Lecture 28: Oct 30

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution

Student's t and F distributions Y has a t_k distribution (t with k degrees of freedom) if its pdf can be written as:

$$f(y) = \frac{\Gamma\left[(v+1)/2\right]}{\sqrt{v\pi}\Gamma(v/2)} \frac{1}{(1+y^2/v)^{(v+1)/2}}, \quad -\infty < y < \infty$$

Y has an $F(v_1, v_2)$ distribution if its pdf can be written as:

$$f(y) = \frac{(v_1/v_2)\Gamma\left[(v_1+v_2)/2\right](v_1 y/v_2)^{v_1/2-1}}{\Gamma(v_1/2)\Gamma(v_2/2)(1+v_1 y/v_2)^{(v_1+v_2)/2}}, \quad 0 \leqslant y < \infty$$

There are many important properties and relationships between these three distributions (e.g., χ^2_k is the distribution of the sum of the squares of k independent standard normals).

Gamma distribution Notation: $Y \sim Gamma(a, \lambda)$.

- pdf:
$$f(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{a-1}}{\Gamma(a)}, \quad y \geqslant 0$$
where $\Gamma(a)$ is the gamma function,
$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

- cdf: In general, there is no closed form, unless a is an integer.
- moments:
$$E(Y) = a/\lambda$$
$$Var(Y) = a/\lambda^2$$
- MGF:
$$M_Y(t) = \left(\frac{1}{1-t/\lambda}\right)^a, \quad t < \theta$$

Another parameterization Same as the exponential distribution, we can let $\beta = \frac{1}{\lambda}$, then we have

- pdf:
$$f(y) = \frac{y^{a-1}e^{-y/\beta}}{\Gamma(a)\beta^a}, \quad y \geqslant 0$$

- moments:
$$EX = \alpha\beta$$
$$Var(X) = \alpha\beta^2$$

- MGF:
$$M_Y(t) = \left(\frac{1}{1-t\beta}\right)^a, \quad t < \frac{1}{\beta}$$

Notes:

- The special case $a = 1$ corresponds to an *exponential*(λ)
- The parameter a is known as the *shape parameter*, since it most influences the peakedness of the distribution.
- The parameter β is called the *scale parameter* since most of its influence is on the spread of the distribution.
- The special case $Gamma(a = n/2, \lambda = 1/2)$, for integer n, corresponds to the χ^2_n distribution with n degrees of freedom.
- The gamma distribution can be derived as the sum of a independent *exponential*(λ) distributions.

Beta distribution Notation: $Y \sim Beta(a, b)$.

- Sample space: $[0, 1]$
- pdf:
$$f(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}, \quad 0 \leqslant y \leqslant 1$$
where $B(a, b)$ is the Beta function,
$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$
and $\Gamma(a)$ is the gamma function. Note that if a and b are integers, then $B(a, b)$ can be calculated in closed form.
- cdf: In general, there is no closed form, except if a and b are integers.

- moments:

$$EY = \frac{a}{a+b}$$
$$Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}$$

The beta distribution is very flexible, and can take a wide variety of shapes by varying its parameters.

- Special case: $Beta(1,1) = U(0,1)$.

Omitted distributions: Weibull distribution, and Cauchy distribution.

Location and Scale families

Let Z be a continuous random variable with pdf $f(z)$. Define the class of rvs

$$X_{\mu,\sigma} = \sigma Z + \mu, \quad \mu \in \mathbb{R}, \sigma > 0$$

Then

1. $X_{\mu,\sigma}$ has pdf
$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

2. $$E(X) = \sigma E(Z) + \mu, \quad Var(X) = \sigma^2 Var(Z)$$

3. The variable $Z = X_{0,1}$ is called the *generator* and is a member of the class.

Location families and scale families

- The family of pdfs $f_{\mu,\sigma}(x)$ is called a *location-scale* family where μ is called the *location parameter*, and σ is called the *scale parameter*.
- The family of pdfs
$$f_{\mu,1}(x) = f(x-\mu)$$
with $\sigma = 1$ is called a *location* family.
- The family of pdfs
$$f_{0,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$
with $\mu = 0$ is called a *scale* family.

Example (Exponential location family) Let $f(x) = e^{-x}, x \geqslant 0$, and $f(x) = 0, x < 0$. To form a location family we replace x with $x - \mu$ to obtain

$$
\begin{aligned}
f(x|\mu) &= \begin{cases} e^{-(x-\mu)} & x-\mu \geqslant 0 \\ 0 & x-\mu < 0 \end{cases} \\
&= \begin{cases} e^{-(x-\mu)} & x \geqslant \mu \\ 0 & x < \mu \end{cases}
\end{aligned}
$$

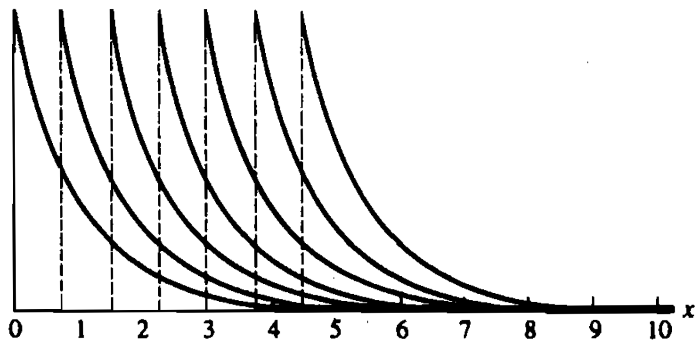

Figure 3.5.2. ***Exponential location densities***

Figure 27.1: Figure 3.5.2. Exponential location densities.

As shown in the above graph, the densities are shifted. Now the positive part of the density starts at μ rather than at 0. If X measures time, then μ might be restricted to be nonnegative so that X will be positive with probability 1 for every value of μ. In this type of model, where μ denotes a bound on the range of X, μ is sometimes called a *threshold parameter*.

The effect of introducing the scale parameter σ is either to stretch ($\sigma > 1$) or to contract ($\sigma < 1$) the graph of $f(x)$ while still maintaining the same basic shape of the graph. This is illustrated in the Figure below.

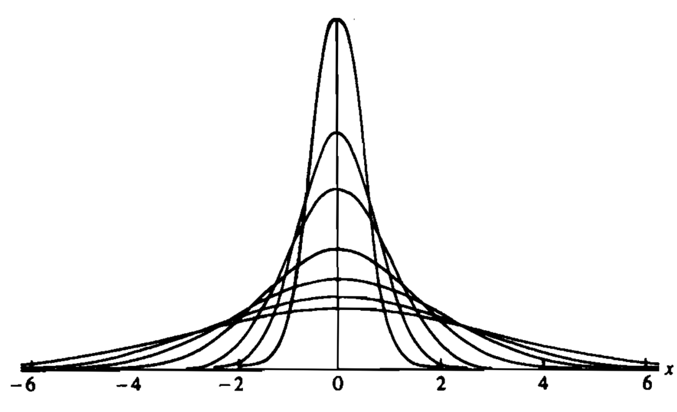

Figure 27.2: Figure 3.5.3. Members of the same scale family

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R} \tag{1}$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geqslant 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x, etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma)$. Then

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{aligned} h(x) &= \tfrac{1}{\sqrt{2\pi}} & c(\mu, \sigma) &= \tfrac{1}{\sigma} \exp\left(-\tfrac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu, \sigma) &= -\tfrac{1}{2\sigma^2} & w_2(\mu, \sigma) &= \tfrac{\mu}{\sigma^2} \\ t_1(x) &= x^2 & t_2(x) &= x \end{aligned}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.