Definition 1 (Downward closed set of restrictions)**.** *A set of restrictions $\mathcal{F}$ is called downward closed if whenever $\rho \in \mathcal{F}$ and $stars(\rho') \subseteq stars(\rho)$, $\rho' \in \mathcal{F}$.*

Lemma 2. *Let $\mathcal{F}$,$\mathcal{F}'$ be two downward closed subsets of restrictions, then $\mathcal{F} \cap \mathcal{F}'$ is also downward closed.*

1 Hastad's proof of SL

Lemma 3 (Switching lemma)**.** *Let f be computed by a depth 2 circuit of bottom fan-in k. Let $\mathcal{F}$ be a downward-closed set of restrictions from $\mathcal{R}_p$. Let $depth(f)$ denote the minimum depth of a decision-tree computing f.*

$$\Pr_{\rho}[depth(f) > s | \rho \in \mathcal{F}] \leq (5pk)^s.$$

Theorem 4 (Criticality of bounded bottom fan-in CNF/DNF)**.** *The criticality of depth 2 circuits with bottom fan-in k is k.*

Proof of lemma 3. Without loss of generality, assume f is a CNF, i.e., f can be written as

$$f = \wedge_{i=1}^{m} C_i$$

where each C_i is disjunction of at most t literals.The proof is by induction on m.

The analysis can be divided into two cases depending on whether ρ sets C_1 to 1. Let us for the sake of convenience, assume that C_1 is a disjunction of the variables $x_1, \ldots, x_{t_o}$ where $t_0 \leq t$. We can bound the probability of the lemma by the max of following two probabilites

$$\Pr_{\rho}[depth(f) > s | \rho \in \mathcal{F} \wedge C_1|_\rho \equiv 1] \tag{1}$$

and

$$\Pr_{\rho}[depth(f) > s | \rho \in \mathcal{F} \wedge C_1|_\rho \not\equiv 1] \tag{2}$$

The first term is taken care of by induction applied to f without its first clause(and thus has at most m-1 cluases). $(\rho | C_1|_\rho \equiv 1)$ is a downward closed set and hence by lemma 2, the conditioning is of the right form.

In the second term however, $(\rho | C_1|_\rho \not\equiv 1)$ is not a downward closed set of restrictions. Therefore bounding in this case needs a bit more work.

□