

Lecture 17: Sept 29

Last time

- Expected Values

Today

- One-page one-sided letter-size hand-written cheat sheet for midterm 1
- Moments

Expected Values

Definition The *expected value* or *mean* of a random variable $g(X)$, denoted by $Eg(X)$, is

$$Eg(X) = \begin{cases} \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} g(x) \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let $g(X) = X$, then we get

$$EX = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} x \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Example (Exponential mean) Suppose X has an *exponential* (λ) *distribution*, $X \sim \text{Exp}(\lambda)$, that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad 0 \leq x < \infty, \lambda > 0.$$

Find out EX .

Solution:

Example (Binomial mean) if X has a *binomial distribution*, $X \sim \text{Binomial}(n, p)$, its pmf is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n,$$

where n is a positive integer, $0 \leq p \leq 1$, and for every fixed pair n and p the pmf sums to 1. Find out EX .

Solution:

The process of taking expectations is a linear operation, which means that the expectation of a linear function of X can be easily evaluated by noting that for any constants a and b , such that

$$E(aX + b) = aEX + b$$

Theorem Let X be a random variable and let a , b , and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

1. $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$.
2. If $g_1(x) \geq 0$ for all x , then $Eg_1(X) \geq 0$.
3. If $g_1(x) \geq g_2(x)$ for all x , then $Eg_1(X) \geq Eg_2(X)$.
4. If $a \leq g_1(x) \leq b$ for all x , then $a \leq Eg_1(X) \leq b$.

Proof:

Example (Method of indicators) An example of how the above properties are useful. Let $X \sim \text{Binomial}(n, p)$ for n positive integer and $0 \leq p \leq 1$ (n is the number of independent identical binary trials and p is the probability of success). We can write

$$X = \sum_{i=1}^n I_i$$

where I_i is the indicator that i^{th} trial is a success (i.e. $I_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$). We have

$$EI_i = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Therefore,

$$EX = \sum_{i=1}^n EI_i = \sum_{i=1}^n p = np.$$

Theorem For a non-negative random variable X (i.e. $f(x) = 0$ for $x < 0$).

$$EX = \begin{cases} \int_0^\infty (1 - F(x))dx, & X \text{ continuous} \\ \sum_{x=0}^\infty (1 - F(x)), & X \in \mathbb{Z} \end{cases}$$

Proof:

Moments

Example (Minimizing distance) The expected value of a random variable has another property, one that we can think of as relating to the interpretation of EX as a good guess at a value of X .

Suppose we measure the distance between a random variable X and a constant b by $(X - b)^2$. The closer b is to X , the smaller this quantity is. We can now determine the value of b that minimizes $E[(X - b)^2]$ and, hence, will provide us with a good predictor of X . (Note that it does no good to look for a value of b that minimizes $(X - b)^2$, since the answer would depend on X , making it a useless predictor of X .)

We could proceed with the minimization of $E(X - b)^2$ by using calculus, but there is a simpler method:

$$\begin{aligned} E(X - b)^2 &= E(X - EX + EX - b)^2 \\ &= E[(X - EX) + (EX - b)]^2 \\ &= E(X - EX)^2 + (EX - b)^2 + 2E[(X - EX)(EX - b)], \end{aligned}$$

where we have expanded the square. Note that $E[(X - EX)(EX - b)] = (EX - b)E(X - EX) = 0$, since $EX - b$ is constant and comes out of the expectation, $E(X - EX) = EX - EX = 0$. This means

$$E(X - b)^2 = E(X - EX)^2 + (EX - b)^2.$$

Such that $E(X - b)^2$ is minimized at $b = EX$. And $E(X - EX)^2$ is actually the variance of X ($Var X = E(X - EX)^2$).

The various moments of a distribution are an important class of expectations.

Definition For each integer n , the n th *moment* of X (or $F_X(x)$), μ'_n , is

$$\mu'_n = EX^n.$$

The n th *central moment* of X , μ_n , is

$$\mu_n = E(X - \mu)^n,$$

where $\mu = \mu'_1 = EX$.

Notes:

- $\mu'_0 = EX^0 = 1$
- μ'_1 is the *mean*, usually denoted by μ .
- $\mu_0 = E(X - \mu)^0 = 1$
- $\mu_1 = 0$
- $\mu_2 = E(X - EX)^2$ is the *variance*
- $\mu_3 = E(X - EX)^3$ is related to the *skewness*.
- $\mu_4 = E(X - EX)^4$ is related to the *kurtosis*.

Definition The *variance* of a random variable X is its second central moment, $\text{Var}(X) = E[(X - EX)^2]$. The positive square root of $\text{Var}(X)$ is the *standard deviation* of X .

The variance gives a measure of the degree of spread of a distribution around its mean. Figure 15.1 shows a plot of two samples, one sample draws 100 numbers from a normal distribution with mean 0 and variance 1, $N(0, 1)$. The other sample draws 100 numbers from a normal distribution with mean 0 and variance 100, $N(0, 100)$.

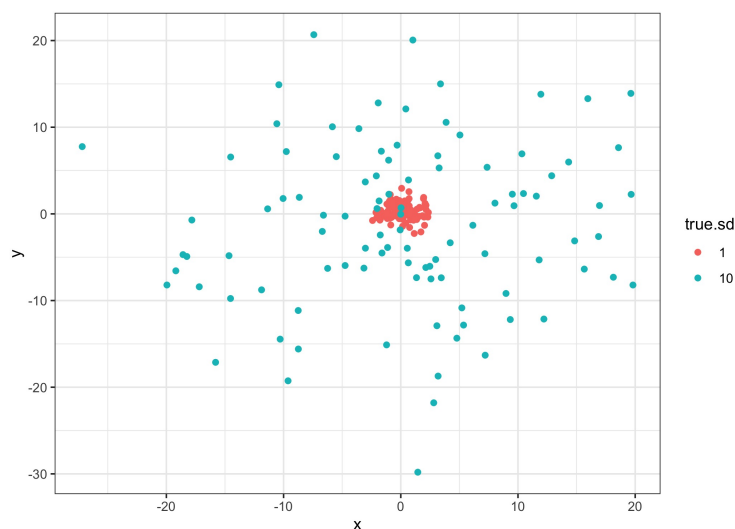


Figure 15.1: Figure 2.1.2. Two samples of 100 numbers drawn from $N(0, 1)$ and $N(0, 100)$.

Example (Exponential variance) Let X have the exponential(λ) distribution. We can calculate the variance of X now.

Solution:

$$\begin{aligned}\text{Var}(X) &= E(X - \lambda)^2 \\ &= \int_0^{\infty} (x - \lambda)^2 \frac{1}{\lambda} e^{-x/\lambda} dx\end{aligned}$$

Theorem If X is a random variable with finite variance, then for any constants a and b ,

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

Proof:

From the definition, we have

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b) - E(aX + b)]^2 \\ &= E(aX - aEX)^2 \\ &= a^2 E(X - EX)^2 \\ &= a^2 \text{Var}(X).\end{aligned}$$

It is sometimes to use an alternative formula for the variance, given by

$$\text{Var}(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}\text{Var}(X) &= E(X - EX)^2 = E[X^2 - 2XEX + (EX)^2] \\ &= EX^2 - 2(EX)^2 + (EX)^2 \\ &= EX^2 - (EX)^2.\end{aligned}$$

Example (Binomial variance) Let $X \sim \text{Binomial}(n, p)$, that is ,

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

What is the variance of X ?

Solutions: