

Lecture 30: Nov 4

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution
- Distribution Families

Location and Scale families

Let Z be a continuous random variable with pdf $f(z)$. Define the class of rvs

$$X_{\mu,\sigma} = \sigma Z + \mu, \quad \mu \in \mathbb{R}, \sigma > 0$$

Then

1. $X_{\mu,\sigma}$ has pdf

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x - \mu}{\sigma}\right)$$

- 2.

$$E(X) = \sigma E(Z) + \mu, \quad \text{Var}(X) = \sigma^2 \text{Var}(Z)$$

3. The variable $Z = X_{0,1}$ is called the *generator* and is a member of the class.

Location families and scale families

- The family of pdfs $f_{\mu,\sigma}(x)$ is called a *location-scale* family where μ is called the *location parameter*, and σ is called the *scale parameter*.
- The family of pdfs

$$f_{\mu,1}(x) = f(x - \mu)$$

with $\sigma = 1$ is called a *location* family.

- The family of pdfs

$$f_{0,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$

with $\mu = 0$ is called a *scale* family.

Example (Exponential location family) Let $f(x) = e^{-x}$, $x \geq 0$, and $f(x) = 0$, $x < 0$. To form a location family we replace x with $x - \mu$ to obtain

$$f(x|\mu) = \begin{cases} e^{-(x-\mu)} & x - \mu \geq 0 \\ 0 & x - \mu < 0 \end{cases}$$

$$= \begin{cases} e^{-(x-\mu)} & x \geq \mu \\ 0 & x < \mu \end{cases}$$

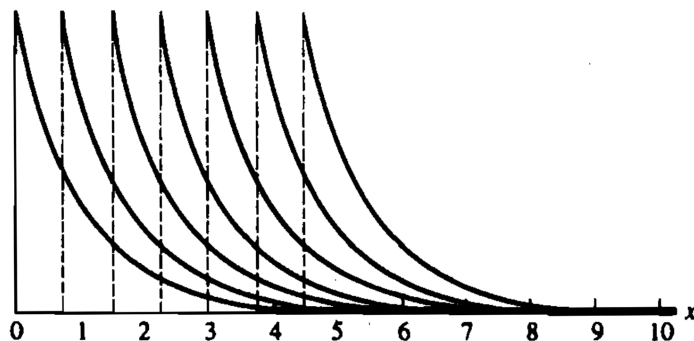


Figure 3.5.2. *Exponential location densities*

Figure 29.1: Figure 3.5.2. Exponential location densities.

As shown in the above graph, the densities are shifted. Now the positive part of the density starts at μ rather than at 0. If X measures time, then μ might be restricted to be nonnegative so that X will be positive with probability 1 for every value of μ . In this type of model, where μ denotes a bound on the range of X , μ is sometimes called a *threshold parameter*.

The effect of introducing the scale parameter σ is either to stretch ($\sigma > 1$) or to contract ($\sigma < 1$) the graph of $f(x)$ while still maintaining the same basic shape of the graph. This is illustrated in the Figure below.

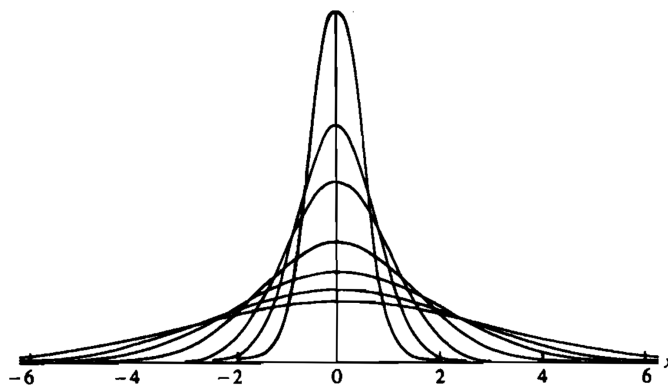


Figure 29.2: Figure 3.5.3. Members of the same scale family

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})\exp\left(\sum_{j=1}^k w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R} \quad (1)$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geq 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x , etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma)$. Then

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{2\pi}} & c(\mu, \sigma) &= \frac{1}{\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu, \sigma) &= -\frac{1}{2\sigma^2} & w_2(\mu, \sigma) &= \frac{\mu}{\sigma^2} \\ t_1(x) &= x^2 & t_2(x) &= x \end{aligned}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.