

Name: Aaron Dumont

Place all written answers from `assignment-01.md` here for easier grading.

1. Asymptotic notation

- 1a **TRUE. Yes**, $2^{n+1} \in O(2^n)$.

Reasoning: Per Big O notation, we must show that there exist positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Let $f(n) = 2^{n+1}$ and $g(n) = 2^n$.

$$f(n) = 2^{n+1} = 2^1 \cdot 2^n = 2 \cdot 2^n.$$

$$2 \cdot 2^n \leq c \cdot 2^n.$$

Divide both sides by 2^n , we get $2 \leq c$.

We can choose $c \geq 2$ and this inequality holds for all $n \geq 0$ (so we can set $n_0 = 0$).

Since we found constants $c \geq 2$ and $n_0 = 0$ that satisfy the definition, 2^{n+1} is in $O(2^n)$.

- 1b **FALSE. No**, $2^{2^n} \in O(2^n)$.

Reasoning: In class we have demonstrated that any polylogarithmic function grows slower than any polynomial function. Expressed differently, $\log^i(n) \in O(n^j) \forall i, j > 0$. Conversely $n^j \in \Omega(\log^i(n)) \forall i, j > 0$.

Let $f(n) = 2^{2^n}$ and $g(n) = 2^n$.

Inequality: $2^{2^n} \leq c \cdot 2^n$. We must check if there is a constant c that satisfies this.

Let $a = 2^n$

Then $2^a \leq c \cdot a$.

Take \log_2 of both sides.

$$a \leq \log_2 c + \log_2 a$$

Therefore there is no c that satisfies this inequality as the left polynomial which grows linearly always grows faster than the right polylogarithmic function.

- 1c **FALSE. No**, $n^{1.01} \in O(\log^2 n)$.

Reasoning: As above, any polylogarithmic function grows slower than any polynomial function. Expressed differently, $\log^i(n) \in O(n^j) \forall i, j > 0$. Conversely $n^j \in \Omega(\log^i(n)) \forall i, j > 0$. The polynomial function $n^{1.01}$ grows much faster than a constant multiple of $\log n$ (in this case $\log^2 n$).

- 1d **TRUE. Yes**, $n^{1.01} \in \Omega(\log^2 n)$.

Reasoning: As above, as we have shown in class, $n^j \in \Omega(\log^i(n)) \forall i, j > 0$. Therefore, the statement is true.

- 1e **FALSE. No**, $\sqrt{n} \in O(\log^3 n)$.

Reasoning: As above, any polylogarithmic function grows slower than any polynomial function. Expressed differently, $\log^i(n) \in O(n^j) \forall i, j > 0$. Conversely $n^j \in \Omega(\log^i(n)) \forall i, j > 0$. The polynomial function \sqrt{n} which is $n^{0.5}$ grows faster than a constant multiple of $\log n$ (in this case $\log^3 n$).

- 1f **TRUE. Yes**, $\sqrt{n} \in \Omega(\log^3 n)$.

Reasoning: As above, as we have shown in class, $n^j \in \Omega(\log^i(n)) \forall i, j > 0$. Therefore, the statement is true. This is a statement of lower bound so it is true.

Test code output for Problem 2

Figure 1: Test code output for Problem 2

- 1g We will assume that there exists a function, $f(n)$, in the intersection of $o(g(n))$ and $\omega(g(n))$. By definition:
 1. $f(n) \in o(g(n))$, for any constant $c_1 > 0$ and there is an n_1 such that $f(n) < c_1 g(n)$ for all $n \geq n_1$.
 2. $f(n) \in \omega(g(n))$, for any constant $c_2 > 0$ and there is an n_2 such that $c_2 g(n) < f(n)$ for all $n \geq n_2$.

The above must hold for every positive constant c . So let $c_1 = c_2$. We can also select $n_0 = \max(n_1, n_2)$. By selecting $n_0 = \max(n_1, n_2)$, we establish a threshold. For any $n \geq n_0$, we are guaranteed that $n \geq n_1$ AND $n \geq n_2$, meaning both inequalities must hold. Therefore:

$c_2 g(n) < f(n) < c_1 g(n)$. $c_1 = c_2$. Therefore, $c_1 g(n) < f(n) < c_1 g(n)$. Since this is a strict inequality definition, there is no $f(n)$ that exists to satisfy this strict inequality.

2. SPARC to Python

- 2a - See main.py
- 2b - This function takes two numbers as input, checks if the base case is true and then recursively returns the maximum of the two numbers and the remainder of y maximum divided by the minimum number. The recursion only works on the second argument, which is repeatedly replaced by $y \% (\text{current_second_arg})$. Eventually, the second argument will become 0, and the function will return the first argument, which has been $\max(a, b)$ all along. In my own words, the function computes the maximum of two non-negative integers. For any inputs a and b , $\text{foo}(a, b)$ will return $\max(a, b)$.

Note: If the function in SPARC had $(\text{foo } x, y \text{ mod } x)$ instead of $(\text{foo } y, y \text{ mod } x)$, this would be the Euclidean algorithm and would return the greatest common divisor (GCD). The function $\text{foo}(a, b)$ calculates the Greatest Common Divisor (GCD) of two non-negative integers a and b under this modified algorithm.

- 2c - **Analysis:**

Algorithm: $y = \max(a, b)$ is constant. $\text{foo}(y, b_i)$ calls $\text{foo}(y, y \% b_i)$.

In each recursive call, a constant number of operations is performed: 2 comparisons, a min, a max, a mod, and the function call itself. This is constant work $O(1)$. In terms of the number of recursive calls, the second argument gets smaller with each step until it becomes 0. The sequence if $b_0 = \min(a, b)$, $b_1 = y \% b_0$, $b_2 = y \% b_1$, etc. The number of steps is logarithmic in the value of the inputs ie. $O(\log(\min(a, b)))$.

Work: The total number of operations, equivalent to the time it would take on a single processor. $\text{Work} = \text{work per step} * \# \text{ of steps}$.

$$W(a, b) = O(1) * O(\log(\min(a, b))) = O(\log(\min(a, b)))$$

Span: Span (or depth) is the longest chain of dependent operations, which represents the execution time on an infinite number of processors. This is a completely sequential algorithm as each call depends on the result of the next one. So the Span will be the same as work.

$$S(a, b) = O(1) * O(\log(\min(a, b))) = O(\log(\min(a, b)))$$

Output from test code in main.py

3. Parallelism and recursion

- 3a - See longest_run in main.py
- 3b - The work and span of this sequential algorithm: The work of an algorithm is the total number of operations it performs. In the for loop, we iterate through the list (all n elements of the array) and inside the loop it performs a constant number of operations (a comparison, an addition, a max operation, an assignment). Therefore total work is linear and proportional to n .

Test code output for Problem 3

Figure 2: Test code output for Problem 3

$$W(n) = O(n)$$

The span is the longest chain of dependent operations, representing the runtime with infinite processors. Our implementation in this algorithm is sequential (no parallelism). Therefore, span is also proportional to n .

$$S(n) = O(n)$$

- 3d

n is the length of mylist.

Work: Work is the total number of operations. This function splits the problem into 2 subproblems of size $n/2$ and then does a constant amount of work ($O(1)$) to combine them.

Therefore $W(n) = 2W(n/2) + O(1)$.

Can use brick method that is leaf dominated. $n = s^h$ so $\log_2(n) = \log_2(2^h)$ and $h = \log_2(n)$. Number of leaves = 2^h which is $2^{\log_2(n)}$ which is n . This simplifies to $W(n) \in O(n)$.

Span: Span is the longest dependency path. As a result of the two recursive calls, `longest_run_recursive(left_half, ...)` and `longest_run_recursive(right_half, ...)` are still being run **sequentially**. Therefore the Span and Work are the same.

$S(n) = 2S(n/2) + O(1)$. This simplifies to $S(n) \in O(n)$.

- 3e **Work:** Work is the total number of operations. This function splits the problem into 2 subproblems of size $n/2$ and then does a constant amount of work ($O(1)$) to combine them. The work **remains the same**.

Therefore $W(n) = 2W(n/2) + O(1)$. This simplifies to $W(n) \in O(n)$ as reasoned above.

Span: Span is the longest dependency path. As a result of the two recursive calls, `longest_run_recursive(left_half, ...)` and `longest_run_recursive(right_half, ...)` are now being run **in parallel**. Therefore the Span is the height of the tree as the algorithm is operating in parallel. We can use the brick method that is balanced with the first level having a cost of 1 and the number of leaves equal to h .

$S(n) = S(n/2) + O(1)$. We have shown above that the height of the tree, h , is $\log_2(n)$ and the first level has a cost of 1. This simplifies to $S(n) \in O(\log n)$.

Output from test code in main.py

4. GCD

No fourth problem was included