Lecture 6: Sept 1

Last time

- Calculus of Probabilities (1.2)

Today

- no class next Monday (Labor day)
- Conditional Probability (1.3)
- Independence (1.3)

Theorem If Pr is a probability function, then

1. $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$ for any partition $C_1, C_2, \ldots$;
2. $\Pr(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \Pr(A_i)$ for any sets $A_1, A_2, \ldots$.

where (1) is also referred to as "Total probability" and (2) is Boole's inequality.
proof:

Conditional Probability

All of the probabilities that we have dealt with thus far have been unconditional probabilities. A sample space was defined and all probabilities were calculated with respect to that sample space. In many instances, however, we are in a position to update the sample space based on new information. In such cases we want to be able to update probability calculations or to calculate *conditional probabilities*.

Definition If A and B are events in S, and $\Pr(B) > 0$, then the *conditional probability* of A given B, written $\Pr(A|B)$, is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Note that B becomes the sample space now: $\Pr(B|B) = 1$.

Example Four cards are dealt from the top of a well-shuffled deck. What is the probability that they are the four aces? What is the probability of getting four aces at the top if knowing the first card is an ace? (there are in total 52 cards)

solution:

Theorem (Bayes' Rule) Let $A_1, A_2, \ldots$ be a partition of the sample space, and let B be any set. Then, for each $i = 1, 2, \ldots,$

$$\Pr(A_i|B) = \frac{\Pr(B|A_i)\Pr(A_i)}{\sum_{j=1}^{\infty}\Pr(B|A_j)\Pr(A_j)}.$$

proof:

Independence

Definition Two events, A and B, are *statistically independent* if

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

Note that independence could have been defined using Bayes' rule by $\Pr(A|B) = \Pr(A)$ or $\Pr(B|A) = \Pr(B)$ as long as $\Pr(A) > 0$ or $\Pr(B) > 0$. More notation, often statisticians omit $\cap$ when writing intersection in a probability function which means $\Pr(AB) = \Pr(A \cap B)$. Sometime, statisticians use comma (,) to replace $\cap$ inside a probability function too, $\Pr(A, B) = \Pr(A \cap B)$.

Theorem If A and B are independent events, then the following pairs are also independent.

1. A and B^c,
2. A^c and B,
3. A^c and B^c.

Theorem If A and B are independent events, then the following pairs are also independent.

1. A and B^c,
2. A^c and B,
3. A^c and B^c.

proof:

Example Let the sample space S consist of the $3!$ permutations of the letters a, b, and c along with the three triples of each letter. Thus,

$$S = \left\{ \begin{array}{ccc} \text{aaa} & \text{bbb} & \text{ccc} \\ \text{abc} & \text{bca} & \text{cba} \\ \text{acb} & \text{bac} & \text{cab} \end{array} \right\}.$$

Furthermore, let each element of S have probability $\frac{1}{9}$. Define

$$A_i = \{i^{th} \text{ place in the triple is occupied by } a\}.$$

What are the values for $\Pr(A_i), i = 1, 2, 3$? Are they pairwise independent?
solution

Definition* A collection of events $A_1, \ldots, A_n$ are *mutually independent* if for any subcollection $A_{i_1}, \ldots, A_{i_k}$, we have

$$\Pr\left(\cap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \Pr(A_{i_j}).$$

Random Variables

In many experiments, it is easier to deal with a summary variable than with the original probability structure.

Example consider an opinion poll, we might decide to ask 50 people whether they agree or disagree with a certain issue. If we record a "1" for agree and "0" for disagree, the sample space for this experiment has 2^{50} elements (all length 50 strings consist of 1s and 0s). However, if we are only interested in the number of people who agree, we may define a variable $X =$ number of 1s recorded out of 50. Then, the sample space for X is the set of integers $\{0, 1, 2, \ldots, 50\}$.

Definition A *random variable* (r.v.) is a function from a sample space S into the real numbers.

Example In some experiments random variables are implicitly used

Examples of random variables

Experiment	Random variable
Toss two dice	$X =$ sum of numbers
Toss a coin 25 times	$X =$ number of heads in 25 tosses
Apply different amounts of fertilizer to corn plants	$X =$ yield / acre

In defining a random variable, we have also defined a new sample space (the range of the random variable).

Induced probability function Suppose we have a sample space $S = \{s_1, s_2, \ldots, s_n\}$ with a probability function Pr defined on the original sample space. We define a random variable X with range $\mathcal{X} = \{x_1, \ldots, x_m\}$. We can define a probability function $\Pr_X$ on $\mathcal{X}$ in the following way. We will observe $X = x_i$ if an only if the outcome of the random experiment is an $s_j \in S$ such that $X(s_j) = x_i$. Therefore,

$$\Pr_X(X = x_i) = \Pr(\{s_j \in S : X(s_j) = x_i\}),$$

defines an *induced* probability function on $\mathcal{X}$, defined in terms of the original function Pr.

We will write $\Pr(X = x_i)$ rather than $\Pr_X(X = x_i)$ for simplicity. Note on notation: random variables will always be denoted with uppercase leeters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters.

Example Consider the experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the sample space is

s	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(s)$	3	2	2	2	1	1	1	0

What is the range of X? What is the induced probability function $\Pr_X$?
solution:

So far, we have seen finite S and finite $\mathcal{X}$, and the definition of $\Pr_X$ is straightforward. If $\mathcal{X}$ is uncountable, we define the induced probability function, $\Pr_X$ for anyset $A \subset \mathcal{X}$,

$$\Pr_X(X \in A) = \Pr(\{s \in S : X(s) \in A\}).$$

This defines a legitimate probability function for which the Kolmogorov Axioms can be verified.