Lecture 36: Dec 2

Last time

- Multiple Random Variables (Chapter 4)

Today

- Course Evaluations (8/33)
- Expectation
- Final exam format
  - Final exam will be take home
  - Open book, open note, not open internet
  - Final exam will be released on Friday (12/06/2024) right after class
  - Final exam due 23:59 pm on Friday 12/13/2024.
  - Scan and submit your exam via email with a single pdf file.
  - Send your email to both your instructor and your TA.
  - Submitted exams should be human-readable to receive non-zero scores.

Example: Buffon's Needle A table is ruled with lines distance 1 unit apart. A needle of length $L \leqslant 1$ is thrown randomly on the table. What is the probability that the needle intersects a line?
Solution:
Define two random variables:

- X: distance from low end of the needle to the nearest line above
- θ: angle from the vertical to the needle.

By "random", we assume X and θ are independent, and

$$X \sim U(0,1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\theta}(x,\theta) = 1/\pi, \quad 0 \leqslant x \leqslant 1, -\pi/2 \leqslant \theta \leqslant \pi/2$$

For the needle to intersect a line, we need $X < L\cos(\theta)$.

Expectations of Independent RVs (Theorem 4.2.10) Let X and Y be independent rvs.

- For any $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$,

  $$\Pr(X \in A, Y \in B) = \Pr(X \in A)\Pr(Y \in B)$$

  i.e., the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

- Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then

$$E\left[g(X)h(Y)\right] = \left[Eg(X)\right]\left[Eh(Y)\right]$$

Proof:

$$\begin{aligned}
E\left[g(X)h(Y)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\
&= \left(\int_{-\infty}^{\infty} g(x)f_X(x)dx\right)\left(\int_{-\infty}^{\infty} h(y)f_Y(y)dy\right) \\
&= \left[Eg(X)\right]\left[Eh(Y)\right]
\end{aligned}$$

Example X, Y are independent

$$\begin{aligned}
E(X^2Y^3) &= (EX^2)(EY^3) \\
E(Y^2Y^3) &\neq (EY^2)(EY^3)
\end{aligned}$$

Bivariate Transformation

Functions of random variables Let (X,Y) be a bivariate rv with known distributions. Define (U,V) by

$$U = g_1(X,Y), \quad V = g_2(X,Y)$$

Probability mapping For any Borel set $B \subset \mathbb{R}^2$,

$$\Pr\left[(U,V) \in B\right] = \Pr\left[(X,Y) \in A\right]$$

where A is the inverse mapping of B, such that

$$A = \{(x,y) \in \mathbb{R}^2 : (g_1(x,y), g_2(x,y)) \in B\}.$$

The inverse is well defined even if the mapping is not bijective.

Example Let $g_1(x,y) = x, g_2(x,y) = x^2 + y^2$.

Discrete RVs Suppose that (X,Y) is a discrete rv, i.e., the pmf is positive on a countable set $\mathcal{A}$. Then (U,V) is also discrete and takes values on a countable set $\mathcal{B}$. Define

$$A_{u,v} = \{(x,y) \in \mathcal{A} : g_1(x,y) = u, g_2(x,y) = v\}$$

Then

$$f_{UV}(u,v) = \Pr(U = u, V = v) = \sum_{(x,y)\in A_{u,v}} f_{XY}(x,y)$$

Sum of two independent Poissons Let $X \sim Poisson(\lambda_1)$, $Y \sim Poisson(\lambda_2)$, independent, and define

$$U = X + Y, \quad V = Y$$

- (X, Y) takes values in $\mathcal{A} = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$
- (U, V) takes values on $\mathcal{B} = \{(u, v) : v = 0, 1, 2, \ldots, u = v, v+1, v+2, \ldots\}$.
- For a particular (u, v), $A_{uv} = \{(x, y) \in \mathcal{A} : x + y = u, y = v\} = (u - v, u)$.

The joint pmf of U and V is

$$f_{UV}(u, v) = f_{XY}(u - v, v) = \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{(v)!}$$

The distribution of $U = X + Y$ is the marginal

$$\begin{aligned} f_U(u) &= \sum_{v=0}^{u} \frac{e^{-\lambda_1}\lambda_1^{u-v}}{(u-v)!}\frac{e^{-\lambda_2}\lambda_2^{v}}{(v)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!}\sum_{v=0}^{u}\binom{u}{v}\lambda_1^{u-v}\lambda_2^{v} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{u!}(\lambda_1+\lambda_2)^u \end{aligned}$$

We obtain that U is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Bivariate Transformations of Continuous RVs Suppose (X, Y) is continuous and the joint transformation

$$u = g_1(x, y), \quad v = g_2(x, y)$$

is one-to-one and differentiable. Define the inverse mapping

$$x = h_1(u, v), \quad y = h_2(u, v)$$

Then

$$f_{UV}(u, v) = f_{XY}\left(h_1(u, v), h_2(u, v)\right) ||J(u, v)||$$

where $J(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$ given by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$