Problem set 02

(1) $\log n! \in \Theta(n \log n)$

$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$$

~~Let Find c & n_0 such that~~

~~$\log n! \leq c.\, n \log n$~~

~~$\log n \,.\, \log(n-1)! \leq c.n \log n$~~

$\log(n-1)! \leq c.n$

Finding constants $c_1, c_2 > 0$ & n_0 such that For all $n \geq n_0$

$$c_1 n \log n \leq \log(n!) \leq c_2 n \log n$$

upperbound

$$n! = n \times (n-1) \times (n-2) \times \dots \dots 1 \leq n^n$$

take log of both sides

$$\log(n!) \leq \log(n^n)$$

$$\log(n!) \leq n \log(n)$$

$\therefore$ when $c_2 = 1$, $\log(n!) \leq n \log(n)$

lower bound

$$\left(\frac{n}{2}\right)^{n/2} \leq n!$$

$$\frac{n}{2} \log\left(\frac{n}{2}\right) \leq \log(n!)$$

$$\frac{n}{2}\underbrace{\left[\log(n) - 1\right]}_{\geq \frac{1}{2}\log n} \leq \log n! \qquad \left[~~\because \log \tfrac{1}{2} = 1~~\right]$$

$\geq \frac{1}{2} \log n$ ~~$\leq \log n!$~~

when $n \geq 4$

$$\frac{n}{2} \times \frac{1}{2} \log n \leq \log n!$$

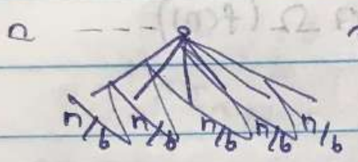
$\therefore$ when $c_1 = \frac{1}{4}$, & $n \geq 4$ $\quad \frac{1}{4} n \log n \leq \log(n!)$

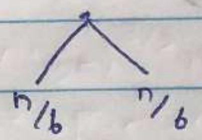
$$\therefore \frac{1}{4} n \log n \leq \log n! \leq n \log n$$

when $c_1 = \frac{1}{4}$, $c_2 = 1$, and for all $n \geq 4$

② (i) $T(n) = 2T(\frac{n}{6}) + 1$

Brick method





Level 0 : c_1

Level 1 : $2c_1$

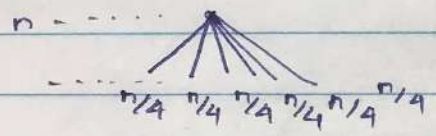
level 2 : $2^2 c_1$

$$w(n) = \sum_{i=1}^{\log_2 n} 2^i c_1 = c_1 \sum_{i=1}^{\log_6 n} 2^i$$

$$= c_1 n^{\log_6 2}$$

∴ ~~T(n) ∈ Θ(n)~~ ∴ $T(n) \in O(n^{\log_6 2})$

(ii) $T(n) = 6T(n/4) + n$



level 0 : $c_1 n + c_2$

level 1 : $6(c_1 n/4 + c_2)$

level 2 : $6^2(c_1 n/4^2 + c_2)$

$$w(n) = \sum_{i=1}^{\log_4 n} 6^i \left(c_1 \frac{n}{4^i} + c_2\right) = c_1 n \sum_{i=1}^{\log_4 n} \left(\frac{6}{4}\right)^i + c_2 \sum_{i=1}^{\log_4 n} 1$$

$$= \cancel{\sum_{i=1}^{\log_4 n}} c_1 n \times \frac{3/2}{1/2} (3/2)^{\log_4 n} + c_2 \log n$$

$$= 3c_1 n \times n^{\log_4 1.5} + c_2 \log n$$

$$= 3c_1 n^{1.029} + c_2 \log n$$

$T(n) \in O(n^{1.029})$ //

(iii) $T(n) = 7\,T(n/7) + n$

level 0 $= c_1 n + c_2$

level 1 $= 7\,(c_1 n/7 + c_2)$

level 2 $= 7^2\,(c_1 n/7^2 + c_2)$

$$W(n) = \sum_{i=0}^{\log_7 n} 7^i\,(c_1 n/7^i + c_2) = c_1 n \log_7 n + \log_7 n$$

$T(n) \in O(n \log n)$ //

(iv) $T(n) = 9\,T(n/4) + n^2$

level 0 $= c_1 n^2 + c_2$

level 1 $= 9\left(c_1\left(\frac{n}{4}\right)^2 + c_2\right)$

level 2 $= 9^2\left(c_1\left(\frac{n}{\cancel{4}}\right)^2 + c_2\right)$

$$W(n) = \sum_{i=0}^{\log_4 n} c_1 n^2 \left(\frac{9}{4}\right)^i + \sum_{i=0}^{\log_4 n} c_2\, 9^i = c_1 n^2 \left(\frac{9/4}{5/4}\right) 9/4^{\log_4 n} + c_2\, 9/8\; 9^{\log_4 n}$$

$= c_1 n^2 \times 9/5 \times n^{1.35} + c_2\, 9/8 \times n^{5.418}$

$T(n) \in O(n^{5.418})$ //

(v) $T(n) = 4\,T(n/2) + n^3$

level 0 $= c_1 n^3 + c_2$

level 1 $= 4\,c_1\left(\frac{n}{2}\right)^3 + c_2 = \frac{c_1 n^3}{2} + c_2 4$

level 2 $= 4^2\left(c_1\left(\frac{n}{2^2}\right)^3 + c_2\right) = c_1 n^3/2^2 + c_2 4^2$

$$\cancel{W(n) = \sum_{i=0}^{\log_2 n} 4^i\left[c_1\left(\frac{n}{2^i}\right)^3 + c_2\right]} = c_1 n^3 \sum_{i=0}^{\log_2 n}\left(\frac{4}{2^3}\right)^i + \sum_{i=0}^{\log_2 n} 4^i c_2$$

$= \cancel{c_1 n^3}$

$$W(n) = \sum_{i=0}^{\log_2 n} c_1 n^3/2^i + \sum_{i=0}^{\log_2 n} c_2 4^i = c_1 n^3 + c_2 n^2$$

$\therefore\ T(n) \in \Theta(n^3)$ //

(vi) $T(n) = 49T\left(\frac{n}{25}\right) + n^{3/2}\log n$

Total work at level i

$$W_i = 49^i \cdot \frac{n^{3/2}}{25^{i\,3/2}}\left(\log n - \log 25^i\right) = n^{3/2}(\log n - i\log 25)\left(\frac{49}{25^{3/2}}\right)^i$$

let $r = \frac{49}{25^{3/2}} = \frac{49}{125} \simeq 0.39 < 1$

$\therefore$ the geometric factor decays exponentially with i

deapth of recursion $= \log_{25} n$

No. of leaves $= 49^{\log_{25} n} = n^{\log_{25} 49} = n^{1.209}$

leaf total cost $= \Theta(n^{\log_{25} 49}) = \Theta(n^{1.209})$

Sum of work

$$S = \sum_{i=0}^{\log_{25} n} n^{3/2}(\log n - i\log 25)\, r^i = \sum_{i=0}^{\log_{25} n}$$

$$= n^{3/2}\left(\log n \sum_{i=0}^{\log_{25} n} r^i - \log 25 \sum_{i=0}^{\log_{25} n} i r^i\right)$$

since $r < 1$, $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, $\sum_{i=0}^{\infty} i r^i = \frac{r}{(1-r)^2}$

$$\therefore\ S \simeq n^{3/2}\left(\log n \cdot \frac{1}{1-r} - \log 25 \frac{r}{(1-r)^2}\right)$$

$\therefore\ T(n) \in \Theta(n^{3/2}\log n)$

(vii) $T(n) = T(n-1) + 2$

$$T(n) = T(n-1) + 2 = (T(n-2) + 2) + 2$$
$$= (T(n-3) + 2) + 4$$
$$= T(n-3) + 6$$
$$\vdots$$
$$= T(n-k) + 2k$$

when $k = n-1$:

$$T(n) = T(1) + 2(n-1) = O(n)$$

when Base case $T(1) = O(1)$

$$\therefore T(n) \in O(n)$$

(viii) $T(n) = T(n-1) + n^c$, with $c \geq 1$

$$\cancel{T(n) = (T(n-2) + (n-1)^c) + n^c} = \cancel{(T(n-2)) + 2n^c}$$
$$= \cancel{T(n-2) + 2n^c} \quad = \cancel{T(n-3) + 3n^c}$$
$$\vdots$$
$$\cancel{T(n-k) + kn^c}$$

level 0 : $c_1 n^c$

level 1 : $c_1 (n-1)^c$

level 2 : $c_1 (n-2)^c$

$\vdots$

level k : $c_1 (n-k)^c$

$$W(n) = \sum_{k=0}^{n-1} c_1 (n-k)^c = c_1 \left[\underbrace{n^c + (n-1)^c + (n-2)^c + \dots 1^c}_{\text{ignoring Bernuli numbers assymptotically}} \right]$$

if $c = 1$, $1 + 2 + \dots n = \frac{n(n+1)}{2}$

if $c = 2$, $1^2 + 2^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$

if $c = 3$, $1^3 + 2^3 + \dots n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$\therefore \quad T(n) = O(n^{c+1})$

(ix) $T(n) = T(\sqrt{n}) + 1$

level 0 : 1

level 1 : 1

level k : 1

$\therefore$ This ~~is rootdomi~~ recurrence is balanced with a work of 1 at each level.

$$n^{\frac{1}{2^k}} = 2$$

$$\log n^{1/2^k} = \log 2$$

$$\frac{1}{2^k} \log(n) = 1$$

$$2^k = \log_2 n$$

$$k = \log \log n$$

$\therefore \quad T(n) = O(\log \log(n))$

(3) (i) $T(n) = 2T(n/5) + O(n^2)$

level 0 : $c_1 n^2 + c_2$

level 1 : $2\left(c_1\left(\frac{n}{5}\right)^2 + c_2\right)$

level 2 : $2^2\left(c_1\left(\frac{n}{5^2}\right)^2 + c_2\right)$

$\vdots$

level i : $2^i\left(c_1\left(\frac{n}{5^i}\right)^2 + c_2\right)$

$$W(n) = \sum_{i=0}^{\log_5 n} 2^i\left(c_1\left(\frac{n}{5^i}\right)^2 + c_2\right) = c_1 n^2 \sum_{i=0}^{\log_5 n} \underbrace{\left(\frac{2}{5^2}\right)}_{r}^i + c_2 \sum_{i=0}^{\log_5 n} 1$$

let $r = \left(\frac{2}{5^2}\right)$

$$= c_1 n^2\left(\frac{1}{1-r}\right) + c_2 \log n$$

$$= O(n^2)$$

$S(n) = S(n/5) + O(n^2)$

recurrence expand as $S(n) = n^2 + \left(\frac{n}{5}\right)^2 + \left(\frac{n}{25}\right)^2 + \ldots\ldots$

$$S(n) = n^2\left(1 + \frac{1}{25} + \frac{1}{25^2} + \ldots\ldots\right)$$

$\therefore S(n) = \Theta(n^2)$

Algorithm B

(ii) $T(n) = T(n-1) + O(\log n)$

level 0 : $c_1 . \log n + c_2$

level 1 : $c_1 . \log(n-1) + c_2$

$\vdots$

level i : $c_1 . \log(n-i) + c_2$

$$W(n) = \sum_{i=0}^{n-1} (c_1 \log(n-i) + c_2) = c_1 (\log(n-1) + \log(n-2) + \dots \log 1) + c_2 \sum_{i=0}^{n-1} 1$$

$$= c_1 (\log n!)$$

$$= O(n \log n)$$

$$S(n) = S(n-1) + O(\log n)$$

$$= S(n-2) + O(\log(n-1)) + O(\log n)$$

$$= S(n-3) + O(\log(n-2)) + O(\log(n-1)) + O(\log n)$$

$$\in O(\log(n!)) = O(n \log n)$$

(iii) Algorithm C

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n^{1.1})$$

level 0 : $c_1 n^{1.1} + c_3$

level 1 : $c_1 \left(\frac{n}{3}\right)^{1.1} + c_2 \left(\frac{2n}{3}\right)^{1.1} + c_3$

level 2 : $c_1 \left(\frac{n}{3^2}\right)^{1.1} + c_2 \left(\frac{2^2 n}{3^2}\right)^{1.1} + c_3$

$\vdots$

level i : $c_1 \left(\frac{n}{3^i}\right)^{1.1} + c_2 \left(n . \left(\frac{2}{3}\right)^i\right)^{1.1} + c_3$

$$W(n) = c_1 n^{1.1} \sum_{i=0}^{\log_3 n} \frac{1}{3^i} + c_2 n^{1.1} \sum_{i=0}^{\log_{2/3} n} \left(\frac{2}{3}\right)^i + c_3$$

$$= O(n^{1.1})$$

$$S(n) = S(2n/3) + O(n^{1.1}) = n^{1.1} + \left(\frac{2}{3}\right)^{1.1} n^{1.1} + \left(\frac{2}{3}\right)^{2.1.1} n^{1.1} + \dots$$

$$= O(n^{1.1})$$

Based on the analysis conducted (work & span) on the three algorithms, Algorithm C performs the best with $O(n^{1.1})$ work & span outperforming other algorithms.

But, For extreme large values $(n > 10^{12})$, algorithm B works well.

(4) (i) Algorithm A

$$T(n) = 5T\left(\frac{n}{2}\right) + n$$

level 0 : $c_1 n + c_2$

level 1 : $5\left(c_1 \frac{n}{2} + c_2\right) = \frac{5}{2} c_1 n + 5c_2$

level 2 : $5^2\left(c_1 \frac{n}{2^2} + c_2\right) = \left(\frac{5}{2}\right)^2 c_1 n + 5^2 c_2$

$\therefore$ this is a leaf dominated recurrence

No of leaves $= n^{\log_2 5}$

$\therefore$ $T(n) \doteq O(n^{\log_2 5})$

Span $S(n) = S\left(\frac{n}{2}\right) + n$

level 0 : n

level 1 : $n/2$

$\therefore$ $S(n) = O(n)$ $\because$ root dominated

(ii) Algorithm B

$$T(n) = 2T(n-1) + O(1)$$
$$= 2(2T(n-2) + O(1)) + O(1)$$
$$\vdots$$
$$= 2T(n-k) + O(1)$$

when $k = n-1$

$$T(n) = 2T(1) + O(1) = O(1)$$

$$\therefore T(n) = O(n)$$

$$S(n) = S(n-1) + O(1)$$
$$S(n) = O(n)$$

(iii) Algorithm C

$$T(n) = 9T\left(\frac{n}{3}\right) + O(n^2)$$

level 0 $= c_1 n^2 + c_2$

level 1 $= 9 c_1 \left(\frac{n}{3}\right)^2 + c_2 = c_1 n^2 + c_2$

level 2 $= 9^2 c_1 \left(\frac{n}{3^2}\right)^2 + c_2 = c_1 n^2 + c_2$

$$W(n) = \sum_{i=0}^{\log_3 n} c_1 n^2 + c_2 = \cancel{O(n^2)} \quad O(n^2 \log n)$$

$$S(n) = S(n/3) + O(n^2)$$

$$= n^2 + \left(\frac{n}{3}\right)^2 + \left(\frac{n}{3^2}\right)^2 + \cdots$$

$$= n^2\left(1 + \frac{1}{9} + \frac{1}{81} + \cdots\right)$$

$$= O(n^2)$$

~~Algorithm A shows the best performance with O(log n) work & span, outperforming other algorithms.~~

(5)

(a) ~~n~~ students

$$T(n) =$$

Algorithm B shows the best performance with $O(n)$ linear work and span, while the algorithms A & C shows polynomial growths