Lecture 29: Nov 1

Last time

- Common Continuous Distribution

Today

- Common Continuous Distribution

Beta distribution Notation: $Y \sim Beta(a, b)$.

- Sample space: $[0, 1]$
- pdf:
$$f(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}, \quad 0 \leqslant y \leqslant 1$$
where $B(a, b)$ is the Beta function,
$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$
and $\Gamma(a)$ is the gamma function. Note that if a and b are integers, then $B(a, b)$ can be calculated in closed form.
- cdf: In general, there is no closed form, except if a and b are integers.
- moments:
$$EY = \frac{a}{a+b}$$
$$Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}$$
The beta distribution is very flexible, and can take a wide variety of shapes by varying its parameters.
- Special case: $Beta(1, 1) = U(0, 1)$.

Omitted distributions: Weibull distribution, and Cauchy distribution.

Location and Scale families

Let Z be a continuous random variable with pdf $f(z)$. Define the class of rvs
$$X_{\mu,\sigma} = \sigma Z + \mu, \quad \mu \in \mathbb{R}, \sigma > 0$$
Then

1. $X_{\mu,\sigma}$ has pdf
$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

2.
$$E(X) = \sigma E(Z) + \mu, \quad Var(X) = \sigma^2 Var(Z)$$

3. The variable $Z = X_{0,1}$ is called the *generator* and is a member of the class.

Location families and scale families

- The family of pdfs $f_{\mu,\sigma}(x)$ is called a *location-scale* family where μ is called the *location parameter*, and σ is called the *scale parameter.*
- The family of pdfs
$$f_{\mu,1}(x) = f(x-\mu)$$
with $\sigma = 1$ is called a *location* family.
- The family of pdfs
$$f_{0,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$
with $\mu = 0$ is called a *scale* family.

Example (Exponential location family) Let $f(x) = e^{-x}, x \geqslant 0$, and $f(x) = 0, x < 0$. To form a location family we replace x with $x - \mu$ to obtain

$$\begin{aligned} f(x|\mu) &= \begin{cases} e^{-(x-\mu)} & x-\mu \geqslant 0 \\ 0 & x-\mu < 0 \end{cases} \\ &= \begin{cases} e^{-(x-\mu)} & x \geqslant \mu \\ 0 & x < \mu \end{cases} \end{aligned}$$

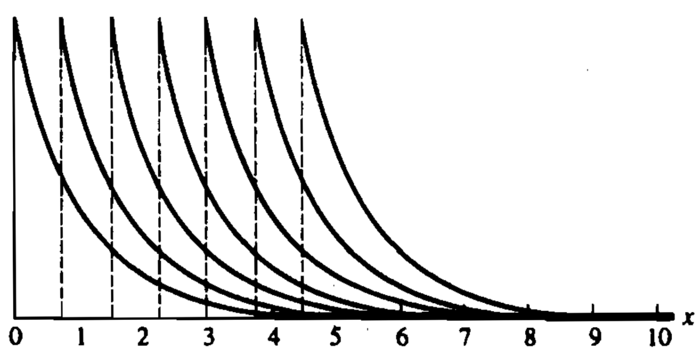

Figure 3.5.2. ***Exponential location densities***

Figure 28.1: Figure 3.5.2. Exponential location densities.

As shown in the above graph, the densities are shifted. Now the positive part of the density starts at μ rather than at 0. If X measures time, then μ might be restricted to be nonnegative so that X will be positive with probability 1 for every value of μ. In this type of model, where μ denotes a bound on the range of X, μ is sometimes called a *threshold parameter*.

The effect of introducing the scale parameter σ is either to stretch ($\sigma > 1$) or to contract ($\sigma < 1$) the graph of $f(x)$ while still maintaining the same basic shape of the graph. This is illustrated in the Figure below.

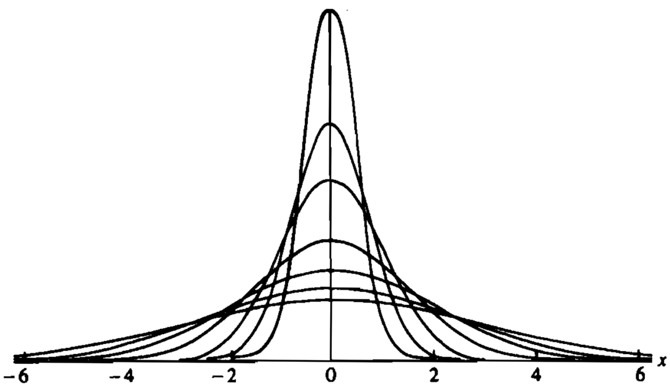

Figure 28.2: Figure 3.5.3. Members of the same scale family

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R} \tag{1}$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geqslant 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x, etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $f(x|\mu, \sigma^2)$ be the $n(\mu, \sigma^2)$ family of pdfs, where $\boldsymbol{\theta} = (\mu, \sigma)$. Then

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{array}{rl} h(x) = \frac{1}{\sqrt{2\pi}} & c(\mu, \sigma) = \frac{1}{\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu, \sigma) = -\frac{1}{2\sigma^2} & w_2(\mu, \sigma) = \frac{\mu}{\sigma^2} \\ t_1(x) = x^2 & t_2(x) = x \end{array}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.