

Lecture 13: Sept 20

Last time

- Transformations of Random Variables

Today

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Transformations of Random Variables

Theorem Suppose a continuous random variable X has cdf $F_X(x)$, let $Y = g(X)$, and let \mathcal{X} and \mathcal{Y} be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If g is an increasing function on \mathcal{X} , $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
2. If g is a decreasing function on \mathcal{X} , $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Proof: We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr(g(X) \leq y) \end{aligned}$$

Theorem Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let \mathcal{X} and \mathcal{Y} be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Suppose that $f_X(x)$ is continuous on \mathcal{X} and that $g^{-1}(y)$ has a continuous derivative on \mathcal{Y} . Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

From last theorem, we have the cdf forms $F_Y(y)$. Then $f_Y(y) = \frac{d}{dy} F_Y(y)$. (finish the proof)

Example (Square transformation) Suppose X is a continuous random variable. For $y > 0$, the cdf of $Y = X^2$ is

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}).$$

Because x is continuous, we can drop the equality from the left endpoint and obtain

$$\begin{aligned} F_Y(y) &= \Pr(-\sqrt{y} < X \leq \sqrt{y}) \\ &= \Pr(X \leq \sqrt{y}) - \Pr(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

The pdf of Y can now be obtained from the cdf by differentiation: where we use the chain rule to differentiate $F_X(\sqrt{y})$ and $F_X(-\sqrt{y})$.

Example (Linear transformation) Suppose X is a continuous random variable with pdf $f_X(x)$. Let

$$Y = a + bX, \quad \frac{dy}{dx} = b.$$

Then

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{dx}{dy} \right| = f_X\left(\frac{y-a}{b}\right) \frac{1}{|b|}.$$

This transformation is often used when X has mean 0 and standard deviation 1. The linear transformation above creates a random variable Y with a distribution that has the same shape as that of X but has mean a and variance b^2 .

Conversely, if Y has mean a and standard deviation b , then $X = (Y - a)/b$ has mean 0 and standard deviation 1. This is called sometimes the “Studentized” transformation.

Example (Normal distribution) Let $X \sim N(0, 1)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

The transformation

$$Y = \mu + \sigma X, \quad X = \frac{Y - \mu}{\sigma}$$

yields

$$f_Y(y) = f_X\left(\frac{y - \mu}{\sigma}\right) \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y - \mu)^2}{2\sigma^2}}.$$

More generally, a distribution is a member of the class of *location-scale* distributions if the distribution of a linear transformation of a random variable with that distribution has the same distribution, but with different parameters.

Example (Square root of an exponential RV) Suppose $X \sim \exp(\lambda)$, so that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and consider the distribution of $Y = \sqrt{X}$. The transformation

$$y = g(x) = \sqrt{x}, \quad x \geq 0$$

is one-to-one and has an inverse $x = y^2$ with $dx/dy = 2y$. Thus

This distribution is a particular form of the Rayleigh distribution and is a special case of the Weibull distribution.