# Lecture 2: Aug 20

## Last time

- Introduction
- Introduce yourself
- Course logistics

## Today

- Continue self-introduction
- Set theory (1.1)
- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)
- Conditional Probability (1.3)

## Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space*.

**Definition** The set, $S$, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

**Example** The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university
- an experiment where the observation is reaction time to a certain stimulus

**Definition** An *event* is any collection of possible outcomes of an experiment, that is, any subset of $S$ (including $S$ itself).

Let $A$ be an event,

- $A$ is a subset of $S$,
- event $A$ occurs if the outcome of the experiment is in the set $A$,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:
$$A \subset B \iff x \in A \implies x \in B$$
- Equality:
$$A = B \iff A \subset B \text{ and } B \subset A$$
- Union: the union of $A$ and $B$, written as $A \cup B$, is the set of elements that belong to either $A$ or $B$ or both
$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$
- Intersection: the intersection of $A$ and $B$, written $A \cap B$, is the set of elements that belong to both $A$ and $B$:
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$
- Complementation: the complement of $A$, written $A^c$, is the set of all elements that are not in $A$:
$$A^c = \{x : x \notin A\}.$$

Theorem For any three events, $A$, $B$, and $C$, defined on a sample space $S$,

1. Commutativity
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A;$$
2. Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C;$$
3. Distributive Laws
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$
4. DeMorgan's Laws
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

We show the proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.
*proof:*

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$:
  Let $x \in (A \cap (B \cup C))$. By definition of intersection, $x \in (B \cup C)$ that is, either $x \in B$ or $x \in C$. Since $x$ also must be in $A$, we have that either $x \in (A \cap B)$ or $x \in (A \cap C)$; therefore, $x \in ((A \cap B) \cup (A \cap C))$.

- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$:
  Let $x \in ((A \cap B) \cup (A \cap C))$. This implies that $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$, then $x$ is in both $A$ and $B$. Since $x \in B$, then $x \in (B \cup C)$and thus $x \in (A \cap (B \cup C))$. It follows the same argument when $x \in (A \cap C)$, we still have $x \in (A \cap (B \cup C))$.

Definition Two events $A$ and $B$ are *disjoint* (or *mutually exclusive*) if $A \cap B = \varnothing$. The events $A_1, A_2, \ldots$ are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Definition If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = S$, then the collection of $A_1, A_2, \ldots$ forms a *partition* of $S$.

Example The sets $A_i = [i, i+1), i = 0, 1, 2, \ldots$ form a partition of $[0, \infty)$.