16 Lecture 16 Feb 24

Last time

- Confidence intervals for SLR
- Multiple linear regression

Today

- Multiple correlation
- Lab review

A multiple linear regression (MLR) model w/ p independent variables

Let p independent variables be denoted by $x_1, \ldots, x_p$.

- Observed values of p independent variables for i^{th} subject from sample denoted by $x_{i1}, \ldots, x_{ip}$
- response variable for i^{th} subject denoted by Y_i
- For $i = 1, \ldots, n$, MLR model for Y_i:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i$$

- As in SLR, $\epsilon_1, \ldots, \epsilon_n \overset{iid}{\sim} N(0, \sigma^2)$

Least squares estimates of regression parameters minimize $SS[E]$:

$$SS[E] = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \cdots - \beta_p x_{ip})^2$$

$$\boxed{\hat{\sigma}^2 = \frac{SS[E]}{n-p-1}}$$

Interpretations of regression parameters:

- σ^2 is unknown <u>error variance</u> parameter
- $\beta_0, \beta_1, \ldots, \beta_p$ are $p+1$ unknown regression parameters:
  - β_0: average response when $x_1 = x_2 = \cdots = x_p = 0$
  - β_i is called a <u>partial slope</u> for x_i. Represents mean change in y per unit increase in x_i *with all other independent variables held fixed.*

Matrix formulation of MLR

Let a vector for p observed independent variables for individual i be defined by

$$x_{i\cdot} = (1, x_{i1}, x_{i2}, \ldots, x_{ip}).$$

The MLR model for $Y_1, \ldots, Y_n$ is given by

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \cdots + \beta_p X_{1p} + \epsilon_1 \\ Y_2 &= \beta_0 + \beta_1 X_{21} + \beta_2 X_{22} + \cdots + \beta_p X_{2p} + \epsilon_2 \\ \vdots &= \vdots \\ Y_n &= \beta_0 + \beta_1 X_{n1} + \beta_2 X_{n2} + \cdots + \beta_p X_{np} + \epsilon_n \end{aligned}$$

This system of n equations can be expressed using matrices:

$$\boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}}$$

where

- $\mathbf{Y}$ denotes a response vector of size $n \times 1$
- $\mathbf{X}$ denotes a design matrix of size $n \times (p+1)$
- $\boldsymbol{\beta}$ denotes a vector of regression parameters of size $(p+1) \times 1$
- $\boldsymbol{\epsilon}$ denotes an error vector of size $n \times 1$

Here, the error vector ϵ is assumed to follow a multivariate normal distribution with variance-covariance matrix $\sigma^2 \mathbf{I}_n$. For individual i,

$$y_i = \mathbf{x}_{i\cdot}\boldsymbol{\beta} + \epsilon_i.$$

Some simplified expressions: ($\mathbf{a}$ is a known $p \times 1$ vector)

$$\boxed{\begin{aligned} \hat{\boldsymbol{\beta}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\ \mathbf{Var}\left(\hat{\boldsymbol{\beta}}\right) &= \sigma^2(\mathbf{X}^T\mathbf{X})^{-1} \\ &= \boldsymbol{\Sigma} \\ \widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}}) &= MS[E](\mathbf{X}^T\mathbf{X})^{-1} \\ &= \widehat{\boldsymbol{\Sigma}} \\ \widehat{\mathrm{Var}}(\mathbf{a}^T\hat{\boldsymbol{\beta}}) &= \mathbf{a}^T\widehat{\boldsymbol{\Sigma}}\mathbf{a} \end{aligned}}$$

Question: what are the dimensions of each of these quantities?

- $(\mathbf{X}^T\mathbf{X})^{-1}$ may be verbalized as " x transposed x inverse"
- $\widehat{\boldsymbol{\Sigma}}$ is the estimated variance-covariance matrix for the estimate of the regression parameter vector $\hat{\beta}$

- $\mathbf{X}$ is assumed to be of full *rank*.

Some more simplified expressions:

$$\begin{aligned}\hat{\mathbf{Y}} &= \mathbf{X}\hat{\boldsymbol{\beta}} \\ &= \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\ &= \mathbf{H}\mathbf{Y} \\ \hat{\boldsymbol{\epsilon}} &= \mathbf{Y} - \hat{\mathbf{Y}} \\ &= \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} \\ &= (\mathbf{I} - \mathbf{H})\mathbf{Y}\end{aligned}$$

- $\hat{\mathbf{Y}}$ is called the vector of fitted or predicted values
- $\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ is called the hat matrix
- $\hat{\boldsymbol{\epsilon}}$ is the vector of residuals

For the Duncan's data example on income, education and prestige, with $p = 2$ independent variables and $n = 45$ observations,

$$\mathbf{X} = \begin{bmatrix} 1 & 62 & 86 \\ 1 & 72 & 76 \\ \vdots & \vdots & \vdots \\ 1 & 8 & 32 \end{bmatrix}$$

and

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 45 & 1884 & 2365 \\ 1884 & 105148 & 122197 \\ 2365 & 122197 & 163265 \end{bmatrix}$$

$$(\mathbf{X}^T\mathbf{X})^{-1} = \begin{bmatrix} 0.10211 & -0.00085 & -0.00084 \\ -0.00085 & 0.00008 & -0.00005 \\ -0.00084 & -0.00005 & 0.00005 \end{bmatrix}$$

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \begin{bmatrix} -6.0646629 \\ 0.5987328 \\ 0.5458339 \end{bmatrix} = ?$$

$$SS[E] = \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = (\mathbf{Y} - \hat{\mathbf{Y}})^T(\mathbf{Y} - \hat{\mathbf{Y}}) = 7506.7$$

$$MS[E] = \frac{SS[E]}{df} = \frac{7506.7}{45 - 2 - 1} = 178.73$$

$$\widehat{\boldsymbol{\Sigma}} = MS[E](\mathbf{X}^T\mathbf{X})^{-1} = \begin{bmatrix} 18.249481 & -0.151845008 & -0.150706025 \\ -0.151845 & 0.014320275 & -0.008518551 \\ -0.150706 & -0.008518551 & 0.009653582 \end{bmatrix}$$

Multiple correlation, JF 5.2.3

The sums of squares in multiple regression are defined in the same manner as in SLR:

$$\begin{aligned} TSS &= \sum (Y_i - \bar{Y})^2 \\ RegSS &= \sum (\hat{Y}_i - \bar{Y})^2 \\ RSS &= \sum (Y_i - \hat{Y}_i)^2 = \sum \hat{\epsilon}_i^2 \end{aligned}$$

Not surprisingly, we have a similar analysis of variance for the regression:

$$TSS = RegSS + RSS$$

The squared multiple correlation R^2, representing the proportion of variation in the response variable captured by the regression, is defined in terms of the sums of squares:

$$R^2 = \frac{RegSS}{TSS} = 1 - \frac{RSS}{TSS}.$$

Because there are several slope coefficients, potentially with different signs, the *multiple correlation coefficient* is, by convention, the positive square root of R^2. The multiple correlation is also interpretable as the simple correlation between the fitted and observed Y values, i.e. $r_{\hat{Y}Y}$.

Adjusted-R^2

Because the multiple correlation can only rise, never decline, when explanatory variables are added to the regression equation (HW1), investigators sometimes penalize the value of R^2 by a "correction" for degrees of freedom. The corrected (or "adjusted") R^2 is defined as:

$$\begin{aligned} R^2_{adj} &= 1 - \frac{\frac{RSS}{n-p-1}}{\frac{TSS}{n-1}} \\ &= 1 - \left[\frac{(1-R^2)(n-1)}{n-p-1} \right] \end{aligned}$$

Confidence intervals

Confidence intervals and hypothesis tests for individual coefficients closely follow the pattern of simple-regression analysis:

1. substitute an estimate of the error variance (MSE) for the unknown σ^2 into the variance term of $\hat{\beta}_i$
2. find the estimated standard error of a slope coefficient $\widehat{SE}(\hat{\beta}_i)$
3. $t = \frac{\hat{\beta}_i - \beta_i}{\widehat{SE}(\hat{\beta}_i)}$ follows a t-distribution with degrees of freedom as associated with SSE.

Therefore, we can construct the $100(1-\alpha)\%$ confidence interval for a single slope parameter by (why?):

$$\hat{\beta}_i \pm t(n-p-1, \alpha/2)\widehat{SE}(\hat{\beta}_i)$$

Hand-waving proof:

Hypothesis tests

We first test the null hypothesis that all population regression slopes are 0:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$$

The test statistics,

$$F = \frac{RegSS/p}{RSS/(n-p-1)}$$

follows an F-distribution with p and $n-p-1$ degrees of freedom.

We can also test a null hypothesis about a *subset* of the regression slopes, e.g.,

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_q = 0.$$

Or more generally, test the null hypothesis

$$H_0 : \beta_{q_1} = \beta_{q_2} = \cdots = \beta_{q_k} = 0$$

where $0 \leqslant q_1 < q_2 < \cdots < q_k \leqslant p$ is a subset of k indices. To get the F-statistic for this case, we generally perform the following steps:

1. Fit the *full* ("unconstrained") model, in other words, model that provides context for H_0. Record SSR_{full} and the associated df_{full}
2. Fit the *reduced* ("constrained") model, in other words, full model constrained by H_0. Record SSR_{red} and the associated df_{red}
3. Calculate the F-statistic by

$$F = \frac{[SSR_{red} - SSR_{full}]/(df_{red} - df_{full})}{SSR_{full}/df_{full}}$$

4. Find p-value (the probability of observing an F-statistic that is at least as high as the value that we obtained) by consulting an F-distribution with numerator $df(ndf) = df_{red} - df_{full}$ and denominator $df(ddf) = df_{full}$. Notation: $F_{ndf,ddf}$, see Figure 16.1.

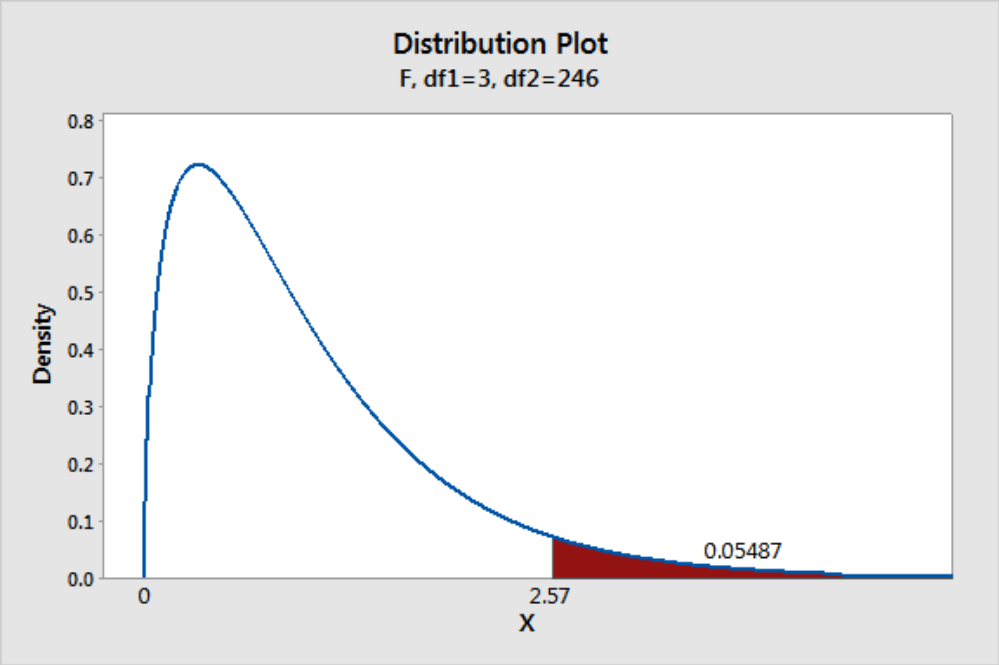

Figure 16.1: An example for p-value for F-statistic value 2.57 with an $F_{3,246}$ distribution