Lecture 15: Sept 25

Last time

- Transformations of Random Variables

Today

- 09/18 attendance still valid (actually today is a Jewish Holiday)
- Transformations of Random Variables
- Expected Values

Transformations of Random Variables

Theorem (Probability integral transformation) Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, that is, $\Pr(Y \leqslant y) = y, 0 < y < 1$.

Before we prove this theorem, we will digress for a moment and look at F_X^{-1}, the inverse of the cdf F_X, in some detail. If F_X is strictly increasing, then F_X^{-1} is well defined by

$$F_X^{-1}(y) = x \iff F_X(x) = y.$$

However, if F_X is constant on some interval, then F_X^{-1} is not well defined as Figure 14.1 illustrates. Any $x_1 \leqslant x \leqslant x_2$ satisfies $F_X(x) = y$

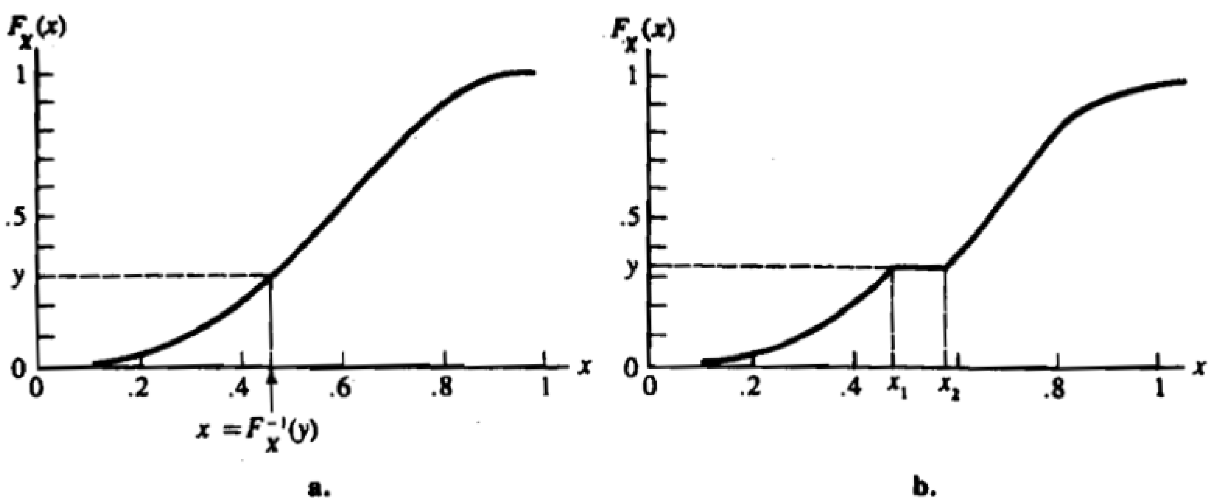

Figure 14.1: Figure 2.1.2. (a) $F_X(x)$ strictly increasing; (b) $F_X(x)$ nondecreasing

This problem is avoided by defining F_X^{-1} for $0 < y < 1$ by

$$F_X^{-1}(y) = \inf\{x : F_X(x) \geqslant y\}.$$

With this definition, for Figure 14.1(b), we have $F_X^{-1}(y) = x_1$.
Proof:

One application of the probability integral transformation is in the generation of random samples from a particular distribution. If it is required to generate an observation X from a population with cdf F_X, we need only generate a uniform random number U, between 0 and 1, and solve for x in the equation $F_X(x) = u$.

Expected Values

Definition The *expected value* or *mean* of a random variable $g(X)$, denoted by $Eg(X)$, is

$$Eg(X) = \begin{cases} \int\limits_{-\infty}^{\infty} g(x) f(x) dx & \text{if } X \text{ is continuous} \\ \sum\limits_{x \in \mathcal{X}} g(x) \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let $g(X) = X$, then we get

$$EX = \begin{cases} \int\limits_{-\infty}^{\infty} x f(x) dx & \text{if } X \text{ is continuous} \\ \sum\limits_{x \in \mathcal{X}} x \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Example (Exponential mean) Suppose X has an *exponential* (λ) *distribution*, $X \sim Exp(\lambda)$, that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad 0 \leqslant x < \infty, \lambda > 0.$$

Find out EX.
Solution:

Example (Binomial mean) if X has a *binomial distribution*, $X \sim Binomial(n, p)$, its pmf is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer, $0 \leqslant p \leqslant 1$, and for every fixed pair n and p the pmf sums to 1.
Find out EX.
Solution:

The process of taking expectations is a linear operation, which means that the expectation of a linear function of X can be easily evaluated by noting that for any constants a and b, such that

$$E(aX + b) = aEX + b$$

Theorem Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

1. $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$.
2. If $g_1(x) \geqslant 0$ for all x, then $Eg_1(X) \geqslant 0$.
3. If $g_1(x) \geqslant g_2(x)$ for all x, then $Eg_1(X) \geqslant Eg_2(X)$.
4. If $a \leqslant g_1(x) \leqslant b$ for all x, then $a \leqslant Eg_1(X) \leqslant b$.

Proof:

Example (Method of indicators) An example of how the above properties are useful. Let $X \sim Binomial(n, p)$ for n positive integer and $0 \leqslant p \leqslant 1$ (n is the number of independent identical binary trials and p is the probability of success). We can write

$$X = \sum_{i=1}^{n} I_i$$

where I_i is the indicator that i^{th} trial is a success (i.e. $I_i \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$). We have

$$EI_i = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Therefore,

$$EX = \sum_{i=1}^{n} EI_i = \sum_{i=1}^{n} p = np.$$

Theorem For a non-negative random variable X (i.e. $f(x) = 0$ for $x < 0$).

$$EX = \begin{cases} \int_0^\infty (1 - F(x))dx, & X \text{ continuous} \\ \sum_{x=0}^\infty (1 - F(x)), & X \in \mathbb{Z} \end{cases}$$

Proof: