Math 3070/6070 Homework 3
Due: Oct 2nd, 2023

1. (1.47 a - d) Prove that the following functions are cdfs.
    1. $\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x), x \in (-\infty, \infty)$
    2. $(1 + e^{-x})^{-1}, x \in (-\infty, \infty)$
    3. $e^{-e^{-x}}, x \in (-\infty, \infty)$
    4. $1 - e^{-x}, x \in (0, \infty)$
2. (1.51) An appliance store receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 ovens at random, without replacement, and tests to see if they are defective. Let X = number of defectives found. Calculate the pmf and cdf of X and plot the cdf.
3. (1.52) Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function
$$g(x) = \begin{cases} f(x)/[1 - F(x_0)] & x \geq x_0 \\ 0 & x < x_0. \end{cases}$$
Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)
4. (1.53) A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function
$$F_Y(y) = \Pr(Y \leq y) = 1 - \frac{1}{y^2}, \quad 1 \leq y < \infty.$$
    1. Verify that $F_Y(y)$ is a cdf.
    2. Find $f_Y(y)$, the pdf of Y.
    3. If the low-water mark is reset at 0 and we use a unit of measurement that is $\frac{1}{10}$ of that given previously, the high-water mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$.
5. (1.54) For each of the following, determine the value of c that makes $f(x)$ a pdf.
    1. $f(x) = c \sin x, \quad 0 < x < \pi/2$
    2. $f(x) = ce^{-|x|}, \quad -\infty < x < \infty$
6. (2.1) In each of the following find the pdf of Y. Show that the pdf integrates to 1.
    1. $Y = X^3$ and $f_X(x) = 42x^5(1 - x), 0 < x < 1$
    2. $Y = 4X + 3$ and $f_X(x) = 7e^{-7x}, 0 < x < \infty$
    3. $Y = X^2$ and $f_X(x) = 30x^2(1 - x)^2, 0 < x < 1$
7. (2.2) In each of the following find the pdf of Y
    1. $Y = X^2$ and $f_X(x) = 1, 0 < x < 1$

2. $Y = -\log(X)$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!m!} x^n (1-x)^m, \quad 0 < x < 1,\ m, n \text{ positive integers}$$

3. $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}, \quad 0 < x < \infty,\ \sigma^2 \text{ a positive constant}$$

8. (2.4) Let λ be a fixed positive constant, and define the function $f(x)$ by $f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$.

   1. Verify that $f(x)$ is a pdf.
   2. If X is a random variable with pdf given by $f(x)$, find $\Pr(X < t)$ for all t. Evaluate all integrals.
   3. Find $\Pr(|X| < t)$ for all t. Evaluate all integrals.