Lecture 27: Oct 30

Last time

- Common Continuous Distribution

Today

- Show HW4 Q9
- Common Continuous Distribution

Common continuous distributions

Shifted exponential Let $X \sim Exp(\lambda)$ and $Y = X + v, v \in \mathbb{R}$. Then, Y has the *shifted exponential distribution* with pdf:

$$f(y) = \begin{cases} \lambda e^{-(y-v)\lambda} & \text{for } y \geqslant v \\ 0 & otherwise \end{cases}$$

Interpretation:

- $v > 0$: Event is delayed
- $v < 0$: The news of the event is delayed

Does the shifted expoenential maintain the memoryless property?

Double exponential The *double exponential distribution* is formed by reflecting an exponential distribution around zero. It has pdf:

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \quad x \in \mathbb{R}$$

Laplace distribution Suppose X has the above distribution with $\lambda = 1$. Now let $Y = \sigma X + \mu, \mu \in \mathbb{R}$ (shifting) and $\sigma > 0$ (scaling). Then Y has the *Laplace distribution* with pdf:

$$f_Y(y) = \frac{1}{2\sigma}\exp\left(-\frac{|y-\mu|}{\sigma}\right)$$

with moments

$$EY = \mu, \quad Var(Y) = 2\sigma^2$$

The Laplace distribution provides an alternative to the normal for centered data with fatter tails but all finite moments.

Normal Distribution Introduced by De Moivre (1667 - 1754) in 1733 as an approximation to the binomial. Later studied by Laplace and others as part of the Central Limit Theorem. Gauss derived the normal as a suitable distribution for outcomes that could be thought of as sums of many small deviations.

- Sample space: $\mathbb{R} = (-\infty, \infty)$
- pdf: For $Y \sim N(\mu, \sigma^2)$,

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \qquad -\infty < y < \infty$$

- cdf: There is no closed form.
- When $\mu = 0$ and $\sigma = 1$, the distribution is called *standard normal*:

$$\Phi(y) = \Pr(Y \leqslant y), \quad \Phi(-y) = 1 - \Phi(y)$$

- Mean:

$$EY = \mu$$

- Variance:

$$Var(Y) = E(Y-\mu)^2 = \sigma^2$$

- Higher central moments:

$$E(Y-\mu)^m = \begin{cases} \frac{m!}{2^{m/2}(m/2)!}\sigma^m & m \text{ is even} \\ 0 & m \text{ is odd} \end{cases}$$

- In particular:

$$\mu_3 = E(Y-\mu)^3 = 0(\text{Skewness})$$
$$\mu_4 = E(Y-\mu)^4 = 3\sigma^4$$

- Moment generating function:

$$M_Y(t) = \exp(\mu t + \sigma^2 t^2/2)$$

Standardization

$$Y \sim N(\mu, \sigma^2) \iff Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$$

Shifting and scaling:

$$Z \sim N(0,1) \iff Y = \sigma Z + \mu \sim N(\mu, \sigma^2)$$

Notes

- Normal distribution is useful in many practical settings. E.g. measurement error.

- Plays an important role in *sampling distributions* in *large samples*, since the Central Limit Theorem syas that the sums of independent identically distributed random variables are approximately normal
- There are many important distributions that can be derived from functions of normal random variables (e.g. χ^2, t, F). We will briefly present the pdf's and sample spaces of these distributions.

χ^2 distribution If $Z \sim N(0,1)$, then $X = Z^2$ has the χ^2 distribution with 1 degree of freedom. More generally, we have the χ^2 distribution with v degrees of freedom with pdf:

$$f(x) = \frac{(x/2)^{\frac{v}{2}-1} e^{-x/2}}{2\Gamma(v/2)}, \quad x > 0$$

where $\Gamma(a)$ is the complete gamma function,

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

The $\chi^2(v)$ distribution is a special case of the gamma distribution, so it is easier to derive its properties from the gamma.

Facts about the Gamma function

- $\Gamma(a+1) = a\Gamma(a), a > 0$
- $\Gamma(1) = 1$
- $\Gamma(n) = (n-1)!$
- $\Gamma(1/2) = \sqrt{\pi}$

Student's t and F distributions Y has a t_k distribution (t with k degrees of freedom) if its pdf can be written as:

$$f(y) = \frac{\Gamma\left[(v+1)/2\right]}{\sqrt{v\pi}\Gamma(v/2)} \frac{1}{(1+y^2/v)^{(v+1)/2}}, \quad -\infty < y < \infty$$

Y has an $F(v_1, v_2)$ distribution if its pdf can be written as:

$$f(y) = \frac{(v_1/v_2)\Gamma\left[(v_1+v_2)/2\right](v_1 y/v_2)^{v_1/2-1}}{\Gamma(v_1/2)\Gamma(v_2/2)(1+v_1 y/v_2)^{(v_1+v_2)/2}}, \quad 0 \leqslant y < \infty$$

There are many important properties and relationships between these three distributions (e.g. χ^2_k is the distribution of the sum of the squares of k independent standard normals).

Gamma distribution Notation: $Y \sim Gamma(a, \lambda)$.

- pdf:
$$f(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{a-1}}{\Gamma(a)}, \quad y \geqslant 0$$
where $\Gamma(a)$ is the gamma function,
$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$$

- cdf: In general, there is no closed form, unless a is an integer.

- moments:
$$E(Y) = a/\lambda$$
$$Var(Y) = a/\lambda^2$$

- MGF:
$$M_Y(t) = \left(\frac{1}{1 - t/\lambda}\right)^a, \quad t < \theta$$

Another parameterization Same as the exponential distribution, we can let $\beta = \frac{1}{\lambda}$, then we have

- pdf:
$$f(y) = \frac{y^{a-1}e^{-y/\beta}}{\Gamma(a)\beta^a}, \quad y \geqslant 0$$

- moments:
$$EX = \alpha\beta$$
$$Var(X) = \alpha\beta^2$$

- MGF:
$$M_Y(t) = \left(\frac{1}{1 - t\beta}\right)^a, \quad t < \frac{1}{\beta}$$

Notes:

- The special case $a = 1$ corresponds to an *exponential*(λ)
- The parameter a is known as the *shape parameter*, since it most influences the peakedness of the distribution.
- The parameter β is called the *scale parameter* since most of its influence is on the spread of the distribution.
- The special case $Gamma(a = n/2, \lambda = 1/2)$, for integer n, corresponds to the χ^2_n distribution with n degrees of freedom.
- The gamma distribution can be derived as the sum of a independent *exponential*(λ) distributions.