35 Lecture 35: April 21

Last time

- nested design
- Two-factor designs

Today

- HW3 deadline extended to Friday 04/23 midnight.
- Theoretical background of linear models

Additional reference

Course notes by Dr. Hua Zhou
"A Primer on Linear Models" by Dr. John F. Monahan

Linear Models in the matrix form

Recall the matrix form of the linear model

$$\underset{n\times 1}{\mathbf{Y}} = \underset{n\times p}{\mathbf{X}} \underset{p\times 1}{\beta} + \underset{n\times 1}{\epsilon}$$

Simple linear regression model

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

Multiple linear regression model

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1,p-1} \\ 1 & x_{21} & \dots & x_{2,p-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \dots & x_{n,p-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

One-way ANOVA model

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1,n_1} \\ y_{21} \\ \vdots \\ y_{2,n_2} \\ \vdots \\ y_{a,1} \\ \vdots \\ y_{a,n_a} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & & & \\ \mathbf{1}_{n_2} & & \mathbf{1}_{n_2} & & \\ \vdots & & & \ddots & \\ \mathbf{1}_{n_a} & & & & \mathbf{1}_{n_a} \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1,n_1} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2,n_2} \\ \vdots \\ \epsilon_{a,1} \\ \vdots \\ \epsilon_{a,n_a} \end{bmatrix}$$

Two-way ANOVA model without interaction Model $y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$, $i = 1, \ldots, a$ (a levels in factor 1), $j = 1, \ldots, b$ (b levels in factor 2), and $k = 1, \ldots, n_{ij}$ (n_{ij} observations in the (i, j)-th cell). In total we have $n = \sum_{i,j} n_{ij}$ observations and $p = a + b + 1$ parameters. For simplicity, we consider the case without replicates, i.e., $n_{ij} = 1$ and only write out $\mathbf{X}\beta$. Note adding more replicates to each cell does *not* change the rank of $\mathbf{X}$.

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\beta = \begin{bmatrix} \mathbf{1}_b & \mathbf{1}_b & & & \mathbf{I}_b \\ \mathbf{1}_b & & \mathbf{1}_b & & \mathbf{I}_b \\ \vdots & & & \ddots & \vdots \\ \mathbf{1}_b & & & \mathbf{1}_b & \mathbf{I}_b \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \\ \beta_1 \\ \vdots \\ \beta_b \end{bmatrix}$$

Two-way ANOVA with interaction Model $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$, $i = 1, \ldots, a$ (a levels in factor 1), $j = 1, \ldots, b$ (b levels in factor 2), and $k = 1, \ldots, n_{ij}$ (n_{ij} observations in the (i, j)-th cell). In total we have $n = \sum_{i,j} n_{ij}$ observations and $p = 1 + a + b + ab$ parameters. For simplicity, we consider the case without replicates, i.e., $n_{ij} = 1$ and only write out $\mathbf{X}\beta$.

Note adding more replicates to each cell does *not* change the rank of $\mathbf{X}$.

$$
\mathrm{E}(\mathbf{y}) = \mathbf{X}\beta = \begin{bmatrix} \mathbf{1}_b & \mathbf{1}_b & & & & \mathbf{I}_b & \mathbf{I}_b & & & \\ \mathbf{1}_b & & \mathbf{1}_b & & & \mathbf{I}_b & & \mathbf{I}_b & & \\ \vdots & & & \ddots & & \vdots & & & \ddots & \\ \mathbf{1}_b & & & & \mathbf{1}_b & \mathbf{I}_b & & & & \mathbf{I}_b \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \\ \beta_1 \\ \vdots \\ \beta_b \\ \gamma_{11} \\ \vdots \\ \vdots \\ \gamma_{ab} \end{bmatrix}
$$

For all the above models, we have 5the most general assumption over the error term, i.e. $\epsilon \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

Mixed effects models For mixed effects models, we generally have

$$
\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{Z}\mathbf{u} + \mathbf{e}
$$

- $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a design matrix for fixed-effects $\mathbf{b} \in \mathbb{R}^p$
- $\mathbf{Z} \in \mathbb{R}^{n\times q}$ is a design matrix for random-effects $\mathbf{u} \in \mathbb{R}^q$
- The most general assumption is $\mathbf{e} \sim \mathcal{N}(\mathbf{0}_n, \mathbf{R})$, $\mathbf{u} \sim \mathcal{N}(\mathbf{0}_q, \mathbf{G})$, and $\mathbf{e}$ is independent of $\mathbf{u}$.

In many applications, $\mathbf{e} \sim \mathcal{N}(\mathbf{0}_n, \sigma^2\mathbf{I}_n)$ and

$$
\mathbf{Z}\mathbf{u} = (\mathbf{Z}_1, \dots, \mathbf{Z}_m) \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{pmatrix} = \mathbf{Z}_1\mathbf{u}_1 + \cdots + \mathbf{Z}_m\mathbf{u}_m,
$$

where $\mathbf{u}_i \sim \mathcal{N}(\mathbf{0}_{q_i}, \sigma_i^2\mathbf{I}_{q_i})$, $\sum_{i=1}^m q_i = q$. $\mathbf{e}$ and $\mathbf{u}_i$, $i = 1, \dots, m$, are jointly independent. Then the covariance of responses $\mathbf{y}$

$$
\mathbf{V}(\sigma^2, \sigma_1^2, \dots, \sigma_m^2) = \sigma^2\mathbf{I} + \sum_{i=1}^m \sigma_i^2\mathbf{Z}_i\mathbf{Z}_i^T
$$

Linear equations and generalized inverse

For the linear model

$$
\underset{n\times 1}{\mathbf{Y}} = \underset{n\times p}{\mathbf{X}}\,\underset{p\times 1}{\mathbf{b}} + \underset{n\times 1}{\mathbf{e}},
$$

we obtain the least square estimator by minimize the objective function $Q(\mathbf{b}) = \sum_{i=1}^{n} e_i^2 = (\mathbf{Y} - \mathbf{X}\mathbf{b})^T(\mathbf{Y} - \mathbf{X}\mathbf{b})$. By taking derivative with respect to $\mathbf{b}$ and setting it to zero, we get

$$\left(\frac{\partial Q}{\partial \mathbf{b}}\right)^T = \left(\frac{\partial Q}{\partial b_1}, \frac{\partial Q}{\partial b_2}, \ldots, \frac{\partial Q}{\partial b_p}\right)^T = \left[\frac{\partial\left(\mathbf{Y}^T\mathbf{Y} - 2\mathbf{Y}^T\mathbf{X}\mathbf{b} + \mathbf{b}^T\mathbf{X}^T\mathbf{X}\mathbf{b}\right)}{\partial \mathbf{b}}\right]^T = -2\mathbf{X}^T\mathbf{Y} + 2\mathbf{X}^T\mathbf{X}\mathbf{b}$$

where we used the fact that for constant vector $\mathbf{a} \in \mathbb{R}^{p\times 1}$, constant matrix $\mathbf{A} \in \mathbb{R}^{p\times p}$ and $\mathbf{x} \in \mathbb{R}^{p\times 1}$, we have the two derivatives:

1. $\frac{\partial \mathbf{a}^T\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}^T$
2. $\frac{\partial \mathbf{x}^T\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{x}^T(\mathbf{A} + \mathbf{A}^T)$

By setting $\left(\frac{\partial Q}{\partial \mathbf{b}}\right)^T = \mathbf{0}_{p\times 1}$, we get the Normal equations

$$\mathbf{X}^T\mathbf{X}\mathbf{b} = \mathbf{X}^T\mathbf{Y}$$

Consistency

Assume $\mathbf{A} \in \mathbb{R}^{m\times n}$
Definition: The linear system $\mathbf{A}\mathbf{x} = c$ is consistent if there exists an $\mathbf{x}^*$ such that $\mathbf{A}\mathbf{x}^* = \mathbf{c}$.

- If $\mathbf{A}$ is square and $\mathbf{A}^{-1}$ exists, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$.
- Proposition (g1): If $\mathbf{A}\mathbf{x} = \mathbf{c}$ is consistent, and if $\mathbf{G}$ is any matrix such that $\underset{m\times n}{\mathbf{A}}\ \underset{n\times m}{\mathbf{G}}\ \underset{m\times n}{\mathbf{A}} = \underset{m\times n}{\mathbf{A}}$, then $\mathbf{x}^\psi = \mathbf{G}\mathbf{c}$ is a solution to $\mathbf{A}\mathbf{x} = \mathbf{c}$.
  Proof:
- A matrix $\mathbf{G}$ satisfying $\mathbf{A}\mathbf{G}\mathbf{A} = \mathbf{A}$ is a generalized inverse of $\mathbf{A}$ with notation $\mathbf{A}^-$.
- If $\mathbf{A}$ is square and $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^- = \mathbf{A}^{-1}$ is unique.

The set of all solutions to $\mathbf{A}\mathbf{x} = \mathbf{c}$

Suppose that $\mathbf{A}\mathbf{x} = \mathbf{c}$ is consistent. Then $\mathbf{x}^*$ is a solution to $\mathbf{A}\mathbf{x} = \mathbf{c}$ if and only if $\mathbf{x}^* = \mathbf{A}^-\mathbf{c} + (\mathbf{I} - \mathbf{A}^-\mathbf{A})\mathbf{z}$ for some $\mathbf{z}$ and $\mathbf{A}^-$.
Proof:

Moore-Penrose inverse

Assume $\mathbf{A} \in \mathbb{R}^{m\times n}$

- The Moore-Penrose inverse of $\mathbf{A}$ is a matrix $\mathbf{A}^+ \in \mathbb{R}^{n\times m}$ with the following properties
  1. $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$ (Generalized inverse, g_1 inverse, or inner pseudo-inverse)
  2. $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$. (outer pseudo-inverse. Any g_1 inverse that satisfies this condition is called a g_2 inverse, or reflexive generalized inverse)

3. $\mathbf{A}^+\mathbf{A}$ is symmetric
4. $\mathbf{A}\mathbf{A}^+$ is symmetric

- $\mathbf{A}^+$ exists and is unique for any matrix $\mathbf{A}$.
- In practice, the Moore-Penrose inverse $\mathbf{A}^+$ is easily computed from the singular value decomposition of $\mathbf{A}$.
- $(\mathbf{A}^-)^T$ is a generalized inverse of $\mathbf{A}^T$

General form of the least squares solution

Now we have derived the general form of the least squares solution with generalized inverse.

$$\hat{\mathbf{b}} = (\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{y} + [\mathbf{I}_p - (\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{X}]\mathbf{q}$$

where $\mathbf{q} \in \mathbb{R}^p$ is arbitrary.

Positive (semi)definite matrix

Assume $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric (i.e. $\mathbf{A} = \mathbf{A}^T$)

- A real symmetrix matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive semi-definite (or nonnegative definite, or p.s.d.) if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0$ for all $\mathbf{x}$. Notation $\mathbf{A} \succeq_{p.s.d.} \mathbf{0}$
- E.g., the Gramian matrix $\mathbf{X}^T\mathbf{X}$ is p.s.d.
- We write $\mathbf{A} \succeq_{p.s.d.} \mathbf{B}$ means $\mathbf{A} - \mathbf{B} \succeq_{p.s.d.} \mathbf{0}$
- Cholesky decomposition. Each positive semidefinite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be factorized as $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ for some lower triangular matrix $\mathbf{L} \in \mathbb{R}^{n\times n}$ with nonnegative diagonal entries.
- $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if $\mathbf{A}$ is a covariance matrix of a random vector.
  Proof:

Estimable function

Assume the linear mean model: $\mathbf{Y} = \mathbf{X}\mathbf{b} + \mathbf{e}$, $\mathrm{E}(\mathbf{e}) = \mathbf{0}$. One main interest is estimation of the underlying parameter $\mathbf{b}$. Can $\mathbf{b}$ be estimated or what functions of $\mathbf{b}$ can be estimated?

- A parametric function $\mathbf{\Lambda}\mathbf{b}$, $\mathbf{\Lambda} \in \mathbb{R}^{m\times p}$ is said to be (linearly) estimable if there exists an affinely unbiased estimator of $\mathbf{\Lambda}\mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^p$. That is there exist constants $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{c} \in \mathbb{R}^m$ such that $\mathrm{E}(\mathbf{A}\mathbf{y} + \mathbf{c}) = \mathbf{\Lambda}\mathbf{b}$ for all $\mathbf{b}$.
- Theorem: Assuming the linear mean model, the parametric function $\mathbf{\Lambda}\mathbf{b}$ is (linearly) estimable if and only if $\mathcal{C}(\mathbf{\Lambda}) \subset \mathcal{C}(\mathbf{X}^T)$, or equivalently $\mathcal{N}(\mathbf{X}) \subset \mathcal{N}(\mathbf{\Lambda})$.
  "$\mathbf{\Lambda}\mathbf{b}$ is estimable $\iff$ the row space of $\mathbf{\Lambda}$ is contained in the row space of $\mathbf{X}$ $\iff$

the null space of $\mathbf{X}$ is contained in the null space of $\mathbf{\Lambda}$."
Proof:

- Corollary: $\mathbf{Xb}$ is estimable.
  "Expected value of any observation $\mathrm{E}(y_i)$ and their linear combinations are estimable."
- Corollary: If $\mathbf{X}$ has full column rank, then any linear combinations of $\mathbf{b}$ are estimable.
- If $\mathbf{\Lambda b}$ is (linearly) estimable, then its *least squares estimator* $\mathbf{\Lambda}\hat{\mathbf{b}}$ is invariant to the choice of the least squares solution $\hat{\mathbf{b}}$.
  Proof:
- The least squares estimator $\mathbf{\Lambda}\hat{\mathbf{b}}$ is a linearly unbiased estimator of $\mathbf{\Lambda b}$. *Proof:*