Lecture 33: Nov 18

Last time

- Midterm exam 2

Today

- Multiple Random Variables (Chapter 4)

Example (Bivariate random variable) A fair coin is flipped 3 times. Define the random vector (X, Y) where X represents the number of heads on the last toss and Y the total number of heads. Then, the probabilities of various outcomes are given in the following table:

Outcome	(x, y)	$\Pr(outcome)$
(H, H, H)	(1, 3)	1/8
(H, T, H), (T, H, H)	(1, 2)	2/8
(H, H, T)	(0, 2)	1/8
(T, T, H)	(1, 1)	1/8
(T, H, T), (H, T, T)	(0, 1)	2/8
(T, T, T)	(0, 0)	1/8

Definition Two random variables X and Y are said to be jointly *discrete* if there is an associated *joint probability mass function*,

$$f_{X,Y}(x, y) = \Pr\{X = x, Y = y\}$$

which sums to 1 over a finite or possibly countable combinations of x and y for which $f_{X,Y}(x, y) > 0$, i.e.,

$$\sum_{x,y} f_{X,Y}(x, y) = 1$$

From this, one can also obtain the marginal pmfs of X and Y as follows:

$$f_X(x) = \Pr(X = x) = \sum_y f_{X,Y}(x, y)$$

$$f_Y(y) = \Pr(Y = y) = \sum_x f_{X,Y}(x, y)$$

Example Back to the fair coin example again. From the definition, we can construct the joint pmf of X and Y:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

The marginal distributions of X and Y are also easy to find. Note: Marginals do not determine joint pmf.

Bivariate cdfs Whether they are discrete or continuous or some combination of the two, we can always define the *joint cdf*. For $n = 2$, the *bivariate cumulative distribution function* is

$$F_{X,Y}(x, y) = \Pr\{X \leqslant x, Y \leqslant y\}$$

Properties:

- $F_{X,Y}(x, y) \geqslant 0$
- $F_{X,Y}(\infty, \infty) = 1$
- $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$
- $F_{X,Y}(-\infty, -\infty) = 0$
- F is non-decreasing and right-continuous in each variable separately.

Joint probabilities All joint probability statements about X and Y can be answered in terms of their joint cdf:

$$\begin{aligned}&\Pr(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) = \\ &\qquad F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)\end{aligned}$$

Example

$$\Pr(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$$

Note: To ensure that a bivariate function $F(x, y)$ is a proper cdf, it must satisfy all the properties mentioned above and the rectangular property above.

Marginal distributions From $F_{X,Y}$, we can derive the univariate distribution functions for X and Y. These are generally called *marginal distributions*.

$$\begin{aligned}F_X(x) &= \Pr\{X \leqslant x\} = \Pr\{X \leqslant x, Y \leqslant \infty\} = F_{X,Y}(x, \infty) \\ F_Y(y) &= \Pr\{Y \leqslant y\} = \Pr\{X < \infty, Y \leqslant y\} = F_{X,Y}(\infty, y)\end{aligned}$$

Note: Although we can obtain $F_X(x)$ and $F_Y(y)$ from the joint cdf, we cannot do the reverse.

Continuous Bivariate RVs The random variables X and Y are said to be *jointly continuous* if there exists a function $f_{X,Y}(x, y)$, such that for any Borel set B of 2-tuples in $\mathbb{R}^2$,

$$\Pr\{(X, Y) \in B\} = \int\int_{(x,y)\in B} f_{X,Y}(x, y)dxdy.$$

The function $f_{X,Y}(x, y)$ is called the *joint probability density function* for X and Y. It follows in this case that

$$\begin{aligned}F_{X,Y}(x, y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s, t)dtds, \\ f_{X,Y}(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y}\end{aligned}$$

Properties of the bivariate pdf

- $f_{X,Y}(x, y) \geqslant 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$
- $f_{X,Y}(x, y)$ is not a probability, but can be thought of as a relative probability of (X, Y) falling into a small rectangle located at (x, y):

$$\Pr\{x < X \leqslant x + dx, y < Y \leqslant y + dy\} \approx f(x, y) dx dy$$

- The *marginal probability density functions* for X and Y can be obtained as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

Example 1

$$F_{X,Y}(x, y) = xy \qquad 0 < x \leqslant 1, 0 < y \leqslant 1$$
$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Example 2

$$F_{X,Y}(x, y) = x - x \log \frac{x}{y} \qquad 0 < x \leqslant y \leqslant 1$$
$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} =$$
$$f_X(x) =$$
$$f_Y(y) =$$

Note: Once we have $f_X(x)$ and $f_Y(y)$, we can obtain $F_X(x)$ and $F_Y(y)$ directly. Double check: $F_X(x) = F_{X,Y}(x, \infty)$.

Conditional Distributions

Conditional Distributions - Discrete Recall if A and B are two events, the probability of A conditional on B is:

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)}$$

Defining the events $A = \{Y = y\}$ and $B = \{X = x\}$, it follows that

$$\begin{aligned}\Pr\{Y = y|X = x\} &= \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \\ &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\ &= f_{Y|X}(y|x)\end{aligned}$$

This is called the *conditional probability mass function* of Y given X.

Example: Discrete Back to the fair coin example. From the joint pmf of X and Y, we can derive all the conditional pmfs:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

Conditional Distribution - Continuous If $F(x, y)$ is absolutely continuous, we define the conditional density of X given Y as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

Example 1

$$\begin{array}{lll} F_{XY}(x, y) = xy & 0 < x < 1, & 0 < y < 1 \\ f_{XY}(x, y) = 1 & 0 < x < 1, & 0 < y < 1 \\ f_X(x) = 1 & 0 < x < 1 & \\ f_Y(y) = 1 & 0 < y < 1 & \\ f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1 & 0 < x < 1 & (0 < y < 1) \\ f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 1 & 0 < y < 1 & (0 < x < 1) \end{array}$$

Note: Here we get that the conditional densities are the same as the marginals. This means X and Y are independent.

Example 2

$$\begin{array}{lll} F_{XY}(x, y) = x - x \log \frac{x}{y} & 0 < x \leqslant y \leqslant 1 & \\ f_{XY}(x, y) = 1/y & 0 < x \leqslant y \leqslant 1 & \\ f_X(x) = -\log x & 0 < x \leqslant 1 & \\ f_Y(y) = 1 & 0 < y \leqslant 1 & \\ f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1/y & 0 < x \leqslant y & (0 < y \leqslant 1) \\ f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = -\frac{1}{y \log x} & x \leqslant y \leqslant 1 & (0 < x \leqslant 1) \end{array}$$

- Y is marginally uniform, but not conditionally uniform.
- X is conditionally uniform, but not marginally uniform.