Lecture 4: Aug 25

Last time

- Set theory (1.1)

Today

- Axiomatic Foundations (1.2)

Axiomatic Foundations

For each event A in the sample space S, we want to associate with A a number between zero and one that will be called the probability of A, denoted by $\Pr(A)$. The domain of Pr is the set where the arguments of the function $\Pr(\cdot)$ are defined. It is natural to define the domain of Pr as all subsets of S, that is for each $A \subset S$, we define $\Pr(A)$ as the probability that A occurs. However, there are some technical difficulties to overcome which requires us to familiarize with the following.

Definition A collection of subsets of S is called a *sigma algebra* (or *Borel field*), denoted by $\mathcal{B}$, if it satisfies the following three properties:

1. $\varnothing \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$).
2. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).
3. If $A_1, A_2, \cdots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions).

From Property (1) and (2), we see that the empty set and its complement S (since $S = \varnothing^c$) are always in a sigma algebra. In fact, they construct the *trivial* algebra $\{\varnothing, S\}$ which is the smallest sigma algebra.

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \cdots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

This is because:

Example If S is finite or countable (where the elements of S can be put into one to one correspondence with a subset of the integers), then these technicalities really do not arise, for we define for a given sample space S,

$$\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}.$$

If S has n elements, there are 2^n sets in $\mathcal{B}$ (why?).[hint: for each element, it is either in or out of a subset, so 2 choices].

Example Let $S = (-\infty, \infty)$, be the real line. Then $\mathcal{B}$ is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

for all real numbers a and b. Also, from the properties of $\mathcal{B}$, it follows that $\mathcal{B}$ contains all sets that can be formed by taking (possibly countably infinite) unions and intersections of sets of the above varieties.

We now define a probability function.

Definition Given a sample space S and an associated sigma algebra $\mathcal{B}$, a *probability function* is a function Pr with domain $\mathcal{B}$ that satisfies

1. $\Pr(A) \geqslant 0$ for all $A \in \mathcal{B}$.
2. $\Pr(S) = 1$.
3. If $A_1, A_2, \dots \in \mathcal{B}$ are pairwise disjoint, then $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$.

The above three properties are usually referred to as the Axioms of Probability (or the Kolmogorov Axioms, after A. Kolmogorov, one of the fathers of probability theory). Any function that satisfies the Axioms of Probability is called a probability function.

Example Consider the simple experiment of tossing a fair coin (just once), so $S = \{H, T\}$. A reasonable probability function is the one that assigns equal probabilities to heads and tails, that is,

$$\Pr(\{H\}) = \Pr(\{T\}).$$

Since $S = \{H\} \cup \{T\}$, we have , from Axiom 2, $\Pr(\{H\} \cup \{T\}) = 1$. Also, $\{H\}$ and $\{T\}$ are disjoint, so $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$. Collectively, we have

$$\begin{aligned}
&\Pr(\{H\}) = \Pr(\{T\}) \\
&\Pr(\{H\} \cup \{T\}) = 1 \\
&\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})
\end{aligned}$$

Therefore, $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.

Example If S is finite or countable (where the elements of S can be put into $1-1$ correspondence with a subset of the integers), then these technicalities really do not arise, for we we define for a given sample space S,

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$$\Pr(\{H\}) = \Pr(\{T\}).$$

Since $S = \{H\} \cup \{T\}$, we have , from Axiom 1, $\Pr(\{H\} \cup \{T\}) = 1$. Also, $\{H\}$ and $\{T\}$ are disjoint, so $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$. Collectively, we have

$$\begin{aligned}
&\Pr(\{H\}) = \Pr(\{T\}) \\
&\Pr(\{H\} \cup \{T\}) = 1 \\
&\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})
\end{aligned}$$

Therefore, $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.

Caculus of Probabilities

We start with some fairly self-evident properties of the probability function when applied to a single event.

Theorem If Pr is a probability function and A is any set in $\mathcal{B}$, then

1. $\Pr(\varnothing) = 0$, where $\varnothing$ is the empty set;
2. $\Pr(A) \leqslant 1$;
3. $\Pr(A^c) = 1 - \Pr(A)$.

proof:

Formula (2) in the above theorem gives a useful inequality for the probability of an intersection (Bonferroni's Inequality):

$$\Pr(A \cap B) \geqslant \Pr(A) + \Pr(B) - 1.$$

Theorem If Pr is a probability function, then

1. $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$ for any partition $C_1, C_2, \ldots$;
2. $\Pr(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \Pr(A_i)$ for any sets $A_1, A_2, \ldots$.

where (1) is also referred to as "Total probability" and (2) is Boole's inequality.
proof: