

## Lecture 2: Aug 21

### Last time

- Introduction
- Introduce yourself
- Course logistics

### Today

- Continue self-introduction
- Set theory (1.1)
- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)
- Conditional Probability (1.3)

### Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space*.

**Definition** The set,  $S$ , of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

**Example** The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university
- an experiment where the observation is reaction time to a certain stimulus

**Definition** An *event* is any collection of possible outcomes of an experiment, that is, any subset of  $S$  (including  $S$  itself).

Let  $A$  be an event,

- $A$  is a subset of  $S$ ,
- event  $A$  occurs if the outcome of the experiment is in the set  $A$ ,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:

$$A \subset B \iff x \in A \implies x \in B$$

- Equality:

$$A = B \iff A \subset B \text{ and } B \subset A$$

- Union: the union of  $A$  and  $B$ , written as  $A \cup B$ , is the set of elements that belong to either  $A$  or  $B$  or both

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- Intersection: the intersection of  $A$  and  $B$ , written  $A \cap B$ , is the set of elements that belong to both  $A$  and  $B$ :

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- Complementation: the complement of  $A$ , written  $A^c$ , is the set of all elements that are not in  $A$ :

$$A^c = \{x : x \notin A\}.$$

**Theorem** For any three events,  $A$ ,  $B$ , and  $C$ , defined on a sample space  $S$ ,

1. Commutativity

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A; \end{aligned}$$

2. Associativity

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C, \\ A \cap (B \cap C) &= (A \cap B) \cap C; \end{aligned}$$

3. Distributive Laws

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C); \end{aligned}$$

4. DeMorgan's Laws

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c; \end{aligned}$$

We show the proof of  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.

*proof:*

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ :  
Let  $x \in (A \cap (B \cup C))$ . By definition of intersection,  $x \in (B \cup C)$  that is, either  $x \in B$  or  $x \in C$ . Since  $x$  also must be in  $A$ , we have that either  $x \in (A \cap B)$  or  $x \in (A \cap C)$ ; therefore,  $x \in ((A \cap B) \cup (A \cap C))$ .
- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ :  
Let  $x \in ((A \cap B) \cup (A \cap C))$ . This implies that  $x \in (A \cap B)$  or  $x \in (A \cap C)$ . If  $x \in (A \cap B)$ , then  $x$  is in both  $A$  and  $B$ . Since  $x \in B$ , then  $x \in (B \cup C)$  and thus  $x \in (A \cap (B \cup C))$ . It follows the same argument when  $x \in (A \cap C)$ , we still have  $x \in (A \cap (B \cup C))$ .

**Definition** Two events  $A$  and  $B$  are *disjoint* (or *mutually exclusive*) if  $A \cap B = \emptyset$ . The events  $A_1, A_2, \dots$  are *pairwise disjoint* (or *mutually exclusive*) if  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ .

**Definition** If  $A_1, A_2, \dots$  are pairwise disjoint and  $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots = S$ , then the collection of  $A_1, A_2, \dots$  forms a *partition* of  $S$ .

**Example** The sets  $A_i = [i, i + 1), i = 0, 1, 2, \dots$  form a partition of  $[0, \infty)$ .