28 Lecture 28: April 5

Last time

- two-way ANOVA

Today

- Announcement: per requested by three students, we will do a third poll on Wednesday for the alternative grading path (the last time).
- Lab session review
- ANCOVA
- Linear contrasts of means

Additional reference

Course notes by Dr. Jason Osborne.

A three-factor example

In a balanced, complete, crossed design, $N = 36$ shrimp were randomized to $abc = 12$ treatment combinations from the factors below:

- A1: Temperature at $25°C$
- A2: Temperature at $35°C$
- B1: Density of shrimp population at 80 shrimp/$40l$
- B2: Density of shrimp population at 160 shrimp/$40l$
- C1: Salinity at 10 units
- C2: Salinity at 25 units
- C3: Salinity at 40 units

The response variable of interest is weight gain Y_{ijkl} after four weeks.

Three-way ANOVA model

$$
\begin{aligned}
Y_{ijkl} &= \mu + \alpha_i + \beta_j + \gamma_k \\
&\quad + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} \\
&\quad + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}
\end{aligned}
$$

$$
\begin{aligned}
i &= 1, 2 \\
j &= 1, 2 \\
k &= 1, 2, 3 \\
l &= 1, 2, 3 \\
\epsilon_{ijkl} &\overset{iid}{\sim} \mathcal{N}(0, \sigma^2)
\end{aligned}
$$

Many constraints such as (over one dimension):

$$
\begin{aligned}
\sum_i \alpha_i &= 0 \\
\sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} &= 0 \quad \text{for all } i, j \\
\sum_i (\alpha\beta\gamma)_{ijk} = \sum_j (\alpha\beta\gamma)_{ijk} = \sum_k (\alpha\beta\gamma)_{ijk} &= 0 \quad \text{for all } i, j, k
\end{aligned}
$$

Now, please finish the table below

Source	df
A	
B	
C	
$A \times B$	
$A \times C$	
$B \times C$	
$A \times B \times C$	
Residual	
Total	

Answer:

The three-way ANOVA model includes parameters for

- Main effects: α_i, β_j and γ_k.
- Two-way interactions between each pair of factors: $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$ and $(\beta\gamma)_{jk}$.
- Three-way interaction among all three factors: $(\alpha\beta\gamma)_{ijk}$.

Readings:

1. JF 8.3.1 on parameter estimates and hypothesis testing for three-way ANOVA model.
2. JF 8.3.2 on Higher-order classifications.

Analysis of Covariance

Analysis of covariance (ANCOVA) is a term used to describe linear models that contain both qualitative and quantitative explanatory variables. The method is, therefore, equivalent to dummy-variable regression, discussed in the previous lectures, although the ANCOVA model is parametrized differently from the dummy-regression model.

Covariate is a variable known to affect the response that

1. differs among EUs
2. reflects differences that exist independently of experimental treatment.

A nutrition example

A nutrition scientist conducted an experiment to evaluate the effects of four vitamin supplements on the weight gain of laboratory animals. The experiment was conducted in a completely randomized design with $N = 20$ animals randomized to $a = 4$ supplement groups, each with sample size $n \equiv 5$. The response variable of interest is weight gain, but calorie intake z was measured simultaneously.

Diet	$y(g)$	Diet	y	Diet	y	Diet	y
1	48	2	65	3	79	4	59
1	67	2	49	3	52	4	50
1	78	2	37	3	63	4	59
1	69	2	75	3	65	4	42
1	53	2	63	3	67	4	34
1	$\bar{y}_{1+} = 63$	2	$\bar{y}_{2+} = 57.8$	3	$\bar{y}_{3+} = 65.2$	4	$\bar{y}_{4+} = 48.8$
1	$s_1 = 12.3$	2	$s_2 = 14.9$	3	$s_3 = 9.7$	4	$s_4 = 10.9$

Question: Is there evidence of a vitamin supplement effect?

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Diet	3	797.8	265.9	1.823	0.184
Residuals	16	2334.4	145.9		

But calorie intake z was measured simultaneously:

Diet	$y(g)$	z	Diet	y	z	Diet	y	z	Diet	y	z
1	48	350	2	65	400	3	79	510	4	59	530
1	67	440	2	49	450	3	52	410	4	50	520
1	78	440	2	37	370	3	63	470	4	59	520
1	69	510	2	75	530	3	65	470	4	42	510
1	53	470	2	63	420	3	67	480	4	34	430

Question: How and why could these new data be incorporated into analysis?
Answer: ANCOVA can be used to reduce unexplained variation.

ANCOVA model,

$$y_{ij} = \mu + \alpha_i + \beta z_{ij} + \epsilon_{ij}$$

where μ is the reference level, α_i is the main effect of treatment, β is the partial regression coefficient, and $\epsilon_{ij} \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2)$. The model is equivalent as the dummy-variable regression model,

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_z z_i + \epsilon_i \quad \text{for } i = 1, \ldots, 20$$

Finish the table below

Source	df
Diet	
Covariate	1
Residual	
Total	

Answer:

To test for difference among treatments. The null hypothesis in terms of α_i is
$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_4 = 0$ v.s. $H_a :$ at least one $\alpha_i \neq 0$
And the null hypothesis in terms of β_i is
$H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ v.s. $H_a :$ at least one $\beta_i \neq 0$

Question: which two models do we compare when testing the above null hypothesis? *Answer:*

Linear contrasts of means

With ANOVA (or ANCOVA) models, we do not generally test hypotheses about individual coefficients (but we can do so if we wish). For dummy-coded regressors in one-way ANOVA, a t-test or F-test of $H_0 : \alpha_1 = 0$, for example, is equivalent to testing for the difference in means between the first group and the baseline group, $H_0 : \mu_1 = \mu_m$.

Consider the one-way ANOVA model:

$$Y_{ij} = \mu_i + \epsilon_{ij}, i = 1, 2, \ldots, t, \text{ and } j = 1, 2, \ldots, n_i$$

with $\epsilon_{ij} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.

A linear function of the group means of the form

$$\theta = c_1\mu_1 + c_2\mu_2 + \cdots + c_t\mu_t$$

is called a linear combination of the treatment means. And the c_i's are the coefficients of the linear combination. If

$$c_1 + c_2 + \cdots + c_t = \sum_{j=1}^{t} c_j = 0,$$

the linear combination is called a contrast. Contrasts with more than two non-zero coefficients are called complex contrasts.

Let two contrasts θ_1 and θ_2 be given by

$$\theta_1 = c_1\mu_1 + \cdots + c_t\mu_t = \sum_{j=1}^{t} c_j\mu_j$$

$$\theta_2 = d_1\mu_1 + \cdots + d_t\mu_t = \sum_{j=1}^{t} d_j\mu_j,$$

then the two contrasts θ_1 and θ_2 are mutually orthogonal if the products of their coefficients sum to zero:

$$c_1d_1 + \cdots + c_td_t = \sum_{j=1}^{t} = 0$$

θ_i and θ_j are orthogonal $\implies$ $\hat{\theta}_i$ and $\hat{\theta}_j$ are statistically independent.

Types of effects

Consider the following two-way ANOVA model:

$$\begin{aligned} Y_{ijk} &= \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk} \\ &\quad i = 1, 2 = a \text{ and } j = 1, 2 = b \text{ and } k = 1, 2, \ldots, 7 = n. \end{aligned}$$

$\epsilon_{ijk} \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$. Parameter constraints: $\sum_i \alpha_i = \sum_j \beta_j = 0$ and $\sum_i (\alpha\beta)_{ij} = 0$ for each j and $\sum_j (\alpha\beta)_{ij} = 0$ for each i.

- Factor A: AGE has $a = 2$ levels - A_1 : younger and A_2 : older
- Factor B: GENDER has $b = 2$ levels - B_1 : female and B_2 : male

Three kinds of effects in this 2×2 design:

1. Simple effects are simple contrasts.
   - $\mu(A_1B) = \mu_{12} - \mu_{11}$ - simple effect of gender for young folks.
   - $\mu(AB_1) = \mu_{21} - \mu_{11}$ - simple effect of age for women.
2. Interaction effects are differences of simple effects: $\mu(AB) = \mu(AB_2) - \mu(AB_1) = (\mu_{22} - \mu_{12}) - (\mu_{21} - \mu_{11})$
   - difference between simple age effects for men and women
   - difference between simple gender effects for old and young folks
   - interaction effect of AGE and GENDER.
3. Main effects are averages or sums of simple effects

$$\mu(A) = \frac{1}{2}(\mu(AB_1) + \mu(AB_2))$$
$$\mu(B) = \frac{1}{2}(\mu(A_1B) + \mu(A_2B))$$