14 Lecture 14: Feb 19

Last time

- HW1 review
- Probability review, cont

Today

- Probability review
- Lab session

Reference:

- Statistical Inference, 2nd Edition, by George Casella & Roger L. Berger
- Review of Probability Theory by Arian Maleki and Tom Do

Chi-square, t-, and F-Distributions

Let $Z_1, Z_2, \ldots, Z_k \overset{iid}{\sim} N(0,1)$, then $X^2 \equiv Z_1^2 + Z_2^2 + \cdots + Z_k^2 \sim \chi_k^2$ (with k degrees of freedom).
If $X \sim \chi_k^2$

$$\mathbf{E}(X) = k$$
$$\mathbf{Var}(X) = 2k.$$

Student's t versus χ^2

If $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

When σ is unknown,

$$\frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} \sim t_{n-1}, \quad \text{where } \hat{\sigma} = \sqrt{\frac{\sum(X_i - \bar{X})^2}{n-1}}.$$

Note that

$$\begin{aligned}\frac{\bar{X} - \mu}{\hat{\sigma}/\sqrt{n}} &= \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \cdot \frac{1}{\frac{\hat{\sigma}}{\sigma}} \\ &= Z \cdot \frac{1}{\sqrt{\frac{\sum(X_i - \bar{X})^2}{(n-1)\sigma^2}}} \\ &= \frac{Z}{\sqrt{\frac{\chi_{n-1}^2}{n-1}}}\end{aligned}$$

F versus χ^2

$$F_{ndf,ddf} \equiv \frac{\chi^2_{ndf}/ndf}{\chi^2_{ddf}/ddf}$$

t versus F

$$\begin{aligned} t_k &= \frac{Z}{\sqrt{\chi^2_k/k}} \\ &= \frac{\sqrt{\chi^2_1/1}}{\sqrt{\chi^2_k/k}} \\ &= \sqrt{F_{1,k}} \end{aligned}$$

or, in other words, $t_k^2 = F_{1,k}$

Random vectors and matrices

The cdf for random vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \text{ is } F_{\mathbf{Y}}(\mathbf{y}) = \Pr(Y_1 \leqslant y_1, Y_2 \leqslant y_2, \ldots, Y_n \leqslant y_n)$$

If a joint pdf exists, then $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{Y}}(y_1, \ldots, y_n)$ and

$$F_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \ldots \int_{-\infty}^{y_n} f_{\mathbf{Y}}(\mathbf{t}) d\mathbf{t}$$

Moments

$$\begin{aligned} \mathbf{E}\left(\mathbf{Y}\right) &= \mu_{\mathbf{Y}} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \\ \mathbf{Var}\left(\mathbf{Y}\right) &= \mathbf{E}\left((\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Y} - \mu_{\mathbf{Y}})^T\right) \\ &= \mathbf{E}\left(\begin{bmatrix} (Y_1 - \mu_1)^2 & (Y_1 - \mu_1)(Y_2 - \mu_2) & \ldots \\ (Y_2 - \mu_2)(Y_1 - \mu_1) & (Y_2 - \mu_2)^2 & \ldots \\ \ldots & & \end{bmatrix}\right) \\ &= \mathbf{E}\left([(Y_i - \mu_i)(Y_j - \mu_j), i = 1, 2, \ldots, n, j = 1, 2, \ldots, n]\right) \\ &= (\sigma_{ij})_{i=1,2,\ldots,n; j=1,2,\ldots,n} \end{aligned}$$

where $\sigma_{ij} = Cov(Y_i, Y_j)$

Linear functions

Let $\mathbf{X} \in \mathbb{R}^{k\times 1}$, $\mathbf{Y} \in \mathbb{R}^{n\times 1}$ and $\mathbf{A} \in \mathbb{R}^{k\times 1}$, $\mathbf{B} \in \mathbb{R}^{k\times n}$ be non-random, then

$$\begin{aligned} \underset{k\times 1}{\mathbf{X}} &= \underset{k\times 1}{\mathbf{A}} + \underset{k\times n}{\mathbf{B}}\,\underset{n\times 1}{\mathbf{Y}} \\ \mathbf{E}\left(\mathbf{X}\right) &= \mathbf{A} + \mathbf{B}\mathbf{E}\left(\mathbf{Y}\right) \\ \mathbf{Var}\left(\mathbf{X}\right) &= \mathbf{B}\mathbf{Var}\left(\mathbf{Y}\right)\mathbf{B}^T \end{aligned}$$

Sums of random vectors

$$\begin{aligned} \underset{n\times 1}{\mathbf{X}} &= \underset{n\times 1}{\mathbf{Y}} + \underset{n\times 1}{\mathbf{Z}} \\ \mathbf{E}\left(\mathbf{X}\right) &= \mathbf{E}\left(\mathbf{Y}\right) + \mathbf{E}\left(\mathbf{Z}\right) = \mathbf{E}\left(\mathbf{Y} + \mathbf{Z}\right) \end{aligned}$$

Note that there is no independence assumed above.

$$\mathbf{Var}\left(\mathbf{X}\right) = \mathbf{Var}\left(\mathbf{Y} + \mathbf{Z}\right) = \mathbf{Var}\left(\mathbf{Y}\right) + \mathbf{Var}\left(\mathbf{Z}\right) + Cov(\mathbf{Y}, \mathbf{Z}) + Cov(\mathbf{Z}, \mathbf{Y})$$

If $\mathbf{Y}, \mathbf{Z}$ are uncorrelated, then $\mathbf{Var}\left(\mathbf{X}\right) = \mathbf{Var}\left(\mathbf{Y}\right) + \mathbf{Var}\left(\mathbf{Z}\right)$