Time-dependent rate model

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The clock model in the partial vectors for one branch

The partial differential equation

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}(t)\mathbf{Q}\mu(t)$$

has the solution involving matrix exponential for one branch of time (t_1, t_2):

$$\log \mathbf{P}(t_2) - \log \mathbf{P}(t_1) = \int_{t_1}^{t_2} \mathbf{Q}\mu(t)dt$$

1. For constant $\mu(t) = r$.

   $$\mathbf{P}(t_2) = \mathbf{P}(t_1)e^{\mathbf{Q}r(t_2-t_1)}$$

   that translates to

   $$r(t_1, t_2) = r$$

2. For a linear (i.e. time dependent) function, $\mu(t) = \beta t$

   $$\begin{aligned}\mathbf{P}(t_2) &= \mathbf{P}(t_1)e^{\frac{1}{2}\mathbf{Q}\beta(t_2^2-t_1^2)} \\ &= \mathbf{P}(t_1)e^{\mathbf{Q}(t_2-t_1)\beta\frac{t_1+t_2}{2}}\end{aligned}$$

   which suggests a mid-point for node heights with

   $$r(t_1, t_2) = \beta\frac{t_1+t_2}{2}$$

3. For a linear function with a upper threshold,

   $$\mu(t) = \begin{cases} \beta t & t \geq T \\ \beta T & t < T \end{cases}$$

   has

   $$\mathbf{P}(t_2) = \begin{cases} \mathbf{P}(t_1)e^{\mathbf{Q}\beta T(t_2-t_1)} & t_1 < t_2 < T \\ \mathbf{P}(t_1)e^{\mathbf{Q}\beta T(T-t_1)+\mathbf{Q}(t_2-T)\beta\frac{t_2+T}{2}} & t_1 < T < t_2 \\ \mathbf{P}(t_1)e^{\mathbf{Q}(t_2-t_1)\beta\frac{t_1+t_2}{2}} & T < t_1 < t_2 \end{cases}$$

that translates into

$$\mathbf{r}(t_1,t_2) = \begin{cases} \beta T & t_1 < t_2 < T \\ \beta \frac{\frac{1}{2}(t_2^2 - T^2) + T(T - t_1)}{t_2 - t_1} & t_1 < T < t_2 \\ \beta \frac{t_1 + t_2}{2} & T < t_1 < t_2 \end{cases}$$