψ and Dirac mass is flipped $m \to -m$, and

$$(CT)^{-1}\Delta\mathcal{H}(CT) = -\eta_T^2 \psi^T \mathcal{C}\psi - \eta_T^{*2}\bar{\psi}\mathcal{C}^{-1}\bar{\psi}^T. \tag{10.168}$$

I'. Dirac fermions in Euclid:

We need to assume that the form of ΔH is unchanged when continued to Euclid.

1') Time reversal T is naturally continued to

$$T : \begin{cases} \psi(\tau,x) \to \lambda_T \gamma^0 \mathcal{C}^{-1}\bar{\psi}^T \\ \bar{\psi}(\tau,x) \to \lambda_T^* \psi^T \mathcal{C}\gamma^0, \end{cases} \tag{10.169}$$

which gives $T^2 = 1$ independing on λ_T that is fixed by the transformation rule of $\Delta\mathcal{H}$:

$$T : \Delta\mathcal{H} \to \lambda_T^{*2}\psi^T \mathcal{C}\psi + \lambda_T^2 \bar{\psi}^T \mathcal{C}^{-1}\bar{\psi}^T, \tag{10.170}$$

which can be the same as its Minkowski counterpart only if

$$\lambda_T^2 = \eta_T^{*2}. \tag{10.171}$$

2') The second time reversal CT is naturally continued to

$$CT : \begin{cases} \psi(\tau,x) \to \lambda_{CT}\gamma^0\psi; \\ \bar{\psi}(\tau,x) \to -\lambda_{CT}^*\bar{\psi}\gamma^0, \end{cases} \tag{10.172}$$

and $(CT)^2 = \exp(2F\ln\lambda_{CT})$ or simply its effect on ψ as $(CT)^2 = \lambda_{CT}^2$, and

$$CT : \Delta\mathcal{H} \to -\lambda_{CT}^2 \psi^T \mathcal{C}\psi - \lambda_{CT}^{*2}\bar{\psi}\mathcal{C}^{-1}\bar{\psi}^T, \tag{10.173}$$

which is the same as the Minkowski case only if

$$\lambda_{CT}^2 = \eta_T^2, \tag{10.174}$$

which means $(CT)^2$ must be opposite after continuation, independing on η_T as expected.

∴ APS b.d. condition

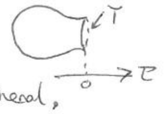
$$\mathcal{L}_M = -\bar{\Psi}(\not{D}_X+m)\Psi \qquad \{\gamma^\mu,\gamma^\nu\} = 2\,\mathrm{diag}(\overset{t}{-1},\underbrace{+1,+1,\cdots}_{\vec{x}_j}) \qquad \bar{\Psi}\equiv\Psi^\dagger\cdot i\gamma^t \qquad \tau\equiv it \Rightarrow \gamma^\tau = i\gamma^t$$

$$\therefore \bar{\Psi} = \Psi\gamma^\tau$$

$$X = Y\times\mathbb{R}_t\ ;\ \not{D}_X = \gamma^\mu\nabla_\mu$$

$$\therefore \mathcal{H} = \Psi^\dagger\cdot i\gamma^0(\gamma^j\nabla_j+m)\Psi = \Psi^\dagger[\gamma^\tau\gamma^j\nabla_j)+m\gamma^\tau]\Psi \qquad \& \quad (\tau,\vec{x}_j)\equiv x_k$$

$$Z_E = \int D(\bar{\Psi},\Psi)\exp(-S_E)\ ;\ S_E = \int\bar{\Psi}(\not{D}_X+m)\Psi \quad \& \quad X:$$



$\therefore \partial X = Y$

If $Y\neq\emptyset$ $\quad i\not{D}_X$ is not self-adjoint in general,

Caveat: $\Psi_1^\dagger,\Psi_2^\dagger\in\bar{S}(E)$

since
$$\int\Psi_1^\dagger i\not{D}_X\Psi_2 - \int(i\not{D}_X\Psi_1)^\dagger\Psi_2$$
$$=\int\Psi_1^\dagger\cdot i(\gamma^k(\partial_k+iA_k))\Psi_2 + \int\Psi_1^\dagger\cdot i\gamma^k(\overleftarrow{\partial_k}+(-i)A_k)\cdot\Psi_2$$
$$=\int_Y\Psi_1^\dagger\cdot i\gamma^\tau\Psi_2$$

∴ We need a b.c. at $\tau=0$, i.e, at Y, to eliminate this bd. term.

$$\gamma^\tau: S(E)\to S(E) \quad \& \ (\gamma^\tau)^2=+1 \Rightarrow S(E) = S_+(E)\oplus S_-(E)$$

$$\therefore \rightarrow = \int_Y[i\,\Psi_{1,+}^\dagger\Psi_{2,+} - i\,\Psi_{1,-}^\dagger\cdot\Psi_{2,-}]$$ as the surface term on Y

Define $D_Y\equiv\gamma^\tau\gamma^j\nabla_j \Rightarrow \mathcal{H} = \Psi^\dagger(D_Y+m\gamma^\tau)\Psi \quad \& \quad \gamma^\tau\cdot D_Y + D_Y\gamma^\tau = 0.$

Restricted on Y, $\quad \gamma^\tau\psi_{\pm,i} = \pm\psi_{\pm,i} \quad D_Y\psi_{\pm,i}\equiv\lambda_{Y,i}\psi_{\mp,i}$ with $\lambda_{Y,i}\geq 0$

i.e., $\lambda_{Y,i}\in\sqrt{\mathrm{Spec}\,D_Y^2}$

(Note $D_Y(\psi_{+,i}\pm\psi_{-,i}) = \pm\lambda_i(\psi_{+,i}\pm\psi_{-,i})$.) $\Rightarrow |\det(D_Y^{\pm\mp})| = |\det D_Y|^{1/2} = \sqrt{\det(D_Y^{-+}D_Y^{+-})}$.

$\gamma^\tau = \begin{pmatrix}\mathbb{1} & \\ & -\mathbb{1}\end{pmatrix};\ D_Y = \begin{pmatrix} & D_Y^{+-} \\ D_Y^{-+} & \end{pmatrix}$ $\quad\exists$ basis s.t.

Caveat: $\det(D_Y^{-+})$ denotes the partition f.c. formally) whose phase can be "adjusted" by $D_Y\psi_{\pm,i} = \lambda_{Y,i}e^{i\alpha_i}\psi_{\mp,i}$

$\because D_Y^\dagger = D_Y \quad \therefore (D_Y^{+-})^\dagger = D_Y^{-+}$

& APS b.c. reads:

$$\Psi_-\big|_Y = U_Y\Psi_+\big|_Y \text{ with } U_Y\equiv\frac{D_Y^{-+}}{\sqrt{D_Y^{-+}D_Y^{+-}}} \Rightarrow U_Y^\dagger U_Y = \mathbb{1}_{S_+(E)}$$

$\Rightarrow$ surface term vanishes $\Rightarrow i\not{D}_X$ hermitian/self-adjoint.

since $\begin{cases}U_Y\psi_{+,i} = \psi_{-,i} \\ U_Y^\dagger\psi_{-,i} = \psi_{+,i}\end{cases}$ (by $\lambda_i>0$)

$\therefore$ if $\Psi\big|_Y \equiv \sum_i A_+\psi_{+,i} + A_-\psi_{-,i}$

then $A_- \underset{\tau=0}{=} A_+$ or $A_- - A_+\big|_{\tau=0} = 0 \Rightarrow \langle APS|(A_- - A_+) = 0$ back to Hamiltonian language.

(Notation: $\bar{\Psi}_\pm\gamma^\tau = \pm\bar{\Psi}_\pm$) $\qquad$ For $\bar{\Psi}\in\bar{S}(E) \Rightarrow \bar{\Psi}_- = \bar{\Psi}_+\cdot U_Y^\dagger \Rightarrow \bar{A}_- - \bar{A}_+\big|_{\tau=0} = 0$

$$\therefore \bar{\Psi} = \sum_i\bar{A}_+\psi_{+,i}^\dagger + \bar{A}_-\psi_{-,i}^\dagger$$

★ But in operator language, (even though we did a Wick rotation), we still have $\bar{\Psi} = \Psi^\dagger\cdot\gamma^\tau = \sum_i A_+^\dagger\psi_{+,i}^\dagger - A_-^\dagger\psi_{-,i}^\dagger$

$$\therefore \bar{A}_+ = A_+^\dagger \ \& \ \bar{A}_- = -A_-^\dagger$$

$$\therefore A_+^\dagger + A_-^\dagger\big|_{\tau=0} = 0 \Rightarrow \langle APS|(A_+^\dagger + A_-^\dagger) = 0$$

Physical meaning of APS b.d. condition: $\because \mathcal{H} = \Psi^\dagger(D_Y+m\gamma^\tau)\Psi$

Alternative justification: $|\Psi\rangle e^{-\int\bar{\Psi}\Psi}\langle\bar{\Psi}|\Psi\rangle$, $\bar{\Psi}$ path-integral of $\Psi^\dagger$ ← notation change, not $\Psi^\dagger\gamma^\tau$

$\Psi = \sum A_i\psi_i$, $\bar{\Psi} = \sum A_i^\dagger\psi_i^\dagger$

$\int DA_i\,DA_i^\dagger\,e^{-\sum A_i^\dagger A_i}\langle A^\dagger|\Psi\rangle$ $\quad$ if $\langle A^\dagger|\Psi\rangle = \delta(A_n^\dagger) = A_n^\dagger$

$A_i\prod_{i\neq n}\delta(A_{i\neq n})$ $\quad\therefore A_n^\dagger|\Psi\rangle = 0 \Rightarrow A_{i\neq n}|\Psi\rangle$ for $\forall i\neq n$

$$\therefore H_Y\equiv\int_Y\mathcal{H} = \sum_i(A_+^\dagger, A_-^\dagger)_i\begin{pmatrix}m & \lambda_i \\ \lambda_i & -m\end{pmatrix}\begin{pmatrix}A_+ \\ A_-\end{pmatrix}_i$$

if $m=0 \Rightarrow H_Y(m=0) = \sum_i\lambda_i(B_{1,i}^\dagger B_{1,i} - B_{2,i}^\dagger B_{2,i})$

$$\langle APS| = \langle G.S.(m=0)| \Leftarrow \begin{cases}B_{1,i} = A_{+,i}+A_{-,i} \\ B_{2,i} = A_{+,i}-A_{-,i}\end{cases}$$

II. Chiral Axial charge. $Q_A \equiv \int_Y \bar{\Psi}^\dagger \bar{\gamma} \Psi + \text{const.}$ with $\bar{\gamma}$ the chiral operator.

It's convenient to change a basis by $\mathrm{Spec}(\bar{\gamma}) = \{\uparrow, \downarrow\} \equiv \sigma$

$\because [\bar{\gamma}, D_Y] = 0 \quad \therefore \bar{\gamma}\psi_\sigma = \sigma\psi_\sigma \ \& \ D_Y \psi_{\sigma,i} = \tilde{\lambda}_i \psi_{\sigma,i}$

Let's restrict to $\bar{\gamma}\psi_{\uparrow,i} = \psi_{\uparrow,i} \quad D_Y\psi_{\uparrow,i} = \tilde{\lambda}_i \psi_{\uparrow,i}$

then $\psi_{\downarrow,i} \equiv \gamma^{\tau}\psi_{\uparrow,i} \Rightarrow D_Y\psi_{\downarrow,i} = -\tilde{\lambda}_i\psi_{\downarrow,i}$

$\Psi = \sum_i A_{\uparrow,i}\psi_{\uparrow,i} + A_{\downarrow,i}\psi_{\downarrow,i} \qquad \therefore H_Y^{(m=0)} = \sum_i \tilde{\lambda}_i (A^\dagger_{\uparrow,i}A_{\uparrow,i} - A^\dagger_{\downarrow,i}A_{\downarrow,i})$

but $\tilde{\lambda}_i$ can be also negative.

Thus, $\begin{pmatrix} A_{\uparrow,i} \\ A^\dagger_{\downarrow,i}\end{pmatrix}|APS\rangle = 0$ if $\tilde{\lambda}_i > 0$

Exactly the same as Witten's RMP (2.45) ← after redefining ~~$\bar{\gamma} = \ldots$~~ $D_Y^{(new)} \equiv -\bar{\gamma} D_Y$ $\ni \& \begin{cases} D_Y^{(new)}\psi_{\uparrow,i} = -\tilde{\lambda}_i\psi_{\uparrow,i} \\ D_Y^{(new)}\psi_{\downarrow,i} = -\tilde{\lambda}_i\psi_{\downarrow,i}\end{cases}$

$\& \ \begin{pmatrix} A^\dagger_{\uparrow,i} \\ A_{\downarrow,i}\end{pmatrix}|APS\rangle = 0$ if $\tilde{\lambda}_i < 0$

$$Q_A\Big|_{|APS\rangle} \simeq \sum_i (A^\dagger_{\uparrow,i}A_{\uparrow,i} - A^\dagger_{\downarrow,i}A_{\downarrow,i}) + \text{const.}$$

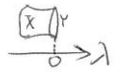
$$\simeq \sum_{\tilde{\lambda}_i>0}(N_{\uparrow,i} - N_{\downarrow,i}) + \sum_{\tilde{\lambda}_i<0}(N_{\uparrow,i} - N_{\downarrow,i}) = \sum_{\tilde{\lambda}_i>0}(-1) + \sum_{\tilde{\lambda}_i<0}(+1) \equiv -2\eta(D_Y)$$

III. Meaning of $\eta(D_Y)$: $2\eta(D_Y)$ from the above Equation is defined as number difference between positive D_Y-eigenvectors and negative ones restricted to $\bar{\gamma} = +1$.

$\therefore$ We should've written as $\eta(D_Y) = \eta(\tilde{D}_Y) \quad \tilde{D}_Y \equiv D_Y\big|_{\bar{\gamma}=1} \left(= D_Y^{(new)}\big|_{\bar{\gamma}=1}\right)$

Recall the path integral on X

$$Z_E = \int \mathcal{D}(\bar{\Psi},\Psi)\exp\left(-\int \bar{\Psi}(\not{D}_X + m)\Psi\right)$$

now we create bd. Y along λ-direction 

$\& \ (1-\gamma^\lambda)\Psi\big|_{\lambda=0} = 0 \ \& \ \bar{\Psi}(1+\gamma^\lambda)\big|_{\lambda=0} = 0, \quad \& \begin{cases} m<0, \text{ if } \lambda<0 \\ m>0, \text{ if } \lambda>0 \end{cases}$

Consistent with everything even if $\lambda \neq \epsilon$! since $\{\gamma^\lambda, \bar{\gamma}\} = 0$

$\therefore \Psi \equiv \chi \exp(-|m|\lambda)/\sqrt{\epsilon}; \ \bar{\Psi} \equiv \exp(-|m|\lambda)\bar{\chi}/\sqrt{\epsilon}$; $\epsilon^{-1} \gtrsim \max(|m|)$

~~Take $\lambda = \epsilon$~~ $\therefore Z_E\Big|_{\lambda=0} \simeq \int \mathcal{D}(\bar{\chi},\chi)\exp\left(-\int_Y \bar{\chi}\gamma^\lambda D_Y\chi\right) \qquad$ (Recall $D_Y = \gamma^\lambda \vec{\gamma}\cdot\vec{\nabla}_Y$)

By a standard construction when $\dim(Y) \in \mathbb{Z}_{odd}$

$\gamma^\lambda = \begin{pmatrix}\mathbb{1} & \\ & -\mathbb{1}\end{pmatrix}; \ \vec{\gamma} = \begin{pmatrix} & \vec{\gamma}_Y \\ \vec{\gamma}_Y & \end{pmatrix}; \quad \bar{\chi} = \begin{pmatrix} 0 \\ \bar{\chi}_Y\end{pmatrix}^T; \ \chi = \begin{pmatrix}\chi_Y \\ 0\end{pmatrix} \qquad = \int \mathcal{D}(\bar{\chi}_Y,\chi_Y)\exp\left(i\int_Y \bar{\chi}_Y \not{D}_Y \chi_Y\right)$ with $\not{D}_Y \equiv i\vec{\gamma}_Y\cdot\vec{\nabla}_Y$

$\bar{\gamma} \propto \gamma^\lambda \prod \gamma^i \ \therefore \bar{\gamma} = \begin{pmatrix} & -i\Gamma_{ch} \\ i\Gamma_{ch} & \end{pmatrix}; \ \Gamma_{ch}^\dagger = \Gamma_{ch}, \ \Gamma_{ch}^2 = \mathbb{1}$ & $[\Gamma_{ch}, \vec{\gamma}_Y] = 0$ $\quad ; D_Y = \begin{pmatrix} & -\vec{\gamma}_Y\cdot\vec{\nabla}_Y \\ \vec{\gamma}_Y\cdot\vec{\nabla}_Y & \end{pmatrix} \quad = \dfrac{\det(\not{D}_Y)}{\det(\not{D}_Y \pm i\mu)} \quad (\mu > 0)$

$\therefore \psi_{\uparrow,i} = \begin{pmatrix}\frac{\chi_i}{2} \\ \frac{\xi_i}{2}\end{pmatrix}$ with $\begin{cases} -i\Gamma_{ch}\xi_i = \chi_i \\ i\Gamma_{ch}\chi_i = \xi_i\end{cases} \qquad = \exp\left(\mp i\frac{\pi}{2}2\eta(\not{D}_Y)\right)\cdot|\det \not{D}_Y|$

$\& \ \psi_{\downarrow,i} \equiv \gamma^\lambda\psi_{\uparrow,i} = \begin{pmatrix}\chi_i/2 \\ -\xi_i/2\end{pmatrix} \& \begin{cases} -\vec{\gamma}_Y\cdot\vec{\nabla}_Y\xi_i = \lambda_i\chi_i \\ \vec{\gamma}_Y\cdot\vec{\nabla}_Y\chi_i = \lambda_i\xi_i\end{cases}$ with $2\eta(\not{D}_Y) \equiv \#(\text{positive-eigenvalue eigenvectors of } \not{D}_Y) - \#(\text{negative} \cdots)$

Since $\exists\eta$ s.t. $\gamma^\lambda(\psi_{\uparrow,i} + \eta\psi_{\downarrow,i}) = (+1)(\psi_{\uparrow,i} + \eta\psi_{\downarrow,i})$ due to $\{\gamma^\lambda, \bar{\gamma}\} = 0$ & η unique $= 1$ if $\lambda = \epsilon$ to be taken.

$\therefore \psi_{\uparrow,i} + \psi_{\downarrow,i} = \begin{pmatrix} * \\ 0\end{pmatrix} \equiv \begin{pmatrix}\chi_i \\ 0\end{pmatrix}$ & $D_Y(\psi_{\uparrow,i} + \psi_{\downarrow,i}) \overset{\text{by def.}}{=} \lambda_i(\psi_{\uparrow,i} - \psi_{\downarrow,i})$

$+ \begin{pmatrix} 0 \\ \vec{\gamma}_Y\vec{\nabla}_Y\cdot\chi_i\end{pmatrix} = \lambda_i\begin{pmatrix}0 \\ \xi_i\end{pmatrix}$

It's consistent to further take $-i\xi_i = \chi_i$ and $i\xi_i = \chi_i$ to diagonalize Γ_{ch} : $\Gamma_{ch}\chi_i = \pm\chi_i$

$\therefore$ if $\Gamma_{ch}\chi_i = +\chi_i \Rightarrow \chi_i = -i\xi_i \quad \therefore \vec{\gamma}_Y\cdot\nabla_Y\cdot\chi_i = i\lambda_i \chi_i \Rightarrow -i\vec{\gamma}_Y\cdot\nabla_Y\chi_i = +\lambda_i\cdot\chi_i$

But $[\Gamma_{ch}, \vec{\gamma}_Y] = 0 \Rightarrow \Gamma_{ch} \propto \mathbb{1}$ $\quad \therefore \mathrm{Spec}(\Gamma_{ch}) = 1$ (or -1) $\therefore$ $\eta(\tilde{D}_Y) = \eta(\tilde{D}_Y)$ or $-\eta(\tilde{D}_Y)$

Clifford algebra @ Y is irreducible

Since $\psi_{\uparrow,i} \longleftrightarrow \chi_i$ (or $-\chi_i$) is invertible mapping, depending on Γ_{ch}'s sign

Caveat: $\eta(D_Y)$ is obtained on the b.d. with local b.d. condition $(1-\gamma^\lambda)\Psi|_{\lambda=0} = \bar{\Psi}(1+\gamma^\lambda)|_{\lambda=0} = 0$ rather than APS b.d.

Take $\lambda = \tau$, such a b.d condition $\Leftrightarrow \langle L|$ & with $\langle L|A_{-,i} = 0$ & $\langle L|A^\dagger_{+,i} = 0$

IV. APS b.d. condition with zero modes of D_Y (or $\tilde{D}_Y$ or $\not{D}_Y$)

Still take $\gamma^\tau = \begin{pmatrix}\mathbb{1} & \\ & -\mathbb{1}\end{pmatrix}$; $\bar{\gamma} = \begin{pmatrix} & -i\mathbb{1} \\ i\mathbb{1} & \end{pmatrix}$ if $\gamma^\tau f_{\pm,\alpha} = \pm f_{\pm,\alpha}$; $D_Y f_{\pm,\alpha} = 0$, $f_{-,\alpha} = \bar{\gamma} f_{+,\alpha}$

$\therefore \bar{\gamma}$ enables pairing of $+$ and $-$ modes on X.

if $\Psi|_Y = \sum_\alpha A_{+,\alpha} f_{+,\alpha} + A_{-,\alpha}\cdot f_{-,\alpha}$ + (non-zero-modes)

& then surface-term contribution $= \int i\,\Psi^\dagger_{+,\text{zero}}\Psi_{+,\text{zero}} - i\,\Psi^\dagger_{-,\text{zero}}\Psi_{-,\text{zero}} = 0$ since $\bar{\gamma}^\dagger\bar{\gamma} = \mathbb{1}$ ($= \bar{\gamma}^2$)

$\therefore$ b.d condition for path integral $\begin{cases}\bar{\gamma}\Psi_{+,\text{zero}} = \Psi_{-,\text{zero}} \\ \bar{\Psi}_{+,\text{zero}}\bar{\gamma}^\dagger = \bar{\Psi}_{-,\text{zero}}\end{cases}$

since in path integral (Euclidean) $\bar{\Psi}$ transforms as "$\Psi^\dagger$" in $S(E)$

$\therefore \quad A_{+,\alpha} = A_{-,\alpha} \Rightarrow \langle APS|(A_{+,\alpha} - A_{-,\alpha}) = 0$

$\bar{\Psi}|_Y = \sum_\alpha \bar{A}_{+,\alpha} f^\dagger_{+,\alpha} + \bar{A}_{-,\alpha} f^\dagger_{-,\alpha} + (\cdots)$

$\Psi^\dagger|_Y\cdot\gamma^\tau = \sum_\alpha A^\dagger_{+,\alpha} f^\dagger_{+,\alpha} - A^\dagger_{-,\alpha} f^\dagger_{-,\alpha} + (\cdots)$

$\bar{A}_{+,\alpha} = \bar{A}_{-,\alpha}$

$\Rightarrow \langle APS|(A^\dagger_{+,\alpha} + A^\dagger_{-,\alpha}) = 0$

Now let's sketch $A_{\sigma,\alpha}$ in terms of $A_{\pm,\alpha}$

$$\begin{cases} A_{\uparrow,\alpha} \equiv A_{+,\alpha} + \eta A_{-,\alpha} \\ f_{\uparrow,\alpha} \equiv f_{+,\alpha} + \eta^* f_{-,\alpha} \end{cases} \Rightarrow \bar{\gamma} f_{\uparrow,\alpha} = \eta^* f_{+,\alpha} + f_{-,\alpha} \Rightarrow \eta = 1$$

Similarly, $\begin{cases} A_{\downarrow,\alpha} = A_{+,\alpha} - A_{-,\alpha} \\ f_{\downarrow,\alpha} = f_{+,\alpha} - f_{-,\alpha}\end{cases}$ up to some $\frac{1}{\sqrt{2}}$ etc.

$\therefore \langle APS|A^\dagger_{\uparrow,\alpha} = \langle APS|A_{\downarrow,\alpha} = 0$ or $A^\dagger_{\downarrow,\alpha}|APS\rangle = A_{\uparrow,\alpha}|APS\rangle = 0$

$\therefore \quad Q_A|_{|APS\rangle} = (\cdots) + \sum_\alpha(-1) \equiv 2\eta(D_Y)$

$\Rightarrow \eta(D_Y)$ modified to be $\sum_i \frac{1}{2}\sum \mathrm{sgn}(\lambda_i)$ (including α's) with $\mathrm{sgn}(0) \equiv 1$.

Similarly, $2\eta(\not{D}_Y)$ is also correspondingly modified, which is physically OK since $|\det \not{D}_Y| = 0$ in the presence zero modes.

V. Dimension Reduction for Majorana: ($D = 4 \bmod 8$ for bulk)

$Z_E = \int D\Psi \exp(-\int \bar{\Psi}(\not{D}_X + m)\Psi)$ $\quad \bar{\Psi} \equiv \Psi^T \mathcal{C}$; $\mathcal{C} = \begin{pmatrix} & C_Y \\ C_Y & \end{pmatrix}$, $C_Y^T = -C_Y$ as C-matrix for $D-1 = 3 \bmod 8$

$(1-\gamma^\lambda)\Psi|_{\lambda=0} = 0 \;(\Leftrightarrow \bar{\Psi}(1+\gamma^\lambda)|_{\lambda=0} = 0)$ then $\Psi = \chi e^{|m|\lambda}$ $\quad$ (since $\mathcal{C}\gamma^i\mathcal{C}^{-1} = -\gamma^{iT}$)

$\therefore Z_E|_{\lambda=0} \simeq \int D\chi \exp(-i\int_Y \chi_Y^T \cdot C_Y\cdot\not{D}_Y\chi) \simeq \mathrm{Pf}(C_Y\not{D}_Y)/\mathrm{Pf}(C_Y(\not{D}_Y \pm i\mu))$

$D = 2 \bmod 8$ for bulk: $\quad \bar{\Psi} = \Psi^T\mathcal{T}$, $Z_E = \int D\Psi \exp(-\int \bar{\Psi}(\not{D}_X + i\bar{\gamma}m)\Psi)$; $\mathcal{T} = \begin{pmatrix} L_Y & \\ & L_Y\end{pmatrix}$, $L_Y^T = L_Y$ as $\mathcal{T}$-matrix in $D-1$.

$\therefore \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi_Y \\ -\chi_Y\end{pmatrix}\exp(-\int_0^\lambda m(s)ds)$ i.e., $(i\bar{\gamma} - \gamma^\lambda)\Psi|_{\lambda=0} = 0 \;(\Leftrightarrow \bar{\Psi}(-i\bar{\gamma} - \gamma^\lambda)|_{\lambda=0} = 0)$

$\therefore Z_E|_{\lambda=0} \simeq \int D\chi \exp(-i\int_Y \chi_Y^T L_Y\not{D}_Y\chi_Y) \simeq \mathrm{Pf}(L_Y\not{D}_Y)/\mathrm{Pf}(L_Y(\not{D}_Y \pm i\mu))$

$\left(D=3 \bmod 8 \text{ for bulk}\right)$: Recall $\gamma^\lambda = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\gamma^Y = \left\{ \begin{pmatrix} & \gamma^{\vec{\omega}} \\ \gamma^{\vec{\omega}} & \end{pmatrix} ; \begin{pmatrix} & -i1 \\ i1 & \end{pmatrix} \right\}$

$\exists$ antisym $\mathcal{C} = \Gamma_{D=2 \bmod 8} \cdot \begin{pmatrix} & -i \\ i & \end{pmatrix}$ $\quad$ $\mathcal{C}$ isn't of $\begin{pmatrix} C & \\ & C \end{pmatrix}$ since $\nexists C$ in $D=1 \bmod 8$.

with $\Gamma_{D=2 \bmod 8} = \begin{pmatrix} L_\omega & \\ & L_\omega \end{pmatrix}$: L_ω is Γ-matrix for $D=1 \bmod 8$ which is sym.

$\therefore \bar{\Psi} = \Psi^T \mathcal{C}$

Boundry modes: $\Psi \simeq \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-\int_0^\lambda m(s)ds}$ satisfying $(1-\gamma^\lambda)\Psi|_{\lambda=0} = 0$

$\therefore \quad \bar{\Psi}(\not{D}_X + m)\Psi = -\bar{\Psi} \begin{pmatrix} & D_Y^{+-} \\ D_Y^{-+} & \end{pmatrix} \Psi$

$= i\chi^T L_\omega D_Y^{-+} \chi$

where $\quad D_Y^{-+} \equiv \gamma^{\vec{\omega}} \cdot \nabla_{\vec{\omega}} + i\partial_{\bar{\omega}} = \left(D_Y^{+-}\right)^\dagger$

$\because (L_\omega D_Y^{-+})^T = -(L_\omega D_Y^{-+}) \quad \therefore \quad \mathcal{Z}_E|_{\lambda=0} \simeq \int \mathcal{D}\chi \exp\left(-i\int \chi^T L_\omega D_Y^{-+} \chi\right) = \mathrm{Pf}(L_\omega D_Y^{-+})$

To see its value, let's add by hand second copy η

$\therefore \Psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$; $\bar{\Psi} = \Psi^T \mathcal{C}$

$\therefore \quad -\bar{\Psi} \begin{pmatrix} & D_Y^{+-} \\ D_Y^{-+} & \end{pmatrix} \Psi = i\chi^T L_\omega D_Y^{-+} \chi - i\eta^T L_\omega D_Y^{+-} \eta$

) We can find a basis by a unitary transformation s.t. $L_\omega = 1$

$= i\chi^T D_Y^{-+} \chi + \eta^T (iD_Y^{-+})^\dagger \eta$

$\therefore \mathcal{Z}_E[\chi, \eta] = \mathrm{Pf}(D_Y^{-+}) \cdot \mathrm{Pf}(D_Y^{+-}) = |\mathrm{Pf}(D_Y^{-+})|^2 = |\mathrm{Pf}(L_\omega D_Y^{-+})|^2$

but we also know that

$\mathcal{Z}_E[\chi, \eta] = \int \mathcal{D}\Psi \exp\left(\int \Psi^T \mathcal{C} D_Y \Psi\right) = \mathrm{Pf}(\mathcal{C} D_Y)$

$\therefore |\mathcal{Z}_E|_{\lambda=0}| = |\mathrm{Pf}(L_\omega D_Y^{-+})| = \sqrt{|\mathrm{Pf}(\mathcal{C} D_Y)|} \simeq |\det D_Y|^{1/4}$

i.e., Maj-Weyl $= \sqrt[4]{\text{Dirac}}$

$\therefore$ APS boundary condition for Majorana fermion $\quad$ (Caveat: $\underset{\text{operator}}{\overset{\text{Grassmann}}{\ }}$)

Majorana condition: $\quad \bar{\Psi} \doteq \Psi^\dagger \gamma^\tau \doteq -\Psi^T \mathcal{C}$

Let's expand Ψ into ψ_i with $D_Y \psi_i = \lambda_i \psi_i$ $(\lambda_i > 0)$

& $\gamma^\tau \mathcal{C}^{-1} \psi_i^*$ $\quad$ $D_Y \gamma^\tau \mathcal{C}^{-1} \psi_i^* = -\lambda_i \psi_i$

$\Rightarrow \Psi = \sum A_i \psi_i + B_i \gamma^\tau \mathcal{C}^{-1} \psi_i^*$

Maj condition gives $\quad B_i^\dagger = A_i$ or $B_i = A_i^\dagger$

APS condition: $\quad \Psi_-|_{\tau=0} = U_Y \Psi_+|_{\tau=0}$ $\quad$ i.e., $\dfrac{D_Y}{|D_Y|} \Psi\Big|_{\tau=0} = \Psi\Big|_{\tau=0}$

$\therefore \langle APS| A_i^\dagger = 0$ $\quad$ since $(-\lambda_i < 0)$

2^{nd} APS condition: $\quad \bar{\Psi}_-|_{\tau=0} = \bar{\Psi}_+|_{\tau=0} U_Y^\dagger \Rightarrow \bar{\Psi}\Big|_{\tau=0}(1-\gamma^\tau) = \bar{\Psi}\Big|_{\tau=0}(1+\gamma^\tau) \cdot \dfrac{D_Y}{|D_Y|}$

$\Rightarrow \Psi^T \mathcal{C}(1-\gamma^\tau) = \Psi^T \mathcal{C}(1+\gamma^\tau) \dfrac{D_Y}{|D_Y|}$

$\cancel{\Psi^T(1+\gamma^{\tau T})\mathcal{C} = \Psi^T(1-\gamma^\tau)\frac{D_Y^T}{|D_Y|}\mathcal{C}}$

$\mathcal{H} = \int \Psi^\dagger (D_Y + m\gamma^\tau) \Psi$

$\therefore H(m=0) = \sum_i \lambda_i A_i^\dagger A_i$

$\therefore \langle APS| = \langle G.S.(m=0)|$ again.

(handwavy, by $** = \mathrm{id}$, since Ψ^* doesn't exist actually as Grassmann) $\Rightarrow \left((1-\gamma^\tau)\mathcal{C}^{-1}\Psi^*\right)^* = \left(\frac{D_Y}{|D_Y|}(1+\gamma^\tau)\mathcal{C}^{-1}\Psi^*\right)^*$

$\left(\mathcal{C}^{-1}(1+\gamma^\tau)\Psi^*\right)^* = \left((1+\gamma^\tau)\dfrac{D_Y^*}{|D_Y|}\Psi^*\right)^* \Rightarrow \Psi = \dfrac{D_Y}{|D_Y|}\Psi$

$\cancel{\left(D_Y \mathcal{C}^{-1}\psi_i^* = \lambda_i \mathcal{C}^{-1}\psi_i\right)_{\lambda_i>0} \Rightarrow (1+\gamma^\tau)\mathcal{C}^{-1}\Psi^* = (1+\gamma^\tau)\mathcal{C}^{-1}\Psi^*}$ $\quad$ $\therefore$ the same as 1^{st} APS condition.

$\cancel{\Rightarrow (1+\gamma^{\tau T})\Psi^* = (1-\gamma^{\tau T})\Psi^*}$

$\cancel{\Rightarrow (1+\gamma^\tau)}$

11 Kramers-Wannier DAILY: 2021.12.21-12.23

In this part, I will try to review Kramers-Wannier duality of quantum transverse Ising model in a formal way. The Hamiltonian of transverse Ising model writes as:

$$\mathcal{H}_{\text{Ising}}(J,h) \equiv \sum_{i=1}^{L} -J\sigma_i\sigma_{i+1} - h\tau_i, \tag{11.1}$$

with a periodic boundary condition and $\sigma_i\tau_j = (-1)^{\delta_{i,j}}\tau_j\sigma_i$, $\sigma^2 = \tau^2 = 1$. The above Hamiltonian possesses a $\mathbb{Z}_2$ symmetry generated by

$$Q \equiv \prod_i \tau_i, \tag{11.2}$$

and $Q = 1$ will be called Q-invariant sector.

11.1 Kramers-Wannier duality

Let us introduce or almost double the degrees of freedom by define Ising degrees of freedom (μ, η) on the dual lattice points $\mathbb{Z} + 1/2$. Then we couple them to the original Ising model to produce a matter-gauge theory:

$$\mathcal{H}_{\text{Ising-gauge}}(J,h) \equiv \sum_i -J\sigma_i\mu_{i+1/2}\sigma_{i+1} - h\tau_i, \tag{11.3}$$

with the following constraint as the Gauss law:

$$\mathcal{Q}_i \equiv \eta_{i-1/2}\tau_i\eta_{i+1/2} = 1. \tag{11.4}$$

The Hilbert subspace satisfies this constraint will be called the gauge-invariant sector. The matter field is (σ, τ) while the gauge field is (η, μ). The local gauge symmetry is generated by $\mathcal{Q}_{i=1,2,\cdots,L}$ which indeed commutes with $\mathcal{H}_{\text{Ising-gauge}}(J,h)$. Therefore, the above constraint significantly kill most degrees of freedom so that the number of degrees of freedom is *exactly* restored as we will see. Anyway, let us make everything quantitatively. We first need to label all the gauge transformation:

$$G_{\text{gauge}} = \left\{ \prod_j (\mathcal{Q})_j^m | m_j \in \pm 1 \right\}, \tag{11.5}$$

which implies that $|G_{\text{gauge}}| = 2^L$. We can label its element as $g_1 \equiv 1, g_2 \equiv Q, g_3 \cdots, g_{2^L}$ where we only specify the first two elements for later use. Then the partition function of the gauge theory under Gauss law: $U(\beta) \equiv \exp(-\beta\mathcal{H}_{\text{Ising-gauge}})$

$$Z_{\text{Ising-gauge}} = \text{Tr}\left[U(\beta) \frac{\sum_{k=1}^{2^L} g_k}{2^L} \right], \tag{11.6}$$

where "Tr" is done over the whole Hilbert space defined on points $i \in \mathbb{Z}$ and $i+1/2 \in \mathbb{Z}+1/2$, of course. The formula above is basis-independent and the factor involving summation above works as a projection operator onto the gauge-invariant sector. Nevertheless, we will choose a convenient basis B as follows. First, we will specify $\sigma \sim \sigma_z$ and $\mu \sim \sigma_z$ numerically, and choose the Pauli-z basis. We define this basis as a *set* S with $|S| = 2^{(L+L)} = 2^{2L}$ since we have totally $2L$ Ising points including matter and gauge fields. It can be checked that G_{gauge} is a group action on this set S. Thus, we can partition S into $|S|/|G_{\text{gauge}}| = 2^L$ orbits. These orbits can be further partitioned into two classes S_0 and S_1 based on the $\bar{Q} \equiv \prod_i \mu_{i+1/2}$ value that measures the *total* twisting number on the bonds which cannot be affected by action of G_{gauge}. The number of orbits in $S_{0,1}$ is equal to $|S_{0,1}| = 2^L/2 = 2^{L-1}$. Then we can choose out the following elements from distinct orbits in S:

$$|v_1\rangle \cdots |v_{2^{L-1}}\rangle, |w_1\rangle \cdots |w_{2^{L-1}}\rangle, \tag{11.7}$$

where $|v\rangle$'s only have a twisting at the bond $L+1/2$: $\mu_{j+1/2}|v\rangle = (-1)^{\delta_{j,L}}|v\rangle$ while $|w\rangle$'s is twisting-free: $\mu_{j+1/2}|w\rangle = |w\rangle$. Then the full basis B of the large Hilbert space traced by "Tr" before can be constructed as:

$$B \equiv \{v_{kl} \equiv g_k|v_l\rangle, w_{kl} \equiv g_k|w_l\rangle : \; k = 1, \cdots, |G_{\text{gauge}}|, l = 1, \cdots, |S_{0,1}|\}. \tag{11.8}$$

It can be checked that

$$\langle v_{kl}|U(\beta)g_m|v_{kl}\rangle = \langle w_{kl}|U(\beta)g_m|w_{kl}\rangle = 0, \text{ if } m > 1; \tag{11.9}$$

and

$$\langle v_{kl}|U(\beta)g_m|v_{kl}\rangle = \langle v_l|U(\beta)g_m|v_l\rangle, \tag{11.10}$$

$$\langle w_{kl}|U(\beta)g_m|w_{kl}\rangle = \langle w_l|U(\beta)g_m|w_l\rangle \tag{11.11}$$

since G_{gauge} is abelian.

Then we can continue to calculate the above function:

$$\begin{aligned} Z_{\text{Ising-gauge}} &= \frac{1}{2^L}\sum_m \sum_{k=1}^{2^L}\sum_{l=1}^{2^{L-1}} \langle v_{kl}|U(\beta)g_m|v_{kl}\rangle + \langle w_{kl}|U(\beta)g_m|w_{kl}\rangle \\ &= \sum_{l=1}^{2^{L-1}} \langle v_l|U(\beta)(1+Q)|v_l\rangle + \langle w_l|U(\beta)(1+Q)|w_l\rangle \\ &= \left[\sum_{l=1}^{2^{L-1}} \langle v_l|U(\beta)(1+Q)/2|v_l\rangle + \langle Qv_l|U(\beta)(1+Q)/2|Qv_l\rangle\right] \\ &\quad + \left[\sum_{l=1}^{2^{L-1}} \langle w_l|U(\beta)(1+Q)/2|w_l\rangle + \langle Qw_l|U(\beta)(1+Q)/2|Qw_l\rangle\right] \\ &= \text{untwisted-sector}|_{Q=1} + \text{twisted-sector}|_{Q=1} \\ &= Z_{\text{Ising-orbifold}}. \end{aligned} \tag{11.12}$$

Furthermore, this constraint can be used to obtain $\tau_i = \eta_{i-1/2}\eta_{i+1/2}$ thereby

$$\mathcal{H}_{\text{Ising-gauge}}(J,h) = \sum_i -J\sigma_i\mu_{i+1/2}\sigma_{i+1} - h\eta_{i-1/2}\eta_{i+1/2}. \tag{11.13}$$

Let us further do a canonical transformation

$$\tilde{\mu}_{i+1/2} \equiv \sigma_i \mu_{i+1/2} \sigma_{i+1}; \;\; \tilde{\eta}_{i+1/2} \equiv \eta_{i+1/2}, \tag{11.14}$$

where it should be noted that $\tilde{\mu}$ is gauge-invariant thereby physical observable: $[\tilde{\mu}_{i+1/2}, \mathcal{Q}_j] = 0$. The transformation is canonical in the sense that $\tilde{\mu}$ and $\tilde{\eta}$ have the same algebra as before. Then,

$$\mathcal{H}_{\text{Ising-gauge}}(J, h) \leftrightarrow \mathcal{H}_{\text{Ising}}(h, J). \tag{11.15}$$

11.2 Dual symmetry

The above gauge theory respects a symmetry generated by

$$\bar{Q} \equiv \prod_i \mu_{i+1/2} = \prod_i \tilde{\mu}_{i+1/2}, \tag{11.16}$$

which implies that it is exactly the corresponding $\mathbb{Z}_2$ symmetry of the dual Ising model written in $(\tilde{\eta}, \tilde{\mu})$. In the viewpoint of the gauge field, $\bar{Q}$ is nothing but the closed $\mathbb{Z}_2$-Wilson line of the gauged $\mathbb{Z}_2$ generated by Q.

11.3 Higgs phase: $J \gg h$

Let us consider $J \gg h$ or for simplicity $h = 0$. Then,

$$\mathcal{H}_{\text{Ising-gauge}}(J, h = 0) \equiv \sum_i -J \sigma_i \mu_{i+1/2} \sigma_{i+1}. \tag{11.17}$$

Due to the absence of the second term, the σ-degrees of freedom in this Hamiltonian can be viewed in a fixed background of μ first, which turns out in the SSB phase satisfying

$$\sigma_i \mu_{i+1/2} \sigma_{i+1} = 1, \tag{11.18}$$

$$\text{or } \tilde{\mu}_{i+1/2} = 1, \tag{11.19}$$

where the later equality means that the gauge sector is trivially a product state in low energy thereby no low-energy dynamics at all. This is the $\mathbb{Z}_2$-analog of Higgs mechanism. It should be noted that μ being treated as a background is possible here because $[\mathcal{H}_{\text{Ising-gauge}}(J, h = 0), \mu_{i+1/2}] = 0$ now so that μ's can be given fixed values. Of course, all the above result can be directly seen by the Kramers-Wannier duality developed before which gives that the gauge-sector gap is $2J$ — $\mathbb{Z}_2$ gauge boson μ obtains a mass after swallowing σ "Goldstone" modes to become $\tilde{\mu}$. To be more generally in higher dimensions, we can also include a "magnetic" energy $\prod_{\vec{r} \in \text{plaquette}} \mu_{\vec{r}}$ which is trivial due to that $\prod_{\vec{r} \in \text{plaquette}} \mu_{\vec{r}} = \prod_{\vec{r} \in \text{plaquette}} \tilde{\mu}_{\vec{r}} = 1$. Thus there is indeed no dynamics of the gauge bosons at low energy.

In constrast, if $J \ll h$, the gauge boson remains "gapless" or more precisely of a (two-fold) degenerate low-energy spectrum.

12 Continuum Limit: 2022.3.22

Let us define a lattice model on a lattice with a **physical** length size L, thereby the number of sites $(L/a)^n$. On each site $\vec{i}$, we define a lattice variable $\phi_{\vec{i}}$. The **physical** position of this operator is $\vec{r} = \vec{i}a$. The lattice model has various coupling constants, compactly denoted as K_a. We assume that K_a depends on a in the way such that existing Δ_ϕ

$$\lim_{a\to 0}[a^{-\Delta_\phi}\phi_{\vec{r}/a}]_{L/a,K_a} = \langle\varphi(\vec{r})\rangle_{L,\mathcal{K}(K_{a_0=1})}, \tag{12.1}$$

for some field theory of variable $\varphi(\vec{r})$ with a coupling constant $\mathcal{K}$ which depends on the lattice coupling K_{a_0} when, e.g., $a_0 \equiv 1$ (See **Explanation** below). It should be noted that we use "$[\cdots]$" to denote average over lattice model which clearly depends on the linear number of sites L/a (rather than L!) and the lattice coupling. On the other side, we use "$\langle\cdots\rangle$" to denote field-theory average which of course depends on the physical length L (rather than our discreteness number) and the field-theory coupling parameter $\mathcal{K}$.

Then let us exploit the scaling rule as follows:

$$\begin{aligned}
&\langle\varphi(b\vec{r})\rangle_{bL,\mathcal{K}(K_1)} \\
&= \lim_{a\to 0}[a^{-\Delta_\phi}\phi_{b\vec{r}/a}]_{bL/a,K_a} \\
&= \lim_{a\to 0}[(ba)^{-\Delta_\phi}\phi_{\vec{r}/a}]_{L/a,K_{ba}} \\
&= b^{-\Delta_\phi}\lim_{a\to 0}[(a)^{-\Delta_\phi}\phi_{\vec{r}/a}]_{L/a,K_{ba}} \\
&= b^{-\Delta_\phi}\langle\varphi(\vec{r})\rangle_{L,\mathcal{K}(K_b)}.
\end{aligned} \tag{12.2}$$

Explanation: For n-dimensional scalar bosons,

$$\begin{aligned}
S &= \int d^n x\mathcal{K}\partial\varphi\partial\varphi \\
&\approx \sum a^n\mathcal{K}\frac{(\phi_{\vec{i}+\hat{x}} - \phi_{\vec{i}})^2}{a^2} \quad \text{(if taking } \Delta_\phi = 0) \\
&= \sum \mathcal{K}a^{n-2}(\phi_{\vec{i}+\hat{x}} - \phi_{\vec{i}})^2,
\end{aligned} \tag{12.3}$$

which means

$$\mathcal{K} = \lim_{a\to 0} K_a a^{2-n}. \tag{12.4}$$

Therefore, we can see $\mathcal{K}$ as a series:

$$\mathcal{K} \leftrightarrow \{K_{1/m}(1/m)^{2-n}\}_{m=1,2,3,\cdots}, \tag{12.5}$$

which of course depends on the starting point $K_{1/m=1}$ when $m = 1$, i.e., $K_{a_0=1}$.

We can also readily obtain:

$$\mathcal{K}(K_b) = \lim_{a\to 0} K_{ba}a^{2-n} = \lim_{a\to 0} K_a(a/b)^{2-n} = b^{n-2}\mathcal{K}(K_1). \tag{12.6}$$

Interestingly, when $n = 2$ (2-d CFT), $\mathcal{K}$ does not depend on the starting point of the continuum limit.

13 SPT and LSM in one dimension: 2022.4.22

I will discuss an interesting relation between SPT and LSM in one dimension.

Theorem 1: If a spin-1/2 chain Hamiltonian H_0 possess $G = \mathrm{U}(1)_z \times \mathbb{Z}_2^x$ and it is gapped with a unique ground state $|\mathrm{GS}\rangle$, then in the long length limit

$$I[|\mathrm{GS}\rangle] \equiv \langle \mathrm{GS}| \exp\left(i\frac{2\pi}{L}\sum_j j\hat{s}_j^z\right)|\mathrm{GS}\rangle = \pm 1, \tag{13.1}$$

where $\hat{s}_j^z$ can be also replaced by s^x or s^y due to symmetry.

Corollary 2: (The same conditions as above) When we have an onsite symmetry K which commutes with G, then the translated Hamiltonian $(TK)H_0(TK)^{-1}$ and H_0 must belong to distinct phases, where T is one-site translational symmetry. T can be also replaced by C_2 the site-centered symmetry.

Corollary 3 (Generalized LSM theorem): When a spin-1/2 chain possesses G and KT (or KC_2), it cannot have a unique gapped ground state.

Statement 4: (The same condition as **Theorem 1** but $G = \mathbb{Z}_2^z \times \mathbb{Z}_2^x$) When we have an onsite symmetry K which commutes with G, then the translated Hamiltonian $(TK)H_0(TK)^{-1}$ and H_0 must belong to distinct phases, where T is one-site translational symmetry. T can be also replaced by C_2 the site-centered symmetry.

Statement 5 (Generalized LSM theorem): It follows from the preceding **Statement 4** that When a spin-1/2 chain possesses G and KT (or KC_2), it cannot have a unique gapped ground state.

Here I use "**Statement**" in that it will be "proven" by a help of a physical assumption. Before proving them, we notice that we need to only prove **Theorem 1** and **Statement 4** since the remaining ones follow directly.

13.0.1 Proof of Theorem 1

Let us denote the unique gapped ground state as $|\mathrm{GS}\rangle$, and consider

$$I[|\mathrm{GS}\rangle] \equiv \langle \mathrm{GS}|\hat{U}(1)|\mathrm{GS}\rangle, \tag{13.2}$$

where

$$\hat{U}(\eta) \equiv \exp\left(i\eta\frac{2\pi}{L}\sum_j j\sigma_j^z/2\right) \tag{13.3}$$

Step 1: I to be proven as a real number.

Since $|\mathrm{GS}\rangle$ possesses $\mathbb{Z}_2^x$ symmetry, we have $\prod_j \sigma_j^x|\mathrm{GS}\rangle \propto |\mathrm{GS}\rangle$ that gives:

$$\begin{aligned} I &= \langle \mathrm{GS}|\prod_j \sigma_j^x|\hat{U}(1)|\prod_k \sigma_k^x|\mathrm{GS}\rangle \quad \leftarrow \mathbb{Z}_2^x \textbf{ used} \\ &= \langle \mathrm{GS}|\hat{U}(-1)|(\prod_j \sigma_j^x)(\prod_k \sigma_k^x)|\mathrm{GS}\rangle \\ &= \langle \mathrm{GS}|\hat{U}(-1)|\mathrm{GS}\rangle = I^*. \end{aligned} \tag{13.4}$$

Step 2: $\lim_{L\to\infty}|I| = 1$ to be proven.

Let us define

$$|\Phi\rangle \equiv \hat{U}(1)|\mathrm{GS}\rangle, \tag{13.5}$$

which means $I = \langle \mathrm{GS}|\Phi\rangle$. Let us simply denote $U \equiv \hat{U}(1)$.

By locality, the Hamiltonian H_0 can be written as $H_0 = \sum_j h_j$ with h_j is supported around position j within an upper bound length $2l$.

$$\begin{aligned}
\Delta E &\equiv \langle\Phi|H_0|\Phi\rangle - \langle \mathrm{GS}|H_0|\mathrm{GS}\rangle \\
&= \sum_j \{\langle\Phi|h_j|\Phi\rangle - \langle \mathrm{GS}|h_j|\mathrm{GS}\rangle\} \quad \leftarrow \textbf{locality used} \\
&= \sum_j \langle \mathrm{GS}|U^\dagger h_j U - h_j|\mathrm{GS}\rangle \\
&= \sum_j \langle \mathrm{GS}|\exp\left(\sum_{m:|m-j|<l} -i\frac{2\pi}{L}m\sigma^z_m/2\right) h_j \exp\left(\sum_{n:|n-j|<l} i\frac{2\pi}{L}n\sigma^z_n/2\right) - h_j|\mathrm{GS}\rangle \\
&= \sum_j \langle \mathrm{GS}|\underbrace{\exp\left(\sum_{m:|m-j|<l} -i\frac{2\pi}{L}(m-j)\sigma^z_m/2\right)}_{U(1)_z \ \textbf{used} \uparrow} h_j \exp\left(\sum_{n:|n-j|<l} i\frac{2\pi}{L}(n-j)\sigma^z_n/2\right) - h_j|\mathrm{GS}\rangle \\
&= \sum_{j=1}^{L} \langle \mathrm{GS}|\underbrace{\exp\left(\sum_{|m|<l} -i\frac{2\pi}{L}m\sigma^z_m/2\right)}_{j\textbf{-independent}} h_j \exp\left(\sum_{|n|<l} i\frac{2\pi}{L}n\sigma^z_n/2\right) - h_j|\mathrm{GS}\rangle
\end{aligned} \tag{13.6}$$

Thence, we can safely Taylor expand the exponential by $O(1/L)$ because the exponentials have become **j-independent**. Otherwise, we cannot do it because j itself can take value around L like $j = L-3$. Therefore, the Taylor expansion reads

$$\begin{aligned}
\Delta E &= \sum_j [(c_j/L) + O(1/L^2)] \\
&= [\sum_j (c_j/L)] + O(1/L) \\
&= O(1/L) \text{ since } c_j = 0 \quad \leftarrow \mathbb{Z}_2^x \ \textbf{used again!}
\end{aligned} \tag{13.7}$$

Due to the fact that $|\mathrm{GS}\rangle$ is the unique and gapped and the lowest energy state,

$$|\Phi\rangle = \exp(i\phi)|\mathrm{GS}\rangle + O(1/L). \tag{13.8}$$

Thus,

$$|I| = |\exp(i\phi)| + O(1/L). \tag{13.9}$$

Step 3: By the proceeding steps, when $L \to \infty$,

$$I = \pm 1. \tag{13.10}$$

13.0.2 "Proof" of Statement 4

We first twist the boundary condition from PBC to TBC by $\mathbb{Z}_2^z$ symmetry and the twisted Hamiltonian is denoted as H_{tw} **still with a unique gapped ground state** (the only assumption) denoted as $(|\text{GS}\rangle)_{\text{tw}}$. We define

$$J[|\text{GS}\rangle] \equiv (\langle\text{GS}|)_{\text{tw}} \prod_j \sigma_j^x (|\text{GS}\rangle)_{\text{tw}} \in \pm 1. \tag{13.11}$$

It is a simple exercise to prove that

$$J[KT|\text{GS}\rangle] = -J[|\text{GS}\rangle], \tag{13.12}$$

because

$$(KT|\text{GS}\rangle)_{\text{tw}} = U_{\text{gauge}} KT (|\text{GS}\rangle)_{\text{tw}}, \tag{13.13}$$

where U_{gauge} is a gauge transformation at the twisting position by $\mathbb{Z}_2^z$.

Hence **Statement 2** follows.

14 Gaussian-integrated Weekly: 2023.1.14

Let us consider the following distribution:

$$\mathcal{P}[\phi_{\vec{q}}] = \frac{1}{\mathcal{Z}} \prod_{\vec{q}} \exp\left[-\frac{K}{2}(q^2+\xi^{-2})\phi^*_{\vec{q}}\phi_{\vec{q}}\right]. \tag{14.1}$$

Since $\phi_{\vec{q}}$ is Fourier components of the real function $\varphi_{\vec{r}}$, we have

$$\phi^*_{\vec{q}} = \phi_{-\vec{q}}. \tag{14.2}$$

Thus the minimal set of independent variables can be chosen as:

$$\{\phi_{\vec{q}} | q_1 > 0\}, \tag{14.3}$$

where $\vec{q} = [q_1, q_2, \cdots]$.

[A careful reader might notice that the zero mode $\phi_{\vec{q}=\vec{0}}$ is not included in the above set. It is generally okay for $\xi^{-1} \neq 0$. However, when $\xi^{-1} = 0$, i.e., at the critical point, such an omission of zero mode has a relation with spontaneously symmetry breaking when ϕ has some internal symmetry, but here I don't want to explain this point here.]

Rewriting $\phi_{\vec{q}}$ as a summation of real part and imaginary part:

$$\phi_{\vec{q}} \equiv \phi'_{\vec{q}} + i\phi''_{\vec{q}}, \tag{14.4}$$

Then by Eq. (14.2), we obtain

$$\phi'_{\vec{q}} = \phi'_{-\vec{q}};\ \phi''_{\vec{q}} = -\phi''_{-\vec{q}}. \tag{14.5}$$

Let us restrict ourselves to the minimal set, we have our familiar

$$\mathcal{P}[\phi_{\vec{q}}] = \frac{1}{\mathcal{Z}} \prod_{\vec{q}:q_1>0} \exp\left\{-K(q^2+\xi^{-2})\left[(\phi'_{\vec{q}})^2 + (\phi''_{\vec{q}})^2\right]\right\}. \tag{14.6}$$

We note that $K/2$ in Eq. (14.1) becomes K in Eq. (14.6) due to the double counting in $\prod_{\vec{q}}$ of independent modes in Eq. (14.1).

Thus, by Eqs. (14.5,14.6) and noting that ϕ' and ϕ'' are stochastically independent, we obtain

$$\begin{aligned}\langle \phi_{\vec{q}}\phi_{\vec{p}}\rangle &= \langle \phi'_{\vec{q}}\phi'_{-\vec{p}}\rangle + \langle \phi''_{\vec{q}}\phi''_{-\vec{p}}\rangle \\ &= \left[\frac{\delta_{\vec{q},-\vec{p}}}{2K(q^2+\xi^{-2})} + \frac{\delta_{\vec{q},-\vec{p}}}{2K(q^2+\xi^{-2})}\right] \\ &= \frac{\delta_{\vec{q},-\vec{p}}}{K(q^2+\xi^{-2})}. \end{aligned} \tag{14.7}$$

15 Ghost's Weekly: 2023.1.31 - 2023.2.3

Today, we will try to figure out how to define string partition functions in a sensible way.

We will take the bosonic string partition function as a concrete practice:

$$Z \approx \int [dgdX] \exp[-S[X,g]], \tag{15.1}$$

where g is a metric applicable to some manifold M and the summation of manifolds of all genus is implicitly done at the end. Here "$\approx$" means formally equal, which is to be made precise below.

Later, we will also deal with conventional gauge theories:

$$Z \approx \int [dA] \exp[-S[A,g]]. \tag{15.2}$$

15.1 Manifold formed by metrics

We need to define the *functional* space W: each *point* (g,X) of W is a metric g that is a tensor field and X that is a scalar field. M is a manifold and the tangent space at any of its point (g,X) can be equipped with a metric form defined as

$$||\delta g||^2 \equiv \int d^2x\sqrt{g}\delta g^{\mu\nu}(G_{\mu\nu\alpha\beta} + Cg_{\mu\nu}g_{\alpha\beta})\delta g^{\alpha\beta}, \tag{15.3}$$

where C is an arbitrary positive constant and G is a projection operator to the traceless part:

$$G^{\mu\nu\alpha\beta} \equiv g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}. \tag{15.4}$$

Therefore, C is a just weight to adjust the relative contributions to $||\delta g||^2$ from the traceless part and the trace part of $\{\delta g_{\mu\nu}\}$. Similarly, the length of a functional change δX that is a vector in the tangent space at (g,X) is

$$||\delta X||_g^2 \equiv \int d^2x\sqrt{g}\delta X\delta X. \tag{15.5}$$

These two equations above defines a Riemannian space W (a manifold equipped with a metric). As we know, any Riemannian manifold automatically has a sensible volume form, e.g., M has a metric g with the volume form as $d^2x\sqrt{g}$. Thus, we have interesting analogs:

$$\text{Manifold: } ||\delta x||^2 = g_{\mu\nu}\delta x^\mu\delta x^\nu \;\; \Rightarrow \int [dx] \equiv \int d^2x\sqrt{g}; \tag{15.6}$$

$$X\text{-functional space: } ||\delta X||_g^2 \;\; \Rightarrow \int [dX]_g; \tag{15.7}$$

$$\text{Metric space: } ||\delta g||^2 \;\; \Rightarrow \int [dg]. \tag{15.8}$$

Remark: It is a good point to stop so that we could check several confusing notations, e.g., δg or δX. Actually, they don't mean infinitesimal. Rather, they are the components of a vector in the tangent space at the point (g,X). Readers who are not familiar with this terminology could think δg as a velocity $\lim_{\delta s\to 0}\delta g(s)/\delta s$. This is the reason that δg is only determined by the infinitesimal change of g as below.

15.2 More on the notations

The gauge group G =Weyl×Diff is still a Lie group of infinite dimensions and it can have various disconnected components. For the connected part G_0 including the identity element, it is natural to have a Lie algebra and define the following notation through exponential mapping:

$$\zeta = \exp_G(\zeta), \tag{15.9}$$

where we simply do not distinguish such a group element and its corresponding Lie algebra element. Since the Lie algebra is a vector space, it has coordinate system and this coordinate system is now used to parameterize the group component G_0. Thus the measure of this coordinate is defined in the usual way respecting the general covariance through the volume form induced from the length:

$$||\delta\zeta||^2 \equiv \int \sqrt{g} A g_{\mu\nu} \delta\xi^\mu \delta\xi^\nu + B\delta\omega^2, \tag{15.10}$$

with arbitrary positive constants A and B.

Let us denote the discrete part of G by D; D contains the identity and one element in each of the other components. It should be noted that D depends on our choice. We will denote the elements of D as q. Thus a general element in G can be written as $q\exp_G(\zeta)$. We also denote ${}^{q\zeta}g$ as:

$${}^{q\zeta}g \equiv q[\exp_G(\zeta)(g)]. \tag{15.11}$$

As a digression that we do not need to use, ζ induces a tangent vector field on the manifold W, so the above action can be understood as the result of the one-parameter group by such a W-vector field when time reaches one.

Eventually, we can integrate any function $\mathcal{F}$ of G over the group G through:

$$\sum_{q\in D} \int [d\zeta]_g \mathcal{F}[q\zeta]. \tag{15.12}$$

Here g will always denote the worldsheet metric rather than any element of G.

15.3 Gauge fixing by FP determinants

Let us first define the concept of slice of some reference metric $\hat{g}$.

$$\mathrm{Slice}[\hat{g}] \equiv \{\hat{g}(t) | t \in \mathrm{Mod}\}, \tag{15.13}$$

where "Mod" means the modular space inaccessible by gauge group on $\hat{g}$.

Similarly, we can define the slice for arbitrary metric ${}^{q\zeta}[\hat{g}(t)]$ as

$$\mathrm{Slice}[{}^{q\zeta}[\hat{g}(t)]] = \mathrm{Slice}[{}^{q\zeta}\hat{g}] = \{{}^{q\zeta}[\hat{g}(t)] | t \in \mathrm{Mod}\}. \tag{15.14}$$

We will suppress "$[\cdots]$" of ${}^{q\zeta}[\hat{g}(t)]$. We have enough tools to define the FP determinant as

$$\Delta_{\mathrm{FP}}^{\mathrm{Slice}[\hat{g}]}(g)^{-1} \equiv \int_{\mathrm{Mod}} dt \sum_{q\in D} \int [d\zeta]_{\hat{g}(t)} \delta(g - {}^{q\zeta}\hat{g}(t)). \tag{15.15}$$

Let us investigate the relation between $\Delta_{\rm FP}^{{\rm Slice}[{}^{q\zeta}\hat{g}]}({}^{q\zeta}\hat{g}(t))$ and $\Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}({}^{q\zeta}\hat{g}(t))$:

$$\begin{aligned}
\Delta_{\rm FP}^{{\rm Slice}[{}^{q\zeta}\hat{g}]}({}^{q\zeta}\hat{g}(t))^{-1} &= \int dl \sum_{p\in D} \int [d\alpha]_{{}^{q\zeta}\hat{g}(l)} \delta[{}^{q\zeta}\hat{g}(t) - {}^{p\alpha}({}^{q\zeta}\hat{g}(l))] \\
&= \int dl \sum_{pq\in D\cdot q} \int [d\tilde{\alpha}\zeta]_{\hat{g}(l)} \left|\frac{\partial(\tilde{\alpha}\zeta)}{\partial(\alpha)}\right|_{\alpha_0,l_0} \delta[{}^{q\zeta}\hat{g}(t) - {}^{pq\tilde{\alpha}\zeta}\hat{g}(l))] \\
&= \left|\frac{\partial(\tilde{\alpha}\zeta)}{\partial(\alpha)}\right|_{\alpha_0,l_0}^{-1} \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}({}^{q\zeta}\hat{g}(t))^{-1},
\end{aligned} \tag{15.16}$$

where $\tilde{\alpha}\zeta$ is defined through $q\exp_G(\tilde{\alpha}\zeta) = \exp_G(\alpha) q \exp_G(\zeta)$. Since l is restricted in a fixed modular space, we have $l_0 = t$ and $\alpha_0 = 0$. Also only $pq = q$ contributes to the summation, so we can simply replace $D\cdot q$ by D because both of them contains q. It should be noted that generally, (α_0, l_0) can take multiple possibilities to make the δ-function non-vanishing, if, for example, $l_0 = t$ and α is a conformal Killing vector field of ${}^{q\zeta}\hat{g}(t)$, but for generically ζ, there is no conformal Killing vector field. Since we will integrate out ζ, such effects are just zero measure that we assume.

To define a meaningful partition function, we need the quantity $V_g \equiv \sum_{q\in D}\int [D\zeta]_g$. Then,

$$\begin{aligned}
Z &\equiv \int [dg] \frac{[dX]_g}{V_g} \exp[-S(g,X)] \\
&= \int [dg] \frac{[dX]_g}{V_g} \exp[-S(g,X)] \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}(g) \int dt \sum_{q\in D} \int [d\zeta]_{\hat{g}(t)} \delta(g - {}^{q\zeta}\hat{g}(t)) \\
&= \int dt \sum_q \int [d\zeta]_{\hat{g}(t)} \frac{[dX]_{{}^{q\zeta}\hat{g}(t)}}{V_{{}^{q\zeta}\hat{g}(t)}} \exp[-S({}^{q\zeta}\hat{g}(t), X)] \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}({}^{q\zeta}\hat{g}(t)) \\
&= \int dt \sum_q \int [d\zeta]_{\hat{g}(t)} \frac{[dX]_{{}^{q\zeta}\hat{g}(t)}}{V_{{}^{q\zeta}\hat{g}(t)}} \left|\frac{\partial(\alpha\zeta)}{\partial(\alpha)}\right|_{\alpha_0,l_0}^{-1} \exp[-S({}^{q\zeta}\hat{g}(t), X)] \Delta_{\rm FP}^{{\rm Slice}[{}^{q\zeta}\hat{g}]}({}^{q\zeta}\hat{g}(t)) \\
&= \int dt \frac{\sum_q \int [d\zeta]_{\hat{g}(t)}}{V_{\hat{g}(t)}} [d\,{}^{q\zeta}X]_{{}^{q\zeta}\hat{g}(t)} \exp[-S({}^{q\zeta}\hat{g}(t), {}^{q\zeta}X)] \Delta_{\rm FP}^{{\rm Slice}[{}^{q\zeta}\hat{g}]}({}^{q\zeta}\hat{g}(t)) \\
&= \int dt \frac{\sum_q \int [d\zeta]_{\hat{g}(t)}}{V_{\hat{g}(t)}} [dX]_{\hat{g}(t)} \exp[-S(\hat{g}(t), X)] \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}(\hat{g}(t)),
\end{aligned} \tag{15.17}$$

where we have take $D = 26$ so that

$$[d\,{}^{q\zeta}X]_{{}^{q\zeta}\hat{g}(t)} \Delta_{\rm FP}^{{\rm Slice}[{}^{q\zeta}\hat{g}]}({}^{q\zeta}\hat{g}(t)) = [dX]_{\hat{g}(t)} \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}(\hat{g}(t)), \tag{15.18}$$

and

$$V_{{}^{q\zeta}\hat{g}(t)} \left|\frac{\partial(\alpha\zeta)}{\partial\alpha}\right|_{\alpha_0,l_0} = \int [d\alpha]_{{}^{q\zeta}\hat{g}(t)} \left|\frac{\partial(\tilde{\alpha}\zeta)}{\alpha}\right| = \int [d\tilde{\alpha}\zeta]_{\hat{g}(t)} = V_{\hat{g}(t)}. \tag{15.19}$$

Then

$$Z = \int_{\rm Mod} dt \int [dX]_{\hat{g}(t)} \exp[-S(\hat{g}(t), X)] \Delta_{\rm FP}^{{\rm Slice}[\hat{g}]}(\hat{g}(t)). \tag{15.20}$$

Let us also make the theory local by evaluating $\Delta_{\rm FP}^{\rm Slice[\hat{g}]}(\hat{g}(t))^{-1}$:

$$\Delta_{\rm FP}^{\rm Slice[\hat{g}]}(\hat{g}(t))^{-1} = \int d\delta t \int [d\zeta]_{\hat{g}(t)} \delta[\hat{g}(t) -^{\zeta} \hat{g}(t+\delta t)]$$
$$= |{\rm CKV}[\hat{g}(t)]| \int d\delta t \int_{{\rm CKV}^{\perp}} [d\delta\bar{\zeta}] \exp\left\{-\int d^2x \sqrt{\hat{g}(t)} \frac{1}{0^+}[\delta\bar{\xi}^T P^T P \delta\bar{\xi} + \delta w \delta w + \delta t^i M_{ij} \delta t^j]\right\}$$
$$= |{\rm CKV}[\hat{g}(t)]| \sqrt{{\rm det}' P^T P}^{-1} \sqrt{{\rm det} M}^{-1}, \tag{15.21}$$

where $\delta\bar{\zeta}$ means the tangent vector orthogonal to the conformal Killing vector (CKV) fields that take into account the zero modes of $P^\dagger P$ thereby $\det'$ omitting the zero modes, and the matrix M_{ij} can be derived from projection procedures that can be found in standard textbooks.

15.4 Consistency conditions

Since "Mod" can be arbitrarily chosen, we must require that the integrand

$$Z[\hat{g}(t)] \equiv \int [dX]_{\hat{g}(t)} \exp[-S(\hat{g}(t), X)] \Delta_{\rm FP}^{\rm Slice[\hat{g}]}(\hat{g}(t)) \tag{15.22}$$

be modular invariance.

As mentioned before, we need $Z[g]$ to be Weyl symmetric, e.g., $D = 26$, so that the calculation before makes sense; otherwise we will have an additional Liouville theory.

One final point that I actually have not fully understood is that $1/V_g$ strongly depends on g and the content of D since our measure is not Haar. Such an ambiguity does not appear in the final result due to a perfect cancellation by Eq. (15.19). It is interesting to study whether it is possible to define a Haar measure for G, which is also general coordinate covariant.

Caveat: The footnote on Page 87 in Polchinski is actually incorrect; the Jacobian of the F-P determinant is actually the conformal anomaly of the ghosts to be cancelled with that of $[dX]_g$. Also, in most textbooks, V_g is taken as a constant, but in our systematic derivation so far, we have no reason or necessity to assume this condition. In Nakahara, I don't think that Eq. (14.39) is correct in that the volume of the V(Diff*Weyl) is not well-defined when not given a metric, and even when given a metric, it would wildly and rather discontinuously fluctuate when that metric varies. The confusion is that the final result Eq. (14.954) in Nakahara sounds correct because the V(CKV) is incorrectly introduced in Eq. (14.86). Double mistakes return correctness. Furthermore, the interpretation of the denominator of Eq. (7.3.3) in Polchinski's Vol. 1 V(CKG) is inappropriate although, of course, it does not affect the final result. A correct calculation of V(CKG) is in Page 146 of *Basic Concepts of String Theory*, If we want to extract out V(CKG) in Polchinski's approach, the normalization of the zero mode insertion $b_0\tilde{b}_0c_0\tilde{c}_0$ should be taken care seriously.

15.5 For nonabelian gauge theory

For nonabelian gauge theory, we fix the background spacetime geometry $g_{\mu\nu}$, so we will reserve g for the element of the local gauge group G. First G may be disconnected, e.g., it can contain large gauge transformations that are not connected to the identity. We first construct D with one element in each connected component. Then, we decompose the group element as

$$g = q \exp_G(\theta), \tag{15.23}$$

where $q \in D$ and θ is a vector in Lie algebra. We use the Haar measure

$$[dg] = [d(hgh')], \tag{15.24}$$

where h and h' are arbitrary elements in G. The integration over arbitrary function $\mathcal{F}$ from G can be written as

$$\int [dg]\mathcal{F}(g) = \sum_{q\in D} \int [d\theta]\mathcal{F}[q\exp(\theta)]. \tag{15.25}$$

15.6 FP procedures

We define the FP determinant as

$$1 = 1_\Omega \equiv \Delta^\Omega_{\rm FP}[A] \int [d\bar{A}]\delta[f_\Omega(\bar{A})] \int [dg]\delta(A - {}^g\bar{A}), \tag{15.26}$$

where

$$f_\Omega(\bar{A}) \equiv \partial_\mu A^\mu - \Omega, \tag{15.27}$$

which is to fix the gauge, but it is not fully done due to Killing groups. The elements in Killing groups are just gauge transformation but does not break the gauge condition above. We also obtain

$$\begin{aligned}
\Delta^\Omega_{\rm FP}[{}^hA]^{-1} &= \int [d\bar{A}]\delta[f_\Omega(\bar{A})] \int [dg]\delta({}^hA - {}^g\bar{A}) \\
&= \int [d\bar{A}]\delta[f_\Omega(\bar{A})] \int [d(h^{-1}g)]\delta[{}^h(A - {}^{h^{-1}g}\bar{A})] \\
&= \int [d\bar{A}]\delta[f_\Omega(\bar{A})] \int [dg]\delta(A - {}^g\bar{A}) \\
&= \Delta^\Omega_{\rm FP}[A]^{-1},
\end{aligned} \tag{15.28}$$

where we have used the left invariance of Haar measure and

$$\begin{aligned}
&\delta[{}^h(A - A')] = \exp[-\frac{1}{0^+}\int {\rm tr}^h(A - A')^h(A - A')] \\
&= \exp[-\frac{1}{0^+}\int {\rm tr}(A - A')(A - A')] = \delta(A - A'),
\end{aligned} \tag{15.29}$$

since

$${}^h(A - A') = U_h^\dagger (A - A')U_h. \tag{15.30}$$

We have suppressed the spacetime metric and we will keep doing so.

Defining $V_G \equiv \int [dg]$, we can further define

$$\begin{aligned}
Z &\equiv \int [d\Omega] \exp\left[-\int \frac{\Omega\Omega}{\xi^2} \right] \frac{1}{V_G} \left\{ 1_\Omega \cdot \int [dA] \exp(-S[A]) \right\} \\
&= \int [d\Omega] \exp\left[-\int \frac{\Omega\Omega}{\xi^2} \right] \int [dA] \Delta_{\mathrm{FP}}^{\Omega}[A] \delta[f_\Omega(A)] \exp(-S[A]) \\
&= \int [dA] \Delta_{\mathrm{FP}}^{\Omega=\partial A}[A] \exp(-S[A] - S_{\mathrm{fix}})
\end{aligned} \tag{15.31}$$

where

$$S_{\mathrm{fix}} \equiv \int \frac{(\partial A)^2}{\xi^2}. \tag{15.32}$$

Let us evaluate the FP determinant

$$\begin{aligned}
\Delta_{\mathrm{FP}}^{\Omega=\partial A}[A]^{-1} &= \int dB \delta[\partial(B - A)] \int [dg] \delta(A - {}^g B) \\
&= \int dB \delta[\partial(B - A)] \int [dg] \delta[({}^{g^{-1}} A - B)] \\
&= \int [dg] \delta[\partial(A - {}^{g^{-1}} A)] \\
&= |\mathrm{KG}[A]| \int_{\mathrm{KG}^\perp} [d\delta\theta] \delta[\partial^\mu D_\mu \theta] \\
&= |\mathrm{KG}[A]| \frac{1}{\sqrt{\det'(\partial^\mu D_\mu)^\dagger \partial^\mu D_\mu}},
\end{aligned} \tag{15.33}$$

where again the zero modes are taken care of by the Killing group (KG) whose elements α are determined by

$$\partial^\mu D_\mu \alpha = 0, \tag{15.34}$$

which depends on A through D_μ in non-abelian cases.

15.7 Final remarks

We should note that the above equations have a loop hole; the δ-function in the line of $[d\delta\theta]$ is from that of A fields, thereby from the A's measure. However, $[d\theta]$ is Haar measure and its δ-function is not necessarily the same as its following δ-function. Thus, there should have been a constant coefficient taking into consideration the ratio of the volume form of Haar measure and conventional measure at the identity of G.

One another remark is the volume $|\mathrm{KG}|$ of Killing group. As we know in abelian gauge field, uniform large gauge transformations classified by $\mathbb{Z}$ on torus belong to $\mathrm{KG} \cong \mathbb{Z} \times \mathrm{U}(1)$, but there is infinitely many members. Therefore, we need to also discard this $|\mathbb{Z}|$ part.

16 Stress tensor no stress: 2023.2.6 - 2023.2.13

To simplify the notations, we will use g to label metric tensor, while when we want to write its determinant, we usually add a square root and use $\sqrt{g}$ without confusion.

We will always take the free boson as a concrete example to think of:

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu X\partial_\nu X \\ &= \frac{1}{4\pi\alpha'}\int dzd\bar{z}\frac{1}{i}g_{z\bar{z}}g^{z\bar{z}}\partial X\bar{\partial}X \\ &= \frac{1}{4\pi i\alpha'}\int dzd\bar{z}\partial X\bar{\partial}X, \end{aligned} \tag{16.1}$$

where $dzd\bar{z} \equiv dz\wedge d\bar{z} = -2idx\wedge dy$ and we have stayed in a conformal coordinate such that

$$g_{zz} = g_{\bar{z}\bar{z}} = 0. \tag{16.2}$$

16.1 Conformal transformation

A conformal transformation involves two manifolds M and M':

$$f : M \to M'. \tag{16.3}$$

We will use coordinate labels x for M and x' for M', *etc.*. M and M' are equipped with g and g' as their metrics. We will assume f is bijective, or bijective locally after we take only a little piece. On M', we could have another metric $\tilde{g}$ induced by f as:

$$\tilde{g}(f(x)) \equiv \frac{dx}{df}\frac{dx}{df}g(x). \tag{16.4}$$

We will relabel this manifold as $(\tilde{g},\tilde{M}) \equiv (\tilde{g}, M')$. By definition, f is an isometry between M and $\tilde{M}$. Such a new manifold $\tilde{M}$ seems useless at this point. The conformal transformation f does the following job:

$$g(x)dxdx = \tilde{g}(f(x))df(x)df(x) = \Omega(f(x))^{-2}g'(f(x))df(x)df(x) \tag{16.5}$$

for some positive function $\Omega : M' \to \mathbb{R}^+$. Thus $\tilde{g} = \Omega^{-2}g'$.

Caveat: Eq. (4.1.2) of String Nutshell is incorrect; $g'_{\mu\nu}(x')$ should have been $g'_{\mu\nu}(x)$.

Without the intermediate auxiliary manifold $\tilde{M}$, the conformal transformation means that the small segment on M with length ds is mapped to a small segment on M' but with length $\Omega^2 ds$.

If we take into account $\tilde{M}$, the interpretation is different; first we do a general coordinate transformation from M to $\tilde{M}$, and then do a Weyl transformation with $\tilde{g} \to \Omega^2\tilde{g} = g'$. Since a general coordinate transformation is doing nothing physically, we can roughly think the conformal transformation as equivalent to a subclass of Weyl transformations.

Exercise: Therefore, it is sufficient to have a classical Weyl invariance of the action:

$$\frac{\delta S[g,X]}{\delta g_{\mu\nu}}g_{\mu\nu} = 0 \Rightarrow T^\mu{}_\mu = 0 \Rightarrow T^z{}_z = T^{\bar{z}}{}_{\bar{z}} = 0 \quad \cdots \text{Using } T^{z\bar{z}} = T^{\bar{z}z}. \tag{16.6}$$

In the path integral, A naive "derivation" could be

$$\begin{aligned}\langle\cdots\rangle_g &= \int [dX]_g \exp(-S[g(1+2\delta\omega)])[\cdots] \\ &\stackrel{?}{=} \int [dX]_g \exp(-S[g])(1-8\pi\int\sqrt{g}T^{\mu}{}_{\mu}\delta\omega)[\cdots],\end{aligned} \tag{16.7}$$

which (wrongly) imply that the insertion of $T_{z\bar{z}}$ in the path integral even has zero contact term with any other insertion $[\cdots]$.

Actually, $T^{\mu}{}_{\mu}$ is only formal since it is a composite operator (although zero classically) and such a zero needs a regularization. Conversely, we can also see the above equation as a partial definition of such a regularization. This regularization will be inconsistent with the general covariance later. In the similar sense, $T^{\mu\nu} - T^{\nu\mu}$ can be also nontrivial when contacted with other insertions due to regularization.

Caveat: In some textbook, the stress tensor is defined through path integral like Eq. (3.4.4) on Page 91 in Polchinski Vol. 1, which also includes the variation of the measure. However, we will always explicitly and distinctly define the stress tensor by classical formulation:

$$T^{\mu\nu}(x) \equiv \frac{4\pi}{\sqrt{g}}\frac{\delta S[\{g_{\alpha\beta}\}, X]}{\delta g_{\mu\nu}}, \tag{16.8}$$

where one should keep in mind that all the metric dependence in $S[g, X]$ is through $g_{\alpha\beta}$ rather than $g^{\alpha\beta}$, i.e., $g^{\alpha\beta}$ depending on $g_{\alpha\beta}$ by a matrix inverse. The advantage of our definition is that everything can be defined already by classical action followed by a proper regularization as we will discussed later around Eq. (16.23). Therefore, we should be cautious about the regularization and we need to take a guess on it at first. The path-integral formula of the stress tensor is more compact and even applicable for the non-Lagrangian cases, although the explicit form of the stress tensor might not be clear. We will reproduce all the following results again in that framework.

When $M' = M$ and $g' = g$ which we will always take to be true, then the conformal transformations form a group, so-called the conformal group. If the action $S[g, X]$ is invariant under the conformal group, we will say that the system is classically conformal invariant.

Take the bosonic model as an example, the general coordinate transformation $\tilde{X}(x)$ of X is defined through

$$\tilde{X}(f(x)) = X(x), \tag{16.9}$$

and

$$S[g, X] \mapsto S[\tilde{g}, \tilde{X}] \mapsto S[g, \tilde{X}], \tag{16.10}$$

where the first one is the general coordinate transformation while the second one is the Weyl transformation that maps the metrics back. If $S[g, \tilde{X}] = S[g, X]$ as it is indeed the case for free massless boson above, the system is conformally invariant. It should be emphasized that the metrics are the same for both ends of Eq. (16.10), so the transformation is actually effecting only dynamical variable like X, rather than the background information like g. Thus it is indeed a symmetry transformation in our common viewpoint.

16.2 Preparations for Conformal Ward Identity

We already know that the measure is defined through the following length element in the functional space:

$$||\delta X||_g^2 = \int d^2x\sqrt{g}\delta X\delta X, \tag{16.11}$$

so the corresponding measure $[dX]_g$ only depends on the determinant of the metric or $\sqrt{g}$ and it is general covariant:

$$[dX]_g = [d\tilde{X}]_{\tilde{g}}. \tag{16.12}$$

For later convenience, when we will also sometimes write $[dX]_g$ as $[dX]_{\sqrt{g}}$ interchangeably, but no confusion rises since the measure indeed only depends on the metric determinant.

However, when we do the second step, i.e., the Weyl rescaling $\tilde{g} \to \Omega^2\tilde{g} = g$, we might have $[dX]_{\tilde{g}} \neq [dX]_g$. Let us take an infinitesimal form of transformation

$$f(z) = z + \epsilon v^z(z), \tag{16.13}$$

where v^z is holomorphic and ϵ is a label to tell us the order of smallness, which will be taken to be space dependent. So we do not take ϵ as a constant. In general coordinate systems and we define

$$\tilde{g}(x) = g(x) + \delta g(x), \tag{16.14}$$

then we have

$$\delta g_{\mu\nu}(x) = -\nabla_\mu \epsilon v_\nu - \nabla_\nu \epsilon v_\mu. \tag{16.15}$$

Exercise: Prove that $\sqrt{\tilde{g}} = \Omega^{-2}\sqrt{g}$, where

$$\Omega^2 = 1 + \nabla\cdot(\epsilon v) \equiv 1 + 2\delta\omega. \tag{16.16}$$

Then, since the measure only depends on the determinant of the metrics

$$[dX]_{\sqrt{g}} = [dX]_{\Omega^2\sqrt{\tilde{g}}} = [dX]_{\sqrt{\tilde{g}}}\left[1 + \frac{c}{24\pi}\int d^2x\sqrt{g}R\delta\omega\right], \tag{16.17}$$

where R is the scalar curvature. We will first develop ϵ to be coordinate dependent $\epsilon(z,\bar{z})$, so the action is not necessarily invariant under this modulating transformation. However, we know that the action is invariant when ϵ is constant. The corresponding general coordinate transformation of dynamical variable is

$$\tilde{X}(x) = X(x) - \epsilon(x)v^\mu(x)\partial_\mu X(x). \tag{16.18}$$

Exercise: Prove that $[dX]_g = [dX]_{\tilde{g}}$ when ϵ in Eq. (16.13) is constant and $f(z)$ is a *globally defined* holographic function. As a corollary without calculation, $\int d^2x\sqrt{g}R\delta\omega = 0$.

Caveat: Our way to define j as above is distinct from Eqs. (2.3.3, 2.3.4) on Page 41 Polchinski Vol. 1, where the coordinate dependence is directly multiplied to the operator transformation. Although these two ways are equivalent for free boson theory, they are different for other non-scalar theories, e.g., bc and $\beta\gamma$ CFTs. One could try to use Polchinski's way, but the resultant incorrect $\bar{T} \neq 0$. Our approach is actually *correct* and systematic in order that Eq. (16.20) can be generally derived.

Thus we could argue, when $\epsilon(x)$ is compactly supported and is nonzero only where $v^z(z)$ is well-defined:

$$\begin{aligned} S[g, \tilde{X}] &= S[g, X] - \frac{i}{2\pi} \int d^2x \sqrt{g} j^\mu \nabla_\mu \epsilon \\ &= S[g, X] + \frac{i}{2\pi} \int d^2x \sqrt{g} \epsilon \nabla \cdot j \end{aligned} \tag{16.19}$$

the first line of which can be thought as the definition of j^μ up to some ambiguity such as a constant shift. For the global symmetry cases, i.e., $v^z(z)$ is globally well-defined, the derivative "$\nabla\epsilon$" is natural since, when ϵ is a constant, the action is invariant. While for locally defined transformations, i.e., $v^z(z)$ is only locally well-defined, the first line can be considered as the definition of the local symmetries. The second line has made us of the fact that ϵ is compactly supported.

Exercise: By Eq. (16.8) and using $S[g, X] = S[\tilde{g}, \tilde{X}]$ in Eq. (16.19), show that

$$j_\mu = iv^\nu T_{\mu\nu} \quad \cdots \text{Using } \nabla_z v^{\bar{z}} = \nabla_{\bar{z}} v^z = 0, \tag{16.20}$$

where $T_{z\bar{z}} = 0$ already obtained before by Weyl invariance of the action and for the free massless boson theory

$$\begin{aligned} T_{zz} &= -\frac{1}{\alpha'} \partial X \partial X; \\ T_{\bar{z}\bar{z}} &= -\frac{1}{\alpha'} \bar{\partial} X \bar{\partial} X. \end{aligned} \tag{16.21}$$

However, the above operators T_{zz} and $T_{\bar{z}\bar{z}}$ are hardly well-defined and they need regularizations. In the flat manifold, we have

$$[T_{zz}]_{g=\delta} = \lim_{z' \to z} -\frac{1}{\alpha'} \partial X(z') \partial X(z) - \frac{1}{2(z'-z)^2}. \tag{16.22}$$

Since the general coordinate covariance is a prerequisite,

$$[T_{zz}]_g = \lim_{z' \to z} -\frac{1}{\alpha'} \partial X(z') \partial X(z) + \frac{1}{2} \partial_{z'} \partial_z \ln ||z', z||_g^2, \tag{16.23}$$

where $||z', z||_g = ||z, z'||_g$ is the geodesic distance between two sufficiently near points with coordinates z' and z under the metric g. In the flat-plane case, $\langle T_{zz}(z_0) \rangle_{g=\delta} = 0$ because T_{zz} transforms nontrivially under *global* conformal symmetry transformations that could fix the z_0 that can actually be any other point. Therefore, generic composite tensor operators $[A_{\mu\cdots}\{X(x_i)\}]_g$ are metric-dependent in a general coordinate covariant way:

$$[A_{\alpha\cdots}\{\tilde{X}(\tilde{x})\}]_{\tilde{g}} d\tilde{x}^{\alpha\cdots} = [A_{\mu\cdots}\{X(x)\}]_g dx^{\mu\cdots}. \tag{16.24}$$

It should be noted that the conformal transformation is only done on the dynamical variable X while keeping the metric g intact, so it is natural to define the following variation

$$[\delta A_{\alpha\cdots}(x)]_g \equiv [A_{\alpha\cdots}\{\tilde{X}(x)\}]_g - [A_{\alpha\cdots}\{X(x)\}]_g. \tag{16.25}$$

16.3 Conformal Ward Identity

Now we have enough tools to write down the conformal Ward identity. Let us choose

$$\epsilon(x) \approx \begin{cases} \epsilon_0, \ x \in B_0; \\ 0, \ x \notin B_0, \end{cases} \tag{16.26}$$

where B_0 is any open set containing the points x_i of the operator $[A\{X(x_i)\}]_g$ as interior points of B_0. Here "$\approx$" means that we will require $\epsilon(x)$ to decay sufficiently rapidly to zero when approaching the boundary of B_0.

$$\begin{aligned}
&\int [dX]_g [A\{X(x_i)\}]_g \exp[-S[g,X]] \\
= &\int [d\tilde{X}]_g [A\{\tilde{X}(x_i)\}]_g \exp[-S[g,\tilde{X}]] \\
= &\int \left\{ [d\tilde{X}]_{\tilde{g}} \left[1 + \frac{c}{24\pi} \int d^2x \sqrt{g} R \delta\omega \right] \right\} \{[A\{X(x_i)\}]_g + [\delta A]_g\} \\
&\left\{ \exp[-S[g,X]] \left[1 - \frac{i}{2\pi} \int d^2x \sqrt{g} \epsilon \nabla \cdot j \right] \right\} \cdots \text{Regularization fixed by this expression} \\
= &\int \left\{ [dX]_g \left[1 - \frac{c}{48\pi} \int d^2x \sqrt{g} \epsilon v^\mu \partial_\mu R \right] \right\} \{[A\{X(x_i)\}]_g + [\delta A]_g\} \\
&\left\{ \exp[-S[g,X]] \left[1 - \frac{i}{2\pi} \int d^2x \sqrt{g} \epsilon \nabla \cdot j \right] \right\} \cdots \text{By } [d\tilde{X}]_{\tilde{g}} = [dX]_g \text{ and the definition of } \delta\omega \\
= &\int \left\{ [dX]_g \left[1 - \frac{c}{48\pi} \epsilon_0 \int_{B_0} d^2x \sqrt{g} v^\mu \partial_\mu R \right] \right\} \{[A\{X(x_i)\}]_g + [\delta A]_g\} \\
&\left\{ \exp[-S[g,X]] \left[1 - \frac{i}{2\pi} \epsilon_0 \int_{B_0} d^2x \sqrt{g} \nabla \cdot j \right] \right\} \cdots \text{Using the definition of } \epsilon(x).
\end{aligned} \tag{16.27}$$

Therefore,

$$\int [dX]_g [\delta A]_g \exp[-S[g,X]] = \epsilon_0 \int [dX]_g [A\{X(x_i)\}]_g \left\{ \int_{B_0} d^2x \sqrt{g} \left[\frac{c}{48\pi} v \cdot \partial R + \frac{i}{2\pi} \nabla \cdot j \right] \right\}. \tag{16.28}$$

Now comes an interesting trick in the complex coordinate:

$$\partial_{\bar{z}} t_{zz} = -\frac{1}{2} g_{z\bar{z}} \partial_z R, \tag{16.29}$$

where

$$\begin{aligned}
t_{zz} &\equiv \partial_z \Gamma^z{}_{zz} - \frac{1}{2} \left(\Gamma^z{}_{zz}\right)^2; \\
\Gamma^z{}_{zz} &= \partial_z \ln g_{z\bar{z}},
\end{aligned} \tag{16.30}$$

similarly for $t_{\bar{z}\bar{z}}$. Then

$$\begin{aligned}
&\int_{B_0} d^2x\sqrt{g}v\cdot\partial R \\
&= \int_{B_0} idzd\bar{z}g_{z\bar{z}}[v^z\partial_z R + v^{\bar{z}}\partial_{\bar{z}}R] \quad \cdots \text{Using complex coordinate} \\
&= \int_{B_0} -2idzd\bar{z}[v^z\partial_{\bar{z}}t_{zz} + v^{\bar{z}}\partial_z t_{\bar{z}\bar{z}}] \cdots \text{Using the trick} \\
&= \int_{B_0} -2idzd\bar{z}\left\{\partial_{\bar{z}}[v^z t_{zz}] + \partial_z[v^{\bar{z}}t_{\bar{z}\bar{z}}]\right\} \cdots \text{Using the (anti-)holomorphicity of } v^z\ (v^{\bar{z}}) \\
&= 2i\oint_{\partial B_0} dz[v^z t_{zz}] - d\bar{z}[v^{\bar{z}}t_{\bar{z}\bar{z}}].
\end{aligned} \tag{16.31}$$

Moreover,

$$\begin{aligned}
&\int_{B_0} d^2x\sqrt{g}\nabla\cdot j \\
&= \int_{B_0} -dzd\bar{z}g_{z\bar{z}}\{\nabla_{\bar{z}}[v^\mu T_{\mu z}g^{z\bar{z}}] + \nabla_z[v^\mu T_{\mu\bar{z}}g^{z\bar{z}}]\} \\
&= \int_{B_0} -dzd\bar{z}\{\partial_{\bar{z}}[v^\mu T_{\mu z}] + \partial_z[v^\mu T_{\mu\bar{z}}]\} \cdots \text{Using } \nabla g = 0 \text{ and mixed connections vanish} \\
&= \oint_{\partial B_0} dz[v^z T_{zz}] - d\bar{z}[v^{\bar{z}}T_{\bar{z}\bar{z}}] \cdots \textbf{Exercise: } \text{Prove } T_{\bar{z}z} = T_{\bar{z}z} - T_{z\bar{z}} = 0 \text{ when contactless.}
\end{aligned} \tag{16.32}$$

It should be remarked that these two results are explicitly metric independent, but one should keep in mind that this phenomenon implicitly depends on the fact that we are using the conformal complex coordinate. Putting all these together, we obtain

$$\begin{aligned}
&\int [dX]_g([\delta A]_g/\epsilon_0)\exp[-S[g,X]] \\
&= \int [dX]_g\left[-\frac{1}{2\pi i}\oint_{\partial B_0}\left(dzv^z\hat{T}_{zz} - d\bar{z}v^{\bar{z}}\hat{T}_{\bar{z}\bar{z}}\right)\right][A\{X(z_i)\}]_g\exp[-S[g,X]],
\end{aligned} \tag{16.33}$$

where

$$\left[\hat{T}_{zz}\right]_g \equiv [T_{zz}]_g + \frac{c}{12}[t_{zz}]_g; \tag{16.34}$$

$$\left[\hat{T}_{\bar{z}\bar{z}}\right]_g \equiv [T_{\bar{z}\bar{z}}]_g + \frac{c}{12}[t_{\bar{z}\bar{z}}]_g\,. \tag{16.35}$$

Exercise: Prove that $\partial_{\bar{z}}[\hat{T}_{zz}(z)]_g = \partial_z[\hat{T}_{\bar{z}\bar{z}}(z)]_g = 0$ by taking $A = 1$.

Since we can insert any other operators away from $[A\{X(z_i)\}]_g$, we have the corresponding OPEs between $[A\{X(z_i)\}]_g$ and $[\hat{T}]_g$ knowing the form of $(\delta A/\epsilon_0)$.

The final question is whether we have the well-known OPE between $\hat{T}$s? It is equivalent to prove whether the following is true: when z is well in B_0 such that ϵ is a constant thereby $f(z)$ being holomorphic in Eq. (16.13),

$$\left[\hat{\hat{T}}_{zz}(f(z))\right]_g dfdf = \left[\hat{T}_{zz}(z)\right]_g dzdz - \frac{c}{12}\{f,z\}dzdz, \tag{16.36}$$

where the tilde in $[\tilde{\hat{T}}_{zz}(z)]_g$ means the replacements of $X(z)$ by $\tilde{X}(z)$ in $[\hat{T}_{zz}(z)]_g$, i.e., $[\tilde{\hat{T}}_{zz}(z)]_g$ being transformed from $[\hat{T}_{zz}(z)]_g$ by conformal symmetry transformation. To prove it, we can use the following useful fact:

$$[t_{zz}]_{\tilde{g}} df df = t_{zz} dz dz - \{f, z\} dz dz. \tag{16.37}$$

Then it is equivalent to prove

$$\begin{aligned} &\left[\tilde{\hat{T}}_{zz}(f(z))\right]_g df df = \left[\tilde{T}_{zz}(f(z))\right]_{\tilde{g}} df df + \frac{c}{12}\left\{[t_{zz}]_{\tilde{g}} df df + \{f,z\} dz dz\right\} - \frac{c}{12}\{f,z\} dz dz \\ &\Leftrightarrow [T_{zz}]_g - [T_{zz}]_{\tilde{g}} = -\frac{c}{12}\left\{[t_{zz}]_g - [t_{zz}]_{\tilde{g}}\right\} \Leftrightarrow [\hat{T}_{zz}]_g - [\hat{T}_{zz}]_{\tilde{g}} = 0 \end{aligned} \tag{16.38}$$

$\cdots$ by the definition of $\tilde{\hat{T}}_{zz}$ and general coordinate covariance,

which can be readily proven in the free boson case [c.f. Eq. (3.6.15c) in Polchinski Vol. 1].

Exercise: Prove that Eq. (16.38) holds for any CFT by studying the Weyl transformations of both sides of Eq. (16.33), using any Weyl transformation that is nontrivial only inside B_0 but still trivial around $\{x_i\}$, the result of which actually generalizes Eq. (16.38).

Therefore, in our bosonic example, we have correctly fixed the constant term and have the final OPE:

$$[\hat{T}_{zz}(z)]_g[\hat{T}_{zz}(w)]_g \sim \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}[\hat{T}(w)]_g + \frac{1}{z-w}\partial_w[\hat{T}(w)]_g. \tag{16.39}$$

Various Virasoro operators follow:

$$[\hat{T}_{zz}(z)]_g = \sum_{m\in\mathbb{Z}} \frac{[L_m]_g}{z^{m+2}}. \tag{16.40}$$

Similarly, $\langle[\hat{T}_{zz}(z_0)]_g\rangle = 0$ when there exist global conformal transformations fixing z_0.

16.4 Weyl weight turning out to be conformal weight

I assume that the reader is familiar with the primary field ϕ in the flat space $g = \delta$ with conformal weight $(h, \bar{h})$:

$$[\tilde{\phi}(f(z))]_\delta (df)^h (d\bar{f})^{\bar{h}} = [\phi(z)]_\delta (dz)^h (d\bar{z})^{\bar{h}}. \tag{16.41}$$

However, this transformation rule has difficulty when we go to curved space in higher dimensions, which will be done below.

In a curved space, we consider the Weyl transformation rule as a more fundamental property of fields since the conformal transformation is actually equivalent to a (subclass of) Weyl transformation up to a general coordinate transformation as discussed in the beginning. The Weyl transformation rule of $[\phi(x)]_g$ is

$$[\phi(x)]_g = [\phi(x)]_{\Omega^{-2}g}\Omega^{-\Delta_\phi}, \tag{16.42}$$

and it has a conventional coordinate transformation:

$$[\tilde{\phi}(f(z))]_{\tilde{g}} (df)^n (d\bar{f})^{\bar{n}} = [\phi(z)]_g (dz)^n (d\bar{z})^{\bar{n}}. \tag{16.43}$$

Here n and $\bar{n}$ are integer counting the numbers of, respectively, z and $\bar{z}$ in the tensor subscripts "$\cdots$" of $\phi_{\dots}$, e.g., $(n, \bar{n}) = (2, 0)$ for T_{zz} (although T_{zz} does not have well-defined $\Delta_{T_{zz}}$). Such ϕ with well-defined Δ_ϕ and integral $(n, \bar{n})$ is called a bosonic *primary field.* Therefore, together with $\tilde{g}(x) = \Omega^{-2} g(x)$ and

$$\Omega(f(z))^2 g_{z\bar{z}}(z) dz d\bar{z} = g_{z\bar{z}}(f(z)) df d\bar{f}, \tag{16.44}$$

we have its conformal transformation property:

$$g_{z\bar{z}}(f(z))^{\Delta_\phi/2} [\tilde{\phi}(f(z))]_g (df)^h (d\bar{f})^{\bar{h}} = g_{z\bar{z}}(z)^{\Delta_\phi/2} [\phi(z)]_g (dz)^h (d\bar{z})^{\bar{h}}, \tag{16.45}$$

where

$$h \equiv \frac{\Delta_\phi}{2} + n, \tag{16.46}$$

$$\bar{h} \equiv \frac{\Delta_\phi}{2} + \bar{n}. \tag{16.47}$$

It motivates us to define a Weyl invariant quantity:

$$[\Phi(z)]_g \equiv g_{z\bar{z}}(z)^{\Delta_\phi/2} [\phi(z)]_g, \tag{16.48}$$

which then transforms under conformal transformations in our familiar way

$$[\tilde{\Phi}(f(z))]_g (df)^h (d\bar{f})^{\bar{h}} = [\Phi(z)]_g (dz)^h (d\bar{z})^{\bar{h}}. \tag{16.49}$$

In general dimensions, the above definition could be generalized as

$$[\Phi(x)]_g \equiv \sqrt{g(x)}^{\Delta_\phi/2} [\phi(x)]_g, \tag{16.50}$$

and the rule under conformal transformations is

$$[\tilde{\Phi}_{\dots}(f(x))]_g df^{\cdots} \left[\det\left(\frac{\partial f^\alpha(x)}{\partial x^\beta}\right)\right]^{\Delta_\phi/2} = [\Phi_{\dots}(z)]_g dx^{\cdots}. \tag{16.51}$$

Therefore, we obtain the curved space correspondence of $[\phi(z)]_\delta$. The above transformation rule directly means that we recover all the corresponding flat space conclusions:

$$[\hat{T}_{zz}(z)]_g [\Phi(w)]_g \sim \frac{h}{(z-w)^2} [\Phi(w)]_g + \frac{1}{z-w} \partial [\Phi(w)]_g. \tag{16.52}$$

Problem: It is possible to discuss the fermionic cases. It would be helpful to review Dirac equations in curved space, where the zweibein is naturally introduced in order to generalize Eq. (16.48). These will be all done later. It is also interesting to further generalize to parafermionic cases, which still remains open.

17 Stress tensor in bosonic path: 2023.2.19-2023.2.21

We now start to reformulate everything in the last weekly through path integral formalism for general bosonic QFT on d-dimensional Riemannian manifold.

17.1 Stress tensor defined

First, we define the stress tensor through a path integral:

$$\langle [T^{\mu\nu}(x)]_g [O(x_i)]_g \rangle_g \equiv -\frac{4\pi}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle [O(x_i)]_g \rangle_g, \tag{17.1}$$

for a general operator insertion $[O]_g$.

Therefore, the stress tensor defined here is different from the last week in that we now also includes the measure change due to the variation of the metric.

Equivalently,

$$\int \sqrt{g} \delta g_{\mu\nu}(x) \langle [T^{\mu\nu}(x)]_g [O(x_i)]_g \rangle_g = -4\pi \delta_g \langle [O(x_i)]_g \rangle_g. \tag{17.2}$$

By the general coordinate covariance:

$$\langle [O(x_i)]_g \rangle_g = \langle [O(\tilde{x}_i)]_{\tilde{g}} \rangle_{\tilde{g}}, \tag{17.3}$$

where

$$\tilde{x}^\mu = x^\mu + \epsilon v^\mu \equiv x^\mu + \xi^\mu. \tag{17.4}$$

Then we have

$$-\frac{1}{2\pi} \int \sqrt{g} \nabla^\mu T_{\mu\nu}(x) \xi^\nu(x) O(x_i) = -\xi^\nu(x_i) \partial_\nu O(x_i), \tag{17.5}$$

when $[O(x_i)]_g$ is a scalar under general coordinate transformation; $[O]_g = [\tilde{O}]_{\tilde{g}}$ where $\tilde{O}$ means the substitution of fundamental field $\phi(x)$ in O by $\tilde{\phi}(\tilde{x})$. For general fields

$$-\frac{1}{2\pi} \int \sqrt{g} \nabla^\mu T_{\mu\nu}(x) \xi^\nu(x) O(x_i) = -\mathcal{L}_\xi O(x_i), \tag{17.6}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ.

Let us define a Weyl transformation of the field as

$$\begin{aligned} \delta_w [O]_g &\equiv \lim_{\delta\omega \to 0} \frac{1}{\delta w} \left\{ [O(\phi)]_g - [O(\phi - \delta\omega \delta_w \phi)]_{g_w} \right\}; \\ g_w &\equiv (1 - 2\delta\omega) g, \end{aligned} \tag{17.7}$$

where ϕ is the fundamental degrees of freedom and $\delta_w \phi$ is either pre-given or determined by

$$S[(1 - 2\delta\omega) g, \phi - \delta\omega \delta_w \phi] = S[g, \phi]. \tag{17.8}$$

For free massless bosonic theory, $\delta_w \phi = 0$.

Exercise: Prove that the conformal transformation for v^μ being locally conformal and ϵ being constant at x_i:

$$\delta O \equiv [\tilde{O}(x_i)]_g - [O(x_i)]_g = -\mathcal{L}_v O(x_i) + \frac{\nabla \cdot v}{d} \delta_w [O]_g. \tag{17.9}$$

17.2 Conformal symmetry defined

So far, we have not used the conformal symmetry of the theory. Since we do not necessarily have the action, we will simply define that the theory is conformal symmetric if its correlation function satisfies:

$$\frac{\langle [O_i(x_i)]_g \rangle_g}{\langle 1 \rangle_g} = \frac{\langle [O_i(x_i)]_{(1-2\delta w)g} + \delta\omega\delta_w O_i \rangle_{(1-2\delta w)g}}{\langle 1 \rangle_{(1-2\delta w)g}}, \tag{17.10}$$

which implies that the mean value of the correlator is insensitive to the length scale. Here we have assumed the fundamental degrees of freedom is Weyl-invariant like in massless bosons, which is however not true in general. We know that the denominator comes from the partition function, so its variation should be local integration:

$$\langle 1 \rangle_g = \langle 1 \rangle_{1-2\delta w g} \left[1 + \int \sqrt{g} F[R] \delta w \right]. \tag{17.11}$$

When $d = 2$, $F[R] = cR/24\pi$.

Exercise: Prove that the trace of the stress tensor is a generator of the Weyl transformation:

$$-\frac{1}{2\pi}\sqrt{g} T^{\mu}{}_{\mu}(x) O(x_i) = \sqrt{g} F[R(x)] O(x_i) + \delta_w O(x_i) \delta(x - x_i). \tag{17.12}$$

Therefore, the stress tensor in the last week can be understood as a modified one such that there is no contactless contribution.

17.3 Conformal Ward-Takahashi Identity

By taking ϵ as that of the last week and noticing that $\partial\epsilon(x_i) = 0$, we derive from Eq. (17.6),using Eq. (17.12) and $\{x_i\}$ being interior points of B_0,

$$\delta O = -\frac{1}{2\pi} \int_{B_0} \sqrt{g} \nabla_\alpha [T^{\alpha\mu} v_\mu] O(x_i) - \int_{B_0} \sqrt{g} F[R(x)] \frac{\nabla \cdot v}{d} O(x_i). \tag{17.13}$$

We can further simplify it as

$$\delta O = -\frac{1}{2\pi} \int_{B_0} \sqrt{g} \nabla_\alpha \left\{ \left[T^{\alpha\mu} + 2\pi F[R(x)] \frac{g^{\alpha\mu}}{d} \right] v_\mu \right\} O(x_i) + \int_{B_0} \sqrt{g} v \cdot \partial \{F[R(x)]\} \frac{1}{d} O(x_i),$$

where the tensor here is subtracted by its trace contribution. Such a subtraction is responsible for the dropout of $T_{z\bar{z}}$ in the following exercise:

Exercise: Taking $d = 2$, prove that

$$[\delta O]_g = -\frac{1}{2\pi i} \oint_{\partial B_0} \left(dz v^z \hat{T}_{zz} - d\bar{z} v^{\bar{z}} \hat{T}_{\bar{z}\bar{z}} \right)_g [O(x_i)]_g, \tag{17.14}$$

where $\hat{T}$ is of the same form as the last week. All the properties of $\hat{T}$, e.g., OPEs, in the last week, will be automatically held now.

Problem: Generalize the stress tensor to fermionic CFTs. [Hint: consider functional derivative upon zweibein.]

17.4 Additional Remarks

In this half week, we use an alternative approach of the stress tensor. It differs from that of the last week by inclusion of the Weyl anomaly into the stress tensor. Of course this is just a matter of definition, but the definition in the last week does not tell us how to regularize the classical stress tensor $T_{\mu\nu}$ in the path integral as $[T]_g$. What we did last week was just to guess.

By a non-Lagrangian definition, we do not need to worry about the regularization since the stress tensor is defined by correlators rather than fields. Inevitably, the measure change is included in the current definition since we do not have a canonical separation of contribution from the classical action and the integral measure (recalling bosonization where anomaly can be transformed into classical action).

So far, our definition automatically gives

$$\langle [T^{\mu\nu} - T^{\nu\mu}] \cdots \rangle = 0. \tag{17.15}$$

Thus, stress tensor for bosonic system is identically symmetric. It will be incorrect as we introduce fermionic degrees of freedom as in the next half of the week.

18 Stress tensor in fermionic path: 2023.2.22-2023.2.23

A natural question is how to further generalize the Ward-Takahashi Identity to fermionic case. To do so, we need a refined definition for the stress tensor by tetrad:

$$\int \sqrt{g}\delta e^a_\mu \langle T^\mu{}_a(x)\psi(x_i)\rangle_e \equiv -2\pi\delta\langle[\psi(x_i)]_e\rangle_e, \tag{18.1}$$

where the tetrad e^a_μ is (not uniquely) defined by

$$e^a_\mu \delta_{ab} e^b_\nu = g_{\mu\nu}, \tag{18.2}$$

up to any local Lorentz transformation on the a, b indices.

18.1 Local Lorentz invariance

Thus, there is a local Lorentz symmetry; if there is an action S for the theory,

$$\psi(x) \to (1 + \frac{i}{2}\lambda^{ab}(x)S_{ab})\psi,\ e^a_\mu \to (\delta^a_b + \lambda^a{}_b(x))e^b_\mu; \tag{18.3}$$

$$\Rightarrow S \to S. \tag{18.4}$$

Here S_{ab} is the ψ-representation of the Lorentz generator, e.g.,$S_{ab} = -i[\gamma_a, \gamma_b]/4$ when ψ is Dirac spinor. In the case of non-Lagrangian model, we have the correlator version:

$$\left\langle \left(1 - \frac{i}{2}\lambda^{ab}(x)S_{ab}\right)\psi(x_i)\right\rangle_{e+\lambda e} = \langle\psi(x_i)\rangle_e\,. \tag{18.5}$$

Exercise: Why the sign above is "$-$" reappearing later? Also by the definition of stress tensor, prove that (trading μ's with a's by e^a_μ)

$$\frac{1}{2\pi}\sqrt{g}(T_{\mu\nu} - T_{\nu\mu})(x)\psi(x_i) = iS_{\mu\nu}\psi(x_i)\delta(x - x_i), \tag{18.6}$$

or

$$\frac{1}{2\pi}\sqrt{g}(T_{ab} - T_{ba})(x)\psi(x_i) = iS_{ab}\psi(x_i)\delta(x - x_i). \tag{18.7}$$

It reproduce our earlier bosonic definition when all the degrees of freedom are bosonic since $S_{ab} = 0$ for bosonic fields.

Due to the local symmetry, there is a corresponding covariant derivative when $\psi(x)$ is an Einstein scalar, e.g., a Dirac spinor:

$$\begin{aligned} D_\nu\psi(x) &\equiv \left(\partial_\nu + \frac{i}{2}S_{ab}\omega^{ab}_\nu\right)\psi(x), \\ \omega^{ab}_\nu &\equiv e^{\mu a}\nabla_\nu e^b_\mu, \end{aligned} \tag{18.8}$$

where ω is antisymmetric about $a \leftrightarrow b$ and called spin connection satisfying a useful condition (but we will not use it):

$$\partial_\mu e^a_\nu + \omega^{ab}_\mu e_{\nu b} - \Gamma^\sigma_{\mu\nu}e^a_\sigma = 0, \tag{18.9}$$

which is commonly written as "$D_\mu e \equiv \left(\nabla_\mu + \frac{i}{2}S^{(e)}_{ab}\omega^{ab}_\mu\right)e = 0$" paralleling to $\nabla_\mu g = 0$.

18.2 General-coordinate invariance

The general-coordinate invariance through $\tilde{x} = x + \xi$ implies that

$$\begin{aligned} \langle \psi(x_i) \rangle_e &= \langle (\psi - \delta_1 \psi)(x_i) \rangle_{e+\delta_1 e} \\ \delta_1 \psi &= -\mathcal{L}_\xi \psi; \\ \delta_1 e^a_\mu &= -\mathcal{L}_\xi e^a_\mu \\ &= -\xi^\nu \nabla_\nu e^a_\mu - e^a_\nu \nabla_\mu \xi^\nu, \end{aligned} \tag{18.10}$$

where we have used the torsion-free condition.

Exercise: Prove that

$$\begin{aligned} \frac{1}{2\pi}\sqrt{g}\xi^\nu \nabla_\mu T^\mu{}_\nu(x)\psi(x_i) &= \delta(x-x_i)\left(\mathcal{L}_\xi + \xi^\nu \frac{i}{2} S_{ab}\omega^{ab}_\nu\right)\psi(x_i) \\ [\text{when } \psi \text{ is Einstein scalar } &= \delta(x-x_i)\xi^\nu D_\nu \psi(x_i).] \end{aligned} \tag{18.11}$$

Therefore, one should notice the additional term on the right-hand side, which is absent in the bosonic case, is to realize the (linear) transformation rule, the same as the left-hand side.

18.3 Conformal Symmetry

The conformal symmetry is defined as

$$\langle [\psi(x_i)]_e \rangle_{e=(1+\delta\omega)\tilde{e}} = \left[1 + \int \sqrt{g} F[R]\delta\omega\right] \langle ([\psi]_{\tilde{e}} + \delta_w \psi)(x_i) \rangle_{\tilde{e}}, \tag{18.12}$$

with

$$\delta_w \psi \equiv \lim_{\delta\omega \to} \frac{1}{\delta\omega}\left\{[\psi(\phi + \delta\omega\delta_w\phi)]_e - [\psi(\phi)]_{\tilde{e}}\right\}, \tag{18.13}$$

where $\delta_w \phi$ is the pre-given transformation of the fundamental field.

Exercise: Prove that

$$-\frac{1}{2\pi}\sqrt{g}T^\mu{}_\mu(x)\psi(x_i) = \sqrt{g}F[R(x)]\psi(x_i) + \delta_w\psi(x_i)\delta(x-x_i). \tag{18.14}$$

Let us see how to define conformal symmetry transformation of field; taking $\xi = v$ a conformal Killing vector. To motivate the definition, we again assume an action $S[e,\psi]$. Firstly,

$$S[e,\psi] = S[e + \delta_1 e, \psi + \delta_1 \psi], \tag{18.15}$$

as in the general coordinate transformation. It will be convenient to decompose δe as

$$\begin{aligned} \delta_1 e^a_\mu &= -\left[v^\nu \omega^{ba}_\nu + \frac{1}{2}(\nabla_\lambda v_\nu - \nabla_\nu v_\lambda)e^\lambda_b e^{a\nu}\right] e^b_\mu - \left(\frac{\nabla \cdot v}{d} e^a_\mu\right) \\ &\equiv -\delta_2 e^a_\mu - \delta_3 e^a_\mu, \end{aligned} \tag{18.16}$$

from which it should be noted that we need to also do an additional local Lorentz transformation before the Weyl transformation in order to restore the tetrad:

$$\begin{aligned} & S[e,\psi] = S[e+\delta_1 e, \psi+\delta_1\psi] \\ = \ & S[e+\delta_1 e+\delta_2 e, \psi+\delta_1\psi+\delta_2\psi] \\ = \ & S[e+\delta_1 e+\delta_2 e+\delta_3 e = e, \psi+\delta_1\psi+\delta_2\psi+\delta_3\psi] \\ \equiv \ & S[e,\psi+\delta\psi] \end{aligned} \tag{18.17}$$

which defines the expression for the conformal transformation of $\delta\psi$ as

$$\delta\psi = [-\mathcal{L}_v\psi] + \left[\frac{i}{2}S_{ab}\left(v^\nu\omega_\nu^{ab} - e^{a\mu}e^{b\nu}\nabla_\mu v_\nu\right)\psi\right] + \left[\frac{\nabla\cdot v}{d}\delta_w\psi\right]. \tag{18.18}$$

18.4 Conformal Ward-Takahashi Identity

So far, we have sufficient tools to derive the Ward-Takahashi Identity by taking

$$\xi = \epsilon(x)v, \tag{18.19}$$

where $\epsilon(x)$ is the compactly supported function defined in the early days. The Ward-Takahashi Identity reads:

$$\delta\psi = -\frac{1}{2\pi}\int_{B_0}\sqrt{g}\nabla_\alpha[T^{(\alpha\mu)}v_\mu]\psi(x_i) - \int_{B_0}\sqrt{g}F[R(x)]\frac{\nabla\cdot v}{d}\psi(x_i). \tag{18.20}$$

We can further simplify it as

$$\delta\psi = -\frac{1}{2\pi}\int_{B_0}\sqrt{g}\nabla_\alpha\left\{\left[T^{\alpha\mu}_{\text{sym}} + 2\pi F[R(x)]\frac{g^{\alpha\mu}}{d}\right]v_\mu\right\}\psi(x_i) + \int_{B_0}\sqrt{g}v\cdot\partial\{F[R(x)]\}\frac{1}{d}\psi(x_i),$$

where

$$T^{\mu\nu}_{\text{sym}} \equiv T^{(\mu\nu)} = \frac{1}{2}[T^{\mu\nu}+T^{\nu\mu}]. \tag{18.21}$$

Exercise: I would encourage the reader to prove the above identities to see where the antisymmetric part of the stress tensor contributes.

Then all the results following Eq. (17.13) in the last half of this week for the bosonic cases still hold just by using $T^{\mu\nu}_{\text{sym}}$ to replace $T^{\mu\nu}$ there.

18.5 Dirac and Majorana fermions as examples in 2D

Let us swiftly review the spinors in 2D Euclidean signature:

$$\{\gamma^a,\gamma^b\} = 2\eta^{ab} = 2\delta^{a,b}, \tag{18.22}$$

and we would use $\gamma^a \equiv [\sigma^x, \sigma^y]$. The Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac,E}} = \bar{\Psi}[\gamma\cdot(\partial+iA)+m]\Psi, \tag{18.23}$$

in which $\bar{\Psi}$ does not mean a complex conjugate and this abuse of notation is due to that Ψ and $\bar{\Psi}$ indeed transform in complex conjugate ways under Lorentz transformation.

The massive Dirac equation is symmetric under

$$\Psi \to \Psi^C \equiv \gamma^1 \Psi^*. \tag{18.24}$$

However, strangely, we cannot impose the corresponding Majorana condition

$$\Psi = \gamma^1 \bar{\Psi}^T \tag{18.25}$$

for the massive Dirac fermion because γ^1 is symmetric. Anyway, this Majorana condition is applicable for the massless case.

On the other hand, for the massless Dirac equation, we can also have the charge conjugation symmetry

$$\Psi^C \equiv \gamma^1 \gamma^2 \Psi^*, \tag{18.26}$$

which, strangely, can be used to define the Majorana condition for the *massive* case (as well as the massless case).

Exercise: Prove that this crossing contradiction does not appear in the Minkowski signature. From this lesson, we need to keep in mind that there is no operator meaning in Dirac fermion in Euclidean signature. In order to have a well-defined operator content, we need to do the Wick rotation on the original Minkowski Dirac fermion, which however explicitly breaks the naive Lorentz invariance.

We can thus decompose a Dirac fermion into two Majorana fermions

$$\psi_{\rm R} \equiv \frac{1}{2}(\Psi + \Psi^C);\ \psi_{\rm Im} \equiv \frac{1}{2i}(\Psi - \Psi^C), \tag{18.27}$$

where $\Psi^C = \gamma^1 \bar{\Psi}^T$ can be used only for the massless case while the other $\Psi^C = \gamma^1 \gamma^2 \bar{\Psi}^T$ can be used for both cases. Then we can concentrate ourselves only on the Majorana fermion.

We will apply $\psi = \gamma^1 \bar{\psi}^T$ to the massless Dirac Lagrangian to kill the degrees of freedom:

$$\begin{aligned} dx^1 dx^2 \mathcal{L}_{\rm Maj,E} &= dx^1 dx^2 \psi^T {\rm diag}\{\partial_1 + i\partial_2, \partial_1 - i\partial_2\}\psi \\ &= i dz d\bar{z} \psi^T {\rm diag}\{\partial_{\bar{z}}, \partial_z\}\psi. \end{aligned} \tag{18.28}$$

Such a Lagrangian is clearly conformal symmetric. Our question is whether this property is still true in the curved case.

18.6 Majorana action on the curved 2D Riemannian manifold

We begin with the following form:

$$S = \int \sqrt{g} \bar{\psi} \left\{ \gamma^a e^\mu_a \left[\partial_\mu + \frac{i}{2} \omega_{\mu ab} S^{ab} \right] \psi \right\} \tag{18.29}$$

with

$$\bar{\psi} = \psi^T \gamma^1;\ S^{ab} = \frac{1}{4i}[\gamma^a, \gamma^b]. \tag{18.30}$$

Let us use the complex coordinate for the local Lorentz system:

$$m = (1) + i(2);\ \bar{m} = (1) - i(2), \eta_{m\bar{m}} = \frac{1}{2} = (\eta^{m\bar{m}})^{-1}. \tag{18.31}$$

Then

$$\eta_{m\bar{m}} e_z^m e_{\bar{z}}^{\bar{m}} = g_{z\bar{z}} \tag{18.32}$$
$$\eta^{m\bar{m}} e_m^z e_{\bar{m}}^{\bar{z}} = g^{z\bar{z}}, \tag{18.33}$$

which gives us

$$e_z^m = (e_m^z)^{-1};\ e_{\bar{z}}^{\bar{m}} = (e_{\bar{m}}^{\bar{z}})^{-1}, \tag{18.34}$$

with all the remaining components vanishing.

The corresponding transformation of γ-matrices is

$$\gamma^m \equiv \gamma^1 + i\gamma^2;\ \gamma^{\bar{m}} \equiv \gamma^1 - i\gamma^2;\ [\gamma^m, \gamma^{\bar{m}}] = 4\sigma_3. \tag{18.35}$$

Recalling that $\omega_{\mu ab} \equiv e_a^\nu \nabla_\mu e_{b\nu}$, we have

$$\frac{1}{8}\omega_{zab}[\gamma^a, \gamma^b] = \sigma_3 e_{m\bar{z}} \partial_z e_{\bar{m}}^{\bar{z}}; \tag{18.36}$$
$$\frac{1}{8}\omega_{\bar{z}ab}[\gamma^a, \gamma^b] = \sigma_3 e_{\bar{m}z} \partial_{\bar{z}} e_m^z. \tag{18.37}$$

Inputting these into the action, we have

$$\begin{aligned} S &= \int idzd\bar{z}(\psi_1\sqrt{e_z^m}, \psi_2\sqrt{e_{\bar{z}}^{\bar{m}}}) \begin{pmatrix} \partial_{\bar{z}} & \\ & \partial_z \end{pmatrix} \begin{pmatrix} \sqrt{e_z^m}\psi_1 \\ \sqrt{e_{\bar{z}}^{\bar{m}}}\psi_2 \end{pmatrix} \\ &\equiv i\int dzd\bar{z}(\psi, \bar{\psi}) \begin{pmatrix} \partial_{\bar{z}} & \\ & \partial_z \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \end{aligned} \tag{18.38}$$

where $\bar{\psi}$ means antiholomorphism, different from $\bar{\Psi}$ before.

The conformal symmetry now can be explicitly guessed as

$$\psi'(f(z)) = (\partial_z f)^{-1/2}\psi(z); \tag{18.39}$$
$$\bar{\psi}'(f(z)) = (\partial_{\bar{z}} \bar{f})^{-1/2}\bar{\psi}(z), \tag{18.40}$$

although it is technically involving to reach through the standard calculation of $\delta_{1,2,3}\psi_{1,2}$.

We could intuitively say that the conformal weights of ψ and $\bar{\psi}$ are partially resulted from $\delta_w\psi_{1,2}$ due to the requirement to cancel the Weyl transformations of the $\sqrt{e_z^m}$ and $\sqrt{e_{\bar{z}}^{\bar{m}}}$, while their conformal spins are due to the local Lorentz rotation to compensate the corresponding $\delta_2 e$ as we have seen before.

It is also tempting to define by force ψ and $\bar{\psi}$ as fundamental degrees of freedom, but this is expensive since ψ and $\bar{\psi}$ are derived based on conformal gauge and they do not have a simple correspondence in arbitrary coordinate systems. If we really want to do so, the prize is a direct conclusion of $T^\mu{}_\mu = 0$ classically.