31 Lecture 31: April 12

Last time

- One last poll on alternative grading path
- Sampling distribution of linear contrasts
- Multiple comparisons

Today

- **Last** poll on alternative grading path result: *passed* with 4 + 4 yes, 4 + 1 no (0 by emails...?)
- Sample size computations for one-way ANOVA
- Lack of fit test
- One-way random effect model (JF Chapter 23 + Dr. Osborne's notes)

Additional reference

Course notes by Dr. Jason Osborne.

Sample size computations for one-way ANOVA

Now consider the null hypothesis in a balanced experiment using one-way ANOVA to compare t treatment means and $\alpha = 0.05$:

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_t = \mu$$

versus the alternative

$$H_a : \mu_i \neq \mu_j \text{ for some } i \neq j$$

Suppose that we intend to use a balanced design. How big does our sample size $n_1 = n_2 = \cdots = n_t = n$ need to be?

The answer depends on lots of things, namely, σ^2 and how many treatment groups t and how much of a difference among the means we hope to be able to detect, and with how big a probability.

Given α, $\mu_1, \ldots, \mu_t$ and σ^2, we can choose n to ensure a power of at least β (i.e. type II error rate) using the noncentral F distribution.

Recall that the critical region for the statistic $F = MS[Trt]/MS[E]$ is everything bigger than $F(\alpha, t-1, N-t) = F^*$.

The power of the F-test conducted using $\alpha = 0.05$ to reject H_0 under this alternative is given by

$$1 - \beta = \Pr(MS[Trt]/MS[E] > F^*; H_1 \text{ is true}). \tag{1}$$

Let $\tau_i = \mu_i - \mu$ for each treatment i so that

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_t = 0$$

When some H_1 is true and the sample size n is used in each group, it can be shown that the F ratio has the noncentral F distribution with noncentrality parameter

$$\gamma = \sum_{j=1}^{t} n_j \left(\frac{\tau_j}{\sigma}\right)^2 = n \sum_{j=1}^{t} \left(\frac{\tau_j}{\sigma}\right)^2$$

This is the parameterization for the F distribution used in both SAS and R.

One way to obtain an adequate sample size is trial and error. Software packages can be used to get probabilities of the form 1 for various values of n.

Example

Suppose we want to test equal mean binding fractions among antibiotics against the alternative

$$H_1 : \mu_P = \mu + 3, \mu_T = \mu + 3, \mu_S = \mu - 6, \mu_E = \mu, \mu_C = \mu$$

so that

$$\tau_1 = \tau_2 = 3, \tau_3 = -6, \tau_4 = \tau_5 = 0.$$

Assume $\sigma = 3$ (is it arbitrary? any idea of how to guess?) and we need to use $\alpha = \beta = 0.05$. The noncentrality parameter is given by

$$\gamma = n[(\frac{3}{3})^2 + (\frac{3}{3})^2 + (\frac{-6}{3})^2]$$

The $\alpha = 0.05$ critical value for H_0 is given by

$$F^* = F(5 - 1, 5n - 5, 0.05).$$

We need the area to the right of F^* for the noncentral F distribution with degrees of freedom 4 and $5(n - 1)$ and noncentrality parameter $\gamma = 6n$ to be greater or equal to the desired power level of $1 - \beta = 0.8$.

We will revisit this example in the lab session on Friday.

Lack-of-fit test

Hiking example: completely randomized experiment involving alpine meadows in the White Mountains of New Hampshire. $N = 20$ lanes of dimension $0.5m \times 1.5m$ randomized to 5 trampling treatments:

i: trt group	x: Number of passes	y_{ij}: Height (cm)			
1	0	20.7	15.9	17.8	17.6
2	25	12.9	13.4	12.7	9.0
3	75	11.8	12.6	11.4	12.1
4	200	7.6	9.5	9.9	9.0
5	500	7.8	9.0	8.5	6.7

Two models for mean plant height:

$$\begin{aligned} \text{SLR model: } & \mu(x) = \beta_0 + \beta_1 x \\ \text{one-factor ANOVA model: } & \mu_{ij} = \mu + \alpha_i \end{aligned}$$

When the t treatments have an interval scale, the SLR model, and all polynomials of degree $p \leqslant t-2$ (why?), are nested in one-factor ANOVA model with t treatment means.

Answer:

F-ratio for lack-of-fit test

To test for lack-of-fit of a polynomial (reduced) model of degree p, use extra sum-of-squares F-ratio on $t-1-p$ and $N-t$ df:

$$F = \frac{SS[\text{lack of fit}]/(t-1-p)}{MS[\text{pure error}]}$$

where

$$MS[\text{pure error}] = MS[E]_{full}$$

and

$$\begin{aligned} SS[\text{lack-of-fit}] &= SS[Trt] - SS[Reg]_{poly} \\ &= SS[E]_{poly} - SS[E]_{full} \end{aligned}$$

What is the SS[lack of fit] for the meadows data?

Next step: either go with the one-factor ANOVA model or specify some other model, such as quadratic.

One-way random effects model

Let's first consider an example.

- Genetics study w/ beef animals. Measure birthweight Y (lbs).
- $t = 5$ sires, each mated to a separate group of $n = 8$ dams.

- $N = 40$, completely randomized.

Sire #	Level	Sample Birthweights								$\bar{y}_{i+}$	s_i
177	1	61	100	56	113	99	103	75	62	83.6	22.6
200	2	75	102	95	103	98	115	98	94	97.5	11.2
201	3	58	60	60	57	57	59	54	100	63.1	15.0
202	4	57	56	67	59	58	121	101	101	77.5	25.9
203	5	59	46	120	115	115	93	105	75	91.0	28.0

Question: Statistical model for these data?
Answer: One-way fixed effects model?

$$Y_{ij} = \mu + \tau_i + \epsilon_{ij}$$

where τ_i denotes the difference between the mean birthweight of population of offspring from sire i and μ, mean of whole population.

<u>The one-way random effects model</u>

$$Y_{ij} = \underbrace{\mu}_{\text{fixed}} + \underbrace{T_i}_{\text{random}} + \underbrace{\epsilon_{ij}}_{\text{random}} \qquad \text{for } i = 1, 2, \ldots, t \text{ and } j = 1, \ldots, n$$

with

- $T_1, T_2, \ldots, T_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_T^2)$
- $\epsilon_{11}, \ldots, \epsilon_{tn} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$
- $T_1, T_2, \ldots, T_t$ independent of $\epsilon_{11}, , \ldots, \epsilon_{tn}$

Features

- $T_1, T_2, \ldots$ denote <u>random effects</u>, drawn from some population of interest. That is $T_1, T_2, \ldots$ is a <u>random sample</u>.
- σ_T^2 and σ^2 are called <u>variance components</u>
- conceptually different from one-way fixed effects model

For beef animal genetic study, with $t = 5$ and $n = 8$, the random effects $T_1, T_2, \ldots, T_5$ reflect sire-to-sire variability.

No particular interest in $\tau_1, \tau_2, \ldots, \tau_5$ from the fixed effects model:

$$Y_{ij} = \underbrace{\mu}_{\text{fixed}} + \underbrace{\tau_i}_{\text{fixed}} + \underbrace{\epsilon_{ij}}_{\text{random}} \qquad \text{for } i = 1, 2, \ldots, t \text{ and } j = 1, \ldots, n$$

with

- $\tau_1, \tau_2, \ldots, \tau_t$ unknown model parameters
- $\epsilon_{11}, \ldots, \epsilon_{tn} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$

Exercise

Using the random effects model, specify

$$E(Y_{ij}) \text{ and } Var(Y_{ij})$$

Recall:

- Two *components* to variability in data: σ^2, σ_T^2
- $T_1, T_2, \ldots, T_5$ a random sample of sire effects
- Sire effects is a population in its own right.
- Model parameters: σ^2, σ_T^2, μ.

Answer:

Sums of squares and mean squares are the same as in one-way fixed effects ANOVA:

$$\begin{aligned} SS[T] &= \sum\sum(\bar{y}_{i+} - \bar{y}_{++})^2 \\ SS[E] &= \sum\sum(y_{ij} - \bar{y}_{i+})^2 \\ SS[Tot] &= \sum\sum(y_{ij} - \bar{y}_{++})^2 \end{aligned}$$

ANOVA table

The ANOVA table is almost the same, it just has a different expected mean squares column:

Source	SS	df	MS	Expected MS
Treatment	$SS[T]$	$t-1$	$MS[T]$	$\sigma^2 + n\sigma_T^2$
Error	$SS[E]$	$N-t$	$MS[E]$	σ^2
Total	$SS[Tot]$	$N-1$		

Estimating parameters of one-way random effects model

1. **Method of moment (M.o.M.) estimation**: Equate EMS with observed MS and solve. Problem: M.o.M estimation can give $\hat{\sigma}^2 < 0$ (estimates of variances < 0)
2. **Maximum likelihood (ML) / Restricted maximum likelihood (REML)**: Numerical procedures that avoid negative variance estimates.

- ML: full maximum-likelihood estimation maximizes the likelihood with respect to all of the parameters of the model simultaneously (i.e., both the fixed-effects parameters and the variance components).
- REML: restricted (or residual) maximum-likelihood estimation integrates the fixed effects out of the likelihood and estimates the variance components; given the resulting estimates of the variance components, estimates of the fixed effects are recovered. REML estimates are the same as M.o.M. estimates with balanced data.

For one-way random-effects model:

$$\begin{aligned}\hat{\mu} &= \bar{y}_{++}\\ \hat{\sigma}^2 &= MS[E]\\ \hat{\sigma}_T^2 &= \frac{MS[T]-MS[E]}{n}\end{aligned}$$

For sires data, $\bar{y}_{++} = 82.6$ and

Source	SS	df	MS	Expected MS
Sire	5591	4	1398	$\sigma^2 + 8\sigma_T^2$
Error	16233	35	464	σ^2
Total	21824	39		

Obtain the parameter estimates. *Answers:*