Lecture 12: Sept 18

Last time

- Continuous Random Variables

Today

- Transformations of Random Variables

Transformations of Random Variables

Theorem If X is a r.v. with sample space $\mathcal{X} \subset \mathbb{R}$ and cdf $F_X(x)$, then any function of X, say $Y = g(X)$ is also a random variable. The new random variable Y has a new sample space $\mathcal{Y} = g(X) \subset \mathbb{R}$. The objective is to find the cdf $F_Y(y)$ of Y.

Probability mapping: For any set $A \subset \mathcal{Y}$:

$$\begin{aligned}\Pr(Y \in A) &= \Pr(g(X) \in A)\\ &= \Pr(\{x \in \mathcal{X} : g(x) \in A\})\\ &= \Pr(X \in g^{-1}(A)),\end{aligned}$$

where we have defined

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}.$$

Notice that $g^{-1}(A)$ is well defined even if $g(\cdot)$ is not necessarily bijective.

Example (Binomial transformation) A discrete random variable X has a *binomial distribution* if its pmf is of the form

$$f_X(x) = \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer and $0 \leqslant p \leqslant 1$. Values such as n and p that can be set to different values, producing different probability distributions, are called *parameters*. Consider a random variable $Y = g(X)$, where $g(x) = n - x$; that is, $Y = n - X$. Here $\mathcal{X} = \{0, 1, \ldots, n\}$ and $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\} = \{0, 1, \ldots, n\}$. For any $y \in \mathcal{Y}$, $n - x = g(x) = y$ if and only if $x = n - y$. Therefore, $g^{-1}(y) = n - y$ and

$$\begin{aligned}f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x)\\ &= f_X(n-y)\\ &= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}\\ &= \binom{n}{y} (1-p)^y p^{n-y}.\end{aligned}$$

Therefore, Y also has a binomial distribution, but with parameters n and $1 - p$.

Example (exercise 2.3) Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}(\frac{2}{3})^x, x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.
Solution:

Theorem Suppose a continuous random variable X has cdf $F_X(x)$, let $Y = g(X)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If g is an increasing function on $\mathcal{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
2. If g is a decreasing function on $\mathcal{X}$, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Proof: We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y) \\ &= \Pr(g(X) \leqslant y) \end{aligned}$$

Theorem Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)| & y \in \mathcal{Y} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof:
From last theorem, we have the cdf forms $F_Y(y)$. Then $f_Y(y) = \frac{d}{dy}F_Y(y)$. (finish the proof)

Example (Square transformation) Suppose X is a continuous random variable. For $y > 0$, the cdf of $Y = X^2$ is

$$F_Y(y) = \Pr(Y \leqslant y) = \Pr(X^2 \leqslant y) = \Pr(-\sqrt{y} \leqslant X \leqslant \sqrt{y}).$$

Because x is continuous, we can drop the equality from the left endpoint and obtain

$$\begin{aligned} F_Y(y) &= \Pr(-\sqrt{y} < X \leqslant \sqrt{y}) \\ &= \Pr(X \leqslant \sqrt{y}) - \Pr(X \leqslant -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

The pdf of Y can now be obtained from the cdf by differentiation: where we use the chain rule to differentiate $F_X(\sqrt{y})$ and $F_X(-\sqrt{y})$.

Example (Linear transformation) Suppose X is a continuous random variable with pdf $f_X(x)$. Let

$$Y = a + bX, \quad \frac{dy}{dx} = b.$$

Then

$$f_Y(y) = f_X\left[g^{-1}(y)\right]|\frac{dx}{dy}| = f_X(\frac{y-a}{b})\frac{1}{|b|}.$$

This transformation is often used when X has mean 0 and standard deviation 1. The linear transformation above creates a random variable Y with a distribution that has the same shape as that of X but has mean a and variance b^2.

Conversely, if Y has mean a and standard deviation b, then $X = (Y-a)/b$ has mean 0 and standard deviation 1. This is called sometimes the "Studentized" transformation.

Example (Normal distribution) Let $X \sim N(0,1)$:

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

The transformation

$$Y = \mu + \sigma X, \quad X = \frac{Y-\mu}{\sigma}$$

yields

$$f_Y(y) = f_X(\frac{y-\mu}{\sigma})\frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

More generally, a distribution is a member of the class of *location-scale* distributions if the distribution of a linear transformation of a random variable with that distribution has the same distribution, but with different parameters.

Example (Square root of an exponential RV) Suppose $X \sim exp(\lambda)$, so that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0 \\ 0 & otherwise \end{cases}$$

and consider the distribution of $Y = \sqrt{X}$. The transformation

$$y = g(x) = \sqrt{x}, \quad x \geqslant 0$$

is one-to-one and has an inverse $x = y^2$ with $dx/dy = 2y$. Thus
This distribution is a particular form of the Rayleigh distribution and is a special case of the Weibull distribution.