11 Lecture 11: Feb 10

Last time

- Introduction of simple linear regression

Today

- HW2 posted
- The statistical model of the SLR (JF chapter 6)
- Properties of the Least-Squares estimator
- Inference of SLR model

Properties of the Least-Squares estimator

Under the strong assumptions of the simple linear regression model, the least squares coefficients $\hat{\beta}_{ls}$ have several desirable properties as estimators of the population regression coefficients β_0 and β_1:

- The least-squares intercept and slope are *linear estimators*, in the sense that they are linear functions of the observations y_i.
  Proof:

- The sample least-squares coefficients are *unbiased estimators* of the population regression coefficients:
$$\begin{aligned}\mathbf{E}\left(\hat{\beta}_0\right) &= \beta_0 \\ \mathbf{E}\left(\hat{\beta}_1\right) &= \beta_1\end{aligned}$$
  Proof:

- Both $\hat{\beta}_0$ and $\hat{\beta}_1$ have simple sampling variances:
$$\begin{aligned}\mathrm{Var}(\hat{\beta}_0) &= \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2} \\ \mathrm{Var}(\hat{\beta}_1) &= \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2}\end{aligned}$$
  Proof:

- Rewrite the formula for $\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{(\mathrm{n}-1)\mathrm{S}_\mathrm{X}^2}$, we see that the sampling variance of the slope estimate will be small when

- The error variance σ_ϵ^2 is small
- The sample size n is large
- The explanatory-variable values are spread out (i.e. have a large variance, S_X^2)

- (Gauss-Markov theorem) Under the assumptions of linearity, constant variance, and independence, the least-squares estimators are BLUE (Best Linear Unbiased Estimator), that is they have the smallest sampling variance and are unbiased. (show this)
  Proof:

- Under the full suite of assumptions, the least-squares coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ are the maximum-likelihood estimators of β_0 and β_1. (show this)
  Proof:

- Under the assumption of normality, the least-squares coefficients are themselves normally distributed. Summing up,

$$\hat{\beta}_0 \sim N(\beta_0, \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2})$$
$$\hat{\beta}_1 \sim N(\beta_1, \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2})$$