Lecture 35: Nov 22

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations (7/33)
- Independence

Conditional Distribution - Continuous If $F(x, y)$ is absolutely continuous, we define the conditional density of X given Y as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

Example 1

$$\begin{array}{lll}
F_{XY}(x, y) = xy & 0 < x < 1, & 0 < y < 1 \\
f_{XY}(x, y) = 1 & 0 < x < 1, & 0 < y < 1 \\
f_X(x) = 1 & 0 < x < 1 & \\
f_Y(y) = 1 & 0 < y < 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1 & 0 < x < 1 & (0 < y < 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = 1 & 0 < y < 1 & (0 < x < 1)
\end{array}$$

Note: Here we get that the conditional densities are the same as the marginals. This means X and Y are independent.

Example 2

$$\begin{array}{lll}
F_{XY}(x, y) = x - x \log \frac{x}{y} & 0 < x \leqslant y \leqslant 1 & \\
f_{XY}(x, y) = 1/y & 0 < x \leqslant y \leqslant 1 & \\
f_X(x) = -\log x & 0 < x \leqslant 1 & \\
f_Y(y) = 1 & 0 < y \leqslant 1 & \\
f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = 1/y & 0 < x \leqslant y & (0 < y \leqslant 1) \\
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = -\frac{1}{y \log x} & x \leqslant y \leqslant 1 & (0 < x \leqslant 1)
\end{array}$$

- Y is marginally uniform, but not conditionally uniform.
- X is conditionally uniform, but not marginally uniform.

Independent Random Variables

Independence The random variable X and Y are said to be *independent* if for any two Borel sets A and B,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B)$$

All events defined in terms of X are independent of all events defined in terms of Y.

Using the Kolmogorov axioms of probability, it can be shown that X and Y are independent if and only if $\forall(x, y)$ (except possibly for sets of probability 0)

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

or in terms of pmfs (discrete) and pdfs (continuous)

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Checking independence

- A necessary condition for independence of X and Y is that their joint pdf/pmf has positive probability on a rectangular domain.
- If the domain is rectangular, one can try to write the joint pdf/pmf as a product of functions of x and y only.

Example Two points are selected randomly on a line of length a so as to be on opposite sides of the mid-point of the line. Find the probability that the distance between them is less than $a/3$.

Solution:
Let X be the coordinate of a point selected randomly in $[0, a/2]$ and Y be the coordinate of a point selected randomly in $[a/2, a]$. Assume X and Y are independent and uniform over its interval. The joint density is

$$f_{X,Y}(x, y) = 4/a^2, \quad 0 \leqslant x \leqslant a/2, a/2 \leqslant y \leqslant a$$

Therefore, the solution is

$$\Pr(Y - X < a/3) =$$

Example: Buffon's Needle A table is ruled with lines distance 1 unit apart. A needle of length $L \leqslant 1$ is thrown randomly on the table. What is the probability that the needle intersects a line?
Solution:
Define two random variables:

- X: distance from low end of the needle to the nearest line above
- θ: angle from the vertical to the needle.

By "random", we assume X and θ are independent, and

$$X \sim U(0, 1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\theta}(x, \theta) = 1/\pi, \quad 0 \leqslant x \leqslant 1, -\pi/2 \leqslant \theta \leqslant \pi/2$$

For the needle to intersect a line, we need $X < L\cos(\theta)$.