

## Lecture 14: Sept 20

Last time

- Transformations of Random Variables

Today

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### Transformations of Random Variables

**Theorem** Suppose a continuous random variable  $X$  has cdf  $F_X(x)$ , let  $Y = g(X)$ , and let  $\mathcal{X}$  and  $\mathcal{Y}$  be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If  $g$  is an increasing function on  $\mathcal{X}$ ,  $F_Y(y) = F_X(g^{-1}(y))$  for  $y \in \mathcal{Y}$ .
2. If  $g$  is a decreasing function on  $\mathcal{X}$ ,  $F_Y(y) = 1 - F_X(g^{-1}(y))$  for  $y \in \mathcal{Y}$ .

*Proof:* We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr(g(X) \leq y) \end{aligned}$$

**Theorem** Let  $X$  have pdf  $f_X(x)$  and let  $Y = g(X)$ , where  $g$  is a monotone function. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Suppose that  $f_X(x)$  is continuous on  $\mathcal{X}$  and that  $g^{-1}(y)$  has a continuous derivative on  $\mathcal{Y}$ . Then the pdf of  $Y$  is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:*

From last theorem, we have the cdf forms  $F_Y(y)$ . Then  $f_Y(y) = \frac{d}{dy} F_Y(y)$ . (finish the proof)

**Example** (Square transformation) Suppose  $X$  is a continuous random variable. For  $y > 0$ , the cdf of  $Y = X^2$  is

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}).$$

Because  $x$  is continuous, we can drop the equality from the left endpoint and obtain

$$\begin{aligned} F_Y(y) &= \Pr(-\sqrt{y} < X \leq \sqrt{y}) \\ &= \Pr(X \leq \sqrt{y}) - \Pr(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

The pdf of  $Y$  can now be obtained from the cdf by differentiation: where we use the chain rule to differentiate  $F_X(\sqrt{y})$  and  $F_X(-\sqrt{y})$ .

**Example** (Linear transformation) Suppose  $X$  is a continuous random variable with pdf  $f_X(x)$ . Let

$$Y = a + bX, \quad \frac{dy}{dx} = b.$$

Then

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{dx}{dy} \right| = f_X\left(\frac{y-a}{b}\right) \frac{1}{|b|}.$$

This transformation is often used when  $X$  has mean 0 and standard deviation 1. The linear transformation above creates a random variable  $Y$  with a distribution that has the same shape as that of  $X$  but has mean  $a$  and variance  $b^2$ .

Conversely, if  $Y$  has mean  $a$  and standard deviation  $b$ , then  $X = (Y - a)/b$  has mean 0 and standard deviation 1. This is called sometimes the “Studentized” transformation.

**Example** (Normal distribution) Let  $X \sim N(0, 1)$ :

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

The transformation

$$Y = \mu + \sigma X, \quad X = \frac{Y - \mu}{\sigma}$$

yields

$$f_Y(y) = f_X\left(\frac{y - \mu}{\sigma}\right) \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y - \mu)^2}{2\sigma^2}}.$$

More generally, a distribution is a member of the class of *location-scale* distributions if the distribution of a linear transformation of a random variable with that distribution has the same distribution, but with different parameters.

**Example** (Square root of an exponential RV) Suppose  $X \sim \exp(\lambda)$ , so that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and consider the distribution of  $Y = \sqrt{X}$ . The transformation

$$y = g(x) = \sqrt{x}, \quad x \geq 0$$

is one-to-one and has an inverse  $x = y^2$  with  $dx/dy = 2y$ . Thus

This distribution is a particular form of the Rayleigh distribution and is a special case of the Weibull distribution.