

# Math 6040/7260 Linear Models

Mon/Wed/Fri 1:00pm - 1:50pm

Instructor: Dr. Xiang Ji, xji4@tulane.edu

## 1 Lecture 1: Jan 13

### Today

- Introduction
- Introduce yourself
- Course logistics

### What is this course about?

The term “linear models” describes a wide class of methods for the statistical analysis of multivariate data. The underlying theory is grounded in linear algebra and multivariate statistics, but applications range from biological research to public policy. The objective of this course is to provide a solid introduction to both the theory and practice of linear models, combining mathematical concepts with realistic examples.

### Prerequisite

- **Must:** Introduction to Probability (Math 3070/6070), Mathematical Statistics (Math 3080/6080)
- **Good to have:** Scientific Computation II

### A hierarchy of linear models

- The linear mean model:

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times p}{\mathbf{X}} \underset{p \times 1}{\boldsymbol{\beta}} + \underset{n \times 1}{\boldsymbol{\epsilon}}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$ . Only assumption is that errors have mean 0.

- Gauss-Markov model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  and  $\mathbf{Var}(\boldsymbol{\epsilon}) = \sigma^2 \mathbf{I}$ . Uncorrelated errors with constant variance.

- Aitken model or general linear model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  and  $\mathbf{Var}(\boldsymbol{\epsilon}) = \sigma^2 \mathbf{V}$ .  $\mathbf{V}$  is fixed and known.

- Variance components models:  $\mathbf{y} \sim N(\mathbf{X}\beta, \sigma_1^2\mathbf{V}_1 + \sigma_2^2\mathbf{V}_2 + \dots + \sigma_r^2\mathbf{V}_r)$  with  $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_r$  known.
- General mixed linear Model:

$$\mathbf{y} = \mathbf{X}\beta + \epsilon$$

where  $\mathbf{E}(\epsilon) = \mathbf{0}$  and  $\mathbf{Var}(\epsilon) = \Sigma(\theta)$ .

- Generalized linear models (GLMs). Logistic regression, probit regression, log-linear model (Poisson regression), ... Note the difference from the general linear model. GLMs are generalization of the *concept* of linear models. They are covered in Math 7360 - Data Analysis class (<https://tulane-math-7360-2023.github.io/>).

## Syllabus

Check course website frequently for updates and announcements.

<https://tulane-math-7260-2025.github.io/>

## HW submission

Through Github with demo on Friday class.

## Presentations

Let me know your pick by the end of Friday (01/17/2025).

## Spring 2023 comments

### 1. Experience in this course

- I enjoyed this class a lot. I enjoyed that the class only had a few students, which made it feel like a community.
- It was a bit hard to follow towards the end of the semester. I think we spent too much time on easier topics at the beginning of the semester but relatively less time on harder topics towards the end
- The course was interesting, and the professor did a good job of thoroughly teaching the topics discussed.

### 2. Strong aspects of this course

- Having the lectures based off of lecture notes was very nice because I could just listen to Dr. Ji and didn't need to worry about copying down a ton of material, which helped me learn better
- I really enjoyed the setup of the course: prewritten notes for lectures, lab sessions for application and learning computing, presentations to learn fun material,

homework graded for completeness, take home exams. I feel like this setup helped me enjoy and learn the material without feeling stressed.

- The expectations were clearly presented, and the exams fairly reflected the materials we discussed in class. The content was also discussed thoroughly.

## Spring 2022 comments

### 1. Experience in this course

- I really enjoyed this class.
- The lecture is dry. Expect Dr. Ji to read through the lecture notes until someone asks a question. Also, having a background in statistics gives context to the lecture notes.

*Response:* I did emphasize on pre-requisite last year, but it still seemed not enough.

- I think this course was well laid out. Even though I did take it as a graduation requirement, I ended up enjoying the course. I also think the work load is manageable and professor Xi does provide all the tools necessary to succeed in the course. It can be overwhelming at first but with a little time and effort you can get the hang of the material.

*Response:* Mind the typos, students.

### 2. Strong aspects of this course

- Professor Ji is a great professor.
- Dr. Ji provides an inhuman amount of course material to help supplement learning, it was extremely helpful to have labs (answered and unanswered) as well as homework keys posted.

*Response:* There will be labs again.

- Notes are very structured and the professor is nice.
- The strongest aspect of this course is the homework. They are a great way to interact with and learn the material. The problems can seem challenging, but are doable with a little effort. Also, he publishes the keys afterward so you can check your work and see what the reasoning behind the answer is. This coupled with the lab sessions are a great way to prepare for the exams.

### 3. There will be an internal mid-term-ish evaluation for this course. Will remember to go over them.

## Spring 2021 comments

### 1. Experience in this course

- Overall, I had a pretty good experience in this course. It moved quickly, but that is expected from this level of course. Sometimes it was hard to stay engaged with the lectures and to really absorb the course material. Because the lectures moved so fast, I really appreciated how Professor made the full notes available at the time of the lecture. I would have liked if there were a few more examples with the notes, as sometimes the homework felt disjoint from the notes.

*Response:* I will try to move slower this semester. I will start lab sessions earlier too.

- The professor is an extremely intelligent, kind, and understanding professor. He prioritizes in making sure that we understand the material and seeing how the material can be applied. His lecture notes were a godsend because the texts could be a bit ambiguous at times but he elucidated the material in such a comprehensible manner.

*Response:* I will try to fix the left-over typos.

- Mentioned in class from other students/internal evaluation, conveying the mathematical concepts through the presentation is not a good idea to follow the class in real-time. Prepared presentation can give rise to a distraction on what we have been going over.

*Response:* I am still delivering this class in hybrid-mode. I found the presentations fit online teaching better. I think the difficulty might be caused by (1) fast moving lecture (2) I only realized the need of reviewing basic concepts of probability almost a quarter into the semester...

- I found the setup of the course not very engaging. Additionally, many of the class notes came directly from the additional sources with no additional information or explanation, which I found to be not very helpful.

*Response:* I actually like them. I was the guinea pig to test them.

- Easily help us to understand the main course, and the notes and details are great.

*Response:* There will be notes.

- Moves very quickly and can be hard to keep up with. Sometimes instructions are unclear.

*Response:* I will try to slow down.

- Both the instructor and the TA were helpful. It was hard to follow along in class though.

*Response:* We don't have TA this time. Make use of the office hour. And I have to say, it needs effort to ace in this class.

## 2. Strong aspects of this course

- Having the lecture notes and labs available was very helpful. Professor was also always very nice and accommodating, and willing to meet with me when I needed help. He also always responded to student feedback, if we asked for an extra day or two on the homework or something like that.

*Response:* Here is an example of correctly using the office hours.

- His lecture notes and the lab sessions.

*Response:* They will be there again.

- Lab session is necessarily required to this class. A lot of computations in the class would be done by computer due to the complexity, and students are expected to handle with the computer programming properly at a desired level. The course can be an introduction to the statistical computation, which does not exist in the mathematics department.

*Response:* Hmm, there is a course Math 7360 Data Analysis that focuses more on the computational side.

- I appreciated the homework reviews in class and felt these helped clarify the material.

*Response:* Of course, the reviews will be there again. The purpose of the course is for you to learn.

- Grading was easy which made up for the rigor.

*Response:* Don't rely on this...

- Really appreciate that Professor Xiang made such a neat and tidy notes for us. It is really helpful for me to review. And notes have a great interaction with us, Professor Xiang also leaves some questions to help us think about the logic behind.

*Response:* Well, Xiang is my first name. Please call me Prof. X.

- Prof. Xiang was highly organized and wanted his students to understand the course content more than he made them worry about grades. I learned a lot about Linear Models and feel confident applying the course content professionally and academically. I wish most of the Math department had his teaching style and implemented his course documents and organization structure. Prof. Xiang made the course content in class digestible and if I needed to review the material I could easily find it through his course notes and textbook. I wish I could say the same about my other courses.

*Response:* Hmm, I like Prof. X. better.

- I really appreciated the emphasis on learning. It allowed for most students to take it at the pace that was good for them.

*Response:* Please don't let your score rely on this comment.

3. There will be an internal mid-term-ish evaluation for this course. Will remember to go over them.

Rate my professor comments

N.A.

## 2 Lecture 2:Jan 15

### Last time

- Introduction
- Course logistics

### Today

- Reply to the “Presentation Dates” thread on Canvas by the end of Friday.
- Git

### What is git?

Git is currently the most popular system for version control according to [Google Trend](#). Git was initially designed and developed by [Linus Torvalds](#) in 2005 for Linux kernel development. Git is the British English slang for unpleasant person.

### Why using git?

- [GitHub](#) is becoming a de facto central repository for open source development.
- **Advertise** yourself through GitHub (e.g., host a free personal webpage on GitHub, [example and tutorial](#)).
- a skill that employers look for (according to [this AmStat article](#)).

### Git workflow

Figure [2.1](#) shows its basic workflow.

### What do I need to use Git?

- A **Git server** enabling multi-person collaboration through a centralized repository.
- A **Git client** on your own machine.
  - Linux: Git client program is shipped with many Linux distributions, e.g., Ubuntu and CentOS. If not, install using a package manager, e.g., `yum install git` on CentOS.
  - Mac: follow instructions at <https://www.atlassian.com/git/tutorials/install-git>.
  - Windows: Git for Windows at <https://gitforwindows.org> (Graphical User Interface, or in short, GUI) aka Git Bash.
- Do **not** totally rely on GUI or IDE (Integrated Development Environment). Learn to use Git on command line, which is needed for cluster and cloud computing.



Figure 2.1

## Git survival commands

- `git pull` synchronize local Git directory with remote repository.
- Modify files in local working directory.
- `git add FILES` add snapshots to staging area
- `git commit -m "message"` store snapshots permanently to (**local**) Git repository
- `git push` push commits to remote repository.

## Git basic usage

Working with your local copy.

- `git pull` : update local Git repository with remote repository (fetch + merge).
- `git log FILENAME` : display the current status of working directory.
- `git diff` : show differences (by default difference from the most recent commit).
- `git add file1 file2 ...` : add file(s) to the staging area.

- `git commit` : commit changes in staging area to Git directory.
- `git push` : publish commits in local Git repository to remote repository.
- `git reset --soft HEAD 1` : undo the last commit.
- `git checkout FILENAME` : go back to the last commit, discarding all changes made.
- `git rm FILENAME` : remove files from git control.

## Git demonstration

Show how to create a private git repository for HW and Exam submissions.

On [GitHub](#)

- Obtain [student developer pack](#).
- Create a private repository `Xiang-Ji-math-6040-2025-spring` (please substitute 6040 by 7260 if you are taking the graduate level and use your own first and last names). Add `xji3` as your collaborators with write permission ([instruction](#)).

On your local machine:

- clone the repository: please refer to [this webpage](#) with instructions for your operating system.
- enter the folder: `cd Xiang-Ji-math-6040-2025-spring`.
- after finishing the rest of the questions, save your file inside your git repository folder `Xiang-Ji-math-6040-2025-spring` with name `hw1.pdf` (for example). Please make it human-readable.
- now using git commands to stage this change: `git add hw1.pdf`
- commit: `git commit -m "hw1 submission"` (remember to replace the quotation mark)
- push to remote server: `git push`
- tag version hw1: `git tag hw1` and push: `git push --tags`.

Take a look at the tags on GitHub ([instructions](#)).

When submitting your hw, please email your instructor ([xji4@tulane.edu](mailto:xji4@tulane.edu)) a link to your tag ([instructions](#)).



### 3 Lecture 3:Jan 17

Last time

- Git

Today

- HW1 posted
- Linear algebra: vector and vector space, rank of a matrix
- Column space and Nullspace (JM Appendix A)

Notations

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times p}{\mathbf{X}} \underset{p \times 1}{\boldsymbol{\beta}} + \underset{n \times 1}{\boldsymbol{\epsilon}}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

- All vectors are column vector
- Write dimensions underneath as in  $\underset{n \times p}{\mathbf{X}}$  or as  $\mathbf{X} \in \mathbb{R}^{n \times p}$
- Bold upper-case letters for Matrices. Bold lower-case letters for Vectors.

### Vector and vector space

(from JM Appendix A)

- A set of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are *linearly dependent* if there exist coefficients  $c_j$  for  $j = 1, 2, \dots, n$  such that  $\sum_{j=1}^n c_j \mathbf{x}_j = \mathbf{0}$  and  $\|\mathbf{c}\|_2 = \sqrt{\sum_{j=1}^n c_j^2} > 0$ . They are *linearly independent* if  $\sum_{j=1}^n c_j \mathbf{x}_j = \mathbf{0}$  implies ( $\implies$ )  $c_j = 0$  for all  $j$ .
- Two vectors are *orthogonal* to each other, written  $\mathbf{x} \perp \mathbf{y}$ , if their inner product is 0, that is  $\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \sum_j x_j y_j = 0$ .
- A set of vectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are mutually orthogonal iff ( $\iff$ )  $\mathbf{x}^{(i)T} \mathbf{x}^{(j)} = 0$  for  $\forall i \neq j$ .
- The most common set of vectors that are mutually orthogonal are the *elementary* vectors  $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(n)}$ , which are all zero, except for one element equal to 1, so that  $\mathbf{e}_i^{(i)} = 1$  and  $\mathbf{e}_j^{(i)} = 0, \forall j \neq i$ .

- A *vector space*  $\mathcal{S}$  is a set of vectors that are closed under addition and scalar multiplication, that is
  - if  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are in  $\mathcal{S}$ , then  $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$  is in  $\mathcal{S}$ .
- A vector space  $\mathcal{S}$  is *generated* or *spanned* by a set of vectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ , written as  $\mathcal{S} = \text{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}\}$ , if any vector  $\mathbf{x}$  in the vector space is a linear combination of  $\mathbf{x}_i, i = 1, 2, \dots, n$ .
- A set of linearly independent vectors that generate or span a space  $\mathcal{S}$  is called a *basis* of  $\mathcal{S}$ .

### Example A.1

Let

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x}^{(3)} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Then  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are linearly independent, but  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ , and  $\mathbf{x}^{(3)}$  are linearly dependent since  $5\mathbf{x}^{(1)} - 2\mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}$

## Rank

Some matrix concepts arise from viewing columns or rows of the matrix as vectors. Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .

- $\text{rank}(\mathbf{A})$  is the maximum number of linearly independent rows or columns of a matrix.
- $\text{rank}(\mathbf{A}) \leq \min\{m, n\}$ .
- A matrix is *full rank* if  $\text{rank}(\mathbf{A}) = \min\{m, n\}$ . It is *full row rank* if  $\text{rank}(\mathbf{A}) = m$ . It is *full column rank* if  $\text{rank}(\mathbf{A}) = n$ .
- a square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is *singular* if  $\text{rank}(\mathbf{A}) < n$  and *non-singular* if  $\text{rank}(\mathbf{A}) = n$ .
- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A} \mathbf{A}^T)$ . (Show this in HW.)
- $\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$ . (Hint: Columns of  $\mathbf{AB}$  are spanned by columns of  $\mathbf{A}$  and rows of  $\mathbf{AB}$  are spanned by rows of  $\mathbf{B}$ .)
- if  $\mathbf{Ax} = \mathbf{0}_m$  for some  $\mathbf{x} \neq \mathbf{0}_n$ , then  $\text{rank}(\mathbf{A}) \leq n - 1$ .

## Column space

*Definition:* The column space of a matrix, denoted by  $\mathcal{C}(\mathbf{A})$  is the vector space spanned by the columns of the matrix, that is,

$$\mathcal{C}(\mathbf{A}) = \{\mathbf{x} : \text{there exists a vector } \mathbf{c} \text{ such that } \mathbf{x} = \mathbf{Ac}\}.$$

This means that if  $\mathbf{x} \in \mathcal{C}(\mathbf{A})$ , we can find coefficients  $c_j$  such that

$$\mathbf{x} = \sum_j c_j \mathbf{a}^{(j)}$$

where  $\mathbf{a}^{(j)} = \mathbf{A}_{\cdot j}$  denotes the  $j^{\text{th}}$  column of matrix  $\mathbf{A}$ .

- The column space of a matrix consists of all vectors formed by multiplying that matrix by any vector.
- The number of basis vectors for  $\mathcal{C}(\mathbf{A})$  is then the number of linearly independent columns of the matrix  $\mathbf{A}$ , and so,  $\dim(\mathcal{C}(\mathbf{A})) = \text{rank}(\mathbf{A})$ .
- The dimension of a space is the number of vectors in its basis.

### Example A.2

Let  $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & 3 \end{bmatrix}$  and  $\mathbf{c} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ . Show that  $\mathbf{Ac}$  is a linear combination of columns in  $\mathbf{A}$ .

*solution:*

$$\mathbf{Ac} = \begin{bmatrix} 1 \times 5 + 1 \times 4 + (-3) \times 3 \\ 1 \times 5 + 2 \times 4 + (-1) \times 3 \\ 1 \times 5 + 3 \times 4 + 1 \times 3 \\ 1 \times 5 + 4 \times 4 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}.$$

You could recognize that

$$\mathbf{Ac} = 5 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 4 \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 3 \times \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} = 5\mathbf{a}^{(1)} + 4\mathbf{a}^{(2)} + 3\mathbf{a}^{(3)} = \begin{bmatrix} 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}.$$

### Result A.1

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})).$$

*proof:* Each column of  $\mathbf{AB}$  is a linear combination of columns of  $\mathbf{A}$  (i.e.  $(\mathbf{AB})_{\cdot j} = \mathbf{A}\mathbf{b}^{(j)}$ ), so the number of linearly independent columns of  $\mathbf{AB}$  cannot be greater than that of  $\mathbf{A}$ . Similarly,  $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{B}^T \mathbf{A}^T)$ , the same argument gives  $\text{rank}(\mathbf{B}^T)$  as an upper bound.

### Result A.2

- (a) If  $\mathbf{A} = \mathbf{BC}$ , then  $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{B})$ .
- (b) If  $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{B})$ , then there exists a matrix  $\mathbf{C}$  such that  $\mathbf{A} = \mathbf{BC}$ .

*proof:* For (a), any vector  $\mathbf{x} \in \mathcal{C}(\mathbf{A})$  can be written as  $\mathbf{x} = \mathbf{Ad} = \mathbf{B}(\mathbf{Cd})$ .

For (b),  $\mathbf{A}_{\cdot j} \in \mathcal{C}(\mathbf{B})$ , so that there exists a vector  $\mathbf{c}^{(j)}$  such that  $\mathbf{A}_{\cdot j} = \mathbf{B}\mathbf{c}^{(j)}$ . The matrix  $\mathbf{C} = (\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \dots, \mathbf{c}^{(n)})$  satisfies that  $\mathbf{A} = \mathbf{BC}$ .

## 4 Lecture 4: Jan 22

### Last time

- Linear algebra: vector and vector space, rank of a matrix, Column space (JM Appendix A)

### Today

- Nullspace
- Intro to R

### Null space

*Definition:* The null space of a matrix, denoted by  $\mathcal{N}(\mathbf{A})$ , is  $\mathcal{N}(\mathbf{A}) = \{\mathbf{y} : \mathbf{A}\mathbf{y} = \mathbf{0}\}$ .

#### Result A.3

If  $\mathbf{A}$  has full-column rank, then  $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ .

*proof:* Matrix  $\mathbf{A}$  has full-column rank means its columns are linearly independent, which means that  $\mathbf{A}\mathbf{c} = \mathbf{0}$  implies  $\mathbf{c} = \mathbf{0}$ .

#### Theorem A.1

Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , then  $\dim(\mathcal{C}(\mathbf{A})) = r$  and  $\dim(\mathcal{N}(\mathbf{A})) = n - r$ , where  $r = \text{rank}(\mathbf{A})$ .

See JM Appendix Theorem A.1 for the proof.

*proof:* Denote  $\dim(\mathcal{N}(\mathbf{A}))$  by  $k$ , to be determined, and construct a set of basis vectors for  $\mathcal{N}(\mathbf{A}) : \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(k)}\}$ , so that  $\mathbf{A}\mathbf{u}^{(i)} = \mathbf{0}$ , for  $i = 1, 2, \dots, k$ . Now, construct a basis for  $\mathbb{R}^n$  by adding the vectors  $\{\mathbf{u}^{(k+1)}, \dots, \mathbf{u}^{(n)}\}$ , which are not in  $\mathcal{N}(\mathbf{A})$ . Clearly,  $\mathbf{A}\mathbf{u}^{(i)} \in \mathcal{C}(\mathbf{A})$  for  $i = k + 1, \dots, n$ , and so the span of these vectors form a subspace of  $\mathcal{C}(\mathbf{A})$ . These vectors  $\{\mathbf{A}\mathbf{u}^{(i)}, i = k + 1, \dots, n\}$  are also linearly independent from the following argument: suppose  $\sum_{i=k+1}^n c_i \mathbf{A}\mathbf{u}^{(i)} = \mathbf{0}$ ; then  $\sum_{i=k+1}^n c_i \mathbf{A}\mathbf{u}^{(i)} = \mathbf{A} [\sum_{i=k+1}^n c_i \mathbf{u}^{(i)}] = \mathbf{0}$ , and hence  $\sum_{i=k+1}^n c_i \mathbf{u}^{(i)}$  is a vector in  $\mathcal{N}(\mathbf{A})$ . Therefore, there exist  $b_i$  such that  $\sum_{i=k+1}^n c_i \mathbf{u}^{(i)} = \sum_{i=1}^k b_i \mathbf{u}^{(i)}$ , or  $\sum_{i=1}^k b_i \mathbf{u}^{(i)} - \sum_{i=k+1}^n c_i \mathbf{u}^{(i)} = \mathbf{0}$ . Since  $\{\mathbf{u}^{(i)}\}$  form a basis for  $\mathbb{R}^n$ ,  $c_i$  must all be zero. Therefore  $\mathbf{A}\mathbf{u}^{(i)}, i = k + 1, \dots, n$  are linearly independent. At this point, since  $\text{span}\{\mathbf{A}\mathbf{u}^{(k+1)}, \dots, \mathbf{A}\mathbf{u}^{(n)}\} \subseteq \mathcal{C}(\mathbf{A})$ ,  $\dim(\mathcal{C}(\mathbf{A}))$  is at least  $n - k$ . Suppose there is a vector  $\mathbf{y}$  that is in  $\mathcal{C}(\mathbf{A})$ , but not in the span; then there exists  $\mathbf{u}^{(n+1)}$  so that  $\mathbf{y} = \mathbf{A}\mathbf{u}^{(n+1)}$  and  $\mathbf{u}^{(n+1)}$  is linearly independent of  $\{\mathbf{u}^{(k+1)}, \dots, \mathbf{u}^{(n)}\}$  (and clearly not in  $\mathcal{N}(\mathbf{A})$ ), making  $n + 1$  linearly independent vectors in  $\mathbb{R}^n$ . Since that is not possible, the span is equal to  $\mathcal{C}(\mathbf{A})$  and  $\dim(\mathcal{C}(\mathbf{A})) = n - k = r = \text{rank}(\mathbf{A})$ , so that  $k = \dim(\mathcal{N}(\mathbf{A})) = n - r$ .

Interpretation: “dimension of column space + dimension of null space = # columns”

Mis-Interpretation: Columns space and null space are orthogonal complement to each other. They are of different orders in general!

# Lecture 5, R Basics

## MATH-7260 Linear Models

Dr. Xiang Ji @ Tulane University

January 27, 2025

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## Announcement

- Just stay warm.

## R basics

### styles

(reading assignment)

Checkout Style guide in Advanced R and the tidyverse style guide.

### Arithmetic

R can do any basic mathematical computations.

symbol	use
+	addition
-	subtraction
*	multiplication
/	division
^	power

symbol	use
%%	modulus
exp()	exponent
log()	natural logarithm
sqrt()	square root
round()	rounding
floor()	flooring
ceiling()	ceiling

## Objects

You can create an R object to save results of a computation or other command.

### Example 1

```
x <- 3 + 5
x
```

```
## [1] 8
```

- In most languages, the direction of passing through the value into the object goes from right to left (e.g. with “=”). However, R allows both directions (which is actually bad!). In this course, we encourage the use of “<-” or “=”. There are people liking “=” over “<-” for the reason that “<-” sometimes break into two operators “< -”.

### Example 2

```
x < - 3 + 5
```

```
## [1] FALSE
```

```
x
```

```
## [1] 8
```

- For naming conventions, stick with either “.” or “\_” (refer to the style guide).

### Example 3

```
sum.result <- x + 5
sum.result
```

```
## [1] 13
```

- *important*: many names are already taken for built-in R functions. Make sure that you don’t override them.

### Example 4

```
sum(2:5)
```

```
## [1] 14
```

```
sum
```

```
## function (... , na.rm = FALSE) .Primitive("sum")
```

```
sum <- 3 + 4 + 5
```

```
sum(5:8)
```

```
## [1] 26
```

```
sum
```

```
## [1] 12
```

- R is case-sensitive. “Math.7260” is different from “math.7260”.

### Locating and deleting objects:

The commands “objects()” and “ls()” will provide a list of every object that you’ve created in a session.

```
objects()
```

```
## [1] "sum"          "sum.result" "x"
```

```
ls()
```

```
## [1] "sum"          "sum.result" "x"
```

The “rm()” and “remove()” commands let you delete objects (tip: always clean-up your workspace as the first command)

```
rm(list=ls()) # clean up workspace
```

### Vectors

Many commands in R generate a vector of output, rather than a single number.

The “c()” command: creates a vector containing a list of specific elements.

Example 1

```
c(7, 3, 6, 0)
```

```
## [1] 7 3 6 0
```

```
c(73:60)
```

```
## [1] 73 72 71 70 69 68 67 66 65 64 63 62 61 60
```

```
c(7:3, 6:0)
```

```
## [1] 7 6 5 4 3 6 5 4 3 2 1 0
```

```
c(rep(7:3, 6), 0)
```

```
## [1] 7 6 5 4 3 7 6 5 4 3 7 6 5 4 3 7 6 5 4 3 7 6 5 4 3 0
```

Example 2 The command “seq()” creates a sequence of numbers.

```
seq(7)
```

```
## [1] 1 2 3 4 5 6 7
```

```
seq(3, 70, by = 6)
```

```
## [1] 3 9 15 21 27 33 39 45 51 57 63 69
```

```
seq(3, 70, length = 6)
```

```
## [1] 3.0 16.4 29.8 43.2 56.6 70.0
```

### Operations on vectors

Use brackets to select element of a vector.

```
x <- 73:60
```

```
x[2]
```

```
## [1] 72
```

```
x[2:5]
```

```
## [1] 72 71 70 69
```

```
x[-(2:5)]
```

```
## [1] 73 68 67 66 65 64 63 62 61 60
```

Can access by “name” (safe with column/row order changes)

```
y <- 1:3
```

```
names(y) <- c("do", "re", "mi")
```

```
y[3]
```

```
## mi
```

```
## 3
```

```
y["mi"]
```

```
## mi
```

```
## 3
```

R commands on vectors

command	usage
sum()	sum over elements in vector
mean()	compute average value
sort()	sort elements in a vector
min(), max()	min and max values of a vector
length()	length of a vector
summary()	returns the min, Q1, median, mean, Q3, and max values of a vector
sample(x, size, replace = FALSE, prob = NULL)	takes a random sample from a vector with or without replacement

**Exercise** Write a command to generate a random permutation of the numbers between 1 and 5 and save it to an object.

## Matrix

matrix() command creates a matrix from the given set of values

```
matrix.example <- matrix(rnorm(100), nrow = 10, ncol = 10, byrow = TRUE)
```

```
matrix.example
```

```
##           [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
## [1,]  0.3666982 -1.7669433  1.7641834 -0.61621887  1.4863224 -0.94479077
## [2,] -0.9857926 -1.2483619 -0.1926700  0.15690203  0.5847165 -1.93870114
## [3,]  1.8374926 -0.1019404  0.3921290  1.66280129 -0.1326306  1.02612476
## [4,] -0.1215171  1.7519705  1.7286263  0.56500359  0.0752892  2.00487245
## [5,]  0.3809785 -1.1764969 -1.3117362 -1.77008120  0.4225604  0.38768775
## [6,] -0.8081790  0.3031515  0.2896139  1.29129656  1.7642979 -0.01015978
## [7,] -0.8699963  1.0416742 -0.2101917 -1.21447486 -1.7583827  0.78481029
## [8,]  0.3492486 -1.5457233  0.9106507  0.07285106 -1.4851710  0.09121639
## [9,] -0.6137160  0.2900285  2.1979912 -1.38086049 -1.5609317  0.35626883
## [10,]  1.9593308 -2.1541732  0.1223442  1.71848814 -1.0178962 -1.81570882
##           [,7]      [,8]      [,9]     [,10]
```



```
## [1,] -1.1364874  2.18514605  0.7735370 -0.03889042
## [2,] -0.3107539  0.95673779 -0.2807409 -0.16059619
## [3,] -0.1206051 -0.23887993  2.2026695 -0.79009467
## [4,]  0.2567208 -0.44896466 -1.1802590  0.82425547
## [5,]  1.1109844 -0.05011302 -0.2438188 -0.19231284
## [6,] -0.2056786 -0.01680366 -1.7923752 -0.92163776
## [7,] -0.4687571  0.80052581  1.8038154 -0.24952519
## [8,]  0.1769619 -1.24786462  1.5204527 -1.98957849
## [9,]  0.6492282  0.84311980 -0.6476299 -0.49847309
## [10,] -1.5319584  0.40464881  0.5350702 -0.17699616
```

## R commands on vector/matrix

command	usage
sum()	sum over elements in vector/matrix
mean()	compute average value
sort()	sort all elements in a vector/matrix
min(), max()	min and max values of a vector/matrix
length()	length of a vector/matrix
summary()	returns the min, Q1, median, mean, Q3, and max values of a vector
dim()	dimension of a matrix
cbind()	combine a sequence of vector, matrix or data-frame arguments and combine by columns
rbind()	combine a sequence of vector, matrix or data-frame arguments and combine by rows
names()	get or set names of an object
colnames()	get or set column names of a matrix-like object
rownames()	get or set row names of a matrix-like object

```
sum(matrix.example)
```

```
## [1] 1.488254
```

```
mean(matrix.example)
```

```
## [1] 0.01488254
```

```
sort(matrix.example)
```

```
## [1] -2.15417318 -1.98957849 -1.93870114 -1.81570882 -1.79237520 -1.77008120
## [7] -1.76694327 -1.75838269 -1.56093167 -1.54572329 -1.53195838 -1.48517104
## [13] -1.38086049 -1.31173623 -1.24836192 -1.24786462 -1.21447486 -1.18025905
## [19] -1.17649695 -1.13648736 -1.01789623 -0.98579258 -0.94479077 -0.92163776
## [25] -0.86999626 -0.80817903 -0.79009467 -0.64762991 -0.61621887 -0.61371604
## [31] -0.49847309 -0.46875710 -0.44896466 -0.31075391 -0.28074086 -0.24952519
## [37] -0.24381878 -0.23887993 -0.21019167 -0.20567863 -0.19267004 -0.19231284
## [43] -0.17699616 -0.16059619 -0.13263062 -0.12151707 -0.12060510 -0.10194035
## [49] -0.05011302 -0.03889042 -0.01680366 -0.01015978  0.07285106  0.07528920
## [55]  0.09121639  0.12234418  0.15690203  0.17696188  0.25672075  0.28961388
## [61]  0.29002852  0.30315153  0.34924861  0.35626883  0.36669820  0.38097853
## [67]  0.38768775  0.39212899  0.40464881  0.42256040  0.53507022  0.56500359
```

```
## [73] 0.58471649 0.64922817 0.77353698 0.78481029 0.80052581 0.82425547
## [79] 0.84311980 0.91065069 0.95673779 1.02612476 1.04167419 1.11098445
## [85] 1.29129656 1.48632237 1.52045266 1.66280129 1.71848814 1.72862626
## [91] 1.75197050 1.76418343 1.76429791 1.80381544 1.83749262 1.95933084
## [97] 2.00487245 2.18514605 2.19799120 2.20266949
```

```
summary(matrix.example)
```

```
##           V1           V2           V3           V4
## Min.      :-0.9858 Min.      :-2.1542 Min.      :-1.3117 Min.      :-1.77008
## 1st Qu.   :-0.7596 1st Qu.   :-1.4714 1st Qu.   :-0.1139 1st Qu.   :-1.06491
## Median    : 0.1139 Median    :-0.6392 Median    : 0.3409 Median    : 0.11488
## Mean      : 0.1495 Mean      :-0.4607 Mean      : 0.5691 Mean      : 0.04857
## 3rd Qu.   : 0.3774 3rd Qu.   : 0.2999 3rd Qu.   : 1.5241 3rd Qu.   : 1.10972
## Max.      : 1.9593 Max.      : 1.7520 Max.      : 2.1980 Max.      : 1.71849
##           V5           V6           V7           V8
## Min.      :-1.75838 Min.      :-1.938701 Min.      :-1.5320 Min.      :-1.2479
## 1st Qu.   :-1.36835 1st Qu.   :-0.711133 1st Qu.   :-0.4293 1st Qu.   :-0.1917
## Median    :-0.02867 Median    : 0.223743 Median    :-0.1631 Median    : 0.1939
## Mean      :-0.16218 Mean      :-0.005838 Mean      :-0.1580 Mean      : 0.3188
## 3rd Qu.   : 0.54418 3rd Qu.   : 0.685530 3rd Qu.   : 0.2368 3rd Qu.   : 0.8325
## Max.      : 1.76430 Max.      : 2.004872 Max.      : 1.1110 Max.      : 2.1851
##           V9           V10
## Min.      :-1.7924 Min.      :-1.9896
## 1st Qu.   :-0.5559 1st Qu.   :-0.7172
## Median    : 0.1456 Median    :-0.2209
## Mean      : 0.2691 Mean      :-0.4194
## 3rd Qu.   : 1.3337 3rd Qu.   :-0.1647
## Max.      : 2.2027 Max.      : 0.8243
```

**Exercise** Write a command to generate a random permutation of the numbers between 1 and 5 and save it to an object.

### Comparison (logic operator)

symbol	use
!=	not equal
==	equal
>	greater
>=	greater or equal
<	smaller
<=	smaller or equal
is.na	is it "Not Available"/Missing
complete.cases	returns a logical vector specifying which observations/rows have no missing values
is.finite	if the value is finite
all	are all values in a logical vector true?
any	any value in a logical vector is true?

```
test.vec <- 73:68
test.vec
```

```
## [1] 73 72 71 70 69 68
```

```
test.vec < 70
## [1] FALSE FALSE FALSE FALSE TRUE TRUE
```

```
test.vec > 70
## [1] TRUE TRUE TRUE FALSE FALSE FALSE
```

```
test.vec[3] <- NA
test.vec
```

```
## [1] 73 72 NA 70 69 68
```

```
is.na(test.vec)
```

```
## [1] FALSE FALSE TRUE FALSE FALSE FALSE
```

```
complete.cases(test.vec)
```

```
## [1] TRUE TRUE FALSE TRUE TRUE TRUE
```

```
all(is.na(test.vec))
```

```
## [1] FALSE
```

```
any(is.na(test.vec))
```

```
## [1] TRUE
```

Now let's do a test of accuracy for doubles in R. Recall that for *Double* precision, we get approximately  $\log_{10}(2^{52}) \approx 16$  decimal point for precision.

```
test.exponent <- -(7:18)
10^test.exponent == 0
```

```
## [1] FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE
```

```
1 - 10^test.exponent == 1
```

```
## [1] FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE FALSE TRUE TRUE
```

```
7360 - 10^test.exponent == 7360
```

```
## [1] FALSE FALSE FALSE FALSE FALSE FALSE TRUE TRUE TRUE TRUE TRUE TRUE
```

```
73600 - 10^test.exponent == 73600
```

```
## [1] FALSE FALSE FALSE FALSE FALSE TRUE TRUE TRUE TRUE TRUE TRUE TRUE
```

## Other operators

%in%, match

```
test.vec
```

```
## [1] 73 72 NA 70 69 68
```

```
66 %in% test.vec
```

```
## [1] FALSE
```

```
match(66, test.vec, nomatch = 0)
```

```
## [1] 0
```

```
70 %in% test.vec
```

```
## [1] TRUE
```

```
match(70, test.vec, nomatch = 0)
```

```
## [1] 4
```

```
match(70, test.vec, nomatch = 0) > 0 # the implementation of %in%
```

```
## [1] TRUE
```

## Control flow

These are the basic control-flow constructs of the R language. They function in much the same way as control statements in any Algol-like (Algol short for “Algorithmic Language”) language. They are all reserved words.

keyword	usage
if	<b>if</b> ( <i>cond</i> ) <i>expr</i>
if-else	<b>if</b> ( <i>cond</i> ) <i>cons.expr</i> <b>else</b> <i>alt.expr</i>
for	<b>for</b> ( <i>var in seq</i> ) <i>expr</i>
while	<b>while</b> ( <i>cond</i> ) <i>expr</i>
break	breaks out of a <i>for</i> loop
next	halts the processing of the current iteration and advances the looping index

## Define a function

Read Function section from Advanced R by Hadley Wickham. We will visit functions in more details.

```
DoNothing <- function() {  
  return(invisible(NULL))  
}  
DoNothing()
```

In general, try to avoid using loops (vectorize your code) in R. If you have to loop, try using **for** loops first. Sometimes, **while** loops can be dangerous (however, a smart *compiler* should detect this).

```
DoBadThing <- function() {  
  result <- NULL  
  while(TRUE) {  
    result <- c(result, rnorm(100))  
  }  
  return(result)  
}  
# DoBadThing()
```

## Install packages

You can install R packages from several places (reference):

- Comprehensive R Archive Network (CRAN)
  - Official R packages repository
  - Some levels of code checks (cross platform support, version control etc)
  - Most common place you will install packages

- Pick a mirror location near you
- `install.packages("packge_name")`
- GitHub
  - May get development version of a package
  - Almost zero level of code checks
  - Common place to develop a package before submitting to CRAN

```
install.packages("devtools")
library(devtools)
install_github("tidyverse/ggplot2")
```

## Load packages

```
library(tidyverse)
```

```
## -- Attaching core tidyverse packages ----- tidyverse 2.0.0 --
## v dplyr      1.1.3      v readr      2.1.4
## v forcats    1.0.0      v stringr    1.5.0
## v ggplot2    3.4.3      v tibble     3.2.1
## v lubridate  1.9.2      v tidyr      1.3.0
## v purrr      1.0.2
```

```
## -- Conflicts ----- tidyverse_conflicts() --
```

```
## x dplyr::filter() masks stats::filter()
```

```
## x dplyr::lag()     masks stats::lag()
```

```
## i Use the conflicted package (<http://conflicted.r-lib.org/>) to force all conflicts to become errors
```

```
require(tidyverse)
```

## 6 Lecture 6: Jan 29

### Last time

- R basics

### Today

- Concepts
- Simple Linear Regression (JF Chapter 5)

### Statistics: its objectives and scope (PVR Chapter 1)

We will use the word *statistics* in a broader sense:

Statistics refers to a body of scientific principles and methodologies that are useful for obtaining information about a phenomenon or a large collection of items. Statistical methods are techniques for using limited amounts of information to arrive at conclusions – called statistical inferences – about the phenomenon or the collection of items of interest.

### Population and sample population

A *population* (sometimes referred to as a statistical population) is a collection (or aggregate) of measurements about which an inference is desired.

**Example:** An investigator is interested in evaluating the relationship between age, blood sugar level, and blood cholesterol level of insulin-dependent diabetics who are on a special experimental diet. The investigator wants to answer the following questions, among others:

1. How does the blood cholesterol level change with age and blood sugar level?
2. Are higher cholesterol levels associated with higher sugar levels?
3. Do older diabetics tend to have higher sugar and cholesterol levels?

*What is the population of interest in this example?* The population of interest is a collection of measurements – each of which consists of three values (age, blood sugar level, and blood cholesterol level) – for an insulin-dependent diabetic who is on the experimental diet.

Note that, in statistics, a measurement is one of the elements form the population. In certain populations, each measurement may consist of several values. Populations in which each measurement

- is a single value are called *univariate* populations
- contains more than one value is called a *multivariate* population.

## Sample and sample size

A *sample* consists of a finite number of measurements chosen from a population. The number of measurements in a sample is called the *sample size*.

**Example:** Answers to the questions about associations between the age, blood sugar level, and blood cholesterol level of diabetics can be based on measurements made on a sample of, say,  $n = 40$  treated insulin-dependent diabetics. Such a sample is a collection of 40 measurements, each of which consists of three values: the age, blood sugar level, and blood cholesterol level of a treated patient.

## Statistical components of a research study

A typical research study consists of three stages. The statistical techniques useful in these three stages are commonly known as *statistical methods in research*, and can be divided into three groups:

1. Methods for designing the research study
2. Methods for organizing and summarizing data
3. Methods for making inferences

In this course, we focus on the third stage that is to use the information in the samples to make conclusions about populations (i.e., making inferences). The key statistical issue in such inferences is their accuracy.

**Example:** Suppose that the average indoor radiation level in a sample of 15 homes built on reclaimed phosphate mine lands is 0.032 WL (working level is a historical unit of concentration of radioactive decay products of radon). Then 0.032 WL could be regarded as an estimate of the average indoor level in all homes built on reclaimed phosphate mine lands.

- How accurate is this estimate?
- Suppose there are a total of 4000 homes built on reclaimed lands, is our small sample representative of the whole population?
- If our sample could be regarded as representative of the population, it would be reasonable to expect that the difference between the estimated value of 0.032 WL and the true mean radiation level for all homes will be small, but how do we get/estimate the actual magnitude of this difference?

The natural question is whether it is possible to assess, with reasonable certainty, the magnitude of the error in our estimate. For example, can we say, with a reasonable degree of confidence, that the average level for the population of all homes will be within 0.001 WL of the average value calculated from sample homes?

## Types of populations (PVR Chapter 2.2)

Statistical populations can be classified into categories depending upon the characteristics of the measurements contained in them.

- Univariate and multivariate populations
  - In a *univariate* population, each measurement consists of a single value
  - In a *multivariate* population, measurements consist of more than one value
- Real and conceptual populations
  - The population of 4000 indoor radon levels is a real population
  - The population of digestibility values for sheep fed June-harvested Pensacola Bahia grass is a conceptual population.
- Finite and infinite populations
  - A population may contain only a finite number of measurements, as in the case of the population of indoor radon levels of 4000 homes
  - A population may have infinitely many measurements, as in the case of a conceptual population of potential digestibility measurements, in which every value in the interval  $[0\%, 100\%]$  is a possible value of a measurement in the population.
- Quantitative and qualitative populations
  - A measurement is said to be *quantitative* if its value can be interpreted on a natural and meaningful scale
  - A measurement is *qualitative* if its value serves the sole purpose of identifying an object or a characteristic. The value of a qualitative measurement has no numerical implications.
- Discrete and continuous populations
  - A population is said to be *discrete* if the distinct values of the measurements contained in it can be arranged in a sequence.
  - A continuous population consists of measurements that take all the values in one or more intervals of a real line.



## Simple linear regression

Figure 7.1 shows Davis's data on the measured and reported weight in kilograms of 101 women who were engaged in regular exercise.



Figure 6.1: Scatterplot of Davis's data on the measured and reported weight of 101 women. The dashed line gives  $y = x$ .

It's reasonable to assume that the relationship between measured and reported weight appears to be linear. Denote:

- measured weight by  $y_i$ : **response variable** or **dependent variable**
- reported weight by  $x_i$ : **predictor variable** or **independent variable**
- intercept:  $\beta_0$
- slope:  $\beta_1$
- error term  $\epsilon_i$ .

Then the simple linear regression model writes:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

For given  $(\hat{\beta}_0, \hat{\beta}_1)$  values, the *fitted value* or *predicted value* for observation  $i$  is:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

Therefore, the residual is

$$\hat{\epsilon}_i = y_i - \hat{y}_i$$

## 7 Lecture 7: Jan 31

### Last time

- R basics
- Simple Linear Regression (JF Chapter 5)

### Today

- Simple Linear Regression (JF Chapter 5)

### Simple linear regression

Figure 7.1 shows Davis's data on the measured and reported weight in kilograms of 101 women who were engaged in regular exercise.

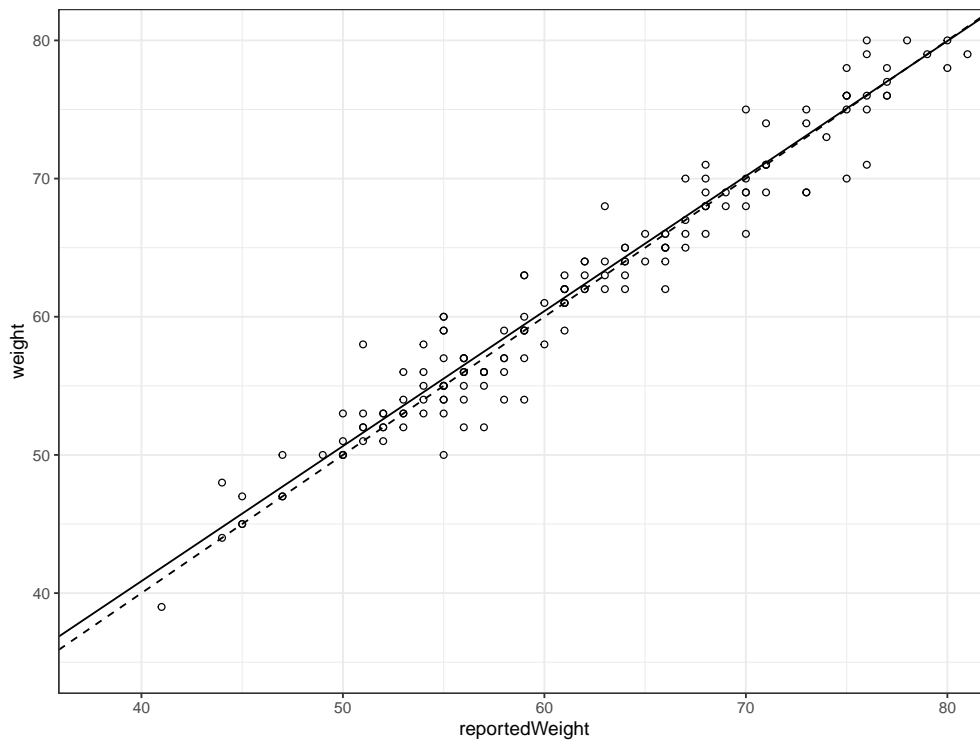


Figure 7.1: Scatterplot of Davis's data on the measured and reported weight of 101 women. The dashed line gives  $y = x$ .

It's reasonable to assume that the relationship between measured and reported weight appears to be linear. Denote:

- measured weight by  $y_i$ : **response variable** or **dependent variable**
- reported weight by  $x_i$ : **predictor variable** or **independent variable**

- intercept:  $\beta_0$
- slope:  $\beta_1$
- residual/error term  $\epsilon_i$ .

Then the simple linear regression model writes:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

For given  $(\hat{\beta}_0, \hat{\beta}_1)$  values, the *fitted value* or *predicted value* for observation  $i$  is:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

Therefore, the residual is

$$\hat{\epsilon}_i = y_i - \hat{y}_i$$

### Fitting a linear model

Choose the “best” values for  $\beta_0, \beta_1$  such that

$$SS[E] = \sum_1^n \left( y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \right)^2 = \sum_1^n (y_i - \hat{y}_i)^2 = \sum_1^n \hat{\epsilon}_i^2$$

is minimized. These are **least squares** (LS) estimates:

$$\begin{aligned}\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ \hat{\beta}_1 &= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}.\end{aligned}$$

*Definition:* The line satisfying the equation

$$y = \hat{\beta}_0 + \hat{\beta}_1 x$$

is called the linear regression of  $y$  on  $x$  which is also called the least squares line.

For Davis’s data, we have

$$\begin{aligned}n &= 101 \\ \bar{y} &= \frac{5780}{101} = 57.228 \\ \bar{x} &= \frac{5731}{101} = 56.743 \\ \sum (x_i - \bar{x})(y_i - \bar{y}) &= 4435.9 \\ \sum (x_i - \bar{x})^2 &= 4539.3,\end{aligned}$$

so that

$$\begin{aligned}\hat{\beta}_1 &= \frac{4435.9}{4539.3} = 0.97722 \\ \hat{\beta}_0 &= 57.228 - 0.97722 \times 56.743 = 1.7776\end{aligned}$$

## Least squares estimates

The simple linear regression (SLR) model writes:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i.$$

The least squares estimates minimizes the sum of squared error (SSE) which is

$$SS[E] = \sum_1^n \left( y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \right)^2 = \sum_1^n (y_i - \hat{y}_i)^2 = \sum_1^n \hat{\epsilon}_i^2.$$

The **least squares** (LS) estimates (in vector form):

$$\hat{\beta}_{ls} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} \bar{y} - \hat{\beta}_1 \bar{x} \\ \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \end{pmatrix}.$$

*Definition:* The line satisfying the equation

$$y = \hat{\beta}_0 + \hat{\beta}_1 x$$

is called the linear regression of  $y$  on  $x$  which is also called the least squares line.

## SLR Model in Matrix Form

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 x_1 \\ \beta_0 + \beta_1 x_2 \\ \vdots \\ \beta_0 + \beta_1 x_n \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

## Jargons

- $\mathbf{X}$  is called the *design matrix*
- $\boldsymbol{\beta}$  is the vector of parameters
- $\boldsymbol{\epsilon}$  is the error vector
- $\mathbf{Y}$  is the response vector.

The Design Matrix

$$\mathbf{X}_{n \times 2} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

Vector of Parameters

$$\boldsymbol{\beta}_{2 \times 1} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

Vector of Error terms

$$\boldsymbol{\epsilon}_{n \times 1} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

Vector of Responses

$$\mathbf{Y}_{n \times 1} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Gramian Matrix

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}$$

Therefore, we have

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}.$$

Assume the Gramian matrix has full rank (which actually should be the case, why?), we want to show that

$$\hat{\boldsymbol{\beta}}_{ls} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}.$$

The inverse of the Gramian matrix is

$$(\mathbf{X}^T \mathbf{X})^{-1} = \frac{1}{n \sum_i (x_i - \bar{x})^2} \begin{bmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{bmatrix}$$

Now we have

$$\begin{aligned}
\hat{\beta}_{ls} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} \\
&= \frac{1}{n \sum_i (x_i - \bar{x})^2} \begin{bmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{bmatrix} \begin{bmatrix} \mathbf{1}_n^T \\ \mathbf{x}^T \end{bmatrix} \mathbf{y} \\
&= \frac{1}{n \sum_i (x_i - \bar{x})^2} \begin{bmatrix} \sum_i x_i^2 & -\sum_i x_i \\ -\sum_i x_i & n \end{bmatrix} \begin{bmatrix} \sum_i y_i \\ \sum_i x_i y_i \end{bmatrix} \\
&= \frac{1}{n \sum_i (x_i - \bar{x})^2} \begin{bmatrix} (\sum_i x_i^2)(\sum_i y_i) - (\sum_i x_i)(\sum_i x_i y_i) \\ n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i) \end{bmatrix} \\
&= \begin{bmatrix} \bar{y} - \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \bar{x} \\ \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \end{bmatrix}
\end{aligned}$$

## 9 Lecture 9: Feb 3

Last time

- Introduction of simple linear regression

Today

- Simple correlation

*From last lecture:* Assume the Gramian matrix has full rank (which actually should be the case, why?)

$$\mathbf{X}^T \mathbf{X} = \begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}$$

*Proof:* By Cauchy-Schwarz inequality, we have

$$n \sum_i x_i^2 \geq (\sum_i x_i)^2$$

where the equality holds only if all  $x_i$  are equal.

Some properties:

- (a)  $\sum x_i \hat{\epsilon}_i = 0$ .
- (b)  $\sum \hat{y}_i \hat{\epsilon}_i = 0$  (HW1).

*Proof:* For (a), we look at

$$\begin{aligned} & \mathbf{X}^T \hat{\epsilon} \\ &= \mathbf{X}^T (\mathbf{Y} - \mathbf{X} \hat{\beta}) \\ &= \mathbf{X}^T [\mathbf{Y} - \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}] \\ &= \mathbf{X}^T \mathbf{Y} - \mathbf{X}^T \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} \\ &= \mathbf{X}^T \mathbf{Y} - \mathbf{X}^T \mathbf{Y} \\ &= \mathbf{0} \end{aligned}$$

### Other quantities in Matrix Form

Fitted values

$$\hat{\mathbf{Y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 + \hat{\beta}_1 x_1 \\ \hat{\beta}_0 + \hat{\beta}_1 x_2 \\ \vdots \\ \hat{\beta}_0 + \hat{\beta}_1 x_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \mathbf{X} \hat{\beta}$$



Hat matrix

$$\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$$

$$\hat{\mathbf{Y}} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$$

$$\hat{\mathbf{Y}} = \mathbf{H}\mathbf{Y}$$

where  $\mathbf{H} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$  is called “hat matrix” because it turns  $\mathbf{Y}$  into  $\hat{\mathbf{Y}}$ .

## Simple correlation

Having calculated the least squares line, it is of interest to determine how closely the line fits the scatter of points. There are many ways of answering it. The standard deviation of the residuals,  $S_E$ , often called the *standard error of the regression* or the *residue standard error*, provides one sort of answer. Because of estimation considerations, the variance of the residuals is defined using *degrees of freedom*  $n - 2$ :

$$S_\epsilon^2 = \frac{\sum \hat{\epsilon}_i^2}{n - 2}.$$

The residual standard error is,

$$S_\epsilon = \sqrt{\frac{\sum \hat{\epsilon}_i^2}{n - 2}}$$

For the Davis’s data, the sum of squared residuals is  $\sum \hat{\epsilon}_i^2 = 418.87$ , and thus the standard error of the regression is

$$S_\epsilon = \sqrt{\frac{418.87}{101 - 2}} = 2.0569\text{kg}.$$

On average, using the least-squares regression line to predict measured weight from reported weight results in an error of about 2 kg.

*Sum of squares:*

- Total sum of squares (TSS) for Y:  $\text{TSS} = \sum (y_i - \bar{y})^2$
- Residual sum of squares (RSS):  $\text{RSS} = \sum (y_i - \hat{y}_i)^2$
- regression sum of squares (RegSS):  $\text{RegSS} = \text{TSS} - \text{RSS} = \sum (\hat{y}_i - \bar{y})^2$
- $\text{RegSS} + \text{RSS} = \text{TSS}$

## 9 Lecture 10: Feb 5

### Last time

- Sum of squares

### Today

- R-square
- Statistical model of SLR

### Sample correlation coefficient

*Definition:* The sample correlation coefficient  $r_{xy}$  of the paired data  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  is defined by

$$r_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y}) / (n - 1)}{\sqrt{\sum (x_i - \bar{x})^2 / (n - 1) \times \sum (y_i - \bar{y})^2 / (n - 1)}} = \frac{s_{xy}}{s_x s_y}$$

$s_{xy}$  is called the sample covariance of  $x$  and  $y$ :

$$s_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n - 1}$$

$s_x = \sqrt{\sum (x_i - \bar{x})^2 / (n - 1)}$  and  $s_y = \sqrt{\sum (y_i - \bar{y})^2 / (n - 1)}$  are, respectively, the sample standard deviations of  $X$  and  $Y$ .

Some properties of  $r_{xy}$ :

- $r_{xy}$  is a measure of the linear association between  $x$  and  $y$  in a dataset.
- correlation coefficients are always between  $-1$  and  $1$ :

$$-1 \leq r_{xy} \leq 1$$

- The closer  $r_{xy}$  is to  $1$ , the stronger the positive linear association between  $x$  and  $y$
- The closer  $r_{xy}$  is to  $-1$ , the stronger the negative linear association between  $x$  and  $y$
- The bigger  $|r_{xy}|$ , the stronger the linear association
- If  $|r_{xy}| = 1$ , then  $x$  and  $y$  are said to be perfectly correlated.
- $\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2} = r_{xy} \frac{s_y}{s_x}$

## R-square

The ratio of RegSS to TSS is called the *coefficient of determination*, or sometimes, simply “r-square”. It represents the proportion of variation observed in the response variable  $y$  which can be “explained” by its linear association with  $x$ .

- In simple linear regression, “r-square” is in fact equal to  $r_{xy}^2$ . (But this isn’t the case in multiple regression.)
- It is also equal to the squared correlation between  $y_i$  and  $\hat{y}_i$ . (This is the case in multiple regression.)

For Davis’s regression of measured on reported weight:

$$\text{TSS} = 4753.8$$

$$\text{RSS} = 418.87$$

$$\text{RegSS} = 4334.9$$

Thus,

$$r^2 = \frac{4334.9}{4753.8} = 1 - \frac{418.87}{4753.8} = 0.9119$$

## The statistical model of Simple Linear Regression

Standard statistical inference in simple regression is based on a *statistical model* that describes the population or process that is sampled:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where the coefficients  $\beta_0$  and  $\beta_1$  are the *population regression parameters*. The data are randomly sampled from some population of interest.

- $y_i$  is the value of the response variable
- $x_i$  is the explanatory variable
- $\epsilon_i$  represents the aggregated omitted causes of  $y$  (i.e., the causes of  $y$  beyond the explanatory variable), other explanatory variables that could have been included in the regression model, measurement error in  $y$ , and whatever component of  $y$  is inherently random.

## Key assumptions of SLR

The key assumptions of the SLR model concern the behavior of the errors, equivalently, the distribution of  $y$  conditional on  $x$ :

- *Linearity*. The expectation of the error given the value of  $x$  is 0:  $\mathbf{E}(\epsilon) \equiv \mathbf{E}(\epsilon|x_i) = 0$ . And equivalently, the expected value of the response variable is a linear function of the explanatory variable:  $\mu_i \equiv \mathbf{E}(y_i) \equiv \mathbf{E}(y_i|x_i) = \mathbf{E}(\beta_0 + \beta_1 x_i + \epsilon_i|x_i) = \beta_0 + \beta_1 x_i$ .

- *Constant variance.* The variance of the errors is the same regardless of the value of  $x$ :  $\mathbf{Var}(\epsilon|x_i) = \sigma_\epsilon^2$ . The constant error variance implies constant conditional variance of  $y$  on given  $x$ :  $\mathbf{Var}(y|x_i) = \mathbf{E}((y_i - \mu_i)^2) = \mathbf{E}((y_i - \beta_0 - \beta_1 x_i)^2) = \mathbf{E}(\epsilon_i^2) = \sigma_\epsilon^2$ . (Question: why the last equal sign?)
- *Normality.* The errors are independent identically distributed with Normal distribution with mean 0 and variance  $\sigma_\epsilon^2$ . Write as  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$ . Equivalently, the conditional distribution of the response variable is normal:  $y_i \stackrel{iid}{\sim} N(\beta_0 + \beta_1 x_i, \sigma_\epsilon^2)$ .
- *Independence.* The observations are sampled independently.
- *Fixed  $X$ , or  $X$  measured without error and independent of the error.*
  - For experimental research where  $X$  values are under direct control of the researcher (i.e.  $X$ 's are fixed). If the experiment were replicated, then the values of  $X$  would remain the same.
  - For research where  $X$  values are sampled, we assume the explanatory variable is measured without error and the explanatory variable and the error are independent in the population from which the sample is drawn.
- *$X$  is not invariant.*  $X$ 's can not be all the same.

Figure 9.1 shows the assumptions of linearity, constant variance, and normality in SLR model.



Figure 9.1: The assumptions of linearity, constant variance, and normality in simple regression. The graph shows the conditional population distributions  $\Pr(Y|x)$  of  $Y$  for several values of the explanatory variable  $X$ , labeled as  $x_1, x_2, \dots, x_5$ . The conditional means of  $Y$  given  $x$  are denoted  $\mu_1, \dots, \mu_5$ .

## 11 Lecture 11: Feb 10

### Last time

- Introduction of simple linear regression

### Today

- HW2 posted
- The statistical model of the SLR (JF chapter 6)
- Properties of the Least-Squares estimator
- Inference of SLR model

### Properties of the Least-Squares estimator

Under the strong assumptions of the simple linear regression model, the least squares coefficients  $\hat{\beta}_{ls}$  have several desirable properties as estimators of the population regression coefficients  $\beta_0$  and  $\beta_1$ :

- The least-squares intercept and slope are *linear estimators*, in the sense that they are linear functions of the observations  $y_i$ .

*Proof:*

method (a)  $\hat{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$

method (b)  $\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\sum (x_i - \bar{x})y_i}{\sum (x_i - \bar{x})^2} - \frac{\sum (x_i - \bar{x})\bar{y}}{\sum (x_i - \bar{x})^2} = \sum \frac{(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} y_i = \sum k_i y_i$  where

$$k_i = \frac{(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}$$

and  $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

- The simple least-squares coefficients are *unbiased estimators* of the population regression coefficients:

$$\mathbf{E}(\hat{\beta}_0) = \beta_0$$

$$\mathbf{E}(\hat{\beta}_1) = \beta_1$$

*Proof:*

method (a)  $\mathbf{E}(\hat{\beta}) = \mathbf{E}((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}) = \mathbf{E}((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \beta) = \beta$ . (note:  $\mathbf{E}(Y) = \mathbf{E}(\mathbf{X}\beta + \epsilon) = \mathbf{E}(\mathbf{X}\beta) + \mathbf{E}(\epsilon) = \mathbf{X}\beta$ )

method (b) recall that  $\hat{\beta}_1 = \sum k_i y_i$  where  $k_i = \frac{(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}$ . First, we want to show

$$1. \sum k_i = 0$$

$$2. \sum k_i x_i = 1$$

They are actually quite easy:  $\sum k_i = \sum_i \frac{(x_i - \bar{x})}{\sum_j (x_j - \bar{x})^2} = \frac{(\sum_i x_i) - n\bar{x}}{\sum_j (x_j - \bar{x})^2} = 0$ , and  $\sum k_i x_i = \sum_i \frac{(x_i - \bar{x})x_i}{\sum_j (x_j - \bar{x})^2} = \frac{(\sum_i x_i^2) - \bar{x}(\sum_i x_i)}{\sum_j (x_j - \bar{x})^2} = \frac{(\sum_i x_i^2) - n\bar{x}^2}{\sum_j (x_j - \bar{x})^2} = 1$ .

Now  $\mathbf{E}(\hat{\beta}_1) = \mathbf{E}(\sum k_i y_i) = \sum [k_i \mathbf{E}(y_i)] = \sum [k_i(\beta_0 + \beta_1 x_i)] = \beta_0 \sum k_i + \beta_1 \sum (k_i x_i) = \beta_1$ , and  $\mathbf{E}(\hat{\beta}_0) = \mathbf{E}(\bar{y} - \hat{\beta}_1 \bar{x}) = \mathbf{E}(\bar{y}) - \bar{x} \mathbf{E}(\hat{\beta}_1) = \mathbf{E}(\frac{1}{n} \sum y_i) - \bar{x} \beta_1 = \frac{1}{n} [\sum \mathbf{E}(y_i)] - \bar{x} \beta_1 = \frac{1}{n} \sum [\beta_0 + x_i \beta_1] - \bar{x} \beta_1 = \beta_0$

- Both  $\hat{\beta}_0$  and  $\hat{\beta}_1$  have simple sampling variances:

$$\text{Var}(\hat{\beta}_0) = \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2}$$

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2}$$

*Proof:*

$$\text{Var}(\hat{\beta}_1) = \text{Var}(\sum k_i y_i) = \sum k_i^2 \text{Var}(y_i) = \sigma_\epsilon^2 \sum k_i^2 = \sigma_\epsilon^2 \frac{\sum_i (x_i - \bar{x})^2}{[\sum_j (x_j - \bar{x})^2]^2} = \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2}, \text{ and}$$

$$\text{Var}(\hat{\beta}_0) = \text{Var}(\bar{y} - \hat{\beta}_1 \bar{x}) = \text{Var}(\bar{y}) + (\bar{x})^2 \text{Var}(\hat{\beta}_1) - 2\bar{x} \text{Cov}(\bar{Y}, \hat{\beta}_1).$$

Now,

$$\text{Var}(\bar{y}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n y_i\right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(y_i) = \frac{\sigma^2}{n},$$

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2},$$

and

$$\begin{aligned} \text{Cov}(\bar{Y}, \hat{\beta}_1) &= \text{Cov}\left\{\frac{1}{n} \sum_{i=1}^n Y_i, \frac{\sum_{j=1}^n (x_j - \bar{x}) Y_j}{\sum_{i=1}^n (x_i - \bar{x})^2}\right\} \\ &= \frac{1}{n \sum_{i=1}^n (x_i - \bar{x})^2} \text{Cov}\left\{\sum_{i=1}^n Y_i, \sum_{j=1}^n (x_j - \bar{x}) Y_j\right\} \\ &= \frac{1}{n \sum_{i=1}^n (x_i - \bar{x})^2} \sum_{i=1}^n (x_j - \bar{x}) \sum_{j=1}^n \text{Cov}(Y_i, Y_j) \\ &= \frac{1}{n \sum_{i=1}^n (x_i - \bar{x})^2} \sum_{i=1}^n (x_j - \bar{x}) \sigma^2 \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} \text{Var}(\hat{\beta}_0) &= \frac{\sigma^2}{n} + \frac{\sigma^2 \bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\sigma^2}{n \sum_{i=1}^n (x_i - \bar{x})^2} \left\{ \sum_{i=1}^n (x_i - \bar{x})^2 + n \bar{x}^2 \right\} \\ &= \frac{\sigma^2 \sum_{i=1}^n x_i^2}{n \sum_{i=1}^n (x_i - \bar{x})^2}. \end{aligned}$$

## 12 Lecture 12 Feb 12

Last time

- Properties of the Least-Squares estimator

Today

- Statistical inference of the SLR model
- Multiple linear regression

### Properties of the Least-Squares estimator

- Rewrite the formula for  $\text{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{(n-1)S_X^2}$ , we see that the sampling variance of the slope estimate will be small when
  - The error variance  $\sigma_\epsilon^2$  is small
  - The sample size  $n$  is large
  - The explanatory-variable values are spread out (i.e., have a large variance,  $S_X^2$ )

- (Gauss-Markov theorem) Under the assumptions of linearity, constant variance, and independence, the least-squares estimators are BLUE (Best Linear Unbiased Estimator), that is they have the smallest sampling variance and are unbiased. (show this)

*Proof:*

Let  $\tilde{\beta}_1$  be another linear unbiased estimator such that  $\tilde{\beta}_1 = \sum c_i y_i$ . For  $\tilde{\beta}_1$  is still unbiased as above,  $\mathbf{E}(\tilde{\beta}_1) = \beta_0 \sum c_i + \beta_1 \sum c_i x_i = \beta_1$  for all  $\beta_1$ , we have  $\sum c_i = 0$  and  $\sum c_i x_i = 1$ .

$$\mathbf{Var}(\tilde{\beta}_1) = \sigma_\epsilon^2 \sum c_i^2$$

Let  $c_i = k_i + d_i$ , then

$$\begin{aligned}\mathbf{Var}(\tilde{\beta}_1) &= \sigma_\epsilon^2 \sum (k_i + d_i)^2 \\ &= \sigma_\epsilon^2 \left[ \sum k_i^2 + \sum d_i^2 + 2 \sum k_i d_i \right] \\ &= \mathbf{Var}(\hat{\beta}_1) + \sigma_\epsilon^2 \sum d_i^2 + 2\sigma_\epsilon^2 \sum k_i d_i\end{aligned}$$

Now we show the last term is 0 to finish the proof.

$$\begin{aligned}\sum k_i d_i &= \sum k_i (c_i - k_i) = \sum c_i k_i - \sum k_i^2 \\ &= \sum_i \left[ c_i \frac{x_i - \bar{x}}{\sum_j (x_j - \bar{x})^2} \right] - \frac{1}{\sum_i (x_i - \bar{x})^2} \\ &= 0\end{aligned}$$

- Under the full suite of assumptions, the least-squares coefficients  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are the maximum-likelihood estimators of  $\beta_0$  and  $\beta_1$ . (show this)

*Proof:*

The log likelihood under the full suite of assumptions is  $\ell = -\log \left[ (2\pi)^{\frac{n}{2}} \sigma_\epsilon^n \right] - \frac{1}{2\sigma_\epsilon^2} (\mathbf{Y} - \mathbf{X}\beta)^T (\mathbf{Y} - \mathbf{X}\beta)$ . Maximizing the likelihood is equivalent as minimizing  $(\mathbf{Y} - \mathbf{X}\beta)^T (\mathbf{Y} - \mathbf{X}\beta) = \epsilon^T \epsilon$  which is the SSE.

- Under the assumption of normality, the least-squares coefficients are themselves normally distributed. Summing up,

$$\hat{\beta}_0 \sim N\left(\beta_0, \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2}\right)$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2}\right)$$



## 14 Lecture 14 Feb 17

Last time

- Inference of SLR model

Today

- Confidence intervals for SLR
- Review Lab 1 and 2
- Multiple linear regression

### Statistical inference of the SLR model

Now we have the distribution of  $\hat{\beta}_0$  and  $\hat{\beta}_1$

$$\begin{aligned}\hat{\beta}_0 &\sim N\left(\beta_0, \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2}\right) \\ \hat{\beta}_1 &\sim N\left(\beta_1, \frac{\sigma_\epsilon^2}{\sum (x_i - \bar{x})^2}\right).\end{aligned}$$

However,  $\sigma_\epsilon$  is never known in practice. Instead, an *unbiased* estimator of  $\sigma_\epsilon^2$  is given by

$$\hat{\sigma}_\epsilon^2 = MS[E] = \frac{SS[E]}{n-2}.$$

show that  $\mathbf{E}(\sum (y_i - \hat{y}_i)^2) = \sigma_\epsilon^2(n-2)$ .

*Proof:*

$$MS[E] = \frac{\sum (y_i - \hat{y}_i)^2}{n-2},$$

we want to show  $\mathbf{E}(\sum (y_i - \hat{y}_i)^2) = \sigma_\epsilon^2(n-2)$ .

LHS:  $\mathbf{E}(\sum (y_i - \hat{y}_i)^2) = \sum_i [\mathbf{E}(y_i - \hat{y}_i)^2]$

$$\text{and } E[(y_i - \hat{y}_i)^2] = \text{Var}(y_i - \hat{y}_i) + [\mathbf{E}(y_i - \hat{y}_i)]^2 = \text{Var}(y_i - \hat{y}_i) = \text{Var}(y_i) + \text{Var}(\hat{y}_i) - 2\text{cov}(y_i, \hat{y}_i)$$

$$\text{Var}(y_i) = \sigma_\epsilon^2$$

$$\text{Var}(\hat{y}_i) = \text{Var}(\bar{y} + \hat{\beta}_1(x_i - \bar{x}))$$

$$= \text{Var}(\bar{y}) + (x_i - \bar{x})^2 \text{Var}(\hat{\beta}_1) + 2(x_i - \bar{x}) \text{Cov}(\bar{y}, \hat{\beta}_1)$$

$$\text{Cov}(\bar{y}, \hat{\beta}_1) = \text{Cov}(\bar{y}, \sum k_i y_i)$$

$$= \sum_i \text{Cov}(\bar{y}, k_i y_i)$$

$$= \sum_i \frac{k_i}{n} \text{Var}(y_i)$$

$$= \frac{1}{n} \sum k_i$$

$$= 0$$

$$\therefore \text{Var}(\hat{y}_i) = \text{Var}(\bar{y}) + (x_i - \bar{x})^2 \text{Var}(\hat{\beta}_1)$$

$$= \frac{1}{n} \sigma_\epsilon^2 + \frac{\sigma_\epsilon^2 (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2}$$

$$= \sigma_\epsilon^2 \left[ \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]$$

Now, we derive the last term  $\text{cov}(y_i, \hat{y}_i)$ :

$$\begin{aligned} \text{cov}(y_i, \hat{y}_i) &= \text{cov}(y_i, \bar{y} + \hat{\beta}_1(x_i - \bar{x})) \\ &= \text{cov}(y_i, \frac{1}{n} \sum_j y_j + (x_i - \bar{x}) \sum_j k_j y_j) \\ &= \text{cov}(y_i, \sum_j \left[ \frac{1}{n} + (x_i - \bar{x}) k_j \right] y_j) \\ &= \sigma_\epsilon^2 \left[ \frac{1}{n} + (x_i - \bar{x}) k_i \right] \\ &= \sigma_\epsilon^2 \left[ \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right] \end{aligned}$$

Therefore, we have for  $i$ th residue

$$\begin{aligned} \text{Var}(y_i - \hat{y}_i) &= \text{Var}(y_i) + \text{Var}(\hat{y}_i) - 2\text{cov}(y_i, \hat{y}_i) \\ &= \sigma_\epsilon^2 + \sigma_\epsilon^2 \left[ \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right] - 2\sigma_\epsilon^2 \left[ \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right] \\ &= \sigma_\epsilon^2 \left[ 1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right]. \end{aligned}$$

And finally, sum over  $i$  we get

$$\sum_i \text{Var}(y_i - \hat{y}_i) = \sigma_\epsilon^2 \sum_i \left[ 1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \right] = (n - 2) \sigma_\epsilon^2$$

## 15 Lecture 15 Feb 19

Last time

- Confidence intervals for SLR

Today

- Review Lab 1 and 2
- Multiple linear regression

### Confidence intervals

Now we substitute  $\hat{\sigma}_\epsilon^2$  into the distribution of  $\hat{\beta}_0$  and  $\hat{\beta}_1$

$$\begin{aligned}\hat{\beta}_1 &\sim N(\beta_1, \frac{\sigma_\epsilon^2}{\sum(x_i - \bar{x})^2}) \\ \hat{\beta}_0 &\sim N(\beta_0, \frac{\sigma_\epsilon^2 \sum x_i^2}{n \sum(x_i - \bar{x})^2})\end{aligned}$$

to get the estimated standard errors:

$$\begin{aligned}\widehat{SE}(\hat{\beta}_1) &= \sqrt{\frac{MS[E]}{\sum(x_i - \bar{x})^2}} \\ \widehat{SE}(\hat{\beta}_0) &= \sqrt{MS[E] \left( \frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i - \bar{x})^2} \right)}\end{aligned}$$

And the  $100(1 - \alpha)\%$  confidence intervals for  $\beta_1$  and  $\beta_0$  are given by

$$\begin{aligned}\hat{\beta}_1 \pm t(n - 2, \alpha/2) \sqrt{\frac{MS[E]}{S_{xx}}} \\ \hat{\beta}_0 \pm t(n - 2, \alpha/2) \sqrt{MS[E] \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)}\end{aligned}$$

where  $S_{xx} = \sum(x_i - \bar{x})^2$

### Confidence interval for $\mathbf{E}(Y|X = x_0)$

The conditional mean  $\mathbf{E}(Y|X = x_0)$  can be estimated by evaluating the regression function  $\mu(x_0)$  at the estimates  $\hat{\beta}_0, \hat{\beta}_1$ . The conditional variance of the expression isn't too difficult (already shown):

$$\text{Var}(\hat{\beta}_0 + \hat{\beta}_1 x_0 | X = x_0) = \sigma^2 \left( \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}} \right)$$

This leads to a confidence interval of the form

$$\hat{\beta}_0 + \hat{\beta}_1 x_0 \pm t(n - 2, \alpha/2) \sqrt{MS[E] \left( \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}} \right)}$$

## Prediction interval

Often, prediction of the response variable  $Y$  for a given value, say  $x_0$ , of the independent variable of interest. In order to make statements about future values of  $Y$ , we need to take into account

- the sampling distribution of  $\hat{\beta}_0$  and  $\hat{\beta}_1$
- the randomness of a future value  $Y$ .

We have seen the predicted value of  $Y$  based on the linear regression is given by  $\hat{Y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ .

The 95% prediction interval has the form

$$\hat{Y}_0 \pm t(n-2, \alpha/2) \sqrt{MS[E] \left( 1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}} \right)}.$$

## Hypothesis test

To test the hypothesis  $H_0 : \beta_1 = \beta_{slope}$  that the population slope is equal to a specific value  $\beta_{slope}$  (most commonly, the null hypothesis has  $\beta_{slope} = 0$ ), we calculate the test statistic ( $T$ -statistics) with  $df = n - 2$

$$t_0 = \frac{\hat{\beta}_1 - \beta_{slope}}{\widehat{SE}(\hat{\beta}_1)} \sim t_{n-2}$$

## Some questions to answer using regression analysis:

1. What is the meaning, in words, of  $\beta_1$ ?  
*Answer:*  $\beta_1$  is the population slope parameter of the SLR model that represents the amount of increase in the mean of the response variable with a unit increase of the explanatory variable.
2. True/False: (a)  $\beta_1$  is a statistic (b)  $\beta_1$  is a parameter (c)  $\beta_1$  is unknown.  
*Answer:* (a) False (b) True (C) True. In reality, the true population parameters are almost never known. However, in simulation studies, we do know them.
3. True/False: (a)  $\hat{\beta}_1$  is a statistic (b)  $\hat{\beta}_1$  is a parameter (c)  $\hat{\beta}_1$  is unknown  
*Answer:* (a) True (b) False (C) False.  $\hat{\beta}_1$  is an estimate of the population parameter  $\beta_1$ .
4. Is  $\hat{\beta}_1 = \beta_1$  ?  
*Answer:* No. However,  $\mathbf{E}(\hat{\beta}_1) = \beta_1$

## Multiple linear regression

JF 5.2+6.2

### Multiple linear regression - an example

An example on the prestige, education, and income levels of 45 U.S. occupations (Duncan's data):

	income	education	prestige
accountant	62	86	82
pilot	72	76	83
architect	75	92	90
author	55	90	76
chemist	64	86	90
minister	21	84	87
professor	64	93	93
dentist	80	100	90
reporter	67	87	52
engineer	72	86	88
lawyer	76	98	89
teacher	48	91	73

“prestige” represents the percentage of respondents in a survey who rated an occupation as “good” or “excellent” in prestige, “education” represents the percentage of incumbents in the occupation in the 1950 U.S. Census who were high school graduates, and “income” represents the percentage of occupational incumbents who earned incomes in excess of \$3,500.

Using the `pairs` command in R, we can look at the pairwise scatter plot between the three variables as in Figure 15.1.

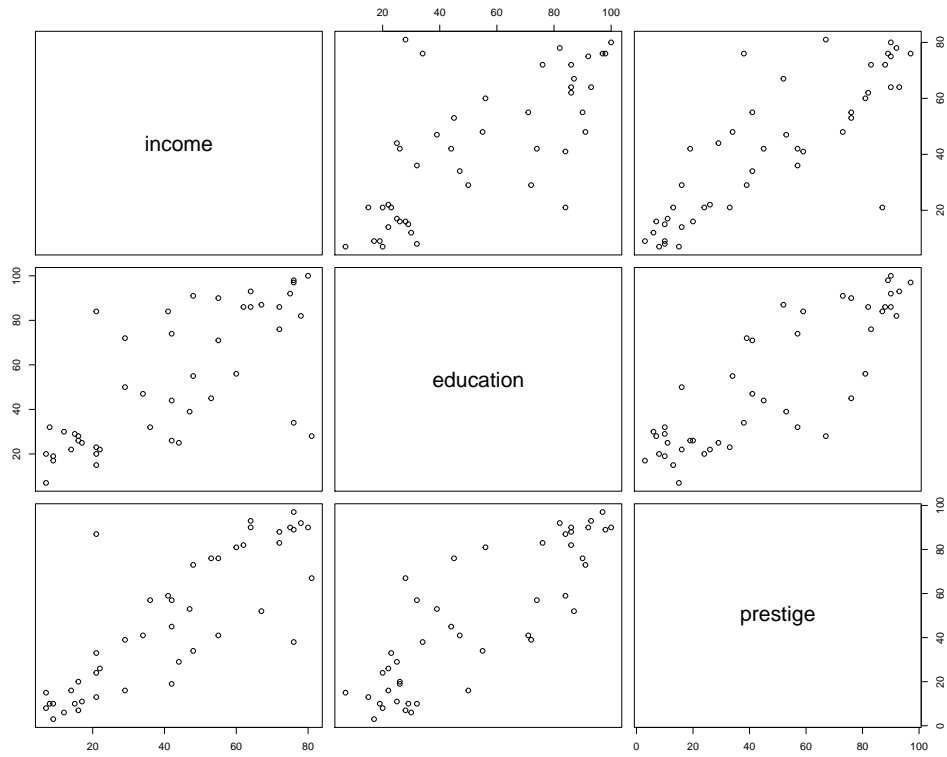


Figure 15.1: Scatterplot matrix for occupational prestige, level of education, and level of income of 45 U.S. occupations in 1950.

Consider a regression model for the “prestige” of occupation  $i$ ,  $Y_i$ , in which the mean of  $Y_i$  is a linear function of two predictor variables  $X_{i1} = \text{income}$ ,  $X_{i2} = \text{education}$  for occupations  $i = 1, 2, \dots, 45$ :

$$Y = \beta_0 + \beta_1 \text{income} + \beta_2 \text{education} + \text{error}$$

or

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i$$

or

$$Y_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \epsilon_1$$

$$Y_2 = \beta_0 + \beta_1 X_{21} + \beta_2 X_{22} + \epsilon_2$$

$$\vdots = \vdots$$

$$Y_{45} = \beta_0 + \beta_1 X_{45,1} + \beta_2 X_{45,2} + \epsilon_{45}$$

## 16 Lecture 16 Feb 24

### Last time

- Confidence intervals for SLR
- Multiple linear regression

### Today

- Multiple correlation
- Lab review

## A multiple linear regression (MLR) model with $p$ independent variables

Let  $p$  independent variables be denoted by  $x_1, \dots, x_p$ .

- Observed values of  $p$  independent variables for  $i^{th}$  subject from sample denoted by  $x_{i1}, \dots, x_{ip}$
- response variable for  $i^{th}$  subject denoted by  $Y_i$
- For  $i = 1, \dots, n$ , MLR model for  $Y_i$ :

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip} + \epsilon_i$$

- As in SLR,  $\epsilon_1, \dots, \epsilon_n \stackrel{iid}{\sim} N(0, \sigma^2)$

Least squares estimates of regression parameters minimize  $SS[E]$ :

$$SS[E] = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i1} - \dots - \beta_p x_{ip})^2$$

$$\hat{\sigma}^2 = \frac{SS[E]}{n-p-1}$$

Interpretations of regression parameters:

- $\sigma^2$  is unknown error variance parameter
- $\beta_0, \beta_1, \dots, \beta_p$  are  $p + 1$  unknown regression parameters:
  - $\beta_0$ : average response when  $x_1 = x_2 = \dots = x_p = 0$
  - $\beta_i$  is called a partial slope for  $x_i$ . Represents mean change in  $y$  per unit increase in  $x_i$  *with all other independent variables held fixed*.

## Matrix formulation of MLR

Let a vector for  $p$  observed independent variables for individual  $i$  be defined by

$$\mathbf{x}_{i\cdot} = (1, x_{i1}, x_{i2}, \dots, x_{ip}).$$

The MLR model for  $Y_1, \dots, Y_n$  is given by

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \dots + \beta_p X_{1p} + \epsilon_1 \\ Y_2 &= \beta_0 + \beta_1 X_{21} + \beta_2 X_{22} + \dots + \beta_p X_{2p} + \epsilon_2 \\ &\vdots \\ Y_n &= \beta_0 + \beta_1 X_{n1} + \beta_2 X_{n2} + \dots + \beta_p X_{np} + \epsilon_n \end{aligned}$$

This system of  $n$  equations can be expressed using matrices:

$$\boxed{\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}}$$

where

- $\mathbf{Y}$  denotes a response vector of size  $n \times 1$
- $\mathbf{X}$  denotes a design matrix of size  $n \times (p + 1)$
- $\boldsymbol{\beta}$  denotes a vector of regression parameters of size  $(p + 1) \times 1$
- $\boldsymbol{\epsilon}$  denotes an error vector of size  $n \times 1$

Here, the error vector  $\boldsymbol{\epsilon}$  is assumed to follow a multivariate normal distribution with variance-covariance matrix  $\sigma^2 \mathbf{I}_n$ . For individual  $i$ ,

$$y_i = \mathbf{x}_{i\cdot} \boldsymbol{\beta} + \epsilon_i.$$

Some simplified expressions: ( $\mathbf{a}$  is a known  $(p + 1) \times 1$  vector)

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} \\ \text{Var}(\hat{\boldsymbol{\beta}}) &= \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1} \\ &= \boldsymbol{\Sigma} \\ \widehat{\text{Var}}(\hat{\boldsymbol{\beta}}) &= MS[E] (\mathbf{X}^T \mathbf{X})^{-1} \\ &= \hat{\boldsymbol{\Sigma}} \\ \widehat{\text{Var}}(\mathbf{a}^T \hat{\boldsymbol{\beta}}) &= \mathbf{a}^T \hat{\boldsymbol{\Sigma}} \mathbf{a} \end{aligned}$$

*Question:* what are the dimensions of each of these quantities?

- $(\mathbf{X}^T \mathbf{X})^{-1}$  may be verbalized as “ $\mathbf{x}$  transposed  $\mathbf{x}$  inverse”
- $\hat{\boldsymbol{\Sigma}}$  is the estimated variance-covariance matrix for the estimate of the regression parameter vector  $\hat{\boldsymbol{\beta}}$



- $\mathbf{X}$  is assumed to be of full *rank*.

Some more simplified expressions:

$$\begin{aligned}
 \hat{\mathbf{Y}} &= \mathbf{X}\hat{\boldsymbol{\beta}} \\
 &= \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
 &= \mathbf{H}\mathbf{Y} \\
 \hat{\boldsymbol{\epsilon}} &= \mathbf{Y} - \hat{\mathbf{Y}} \\
 &= \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} \\
 &= (\mathbf{I} - \mathbf{H})\mathbf{Y}
 \end{aligned}$$

- $\hat{\mathbf{Y}}$  is called the vector of fitted or predicted values
- $\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$  is called the hat matrix
- $\hat{\boldsymbol{\epsilon}}$  is the vector of residuals

For the Duncan's data example on income, education and prestige, with  $p = 2$  independent variables and  $n = 45$  observations,

$$\mathbf{X} = \begin{bmatrix} 1 & 62 & 86 \\ 1 & 72 & 76 \\ \vdots & \vdots & \vdots \\ 1 & 8 & 32 \end{bmatrix}$$

and

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} 45 & 1884 & 2365 \\ 1884 & 105148 & 122197 \\ 2365 & 122197 & 163265 \end{bmatrix}$$

$$(\mathbf{X}^T\mathbf{X})^{-1} = \begin{bmatrix} 0.10211 & -0.00085 & -0.00084 \\ -0.00085 & 0.00008 & -0.00005 \\ -0.00084 & -0.00005 & 0.00005 \end{bmatrix}$$

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} = \begin{bmatrix} -6.0646629 \\ 0.5987328 \\ 0.5458339 \end{bmatrix} = ?$$

$$SS[E] = \hat{\boldsymbol{\epsilon}}^T\hat{\boldsymbol{\epsilon}} = (\mathbf{Y} - \hat{\mathbf{Y}})^T(\mathbf{Y} - \hat{\mathbf{Y}}) = 7506.7$$

$$MS[E] = \frac{SS[E]}{df} = \frac{7506.7}{45 - 2 - 1} = 178.73$$

$$\hat{\boldsymbol{\Sigma}} = MS[E](\mathbf{X}^T\mathbf{X})^{-1} = \begin{bmatrix} 18.249481 & -0.151845008 & -0.150706025 \\ -0.151845 & 0.014320275 & -0.008518551 \\ -0.150706 & -0.008518551 & 0.009653582 \end{bmatrix}$$

### Multiple correlation, JF 5.2.3

The sums of squares in multiple regression are defined in the same manner as in SLR:

$$\begin{aligned}TSS &= \sum (Y_i - \bar{Y})^2 \\RegSS &= \sum (\hat{Y}_i - \bar{Y})^2 \\RSS &= \sum (Y_i - \hat{Y}_i)^2 = \sum \hat{\epsilon}_i^2\end{aligned}$$

Not surprisingly, we have a similar analysis of variance for the regression:

$$TSS = RegSS + RSS$$

The squared multiple correlation  $R^2$ , representing the proportion of variation in the response variable captured by the regression, is defined in terms of the sums of squares:

$$R^2 = \frac{RegSS}{TSS} = 1 - \frac{RSS}{TSS}.$$

Because there are several slope coefficients, potentially with different signs, the *multiple correlation coefficient* is, by convention, the positive square root of  $R^2$ . The multiple correlation is also interpretable as the simple correlation between the fitted and observed  $Y$  values, i.e.,  $r_{\hat{Y}Y}$ .

#### Adjusted- $R^2$

Because the multiple correlation can only rise, never decline, when explanatory variables are added to the regression equation (HW1), investigators sometimes penalize the value of  $R^2$  by a “correction” for degrees of freedom. The corrected (or “adjusted”)  $R^2$  is defined as:

$$\begin{aligned}R_{adj}^2 &= 1 - \frac{\frac{RSS}{n-p-1}}{\frac{TSS}{n-1}} \\&= 1 - \left[ \frac{(1 - R^2)(n - 1)}{n - p - 1} \right]\end{aligned}$$

### Confidence intervals

Confidence intervals and hypothesis tests for individual coefficients closely follow the pattern of simple-regression analysis:

1. substitute an estimate of the error variance (MSE) for the unknown  $\sigma^2$  into the variance term of  $\hat{\beta}_i$
2. find the estimated standard error of a slope coefficient  $\widehat{SE}(\hat{\beta}_i)$
3.  $t = \frac{\hat{\beta}_i - \beta_i}{\widehat{SE}(\hat{\beta}_i)}$  follows a  $t$ -distribution with degrees of freedom as associated with SSE.

Therefore, we can construct the  $100(1 - \alpha)\%$  confidence interval for a single slope parameter by (why?):

$$\hat{\beta}_i \pm t(n - p - 1, \alpha/2) \widehat{SE}(\hat{\beta}_i)$$

*Hand-waving proof:*

we know that  $t = \frac{\hat{\beta}_i - \beta_i}{\widehat{SE}(\hat{\beta}_i)} \sim t_{n-p-1}$ , such that

$$\begin{aligned} 1 - \alpha &= \Pr(-t_c < t < t_c) \\ &= \Pr\left(t_c < \frac{\hat{\beta}_i - \beta_i}{\widehat{SE}(\hat{\beta}_i)} < t_c\right) \\ &= \Pr\left(\hat{\beta}_i - t_c \cdot \widehat{SE}(\hat{\beta}_i) < \beta_i < \hat{\beta}_i + t_c \cdot \widehat{SE}(\hat{\beta}_i)\right) \end{aligned}$$

where  $t_c = t(n - p - 1, \alpha/2)$  is the critical value.

## 17 Lecture 17: February 26

### Last time

- Multiple linear regression

### Today

- HW1 review
- Lab review
- Dummy-Variable regression
- Interactions

## 18 Lecture 18: March 10

### Last time

- HW1 review
- Lab review

### Today

- Dummy-Variable regression
- Interactions
- Unusual and influential data (JF chapter 11)

### Hypothesis tests

We first test the null hypothesis that all population regression slopes are 0:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_p = 0$$

The test statistics,

$$F = \frac{RegSS/p}{RSS/(n - p - 1)}$$

follows an  $F$ -distribution with  $p$  and  $n - p - 1$  degrees of freedom.

We can also test a null hypothesis about a *subset* of the regression slopes, e.g.,

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_q = 0.$$

Or more generally, test the null hypothesis

$$H_0 : \beta_{q_1} = \beta_{q_2} = \cdots = \beta_{q_k} = 0$$

where  $0 \leq q_1 < q_2 < \cdots < q_k \leq p$  is a subset of  $k$  indices. To get the  $F$ -statistic for this case, we generally perform the following steps:

1. Fit the *full* (“unconstrained”) model, in other words, model that provides context for  $H_0$ . Record  $SSR_{full}$  and the associated  $df_{full}$
2. Fit the *reduced* (“constrained”) model, in other words, full model constrained by  $H_0$ . Record  $SSR_{red}$  and the associated  $df_{red}$
3. Calculate the  $F$ -statistic by

$$F = \frac{[SSR_{red} - SSR_{full}]/(df_{red} - df_{full})}{SSR_{full}/df_{full}}$$

4. Find  $p$ -value (the probability of observing an  $F$ -statistic that is at least as high as the value that we obtained) by consulting an  $F$ -distribution with numerator  $df(ndf) = df_{red} - df_{full}$  and denominator  $df(ddf) = df_{full}$ . Notation:  $F_{ndf,ddf}$ , see Figure 18.1.



Figure 18.1: An example for  $p$ -value for F-statistic value 2.57 with an  $F_{3,246}$  distribution

## Dummy-variable regression

For categorical data (factor), we use dummy variable regression:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \epsilon_i$$

where  $D$ , called a dummy variable regressor or an indicator variable, is coded 1 for one level and 0 for all others,

$$D_i = \begin{cases} 1 & \text{for men} \\ 0 & \text{for women} \end{cases}.$$

Therefore, for women, the model becomes

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

and for men

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 + \epsilon_i = (\beta_0 + \beta_2) + \beta_1 X_i + \epsilon_i$$

For example, Figure 18.2 (a) and (b) represents two small (idealized) populations. In both cases, the within-gender regressions of income on education are parallel. Parallel regressions imply additive effects of education and gender on income: Holding education constant, the “effect” of gender is the vertical distance between the two regression lines, which, for parallel lines, is everywhere the same.



Figure 18.2: Idealized data representing the relationship between income and education for populations of men (filled circles) and women (open circles). In (a), there is no relationship between education and gender; in (b), women have a higher average level of education than men. In both (a) and (b), the within-gender (i.e., partial) regressions (solid lines) are parallel. In each graph, the overall (i.e. marginal) regression of income on education (ignoring gender) is given by the broken line. JF Figure 7.1.

### Multi-level factor

We can model the effects of classification factors with  $m$  categories (levels) by using  $m - 1$  indicator variables.

For example, the three-category occupational-type factor can be represented in the regression equation by introducing two dummy regressors:

Category	$D_1$	$D_2$
Professional and managerial	1	0
White collar	0	1
Blue collar	0	0

A model for the regression of prestige on income, education, and type of occupation is then

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \gamma_1 D_{i1} + \gamma_2 D_{i2} + \epsilon_i$$

where  $X_1$  is income and  $X_2$  is education. This model describes three parallel regression planes, which can differ in their intercepts:

$$\begin{aligned} \text{Professional:} \quad Y_i &= (\beta_0 + \gamma_1) + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \\ \text{White collar:} \quad Y_i &= (\beta_0 + \gamma_2) + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \\ \text{Blue collar:} \quad Y_i &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \end{aligned}$$

Therefore, the coefficient  $\beta_0$  gives the intercept for blue-collar occupations;  $\gamma_1$  represents the constant vertical difference between the parallel regression planes for professional and blue-collar occupations (fixing the values of education and income); and  $\gamma_2$  represents the constant vertical distance between the regression planes for white-collar and blue-collar occupations (again, fixing education and income).

In the above prestige example, we chose “blue collar” as the baseline category. Sometimes, it is natural to pick a particular category as the baseline category, for example, the “control group” in an experiment. However, in most applications, the choice of a baseline category is entirely arbitrary.

### Matrix representation

For the above prestige model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \gamma_1 D_{i1} + \gamma_2 D_{i2} + \epsilon_i$$

we have the design matrix  $\mathbf{X}$  as

$$\mathbf{X} = \begin{bmatrix} 1 & X_{11} & X_{12} & D_{11} & D_{12} \\ 1 & X_{21} & X_{22} & D_{21} & D_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & D_{n1} & D_{n2} \end{bmatrix}$$

and the vector of coefficients  $\beta$  is

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$

such that we have (again) the linear model in matrix form:

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

where  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$ , in other words,  $\epsilon \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ .

### Interactions

Two explanatory variables are said to interact in determining a response variable when the partial effect of one depends on the value of the other. Consider the hypothetical data shown in Figure 18.3.



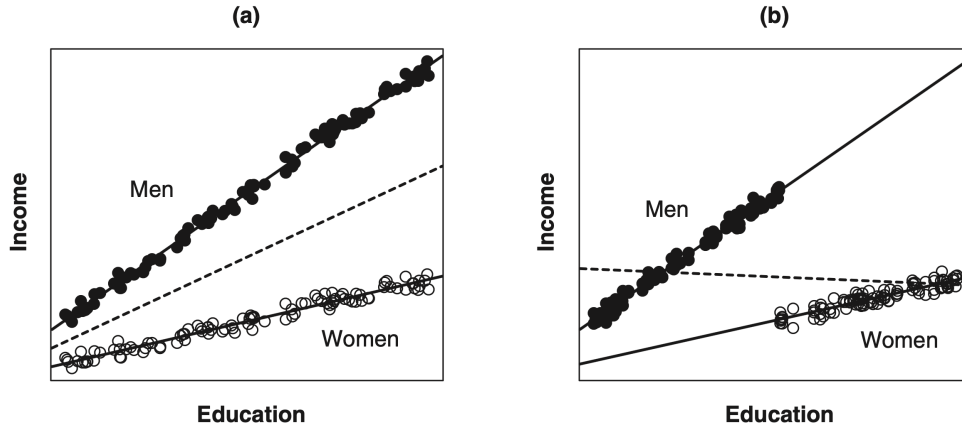


Figure 18.3: Idealized data representing the relationship between income and education for populations of men (filled circles) and women (open circles). In (a), there is no relationship between education and gender; in (b), women have a higher average level of education than men. In both (a) and (b), the within-gender (i.e., partial) regressions (solid lines) are not parallel. The slope for men is greater than the slope for women, and consequently education and gender interact in affecting income. In each graph, the overall regression of income on education (ignoring gender) is given by the broken line. JF Figure 7.7.

It is apparent in both Figure 18.3 (a) and (b) the within-gender regressions of income on education are not parallel: In both cases, the slope for men is larger than the slope for women.

### Modeling interactions

We accommodate the interaction of education and gender by:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 (X_i D_i) + \epsilon_i$$

where we introduce the interaction regressor  $XD$  into the regression equation. For women, the model becomes

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \beta_2 \cdot 0 + \beta_3 (X_i \cdot 0) + \epsilon_i \\ &= \beta_0 + \beta_1 X_i + \epsilon_i \end{aligned}$$

and for men

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \beta_2 \cdot 1 + \beta_3 (X_i \cdot 1) + \epsilon_i \\ &= (\beta_0 + \beta_2) + (\beta_1 + \beta_3) X_i + \epsilon_i \end{aligned}$$

The parameters  $\beta_0$  and  $\beta_1$  are, respectively, the intercept and slope for the regression of income on education among women (the baseline category for gender);  $\beta_2$  gives the difference in intercepts between the male and female groups; and  $\beta_3$  gives the difference in slopes between the two groups.

*Usual guidance:* Models that include an interaction between two predictors should also include the individual predictors by themselves regardless of the statistical significance of the associated  $\beta$ 's.

### Test for the interaction

We can simply test the hypothesis  $H_0 : \beta_3 = 0$  and construct the test statistic  $t = \frac{\hat{\beta}_i - 0}{\widehat{SE}(\hat{\beta}_i)} \sim t_{n-4}$  ( $p = 3$ ).

### Interactions with multi-level factor

We can easily extend the method for modeling interactions by forming product regressors to multi-level factors, to several factors, and to several quantitative explanatory variables. Using the occupational prestige example, the occupational type could possibly interact both with income ( $X_1$ ) and with education ( $X_2$ ):

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \gamma_1 D_{i1} + \gamma_2 D_{i2} \\ + \delta_{11} X_{i1} D_{i1} + \delta_{12} X_{i1} D_{i2} + \delta_{21} X_{i2} D_{i1} + \delta_{22} X_{i2} D_{i2} + \epsilon_i$$

The model therefore permits different intercepts and slopes for the three types of occupations:

Professional:	$Y_i =$	$(\beta_0 + \gamma_1) +$	$(\beta_1 + \delta_{11})X_{i1} +$	$(\beta_2 + \delta_{21})X_{i2} +$	$\epsilon_i$
White collar:	$Y_i =$	$(\beta_0 + \gamma_2) +$	$(\beta_1 + \delta_{12})X_{i1} +$	$(\beta_2 + \delta_{22})X_{i2} +$	$\epsilon_i$
Blue collar:	$Y_i =$	$\beta_0 +$	$\beta_1 X_{i1} +$	$\beta_2 X_{i2} +$	$\epsilon_i$

## 19 Lecture 19: March 12

### Last time

- Dummy-Variable regression
- Interaction

### Today

- Midterm exam starts next Friday
- Unusual and influential data

### Unusual and influential data

Linear models make strong assumptions about the structure of data, assumptions that often do not hold in applications. The method of least squares can be very sensitive to the structure of the data and may be markedly influenced by one or a few unusual observations.

### Outliers

In simple regression analysis, an outlier is an observation whose response-variable value is *conditionally* unusual *given* the value of the explanatory variable: see Figure 19.1.



Figure 19.1: The black point is a regression outlier because it combines a relatively large value of  $Y$  with a relatively small value of  $X$ , even though neither its  $X$ -value nor its  $Y$ -value is unusual individually. Because of the positive relationship between  $Y$  and  $X$ , points with small  $X$ -values also tend to have small  $Y$ -values, and thus the black point is far from other points with similar  $X$ -values. JF Figure 11.1.

Unusual data are problematic in linear models fit by least squares because they can unduly

influence the results of the analysis. Their presence may be a signal that the model fails to capture important characteristics of the data.

Figure 19.2 illustrates some distinctions for the simple-regression model  $Y = \beta_0 + \beta_1 X + \epsilon$ .

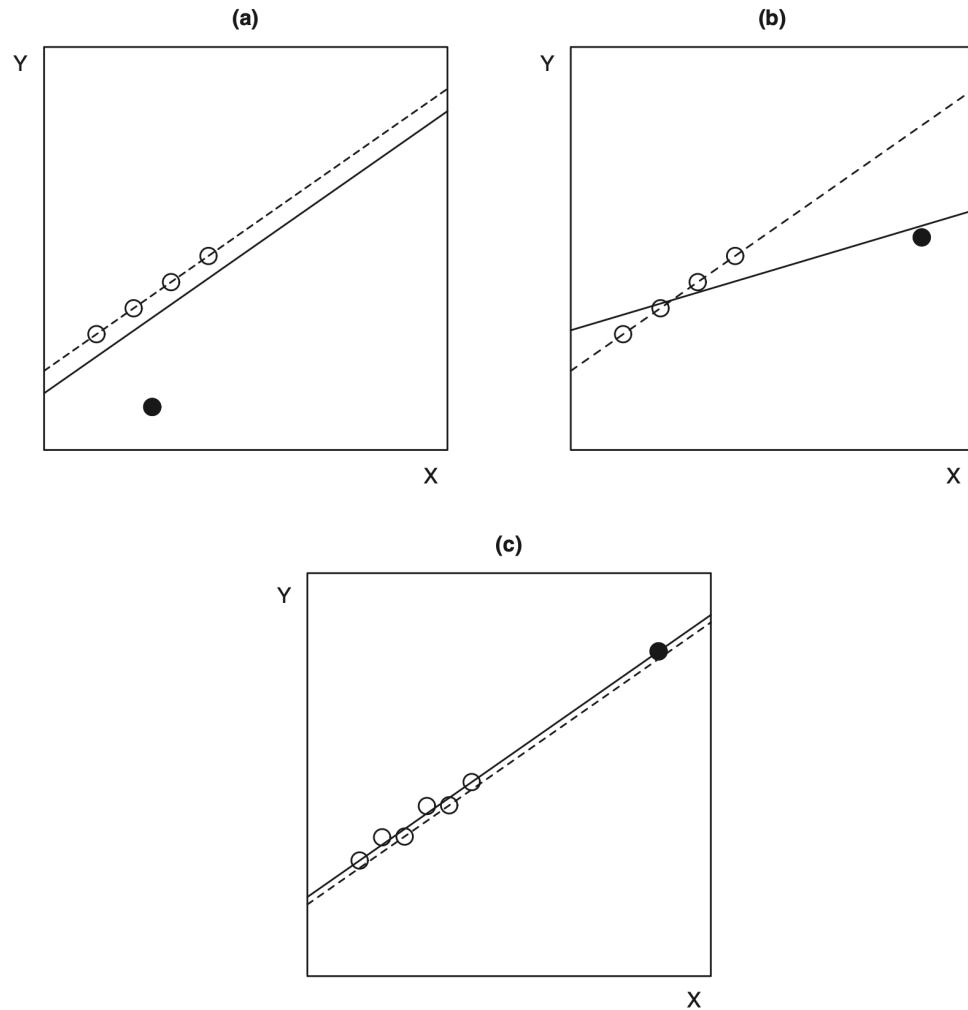


Figure 19.2: Leverage and influence in simple regression. In each graph, the solid line gives the least-squares regression for all the data, while the broken line gives the least-squares regression with the unusual data point (the black circle) omitted. (a) An outlier near the mean of  $X$  has low leverage and little influence on the regression coefficients. (b) An outlier far from the mean of  $X$  has high leverage and substantial influence on the regression coefficients. (c) A high-leverage observation in line with the rest of the data does not influence the regression coefficients. In panel (c), the two regression lines are separated slightly for visual effect but are, in fact, coincident JF Figure 11.2.

Some qualitative distinctions between outliers and high leverage observations:

- An outlier is a data point whose response  $Y$  does not follow the general trend of the rest of the data.

- A data point has high leverage if it has “extreme” predictor  $X$  values:
  - With a single predictor, an extreme  $X$  value is simply one that is particularly high or low.
  - With multiple predictors, extreme  $X$  values may be particularly high or low for one or more predictors, or may be “unusual” combinations of predictor values.

And the influence of a data point is the combination of leverage and discrepancy (“outlyingness”) through the following heuristic formula:

$$\text{Influence on coefficients} = \text{Leverage} \times \text{Discrepancy}.$$

### Assessing leverage: hat-values

The hat-value  $h_i$  is a common measure of leverage in regression. They are named because it is possible to express the fitted values  $\hat{Y}_j$  (“Y-hat”) in terms of the observed values  $Y_i$ :

$$\hat{Y}_j = h_{1j}Y_1 + h_{2j}Y_2 + \cdots + h_{jj}Y_j + \cdots + h_{nj}Y_n = \sum_{i=1}^n h_{ij}Y_i.$$

The weight  $h_{ij}$  captures the contribution of observation  $Y_i$  to the fitted value  $\hat{Y}_j$ : If  $h_{ij}$  is large, then the  $i$ th observation can have a considerable impact on the  $j$ th fitted value. With the least square solutions, for the fitted values:

$$\hat{\mathbf{Y}} = \mathbf{X}\boldsymbol{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}$$

we (already) get the hat matrix:

$$\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$$

*Properties:*

- (idempotent)  $\mathbf{H} = \mathbf{H}\mathbf{H}$
- $h_i \equiv h_{ii} = \sum_{j=1}^n h_{ij}^2$
- $\frac{1}{n} \leq h_i \leq 1$  ([a proof](#) by Mohammad Mohammadi)
- $\bar{h} = (p+1)/n$

In the case of SLR, the hat-values are:

$$h_i = \frac{1}{n} + \frac{(X_i - \bar{X})^2}{\sum_{j=1}^n (X_j - \bar{X})^2}$$

### Detecting outliers: studentized residuals

The variance of the residuals ( $\hat{\epsilon}_i = Y_i - \hat{Y}_i$ ) do not have equal variances (even if the errors  $\epsilon_i$  have equal variances):

$$\text{Var}(\hat{\epsilon}) = \text{Var}(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) = \text{Var}[(\mathbf{I} - \mathbf{H})\mathbf{Y}] = (\mathbf{I} - \mathbf{H})\text{Var}(\mathbf{Y})(\mathbf{I} - \mathbf{H}) = \sigma^2(\mathbf{I} - \mathbf{H})$$

so that for  $\hat{\epsilon}_i$ ,

$$\text{Var}(\hat{\epsilon}_i) = \sigma^2(1 - h_i).$$

High-leverage observations tend to have small residuals (in other words, these observations can pull the regression surface toward them).

The standardized residual (sometimes called internally studentized residual)

$$\hat{\epsilon}'_i \equiv \frac{\hat{\epsilon}_i}{\hat{\sigma}\sqrt{1 - h_i}},$$

however, does not follow a  $t$ -distribution, because the numerator and denominator are not independent.

Suppose, we refit the model deleting the  $i$ th observation, obtaining an estimate  $\hat{\sigma}_{(-i)}$  of  $\sigma$  that is based on the remaining  $n - 1$  observations. Then the studentized residual (sometimes called externally studentized residual)

$$\hat{\epsilon}^*_i \equiv \frac{\hat{\epsilon}_i}{\hat{\sigma}_{(-i)}\sqrt{1 - h_i}}$$

has an independent numerator and denominator and follows a  $t$ -distribution with  $n - p - 2$  degrees of freedom.

The studentized and the standardized residuals have the following relationship (Beckman and Trussell, 1974):

$$\hat{\epsilon}^*_i = \hat{\epsilon}'_i \sqrt{\frac{n - p - 2}{n - p - 1 - \hat{\epsilon}'^2_i}}$$

For large  $n$ ,

$$\hat{\epsilon}^*_i \approx \hat{\epsilon}'_i \approx \frac{\hat{\epsilon}}{\hat{\sigma}}$$

### Test for outlier

It is of our interest to pick the studentized residual  $\hat{\epsilon}^*_{max}$  with the largest absolute value among  $\hat{\epsilon}^*_1, \hat{\epsilon}^*_2, \dots, \hat{\epsilon}^*_n$  to test for outlier. However, by doing so, we are effectively picking the biggest of  $n$  test statistics such that it is not legitimate simply to use  $t_{n-p-2}$  to find a  $p$ -value. We need a correction on the  $p$ -value because of multiple-comparisons.

Suppose that we have  $p' = \Pr(t_{n-p-2} > |\hat{\epsilon}^*_{max}|)$ , the  $p$ -value before correction. Then the Bonferroni adjusted  $p$ -value is  $p = np'$ .

## Measuring influence

Influence on the regression coefficients combines leverage and discrepancy. The most direct measure of influence simply expresses the impact on each coefficient of deleting each observation in turn:

$$D_{ij} = \hat{\beta}_j - \tilde{\beta}_{j(-i)} \quad \text{for } i = 1, \dots, n \text{ and } j = 0, 1, \dots, p$$

where  $\hat{\beta}_j$  are the least-squares coefficients calculated for all the data, and the  $\tilde{\beta}_{j(-i)}$  are the least-squares coefficients calculated with the  $i$ th observation omitted. To assist in interpretation, it is useful to scale the  $D_{ij}$  by (deleted) coefficient standard errors:

$$D_{ij}^* = \frac{D_{ij}}{\widehat{SE}_{(-i)}(\tilde{\beta}_{j(-i)})}$$

Following Belsley, Kuh, and Welsh (1980), the  $D_{ij}$  are often termed  $\text{DFBETA}_{ij}$ , and  $D_{ij}^*$  are called  $\text{DFBETAS}_{ij}$ . One problem associated with using  $D_{ij}$  or  $D_{ij}^*$  is their large number:  $n(p+1)$  of each.

Cook's distance is another popular measure, calculated as

$$D_i = \frac{\sum_{j=1}^n (\tilde{y}_{j(-i)} - \hat{y}_j)^2}{(p+1)\hat{\sigma}^2} = \frac{\hat{\epsilon}_i'^2}{p+1} \times \frac{h_i}{1-h_i}$$

In effect, the first term in the formula for Cook's  $D$  is a measure of discrepancy, and the second is a measure of leverage. We look for values of  $D_i$  that stand out from the rest.

A similar measure suggested by Belsley et al. (1980)

$$\text{DFFITS}_i = \hat{\epsilon}_i^* \frac{h_i}{1-h_i}$$

Except for unusual data configurations, Cook's  $D_i \approx \text{DFFITS}_i^2/(p+1)$ .

## Numerical cutoffs (suggested)

Diagnostic statistic	Cutoff value
$h_i$	$2\bar{h} = \frac{2(p+1)}{n}$ , ( $3\bar{h}$ for small sample)
$D_{ij}^*$	$ D_{ij}^*  > 1$ or $2$ ( $2/\sqrt{n}$ for large samples)
Cook's $D_i$	$D_i > \frac{4}{n-p-1}$
DFFITS	$ \text{DFFITS}_i  > 2\sqrt{\frac{p+1}{n-p-1}}$

## 20 Lecture 20: March 17

### Last time

- Unusual and influential data (JF chapter 11)

### Today

- Anonymous internal midterm evaluations on canvas
- Added-variable plots
- Should unusual data be discarded
- Non-normality

### Added-variable plots

Unlike the case of SLR, the scatterplot with the response variable and one predictor gives only the marginal effect in MLR. Instead, the added-variable plot (also called a partial-regression plot or a partial-regression leverage plot) gives a graphical inspection over each dimension.

Let  $\tilde{Y}_i^{(1)}$  represent the residuals from the least-squares regression of  $Y$  on all the  $X$ s except  $X_1$ , in other words, the residuals from the following fitted regression equation:

$$Y_i = \tilde{\beta}_0^{(1)} + \tilde{\beta}_2^{(1)} X_{i2} + \cdots + \tilde{\beta}_p^{(1)} X_{ip} + \tilde{Y}_i^{(1)}$$

where the parenthetical superscript (1) indicates the omission of  $X_1$  from the right-hand side of the regression equation. Likewise,  $\tilde{X}_i^{(1)}$  is the residual from the least-squares regression of  $X_1$  on all the other  $X$ s:

$$X_{i1} = \tilde{\beta}_0^{(1)} + \tilde{\beta}_2^{(1)} X_{i2} + \cdots + \tilde{\beta}_p^{(1)} X_{ip} + \tilde{X}_i^{(1)}$$

Then, the residuals  $\tilde{Y}_i^{(1)}$  and  $\tilde{X}_i^{(1)}$  have the following interesting properties:

1. The slope from the least-squares regression of  $\tilde{Y}_i^{(1)}$  on  $\tilde{X}_i^{(1)}$  is simply the least-squares slope  $\hat{\beta}_1$  from the *full* multiple regression.
2. The residuals from the simple regression of  $\tilde{Y}_i^{(1)}$  on  $\tilde{X}_i^{(1)}$  are the same as those from the full regression, that is

$$\tilde{Y}_i^{(1)} = \hat{\beta}_1 \tilde{X}_i^{(1)} + \hat{\epsilon}_i$$

3. The variation of  $\tilde{X}_i^{(1)}$  is the *conditional variation* of  $X_1$  holding the other  $X$ s constant.

Figure 20.1 shows that the conditional variation is smaller than its marginal variation – much smaller when  $X_1$  is strongly collinear with other  $X$ s,



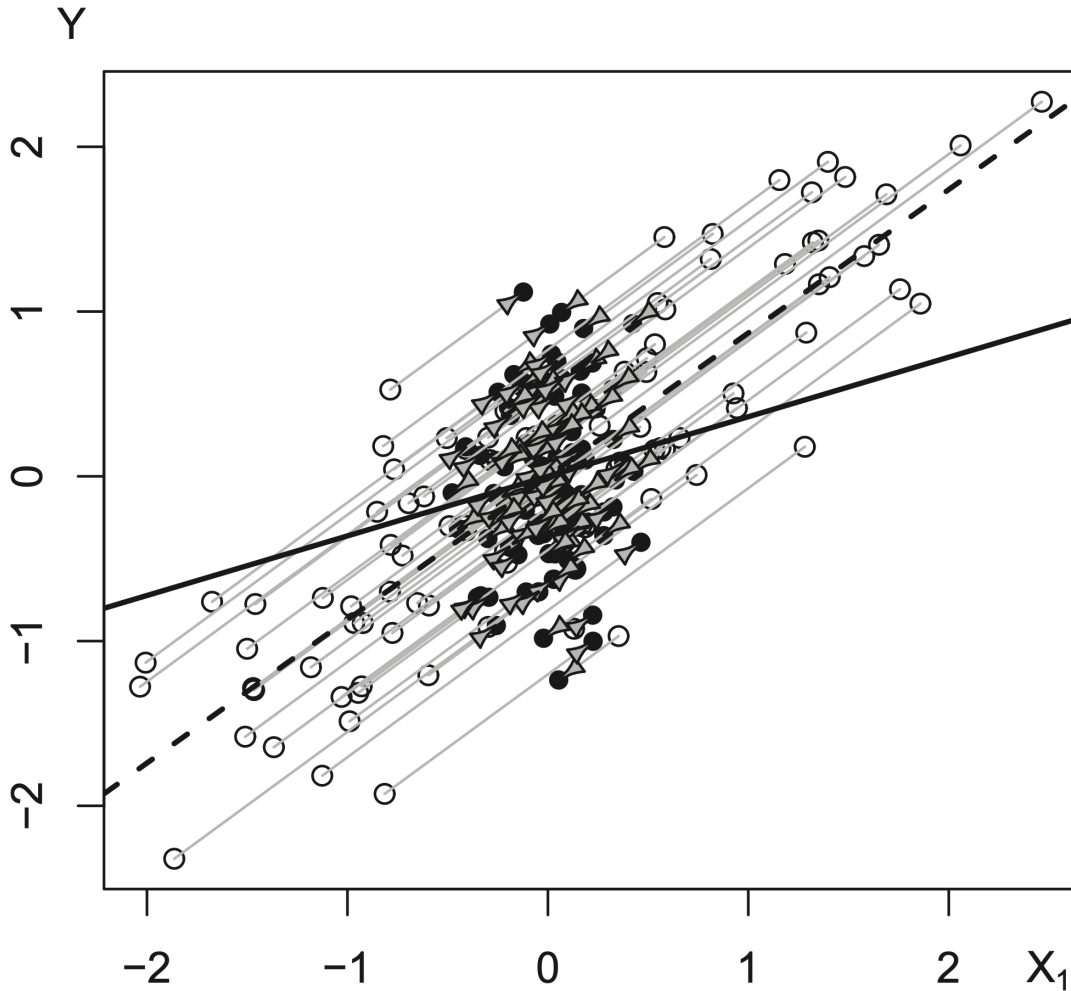


Figure 20.1: The marginal scatterplot (open circles) for  $Y$  and  $X_1$  superimposed on the added-variable plot (filled circles) for  $X_1$  in the regression of  $Y$  on  $X_1$  and  $X_2$ . The variables  $Y$  and  $X_1$  are centered at their means to facilitate the comparison of the two sets of points. The arrows show how the points in the marginal scatterplot map into those in the AV plot. In this contrived data set,  $X_1$  and  $X_2$  are highly correlated ( $r_{12} = 0.98$ ), and so the conditional variation in  $X_1$  (represented by the horizontal spread of the filled points) is much less than its marginal variation (represented by the horizontal spread of the open points). The broken line gives the slope of the marginal regression of  $Y$  on  $X_1$  alone, while the solid line gives the slope  $\hat{\beta}_1$  of  $X_1$  in the MLR of  $Y$  on both  $X$ s. JF Figure 11.9.

Figure 20.2 illustrates the added-variable plots using the Duncan's data.

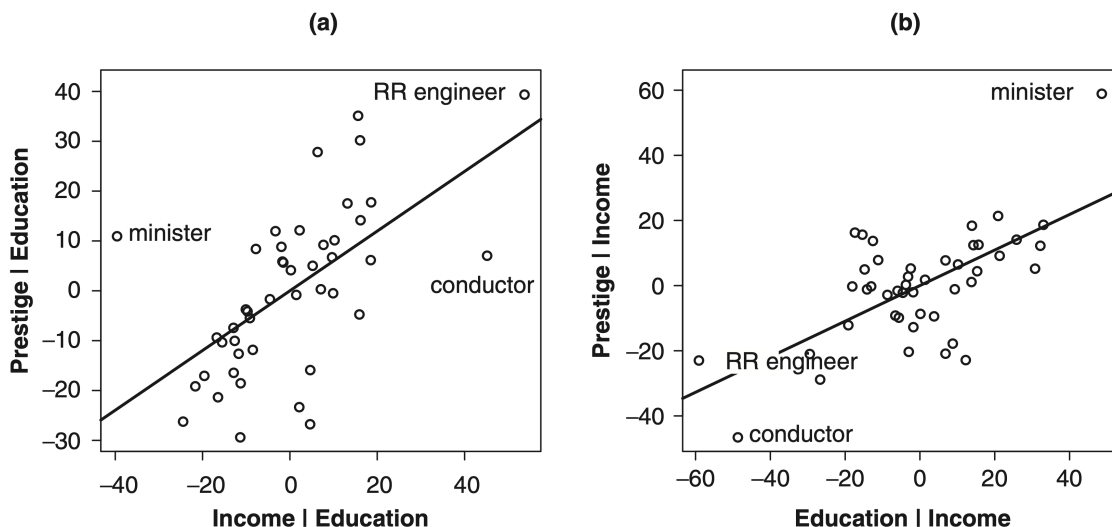


Figure 20.2: Added-variable plots for Duncan’s regression of occupational prestige on the (a) income and (b) education levels of 45 US occupations in 1950. Three unusual observations, *ministers*, *conductors*, and *railroadengineers*, are identified on the plots. The added-variable plot for the intercept  $\hat{\beta}_0$  is not shown. JF Figure 11.10.

The added-variable plot for income in Figure 20.2(a) reveals three observations that exert substantial leverage on the income coefficient:

- *minister*, whose income is unusually low given the educational level of the occupation
- *conductor*, whose income is unusually high given education
- *railroad engineer*, whose income is relatively high given education.

Remember that the horizontal variable in this added-variable plot is the residual from the regression of income on education, and thus values far from 0 in this direction are for occupations with incomes that are unusually high or low given their levels of education.

## Should unusual data be discarded?

In practice, although problematic data should not be ignored, they also should not be deleted automatically and without reflection:

- It is important to investigate *why* an observation is unusual. Truly “bad” data (e.g., an error in data entry ) can often be corrected or, if correction is not possible, thrown away. When a discrepant data point is correct, we may be able to understand why the observation is unusual. For Duncan’s data, for example, it makes sense that ministers enjoy prestige not accounted for by the income and educational levels of the occupation and for a reason not shared by other occupations. In a case like this, where an outlying observation has characteristics that render it unique, we may choose to set it aside from the rest of the data.

- Alternatively, outliers, high-leverage points, or influential data may motivate model respecification, and the pattern of unusual data may suggest the introduction of additional explanatory variables. We noticed, for example, that both conductors and railroad engineers had high leverage in Duncan's regression because these occupations combined relatively high income with relatively low education. Perhaps this combination of characteristics is due to a high level of unionization of these occupations in 1950, when the data were collected. If so, and if we can ascertain the levels of unionization of all of the occupations, we could enter this as an explanatory variable, perhaps shedding further light on the process determining occupational prestige.
- Except in clear-cut cases, we are justifiably reluctant to delete observations or to re-specify the model to accommodate unusual data. Some researchers reasonably adopt alternative estimation strategies, such as robust regression, which continuously down-weights outlying data rather than simply discarding them. Because these methods assign zero or very small weight to highly discrepant data, however, the result is generally not very different from careful application of least squares, and, indeed, robust-regression weights can be used to identify outliers.
- Finally, in large samples, unusual data substantially alter the results only in extreme instances. Identifying unusual observations in a large sample, therefore, should be regarded more as an opportunity to learn something about the data not captured by the model that we have fit, rather than as an occasion to reestimate the model with the unusual observations removed.

## Non-normally distributed errors

The assumption of normally distributed errors is almost always arbitrary. Nevertheless, the central limit theorem ensures that, under very broad conditions, inference based on the least-squares estimator is approximately valid in all but small samples. Why concern about non-normal errors?

- For some types of error distributions, particularly those with heavy tails, the efficiency of least-squares estimation decreases markedly.
- Highly skewed error distributions, aside from their propensity to generate outliers in the direction of the skew, compromise the interpretation of the least-squares fit. This fit is a conditional mean (of  $Y$  given the  $X$ s), and the mean is not a good measure of the center of a highly skewed distribution.
- A multimodal error distribution suggests that omission of one or more discrete explanatory variables that divide the data naturally into groups. An examination of the distribution of the residuals may motivate respecification of the model.

Note: The skewness  $\alpha_3$  is defined as  $\alpha_3 \equiv \frac{\mu_3}{(\mu_2)^{3/2}}$  where  $\mu_n$  denotes the  $n$ th central moment of a random variable  $X$ . The skewness measures the lack of symmetry in the pdf.

## 21 Lecture 21: March 19

### Last time

- Added-variable plots
- Should unusual data be discarded

### Today

- Diagnosing non-normality, non-constant error variance, and nonlinearity (JF chapter 12)
- review HW2 (if time permits)

### Quantile-comparison plot, JF 3.1.3

*Quantile-comparison plots* are useful for comparing an empirical sample distribution with a theoretical distribution, such as the normal distribution.

Let  $P(x)$  represent the theoretical cumulative distribution function (cdf) with which we want to compare the data, that is  $P(x) = \Pr(X \leq x)$ . The quantile-comparison plot is constructed by:

1. Order the data values from smallest to largest,  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ . The  $X_{(i)}$  are called the order statistics of the sample.
2. By convention, the cumulative proportion of the data “below”  $X_{(i)}$  is given by

$$P_i = \frac{i - \frac{1}{2}}{n}$$

3. Use the inverse of the cdf to find the value  $z_i$  corresponding to the cumulative probability  $P_i$ , that is

$$z_i = P^{-1}\left(\frac{i - \frac{1}{2}}{n}\right)$$

4. Plot the  $z_i$  as horizontal coordinates against the  $X_{(i)}$  as vertical coordinates. If  $X$  is sampled from the distribution  $P$ , then  $X_{(i)} \approx z_i$ .
  - if the distributions are identical except for location, then the plot is approximately linear with nonzero intercept,  $X_{(i)} \approx \mu + z_i$
  - if the distributions are identical except for scale, then the plot is approximately linear with a slope different from 1,  $X_{(i)} \approx \sigma z_i$
  - if the distributions differ both in location and scale but have the same shape, then  $X_{(i)} \approx \mu + \sigma z_i$

5. It is often helpful to place a comparison line on the plot to facilitate the perception of departures from linearity. For a normal quantile-comparison plot (comparing the distribution of the data with the standard normal distribution), we can alternatively use the median as a robust estimator of  $\mu$  and the interquartile range/1.39 as a robust estimator of  $\sigma$ .
6. We expect some departure from linearity because of sampling variation. It therefore assists interpretation to display the expected degree of sampling error in the plot. The standard error of the order statistic  $X_{(i)}$  is

$$\text{SE}(X_{(i)}) = \frac{\hat{\sigma}}{p(z_i)} \sqrt{\frac{P_i(1 - P_i)}{n}}$$

where  $p(z_i)$  is the probability density function, pdf, corresponding to the CDF  $P(z)$ . The values along the fitted line are given by  $\hat{X}_{(i)} = \hat{\mu} + \hat{\sigma}z_i$ . An approximate 95% confidence “envelope” around the fitted line is, therefore,

$$\hat{X}_{(i)} \pm 2 \times \text{SE}(X_{(i)})$$

- Figure 21.1 plots a sample of  $n = 100$  observations from a normal distribution with mean  $\mu = 50$  and standard deviation  $\sigma = 10$ .

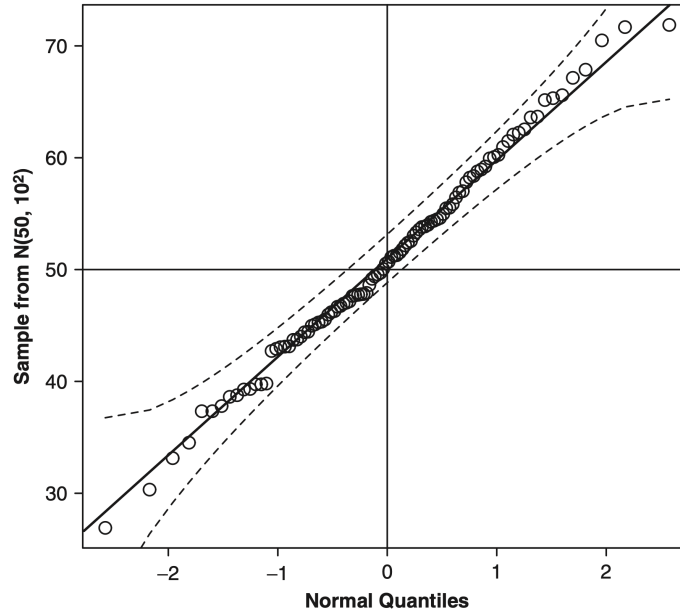


Figure 21.1: Normal quantile-comparison plot for a sample of 100 observations drawn from a normal distribution with mean 50 and standard deviation 10. The fitted line is through the quartiles of the distribution, the broken lines give a pointwise 95% confidence interval around the fit. JF Figure 3.8.

The plotted points are reasonably linear and stay within the rough 95% confidence envelope.

- Figure 21.2 plots a sample of  $n = 100$  observations from the positively skewed chi-square distribution with 2 degrees of freedom. The positive skew of the data is reflected in points that lie *above* the comparison line in both tails of the distribution. (In contrast, the tails of negatively skewed data would lie *below* the comparison line.)

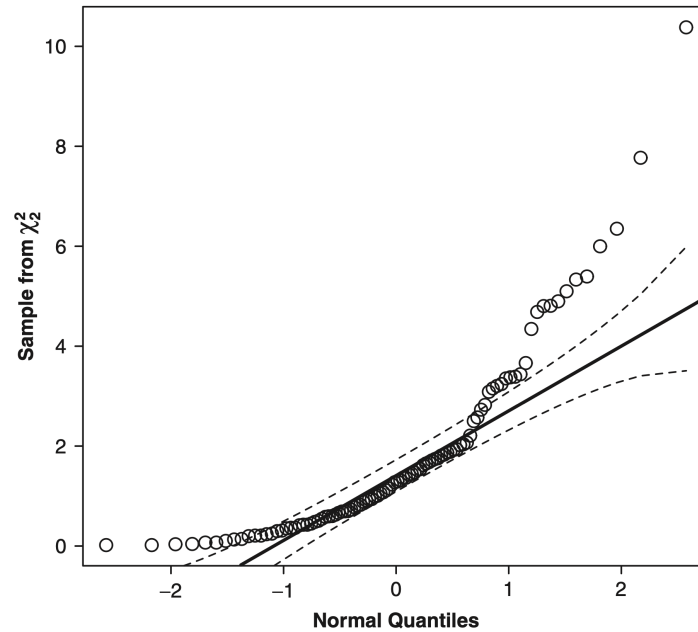


Figure 21.2: Normal quantile-comparison plot for a sample of 100 observations drawn from the positively skewed chi-square distribution with 2 degrees of freedom. JF Figure 3.9.

- Figure 21.3 plots a sample of  $n = 100$  observations from the heavy-tailed  $t$  distribution with 2 degrees of freedom. In this case, values in the upper tail lie above the corresponding normal quantiles, the values in the lower tail below the corresponding normal quantiles.

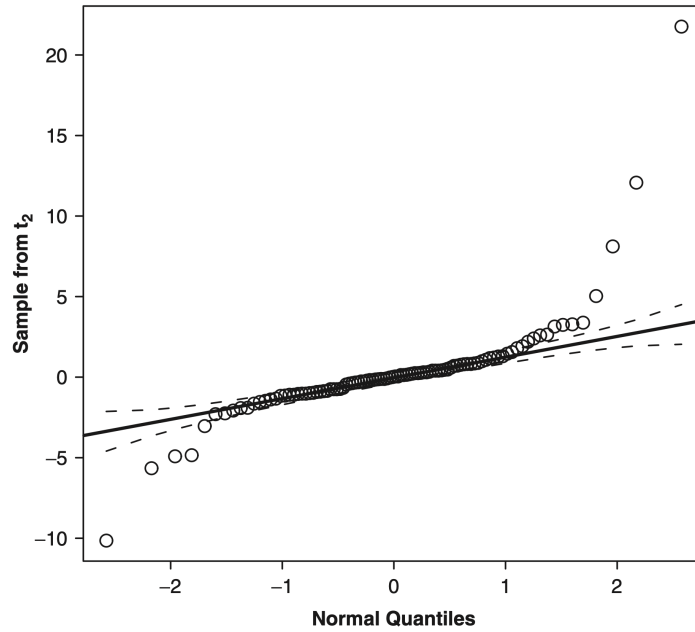


Figure 21.3: Normal quantile-comparison plot for a sample of 100 observations drawn from heavy-tailed  $t$ -distribution with 2 degrees of freedom. JF Figure 3.10.

- Figure 21.4 shows the normal quantile-comparison plot for the distribution of infant mortality. The positive skew of the distribution is readily apparent.

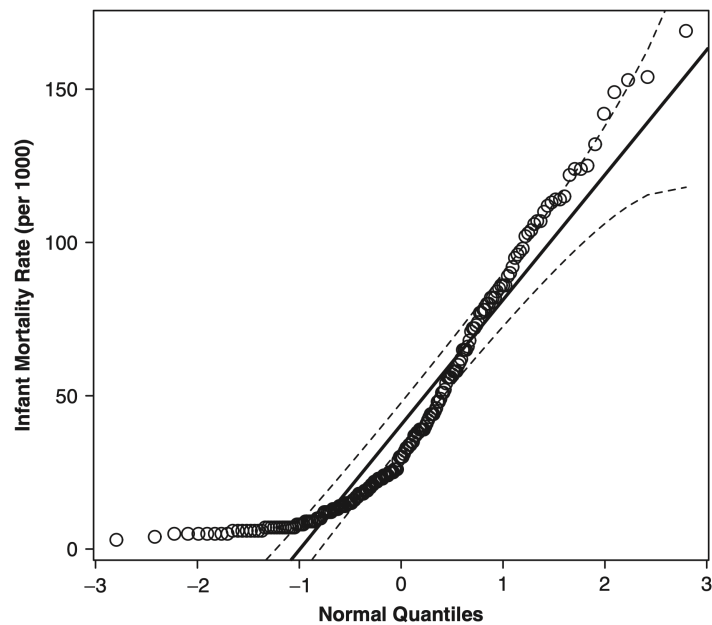


Figure 21.4: Normal quantile-comparison plot for the distribution of infant mortality. Note the positive skew. JF Figure 3.11.

## Nonconstant error variance

One of the assumptions of the regression model is that the variation of the response variable around the regression surface (the error variance) is everywhere the same:

$$\text{Var}(\epsilon) = \text{Var}(Y|x_1, \dots, x_p) = \sigma_\epsilon^2$$

Constant error variance is often termed homoscedasticity, and similarly, nonconstant error variance is termed heteroscedasticity. We detect nonconstant error variances through graphical methods.

### Residual plots

Because the least square residuals have unequal variance even when the constant variance assumption is correct:

$$\text{Var}(\hat{\epsilon}_i) = \sigma^2(1 - h_i).$$

It is preferable to plot studentized residuals against fitted values. A pattern of changing spread is often more easily discerned in a plot of absolute studentized residuals,  $|\hat{\epsilon}_i^*|$ , or squared studentized residuals,  $\hat{\epsilon}_i^{*2}$ , against  $\hat{Y}$ . If the values of  $\hat{Y}$  are all positive, then we can plot  $\log|\hat{\epsilon}_i^*|$  against  $\log \hat{Y}$ . Figure 21.5 shows a plot of studentized residuals against fitted values and spread-level plot of studentized residuals, several points with negative fitted values were omitted.

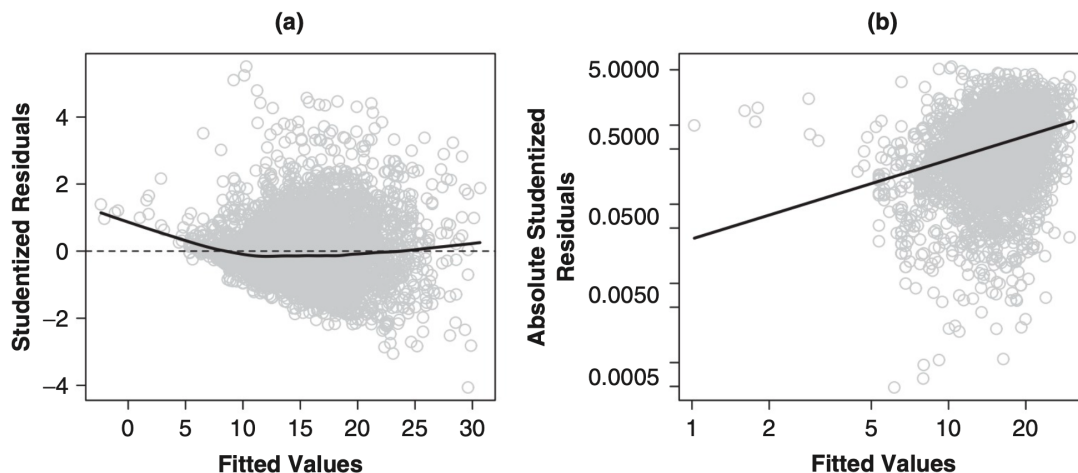


Figure 21.5: (a) Plot of studentized residuals versus fitted values and (b) spread-level plot for studentized residuals. JF Figure 12.3.

It is apparent from both graphs that the residual spread tends to increase with the level of the response, suggesting a violation of constant error variance assumption.



## Weighted-least-squares estimation

Weighted-least-squares (WLS) regression provides an alternative approach to estimation in the presence of nonconstant error variance. Suppose that the errors from the linear regression model  $\mathbf{Y} = \mathbf{X}\beta + \epsilon$  are independent and normally distributed, with zero means but *different* variances:  $\epsilon_i \sim N(0, \sigma_i^2)$ . Suppose further that the variances of the errors are known up to a constant of proportionality  $\sigma_\epsilon^2$ , so that  $\sigma_i^2 = \sigma_\epsilon^2/w_i^2$ . Then the likelihood for the model is

$$L(\beta, \sigma_\epsilon^2) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp \left[ -\frac{1}{2} (\mathbf{Y} - \mathbf{X}\beta)^T \Sigma^{-1} (\mathbf{Y} - \mathbf{X}\beta) \right]$$

where  $\Sigma$  is the covariance matrix of the errors,

$$\Sigma = \sigma_\epsilon^2 \times \text{diag}\{1/w_1^2, \dots, 1/w_n^2\} \equiv \sigma_\epsilon^2 \mathbf{W}^{-1}$$

The maximum-likelihood estimators of  $\beta$  and  $\sigma_\epsilon^2$  are then

$$\begin{aligned} \hat{\beta} &= (\mathbf{X}^T \mathbf{W} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{W} \mathbf{Y} \\ \hat{\sigma}_\epsilon^2 &= \frac{\sum (w_i \hat{\epsilon}_i)^2}{n} \end{aligned}$$

## Correcting OLS standard errors for nonconstant variance

The covariance matrix of the ordinary-least-squares (OLS) estimator is

$$\begin{aligned} \mathbf{Var}(\hat{\beta}) &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Var}(\mathbf{Y}) \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \\ &= \sigma_\epsilon^2 (\mathbf{X}^T \mathbf{X})^{-1} \end{aligned}$$

under the standard assumptions, including the assumption of constant error variance,  $\mathbf{Var}(\mathbf{Y}) = \sigma_\epsilon^2 \mathbf{I}_n$ . If, however, the errors are heteroscedastic but independent then  $\Sigma \equiv \mathbf{Var}(\mathbf{Y}) = \text{diag}\{\sigma_1^2, \dots, \sigma_n^2\}$ , and

$$\mathbf{Var}(\hat{\beta}) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \Sigma \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1}$$

White (1980) shows that the following is a consistent estimator of  $\mathbf{Var}(\hat{\beta})$

$$\tilde{\mathbf{Var}}(\hat{\beta}) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \hat{\Sigma} \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1}$$

with  $\hat{\Sigma} = \text{diag}\{\hat{\sigma}_1^2, \dots, \hat{\sigma}_n^2\}$ , where  $\hat{\sigma}_i^2$  is the OLS residual for observation  $i$ .

Subsequent work suggested small modifications to White's coefficient-variance estimator, and in particular simulation studies by Long and Ervin (2000) support the use of

$$\tilde{\mathbf{Var}}^*(\hat{\beta}) = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \hat{\Sigma}^* \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1}$$

where  $\hat{\Sigma}^* = \text{diag}\{\hat{\sigma}_i^2/(1 - h_i)^2\}$  and  $h_i$  is the hat-value associated with observation  $i$ . In large samples, where  $h_i$  is small, the distinction between  $\tilde{\mathbf{Var}}(\hat{\beta})$  and  $\tilde{\mathbf{Var}}^*(\hat{\beta})$  essentially disappears.

A rough *rule* is that nonconstant error variance seriously degrades the least-squares estimator only when the ratio of the largest to smallest variance is about 10 or more (or, more conservatively, about 4 or more).

## 22 Lecture 22: March 26

### Last time

- Midterm exam

### Today

- Friday lecture
- Nonlinearity
- Data transformation

### Nonlinearity

If  $\mathbf{E}(\mathbf{Y}|\mathbf{X})$  is not linear in  $\mathbf{X}$  (in other words,  $\mathbf{E}(\epsilon|\mathbf{X}) \neq 0$  for some  $x$ ),  $\hat{\beta}$  may be biased and inconsistent. Usually we employ “linearity by default” but we should try to make sure this is appropriate: **detect** non-linearities and **model** them accurately.

### Lowess smoother, JF 2.3

We can employ local averaging plots to help with diagnostics. Lowess method is in many respects similar to local-averaging smoothers, except that instead of computing an average  $Y$ -value within the neighborhood of a focal  $x$ , the lowess smoother computes a *fitted* value based on a locally weighted least-squares line, giving more weight to observations in the neighborhood that are close to the focal  $x$  than to those relatively far away. The name “lowess” is an acronym for *locally weighted scatterplot smoother* and is sometimes rendered as *loess*, for *local regression*.

### Component-plus-residual plots

Added-variable plots, introduced for detecting influential data, can reveal nonlinearity. However, the added-variable plots are not always useful for locating a transformation:

- The added-variable plot adjusts  $X_j$  for the other  $X$ s.
- The *unadjusted*  $X_j$  is transformed in respecifying the model.

Moreover, Cook (1998, Section 14.5) shows that added-variable plots are biased toward linearity when the correlations among the explanatory variables are large.

Component-plus-residual plots (also called *partial-residual plots*) are often an effective alternative. The component-plus-residual plots are not as suitable as added-variable plots for revealing leverage and influence, though. The component-plus-residual plots are constructed by

1. Compute residuals from full regression:

$$\hat{\epsilon}_i = Y_i - \hat{Y}_i$$

2. Compute “linear component” of the partial relationship:

$$C_i = \hat{\beta}_j X_{ij}$$

3. Add linear component to residual to get partial residual for the  $j$ th explanatory variable

$$\hat{\epsilon}_i^{(j)} = \hat{\epsilon}_i + C_i = \hat{\epsilon}_i + \hat{\beta}_j X_{ij}$$

4. Plot  $\hat{\epsilon}_i^{(j)}$  against  $X_{.j}$

Figure 22.1 shows the component-plus-residual plots for the regression of log wages on variables (age, education and sex) of the 1994 wave of Statistics Canada’s Survey of Labour and Income Dynamics (SLID) data. The SLID data set includes 3997 employed individuals who were between 16 and 65 years of age and who resided in Ontario.

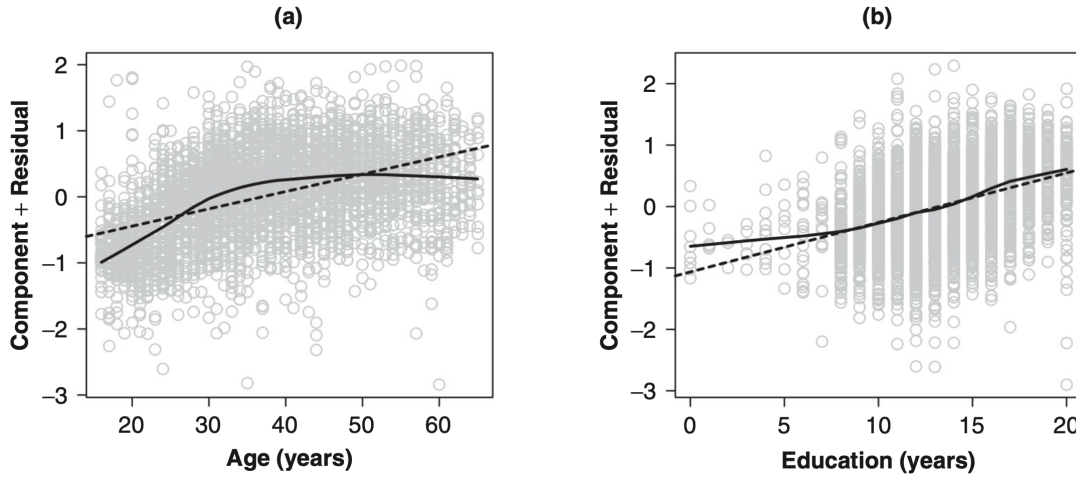


Figure 22.1: Component-plus-residual plots for age and education in SLID regression of log wages on these variables and sex. The solid lines are for lowess smooths with spans of 0.4, and the broken lines are for linear least-squares fits. JF Figure 12.6.

## Data transformation

The family of powers and Roots, JF 4.1

A particularly useful group of transformations is the “family” of powers and roots:

$$X \rightarrow X^p$$

where the arrow indicates that we intend to replace  $X$  with the transformed variable  $X^p$ . If  $p$  is negative, then the transformation is an inverse power. For example,  $X^{-1} = 1/X$ . If  $p$  is a fraction, then the transformation represents a root. For example,  $X^{1/3} = \sqrt[3]{X}$ .

It is more convenient to define the family of power transformations in a slightly more complex manner, called the Box-Cox family of transformations (introduced in a seminal paper on transformations by Box & Cox, 1964):

$$X \rightarrow X^{(p)} = \frac{X^p - 1}{p}$$

Because  $X^{(p)}$  is a linear function of  $X^p$ , the two transformations have the same essential effect on the data, but, as is apparent in Figure [22.2](#)

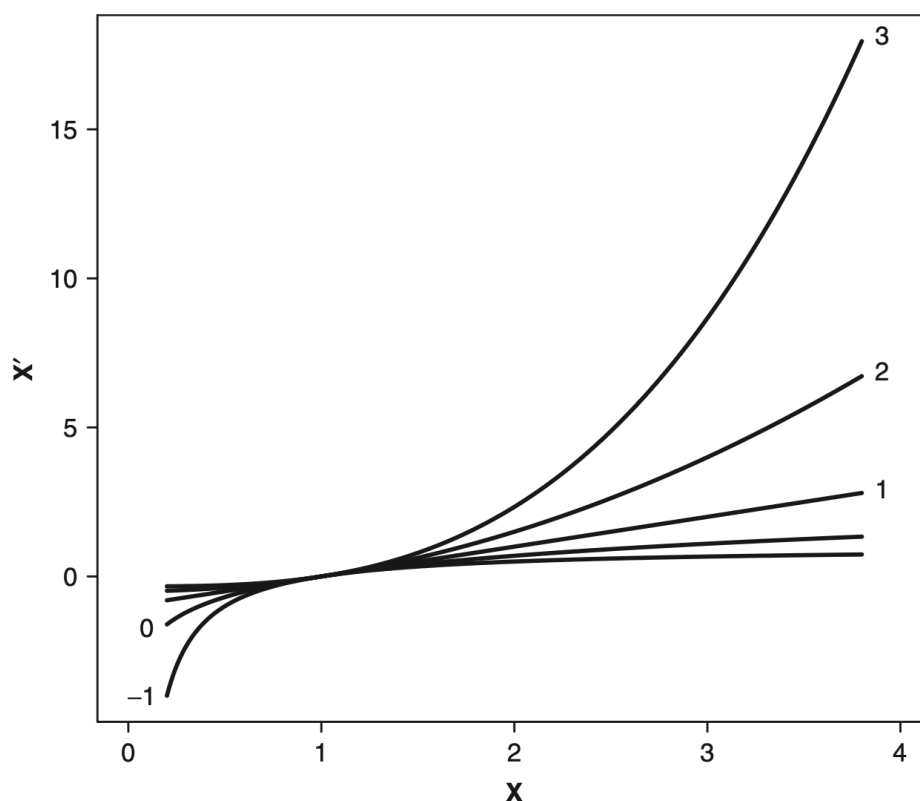


Figure 22.2: The Box-Cox family of power transformations  $X'$  of  $X$ . The curve labeled  $p$  is the transformation  $X^{(p)}$ , that is  $(X^p - 1)/p$ ;  $X^{(0)}$  is  $\log_e(X)$ . JF Figure 4.1.

- Dividing by  $p$  preserves the direction of  $X$ , which otherwise would be reversed when  $p$  is negative.
- The transformations  $X^{(p)}$  are “matched” above  $X = 1$  both in level and in slope:
  1.  $X^{(p)} = 0$ , for all values of  $p$
  2. each transformation has a slope of 1 at  $X = 1$ .
- Descending the “ladder” of powers and roots towards  $X^{(-1)}$  compresses the large values of  $X$  and spreads out the small ones. Ascending the ladder of powers and roots towards

$X^{(2)}$  has the opposite effect. As  $p$  moves further from  $p = 1$  (i.e., no transformation) in either direction, the transformation grows more powerful, increasingly “bending” the data.

- The power transformation  $X^0$  is useless because it changes all values to 1, but we can think of the log transformation as a kind of “zeroth” power:

$$\lim_{p \rightarrow 0} \frac{X^p - 1}{p} = \log_e X$$

and by convention,  $X^{(0)} \equiv \log_e X$ .

### Box-Cox transformation of $Y$

Box and Cox (1964) suggested a power transformation of  $Y$  with the objective of normalizing the error distribution, stabilizing the error variance, and straightening the relationship of  $Y$  to the  $X$ s. The general Box-Cox model is

$$Y_i^{(\lambda)} = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \epsilon_i$$

where  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$ , and

$$Y_i^{(\lambda)} = \begin{cases} \frac{Y_i^\lambda - 1}{\lambda} & \text{for } \lambda \neq 0 \\ \log_e Y_i & \text{for } \lambda = 0 \end{cases}$$

Note: in statistics,  $\log_e$  is often written as  $\log$ .

For a particular choice of  $\lambda$ , the conditional maximized log-likelihood (see JF 12.5.1 p.324 footnote 55) is

$$\begin{aligned} \log_e L(\beta_0, \beta_1, \dots, \beta_p, \sigma_\epsilon^2 | \lambda) &= -\frac{n}{2} (1 + \log_e 2\pi) \\ &\quad - \frac{n}{2} \log_e \hat{\sigma}_\epsilon^2(\lambda) + (\lambda - 1) \sum_{i=1}^n \log_e Y_i \end{aligned}$$

where  $\hat{\sigma}_\epsilon^2(\lambda) = \sum \hat{\epsilon}_i^2(\lambda)/n$  and where  $\hat{\epsilon}_i(\lambda)$  are the residuals from the least-squares regression of  $Y^{(\lambda)}$  on  $X$ s. The least-squares coefficients from this regression are the maximum-likelihood estimates of  $\beta$ s conditional on the values of  $\lambda$ .

A simple procedure for finding the maximum-likelihood estimator  $\hat{\lambda}$  is to evaluate the maximized  $\log_e L$  (called the profile log-likelihood) for a range of values of  $\lambda$ . To test:  $H_0 : \lambda = 1$ , calculated the likelihood-ratio statistic

$$G_0^2 = -2[\log_e L(\lambda = 1) - \log_e L(\lambda = \hat{\lambda})]$$

which is asymptotically distributed as  $\chi_1^2$  with one degree of freedom under  $H_0$ . A 95% confidence interval for  $\lambda$  includes those values for which

$$\log_e L(\lambda) > \log_e L(\lambda = \hat{\lambda}) - 1.92$$

The number 1.92 comes from  $\frac{1}{2}\chi_{1,0.05}^2 = 0.5 \times 1.96^2$ .

Figure 22.3 shows a plot of the profile log-likelihood against  $\lambda$  for the original SLID regression of composite hourly wages on sex, age, and education. The maximum-likelihood estimate of  $\lambda$  is  $\hat{\lambda} = 0.09$ , and a 95% confidence interval runs from 0.04 to 0.13. Although 0 is outside of the CI (confidence interval), it is essentially the same as log transformation of wages (the correlation between log wages and wages<sup>0.09</sup> is 0.9996).

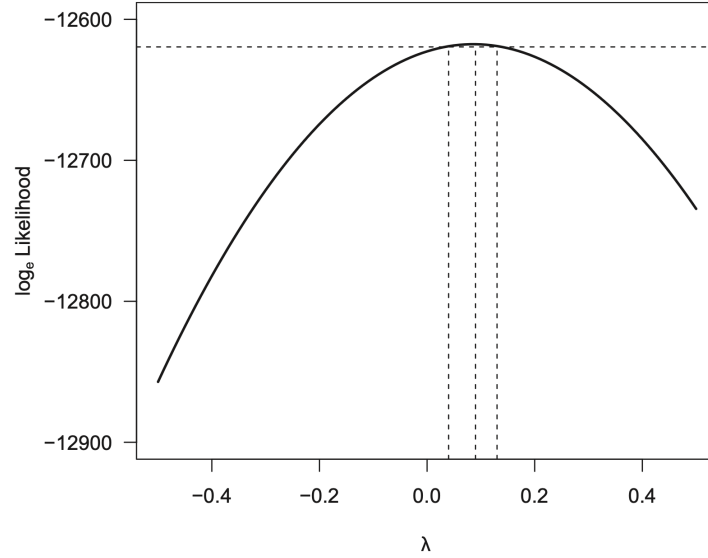


Figure 22.3: Box-Cox transformations for the SLID regression of wages on sex, age, and education. The maximized (profile) log-likelihood is plotted against the transformation parameter  $\lambda$ . The intersection of the line near the top of the graph with the profile log-likelihood curve marks off a 95% confidence interval for  $\lambda$ . The maximum of the log-likelihood corresponds to the MLE of  $\lambda$ . JF Figure 12.14.

### Box-Tidwell transformation of $X$ s

Now, consider the model

$$Y_i = \beta_0 + \beta_1 X_{i1}^{\gamma_1} + \cdots + \beta_p X_{ip}^{\gamma_p} + \epsilon_i$$

where the errors are independently distributed as  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$  and all the  $X_{ij}$  are positive.

The parameters of this model ( $\beta_0, \beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_p$ , and  $\sigma_\epsilon^2$ ) could be estimated by general nonlinear least squares. Box and Tidwell (1962) suggested the following computationally more efficient procedure (also yields a constructed-variable diagnostic):

1. Regress  $Y$  on  $X_1, \dots, X_p$ , obtaining  $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_p$ . (“Regress  $A$  on  $B$ s” is the same as “fitting the linear regression model with  $A$  as the response variable and  $B$ s as the explanatory variables”.)
2. Regress  $Y$  on  $X_1, \dots, X_p$  and the constructed variables  $X_1 \log_e X_1, \dots, X_p \log_e X_p$  (again, by fitting the model of  $Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \delta_1 X_1 \log_e X_1 + \cdots + \delta_p X_p \log_e X_p + \epsilon_i$ )

to obtain  $\tilde{\beta}_0, \tilde{\beta}_1, \dots, \tilde{\beta}_p, \tilde{\delta}_1, \dots, \tilde{\delta}_p$ . In general  $\hat{\beta}_i \neq \tilde{\beta}_i$ . (The constructed variables result from the first-order Taylor-series approximation to  $X_j^{\gamma_j}$  evaluated at  $\gamma_j = 1$ :  $X_j^{\gamma_j} \approx X_1 + (\gamma_j - 1)X_1 \log_e X_1$ .)

3. The constructed variable  $X_j \log_e X_j$  can be used to assess the need for a transformation of  $X_j$  by testing the null hypothesis  $H_0 : \delta_j = 0$ . Added-variable plots for the constructed variables are useful for assessing leverage and influence on the decision to transform the  $X$ s.
4. A preliminary estimate of the transformation parameter  $\gamma_j$  (not the MLE) is

$$\tilde{\gamma}_j = 1 + \frac{\tilde{\delta}_j}{\hat{\beta}_j}$$

where  $\tilde{\delta}_j$  is from step 2 and  $\hat{\beta}_j$  is from step 1.

## Polynomial regression

A machinery of multiple regression to fit non-linear relationships between predictor(s) and response.

- Linear:  $y = \beta_0 + \beta_1 x + \epsilon$
- Quadratic:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$
- Cubic:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \epsilon$
- $k^{th}$  order polynomial:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_k x^k + \epsilon$

Question:

Does quadratic model provide a significantly better fit than linear model?

*Solution:* Test  $H_0 : \beta_2 = 0$  vs.  $H_a : \beta_2 \neq 0$ .

Alternatively, compare the corresponding adjusted- $R^2$  values.

## 23 Lecture 23: March 31

### Last time

- Nonlinearity
- midterm review

### Today

- Polynomial regression
- Collinearity
- Principal component

### Polynomial regression

A machinery of multiple regression to fit non-linear relationships between predictor(s) and response.

- Linear:  $y = \beta_0 + \beta_1 x + \epsilon$
- Quadratic:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$
- Cubic:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \epsilon$
- $k^{th}$  order polynomial:  $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k + \epsilon$

Question:

Does quadratic model provide a significantly better fit than linear model?

*Solution:* Test  $H_0 : \beta_2 = 0$  vs.  $H_a : \beta_2 \neq 0$ .

Alternatively, compare the corresponding adjusted- $R^2$  values.

### Additional reference

- “A First Course in Linear Model Theory” by Nalini Ravishanker and Kipak K. Dey

### Collinearity

In linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$
$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$$

Collinearity (or multicollinearity) exists when there is “near-dependency” between the columns of the design matrix  $\mathbf{X}$ .

- Two or more columns.
- In other words, high correlation between explanatory variables.



- the data/model pair is ill-conditioned when  $\mathbf{X}^T\mathbf{X}$  is nearly singular.

Perfect collinearity leads to rank-deficiency in  $\mathbf{X}$  such that  $\mathbf{X}^T\mathbf{X}$  is singular. In the case of perfect collinearity, two or more columns are linear-dependent.

#### An example of perfect collinearity

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i4} + \beta_5 X_{i5} + \epsilon_i$$

Consider the case, where

- $Y_i$  represents the amount of sales.
- $X_{i1}, X_{i2}, \dots, X_{i4}$  are categorical that represent the quarter in which the sample is collected:  $X_{ij} = \mathbf{1}(\text{sample } i \text{ collected in quarter } j)$ .
- $X_{i5}$  represents expense spent in advertising.

The dummy variable trap  $X_{i4} = 1 - X_{i1} - X_{i2} - X_{i3}$ . Recall that we need  $m - 1$  dummy variables for  $m$  categories.

#### An example of high correlation between predictors

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i$$

Consider the case, where

- $Y_i$  represents the salary of individual  $i$ .
- $X_{i1}$  represents the age of individual  $i$ .
- $X_{i2}$  represents the experience of individual  $i$ .

How to interpret  $\beta_1$ ?

We expect high correlation between age and experience.

#### Problems caused by multicollinearity

1. large standard errors of the regression coefficients
  - small associated t-statistics
  - conclusion that truly useful explanatory variables are insignificant in explaining the regression
2. the sign of regression coefficients may be the opposite of what a mechanistic understanding of the problem would suggest
3. deleting a column of the predictor matrix will cause large changes in the coefficient estimates for other variables

However, multicollinearity does **not** greatly affect the **predicted values**.

## Signs and detections of multicollinearity

Some signs for multicollinearity:

1. Simple correlation between a pair of predictors exceeds 0.9 or  $R^2$ .
2. High value of the multiple correlation coefficient with some high partial correlations between the explanatory variables.
3. Large  $F$ -statistics with some small  $t$ -statistics for individual regression coefficients

Some approaches for detecting multicollinearity:

1. Pairwise correlations among the explanatory variables
2. Variance inflation factor
3. Condition number

## Variance inflation factor

For a multiple linear regression with  $k$  explanatory variables. We can regress  $X_j$  on the  $(k - 1)$  other explanatory variables and denote  $R_j$  as the coefficient of determination.

Then the variance inflation factor (VIF) is defined as

$$\text{VIF}_j = \frac{1}{1 - R_j^2}$$

- $\text{VIF}_j \in [1, +\infty)$
- A suggested threshold is 10
- May use the averaged  $\overline{\text{VIF}} = \sum_{j=1}^k \text{VIF}_j / k$ .

## 24 Lecture 24: April 2

### Last time

- Variance inflation factor

### Today

- Condition index and condition number
- Principal component analysis (JF 13.1.1, RD 8.3.4)
- Penalized regression

### Condition index and condition number

We first scale the design matrix  $\mathbf{X}$  into column-equilibrated predictor matrix  $\mathbf{X}_E$  such that  $\{X_E\}_{ij} = X_{ij}/\sqrt{\mathbf{X}_j^T \mathbf{X}_j}$ .

Let  $\mathbf{X}_E = \mathbf{U}\mathbf{D}\mathbf{V}^T$  be the singular-value decomposition (SVD) of the  $n \times p$  matrix  $\mathbf{X}_E$  where  $\mathbf{U}^T \mathbf{U} = \mathbf{V}^T \mathbf{V} = \mathbf{I}_p$  and  $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_p)$  is a diagonal matrix with  $d_j \geq 0$ .

The  $j^{\text{th}}$  condition index is defined as

$$\eta(\mathbf{X}_E) = d_{\max}/d_j, \quad j = 1, 2, \dots, p$$

The condition number is defined as

$$C = d_{\max}/d_{\min}$$

$$C \geq 1, \quad d_{\max} = \max_{1 \leq j \leq p} d_j \quad \text{and} \quad d_{\min} = \min_{1 \leq j \leq p} d_j$$

Some properties of the condition number

- Large condition number indicates evidence of multicollinearity
- Typical cutoff values, 10, 15 to 30.

Some problems with the condition number

- practitioners have different opinions of whether  $\mathbf{X}$  should be centered around their means for SVD.
  - centering may remove nonessential ill conditioning, e.g.  $\text{Cor}(X, X^2)$
  - centering may mask the role of the constant term in any underlying near-dependencies
- the degree of multicollinearity with dummy variables may be influenced by the choice of reference category

- condition number is affected by the scale of the  $\mathbf{X}$  measurements
  - By scaling down any column of  $\mathbf{X}$ , the condition number can be made arbitrarily large
  - Known as *artificial ill-conditioning*
  - The condition number of the scaled matrix  $\mathbf{X}_E$  is also referred to as the *scaled condition number*

Recall that  $\mathbf{X}_E = \mathbf{U}\mathbf{D}\mathbf{V}^T$  is the singular-value decomposition (SVD) of  $\mathbf{X}_E$ , where  $\mathbf{U}^T\mathbf{U} = \mathbf{V}^T\mathbf{V} = \mathbf{I}_p$  and  $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_p)$  is a diagonal matrix with  $d_j \geq 0$ .

Then

$$\begin{aligned}\mathbf{X}_E^T\mathbf{X}_E &= \mathbf{V}\mathbf{D}\mathbf{U}^T\mathbf{U}\mathbf{D}\mathbf{V}^T \\ &= \mathbf{V}\mathbf{D}^2\mathbf{V}^T\end{aligned}$$

is the spectral decomposition of the Gramian matrix  $\mathbf{X}_E^T\mathbf{X}_E$  with  $\{d_j^2\}$  being the eigenvalues and  $\mathbf{V}$  being the corresponding eigen vector matrix. This relationship links the condition numbers to the eigen values of the Gramian matrix.

### Variance decomposition method

The variance-covariance matrix of the coefficient

$$\begin{aligned}\text{Cov}(\hat{\beta}) &= \sigma^2(\mathbf{X}_E^T\mathbf{X}_E)^{-1} \\ &= \sigma^2\mathbf{V}\mathbf{D}^{-2}\mathbf{V}^T\end{aligned}$$

Its  $j^{\text{th}}$  diagonal element is the estimated variance of the  $j^{\text{th}}$  coefficient,  $\hat{\beta}_j$ . Then

$$\text{Var}(\hat{\beta}_j) = \sigma^2 \sum_{h=1}^p \frac{v_{jh}^2}{d_h^2}$$

- Let  $q_{jh} = \frac{v_{jh}^2}{d_h^2}$  and  $q_j = \sum_{h=1}^p q_{jh}$ .
- The variance decomposition proportion is  $\pi_{jh} = q_{jh}/q_j$ .
- $\pi_{jh}$  denotes the proportion of the variance of the  $j^{\text{th}}$  regression coefficient associated with the  $h^{\text{th}}$  component of its decomposition.
- The variance decomposition proportion matrix is  $\mathbf{\Pi} = \{\pi_{jh}\}$ .

In practice, it is suggested to combine condition index and proportions of variance for multicollinearity diagnostic. Identify multicollinearity if

Condition	Proportions of variance			
Index	$Var(\hat{\beta}_1)$	$Var(\hat{\beta}_2)$	...	$Var(\hat{\beta}_p)$
$\eta_1$	$\pi_{11}$	$\pi_{12}$	...	$\pi_{1p}$
$\eta_2$	$\pi_{21}$	$\pi_{22}$	...	$\pi_{2p}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\eta_p$	$\pi_{p1}$	$\pi_{p2}$	...	$\pi_{pp}$

Table 1: Table of condition index and proportions of variance

- Two or more elements in the  $j^{th}$  row of matrix  $\mathbf{\Pi}$  are relatively large
- And its associated condition index  $\eta_j$  is large too

## Principal Components

The method of principal components, introduced by Karl Pearson (1901) and Harold Hotelling (1933), provides a useful representation of the correlational structure of a set of variables. Some advantages of the principal component analysis include

- more unified
- linear transformation of the original predictors into a new set of orthogonal predictors
- the new orthogonal predictors are called principal components

Principal components regression is an approach that inspects the sample data  $(\mathbf{Y}, \mathbf{X})$  for directions of variability and uses this information to reduce the dimensionality of the estimation problem. The procedure is based on the observation that every linear regression model can be restated in terms of a set of orthogonal predictor variables, which are constructed as linear combinations of the original variables. The new orthogonal variables are called the principal components of the original variables.

Let  $\mathbf{X}^T \mathbf{X} = \mathbf{Q} \mathbf{\Delta} \mathbf{Q}^T$  denote the spectral decomposition of  $\mathbf{X}^T \mathbf{X}$ , where  $\mathbf{\Delta} = \text{diag}\{\lambda_1, \dots, \lambda_p\}$  is a diagonal matrix consisting of the (real) eigenvalues of  $\mathbf{X}^T \mathbf{X}$ , with  $\lambda_1 \geq \dots \geq \lambda_p$  and  $\mathbf{Q} = (\mathbf{q}_1, \dots, \mathbf{q}_p)$  denotes the matrix whose columns are the orthogonal eigenvectors of  $\mathbf{X}^T \mathbf{X}$  corresponding to the ordered eigenvalues. Consider the transformation

$$\mathbf{Y} = \mathbf{X} \mathbf{Q} \mathbf{Q}^T \boldsymbol{\beta} + \boldsymbol{\epsilon} = \mathbf{Z} \boldsymbol{\theta} + \boldsymbol{\epsilon},$$

where  $\mathbf{Z} = \mathbf{X} \mathbf{Q}$ , and  $\boldsymbol{\theta} = \mathbf{Q}^T \boldsymbol{\beta}$ .

The elements of  $\boldsymbol{\theta}$  are known as the regression parameters of the principal components. The matrix  $\mathbf{Z} = \{\mathbf{z}_1, \dots, \mathbf{z}_p\}$  is called the matrix of principal components of  $\mathbf{X}^T \mathbf{X}$ .  $\mathbf{z}_j = \mathbf{X} \mathbf{q}_j$  is the  $j$ th principal component of  $\mathbf{X}^T \mathbf{X}$  and  $\mathbf{z}_j^T \mathbf{z}_j = \lambda_j$ , the  $j$ th largest eigenvalue of  $\mathbf{X}^T \mathbf{X}$ .

Principal components regression consists of deleting one or more of the variables  $\mathbf{z}_j$  (which correspond to small values of  $\lambda_j$ ), and using OLS estimation on the resulting reduced regression model.

### Derivation under standardized predictors, JF 13.1.1

Consider the vectors of standardized predictors,  $\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_p^*$  (obtained by subtracting the mean and divided by standard deviation of the original predictor vectors). Because the principal components are linear combinations of the original predictors, we write the first principal component as

$$\begin{aligned}\mathbf{w}_1 &= A_{11}\mathbf{x}_1^* + A_{21}\mathbf{x}_2^* + \dots + A_{p1}\mathbf{x}_p^* \\ &= \mathbf{X}^* \mathbf{a}_1\end{aligned}$$

The variance of the first component becomes

$$\begin{aligned}S_{w_1}^2 &= \frac{1}{n-1} \mathbf{w}_1^T \mathbf{w}_1 \\ &= \frac{1}{n-1} \mathbf{a}_1^T \mathbf{X}^{*T} \mathbf{X}^* \mathbf{a}_1 \\ &= \mathbf{a}_1^T \mathbf{R}_{XX} \mathbf{a}_1\end{aligned}$$

where  $\mathbf{R}_{XX} = \frac{1}{n-1} \mathbf{X}^{*T} \mathbf{X}^*$ . We want to maximize  $S_{w_1}^2$  under the normalizing constraint  $\mathbf{a}_1^T \mathbf{a}_1 = 1$  (otherwise  $S_{w_1}^2$  can be arbitrarily large by inflating  $\mathbf{a}_1$ ). Consider

$$F_1 \equiv \mathbf{a}_1^T \mathbf{R}_{XX} \mathbf{a}_1 - L_1(\mathbf{a}_1^T \mathbf{a}_1 - 1)$$

where  $L_1$  is a Lagrange multiplier. By differentiating this equation with respect to  $\mathbf{a}_1$  and  $L_1$ ,

$$\begin{aligned}\frac{\partial F_1}{\partial \mathbf{a}_1} &= 2\mathbf{R}_{XX} \mathbf{a}_1 - 2L_1 \mathbf{a}_1 \\ \frac{\partial F_1}{\partial L_1} &= -(\mathbf{a}_1^T \mathbf{a}_1 - 1)\end{aligned}$$

Setting the partial derivatives to 0 produces

$$\begin{aligned}(\mathbf{R}_{XX} - L_1 \mathbf{I}_p) \mathbf{a}_1 &= \mathbf{0} \\ \mathbf{a}_1^T \mathbf{a}_1 &= 1\end{aligned}$$

From the first equation, we see that  $L_1$  is an eigenvalue of  $\mathbf{R}_{XX}$  such that  $\mathbf{R}_{XX} \mathbf{a}_1 = L_1 \mathbf{a}_1$  and

$$S_{w_1}^2 = \mathbf{a}_1^T \mathbf{R}_{XX} \mathbf{a}_1 = L_1 \mathbf{a}_1^T \mathbf{a}_1 = L_1$$

To maximize  $S_{w_1}^2$ , we only need to pick the largest eigenvalue of  $\mathbf{R}_{XX}$ .

### Additional reference

- “A First Course in Linear Model Theory” by Nalini Ravishanker and Kipak K. Dey
- [Lecture notes](#) by Cedric Ginestet

## Ridge Regression

Ridge regression and the Lasso regression are two forms of regularized regression. These methods can be used to alleviate the consequences of multicollinearity.

1. When variables are highly correlated, a large coefficient in one variable may be alleviated by a large coefficient in another variable, which is negatively correlated to the former.
2. Regularization imposes an upper threshold on the values taken by the coefficients, thereby producing a more parsimonious solution, and a set of coefficients with smaller variance.

### Constrained optimization

Ridge regression is motivated by a constrained minimization problem, which can be formulated as

$$\begin{aligned}\hat{\boldsymbol{\beta}}^{ridge} &= \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \sum_{i=1}^n (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2 \\ \text{subject to } \|\boldsymbol{\beta}\|_2^2 &= \sum_{j=1}^p \beta_j^2 \leq t\end{aligned}$$

for  $t \geq 0$ .

Use a Lagrange multiplier, we can rewrite the formula as

$$\hat{\boldsymbol{\beta}}^{ridge} = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left\{ \sum_{i=1}^n (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2 + \lambda \sum_{j=1}^p \beta_j^2 \right\}$$

for  $\lambda \geq 0$  and where there is a one-to-one correspondence between  $t$  and  $\lambda$ .  $\lambda$  is an arbitrary constant usually referred to as the “ridge constant”.

### Analytical solutions

The ridge-regression estimator has analytical solution

$$\hat{\boldsymbol{\beta}}^{ridge} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{Y}$$

This is obtained by differentiating the objective function with respect to  $\boldsymbol{\beta}$  and set it to 0:

$$\begin{aligned}& \frac{\partial}{\partial \boldsymbol{\beta}} \{ (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^T (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) + \lambda \boldsymbol{\beta}^T \boldsymbol{\beta} \} \\ &= 2(\mathbf{X}^T \mathbf{X})\boldsymbol{\beta} - 2\mathbf{X}^T \mathbf{Y} + 2\lambda \boldsymbol{\beta} \\ &= 0\end{aligned}$$

Therefore,

$$(\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})\boldsymbol{\beta} = \mathbf{X}^T \mathbf{Y}$$

Since we are adding a positive constant to the diagonal of  $\mathbf{X}^T \mathbf{X}$ , we are, in general, producing an invertible matrix,  $\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}$  even if  $\mathbf{X}^T \mathbf{X}$  is singular. Historically, this particular aspect of ridge regression was the main motivation behind the adoption of this particular extension of OLS theory.

The ridge regression estimator is related to the classical OLS estimator,  $\hat{\boldsymbol{\beta}}^{OLS}$ , in the following manner

$$\hat{\boldsymbol{\beta}}^{ridge} = [\mathbf{I} + \lambda(\mathbf{X}^T \mathbf{X})^{-1}]^{-1} \hat{\boldsymbol{\beta}}^{OLS},$$

assuming  $\mathbf{X}^T \mathbf{X}$  is non-singular. This relationship can be verified by applying the definition of  $\hat{\boldsymbol{\beta}}^{OLS}$ ,

$$\begin{aligned} \hat{\boldsymbol{\beta}}^{ridge} &= [\mathbf{I} + \lambda(\mathbf{X}^T \mathbf{X})^{-1}]^{-1} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} \\ &= (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{Y} \end{aligned}$$

using the fact  $\mathbf{B}^{-1} \mathbf{A}^{-1} = (\mathbf{A} \mathbf{B})^{-1}$ .

Moreover, when  $\mathbf{X}$  is composed of orthonormal variables, such that  $\mathbf{X}^T \mathbf{X} = \mathbf{I}_p$ , it then follows that

$$\hat{\boldsymbol{\beta}}^{ridge} = \frac{1}{1 + \lambda} \hat{\boldsymbol{\beta}}^{OLS}$$

#### Bias and variance of ridge estimator

Ridge estimation produces a biased estimator of the true parameter  $\boldsymbol{\beta}$ . With the definition of  $\hat{\boldsymbol{\beta}}^{ridge}$  and the model assumption  $\mathbf{E}(\mathbf{Y}|\mathbf{X}) = \mathbf{X}\boldsymbol{\beta}$ , we obtain,

$$\begin{aligned} \mathbf{E}(\hat{\boldsymbol{\beta}}^{ridge}|\mathbf{X}) &= (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{X} \boldsymbol{\beta} \\ &= (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I} - \lambda \mathbf{I}) \boldsymbol{\beta} \\ &= \boldsymbol{\beta} - \lambda (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \boldsymbol{\beta} \end{aligned}$$

where the bias of the ridge estimator is proportional to  $\lambda$ . The variance of the ridge estimator is

$$\mathbf{Var}(\hat{\boldsymbol{\beta}}^{ridge}|\mathbf{X}) = \sigma^2 (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} (\mathbf{X}^T \mathbf{X}) (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1}.$$

When  $\lambda$  increases, the inverted term  $(\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1}$  is increasingly dominated by  $\lambda \mathbf{I}$ . The variance of the ridge estimator, therefore, is a decreasing function of  $\lambda$ . This result is intuitively reasonable because the estimator itself is driven toward  $\mathbf{0}$ .

#### Variance-bias tradeoff

The mean-squared error of an estimator can be decomposed into the sum of its squared bias and sampling variance.

$$\begin{aligned} MSE(\hat{\theta}) &= \mathbf{E}((\hat{\theta} - \theta)^2) = \mathbf{E}(\hat{\theta}^2) + \theta^2 - 2\theta \mathbf{E}(\hat{\theta}) \\ Bias^2(\hat{\theta}) &= [\mathbf{E}(\hat{\theta}) - \theta]^2 = \mathbf{E}^2(\hat{\theta}) + \theta^2 - 2\theta \mathbf{E}(\hat{\theta}) \\ Var(\hat{\theta}) &= \mathbf{E}(\hat{\theta}^2) - \mathbf{E}^2(\hat{\theta}) \end{aligned}$$



Therefore

$$MSE(\hat{\theta}) = Bias^2(\hat{\theta}) + Var(\hat{\theta})$$

The essential idea here is to trade a small amount of bias in the coefficient estimates for a large reduction in coefficient sampling variance. Hoerl and Kennard (1970) prove that it is always possible to choose a positive value of the ridge constant  $\lambda$  so that the mean-squared error of the ridge estimator is less than the mean-squared error of the least-squares estimator. These ideas are illustrated heuristically in Figure 21.1

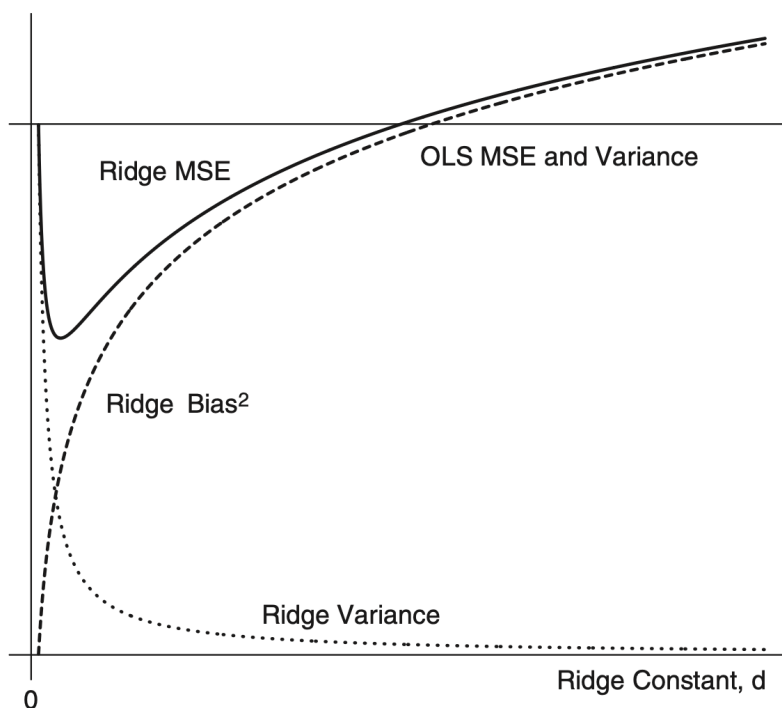


Figure 24.1: Trade-off of bias and against variance for the ridge-regression estimator. The horizontal line gives the variance of the least-squares (OLS) estimator; because the OLS estimator is unbiased, its variance and mean-squared error are the same. The broken line shows the squared bias of the ridge estimator as an increasing function of the ridge constant  $d$  (i.e.  $\lambda$  in our notes). The dotted line shows the variance of the ridge estimator. The mean-squared error (MSE) of the ridge estimator, given by the heavier solid line, is the sum of its variance and squared bias. For some values of  $d$ , the MSE error of the ridge estimator is below the variance of the OLS estimator. JF Figure 13.9.