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Place all written answers from `assignment-01.md` here for easier grading.

1. **Asymptotic notation**

- 1a **TRUE. Yes**,  $2^{n+1} \in O(2^n)$ .

**Reasoning:** Per Big O notation, we must show that there exist positive constants  $c$  and  $n_0$  such that  $f(n) \leq c \cdot g(n)$  for all  $n \geq n_0$ .

Let  $f(n) = 2^{n+1}$  and  $g(n) = 2^n$ .

$$f(n) = 2^{n+1} = 2^1 \cdot 2^n = 2 \cdot 2^n.$$

$$2 \cdot 2^n \leq c \cdot 2^n.$$

Divide both sides by  $2^n$ , we get  $2 \leq c$ .

We can choose  $c \geq 2$  and this inequality holds for all  $n \geq 0$  (so we can set  $n_0 = 0$ ).

Since we found constants  $c \geq 2$  and  $n_0 = 0$  that satisfy the definition,  $2^{n+1}$  is in  $O(2^n)$ .

- 1b **FALSE. No**,  $2^{2^n} \in O(2^n)$ .

**Reasoning:** In class we have demonstrated that any polylogarithmic function grows slower than any polynomial function. Expressed differently,  $\log^i(n) \in O(n^j) \forall i, j > 0$ . Conversely  $n^j \in \Omega(\log^i(n)) \forall i, j > 0$ .

Let  $f(n) = 2^{2^n}$  and  $g(n) = 2^n$ .

Inequality:  $2^{2^n} \leq c \cdot 2^n$ . We must check if there is a constant  $c$  that satisfies this.

Let  $a = 2^n$

Then  $2^a \leq c \cdot a$ .

Take  $\log_2$  of both sides.

$$a \leq \log_2 c + \log_2 a$$

Therefore there is no  $c$  that satisfies this inequality as the left polynomial which grows linearly always grows faster than the right polylogarithmic function.

- 1c **FALSE. No**,  $n^{1.01} \in O(\log^2 n)$ .

**Reasoning:** As above, any polylogarithmic function grows slower than any polynomial function. Expressed differently,  $\log^i(n) \in O(n^j) \forall i, j > 0$ . Conversely  $n^j \in \Omega(\log^i(n)) \forall i, j > 0$ . The polynomial function  $n^{1.01}$  grows much faster than a constant multiple of  $\log n$  (in this case  $\log^2 n$ ).

- 1d **TRUE. Yes**,  $n^{1.01} \in \Omega(\log^2 n)$ .

**Reasoning:** As above, as we have shown in class,  $n^j \in \Omega(\log^i(n)) \forall i, j > 0$ . Therefore, the statement is true.

- 1e **FALSE. No**,  $\sqrt{n} \in O(\log^3 n)$ .

**Reasoning:** As above, any polylogarithmic function grows slower than any polynomial function. Expressed differently,  $\log^i(n) \in O(n^j) \forall i, j > 0$ . Conversely  $n^j \in \Omega(\log^i(n)) \forall i, j > 0$ . The polynomial function  $\sqrt{n}$  which is  $n^{0.5}$  grows faster than a constant multiple of  $\log n$  (in this case  $\log^3 n$ ).

- 1f **TRUE. Yes**,  $\sqrt{n} \in \Omega(\log^3 n)$ .

**Reasoning:** As above, as we have shown in class,  $n^j \in \Omega(\log^i(n)) \forall i, j > 0$ . Therefore, the statement is true. This is a statement of lower bound so it is true.

- 1g We will assume that there exists a function,  $f(n)$ , in the intersection of  $o(g(n))$  and  $\omega(g(n))$ . By definition:

1.  $f(n) \in o(g(n))$ , for any constant  $c_1 > 0$  and there is an  $n_1$  such that  $f(n) < c_1 g(n)$  for all  $n \geq n_1$ .
2.  $f(n) \in \omega(g(n))$ , for any constant  $c_2 > 0$  and there is an  $n_2$  such that  $c_2 g(n) < f(n)$  for all  $n \geq n_2$ .

The above must hold for every positive constant  $c$ . So let  $c_1 = c_2$ . We can also select  $n_0 = \max(n_1, n_2)$ . By selecting  $n_0 = \max(n_1, n_2)$ , we establish a threshold. For any  $n \geq n_0$ , we are guaranteed that  $n \geq n_1$  AND  $n \geq n_2$ , meaning both inequalities must hold. Therefore:

$c_2 g(n) < f(n) < c_1 g(n)$ .  $c_1 = c_2$ . Therefore,  $c_1 g(n) < f(n) < c_1 g(n)$ . Since this is a strict inequality definition, there is no  $f(n)$  that exists to satisfy this strict inequality.

## 2. SPARC to Python

- 2a - See main.py
- 2b - This function takes two numbers as input, checks if the base case is true and then recursively returns the maximum of the two numbers and the remainder of y maximum divided by the minimum number. The recursion only works on the second argument, which is repeatedly replaced by  $y \% (\text{current\_second\_arg})$ . Eventually, the second argument will become 0, and the function will return the first argument, which has been  $\max(a, b)$  all along. In my own words, the function computes the maximum of two non-negative integers. For any inputs  $a$  and  $b$ ,  $\text{foo}(a, b)$  will return  $\max(a, b)$ .

Note: If the function in SPARC had  $(\text{foo } x, y \bmod x)$  instead of  $(\text{foo } y, y \bmod x)$ , this would be the Euclidean algorithm and would return the greatest common divisor (GCD). The function  $\text{foo}(a, b)$  calculates the Greatest Common Divisor (GCD) of two non-negative integers  $a$  and  $b$  under this modified algorithm.

- 2c - Analysis:

Algorithm:  $y = \max(a, b)$  is constant.  $\text{foo}(y, b_i)$  calls  $\text{foo}(y, y \% b_i)$ .

In each recursive call, a constant number of operations is performed: 2 comparisons, a min, a max, a mod, and the function call itself. This is constant work  $O(1)$ . In terms of the number of recursive calls, the second argument gets smaller with each step until it becomes 0. The sequence is  $b_0 = \min(a, b)$ ,  $b_1 = y \% b_0$ ,  $b_2 = y \% b_1$ , etc. The number of steps is logarithmic in the value of the inputs ie.  $O(\log(\min(a, b)))$ .

**Work:** The total number of operations, equivalent to the time it would take on a single processor.  $\text{Work} = \text{work per step} * \# \text{ of steps}$ .

$$W(a, b) = O(1) * O(\log(\min(a, b))) = O(\log(\min(a, b)))$$

**Span:** Span (or depth) is the longest chain of dependent operations, which represents the execution time on an infinite number of processors. This is a completely sequential algorithm as each call depends on the result of the next one. So the Span will be the same as work.

$$S(a, b) = O(1) * O(\log(\min(a, b))) = O(\log(\min(a, b)))$$

## Output from test code in main.py

### 3. Parallelism and recursion

- 3a - See longest\_run in main.py
- 3b - The work and span of this sequential algorithm: The work of an algorithm is the total number of operations it performs. In the for loop, we iterate through the list (all  $n$  elements of the array) and inside the loop it performs a constant number of operations (a comparison, an addition, a max operation, an assignment). Therefore total work is linear and proportional to  $n$ .

$$W(n) = O(n)$$

The span is the longest chain of dependent operations, representing the runtime with infinite processors. Our implementation in this algorithm is sequential (no parallelism). Therefore, span is also proportional to  $n$ .

$$S(n) = O(n)$$

- 3d

$n$  is the length of mylist.

**Work:** Work is the total number of operations. This function splits the problem into 2 subproblems of size  $n/2$  and then does a constant amount of work ( $O(1)$ ) to combine them.

Therefore  $W(n) = 2W(n/2) + O(1)$ .

Can use brick method that is leaf dominated.  $n = s^h$  so  $\log_2(n) = \log_2(2^h)$  and  $h = \log_2(n)$ . Number of leaves =  $2^h$  which is  $2^{\log_2(n)}$  which is  $n$ . This simplifies to  $W(n) \in O(n)$ .

**Span:** Span is the longest dependency path. As a result of the two recursive calls, `longest_run_recursive(left_half, ...)` and `longest_run_recursive(right_half, ...)` are still being run **sequentially**. Therefore the Span and Work are the same.

$S(n) = 2S(n/2) + O(1)$ . This simplifies to  $S(n) \in O(n)$ .

- 3e **Work:** Work is the total number of operations. This function splits the problem into 2 subproblems of size  $n/2$  and then does a constant amount of work ( $O(1)$ ) to combine them. The work **remains the same**.

Therefore  $W(n) = 2W(n/2) + O(1)$ . This simplifies to  $W(n) \in O(n)$  as reasoned above.

**Span:** Span is the longest dependency path. As a result of the two recursive calls, `longest_run_recursive(left_half, ...)` and `longest_run_recursive(right_half, ...)` are now being run **in parallel**. Therefore the Span is the height of the tree as the algorithm is operating in parallel. We can use the brick method that is balanced with the first level having a cost of 1 and the number of leaves equal to  $n$ .

$S(n) = S(n/2) + O(1)$ . We have shown above that the height of the tree,  $h$ , is  $\log_2(n)$  and the first level has a cost of 1. This simplifies to  $S(n) \in O(\log n)$ .

**Output from test code in main.py**

#### 4. GCD

**No fourth problem was included**