

Lecture 2: Aug 21

Last time

- Introduction
- Introduce yourself
- Course logistics

Today

- Set theory (1.1)
- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)
- Conditional Probability (1.3)

Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space*.

Definition The set, S , of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

Example The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university
- an experiment where the observation is reaction time to a certain stimulus

Definition An *event* is any collection of possible outcomes of an experiment, that is, any subset of S (including S itself).

Let A be an event,

- A is a subset of S ,
- event A occurs if the outcome of the experiment is in the set A ,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:

$$A \subset B \iff x \in A \implies x \in B$$

- Equality:

$$A = B \iff A \subset B \text{ and } B \subset A$$

- Union: the union of A and B , written as $A \cup B$, is the set of elements that belong to either A or B or both

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- Intersection: the intersection of A and B , written $A \cap B$, is the set of elements that belong to both A and B :

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- Complementation: the complement of A , written A^c , is the set of all elements that are not in A :

$$A^c = \{x : x \notin A\}.$$

Theorem For any three events, A , B , and C , defined on a sample space S ,

1. Commutativity

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A; \end{aligned}$$

2. Associativity

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C, \\ A \cap (B \cap C) &= (A \cap B) \cap C; \end{aligned}$$

3. Distributive Laws

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C); \end{aligned}$$

4. DeMorgan's Laws

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c; \end{aligned}$$

We show the proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.

proof:

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$:

Let $x \in (A \cap (B \cup C))$. By definition of intersection, $x \in (B \cup C)$ that is, either $x \in B$ or $x \in C$. Since x also must be in A , we have that either $x \in (A \cap B)$ or $x \in (A \cap C)$; therefore, $x \in ((A \cap B) \cup (A \cap C))$.

- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$:
Let $x \in ((A \cap B) \cup (A \cap C))$. This implies that $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$, then x is in both A and B . Since $x \in B$, then $x \in (B \cup C)$ and thus $x \in (A \cap (B \cup C))$. It follows the same argument when $x \in (A \cap C)$, we still have $x \in (A \cap (B \cup C))$.

Definition Two events A and B are *disjoint* (or *mutually exclusive*) if $A \cap B = \emptyset$. The events A_1, A_2, \dots are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Definition If A_1, A_2, \dots are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots = S$, then the collection of A_1, A_2, \dots forms a *partition* of S .

Example The sets $A_i = [i, i + 1), i = 0, 1, 2, \dots$ form a partition of $[0, \infty)$.

Basics of Probability Theory

When an experiment is performed, the realization of the experiment is an outcome in the sample space. If the experiment is performed a number of times, then

- different outcomes may occur each time
- some outcomes may repeat
- the “frequency of occurrence” of an outcome can be thought of as a probability

However, we **do not** define probabilities in terms of frequencies but instead take the mathematically simpler axiomatic approach. The axiomatic approach is not concerned with the interpretations of probabilities, but is concerned only that the probabilities are defined by a function satisfying the axioms. Interpretations of the probabilities are quite another matter:

- The “frequency of occurrence” of an event is one example of a particular interpretation of probability.
- Another possible interpretation is a subjective one, where we can think of the probability as a belief in the chance of an event occurring.