

Lecture 7: Sept 6

Last time

- Conditional Probability (1.3)
- Independence (1.3)

Today

- Independence (1.3)
- Random variables
- Distribution Functions

Independence

Definition Two events, A and B , are *statistically independent* if

$$\Pr(A \cap B) = \Pr(A) \Pr(B)$$

Note that independence could have been defined using Bayes' rule by $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$ as long as $\Pr(A) > 0$ or $\Pr(B) > 0$. More notation, often statisticians omit \cap when writing intersection in a probability function which means $\Pr(AB) = \Pr(A \cap B)$. Sometime, statisticians use comma (,) to replace \cap inside a probability function too, $\Pr(A, B) = \Pr(A \cap B)$.

Theorem If A and B are independent events, then the following pairs are also independent.

1. A and B^c ,
2. A^c and B ,
3. A^c and B^c .

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proof:

Example Let the sample space S consist of the $3!$ permutations of the letters a , b , and c along with the three triples of each letter. Thus,

$$S = \left\{ \begin{array}{ccc} aaa & bbb & ccc \\ abc & bca & cba \\ acb & bac & cab \end{array} \right\}.$$

Furthermore, let each element of S have probability $\frac{1}{9}$. Define

$$A_i = \{i^{th} \text{ place in the triple is occupied by } a\}.$$

What are the values for $\Pr(A_i), i = 1, 2, 3$? Are they pairwise independent?

solution

Definition* A collection of events A_1, \dots, A_n are *mutually independent* if for any subcollection A_{i_1}, \dots, A_{i_k} , we have

$$\Pr\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k \Pr(A_{i_j}).$$

Random Variables

In many experiments, it is easier to deal with a summary variable than with the original probability structure.

Example consider an opinion poll, we might decide to ask 50 people whether they agree or disagree with a certain issue. If we record a “1” for agree and “0” for disagree, the sample space for this experiment has 2^{50} elements (all length 50 strings consist of 1s and 0s). However, if we are only interested in the number of people who agree, we may define a variable $X =$ number of 1s recorded out of 50. Then, the sample space for X is the set of integers $\{0, 1, 2, \dots, 50\}$.

Definition A *random variable* (r.v.) is a function from a sample space S into the real numbers.

Example In some experiments random variables are implicitly used

Examples of random variables

Experiment	Random variable
Toss two dice	$X =$ sum of numbers
Toss a coin 25 times	$X =$ number of heads in 25 tosses
Apply different amounts of fertilizer to corn plants	$X =$ yield / acre

In defining a random variable, we have also defined a new sample space (the range of the random variable).

Induced probability function Suppose we have a sample space $S = \{s_1, s_2, \dots, s_n\}$ with a probability function \Pr defined on the original sample space. We define a random variable X with range $\mathcal{X} = \{x_1, \dots, x_m\}$. We can define a probability function \Pr_X on \mathcal{X} in the following way. We will observe $X = x_i$ if and only if the outcome of the random experiment is an $s_j \in S$ such that $X(s_j) = x_i$. Therefore,

$$\Pr_X(X = x_i) = \Pr(\{s_j \in S : X(s_j) = x_i\}),$$

defines an *induced* probability function on \mathcal{X} , defined in terms of the original function \Pr .

We will write $\Pr(X = x_i)$ rather than $\Pr_X(X = x_i)$ for simplicity. Note on notation: random variables will always be denoted with uppercase letters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters.

Example Consider the experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the sample space is

s	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(s)$	3	2	2	2	1	1	1	0

What is the range of X ? What is the induced probability function \Pr_X ?
solution:

So far, we have seen finite S and finite \mathcal{X} , and the definition of \Pr_X is straightforward. If \mathcal{X} is uncountable, we define the induced probability function, \Pr_X for any set $A \subset \mathcal{X}$,

$$\Pr_X(X \in A) = \Pr(\{s \in S : X(s) \in A\}).$$

This defines a legitimate probability function for which the Kolmogorov Axioms can be verified.

Distribution Functions

Distribution Functions are used to describe the behavior of a r.v.

Cumulative distribution function

Definition The *cumulative distribution function* or *cdf* of a random variable X , denoted by $F_X(x)$, is defined by

$$F_X(x) = \Pr_X(X \leq x), \text{ for all } x.$$

Definition The *survival function* of a random variable X , is defined by

$$S_X(x) = 1 - F_X(x) = \Pr_X(X > x).$$

Example Consider the experiment of tossing three fair coins, and let X = number of heads observed. The cdf of X is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x < 2 \\ \frac{7}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x < \infty \end{cases}$$

Some properties of the cdf:

Let $F(x)$ be a cdf. Then

1. $0 \leq F(x) \leq 1$
2. $\lim_{x \rightarrow -\infty} F(x) = 0$
3. $\lim_{x \rightarrow \infty} F(x) = 1$
4. F is nondecreasing: if $a < b$, then $F(a) \leq F(b)$
5. F is right-continuous: $\lim_{x \downarrow b} F(x) = F(b)$, or $\lim_{x \rightarrow b^+} F(x) = F(b)$
6. $\Pr(a < X \leq B) = F(b) - F(a)$

Theorem The function $F(x)$ is a cdf if and only if the following three conditions hold:

1. $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$
2. F is nondecreasing: if $a < b$, then $F(a) \leq F(b)$
3. F is right-continuous: $\lim_{x \downarrow b} F(x) = F(b)$, or $\lim_{x \rightarrow b^+} F(x) = F(b)$

The cdf does not contain information about the original sample space.

Definition Two random variables X and Y are identically distributed if, for every Borel set $A \subset \mathbb{R}$, $\Pr(X \in A) = \Pr(Y \in A)$.

Example Toss a fair coin n times. The number of heads and the number of tails have the same distribution.

Theorem The following two statements are equivalent:

1. The random variables X and Y are *identically distributed*.
2. $F_X(x) = F_Y(x)$ for every x .

Types of Random Variables

Definition A random variable X can be

- *discrete*:
 - X takes on a finite or countably infinite number of values
 - $F_X(x)$ is step-wise constant
- *continuous*:
 - the range of X consists of subsets of the real line
 - $F_X(x)$ is continuous.
- *mixed*: $F_X(x)$ is piecewise continuous.

Example A random variable has cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \leq x < 1 \\ 2/3 & 1 \leq x < 2 \\ 11/12 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

Is this a valid cdf? Is it a discrete random variable or continuous random variable or mixed?
solution: