Lecture 16: Sept 25

Last time

- Transformations of Random Variables

Today

- One-page one-sided letter-size cheat sheet for midterm 1
- Expected Values
- Moments

Expected Values

Definition The *expected value* or *mean* of a random variable $g(X)$, denoted by $Eg(X)$, is

$$Eg(X) = \begin{cases} \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x\in\mathcal{X}} g(x)\Pr(X=x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let $g(X) = X$, then we get

$$EX = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x\in\mathcal{X}} x\Pr(X=x) & \text{if } X \text{ is discrete} \end{cases}$$

Example (Exponential mean) Suppose X has an *exponential distribution* with parameter λ, $X \sim Exp(\lambda)$, that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda}e^{-x/\lambda}, \quad 0 \leqslant x < \infty, \lambda > 0.$$

Find out EX.
Solution:

Example (Binomial mean) if X has a *binomial distribution*, $X \sim Binomial(n, p)$, its pmf is given by

$$\Pr(X = x) = \binom{n}{x} p^x(1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer, $0 \leqslant p \leqslant 1$, and for every fixed pair n and p the pmf sums to 1. Find out EX.

Solution:

The process of taking expectations is a linear operation, which means that the expectation of a linear function of X can be easily evaluated by noting that for any constants a and b, such that

$$E(aX + b) = aEX + b$$

Theorem Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

1. $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$.
2. If $g_1(x) \geqslant 0$ for all x, then $Eg_1(X) \geqslant 0$.
3. If $g_1(x) \geqslant g_2(x)$ for all x, then $Eg_1(X) \geqslant Eg_2(X)$.
4. If $a \leqslant g_1(x) \leqslant b$ for all x, then $a \leqslant Eg_1(X) \leqslant b$.

Proof: