

## Lecture 20: Oct 13

### Last time

- Distribute Midterm Exam 1
- Alternative weighting passed
- Moments

### Today

- Internal “evaluation” open where you can make anonymous suggestions
- Moments
- Moment Generating Function

### Moments

The various moments of a distribution are an important class of expectations.

**Definition** For each integer  $n$ , the  $n$ th *moment* of  $X$  (or  $F_X(x)$ ),  $\mu'_n$ , is

$$\mu'_n = EX^n.$$

The  $n$ th *central moment* of  $X$ ,  $\mu_n$ , is

$$\mu_n = E(X - \mu)^n,$$

where  $\mu = \mu'_1 = EX$ .

**Theorem** If  $X$  is a random variable with finite variance, then for any constants  $a$  and  $b$ ,

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

It is sometimes to use an alternative formula for the variance, given by

$$\text{Var}(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}\text{Var}(X) &= E(X - EX)^2 = E[X^2 - 2XEX + (EX)^2] \\ &= EX^2 - 2(EX)^2 + (EX)^2 \\ &= EX^2 - (EX)^2.\end{aligned}$$

**Example** (Binomial variance) Let  $X \sim \text{Binomial}(n, p)$ , that is ,

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

What is the variance of  $X$ ?

*Solutions:*

## Moment Generating Function

**Definition** Let  $X$  be a random variable with cdf  $F_X$ . The *moment generating function (mgf)* of  $X$  (or  $F_X$ ), denoted by  $M_X(t)$ , is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for  $t$  in some neighborhood of 0. That is, there is an  $h > 0$  such that, for all  $t$  in  $-h < t < h$ ,  $Ee^{tX}$  exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of  $X$  as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

**Theorem** If  $X$  has mgf  $M_X(t)$ , then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(n)}(0) = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}.$$

That is, the  $n^{\text{th}}$  moment is equal to the  $n^{\text{th}}$  derivative of  $M_X(t)$  evaluated at  $t = 0$ .

*Proof:*

**Example** (Binomial mgf) Let  $X \sim \text{Binomial}(n, p)$ , then its mgf is

$$\begin{aligned} M_X(t) &= \sum_{x=0}^n e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^n \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= [pe^t + (1-p)]^n. \end{aligned}$$

**Theorem** Let  $F_X(x)$  and  $F_Y(y)$  be two cdfs all of whose moments exist.

1. If  $X$  and  $Y$  have **bounded support**, then  $F_X(u) = F_Y(u)$  for all  $u$  if and only if  $EX^r = EY^r$  for all integers  $r = 0, 1, 2, \dots$ .
2. If the moment generating functions exist and  $M_X(t) = M_Y(t)$  for all  $t$  in some neighborhood of 0, then  $F_X(u) = F_Y(u)$  for all  $u$ .

**Theorem** (Convergence of mgfs) Suppose  $\{X_i, i = 1, 2, \dots\}$  is a sequence of random variables, each with mgf  $M_{X_i}(t)$ . Furthermore, suppose that

$$\lim_{i \rightarrow \infty} M_{X_i}(t) = M_X(t), \quad \text{for all } t \text{ in a neighborhood of } 0,$$

and  $M_X(t)$  is an mgf. Then there is a unique cdf  $F_X$  whose moments are determined by  $M_X(t)$  and, for all  $x$  where  $F_X(x)$  is continuous, we have

$$\lim_{i \rightarrow \infty} F_{X_i}(x) = F_X(x).$$

That is, *convergence*, for  $|t| < h$ , of mgfs to an mgf implies *convergence* of cdfs.

**Poisson approximation** One approximation that is usually taught in elementary statistics courses is that binomial probabilities can be approximated by Poisson probabilities. It is taught that the Poisson approximation is valid “when  $n$  is large and  $np$  is small”, and rules of thumb are sometimes given.

The *Poisson*( $\lambda$ ) pmf is given by

$$\Pr(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots,$$

where  $\lambda$  is a positive constant. The approximation states that if  $X \sim \text{Binomial}(n, p)$  and  $Y \sim \text{Poisson}(\lambda)$ , with  $\lambda = np$ , then

$$\Pr(X = x) \approx \Pr(Y = x)$$

for large  $n$  and small  $np$ . We now show that the mgf converge, lending credence to this approximation. Recall that

$$M_X(t) = [pe^t + (1 - p)]^n.$$

For the *Poisson*( $\lambda$ ) distribution, we can calculate (HW4, exercise 2.33)

$$M_Y(t) = e^{\lambda(e^t - 1)},$$

and if we define  $p = \lambda/n$ , then  $M_X(t) = [1 + (e^t - 1)\lambda/n]^n$  such that  $M_X(t) \rightarrow M_Y(t)$  as  $n \rightarrow \infty$ .

**Theorem** For any constant  $a$  and  $b$ , the mgf of the random variable  $aX + b$  is given by

$$M_{aX+b} = e^{bt} M_X(at).$$

*Proof:*