

Lecture 19: Oct 11

Last time

- Distribute Midterm Exam 1
- Alternative weighting passed
- Moments

Today

- Moments
- Moment Generating Function

Moments

The various moments of a distribution are an important class of expectations.

Definition For each integer n , the n th *moment* of X (or $F_X(x)$), μ'_n , is

$$\mu'_n = EX^n.$$

The n th *central moment* of X , μ_n , is

$$\mu_n = E(X - \mu)^n,$$

where $\mu = \mu'_1 = EX$.

Theorem If X is a random variable with finite variance, then for any constants a and b ,

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

It is sometimes to use an alternative formula for the variance, given by

$$\text{Var}(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}\text{Var}(X) &= E(X - EX)^2 = E[X^2 - 2XEX + (EX)^2] \\ &= EX^2 - 2(EX)^2 + (EX)^2 \\ &= EX^2 - (EX)^2.\end{aligned}$$

Example (Binomial variance) Let $X \sim \text{Binomial}(n, p)$, that is ,

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

What is the variance of X ?

Solutions:

Moment Generating Function

Definition Let X be a random variable with cdf F_X . The *moment generating function* (mgf) of X (or F_X), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, Ee^{tX} exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Theorem If X has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(n)}(0) = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}.$$

That is, the n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$.

Proof:

Example (Binomial mgf) Let $X \sim \text{Binomial}(n, p)$, then its mgf is

$$\begin{aligned} M_X(t) &= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= [pe^t + (1-p)]^n. \end{aligned}$$

Theorem Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. If X and Y have **bounded support**, then $F_X(u) = F_Y(u)$ for all u if and only if $EX^r = EY^r$ for all integers $r = 0, 1, 2, \dots$.
2. If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u .

Theorem (Convergence of mgfs) Suppose $\{X_i, i = 1, 2, \dots\}$ is a sequence of random variables, each with mgf $M_{X_i}(t)$. Furthermore, suppose that

$$\lim_{i \rightarrow \infty} M_{X_i}(t) = M_X(t), \quad \text{for all } t \text{ in a neighborhood of } 0,$$

and $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have

$$\lim_{i \rightarrow \infty} F_{X_i}(x) = F_X(x).$$

That is, *convergence*, for $|t| < h$, of mgfs to an mgf implies *convergence* of cdfs.

Poisson approximation One approximation that is usually taught in elementary statistics courses is that binomial probabilities can be approximated by Poisson probabilities. It is taught that the Poisson approximation is valid “when n is large and np is small”, and rules of thumb are sometimes given.

The *Poisson*(λ) pmf is given by

$$\Pr(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots,$$

where λ is a positive constant. The approximation states that if $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Poisson}(\lambda)$, with $\lambda = np$, then

$$\Pr(X = x) \approx \Pr(Y = x)$$

for large n and small np . We now show that the mgf converge, lending credence to this approximation. Recall that

$$M_X(t) = [pe^t + (1 - p)]^n.$$

For the *Poisson*(λ) distribution, we can calculate (HW4, exercise 2.33)

$$M_Y(t) = e^{\lambda(e^t - 1)},$$

and if we define $p = \lambda/n$, then $M_X(t) = [1 + (e^t - 1)\lambda/n]^n$ such that $M_X(t) \rightarrow M_Y(t)$ as $n \rightarrow \infty$.

Theorem For any constant a and b , the mgf of the random variable $aX + b$ is given by

$$M_{aX+b} = e^{bt} M_X(at).$$

Proof: