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Poincaré Duality for Smooth Surfaces

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Dualidade de Poincaré para Superfícies Diferenciáveis

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Abstract

The thesis at hand aims to study the de Rham cohomology of smooth surfaces and present a proof of a duality result, dating from 1895, due to H. Poincaré, namely, the Poincaré Duality Theorem. We also look into some applications of such duality involving the Euler-Poincaré characteristic and the signature of compact surfaces, and discuss its connections to the Hodge deomposition theorem. In order to do so, we develop some preliminary tools by providing an overview of basic concepts in the language of categories and functors, homological algebra and differential forms on surfaces in Euclidean spaces.

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Resumo

A presente monografia tem por objetivo estudar a cohomologia de *de Rham* para superfícies diferenciáveis e apresentar uma prova de um resultado de dualidade, datado de 1895, devido a H. Poincaré, a saber, o Teorema de Dualidade de Poincaré. Também examinamos algumas aplicações de tal dualidade, que envolvem a característica de Euler-Poincaré e a assinatura de superfícies compactas, e discutimos ainda sua conexão com o teorema de decomposição de Hodge. Para isso, desenvolvemos algumas ferramentas preliminares fornecendo uma visão geral de conceitos básicos da linguagem de categorias e functores, álgebra homológica e formas diferenciais em superfícies nos espaços Euclidianos.

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Introduction

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The concept of duality is surely ubiquitous in mathematics, yet there is no precise definition of such term that encompasses all instances in which it is used. One encounters it while studying the language of sets by learning that the complement of a union is the intersection of complements, or even during a first course in linear algebra when one learns that a finite-dimensional vector space is linearly isomorphic to its dual and bi-dual spaces.

The result chosen to be studied in this thesis possesses a duality nature, namely, the famous Duality Theorem of H. Poincaré (1854-1912), dating from 1895. However, the context adressed here is not as broad as the one in which Poincaré was working at the time. More precisely, we prove (following [**7, 12**]) his duality result for the de Rham cohomology of differentiable surfaces in Euclidean spaces (Theorem 3.17), explore some of its applications and discuss its connection to Hodge theory. In order to do so, the preliminary key concepts such as cohomology and differentiable surfaces shall be properly introduced and studied.

Let us dive into an informal description of the key topics to be treated here, so that it can shed some light on Poincaré's theorem.

Roughly speaking, a (differentiable) surface of dimension m is a set sitting in some ambient space $\mathbf{R}^n$ $(n \geq m)$ that locally resembles the Euclidean space $\mathbf{R}^m$, in the sense that it can be "built" by gluing together, with sufficient regularity, pieces of $\mathbf{R}^m$. For instance, considering only spacial coordinates, the Earth (assumed to be a spherical shell) and its equator are, respectively, surfaces of dimensions 2 and 1 in $\mathbf{R}^3$.

Recall from your calculus classes that a (smooth) vector field $F : U \to \mathbf{R}^2$, where $U \subseteq \mathbf{R}^2$ is an open set (which is a 2-dimensional surface in $\mathbf{R}^2$), is said to be irrotational if $\operatorname{rot} F = 0$ and conservative if there is a function $f : U \to R$ such that $F = \operatorname{grad} f$. Very well, the vector field

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right),$$

defined on $U = R^2 - \{(0,0)\}$ is irrotational, but not conservative, since $\oint_{S^1} F \neq 0$.

Now, the linear space of (smooth) vector fields on U has two important linear subspaces, one of which consisting of all irrotational fields and the other of conservative fields. Additionally, every conservative field is irrotational. Thus, in order to facilitate the search for irrotational fields on U which are not conservative, we may form a new vector space (not necessarily finite-dimensional) by declaring that two irrotational fields are equivalent (or cohomologous) if, and only if, their difference is conservative. This is in fact an equivalence relation on the space of irrotational fields. The resulting vector space of cohomology classes

$$H^1(U) := \frac{\{\text{irrotational fields}\}}{\{\text{conservative fields}\}} \tag{0.1}$$

is called the 1st de Rham cohomology group of U. Note that $H^1(U) = \{0\}$ if, and only if, every irrotational field is conservative. Since the vector field F described above is irrotational, but not conservative, it follows that $H^1(U)$ is non-trivial.

A known result from calculus is that, on simply connected open sets, every irrotational field is conservative. Thus, even if we couldn't exhibit the field F explicitly, the fact that $H^1(U) \neq \{0\}$ tells us that $U = \mathbf{R}^2 - \{(0,0)\}$ is not simply connected, which is fairly intuitive, since closed paths in U surrounding the origin cannot be contracted to a point in U without going through the origin.

Going back to (0.1) one may restrict his attention to irrotational and conservative fields vanishing outside a compact subset of U. The resulting quotient space

$$H^1_c(U) := \frac{\{\text{irrotational fields with compact support}\}}{\{\text{conservative fields with compact support}\}}$$

is called 1st de Rham cohomology group with compact support. (Note that, if U was a compact set, then every vector field on U would be compactly supported, whence $H^1(U) = H^1_c(U)$.)

In this particular case, the Poincaré duality theorem asserts that each cohomology class of irrotational fields on U corresponds to an unique linear functional on cohomology classes of compactly supported irrotational fields on U. Put precisely: there is a linear isomorphism

$$H^1(U) \approx (H^1_c(U))^*.$$

More generally, in order to obtain topological information about smooth surfaces, one introduces the de Rham cohomology, which is a way of assigning to each surface a sequence of real vector spaces, which are constructed through a similar process as the one described for $H^1(U)$ and $H^1_c(U)$. However, the machinery of differential forms needs to be developed, since such objects work better in higher dimensions than vector fields. In this general context, Poincaré's theorem states that, if M is an oriented m-dimensional surface and $0 \leq r \leq m$, then

$$H^r(M) \approx (H_c^{m-r}(M))^*\,,$$

where this isomorphism is given by the map

$$D_M : H^r(M) \to (H_c^{m-r}(M))^*\,,$$

where

$$D_M[\alpha] \cdot [\beta] = \int_M \alpha \wedge \beta$$

for $[\alpha] \in H^r(M)$ and $[\beta] \in H_c^{m-r}(M)$. All of this will be made precise throughout the text.

The decision to study the Poincaré duality theorem in this thesis was made mainly because the author was (and still is) interested in studying topics lying in the intersection of analysis, algebra and topology. Also, it is expected that this work be of use to any undergraduate student interested in said topics.

The overall prerequisites for reading the monograph at hand are: a solid course in undergraduate algebra, an undergraduate course in analysis on $\mathbf{R}^n$, some basic notions of general topology and some basic multilinear algebra over finite-dimensional real vector spaces. The work is structured as follows.

The first two chapters set the algebraic and analytic preliminaries for Chapter 3, which contains the main results. The first chapter begins with a brief exposition of some basic notions in the language of categories, e.g., categories, functors and natural transformations. In § 1.2 the cohomology groups and the cohomology functor are introduced. Finally, the chapter finishes with § 1.3 in which exact sequences are introduced, some properties of the Hom functor are discussed and important results regarding exact sequences and cohomology are proved, e.g., the Mayer-Vietoris Theorem and the Five Lemma.

The second chapter aims to define differential forms, integration of forms on surfaces and study the necessary results regarding such concepts. It begins with § 2.1, which contains the definition of surfaces in Euclidean spaces (with and without boundary), orientability, homotopy and partitions of unity. Next, § 2.2 introduces differential forms on surfaces and other concepts surrounding it, e.g., the pullback of forms and the exterior derivative. Finally, in § 2.3, the integral calculus on surfaces is introduced; some useful properties of integrals are discussed and Stokes' Theorem is stated.

The final chapter puts together the subjects developed in the first two chapters in order to present a proof of the Poincaré duality theorem for smooth surfaces (Theorem 3.17) and discuss some applications involving the Euler-Poincaré characteristic. It starts off with the definition of the de Rham complex in § 3.1. Some results regarding the de Rham cohomology are proved, e.g., Poincaré's Lemma (Theorem 3.3) and the homotopy invariance property. Next, in § 3.2, the de Rham cohomology with compact support is introduced and some crucial results regarding such cohomology are discussed. Lastly, § 3.3 is entirely devoted to the proof of Poincaré duality, which was broken into several lemmas. The chapter ends with a section (§ 3.4) presenting some applications of said duality.

Appendix A consists of various results used at some point throughout the text. Some of these results have been proved and some have been stated without a proof.

Chapter 1

Homological Algebra

This chapter begins with a brief introduction to the language of categories, following [**14**]. Right after, we give an overview of homology and cohomology in a more general setting, also following [**14**]. The last section was based on [**7, 14**] and it deals with exactness and the Mayer-Vietoris exact sequence. The only prerequisite for this chapter is a basic course in abstract algebra. **Throughout the text, unless otherwise stated, all rings considered are nonzero commutative rings with identity and no nonzero divisors. In particular, during this chapter, R denotes such a ring.**

1.1. Categories and Functors

Some of the main preliminary notions about categories used throughout the text will be given during this section. We begin with the concept of category.

A *category* $\mathscr{C}$ consists of

1. a class of objects;
2. for every two objects X and Y, a set $\mathrm{Mor}(X,Y)$ (also denoted $\mathrm{Mor}_{\mathscr{C}}(X,Y)$) of *morphisms*, enjoying the following properties:
   a) For every ordered triple of objects X, Y, Z of $\mathscr{C}$, there exists a function of sets assigning to a pair of morphisms $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Y,Z)$ a morphism $gf \in \mathrm{Mor}(X,Z)$, called composite morphism.
   b) For every object X, there is at least one morphism $1_X \in \mathrm{Mor}(X,X)$, called the identity morphism, such that, for every $f \in \mathrm{Mor}(X,Y)$,

   $$1_Y f = f \quad \text{and} \quad f 1_X = f\,.$$

   c) If $f \in \mathrm{Mor}(X,Y)$, $g \in \mathrm{Mor}(Y,Z)$ and $h \in \mathrm{Mor}(Z,W)$, then

   $$h(gf) = (hg)f\,.$$

Remark 1.1. Note that in the definition above the term "class" is used instead of "set". We will not digress into the difference between classes and sets; this can be found in books on axiomatic set theory. The bottom line is that not every class is a set, e.g., there is no set of all sets[1], but there is a class of sets.

It is common (and this is what we will do) to specify a category by its objects and morphisms. For instance, the category of sets and functions of sets; the category groups and group homomorphisms; the category of R-modules and R-module homomorphisms; the category of topological spaces and continuous maps.

We usually denote morphisms $f \in \mathrm{Mor}(X, Y)$ by

$$f : X \to Y \quad \text{and} \quad X \xrightarrow{f} Y\,.$$

Note that the identity morphism is unique. This follows from b) and c).

We say that a morphism $g : Y \to X$ is a left inverse of $f : X \to Y$ if $gf : X \to X$ is the identity morphism 1_X. Analogously, a morphism $h : Y \to X$ is said to be a right inverse of f when $fh : Y \to Y$ is the identity morphism 1_Y. In case h and g are, respectively, right and left inverses of f, we have $g = h$.

By an inverse morphism of $f : X \to Y$, we mean a morphism $f' : Y \to X$ which is a left and right inverse of f. We say that f is an *isomorphism* in $\mathscr{C}$ (the objects X and Y are said to be isomorphic in $\mathscr{C}$), and write $X \approx Y$, if there exists such a morphism f'. Note that every morphism f having a left and a right inverse is an isomorphism, since both inverses coincide. In this case we donoted the (unique) inverse morphism by f^{-1}.

Objects we will frequently deal with are commutative diagrams. A *diagram* consists of pictorial concatenations of morphisms in a category representing compositions (when they make sense) as the one below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "h"] & U \arrow[d, "i"] \\
Z \arrow[r] & W \arrow[r, "j"] & V
\end{tikzcd}$$

A diagram as the one above is said to *commute* if any two paths in the diagram chosen to be traversed by means of composition of morphisms, beginning and ending at the same objects, give the same result. For instance, if the diagram above is commutative, then $igf = jhf$.

Let $\mathscr{C}$ be a category. A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ is a category enjoying the following properties.

1. Objects of $\mathscr{D}$ are also objects of $\mathscr{C}$.
2. $\mathrm{Mor}_{\mathscr{D}}(X, Y) \subseteq \mathrm{Mor}_{\mathscr{C}}(X, Y)$ whenever X and Y are objects of $\mathscr{D}$.
3. The composite of two morphisms $f \in \mathrm{Mor}_{\mathscr{D}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathscr{D}}(Y, Z)$ in $\mathscr{D}$ equals their composite in $\mathscr{C}$.

[1] The existence of such set leads to Russell's paradox.

The category of sets and bijections is a subcategory of the category of sets and functions, but it is not a full subcategory, since not every function is a bijection. If the equality holds in item 2., we say that $\mathscr{D}$ is a full subcategory of $\mathscr{C}$. An example of full subcategory of the category of sets an functions is the one consisting of finite sets and functions. Also, the category of abelian groups and homomorphisms is a full subcategory of the category of groups.

Two useful types of objects in a category are initial and final objects. They help in the generalization of cartesian products and direct sums to the context of categories. They are defined as follows.

An object I in a category $\mathscr{C}$ is said to be *intial* in $\mathscr{C}$ if for every object X in $\mathscr{C}$ there exists only one morphism in $\mathrm{Mor}(I, X)$. Any two initial objects in a category are isomorphic. Indeed, if I and I' are initial in $\mathscr{C}$, then $\mathrm{Mor}(I, I) = \{1_I\}$, $\mathrm{Mor}(I', I') = \{1_{I'}\}$, $\mathrm{Mor}(I, I') = \{f\}$ and $\mathrm{Mor}(I', I) = \{g\}$. Thus $gf = 1_I$ and $fg = 1_{I'}$.

Example 1.2. In the category of commutative rings and ring homomorphisms, $\mathbf{Z}$ is an example of an initial object. During a first course in abstract algebra we learn that, for every ring R, there exists a unique ring homomorphism from $f : \mathbf{Z} \to R$, namely, $f(n) = 1_R + \cdots + 1_R$ (n-times).

Similarly, we say that an object F is *final* in $\mathscr{C}$ if, for every object X, the set $\mathrm{Mor}(X, F)$ is a singleton. Note that any two final objects in a category are isomorphic.

Example 1.3. Singletons are examples of final objects in the category of sets and functions. The subcategory of sets and bijections does not admit intial nor final objects.

Let us fix a family of objects $(Y_n)_{n\in L}$ in a category $\mathscr{C}$. We then produce a new category $\mathscr{S}(Y_n)_{n\in L}$ whose objects are families of morphisms having the same range $(f_n : Y_n \to Z)_{n\in L}$ and whose morphisms from an object $(f_n : Y_n \to Z)_{n\in L}$ to $(f'_n : Y_n \to Z')_{n\in L}$ is a morphism $h : Z \to Z'$ such that $hf_n = f'_n$ for every $n \in L$. The *sum* (or *coproduct*) of the family $(Y_n)_{n\in L}$ is an initial object in $\mathscr{S}(Y_n)_{n\in L}$ and is denoted by

$$\bigoplus_{n\in L} Y_n \,.$$

Bottom line: the sum of $(Y_n)_{n\in L}$ in $\mathscr{C}$ consists of an object $\bigoplus_{n\in L} Y_n$ of $\mathscr{C}$ together with a family of morphisms $(i_k : Y_k \to \bigoplus Y_n)_{k\in L}$ (called inclusions) such that, for any other family $(f'_k : Y_k \to Z')_{k\in L}$, there exists an unique morphism $h : \bigoplus Y_n \to Z'$ making the diagram below commute for each $k \in L$.

$$\begin{tikzcd} & \bigoplus Y_n \arrow[dd, dashed, "\exists! h"] \\ Y_k \arrow[ur, "i_k"] \arrow[dr, "f'_k"'] & \\ & Z' \end{tikzcd}$$

Example 1.4. The disjoint union of sets $\bigcup_n Y_n \times \{n\}$ together with the inclusion maps $i_k : y \mapsto (y, k)$ is a coproduct in the category of sets and functions. Also, the directs sum of R-modules and the direct sum of abelian groups are examples of coproducts. (See [1].)

Remark 1.5. The standard notation for coproducts is $\coprod Y_n$, yet we shall use $\bigoplus Y_n$ since it fits the context better.

In a similar fashion, we produce a new category $\mathscr{P}(Y_n)_{n\in L}$ whose objects are families of morphisms having the same domain $(g_n : Z \to Y_n)_{n\in L}$ and whose morphisms from an object $(g_n : Z \to Y_n)_{n\in L}$ to $(g'_n : Z' \to Y_n)_{n\in L}$ is a morphism $h : Z \to Z'$ such that $g_n = g'_n h$ for every $n \in L$. The *product* of the family $(Y_n)_{n\in L}$ is a final object in $\mathscr{P}(Y_n)_{n\in L}$ and is denoted by

$$\prod_{n\in L} Y_n \,.$$

In other words, the product of $(Y_n)_{n\in L}$ in $\mathscr{C}$ consists of an object $\prod_{n\in L} Y_n$ of $\mathscr{C}$ together with a family of morphisms $(p_k : \prod Y_n \to Y_k)_{k\in L}$ (called projections) such that, for any other family $(g'_k : Z' \to Y_k)_{k\in L}$, there exists an unique morphism $h : Z' \to \prod Y_n$ making the diagram below commute for each $k \in L$.

$$\begin{tikzcd}
Z' \arrow[dd, dashed, "\exists! h"'] \arrow[dr, "g'_k"] & \\
& Y_k \\
\prod Y_n \arrow[ur, "p_k"'] &
\end{tikzcd}$$

Example 1.6. The cartesian product of sets together with the usual projections is a product in the category of sets and functions. Also, the direct product of groups and the direct product of R-modules are examples of products in their respective categories. (See [1].)

1.1.1. Functors

To conclude this section we define "maps" between categories, called functors.

A *covariant functor* (resp.*contravariant*) T between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of an object function which assings to every object X of $\mathscr{C}$ an object $T(X)$ of $\mathscr{D}$ and a morphism function assigning to every morphism $f : X \to Y$ of $\mathscr{C}$ a morphism $T(f) : T(X) \to T(Y)$ (resp. $T(f) : T(Y) \to T(X)$) of $\mathscr{D}$ such that

1. $T(1_X) = 1_{T(X)}$
2. $T(fg) = T(f)T(g)$ (resp. $T(fg) = T(g)(f)$).

We say that covariant functors act on morphisms by preserving the arrows and that contravariant ones act by reversing the arrows.

Example 1.7. Let $\mathscr{C}$ be a category. The covariant functor $1_{\mathscr{C}}$ assigning each object X to itself and each morphism $f : X \to Y$ to itself is called identity functor.

Example 1.8. There is covariant functor from the category of groups to the category of sets which maps each group to its underlying set and each group homomorphism to its underlying set-function. This functor is called forgetful, since objects and morphisms lose structure via such correspondence.

Example 1.9. Another useful example is the dualization functor $\mathrm{Hom}(\cdot, R)$, which is a contravariant functor from the category of R-modules to itself. It maps each module M to its dual module $\mathrm{Hom}(M, R)$ and a homomorphism $f : M \to N$ to its transpose homomorphism $\mathrm{Hom}(f, R) : \mathrm{Hom}(N, R) \to \mathrm{Hom}(M, R)$. In the context vector spaces, we usually write $E^* = \mathrm{Hom}(E, \mathbf{R})$ and $f^* = \mathrm{Hom}(f, R)$ (or $f^{\#}$ depending on the context).

During this chapter, we will deal mostly with $\mathrm{Hom}(\cdot, R)$ and the cohomology functor H (to be defined in next section).

Functors are important tools for finding algebraic invariants since they preserve isomorphisms, as shown in the proposition below.

Proposition 1.1. *Let T be a covariant (or contravariant) functor from a category $\mathscr{C}$ to a category $\mathscr{D}$. Then T maps isomorphisms in $\mathscr{C}$ to isomorphisms in $\mathscr{D}$.*

Proof. Let T be covariant. If $f : X \to Y$ is an isomorphism in $\mathscr{C}$, then T applied to $ff^{-1} = 1_Y$ and $f^{-1}f = 1_X$ yields $T(f)T(f^{-1}) = 1_{T(Y)}$ and $T(f^{-1})T(f) = 1_{T(X)}$, respectively. Thus $T(f^{-1}) = T(f)^{-1}$, which shows that $T(f)$ is an isomorphism in $\mathscr{D}$. A similar argument can be applied to the contravariant case. Q.E.D.

If $T : \mathscr{C} \to \mathscr{D}$ and $F : \mathscr{D} \to \mathscr{E}$, we define the composite functor FT from $\mathscr{C} \to \mathscr{E}$ as follows:

1. For every object X of $\mathscr{C}$, $FT(X) = F(T(X))$.
2. For every morphism $f \in \mathrm{Mor}_{\mathscr{C}}(X, Y)$, $FT(f) = F(T(f))$.

The next proposition is easily verified.

Proposition 1.2. *The composition of contravariant and covariant functors (resp. covariant and contravariant) is a contravariant functor. Similarly, the composition of two contravariant functors (resp. covariant) is a covariant functor.*

Example 1.10. There is covariant functor $(\,\cdot\,)^{**} = \mathrm{Hom}(\mathrm{Hom}(\,\cdot\,, \mathbf{R}), \mathbf{R})$ on the category of finite-dimensional real vector spaces which assigns to each vector space E its double dual E^{**} and to each linear map $A : E \to F$ the induced linear map $A^{**} : E^{**} \to F^{**}$ given by $A^{**}(\xi) = \xi A^*$. This is called the double dual functor.

If functors are maps between categories, we can go further and define maps between functors, which are called natural transformations.

Let T_1 and T_2 be two covariant functors from a category $\mathscr{C}$ to a category $\mathscr{D}$. A *natural transformation* φ between T_1 and T_2 consists of a function from the objects of $\mathscr{C}$ to morphisms of $\mathscr{D}$ which assigns to each object X a morphism $\varphi(X) : T_1(X) \to T_2(X)$

such that, for every morphism $f : X \to Y$ of $\mathscr{C}$, the following diagram commutes:

$$\begin{array}{ccc} T_1(X) & \xrightarrow{T_1(f)} & T_1(Y) \\ {\scriptstyle \varphi(X)}\downarrow & & \downarrow{\scriptstyle \varphi(Y)} \\ T_2(X) & \xrightarrow{T_2(X)} & T_2(Y) \end{array}$$

Example 1.11. Let $\mathscr{C}$ be the category of finite-dimensional real vector spaces. There is a natrual transformation φ between the identity functor $1_{\mathscr{C}}$ (Example 1.7) and the double dual functor $(\cdot)^{**}$ (Example 1.10) which assigns to each vector space E the map $\varphi(E) : E \to E^{**}$ defined by $\varphi(E)(v) \cdot \xi = \xi(v)$ (evaluation at v). Given a linear map $A : E \to F$, the commutativity of the diagram

$$\begin{array}{ccc} E & \xrightarrow{A} & F \\ {\scriptstyle \varphi(E)}\downarrow & & \downarrow{\scriptstyle \varphi(F)} \\ E^{**} & \xrightarrow{A^{**}} & F^{**} \end{array}$$

follows from the definitions of $\varphi(E)$ and A^{**}.

We shall derive an example of a natural transformation between cohomology functors in § 1.3. (See Lemma 1.16.)

1.2. Cohomology

The key concept in Chapter 3 is that of cohomology group. This section aims to define such groups and prove some crucial results concerning these groups such as Lemma 1.16 and Theorem 1.17. We begin by defining some graded structures.

A *graded R-module* M consists of a family of R-modules M_q indexed by the integers.[2] Elements belonging to the module M_q are said to have degree q.

A morphism $f : M \to N$ of degree d between graded modules consists of a family $(f_q : M_q \to N_{q+d})_{q\in\mathbf{Z}}$ of homomorphisms of R-modules. If $f : M \to N$ and $g : N \to P$ are morphisms of degree d and d', respectively, then it follows from the diagram

$$M_q \xrightarrow{f_q} N_{q+d} \xrightarrow{g_{q+d}} P_{q+d+d'}$$

that the composite morphism gf has degree $d + d'$. Therefore, there is a category of graded R-modules and morphisms of graded R-modules, with each morphism having some degree $d \in \mathbf{Z}$. This category has a subcategory of graded R-modules and morphisms of R-modules with fixed degree 0.

[2] In standard literature, a graded S-module consists of an S-module N, over a graded ring $S = \bigoplus S_q$, which admits a direct sum decomposition of abelian groups $N = \bigoplus N_q$ such that $S_p N_q \subseteq N_{p+q}$. However, in the sense of our definition, if we consider R as the result of a gradation $R = \bigoplus R_q$ which is nontrivial only in degree 0, we may view $M = (M_q)_{q\in\mathbf{Z}}$ as a the module $\bigoplus M_q$ over $R = \bigoplus R_q$.

A graded module M is said to be *finitely generated* if each M_q is finitely generated and $M_q = \{0\}$ except for a finite set of indices $F \subseteq \mathbf{Z}$. For a finitely generated graded R-module, its *Euler-Poincaré characteristic* $\chi(M)$ is defined by

$$\chi(M) = \sum_{q \in F} (-1)^q r_q(M)\,,$$

where $r_q(M) = \operatorname{rank}(M_q)$. We shall compute the Euler characteristc of specific graded modules in Chapter 3.

A *differential graded R-module* consists of a graded R-module M together with a morphism (of some degree d) $\partial : M \to M$, called differential, such that $\partial \circ \partial = 0$, that is, the composite

$$M_{q-d} \xrightarrow{\partial_{q-d}} M_q \xrightarrow{\partial_q} M_{q+d}$$

is the trival homomorphism for every $q \in \mathbf{Z}$.

A *cochain complex* over a ring R is a differential graded R-module whose differential, called *coboundary operator*, has degree $+1$. More explicitly, a cochain complex C^* consists of a family of R-modules $(C^q)_{q\in\mathbf{Z}}$ together with family of homomorphisms $(\delta^q : C^q \to C^{q+1})_{q\in\mathbf{Z}}$ such that the composite

$$C^{q-1} \xrightarrow{\delta^{q-1}} C^q \xrightarrow{\delta^q} C^{q+1}$$

is the trival homomorphism; we write $C^* = (C^q, \delta^q)_{q\in\mathbf{Z}}$. The elements of C^q are called the *q-cochains* of the complex C^*. If $C^q = \{0\}$ for all $q < 0$ the complex is said to be nonnegative.

For a cochain complex C^* we define the *group of cocycles* to be the graded R-module

$$Z(C^*) = \big(Z^q(C) = \ker \delta^q\big)_{q\in\mathbf{Z}}$$

and the *group of coboundaries* to be the graded R-module

$$B(C^*) = \big(B^q(C^*) = \operatorname{im} \delta^{q-1}\big)_{q\in\mathbf{Z}}\,.$$

Elements of the group $Z^q(C^*)$ are called *q-cocycles* and elements of $B^q(C^*)$ are called *q-coboundaries*.

Note that $B^q(C^*) \subseteq Z^q(C^*)$ for each $q \in \mathbf{Z}$, since $\delta \circ \delta = 0$. This allows us to define the *cohomology group* (over R) of the complex as the graded R-module

$$H(C^*) = \big(H^q(C^*) = Z^q(C^*)/B^q(C^*)\big)_{q\in\mathbf{Z}}\,.$$

We call $H^q(C^*)$ the q-th cohomology group (or q-dimensional cohomology group) of the complex C^*. An element $[z] \in H^q(C^*)$ is called qth *cohomology class* of $z \in Z^q(C^*)$. Two q-cocycles z and z' are said to be *cohomologous* if they belong to the same cohomology class; this means they differ by a coboundary.

Let C^* be a cochain complex. In case the cohomology module $H(C^*)$ is finitely generated, the numbers $b_q(C^*) = \operatorname{rank}(H^q(C^*))$ are called the *Betti numbers* of C^*. In

Chapter 3, we shall relate, via Poincaré dualilty, Betti numbers of two special examples of cochain complexs.

By a *cochain map* (or cochain transformation) between cochain complexes we mean a morphism of graded R-modules $f : C^* \to \bar{C}^*$ of degree 0 commuting with the differentials, that is, commutativity holds in each square

$$\begin{tikzcd} C^q \arrow[r, "\bar{\delta}^q"] \arrow[d, "f_q"'] & C^{q+1} \arrow[d, "f_{q-1}"] \\ \bar{C}^q \arrow[r, "\bar{\delta}^q"] & \bar{C}^{q+1} \end{tikzcd}$$

Therefore, there is a category of cochain complexes (over R) whose objects are cochain complexes over R and whose morphisms are cochain maps.

Remark 1.12. When it comes cochain maps, the usual notation in the literature is f^q instead of f_q. Nevertheless, we will keep using subscripts so that the notation gets lighter during Chapter 3.

From the commutative rectangle

$$\begin{tikzcd} C^{q-1} \arrow[r, "\delta^{q-1}"] \arrow[d, "f_{q+1}"'] & C^q \arrow[r, "\delta^q"] \arrow[d, "f_q"'] & C^{q+1} \arrow[d, "f_{q-1}"] \\ \bar{C}^{q-1} \arrow[r, "\bar{\delta}^{q-1}"] & \bar{C}^q \arrow[r, "\bar{\delta}^q"] & \bar{C}^{q+1} \end{tikzcd}$$

we see that, for $z \in Z^q(C^*)$,

$$\bar{\delta}^q(f_q(z)) = f_{q+1}(\delta^q(z)) = f_{q+1}(0) = 0$$

and

$$f_q(c) = f_q(\delta^{q-1}(z)) = \bar{\delta}^{q-1}(f_{q+1}(z))$$

for $c = \delta^{q-1}(z) \in B^q(C^*)$. This means that, for each q, the homomorphism $f_q : C^q \to \bar{C}^q$ maps q-cocycles of C to q-cocycles of $\bar{C}$ and q-coboundaries of C to q-coboundaries of $\bar{C}$. Thus, each f_q induces (Theorem A.5) a homomorphism $f'_q : H^q(C^*) \to H^q(\bar{C}^*)$ between homology groups, given by $f'_q([z]) = [f_q(z)]$ for $z \in Z^q(C^*)$. This shows that a cochain map $f : C^* \to \bar{C}^*$ induces a morphism

$$f^* : H(C^*) \to H(\bar{C}^*)$$

of degree 0 between the respective cohomology groups, where $f^*_q = (f^*)_q = f'_q$.

Given two cochain maps $f : C^* \to \bar{C}^*$ and $g : \bar{C}^* \to \tilde{C}^*$, we have $(g_q f_q)^* = (g_q)^*(f_q)^*$, whence $(gf)^* = g^* f^*$. It then follows that there exists a covariant functor, called *cohomology functor*, from the category of cochain complexes over R and cochain maps to the category of graded R-modules and morphisms of degree 0 which assigns to a cochain complex C^* its homology group $H(C^*)$ and to a cochain map f its induced morphism $H(f) = f^*$.

Let C^* be a cochain complex. By a *subcomplex* of $C^* = (C^q, \delta^q)_{q\in\mathbf{Z}}$ we mean a cochain complex $\bar{C}^* = (\bar{C}^q, \bar{\delta}^q)_{q\in\mathbf{Z}}$ such that $\bar{C}^q \subseteq C^q$, with $\delta^q(\bar{C}^q) \subseteq \bar{C}^{q+1}$ for every q, and $\bar{\delta}^q = \delta^q|_{\bar{C}^q}$. In this case we write $\bar{C}^* \subseteq C^*$.

Example 1.13. If $\bar{C}^*, \tilde{C}^* \subseteq C^*$ are subcomplexes, then the intersections $\bar{C}^q \cap \tilde{C}^q$ give rise to a subcomplex of C^*, denoted by $\bar{C}^* \cap \tilde{C}^*$. In case $C^q = \bar{C}^q + \tilde{C}^q$ for every q, we write $C^* = \bar{C}^* + \tilde{C}^*$; this is called a decomposition of the cochain complex C^*.

If $(C_j^*)_{j\in J}$ is a family of cochain complexes, we define the sum cochain complex $\bigoplus C_j^*$ by setting

$$\Big(\bigoplus_{j\in J} C_j^*\Big)^q = \bigoplus_{j\in J} C_j^q\,.$$

Its coboundary operators $\delta_\oplus^q : \bigoplus_{j\in J} C_j^q \to \bigoplus_{j\in J} C_j^{q+1}$ are given by $\delta_\oplus^q((z_j)_{j\in J}) = (\delta_j^q(z_j))_{j\in J}$. Using the fact that $\bigoplus_{j\in J} C_j^q$ are coproducts (sums) in the category of R-modules one shows that $\bigoplus C_j^*$ is in fact a coproduct in the category of cochain complexes.

From the definition of $\delta_\oplus^q$ it follows that

$$Z^q\Big(\bigoplus_{j\in J} C_j^*\Big) = \bigoplus_{j\in J} Z^q(C_j^*) \quad \text{and} \quad B^q\Big(\bigoplus_{j\in J} C_j^*\Big) = \bigoplus_{j\in J} B^q(C_j^*)\,,$$

whence

$$H^q\Big(\bigoplus_{j\in J} C_j^*\Big) = \bigoplus_{j\in J} H^q(C_j^*)$$

for every $q \in \mathbf{Z}$. Analogously, for products of cochain complexes we have

$$H^q\Big(\prod_{j\in J} C_j^*\Big) = \prod_{j\in J} H^q(C_j^*)\,.$$

Therefore,

$$H\Big(\bigoplus_{j\in J} C_j^*\Big) = \bigoplus_{j\in J} H(C_j^*) \quad \text{and} \quad H\Big(\prod_{j\in J} C_j^*\Big) = \prod_{j\in J} H(C_j^*)\,.$$

Dual to the notion of cochain complex is that of a *chain complex* (over R). By that we mean a differential graded R-module $C_* = (C_q)_{q\in\mathbf{Z}}$ whose differential, called boundary operator, has degree -1, that is, C_* consists of a family of R-modules $(C_q)_{q\in\mathbf{Z}}$ together with family of homomorphisms $(\partial_q : C_q \to C_{q-1})_{q\in\mathbf{Z}}$ such that the composite

$$C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1}$$

is the trival homomorphism. The *homology group* (over R) of C_* is defined as

$$H(C_*) = \big(H_q(C_*) = Z_q(C_*)/B_q(C_*)\big)_{q\in\mathbf{Z}}\,,$$

where $Z_q(C_*) = \ker \partial_q$ is the *group of cycles* of C_* and $Z_q(C_*) = \operatorname{im} \partial_{q+1}$ is the *group of boundaries* of C_*. The R-module $H_q(C_*)$ the q-th homology group (or q-dimensional homology group) of the complex C_*. An element $[z] \in H_q(C_*)$ is called qth *homology* *class* of z. Two q-cycles z and z' are said to be *homologous* if they belong to the same homology class.

We shall encounter, during § 3.4, an example of chain complex in the context of differential forms.

For chain complexes we have facts analogous to those about cochain complexes stated above. For instance, *chain map* between chain complexes is a morphism of graded R-modules $f : C_* \to \bar{C}_*$ of degree 0 commuting with the differentials (in this case boundary operators). Also, a chain map $f : C_* \to \bar{C}_*$ induces a morphism of degree 0 at the homology level. In this case we write $f_* : H(C_*) \to H(\bar{C}_*)$, where $(f_*)_q([z]) = [f_q(z)]$.

There exists a covariant functor, called *homology functor*, from the category of chain complexes over R and chain maps to the category of graded R-modules and morphisms of degree 0 which assigns to a chain complex C_* its homology group $H(C_*)$ and to a chain map f its induced morphism $H(f) = f_*$.

1.3. Exact Sequences

A three-term sequence of R-modules and R-module homomorphisms

$$\tilde{C} \xrightarrow{\ g\ } C \xrightarrow{\ f\ } \bar{C}$$

is said to be exact at C when $\operatorname{im} g = \ker f$. In this case, it follows that f is a injective if, and only if, $g \equiv 0$. Also, g is surjective if, and only if, $f \equiv 0$. Thus, the exactness of

$$0 \longrightarrow C \xrightarrow{\ f\ } \bar{C}$$

is equivalent to the injectivity of f, and the exactness of

$$\tilde{C} \xrightarrow{\ g\ } C \longrightarrow 0$$

is equivalent to g being onto.

A morphism of degree $+1$ on a graded R-module $(f_q : C_q \to C_{q+1})_{q\in\mathbf{Z}}$ is said to be an *exact sequence* if every three-term subsequence of consecutive R-module homomorphisms is exact at its middle term, i.e., the kernel of each homomorphism coincides with the image of the preceding one. Note that (co)chain complex is exact if, and only if, its (co)homology groups are trivial in every dimension.

There is a category of exact sequences of R-modules, with morphisms between exact sequences $(f_q : C_q \to C_{q+1})_{q\in\mathbf{Z}}$ and $(g_q : \bar{C}_q \to \bar{C}_{q+1})_{q\in\mathbf{Z}}$ being another sequence $(\varphi_q : C_q \to \bar{C}_q)_{q\in\mathbf{Z}}$ such that the diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{q-1} & \xrightarrow{f_{q-1}} & C_q & \xrightarrow{f_q} & C_{q+1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \varphi_{q-1}} & & \big\downarrow{\scriptstyle \varphi_q} & & \big\downarrow{\scriptstyle \varphi_{q+1}} & & \\
\cdots & \longrightarrow & \bar{C}_{q-1} & \xrightarrow{g_{q-1}} & \bar{C}_q & \xrightarrow{g_q} & \bar{C}_{q+1} & \longrightarrow & \cdots
\end{array}$$

Theorem 1.17 describes an important functor from the category of cochain complexes to the one above.

Applying the (contravariant) functor $\operatorname{Hom}(\cdot\,,R)$ to a sequence of R-module homomorphisms

$$\cdots \xrightarrow{f} C_{q-1} \xrightarrow{f_{q-1}} C_q \xrightarrow{f_q} C_{q+1} \xrightarrow{f} \cdots$$

one obtains a sequence of duals, namely,

$$\cdots \xrightarrow{f^{\#}} \operatorname{Hom}(C_{q+1},R) \xrightarrow{f^{\#}_{q+1}} \operatorname{Hom}(C_q,R) \xrightarrow{f^{\#}_{q}} \operatorname{Hom}(C_{q-1},R) \xrightarrow{f^{\#}} \cdots ,$$

where $f^{\#}_{q+1} = \operatorname{Hom}(f_q,R)$[3] is given by $(f^{\#}_{q+1}\varphi)(v) = \varphi(f_q(v))$ for $v \in C_q$. Such dualizations will be useful during § 3.2.

An exact five-term sequence of R-modules and R-module homomorphisms

$$0 \longrightarrow \tilde{C} \xrightarrow{f} C \xrightarrow{g} \bar{C} \longrightarrow 0 \tag{1.1}$$

is called a *short exact sequence*. Note that a short sequence as this one is exact if, and only if, $\operatorname{im} f = \ker g$, f is injective and g is surjective.

Example 1.14. Two useful examples of a short exact sequences of R-modules are those originating from an R-module homomorphism $f : C \to \bar{C}$, namely,

$$0 \longrightarrow \ker f \xhookrightarrow{i} C \xrightarrow{f} f(C) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \xhookrightarrow{i} \bar{C} \xrightarrow{\pi} \operatorname{coker} f \longrightarrow 0\,,$$

where $\operatorname{coker} g = \bar{C}/\operatorname{im} f$.

A short exact sequence of R-modules as the one in (1.1) is said to be *split* if g has a right inverse. In this case, there is an isomorphism $C \approx \tilde{C} \oplus \bar{C}$. For a proof, see [**14**, p. 217].

Example 1.15. In the context of vector spaces ($\mathbf{R}$-modules) every short exact splits; this fact will prove useful during § 3.1. It follows from a more general fact: every short exact sequence of R-modules (as the one in (1.1)) splits whenever $\bar{C}$ is free. Indeed, given a basis $(\bar{c}_i)_{i\in I}$, for each $i \in I$ we can choose (since g is onto) $c_i \in C$ so that $g(c_i) = \bar{c}_i$. There exists a unique homomorphism $h : \bar{C} \to C$ such that $h(\bar{c}_i) = c_i$. Thus $gh = 1_{\bar{C}}$, as we wanted to show.

Proposition 1.3. *Let R be a field. The contravariant functor* $\operatorname{Hom}(\cdot, R)$ *on the category of R-modules to itself is exact. In particular, if the sequence*

$$\tilde{C} \longrightarrow C \longrightarrow \bar{C}$$

is exact, then the same holds for the dual sequence

$$\operatorname{Hom}(\bar{C},R) \longrightarrow \operatorname{Hom}(C,R) \longrightarrow \operatorname{Hom}(\tilde{C},R)\,.$$

[3] To simplify notation, when we apply $\operatorname{Hom}(\cdot\,,R)$ to an R-module homomorphism f we will write $f^{\#}$ instead of $\operatorname{Hom}(f\,,R)$. We will also write $C^* = \operatorname{Hom}(C,R)$ whenever it is convenient.

Proof. When S is an arbitrary ring, the contravariant functor $\operatorname{Hom}(\cdot, S)$ on the category of S-modules is not necessarily exact. However, every short exact sequence of S-modules

$$0 \longrightarrow \tilde{C} \xrightarrow{\ i\ } C \xrightarrow{\ j\ } \bar{C} \longrightarrow 0\,,$$

in which $\bar{C}$ is a free module, the dual sequence

$$0 \longrightarrow \operatorname{Hom}(\bar{C}, S) \xrightarrow{\ j^{\#}\ } \operatorname{Hom}(C, S) \xrightarrow{\ i^{\#}\ } \operatorname{Hom}(\tilde{C}, S) \longrightarrow 0\,,$$

is exact; see [7, p. 18]. Thus, in the case at hand, $\operatorname{Hom}(\cdot, R)$ is exact, since every R-vector space is a free module.

For the second part of the statement above, we use a more general argument. A three-term exact sequence of R-modules $\tilde{C} \to C \to \bar{C}$ gives rise to commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \ker f \arrow[dr] & & & & \operatorname{im} g \arrow[ur] \arrow[dr] & & & \\
& & \tilde{C} \arrow[rr, "f"] \arrow[dr] & & C \arrow[rr, "g"] \arrow[ur] & & \bar{C} \arrow[dr] & & \\
& & & \operatorname{im} f \arrow[ur] \arrow[dr] & & & & \operatorname{coker} g \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

where each diagonal is a short exact sequence. Thus, if T is an exact contravariant functor (i.e., preserves short exact sequence and reverses the arrows) on the category of R-modules to itself, we obtain another commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& T(\operatorname{coker} g) \arrow[dr] & & & & T(\operatorname{im} f) \arrow[ur] \arrow[dr] & & & \\
& & T(\bar{C}) \arrow[rr, "T(f)"] \arrow[dr] & & T(C) \arrow[rr, "T(g)"] \arrow[ur] & & T(\tilde{C}) \arrow[dr] & & \\
& & & T(\operatorname{im} g) \arrow[ur] \arrow[dr] & & & & T(\ker f) \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

where the diagonals are again exact. We then conclude that

$$\begin{aligned}
\operatorname{im} T(f) &= \operatorname{im}(T(\bar{C}) \to T(\operatorname{im} g) \to T(C)) \\
&= \operatorname{im}(T(\operatorname{im} g) \to T(C)) \\
&= \ker(T(C) \to T(\operatorname{im} f)) \\
&= \ker(T(C) \to T(\operatorname{im} f) \to T(\tilde{C})) \\
&= \ker T(g)\,.
\end{aligned}$$

Therefore, the sequence $T(\bar{C}) \to T(C) \to T(\tilde{C})$ is exact.

Q.E.D.

Another useful property of $\mathrm{Hom}(\cdot\,, R)$ is the following.

Proposition 1.4. *Let R be a ring. If $(C_q)_{q\in L}$ is a family of R-modules, then*

$$\mathrm{Hom}\Big(\bigoplus_{q\in L} C_q, R\Big) \approx \prod_{q\in L} \mathrm{Hom}(C_q, R)\,.$$

Proof. Let $i_k : A_k \to \bigoplus C_q$ be the natural inclusion. It suffices to see that the R-module homomorphism

$$\begin{aligned} \xi : \mathrm{Hom}\Big(\bigoplus_{q\in L} C_q, R\Big) &\longrightarrow \prod_{q\in L} \mathrm{Hom}(C_q, R) \\ f &\longmapsto (f \circ i_q)_{q\in L} \end{aligned}$$

is an isomorphism whose inverse homomorphism is

$$\begin{aligned} \prod_{q\in L} \mathrm{Hom}(C_q, R) &\longrightarrow \mathrm{Hom}\Big(\bigoplus_{q\in L} C_q, R\Big) \\ (f_q)_{q\in L} &\longmapsto \quad f : \bigoplus_{q\in L} C_q \longrightarrow R \\ &\qquad\quad (c_q)_{q\in L} \longmapsto \sum_{q\in L} f_q(c_q) \end{aligned}$$

Q.E.D.

A short exact sequence of cochain complexes is a five-term sequence of cochain complexes and cochain maps

$$0 \longrightarrow \tilde{C}^* \xrightarrow{\ f\ } C^* \xrightarrow{\ g\ } \bar{C}^* \longrightarrow 0$$

such that, for each $q \in Z$,

$$0 \longrightarrow \tilde{C}^q \xrightarrow{\ f_q\ } C^q \xrightarrow{\ g_q\ } \bar{C}^q \longrightarrow 0$$

is a short exact sequence of R-modules. A morphism between short exact sequences of cochain complexes is just a commutative diagram of cochain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{C}^* & \xrightarrow{\ f\ } & C^* & \xrightarrow{\ g\ } & \bar{C}^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\tilde{\varphi}} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} & & \\ 0 & \longrightarrow & \tilde{E}^* & \xrightarrow{\ i\ } & E^* & \xrightarrow{\ j\ } & \bar{E}^* & \longrightarrow & 0 \end{array} \tag{1.2}$$

There is a category of short exact sequences of cochain complexes. We define three covariant functors $\tilde{H}$, H and $\bar{H}$ from this category to the category of graded R-modules which assign to each short exact sequence of cochain complexes

$$0 \longrightarrow \tilde{C}^* \xrightarrow{\ f\ } C^* \xrightarrow{\ g\ } \bar{C}^* \longrightarrow 0$$

the graded R-modules $H(\tilde{C}^*)$, $H(C^*)$ and $H(\bar{C}^*)$, respectively. Following this notation, we have the result below.

Lemma 1.16. *On the category of short exact sequences of cochain complexes*

$$S: \quad 0 \longrightarrow \tilde{C}^* \xrightarrow{f} C^* \xrightarrow{g} \bar{C}^* \longrightarrow 0$$

there is a natural transformation $\delta^*: \bar{H} \to \tilde{H}$ *such that* $\delta^*(S): H(\bar{C}^*) \to H(\tilde{C}^*)$ *is a morphism of degree* $+1$ *and* $(\delta^*(S))_q[\bar{z}] = [(f_{q+1})^{-1}\delta^q g_q^{-1}(\bar{z})] \in H^q(\tilde{C}^*)$ *for* $[\bar{z}] \in H^q(\bar{C}^*)$.[4]

Proof. From the short exact sequence S we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{C}^{q+1} \arrow[r, "f_{q+1}"] & C^{q+1} \arrow[r, "g_{q+1}"] & \bar{C}^{q+1} \arrow[r] & 0 \\
0 \arrow[r] & \tilde{C}^{q} \arrow[r, "f_{q}"] \arrow[u, "\tilde{\delta}^q"] & C^{q} \arrow[r, "g_{q}"] \arrow[u, "\delta^q"] & \bar{C}^{q} \arrow[r] \arrow[u, "\bar{\delta}^q"] & 0 \\
0 \arrow[r] & \tilde{C}^{q-1} \arrow[r, "f_{q-1}"] \arrow[u, "\tilde{\delta}^{q-1}"] & C^{q-1} \arrow[r, "g_{q-1}"] \arrow[u, "\delta^{q-1}"] & \bar{C}^{q-1} \arrow[r] \arrow[u, "\bar{\delta}^{q-1}"] & 0
\end{tikzcd}$$

where each row is a short exact sequence of R-modules. Let $\bar{z} \in Z^q(\bar{C})$, where $\bar{z} = g_q(c)$ for some $c \in C^q$, since g_q is surjective. From commutativity we have

$$g_{q+1}(\delta^q(c)) = \bar{\delta}^q(g_q(c)) = \bar{\delta}^q(\bar{z}) = 0\,,$$

whence $\delta^q(c) \in \ker g_{q+1}$. Thus $\delta^q(c) = f_{q+1}(\tilde{c})$ for some $\tilde{c} \in \tilde{C}^{q+1}$. Again, commutativity gives us

$$0 = \delta^{q+1}(\delta^q(c)) = \delta^{q+1}(f_{q+1}(\tilde{c})) = f_{q+2}(\tilde{\delta}^{q+1}(\tilde{c}))\,,$$

and by the injectivity of f_{q+2}, it follows that $\delta^{q+1}(\tilde{c}) = 0$. Therefore, $\tilde{c} \in Z_{q+1}(\tilde{C}^*)$ and $\tilde{c} = f_{q+1}^{-1}(\delta^q(g_q^{-1}(\bar{z})))$. We then define $\delta^*(S)$ by $(\delta^*(S))_q[\bar{z}] = [\tilde{c}]$. Now, to show that this homomorphism is well defined, let $\bar{w} \sim \bar{z} = g_q(c)$, where $\bar{w} = g_q(d)$ for some $d \in C^q$. This means that $g_q(d) = g_q(c) + \bar{\delta}^{q-1}(\bar{u})$ for some $\bar{u} \in \bar{C}^{q-1}$. Since g_{q-1} is surjective, we have $\bar{u} = g_{q-1}(u)$ for some $u \in C^{q-1}$. Commutativity then gives

$$g_q(d) = g_q(c + \delta^{q-1}(u)) \implies d - (c + \delta^{q-1}(u)) \in \ker g_q\,.$$

Thus, there exists $\tilde{d} \in \tilde{C}^q$ such that $f_q(\tilde{d}) = d-(c+\delta^{q-1}(u))$. Applying δ^q to this equality, using $\delta^q(c) = f_{q+1}(\tilde{c})$ and commutativity, it follows that

$$\delta^q(d) = f_{q+1}(\tilde{c} + \tilde{\delta}^q(\tilde{d}))\,,$$

where $\tilde{c}$ is the same as above. From this we see that $f_{q+1}^{-1}(\delta^q(d)) = \tilde{c} + \tilde{\delta}^q(\tilde{d})$, that is, $f_{q+1}^{-1}(\delta^q(d)) \sim \tilde{c}$, whence $f_{q+1}^{-1}(\delta^q(g_q^{-1}(\bar{w}))) \sim \tilde{c}$, as we wanted. Lastly we show that $S \mapsto \delta^*(S)$ is a natural trasnformation. Consider a morphism between short exact sequences

[4] When it is clear from context we write $(\delta^*(S))_q = \delta^*_q$.

of cochain complexes as shown in (1.2) and label the second short sequence as T. We must show that the diagram

$$\begin{array}{ccc} H(\bar{C}^*) & \xrightarrow{\bar{\varphi}^*} & H(\bar{E}^*) \\ {\scriptstyle \delta^*(S)}\downarrow & & \downarrow{\scriptstyle \delta^*(T)} \\ H(\tilde{C}^*) & \xrightarrow{\tilde{\varphi}^*} & H(\tilde{E}^*) \end{array}$$

commutes. Indeed, from the commutativity of (1.2) and the definition of δ^*, it follows that

$$\begin{aligned}
(\tilde{\varphi}^*)_{q+1}(\delta^*(S))_q[\bar{z}] &= (\tilde{\varphi}^*)_{q+1}[f_{q+1}^{-1}\delta^q g_q^{-1}(\bar{z})] \\
&= [\tilde{\varphi}_{q+1} f_{q+1}^{-1}\delta^q g_q^{-1}(\bar{z})] \\
&= [i_{q+1}^{-1}\varphi_{q+1}\delta^q g_q^{-1}(\bar{z})] \\
&= [i_{q+1}^{-1}\delta^q\varphi_q g_q^{-1}(\bar{z})] \\
&= [i_{q+1}^{-1}\delta^q j_q^{-1}\bar{\varphi}_q(\bar{z})] \\
&= (\delta^*(T))_q(\bar{\varphi}^*)_q[\bar{z}]\,,
\end{aligned}$$

where δ in the fourth and fifith lines denotes the coboundary operator of E^*.

Q.E.D.

The lemma above tells us that the connecting morphism δ^* allows one to pass from a short exact sequence of cochain complexes to a sequence at the cohomology level, namely,

$$\cdots \xrightarrow{\delta^*_{q-1}} H^q(\tilde{C}^*) \xrightarrow{f^*_q} H^q(C^*) \xrightarrow{g^*_q} H^q(\bar{C}^*) \xrightarrow{\delta^*_q} H^{q+1}(\tilde{C}^*) \xrightarrow{f^*_{q+1}} \cdots$$

In the next theorem we see that this cohomology sequence is actually exact and that such correspondence extends to a functor, that is, the cohomology sequence is functorial on short exact sequences.

Theorem 1.17 (Mayer-Vietoris)**.** *There is a covariant functor from the category short exact sequence of cochain complexes to the category of exact sequences of R-modules which assings to a short exact sequence*

$$0 \longrightarrow \tilde{C}^* \xrightarrow{f} C^* \xrightarrow{g} \bar{C}^* \longrightarrow 0$$

the sequence

$$\cdots \xrightarrow{\delta^*_{q-1}} H^q(\tilde{C}^*) \xrightarrow{f^*_q} H^q(C^*) \xrightarrow{g^*_q} H^q(\bar{C}^*) \xrightarrow{\delta^*_q} H^{q+1}(\tilde{C}^*) \xrightarrow{f^*_{q+1}} \cdots$$

Proof. From the naturality of δ^*, established in Lemma 1.16, it follows that each morphism between short exact sequences of cochain complexes is assigned to a morphism between the correponding cohomology sequences. Thus, we only need to show that such a sequence is exact, that is, we must prove exactness at $H^q(\tilde{C}^*)$, $H^q(C^*)$ and $H^q(\bar{C}^*)$. We will do so only at $H^q(\bar{C}^*)$. It is easy to see that $\operatorname{im} g^*_q \subseteq \ker \delta^*_q$:

$$\delta^*_q(g^*_q[z]) = \delta^*_q[g_q(z)] = [f_{q+1}^{-1}\delta^q g_q^{-1}(g_q(z))] = [f_{q+1}^{-1}\delta^q(z)] = [f_{q+1}^{-1}(0)] = [0]\,.$$

Now, to show that $\ker \delta_q^* \subseteq \operatorname{im} g_q^*$, let $[\bar{z}] \in \ker \delta_q^*$, where $\bar{z} \in Z^q(\bar{C}^*)$, and consider the commutative diagram in the proof of Lemma 1.16. We have $\bar{z} = g_q(c)$ for some $c \in C^q$ and $[0] = \delta_q^*[\bar{z}] = [f_{q+1}^{-1}\delta^q g_q^{-1}(g_q(c))]$, whence $f_{q+1}^{-1}\delta^q(c) = \tilde{\delta}^q(\tilde{d})$, where $\tilde{d} \in \tilde{C}^q$. Thus

$$\delta^q(c) = f_{q+1}\tilde{\delta}^q(\tilde{d}) = \delta^q f_q(\tilde{d})$$

implies $c - f_q(\tilde{d}) \in C^q$. Therefore

$$\delta^q(c - f_q(\tilde{d})) = \delta^q(c) - \delta^q f_q(\tilde{d}) = \delta^q(c) - f_{q+1}\tilde{\delta}^q(\tilde{d}) = 0\,,$$

whence $c - f_q(\tilde{d}) \in Z^q(C^*)$ and $[c - f_q(\tilde{d})] \in H^q(C^*)$. Finally,

$$g_q^*[c - f_q(\tilde{d})] = [g_q(c) - g_q f_q(\tilde{d})] = [g_q(c)] = [\bar{z}]\,.$$

Q.E.D.

Lemma 1.16 and Theorem 1.17 have analogues, with obvious modifications, to chain complexes and homology. (See [**14**, p. 181].)

Now we present an application of Theorem 1.17; similar applications will be presented in Chapter 3. Let $C^* = \bar{C}^* + \tilde{C}^*$ be a decomposition (Example 1.13) of a cochain complex $C^* = (C^q, \delta^q)_{q\in\mathbf{Z}}$. Define chain maps $i : \bar{C}^* \cap \tilde{C}^* \to \bar{C}^* \oplus \tilde{C}^*$ and $j : \bar{C}^* \oplus \tilde{C}^* \to C^*$ by

$$i_q(z) = (z, z) \quad \text{and} \quad j_q(z, w) = z - w\,.$$

For each q, the homomorphism i_q is clearly injective and j_q is surjective. Also, it is obvious that $\operatorname{im}(i) \subseteq \ker(j)$. If $(z, w) \in \ker(j)$, then $z = w$, whence $(z, w) \in \operatorname{im}(i)$. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow \bar{C}^* \cap \tilde{C}^* \xrightarrow{\ i\ } \bar{C}^* \oplus \tilde{C}^* \xrightarrow{\ j\ } C^* \longrightarrow 0$$

From Theorem 1.17, the resulting exact sequence in cohomology is then

$$\cdots \xrightarrow{\Delta^*} H^q(\bar{C}^* \cap \tilde{C}^*) \xrightarrow{i^*} H^q(\bar{C}^*) \oplus H^q(\tilde{C}^*) \xrightarrow{j^*} H^q(C^*) \xrightarrow{\Delta^*} H^{q+1}(\bar{C}^* \cap \tilde{C}^*) \xrightarrow{i^*} \cdots$$

This exact sequence is called the *Mayer-Vietoris sequence* of the decomposition $C^* = \bar{C}^* + \tilde{C}^*$. We have

$$i_q^*[z] = ([z], [z]) \quad \text{and} \quad j_q^*([z], [w]) = [z - w]\,.$$

From the definition of the natural transformation Δ^* given in Lemma 1.16, it follows that

$$\Delta_q^*[z] = [\delta^q(x)] = [\delta^q(y)]\,,$$

where $z = x - y$, $x \in \bar{C}^q$, $y \in \tilde{C}^q$ and $\delta^q(x) = \delta^q(y)$.

We end this chapter presenting yet another important result to be used later.

Theorem 1.18 (The Five Lemma)**.** *Given a commutative diagram of R-modules and homomorphisms*

$$\begin{array}{ccccccccc} M_5 & \xrightarrow{\alpha_5} & M_4 & \xrightarrow{\alpha_4} & M_3 & \xrightarrow{\alpha_3} & M_2 & \xrightarrow{\alpha_2} & M_1 \\ \downarrow{\scriptstyle \gamma_5} & & \downarrow{\scriptstyle \gamma_4} & & \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} \\ N_5 & \xrightarrow{\beta_5} & N_4 & \xrightarrow{\beta_4} & N_3 & \xrightarrow{\beta_3} & N_2 & \xrightarrow{\beta_2} & N_1 \end{array}$$

in which each row is an exact sequence and γ_1, γ_2, γ_4 and γ_5 are isomorphisms, then γ_3 is an isomorphism.

Proof. We first prove that γ_3 is injective. Let $x_3 \in M_3$ be such that $\gamma_3(x_3) = 0$. Then the commutativity of the diagram above gives us $\gamma_2(\alpha_3(x_3)) = 0$, whence $\alpha_3(x_3) = 0$. Exactness then implies $x_3 = \alpha_4(x_4)$ for some $x_4 \in M_4$. Thus $\beta_4(\gamma_4(x_4)) = \gamma_3(\alpha_4(x_4)) = \gamma_3(x_3) = 0$, whence $\gamma_4(x_4) = \beta_5(y_5)$ for some $y_5 \in N_5$. Since γ_5 is surjective, we have $y_5 = \gamma_5(x_5)$ for some $x_5 \in M_5$. It follows that $\gamma_4(x_4) = \beta_5(y_5) = \beta_5(\gamma_5(x_5)) = \gamma_4(\alpha_5(x_5))$, whence $x_4 = \alpha_5(x_5)$. Thus

$$x_3 = \alpha_4(x_4) = \alpha_4(\alpha_5(x_5)) = 0\,.$$

Now to see that γ_5 is surjective, let $y_3 \in N_3$. Since γ_2 is surjective, we have $\beta_3(y_3) = \gamma_2(x_2)$ for some $x_2 \in M_2$. Commutativity gives $\gamma_1(\alpha_2(x_2)) = \beta_2(\gamma_2(x_2)) = \beta_2(\beta_3(y_3)) = 0$, and we see that $\alpha_2(x_2) = 0$, which implies $x_2 = \alpha_3(x_3)$ for some $x_3 \in M_3$. Again by commutativity, we have $\beta_3(\gamma_3(x_3)) = \gamma_2(\alpha_3(x_3)) = \gamma_2(x_2) = \beta_3(y_3)$. From this we see that $\beta_3(y_3 - \gamma_3(x_3)) = 0$, and exactness implies

$$y_3 - \gamma_3(x_3) = \beta_4(y_4)$$

for some $y_4 \in N_4$. Note that $y_4 = \gamma_4(x_4)$ for some $x_4 \in M_4$. Since $x_3 + \alpha_4(x_4) \in M_3$, it follows that

$$\begin{aligned} \gamma_3(x_3 + \alpha_4(x_4)) &= \gamma_3(x_3) + \gamma_3(\alpha_4(x_4)) \\ &= \gamma_3(x_3) + \beta_4(\gamma_4(x_4)) \\ &= \gamma_3(x_3) + \beta_4(y_4) \\ &= y_3 \end{aligned}$$

Q.E.D.

Differential Forms

We begin this chapter introducing the concept of a surface in Euclidean space. Right after, we define differential forms on surfaces and discuss some of its properties. In the last section we develop the integral calculus of forms on surfaces. Main references for this chapter are [**6, 8, 11**].

2.1. Surfaces in Euclidean spaces

In this section we introduce the concept of surfaces in Euclidean spaces and go through some other notions regarding such objects. We shall assume basic knowledge of general topology and real analysis on $\mathbf{R}^n$ (e.g. integration and differentiation in the sense of Fréchet and Stolz). Natural references for such topics are [**9, 11**].

Before diving into the precise definition of a surface, we need the concept of *immersion*. By that we mean a differentiable map[1] f from an open set $U \subseteq \mathbf{R}^m$ into $\mathbf{R}^n$ such that, for every point $p \in U$, $f'(p) : \mathbf{R}^m \to \mathbf{R}^n$ is an injective linear map. In this case we see that $m \leq n$ by the Rank-Nullity Theorem.

A *parametrization* (or *chart*) of class C^k and dimension m of a subset $X \subseteq \mathbf{R}^n$ is a homeomorphism $\varphi : V_0 \to X$ from an open set $V_0 \subseteq \mathbf{R}^m$ which is also an immersion of class C^k. Given a point $p \in X$, an open neighborhood $V \subseteq X$ of p in X is called a *parametrized neighborhood* of class C^k and dimension m if it admits a parametrization of class C^k and dimension m. In this case, we say that V is a parametrized neighborhood of p.

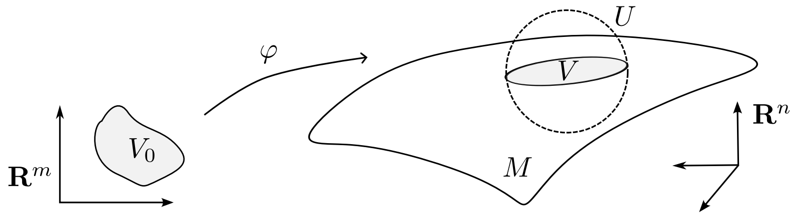
With that being said, a *differentiable surface* of class[2] C^k and dimension m in $\mathbf{R}^n$ (or *codimension* $n - m$) is a subset $M \subseteq \mathbf{R}^n$ such that each point $p \in M$ has parametrized

[1] Throughout the text we use the term "map" for a total functional binary relation with values on an arbitrary set Y (not necessarily a numeric set). The term "function" will be used only in cases where Y is a numeric set, e.g., $\mathbf{R}$ or $\mathbf{C}$.

[2] Unless otherwise stated, we consider surfaces of class C^1, at least.

neighborhood of class C^k and dimension m (a surface that is C^k for each $k \in \mathbf{N}$ is said to be C^∞ or *smooth*).

More explicitly, M can be covered by a collection of open set $U \subseteq \mathbf{R}^n$ such that each $V = U \cap M$ has a parametrization $\varphi : V_0 \to V$ of class C^k and dimension m. Intuitively, this means that, near each point, M is a copy of the m-dimensional Euclidean space. Note that even though n can be a very large number, the "position" of a point $p \in V$ is entirely determined by its m coordinates $(a_1, \dots, a_m) = \varphi^{-1}(p)$.[3]



By taking restrictions one readily sees that each point in a surface admits arbitrally small parametrized neighborhoods.

Remark 2.1. Henceforth, surfaces will be considered as topological spaces with the topology induced by that of the ambient Euclidean space.

A family of C^k parametrizations $\varphi : V_0 \to V \subseteq M$ whose images cover M is called an *atlas* of class C^k for M. The concept of atlas constitutes an essential piece to the definition of orientability, which will be given later on.

Example 2.2. If $U \subseteq M$ is an open set of an m-dimensional surface $M \subseteq \mathbf{R}^n$, then U an m-dimensional surface belonging the same differentiability class as M. Indeed, given a point $p \in U$, there exists a parametrization $\varphi : V_0 \to V$ of $p \in V = A \cap M$, where $A \subseteq \mathbf{R}^n$ and $V_0 \subseteq \mathbf{R}^m$ are open. Thus $p \in W \subseteq V$, where $W = A \cap U$ is an open set of U. Setting $W_0 = \varphi^{-1}(A \cap U)$ it follows that $\varphi : W_0 \to W$ is an m-dimensional parametrization of $W \ni p$. In particular, this result holds for parametrized neighborhoods.

Example 2.3. An easy example of a surface[4] is given by an open set of the n-dimensional Euclidean space. More precisely, every open set $U \subseteq \mathbf{R}^n$ is a smooth (C^∞) n-dimensional surface (or a surface of codimension 0). Indeed, taking $\varphi : U \to U$ to be the identity map, we see that U is a parametrized neighborhood of class C^∞ and dimension n to each point $p \in U$. Actually, every n-dimensional surface of class C^1 in $\mathbf{R}^n$ is an open set. This follows directly from the Inverse Mapping Theorem (Theorem A.2) and the fact that every surface is the union of its parametrized neighborhoods. On the other hand, since $\mathbf{R}^0 = \{0\}$, surfaces of codimension n (dimension 0) in $\mathbf{R}^n$ are precisely the discrete sets.

[3] There is a far more general concept than that of a surface, namely, a smooth manifold. Roughly speaking, it is a sufficiently good (depending on the context) topological space together with parametrizations such as the ones defined above in such a way the one can pass "smoothly" between intersecting charts. We usually say that manifolds are defined intrinsically, since they need not be subsets of some Euclidean space. There is a theorem due to a mathematician called Hassler Whitney (1907-1989) which states that every smooth manifold can be thought of as a surface in some Euclidean space. For further details, see [6].

[4] Note that according our definition the empty set is itself a surface.

Example 2.4. The cartesian product of two surfaces $M_1 \subseteq \mathbf{R}^{n_1}$ and $M_2 \subseteq \mathbf{R}^{n_2}$, where $\dim M_1 = m_1$ and $\dim M_2 = m_2$, is a surface of dimension $m_1 + m_2$. Indeed, given two parametrizations $\varphi : V_0 \to V \subseteq M_1$ and $\psi : W_0 \to W \subseteq M_2$, the map $\zeta : V_0 \times W_0 \to V \times W$ defined by $\zeta(u_0, v_0) = (\varphi(u_0), \psi(v_0))$ is a parametrization of class C^k and dimension $m_1 + m_2$. Thus the product of a finite number of surfaces is still a surface. For instance, since the unit circle S^1 is a smooth 1-dimensional surface[5] in $\mathbf{R}^2$, we see that the torus $T^n = S^1 \times \cdots \times S^1$ is an n-dimensional in $\mathbf{R}^{2n}$.

Example 2.5. Every surface is a locally compact space[6]. Indeed, given a point $x \in M$, there exists an open neighborhood $V \ni x$ and a homeomorphism $\varphi : V_0 \to V$, where $V_0 \subseteq \mathbf{R}^m$ is an open set and $\varphi(a) = x$. There is then a closed (compact) ball B centered at a with $B \subseteq V_0$. Thus the map φ restricts to a homeomorphism $\varphi : B \to \varphi(B)$, whence $\varphi(B) \subseteq V$ is compact and $x \in \operatorname{int} \varphi(B)$, as we wanted to prove.

Example 2.6. From the fact that each parametrization $\varphi : V_0 \to V \subseteq M$ is a homeomorphism, one readily sees that every surface is a locally connected topological space[7]. Therefore, every connected component of a surface M is an open set of M.

If $\varphi : V_0 \to V \subseteq \mathbf{R}^n$ is a C^k parametrization, the set V might not be open, thus we cannot state anything for sure on the differentiability of φ. At any rate, there is the following important theorem. For a proof see [**11**, p. 246].

Theorem 2.7. *Let $M \subseteq \mathbf{R}^n$ be an m-dimensional surface of class C^k and $f : U \to \mathbf{R}^n$ a C^k map (resp. differentiable at $a \in U$), defined on an open set $U \subseteq \mathbf{R}^p$. If $f(U) \subseteq V$, where $V \subseteq M$ is the image of a C^k parametrization $\varphi : V_0 \to V$, then the composite map $\varphi^{-1} \circ f : U \to \mathbf{R}^m$ is also C^k (resp. differentiable at a).*

$$\begin{tikzcd} & V & \\ U \arrow[ur, "f"] \arrow[rr, "\varphi^{-1}\circ f"'] & & V_0 \arrow[ul, "\varphi"'] \end{tikzcd}$$

Furthermore, for $a \in U$ and $b = (\varphi^{-1} \circ f)(a)$,

$$(\varphi^{-1} \circ f)'(a) = [\varphi'(b)]^{-1} \cdot f'(a) : \mathbf{R}^p \to \mathbf{R}^m.$$

The previous theorem has the following consequences: the definition of tangent space to a surface at a point, change of coordinates between parametrizations and the generalization of differentiability to maps between surfaces. Let us begin with the first one.

Let $M \subseteq \mathbf{R}^n$ be a surface of class C^k and dimension m. We define the *tangent space to M at a point $p \in M$* as the m-dimensional linear subspace

$$T_pM = \varphi'(a) \cdot \mathbf{R}^m \subseteq \mathbf{R}^n\,,$$

[5] To see this, use two smooth parametrizations to cover S^1, one of which is the inverse of stereographic projection from the north pole and the other is is the inverse of stereographic projection from the south pole.

[6] A topological space is said to be locally compact if each point lies in a compact neighborhood.

[7] A topological space X is said to be locally connected when, given $x \in X$, each neighborhood $U \ni x$ contains a connected neighborhood $V \ni x$. The connected components of a locally connected space are open. See [**9**, p. 96].

where $\varphi : V_0 \to V$ is a C^k parametrization of an open neighborhood $V \ni p$ in M.

One can also define the tangent space T_pM as the set of all tangent vectors $\lambda'(0)$, where $\lambda :]-\varepsilon, \varepsilon[\to M$ is a differentiable path at 0 with $\lambda(0) = p$. These two definitions are in fact equivalent since they define the same set. We skip the details, which can be found in [**11**, p. 247].

Since parametrizations are immersions, from the first definition of tangent space it folllows that each parametrization $\varphi : V_0 \to V$ determines a basis for T_pM, namely,

$$B_\varphi = \left\{ \frac{\partial \varphi}{\partial x_1} = \varphi'(a) \cdot e_1, \ldots, \frac{\partial \varphi}{\partial x_m} = \varphi'(a) \cdot e_m \right\} .$$

Note that a parametrized neighborhood $\varphi : V_0 \to V = U \cap M$ of an m-dimensional surface is an m-dimensional surface that can be covered by a single parametrization, namely, $\varphi : V_0 \to V$. Therefore, for every $x \in V$, we have

$$T_xV = T_xM .$$

This fact will prove useful in the next section, since it allows one to define the exterior derivative of forms.

Moreover, if M and N are surfaces of dimensions m and n, respectively, such that $M \subseteq N \subseteq \mathbf{R}^p$, then $T_xM \subseteq T_xN$. Indeed, let $\psi : U_0 \to U \subseteq N$ and $\varphi : V_0 \to V \subseteq M$ be parametrizations for neighborhoods $U \ni x$ and $V \ni x$, respectively, such that $x = \varphi(v) = \psi(u)$. The restriction $\psi : \psi^{-1}(U \cap V) \to U \cap V$ is yet another parametrization of $U \cap V \ni x$ in M. Thus, Theorem 2.7 applied to $\psi = \varphi \circ (\varphi^{-1} \circ \psi) : \psi^{-1}(U \cap V) \to \varphi^{-1}(U \cap V)$ gives $\psi'(u) = \varphi'(v) \circ (\varphi^{-1} \circ \psi)'(u)$. Therefore, if $z = \psi'(u) \cdot u_0$ for some $u_0 \in \mathbf{R}^n$, then $z = \varphi'(v) \cdot v_0$, where $v_0 = (\varphi^{-1} \circ \psi)'(u) \cdot u_0 \in \mathbf{R}^m$. This shows that

$$T_xM \subseteq T_xN .$$

As to the second consequence of Theorem 2.7, it says that any two m-dimensinal parametrizations of the same neighborhood "differ" by a diffeomorphism. This is the content of the next theorem.

Theorem 2.8. *Let $\psi : W_0 \to V \subseteq \mathbf{R}^n$ be an m-dimensional parametrization of class C^k. A necessary and suficient condition for a C^k-map $\varphi : V_0 \to V$ to be a parametrization of V is that $\varphi = \psi \circ \xi$, where $\xi : V_0 \to W_0$ is a diffeomorphism of class C^k. In particular, φ is an m-dimensional parametrization.*

$$\begin{tikzcd} & V & \\ V_0 \arrow[ur, "\varphi"] \arrow[rr, "\xi"'] & & W_0 \arrow[ul, "\psi"'] \end{tikzcd}$$

Proof. Clearly the above condition is sufficient. To see that it is also necessary, let $\varphi : V_0 \to V$ to be a parametrization of V. It follows from Theorem 2.7 that $\xi = \psi^{-1} \circ \varphi : V_o \to W_0$ and $\xi^{-1} : \varphi^{-1} \circ \psi : W_0 \to V_0$ are C^k maps, whence C^k diffeomorphisms with $\varphi = \psi \circ \xi$.

Q.E.D.

Using this last result, one can make sense of a change of coordinates for points lying on the intersection of two parametrized neighborhoods. This is what we will do now. Let $\varphi : V_0 \to V$ and $\psi : W_0 \to W$ be parametrizations of two neighborhoods $V, W \subseteq M$ with $V \cap W \neq \emptyset$. Given a point $p = \varphi(a) = \psi(b) \in V \cap W$, we have two bases for T_pM, namely,

$$B_\varphi(a) = \left\{ \frac{\partial \varphi}{\partial x_1}(a), \dots, \frac{\partial \varphi}{\partial x_m}(a) \right\} \quad \text{and} \quad B_\psi(b) = \left\{ \frac{\partial \psi}{\partial y_1}(b), \dots, \frac{\partial \psi}{\partial y_m}(b) \right\}.$$

In order to determine the change of basis matrix $[\alpha_{ij}]$ given by

$$\frac{\partial \varphi}{\partial x_j}(a) = \sum_{i=1}^m \alpha_{ij} \frac{\partial \psi}{\partial y_i}(b)$$

we apply the chain rule (Theorem A.1) to $\varphi = \psi \circ \xi$, where $\xi = \psi^{-1} \circ \varphi : \varphi^{-1}(V \cap W) \to \psi^{-1}(V \cap W)$ is a diffeomorphism (by Theorem 2.7) with $\xi(x) = y$. We obtain

$$\frac{\partial \varphi}{\partial x_j}(a) = \sum_{i=1}^m \frac{\partial \psi}{\partial y_i}(b) \frac{\partial \xi_i}{\partial x_j}(a),$$

whence

$$\alpha_{ij} = \frac{\partial \xi_i}{\partial x_j}(a).$$

This shows that change of basis matrix from B_ψ to B_φ in T_pM is precisely the jacobian matrix of $\xi = \psi^{-1} \circ \varphi$ at $a = \varphi^{-1}(p)$.

Finally, for the last consequence of Theorem 2.7 we have the definition of differentiability of maps between surfaces.

Let $M \subseteq \mathbf{R}^n$ an m-dimensional surface of class C^k. A map $f : M \to \mathbf{R}^d$ is said to be *differentiable* at a point $p \in M$ if there exists a C^k parametrization $\varphi : V_0 \to V$ of a neighborhood $V \ni p$ such that the composite map $f \circ \varphi : V_0 \to \mathbf{R}^d$ is differentiable at $a = \varphi^{-1}(p)$.

$$\begin{tikzcd} M \arrow[r, "f"] & \mathbf{R}^d \\ V_0 \arrow[u, "\varphi"] \arrow[ur, "f\circ\varphi"'] & \end{tikzcd}$$

This definition does not depend on the choice of parametrized neighborhoods of p. Indeed, if $\psi : W_0 \to W$ is a parametrization of a neighborhood $W \ni p$, then by Theorem 2.7 and the identity $f \circ \psi = (f \circ \varphi) \circ (\varphi^{-1} \circ \psi)$ we see that $f \circ \psi$ is differentiable at $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is differentiable at $\varphi^{-1}(p)$.

We say that $f \in C^s$ $(0 \leq s \leq k)$ if for each $p \in M$ there exists a C^k parametrization $\varphi : V_0 \to V$ of a neighborhood $V \ni p$ such that $f \circ \varphi \in C^s$. Again, this does not depend on the chosen neighborhood.

If $f : M \to \mathbf{R}^d$ is differentiable at a point $p \in M$, we define its derivative at p as the linear map $f'(p) : T_pM \to \mathbf{R}^d$ such that

$$f'(p) \cdot v = (f \circ \varphi)'(a) \cdot v_0 \,,$$

where φ is a parametrization of a neighborhood $V \ni p$, $a = \varphi^{-1}(p)$ and $v = \varphi'(a) \cdot v_0$. This linear map is well defined as it does not depend on the parametrized neighboord. (See [11, p. 249].)

$$\begin{tikzcd}
T_pM \arrow[r, "f'(p)"] \arrow[d, "\varphi^{-1}(a)"'] & \mathbf{R}^d \\
\mathbf{R}^m \arrow[ur, "(f\circ\varphi)'(a)"'] &
\end{tikzcd}$$

Now, if $N \subseteq \mathbf{R}^d$ is another surface of class C^k, we say that a map $f : M \to N$ is *differentiable* at $p \in M$ when the corresponding map[8] $f : M \to \mathbf{R}^d$ is differentiable in the sense defined above.

$$\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[r, hook, "i"] & \mathbf{R}^d \\
V_0 \arrow[u, "\varphi"] \arrow[urr, "i\circ f\circ\varphi"'] & &
\end{tikzcd}$$

Moreover, we say that $f \in C^s$ $(0 \leq s \leq k)$ when $f : M \to \mathbf{R}^d$ is of class C^s.

In case $f : M \to N$ is a differentiable bijection such that f^{-1} is differentiable, we say that f is a *diffeomorphism*. In this case we say that f is a diffeomorphism of class C^k if $f \in C^k$.

Example 2.9. Let M be an m-dimensional smooth surface and $\varphi : V_0 \to V$ a smooth parametrization of an open set $V \subseteq M$. Then φ is a smooth diffeomorphism between the surfaces $V_0 \subseteq \mathbf{R}^m$ and V.

As it was refered above, given a vector $v \in T_pM$ we have $v = \lambda'(0)$, where $\lambda :]-\varepsilon, \varepsilon[\to M$ is a differentiable path at 0 with $\lambda(0) = p$. Thus for a differentiable map $f : M \to \mathbf{R}^d$ at p we see that

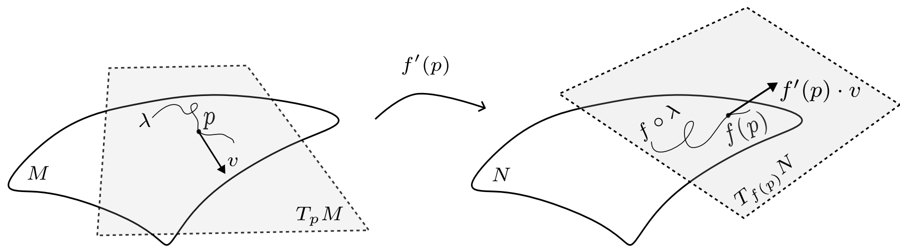
$$f'(p) \cdot v = (f \circ \varphi)'(a) \cdot v_0 = D_{v_0}(f \circ \varphi)(a) = ((f \circ \varphi) \circ (\varphi^{-1} \circ \lambda))'(0) = (f \circ \lambda)'(0) \,,$$

that is, $f'(p) \cdot v$ is a tangent vector to the path $f \circ \lambda :]-\varepsilon, \varepsilon[\to \mathbf{R}^d$ at 0, with $(f \circ \lambda)(0) = f(p)$. From this we conclude that the derivative at a point $p \in M$ of a map $f : M \to N$ between surfaces is a linear transformation

$$f'(p) : T_pM \to T_{f(p)}N \,.$$

Pictorially:

[8] More precisely, $i \circ f : M \to \mathbf{R}^d$, where $i : N \hookrightarrow \mathbf{R}^d$ is the inclusion.



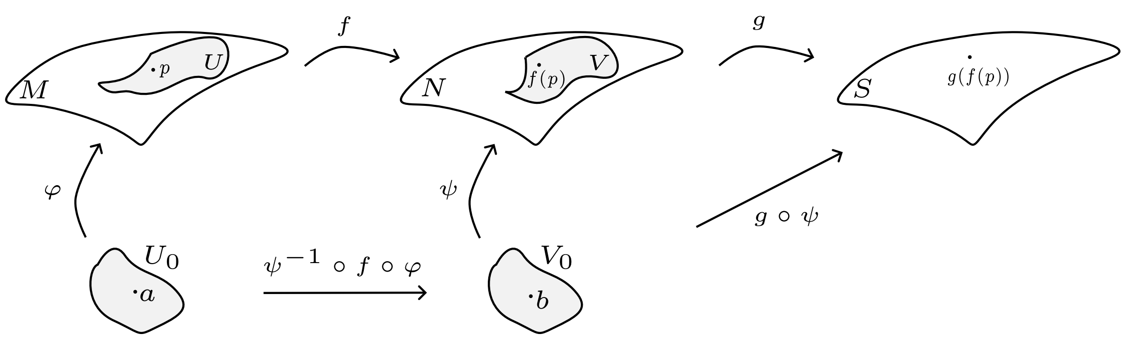
Example 2.10. Let $m \geq 1$. The antipodal map $A : S^m \to S^m$, given by $A(p) = -p$, is smooth map. For $p \in S^m$ and a smooth parametrization $\varphi : V_0 \to V \subseteq S^m$ of $V \ni p$, with $\varphi(a) = p$ and $v = \varphi'(a) \cdot v_0 \in T_pM$, we have

$$A'(x) \cdot v = (A \circ \varphi)'(a) \cdot v_0 = (-\varphi)'(a) \cdot v_0 = -\varphi'(a) \cdot v_0 = -v\,.$$

Thus $f'(p) : T_pS^m \to T_{-p}S^m$ is just the multiplication by -1. (Note that $T_pS^m = T_{-p}S^m \subseteq \mathbf{R}^{m+1}$ since both are the orthogonal complements of x and $-x$; think about the case $m = 1$.)

It follows from Theorem 2.7 that a map $f : M \to N$ between C^k-surfaces is of class C^k if and only if there exists C^k-parametrizations $\psi : V_0 \to V \subseteq N$ and $\varphi : U_0 \to U \subseteq M$, $p \in U$, such that $f(U) \subseteq V$ and $\psi^{-1} \circ f \circ \varphi : U_0 \to V_0$ is of class C^k. (See [**6**, p. 40].)

If $f : M \to N$ is differentiable at $p \in M$ and $g : N \to \mathbf{R}^d$ is differentiable at $f(p)$, then $g \circ f$ is differentiable at p and $(g \circ f)'(p) = g'(f(p)) \cdot f'(p)$. This follows from Theorem 2.7 and the chain rule (Theorem A.1) applied to $((g \circ f) \circ \varphi)(p) = ((g \circ \psi) \circ (\psi^{-1} \circ (f \circ \varphi))(p)$. Pictorially:



From the above, we see that there exists a category whose objects are surfaces of a fixed class of differentiability C^k and whose morphisms are maps of class C^k between surfaces. In Chapter 3, we will focus on the category for which $k = \infty$, that is, smooth surfaces and smooth maps between them.

If $f : M \to N$ is a diffeomorphism, then $f'(p) : T_pM \to T_{f(p)}N$ is an isomorphism for each $p \in M$. Therefore, T_pM and $T_{f(p)}N$ have the same dimension as vector spaces, whence M and N also have the same dimension as surfaces. If, in addition, $f \in C^k$, it follows that $f \circ \varphi : V \to f(V) \subseteq N$ is a C^k parametrization $f(V)$, for every C^k parametrization $\varphi : V_0 \to V \subseteq M$.

Let M and N be surfaces with dimensions m and n respectively. Given a differentiable map $f : M \to N$ we can compute the matrix of its derivative $f'(p) : T_pM \to T_{f(p)}N$ with respect to bases $B_\varphi(u) \subseteq T_pM$ and $B_\psi(v) \subseteq T_{f(p)}N$, where $\varphi : U_0 \to U \ni p$ and $\psi : V_0 \to V \ni f(p)$ are parametrizations such that $p = \varphi(u)$ and $f(p) = \psi(v)$. Since $(f \circ \varphi)(u) = (\psi \circ (\psi^{-1} \circ f \circ \varphi))(u)$ and

$$f'(p) \cdot \left(\frac{\partial \varphi}{\partial u_j}(u)\right) = (f \circ \varphi)'(u) \cdot e_j\,,$$

where e_j is the jth canonical vector in $\mathbf{R}^m$, the chain rule gives us

$$f'(p) \cdot \left(\frac{\partial \varphi}{\partial u_j}(u)\right) = \frac{\partial (f \circ \varphi)}{\partial u_j}(u) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial v_i}(v) \cdot \frac{\partial (\psi^{-1} \circ f \circ \varphi)_i}{\partial u_j}(u)\,. \tag{2.1}$$

Thus the matrix of the linear map $f'(p)$ is precisely the jacobian matrix of $\psi^{-1} \circ f \circ \varphi$ at $u = \varphi^{-1}(p)$.

2.1.1. Partitions of Unity

Now we shall introduce the concept of partitions of unity on a surface. This will be necessary to define integral of forms on surfaces in § 2.3.

A family of subsets A_λ of a topological space X is said to be *locally finite* when each $x \in X$ belongs to a neighborhood which intersects finitely many A_λ's. If, additionally, $X = \bigcup A_\lambda$, we say that $(A_\lambda)_{\lambda \in L}$ is a locally finite cover for X.

When it comes to surfaces, locally finite families enjoy the following properties:

i) Every locally finite familiy on a surface is countable. ([**11**, p. 349])

ii) Compact subsets of a surface intersect only finitely many members of a locally finite family. ([**11**, p. 350])

A *partition of unity* of class C^k on a surface (of class C^k) consists of a family of C^k functions $(\xi_\lambda)_{\lambda \in L}$, $\xi_\lambda : M \to \mathbf{R}$, such that

1. For every $\lambda \in L$, $\xi_\lambda \geq 0$ on M;
2. $(\operatorname{supp} \xi_\lambda)_{\lambda \in L}$ is locally finite on M;
3. For every $x \in M$, $\sum \xi_\lambda(x) = 1$.

Given a cover $A = (A_\lambda)_{\lambda \in L}$ for a surface M, a partition of unity $\sum \xi_\lambda = 1$ is said to be *strictly subordinated* to A if $\operatorname{supp} \xi_\lambda \subseteq A_\lambda$ for every $\lambda \in L$. The next theorem (whose proof can be found in [**11**, p. 351]) tells us that we can always find such partitions of unity.

Theorem 2.11. *Given an open cover A for a surface M of class C^k, there exists a (countable) partition of unity, of class C^k, strictly subordinated to A.*

In the next chapter, this result will help us define a particular example of short exact sequence of cochain complexes. Its proof is based on is based on two lemmas, one of which we state now (to be used in § 3.3).

Lemma 2.12. *Every surface can be written as a countable union of compact sets K_i such that $K_i \subseteq \operatorname{int} K_{i+1}$.*

The lemma above has a rather important consequence besides Theorem 2.11, namely

Proposition 2.1. *Let M be a surface of class C^k and $\mathcal{B}$ a basis for the topology of M which is closed under finite intersections. Then $M = \bigcup_{i\in\mathbf{N}} V_i$, where each V_i is a finite union of open sets belonging to $\mathcal{B}$ such that $V_i \cap V_j = \emptyset$ for every $j \geq i+2$.*

Proof. By Lemma 2.12 we have $M = \bigcup_{i\in\mathbf{N}} K_i$ where each K_i is a compact set and $K_i \subseteq \operatorname{int}_M K_{i+1}$. We argue inductively, making use of Borel-Lebesgue.

1. Define $V_1 = \bigcup A_{\lambda_1}$, where $K_1 \subseteq \bigcup A_{\lambda_1}$ is a finite open cover by sets $A_{\lambda_1} \in \mathcal{B}$ such that $\overline{V}_1 \subseteq \operatorname{int}_M K_2$.

Since $K_2 - \operatorname{int}_M K_1$ is compact (closed subseteq of a compact space):

2. Set $V_2 = \bigcup A_{\lambda_2}$, where $K_2 - \operatorname{int}_M K_1 \subseteq \bigcup A_{\lambda_2}$ is a finite open cover by sets $A_{\lambda_2} \in \mathcal{B}$ such that $\overline{V}_2 \subseteq \operatorname{int}_M K_3$.

Moreover, $K_{i+1} - \operatorname{int}_M K_i$ is compact for every $i \in \mathbf{N}$. Thus

3. For every $i \geq 3$, set $V_i = \bigcup A_{\lambda_i}$, where $K_i - \operatorname{int}_M K_{i-1} \subseteq \bigcup A_{\lambda_2}$ is a finite open cover by sets $A_{\lambda_i} \in \mathcal{B}$, chosen so that $\overline{V}_i \subseteq \operatorname{int}_M K_{i+1}$ and $V_i \cap \overline{V}_{i-2} = \emptyset$.

Q.E.D.

The previous proposition will be crucial during the proof of Poincaré duality in § 3.3. The next one will be useful to compute cohomology groups of the punctured sphere in § 3.2. For a proof see [**11**, p. 348].

Proposition 2.2. *Let M be a surface of class C^k, $p \in M$ and $A \subseteq M$ an open set with $p \in A$. There exists a compactly supported function $\xi : M \to [0,1]$ and open sets $U, W \subseteq A$ such that $p \in U \subseteq W$, $\operatorname{supp}\xi \subseteq W$ and $\xi = 1$ on U.*

2.1.2. Orientation

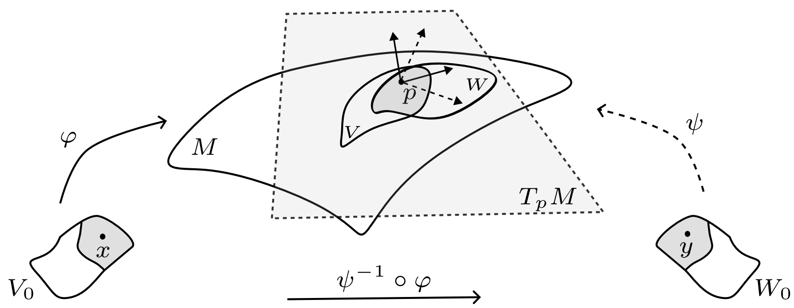
The concept of orientability is crucial ([**11**, p. 372]) to the definition of surface integrals, to be defined in § 2.3 below. We introduce it now.

Given a surface M, two C^1 parametrizations $\varphi : V_0 \to V$ and $\psi : W_0 \to W$ on M are said to be coherent if $V \cap W = \emptyset$ or $V \cap W \neq \emptyset$ and $\operatorname{jac}(\psi^{-1}\circ\varphi) > 0$ [9] on $\varphi^{-1}(V \cap W)$. Setting $\xi = \psi^{-1}\circ\varphi$, this last condition translates into

$$\frac{\partial\varphi}{\partial x_j}(x) = \sum_{i=1}^{m} \frac{\partial\psi}{\partial y_i}(y)\frac{\partial\xi_i}{\partial x_j}(x)\,, \quad \text{where} \quad \det\left[\frac{\partial\xi_i}{\partial x_j}(x)\right] > 0\,,$$

for $p = \varphi(x) = \psi(y) \in V \cap W$ and $1 \leq j \leq m$. In other words, for every $p \in V \cap W$, the maps φ and ψ determine two positive bases on T_pM, where the corresponding change of basis matrix is precisely the jacobian matrix of the diffeomorphism $\psi^{-1}\circ\varphi$ at $x = \psi^{-1}(p)$.

[9] We use $\operatorname{jac} f(x)$ to denote the function $x \mapsto \det f'(x)$ (the jacobian of f at x).



Let M be a surface. By a *coherent* C^k*-atlas* on M we mean an atlas $\mathcal{A}$ (of class C^k) such that any two parametrizations $\varphi, \psi \in \mathcal{A}$ are coherent. In this case $\mathcal{A}$ is said to be *maximal* when it is a maximal element with respect to the inclusion relation in the set of coherent C^k-atlases on M. One can pass from a coherent C^k-atlas $\mathcal{A}$ to a maximal coherent C^k-atlas by including in $\mathcal{A}$ all parametrization φ such that φ and ψ are coherent for all $\psi \in \mathcal{A}$.

We say that a surface M of class C^k is *orientable* when it admits a coherent C^k-atlas. In this case, there is also a maximal coherent C^k-atlas, which is called an *orientation*. An *oriented* surface is an orientable surface on which an oriention $\mathcal{A}$ has been chosen; elements $\varphi \in \mathcal{A}$ are called *positive parametrizations*. We will write $\varphi > 0$ to denote a positive parametrization.

Example 2.13. The product of two orientable surfaces is also an orientable surface ([**11**, p. 256]). Thus, arguing indutively, one verifies that a finete product of orientable surfaces is yet another orientable surface.

Example 2.14. Surfaces that can be covered by only one parametrization are orientable, since the singleton consisting of such parametrization is a coherent atlas. Thus, open subsets $U \subseteq \mathbf{R}^m$ can be (and will be) regarded as oriented smooth surfaces $(U, \mathcal{A})$, where $\mathcal{A}$ is the maximal coherent C^∞-atlas originating from the coherent atlas $\{\mathrm{id}_U\}$. (This will be the case in Example 2.18 below.)

Example 2.15. If U is an open subset of an orientable surface M, then U is an orientable surface itself. Indeed, choosing a coherent atlas $\mathcal{A}$ on M we form a coherent atlas on U consisting of the restrictions $\varphi|_{\varphi^{-1}(U\cap V)}$ of parametrizations $\varphi : V_0 \to V$ belonging to $\mathcal{A}$.

Remark 2.16. In view of the previous example, open sets of an oriented surface M will always be considered as oriented surfaces with the orientation induced by that of M.

Example 2.17. For $m \geq 1$, S^m is an orientable smoooth surface ([**11**, p.254]). The 4-dimensional $S \subseteq \mathbf{R}^6$ consisting of 2×3 matrices of rank 1 is non-orientable. A widely known example of non-orientable surface is the Möbius strip (Figure 2.17), which can be obtained from the rectangle $[0, 2\pi]\times]0, 1[$ by identifying the points $(0, t)$ and $(2\pi, 1 - t)$ for all $0 < t < 1$. The Möbius strip can also be defined as smooth surface of codimension 1 in $\mathbf{R}^3$. (See [**11**, p. 258-259].)

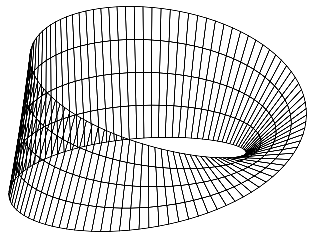


Figure 1. The Möbius strip as a surface of codimension 1 in $\mathbf{R}^3$. Source: https://pgfplots.net/moebius-strip/.

If M and N are diffeomorphic surfaces, then M is orientable if and only if N is orientable. For instance, to see that the orientability of M implies that N is orientable, one constructs a coherent C^k-atlas on N using the composites $f \circ \varphi$, where φ is a positive parametrization on M. For the converse, the same argument applies with f^{-1}.

Let us fix an orientation $\mathcal{A}$ on an orientable surface M. A parametrization $\xi : V_0 \to V$ is said to be *negative* whenever $\operatorname{jac}(\varphi^{-1} \circ \xi) < 0$ on $\xi^{-1}(V \cap W)$, for every $\varphi : W_0 \to W$ in $\mathcal{A}$; we write $\xi < 0$. Any two negative parametrizations are coherent, and thus form a maximal coherent C^k-atlas denoted by $\mathcal{A}^*$. If we consider M together with the orientation given by $\mathcal{A}^*$, we say that M has the *opposite orientation* to $\mathcal{A}$; we write $-M$.

A diffeomorphism $f : M \to N$ between oriented surfaces is said to *preseve orientation* if $f \circ \varphi$ is positive parametrization on N whenever φ is a positive parametrization on M. If f does not preserve orientation we say that f is an orientation-reversing diffeomorphism.

Example 2.18. Each open subset $]a_i, b_i[\subseteq \mathbf{R}$ is diffeomorphic to $\mathbf{R}$ since $f :]a_i, b_i[\to \mathbf{R}$ given by

$$f(x) = \tanh^{-1}\left(\frac{2(x - a_i)}{b_i - a_i} - 1\right)$$

is a smooth diffeomorphism whose inverse is

$$g(x) = a_i + \frac{(\tanh(x) + 1)(b_i - a_i)}{2}\,.$$

If $C = \prod_{i=1}^{m}]a_i, b_i[$ is an open rectangle in $\mathbf{R}^m$, we then have a smooth diffeomorphism $F = f_1 \times \cdots \times f_m : C \to \mathbf{R}^m$ whose coordinate functions are $F_i = f_i \circ p_i$, where p_i denotes the ith projection, that is, $F_i(x_1, \ldots, x_m) = f_i(x_i)$. Moreover, F is orientation-preserving. Indeed, let $\varphi : U_0 \to U \subseteq C$ be a smooth positive parametrization on C. From the convention established in Example 2.14, we only need to show that

$$\operatorname{jac}((\mathrm{id}_{\mathbf{R}^m})^{-1} \circ (F \circ \varphi))(x) > 0$$

for $x \in U_0$, which is the same as $\operatorname{jac}(F \circ \varphi)(x) > 0$. Note that $\operatorname{jac} \varphi(x) > 0$ since φ and id_C are coherent. Also

$$\operatorname{jac} F(u) = \prod_{i=1}^{m} f_i'(u_i) = \prod_{i=1}^{m} \frac{b_i - a_i}{2(u_i - a_i)(b_i - u_i)} > 0\,,$$

for every $u \in C$. Thus $\operatorname{jac}(F \circ \varphi)(x) = \operatorname{jac} F(\varphi(x)) \operatorname{jac} \varphi(x) > 0$. The existence of such diffeomorphism will be useful during § 3.3.

An oriented smooth surface M is said to be *reversible* if there exists a smooth orientation-reversing diffeomorphism $f : M \to M$. In case M does not admit such diffeomorphisms we say that M is *irreversible*. In § 3.4 we present an example of irreversible surface, as an application of Poincaré duality.

Example 2.19. If m is even, then S^m is reversible since the antipodial map (Example 2.10) reverses orientation. For further details, see [**11**, p. 264].

2.1.3. Homotopy

In order to define homotopy in the context of surfaces, we now give a brief overview on surfaces with boundary.

Let $\varphi : E \to \mathbf{R}$ be a nonvanishing linear functional on a real vector space E. A half-space $H \subseteq E$ is a set $H = \{x \in E\,;\, \varphi(x) \leq 0\}$. The boundary of H, denoted by ∂H, is the $\partial H = \{x \in E\,;\, \varphi(x) = 0\}$. Since $\partial H = \operatorname{fr} H$ [10], we have $H = \operatorname{int} H \cup \partial H$. This means that an open set A of H is either a subset of $\operatorname{int} H$ or it intersects the boundary ∂H. For an open set $A \subset H$, its boundary is $\partial A = A \cap \partial H$.

One can extend the definition of differentiability (in the sense of Fréchet and Stolz) to maps defined on open sets of some half-space $H \subseteq \mathbf{R}^m$. A map $f : A \to \mathbf{R}^n$, where $A \subseteq H$ is an open in H, is said to be *differentiable* (resp. of class C^k, $k \geq 1$), if it is the restriction of a differentiable map (resp. of class C^k) $F : U \to \mathbf{R}^n$ on an open set $U \subseteq \mathbf{R}^m$, where $U \supseteq X$. In this case, all differentiable extensions F of f have the same derivative $F'(x) : \mathbf{R}^m \to \mathbf{R}^n$, and we set $f'(x)$ to be this common linear map. (See [**11**, p. 378].)

This generalization allows one to redefine a parametrization $\varphi : V_0 \to V \subseteq \mathbf{R}^n$ requiring only that V_0 be an open set of some half-space in $\mathbf{R}^n$. Thus, a set $M \subseteq \mathbf{R}^n$ is said to be an m-dimensional *surface with boundary* (of class C^k) if every point $x \in M$ belongs to an open neighborhood $V \subseteq M$ which is the image of a C^k-parametrization $\varphi : V_0 \to V$ defined on an open set V_0 of some half-space in $\mathbf{R}^n$.

Let M be an $(m+1)$-dimensional surface with boundary. The *boundary* of M, denoted by ∂M, is defined as the set points $x \in M$ such that $x = \varphi(u) \implies u \in \partial U_0$, whenever $\varphi : U_0 \to U$ is a C^1-parametrization of $U_0 \ni x$. The boundary ∂M is an m-dimensional surface (without boundary) having the same differentiability class as M. (See [**11**, p. 381].)

[10] The symbol $\operatorname{fr} X$ denotes the topological frontier.

Remark 2.20. The definition of surfaces given in § 2.1 corresponds to the case where $\partial M = \emptyset$. Whenever we refer to M as "surface" we mean a surface without boundary, in the sense defined before.

Orientability is defined in the same way for surfaces with boundary, but in this case the orientation on M induces an orientation on ∂M. For further details, see [**11**, p. 384]. Also, differentiability of maps between surfaces with boundary is defined in the same way as before, the only main difference being that $f \circ \varphi$ is differentiable in the sense defined above.

For surfaces with boundary, we usually have similar results and definitons to those given before. (See [**11**].) For instance, the product of two surface with boundary is not a surface with boundary ([**11**, p. 387]). Nevertheless, if M is a surface and N is a surface with boundary, then their product $M \times N$ is a surface with boundary and

$$\partial(M \times N) = M \times \partial N\,.$$

Also $M \times N$ is orientable whenever M and N are orientable. The case of interest here is the product $M \times I$, where M is a surface and $I = [0, 1]$.

Let M and N be surfaces and $I = [0, 1]$. Two C^k-maps $f, g : M \to N$ are said to be *homotopic* (or C^0-homotopic) if there exists a continuous map $H : M \times I \to N$, called *homotopy*, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$; we write

$$H : f \simeq g \quad \text{or} \quad f \simeq g\,.$$

If $H \in C^k$, f and g are said to be *C^k-homotopic*. Homotopy of class C^k defines an equivalence relation on the set of C^k between M and N.(See [**11**, p. 397].)

A surface M is said to be *contractible* if the identity map $\mathrm{id}_M : M \to M$ is homotopic to a constant map on M. If such homotopy is C^k, then M is said to be C^k-contractible.

The next theorem states two important facts regarding homotopies which will be useful later on.

Theorem 2.21. *The following are true:*

1. *Every continuous map $f : M \to N$ between surfaces of class C^k is homotopic to a C^k-map $g : M \to N$.*
2. *Any two C^k-maps $f, g : M \to N$ which are homotopic, are also C^k-homotopic.*

The proof of such facts rely on the existence of tubular neighborhoods of surfaces and can be seen in [**8, 6**]. In particular, we have the following

Proposition 2.3. *Every contractible surface of class C^k is also C^k-contractible.*

Let M be a surface. An open cover $M = \bigcup_{\lambda \in L} A_\lambda$ is said to be *simple* when every finite intersection of open sets A_λ is contractible. Using tubular neighborhoods and convexity arguments it can be shown ([**7**, p. 37]) that every open cover of a surface can be refined[11] by a simple cover. In particular, from Borel-Lebesgue, we obtain

[11] A cover $(A_\lambda)_{\lambda \in L}$ is refined by another cover $(B_\mu)_{\mu \in J}$ when each $B_\mu \subseteq A_\lambda$ for some $\lambda \in L$.

Proposition 2.4. *Every compact surface admits a finite simple covering.*

This last result tells us that every compact surface is of *finite type*. By that we mean a surface which admits a finite simple cover.

More generally, the concept of homotopy is defined (and of contractible space), in the same way as above, for topological spaces, except there is no differential structure in this context, that is, the maps are just continuous. Homotopy is still an equivalence relation in the set of continuous maps between topological spaces X and Y; the quotient by this relation is denoted by $[X;Y]$ and elements therein are denoted by $[f]$ and called homotopy class of f.

It can be shown ([**14**, p. 24]) that the composite of homotopic maps is homotopic, that is, $g_0 f_0 \simeq g_1 f_1$ whenever $f_0 \simeq f_1$ and $g_0 \simeq g_1$. This shows that there exists a category, called *homotopy category*, whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps between topological spaces.

We say that two spaces have the same *homotopy type* if they are isomorphic in the homotopy category. Thus X and Y have the same homotopy type if, and only if, we can find maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. In particular, homeomorphic spaces have the same homotopy type.

In § 3.1 we describe a functor from the homotopy category of smooth surfaces to the category of graded $\mathbf{R}$-modules.

Example 2.22. Let $p \in S^m \subseteq \mathbf{R}^{m+1}$ $(m \geq 1)$ be the north pole. Since the sterographic projection $\pi : S^m - \{p\} \to \mathbf{R}^m$ is a homeomorphism, it follows that $S^m - \{p\}$ and $\mathbf{R}^m$ have the same homotopy type.

Example 2.23. Let $i : S^m \hookrightarrow \mathbf{R}^{m+1}$ be inclusion map. The unit sphere S^m is isomorphic to $R^{m+1} - \{0\}$ in the homotopy category. Indeed, the map $f : R^{m+1} - \{0\} \to S^m$ as $f(x) = x/|x|$ is such that $f \circ i = \mathrm{id}_{S^m}$ and $H : i \circ f \simeq \mathrm{id}_{R^{m+1}-\{0\}}$, where $H(x,t) = (1-t)x/|x| + tx$.

Example 2.24. If $p \in S^m$ $(m \geq 2)$ is the north pole, the sterographic projection $\pi : S^m - \{p\} \to \mathbf{R}^m$ maps the south pole q to 0. Thus, it restricts to a homeomorphism $\pi : S^m - \{p,q\} \to \mathbf{R}^m - \{0\}$. Therefore, $S^m - \{p,q\}$ and $\mathbf{R}^m - \{0\}$ have the same homotopy type. From the previous example, it follows that $S^m - \{p,q\}$ has the same homotopy type S^{m-1}.

Example 2.25. If Y is a contractible topological space, then $X \times Y$ and X have the same homotopy type. Indeed, there is a homotopy $\tilde{H} : Y \times I \to Y$ from a constant map $c : Y \to Y$ to id_Y. We take $f : X \times Y \to X$ and $g : X \to X \times Y$ to be $f(x,y) = x$ and $g(x) = (x,c)$. Thus $f \circ g = \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_{X \times Y}$, where the homotopy $H : (X \times Y) \times I \to X \times Y$ is given by $H((x,y),t) = (x, \tilde{H}(y,t))$.

To finish up this section, we now state a result to be used in § 3.2. A proof can be found in [**11**, p.406].

Proposition 2.5. *Let $p \in A \subseteq S^m$, where A is open in S^m. There exists an open set $V \subseteq S^m$ such that $p \in V \subseteq A$ and V is diffeomorphic to $\mathbf{R}^m$.*

2.2. Differential Forms

In order to define differential forms on surfaces, we assume basic prior knowledge of multilinear algebra over finite-dimensional real vector spaces. (Natural references are [**11**, **10**].)

Let $M \subseteq \mathbf{R}^n$ be an m-dimensional surface. An *r-form* on M is a map

$$\omega : M \to \bigcup_{x \in M} \mathcal{A}_r(T_x M) \tag{2.2}$$

which assigns to each point $x \in M$ an alternating r-linear form $\omega(x) \in \mathcal{A}_r(T_x M)$.[12] Although there might be no concept regarding differentiability involved, it is common to refer to a map as the one in (2.2) as a *differential form of degree* r on M.

Now let us fix parametrization $\varphi : U_0 \to U \subseteq M$ of a neighboord $U \ni x$, where $x = \varphi(u)$. As we know, this determines a basis

$$\left\{ \frac{\partial \varphi}{\partial u_1}(u), \ldots, \frac{\partial \varphi}{\partial u_m}(u) \right\} \subseteq T_x M\,,$$

which in turn, determines a dual basis

$$\{du_1(x), \ldots, du_m(x)\} \subseteq (T_x M)^*\,.$$

The exterior products

$$du_{i_1}(x) \wedge \cdots \wedge du_{i_r}(x)\,,$$

with $1 \le i_1 < \cdots < i_r \le m$, constitute a basis for $\mathcal{A}_r(T_x M)$. Therefore, we can write

$$\omega(x) = \sum_{1 \le i_1 < \cdots < i_r \le m} a_{i_1 \cdots i_r}(u)\, du_{i_1}(x) \wedge \cdots \wedge du_{i_r}(x) \ \in \mathcal{A}_r(T_x M)\,,$$

where

$$a_{i_1 \cdots i_r}(u) = \omega(\varphi(u)) \cdot \left(\frac{\partial \varphi}{\partial u_{i_1}}(u), \ldots, \frac{\partial \varphi}{\partial u_{i_r}}(u) \right)\,.$$

This means that, relative to each parametrization $\varphi : U_0 \to U$, the r-form ω determines $\binom{m}{r}$ functions $a_{i_1 \cdots i_r} : U_0 \to \mathbf{R}$ given by the last equality above, called coordinate functions with respect to φ. Also, note that, for each $i = 1, \ldots, m$, du_i is a differential 1-form on U which assigns to every $x \in U$ the 1-linear form $du_i(x) \in (T_x U)^* = (T_x M)^*$. Thus, ω can be written as

$$\omega \ = \sum_{1 \le i_1 < \cdots < i_r \le m} a_{i_1 \cdots i_r}\, du_{i_1} \wedge \cdots \wedge du_{i_r}\,, \tag{2.3}$$

where $du_{i_1} \wedge \cdots \wedge du_{i_r}$ denotes the exterior product of 1-forms du_i. The exterior product $\alpha \wedge \beta$ of a differential r-form α and a differential s-form β is defined in the obvious way as the $(r+s)$-form given by $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x)$.

[12] We shall denote the space of alternating r-linear forms on a vector space E by $\mathcal{A}_r(E)$.

In order to avoid the heavy notation in (2.3), we write

$$\omega = \sum_I a_I \, du_I \quad \text{or} \quad \omega(x) = \sum_I a_I(u) \, du_I \, ,$$

where the sum is taken over all sets $I = \{i_1 < \cdots < i_r\} \subseteq \{1, 2, \ldots, m\}$. Most of the time, we will use the notation du_I to represent the r-linear form $du_I(x)$, there is no harm in doing so.

For a point $x = \varphi(u) = \psi(v)$, with $v = (\psi^{-1} \circ \varphi)(u)$, lying on the intersection of two parametrized neighborhoods $U \cap V \subseteq M$, we have

$$\omega(x) = \sum_J a_J(u) \, du_J = \sum_I b_I(v) \, dv_I \, .$$

These coordinate functions satisfy the change of coordinates formula ([**11**, p. 412])

$$a_J(u) = \sum_J \mathrm{jac}_{IJ}(\psi^{-1} \circ \varphi)(u) b_I(v) \, , \tag{2.4}$$

where $\mathrm{jac}_{IJ}(\psi^{-1} \circ \varphi)(u)$ represents the determinant of the submatrix obtained from the jacobian matrix of $\psi^{-1} \circ \varphi$ at u by selecting rows and columns with indices in $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$, respectively.

It follows from the definition above that the sum of two r-forms on an m-dimensional surface is yet an r-form on the same surface. Also, the product of an r-form by a real number is an r-form. This means that the set of differential r-forms on an m-dimensional surface has a natural (real) vector space structure. Note that if $r > m$, then such vector space is trivial, since $\mathcal{A}_r(T_x M) = \{0\}$ for every $x \in M$.

The next proposition summarizes some useful properties of the exterior product of forms. The proof follows from the same properties applied to multilinear forms. (See [**10**, p. 53].)

Proposition 2.6. *Let α, β, γ and η be differential forms on a surface M such that $\deg \alpha = \deg \eta = r$, $\deg \beta = s$ and $\deg \gamma = t$. The exterior product of forms enjoys the following properties:*

1. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$;
2. $(\alpha + \eta) \wedge \beta = \alpha \wedge \beta + \eta \wedge \beta$;
3. $c\alpha \wedge \beta = \alpha \wedge c\beta = c(\alpha \wedge \beta)$ *for* $c \in \mathbf{R}$;
4. $\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha$.

In case $M \subseteq \mathbf{R}^n$ is an n-dimensional surface, we already know that M is an open set in $\mathbf{R}^n$ (Example 2.3). This means that we need only one parametrization to cover M, namely, the identity map. Thus, for every point $x \in M$, $\mathcal{A}_r(T_x M) = \mathcal{A}_r(\mathbf{R}^n)$ and the dual basis is the same, denoted by $\{dx_1, \ldots, dx_n\} \subseteq (\mathbf{R}^n)^*$. Therefore, an r-form on M is a map $\omega : M \to \mathcal{A}_r(\mathbf{R}^n)$ such that

$$\omega(x) = \sum_I a_I(x) \, dx_I \in \mathcal{A}_r(\mathbf{R}^n) \, ,$$

where $a_I : M \to \mathbf{R}$ are functions.

A *differential r-form* ω on an m-dimensional C^s-surface M is said to be of *class* C^k, for $0 \le k < s$, if there exists a covering of M by images of C^k-parametrizations $\varphi : U_0 \to U$ for which the coordinate functions $a_I : U_0 \to \mathbf{R}$ are C^k functions; we write $\omega \in C^k$. Note that the change of coordinates formula above allows r-forms on C^s-surface to have differentiability class, at most, $s-1$. If M is smooth (C^∞), then we say that $\omega \in C^\infty$ whenever $\omega \in C^k$ for every k.

The set of differential r-forms of class C^∞ on a smooth surface M, denoted by $\Omega^r(M)$, is closed under addition and scalar multiplication, thus is a linear subspace of the space of r-forms on M. Note that $\Omega^r(M) = \{0\}$ for $r > m$.

Example 2.26. Every vector field $F = (F_1, \dots, F_m)$ of class C^k on an open set $U \subseteq \mathbf{R}^m$ corresponds to a 1-form $\omega_F \in C^k$ on U, namely,

$$\omega_F = \sum_{i=1}^{m} F_i \, dx_i \, .$$

Given an r-form ω on a surface M, we define its *support* as[13]

$$\operatorname{supp} \omega = \operatorname{cl}_M(\{x \in M \, ; \, \omega(x) \neq 0\}) \, ,$$

where cl_M denotes the topological closure in M. Thus $\operatorname{supp} \omega$ is always a closed set of M. Throughout the text, we will be mostly interested in differential forms with compact support, since integrals will only be defined for such forms.

Let α and β be compactly supported r-forms on a surface M and $c \in \mathbf{R} - \{0\}$. We have

$$\{\alpha + \beta \neq 0\} \subseteq \{\alpha \neq 0\} \cup \{\beta \neq 0\} \quad \text{and} \quad \{c\alpha \neq 0\} = \{\alpha \neq 0\} \, .$$

Thus $\operatorname{supp}(\alpha + \beta)$ and $\operatorname{supp}(c\alpha)$ are compact sets. Since the support of the zero form is empty, it follows that the set of compactly supported r-forms is a linear subspace of the spaces of r-forms on M.

From the above we see that the space of compactly supported differential r-forms of class C^∞ on M, denoted by $\Omega^r_c(M)$, is a linear subspace of $\Omega^r(M)$.

Let $(\omega_i)_{i \in I}$ be a family of r-forms of class C^k on a surface M. If each point $x \in M$ belongs only to a finite number of supports $\operatorname{supp} \omega_i$ then we can define a new r-form by setting

$$\omega = \sum_{i \in I} \omega_i \, . \tag{2.5}$$

The definition of ω goes as follows. Given $x \in M$, we have $x \in \bigcap_{s=1}^{n} \operatorname{supp} \omega_{i_s}$ for some $n \in \mathbf{N}$. Thus, we define

$$\omega(x) = \sum_{s=1}^{n} \omega_{i_s}(x) \, .$$

[13] We also use the shorter notation $\{\omega \neq 0\}$ instead of $\{x \in M \, ; \, \omega(x) \neq 0\}$.

Note that it might not be $\omega \in C^k$. Nevertheless, in § 2.3 we shall overcome this problem.

A differential m-form ω on an oriented m-dimensional surface M is said to be *positive* if, for every $x \in M$ and each positive basis $\{v_1, \ldots, v_m\} \subseteq T_x M$, $\omega(x) \cdot (v_1, \ldots, v_m) > 0$.

Example 2.27. If M is an oriented m-dimensional surface of class C^k, then there exists a positive m-form ν on M of class C^{k-1}, called *volume form* (see [**11**, p. 336]). Relative to a positive C^k-parametrization $\varphi : U_0 \to U \subseteq M$, ν is given by

$$\nu(x) = \sqrt{g(u)} du_1 \wedge \cdots \wedge du_m$$

for every $x = \varphi(u) \in U$, where $g(u) = \det(g_{ij}(u))$ and

$$g_{ij}(u) = \left\langle \frac{\partial \varphi}{\partial u_i}(u), \frac{\partial \varphi}{\partial u_j}(u) \right\rangle .$$

Moreover, orientability of surfaces translates into the existence of nonvanishing continuous forms as shown by the result whose proof can be seen in [**11**, p. 339].

Proposition 2.7. *Let M be an m-dimensional surface. Then M is orientable if, and only if, there exists a continuous m-form on M such that $\omega(x) \neq 0$ for every $x \in M$.*

2.2.1. Pullback

We will now define the pullback of forms and state some of its properties. It is well known that a linear map $A : E \to F$ between finite-dimensional vector spaces induces a map $A^* : \mathcal{A}_r(F) \to \mathcal{A}_r(E)$ given by

$$(A^*\alpha)(v_1 \ldots, v_r) = \alpha(A(v_1), \ldots, A(v_r))$$

for every $\alpha \in \mathcal{A}_r(F)$ and $v_1 \ldots, v_r \in F$. Now, if $f : M \to N$ is a C^k-map ($k \geq 1$) between surfaces, for each point $x \in M$ there is a linear map $f'(x) : T_x M \to T_{f(x)} N$, which induces a map

$$[f'(x)]^* : \mathcal{A}_r(T_x N) \to \mathcal{A}_r(T_{f(x)} M) .$$

The *pullback* of r-forms by f is a map

$$f^* = f_r^* : \{r\text{-forms on } N\} \to \{r\text{-forms on } M\}$$

which assings to each r-form ω on N an r-form $f^*\omega$ on M given by

$$(f^*\omega)(x) = [f'(x)]^* \cdot \omega(f(x)) . \tag{2.6}$$

More explicitly,

$$(f^*\omega)(x) \cdot (v_1, \ldots, v_r) = \omega(f(x)) \cdot (f'(x) \cdot v_1, \ldots, f'(x) \cdot v_r)$$

for every $x \in M$ and $v_1, \ldots, v_r \in T_x M$. Note that the pullback of a 0-form $g : N \to \mathbf{R}$ is just the composition of $g \circ f$.

Example 2.28. An important example of pullback is the one given by a parametrization, since it allows one to remain with the original coordinate functions. More precisely, let $\varphi : U_0 \to U \subseteq M$ be a parametrization, $M \subseteq \mathbf{R}^n$ an m-dimensional surface and ω an r-form on M. Regarding φ as a map between surfaces $\varphi : U_0 \to M$ and writing, for $x = \varphi(u) \in U$,

$$\omega(x) = \sum_I a_I(u)\, du_I\,,$$

it can be shown ([11, p. 335]) that

$$(\varphi^*\omega)(u) = \sum_I a_I(u)\, dx_I$$

for every $u = \varphi^{-1}(x) \in U_0$, where $\{dx_1, \ldots, dx_m\} \subseteq (\mathbf{R}^m)^*$ denotes the dual basis.

Example 2.29. The inclusion map $i : M \hookrightarrow N$ yields yet another useful instance of the pullback, called the restriction. Given an r-form on N, we write $\omega|_M = i^*\omega$. For $x \in M$, we already know that $T_xM \subseteq T_xN$. Thus, the definition of i^* tells us that $(i^*\omega)(x)$ is the restriction of the r-linear form $\omega(x)$ to $T_xM \times \cdots \times T_xM$.

Remark 2.30. Dual to the concept of restriction of a form is that of extension. We are more interested in the *zero extension* of a compactly support smooth forms; such extension will be crucial in § 3.2. Let $\omega \in \Omega_c^r(U)$, where $U \subseteq M$ is open. From the fact that $\omega = 0$ on the open set $U - \operatorname{supp}\omega$, we can define a smooth r-form ω_M with compact support ($\operatorname{supp}\omega$) by setting $\omega_M = \omega$ on U and $\omega_M = 0$ on $M - U$. To check that $\omega_M \in C^\infty$, the only parametrized neighborhoods that pose a problem are those originating from points $\operatorname{fr} U$, but this can be overcome since $\operatorname{dist}(\operatorname{fr} U, \operatorname{supp}\omega) > 0$ and $\omega = 0$ on $U - \operatorname{supp}\omega$.

The theorem below summararizes some of the important properties of pullbacks. The first three items follow directly fom the definitions. Item 4. can be proved using the change of coordinates formula from ω to $f^*\omega$ ([11, p. 334]), which in turn can be proved using (2.6) and (2.1).

Theorem 2.31. *Let $f : M \to N$ and $g : N \to P$ be C^k-maps ($k \geq 1$), α, β r-forms on N and $c \in \mathbf{R}$. Then*

1. $f^*(c\alpha + \beta) = cf^*(\alpha) + f^*(\beta)\,;$
2. $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta\,;$
3. $(g \circ f)^*(\omega) = f^*(g^*\omega)\,;$
4. *If $r \geq 1$, $\alpha \in C^s$ and $f \in C^{s+1}$, with $s \geq 0$, then $f^*\alpha \in C^s$.*

When it comes to compactly supported r-forms, the pullback f^* might not preserve the compactness of $\operatorname{supp}\omega$, that is, $f^*\omega$ might not have compact support if $\operatorname{supp}\omega$ is compact ([7, p. 40]). In order for compactness to be preserved, it is necessary to impose a new condition on f, namely, f must be a *proper map*. By that we mean a continuous map $f : X \to Y$ between topological spaces such that $f^{-1}(K) \subseteq X$ is compact for every compact set $K \subseteq Y$.

Clearly, the identity map and the composite of proper maps are also proper. There is then a category whose objects are smooth surfaces and whose morphisms are smooth proper maps between smooth surfaces.

Remark 2.32. If Y is a locally compact Hausdorff space, then $f : X \to Y$ is a proper map if, and only if, f is a closed map and, for every $y \in Y$, $f^{-1}(\{y\})$ is compact. From this characterization we see that the stereographic projection $\pi : S^m - \{p\} \to \mathbf{R}^m$ is a proper map. More generally, homeomorphisms $f : X \to M$ are proper whenever M is a surface (Example 2.5). This will come in handy during § 3.2 and § 3.3.

In case X and Y are metrizable, the property of being proper is equivalent to saying that, for every sequence $(x_n)_{n\in\mathbf{N}}$ of points $x_n \in X$, the sequence $(f(x_n))_{n\in\mathbf{N}}$ has no converging subsequence, whenever the same holds for $(x_n)_{n\in\mathbf{N}}$. In particular, the inclusion map $i : X \to Y \subseteq \mathbf{R}^m$ is proper.

If X is compact and Y is Hausdorff, then continuous maps $f : X \to Y$ are proper, since compact sets are also closed in Hausdorff spaces and closed sets are compact in compact spaces.

Now, let $f : M \to N$ be a C^{k+1}-map ($k \geq 0$) and ω a compactly supported r-form on N, where $r \geq 0$ and $\omega \in C^k$. If $x \in M$ is such $f(x) \notin \operatorname{supp}\omega$, then $\omega(f(x)) = 0$, whence $(f^*\omega)(x) = 0$. Thus

$$\{f^*\omega \neq 0\} \subseteq f^{-1}(\operatorname{supp}\omega)\,.$$

From this, we see that

$$\operatorname{supp} f^*\omega = \operatorname{cl}_M(\{f^*\omega \neq 0\}) \subseteq \operatorname{cl}_M(f^{-1}(\operatorname{supp}\omega)) = f^{-1}(\operatorname{supp}\omega)\,. \tag{2.7}$$

Therefore, $\operatorname{supp} f^*\omega$ is compact whenever f is proper.

Remark 2.33. From Theorem 2.31(1) and (4), it follows that the pullback

$$f^* : \Omega^r_c(N) \to \Omega^r_c(M)\,.$$

by a smooth proper map $f : M \to N$ between smooth surfaces is a linear map.

2.2.2. Exterior Derivative

We end this section defining the exterior derivative of a differential form and stating some of its properties. As we shall see in the next chapter, exterior differentiation is of fundamental importance in extending cohomology to the context of smooth surfaces.

Let M be an m-dimensional surface of class C^2 and ω an r-form on M such that $\omega \in C^1$. For a paramatrization $\varphi : U_0 \to U \subseteq M$, relative to which $\omega = \sum a_I du_I$, there exists a differential $(r+1)$-form ω'_φ on U given by

$$\omega'_\varphi = \sum_I \sum_{j=1}^m \frac{\partial a_I}{\partial u_j}\, du_j \wedge du_I\,.$$

This $(r+1)$-form is independent of the choice of parametrization; see [**11**, p. 344] for further details.

The *exterior derivative* of ω is then defined as the $(r+1)$-form $d\omega$ on M which assings to each $x \in M$ the $(r+1)$-linear form

$$(d\omega)(x) = \omega'_\varphi(x) \in \mathcal{A}_{r+1}(T_xU) = \mathcal{A}_{r+1}(T_xM)\,,$$

where $\varphi : U_0 \to U \subseteq M$ is a parametrization of $U \ni x$. Thus, for every $x = \varphi(u) \in U$, we can write

$$(d\omega)(x) = \sum_I \sum_{j=1}^{m} \frac{\partial a_I}{\partial u_j}(u)\, du_j \wedge du_I\,.$$

We also use a shorter notation:

$$d\omega = \sum_I da_I \wedge du_I\,,$$

where

$$da_I = \sum_{j=1}^{m} \frac{\partial a_I}{\partial u_j}\, du_j\,.$$

When there is need to be precise we will write $d_r\omega$ instead of $d\omega$.

From the above expression for $d\omega$ we see that process of passing from ω to $d\omega$ decreases the diferentiability class by 1. Thus, for $\omega \in C^\infty$, we have $d\omega \in C^\infty$.

Also, note that if ω is an m-form on an m-dimensional surface M, then $d\omega = 0$, since $\mathcal{A}_{m+1}(T_xM) = \{0\}$.

Example 2.34. If $F = (P, Q)$ is a smooth vector field on an open subset $U \subseteq \mathbf{R}^2$ and ω_F its corresponding 1-form (Example 2.26), then

$$d\omega_F = \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du \wedge dv = (\operatorname{rot} F)\, du \wedge dv\,.$$

Example 2.35. In particular, for a 0-form $f : M \to \mathbf{R}^n$ of class C^1, we see that

$$df(x) = \sum_{j=1}^{m} \frac{\partial (f \circ \varphi)}{\partial u_j}(u)\, du_j\,. \tag{2.8}$$

where $\varphi : U_0 \to U \subseteq M$ is a parametrization of $U \ni x$ and $x = \varphi(u)$. From this, it follows that f is a constant function whenever M is connected.

The next theorem summarizes the main properties of the exterior derivative. For a proof, see [**11**, p. 343].

Theorem 2.36. *Let $f : M \to N$ be a C^2-map between surfaces, α, β r-forms on N, with $\alpha, \beta \in C^1$, and $c \in \mathbf{R}$. Then*

1. $d(c\alpha + \beta) = c\, d\alpha + d\beta\,;$
2. *If $\alpha \in C^2$, then $d(d\alpha) = 0\,;$*
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta\,;$
4. $d_r(f_r^* \alpha) = f_{r+1}^*(d_r\alpha)\,.$

From the first item above, it follows that the exterior derivative defines, for $r \geq 0$, a linear map

$$d_r : \Omega^r(M) \to \Omega^{r+1}(M)\,.$$

Also, since $\{d\omega \neq 0\} \subseteq \{\omega \neq 0\}$, the exterior derivative, restricts to a linear map

$$d_r : \Omega^r_c(M) \to \Omega^{r+1}_c(M)\,.$$

Such linear maps will be crucial during the next chapter (§ 3.1), since they constitute the coboundary operator of a special cochain complex.

Note that zero extensions of forms (Remark 2.30) commute with exterior derivatives, that is, if M is a smooth surface and $U \subseteq M$ is open in M, then

$$d_r(\omega_M) = (d_r\omega)_M$$

for $\omega \in \Omega^r(U)$. This will be useful in § 3.2.

Let ω be an r-form of class C^1 on a surface M. We say that ω is *closed* when $d\omega = 0$. If there exists an $(r-1)$-form on M such that $\omega = d\alpha$, ω is said to be *exact*.

From item 4. above we see that a map $f \in C^2$ between surfaces sends closed forms to closed forms and exact forms to exact forms.

Also, it follows from item 3. every exact form is closed. Although the converse is not always true, we shall now see an instance in which it is.

Theorem 2.37. *Every closed differential r-form ($r \geq 1$) on a contractible surface of class C^2 is also exact.*

Proof. Suppose first that $M \subseteq \mathbf{R}^p$ is an m-dimensional surface of class C^2. Parametrizations for the $m+1$-dimensional surface $M \times \mathbf{R}$ are precisely the maps $\varphi \times \mathrm{id}_{\mathbf{R}} : U_0 \times \mathbf{R} \to U \times \mathbf{R}$, where $\varphi : U_0 \to U \subseteq M$ ($U_0 \subseteq \mathbf{R}^m$) are parametrizations of $U \ni x$, given by $(\varphi \times \mathrm{id}_{\mathbf{R}})(u,t) = (\varphi(u),t) = (x,t)$. Thus, representing points of $U_0 \times \mathbf{R} \subseteq \mathbf{R}^{m+1}$ as $(u,t) = (u_1,\ldots,u_m,t)$ and $du_{m+1} = dt$, for any r-form ω on $M \times \mathbf{R}$ we can write[14], with respect to such a parametrization of $U \times \mathbf{R} \ni (x,t)$,

$$\begin{aligned}\omega(x,t) = \sum_S a_S(u,t)du_S &= \sum_{m+1\in S} a_S(u,t)du_S + \sum_{m+1\notin S} a_S(u,t)du_S \\ &= \sum_I (-1)^{r-1}a_I(u,t)dt \wedge du_I + \sum_J a_J(u,t)du_J \\ &= dt \wedge \sum_I (-1)^{r-1}a_I(u,t)du_I + \sum_J a_J(u,t)du_J \\ &= dt \wedge \alpha(u,t) + \beta(u,t)\,,\end{aligned}$$

where $I = \{p_1 < \ldots < p_{r-1}\}$ and $J = \{j_1 < \ldots < j_r\}$ do not contain the index $m+1$. Such decomposition is unique, because if $\omega = dt \wedge \alpha_0 + \beta_0$ is any other decomposition, then $dt \wedge (\alpha - \alpha_0) + (\beta - \beta_0) = 0$, whence $dt \wedge (\beta - \beta_0) = 0$, which implies $\beta = \beta_0$, from which we conclude that $\alpha = \alpha_0$.

[14] Note that the hypotesis of M being a C^2-surface does not play a role here.

Let us denote $\Omega_0^r(X)$, $r \geq 1$, as the space of continuous differential r-forms on a surface X. Very well, the composite map

$$A_r : \Omega_0^r(M \times \mathbf{R}) \longrightarrow \Omega_0^{r-1}(M \times \mathbf{R}) \longrightarrow \Omega_0^{r-1}(M)$$

$$\omega \mapsto \alpha = \sum_I c_I dx_I \mapsto \sum_I (\textstyle\int_0^1 c_I(\cdot, t)\, dt) dx_I$$

is well-defined by the uniqueness of the decomposition of r-forms. Clearly, this map is linear. Now consider the family of maps $i_t : M \to M \times \mathbf{R}$, where $t \in \mathbf{R}$, defined by $i_t(x) = (x, t)$. We will show that $A_{r+1}(d_r\omega) + d_{r-1}(A_r\omega) = i_1^*\omega - i_0^*\omega$ for every r-form ω of class C^1 on $M \times \mathbf{R}$. Indeed, the decomposition $\omega = dt \wedge \alpha + \beta$ gives $d\omega = dt \wedge (-d\alpha) + d\beta$, where $\deg(-d\alpha) = r$ e $\deg(d\beta) = r + 1$. We compute this expression explicitly:

$$\begin{aligned} d\omega(x,t) &= dt \wedge \left((-1)^r \sum_{k,I} \frac{\partial a_I}{\partial u_k}(u,t) dx_k \wedge du_I \right) + \sum_{k,J} \frac{\partial a_J}{\partial u_k}(u,t) du_k \wedge du_J \\ &= dt \wedge \left((-1)^r \sum_{1 \leq k \leq m, I} \frac{\partial a_I}{\partial u_k}(u,t) du_k \wedge du_I + \sum_J \frac{\partial a_J}{\partial t}(u,t) du_J \right) \\ &\qquad\qquad + \sum_{1 \leq k \leq m, J} \frac{\partial a_J}{\partial u_k}(u,t) du_k \wedge du_J\,, \end{aligned}$$

where $I = \{p_1 < \ldots < p_{r-1}\}$ e $J = \{j_1 < \ldots < j_r\}$ do not contain the index $m + 1$. Thus

$$\begin{aligned} A(d\omega)(x) &= (-1)^r \sum_{1 \leq k \leq m, I} \left(\int_0^1 \frac{\partial a_I}{\partial u_k}(u,t)\, dt \right) du_k \wedge du_I + \\ &\qquad + \sum_J \left(\int_0^1 \frac{\partial a_J}{\partial t}(u,t)\, dt \right) du_J\,. \end{aligned} \tag{2.9}$$

On the other hand,

$$\begin{aligned} d(A\omega)(x) &= (-1)^{r-1} \sum_{1 \leq k \leq m, I} \frac{\partial}{\partial u_k} \left(\int_0^1 a_I(u,t)\, dt \right) du_k \wedge du_I \\ &= (-1)^{r-1} \sum_{1 \leq k \leq m, I} \left(\int_0^1 \frac{\partial a_I}{\partial u_k}(u,t)\, dt \right) du_k \wedge du_I\,, \end{aligned}$$

since $a_I \in C^1$ on U_0. Note that in the sum $A(d\omega) + d(A\omega)$ the first term in (2.9) cancels out $d(A\omega)$, leaving just the r-form

$$\sum_J \left(\int_0^1 \frac{\partial a_J}{\partial t}(u,t)\, dt \right) du_J = \sum_J (a_J(u,1) - a_J(u,0)) du_J\,.$$

This is precisely the expression for $i_1^*\omega - i_0^*\omega$. Indeed, from the definition of pullback, it follows that $i_p^*(dt) = 0$ and

$$\begin{aligned}(i^*\omega)(x)\cdot\left(\frac{\partial\varphi}{\partial u_{k_1}}(u),\ldots,\frac{\partial\varphi}{\partial u_{k_r}}(u)\right) &= \omega(x,p)\cdot\left(\left(\frac{\partial\varphi}{\partial u_{k_1}}(u),0\right),\ldots,\left(\frac{\partial\varphi}{\partial u_{k_r}}(u),0\right)\right)\\ &= a_I(u,p)\end{aligned}$$

for $p = 0, 1$. Thus, i_p^* applied to $\omega = \sum_S a_S(u,t)du_S$ yields

$$i_p^*\omega = \sum_J a_J(u,p)du_J\,,$$

where $J = \{j_1 < \ldots < j_r\}$ does not contain $m+1$. Therefore,

$$A_{r+1}(d_r\omega) + d_{r-1}(A_r\omega) = i_1^*\omega - i_0^*\omega\,,$$

as we wanted to show.

Next we consider two differentiable maps $f, g : M \to N$, where N is another surface of class C^2. Suppose that f and g are C^2-homotopic; let $H : M \times [0,1] \to N$ be such homotopy. Using Lemma A.4 we can extend H to a C^2-map $\tilde{H} : M \times \mathbf{R} \to N$ defined by $\tilde{H}(x,t) = H(x,\xi(t))$, where $\xi : \mathbf{R} \to \mathbf{R}$ ($\xi \in C^\infty$) is such that $0 \le \xi \le 1$, $\xi(t) = 0$ for $t \le 0$ and $\xi = 1$ for $t \ge 1$. Thus, if A_r is the linear map defined above, then the map T_r given by the composition

$$\Omega_0^r(N) \xrightarrow{\tilde{H}^*} \Omega_0^r(M \times \mathbf{R}) \xrightarrow{A_r} \Omega_0^{r-1}(M)\,,$$

is well-defined. Since $\tilde{H}(x,0) = f(x)$ and $\tilde{H}(x,1) = g(x)$, then $f = \tilde{H} \circ i_0$ and $g = \tilde{H} \circ i_1$, where we take i_0 and i_1 as above. Therefore, $f^* = i_0^* \circ \tilde{H}^*$ and $g^* = i_1^* \circ \tilde{H}^*$. Thus, if $\omega \in \Omega_0^r(V)$, with $\omega \in C^1$, then $\tilde{H}^*\omega \in C^1$ and we have

$$\begin{aligned}T_{r+1}(d_r\omega) + d_{r-1}(T_r\omega) &= (A_{r+1} \circ \tilde{H}^*)(d_r\omega) + d_{r-1}((A_r \circ \tilde{H}^*)\omega)\\ &= A_{r+1}(\tilde{H}^*(d_r\omega)) + d_{r-1}(A_r(\tilde{H}^*\omega))\\ &= A_{r+1}(d_r(\tilde{H}^*\omega)) + d_{r-1}(A_r(\tilde{H}^*\omega))\\ &= i_1^*(\tilde{H}^*\omega) - i_0^*(\tilde{H}^*\omega)\\ &= (i_1^* \circ \tilde{H}^*)\omega - (i_0^* \circ \tilde{H}^*)\omega\\ &= g^*\omega - f^*\omega\,.\end{aligned}$$

It follows from this equality that if ω is a closed r-form on N, then $g^*\omega - f^*\omega = d_{r-1}(T_r\omega)$, whence $g^*\omega - f^*\omega$ is an exact r-form on M. Finally, if M is contractible, then (by Proposition 2.3) id_M is C^2-homotopic to a constant map c on M. Thus, taking $g = \mathrm{id}_M$ and $f = c$, if ω is a closed r-form on M, then $g^*\omega - f^*\omega = \omega$ is exact.

Q.E.D.

2.3. Integration of Forms on Surfaces

We now define the integral of a compactly supported differentiable m-form over an oriented m-dimensional surface. This concept can be further extended to a differential form whose support is not necessarily compact, but such generalization will not be needed for our purposes.

Let us fix an oriented m-dimensional surface M of class C^1. We begin by defining the integral over M of a compactly supported continuous m-form ω such that $\operatorname{supp}\omega$ is contained in the image of some positive parametrization $\varphi : U_0 \to U \subseteq M$. Relative to φ, we can write $\omega(x) = a(u)du_1 \wedge \cdots \wedge du_m \in \mathcal{A}_m(T_xM)$ for every $x = \varphi(u) \in U$, where $a : U_0 \to \mathbf{R}$ is a continuous function such that $\operatorname{supp}(a) = \varphi^{-1}(\operatorname{supp}\omega)$ (this equality can be verified directly by the definitons). Then we define the integral of ω over M by

$$\int_M \omega = \int_K a\,,$$

where $K \subseteq \mathbf{R}^m$ is any compact Jordan-measurable[15] set such that $\operatorname{supp}(a) \subseteq K \subseteq U_0$.

Clearly, this definition does not depend on the chosen compact set K since a is zero outside its support. Also, the value of the above integral remains the same for any other choice of positive parametrization (this follows from the change of variables formula; see [**11**, p. 333]).

What if $\operatorname{supp}\omega$ is contained in the image of a negative parametrization? Well, to see what happens, let $\psi : V_0 \to U \subseteq M$ be a negative parametrization, with $\operatorname{supp}\omega \subseteq U$, relative to which $\omega(x) = b(v)\,dv_1 \wedge \cdots \wedge dv_m$, for $x = \psi(v) \in U$. We obtain a new parametrization $\varphi : U_0 \to U$ by setting $\varphi = \psi \circ f$ and $U_0 = f^{-1}(V_0)$, where $f : \mathbf{R}^m \to \mathbf{R}^m$ is a diffeomorphism given by

$$f(y_1, \ldots, y_m) = (-y_1, y_2, \ldots, y_m)\,.$$

Note that $\operatorname{jac} f = -1$ on U_0. Since

$$\operatorname{jac}(\zeta^{-1} \circ \varphi) = \operatorname{jac}(\zeta^{-1} \circ \psi)\operatorname{jac} f > 0$$

for any positive parametrization $\zeta : W_0 \to W$, with $W \cap U \neq \emptyset$, it follows that φ is a positive parametrization. From the change of coordinates formula (2.4), we see that

$$\omega(x) = \operatorname{jac}(\psi^{-1} \circ \varphi)(u)\, b(f(u))\, du_1 \wedge \cdots \wedge du_m = -b(f(u))\, du_1 \wedge \cdots \wedge du_m$$

for $x = \varphi(u) \in U$. For a compact Jordan-measurable $L \subseteq \mathbf{R}^m$ such that $\psi^{-1}(\operatorname{supp}\omega) \subseteq L \subseteq V_0$, we have a compact Jordan-measurable $K = f^{-1}(L)$ such that $\varphi^{-1}(\operatorname{supp}\omega) \subseteq K \subseteq U_0$. Thus, Theorem A.3 gives

$$\int_M \omega = \int_K -(b \circ f) = -\int_L b\,.$$

Summarizing: the *integral* over M of a compactly supported continuous m-form ω whose support is contained in the image of a positive or negative parametrization φ :

[15]By a Jordan-measurable set, we mean a set $X \subseteq \mathbf{R}^m$ whose frontier $\operatorname{fr}(X)$ has null Lebesgue measure.

$U_0 \to U \subseteq M$, relative to which $\omega(x) = a(u)du_1 \wedge \cdots \wedge du_m$ for every $x = \varphi(u) \in U$, is defined as

$$\int_M \omega = \pm \int_K a \,,$$

where $K \subseteq \mathbf{R}^m$ is any compact Jordan-measurable set such that $\operatorname{supp}(a) \subseteq K \subseteq U_0$, with positive sign if $\varphi > 0$ and negative sign if $\varphi < 0$.

From the discussion above, it follows that the integral of a differential form is a signed integral, in the sense that its signal flips whenever the orientation of the surface is flipped. More precisely,

$$\int_{-M} \omega = - \int_M \omega \,. \tag{2.10}$$

Let ω be an m-form on M and $\varphi : U_0 \to U \subseteq M$ a positive or negative parametrization, with $\operatorname{supp}\omega \subseteq U$, relative to which $\omega(x) = a(u)du_1 \wedge \cdots \wedge du_m$ for every $x = \varphi(u) \in U$. If $i : U \hookrightarrow M$ denotes the inclusion, then, relative to φ, we can write $i^*\omega = a\, du_1 \wedge \cdots \wedge du_m$. Thus

$$\int_M \omega = \int_U i^*\omega \tag{2.11}$$

A more general result is

Proposition 2.8. *Let $(M, \mathcal{A})$ be an oriented m-dimensional surface of class C^1, $A \subseteq M$ an open set, $\varphi : U_0 \to U \subseteq A$ a positive (or negative) parametrization on A and ω a compactly supported continuous m-form on M such that* $\operatorname{supp}\omega \subseteq U$*. Then*

$$\int_M \omega = \int_A i^*\omega \,,$$

where $i : A \hookrightarrow M$ is the inclusion map.

Proof. We only prove the case where φ is positive; the negative case is analogous. Since A is open in M, it follows that $U \subseteq M$ is a parametrized neighborhood in M. Also, $\varphi \in \mathcal{A}$. To see this, let $\xi : W_0 \to W \subseteq M$ a parametrization on M belonging to $\mathcal{A}$ such that $W \cap V \neq \emptyset$. Since φ is a positive parametrization on A, we see (Remark 2.16) that φ and $\zeta = \xi|_{\xi^{-1}(W\cap A)} : \xi^{-1}(W \cap A) \to W \cap A$ are coherent parametrizations on A, that is,

$$\zeta^{-1} \circ \varphi : \varphi^{-1}(W \cap A \cap U) \to \zeta^{-1}(W \cap A \cap U)$$

has positive jacobian on $\varphi^{-1}(W \cap A \cap U)$. Since $W \cap A \cap U = W \cap U$ and the map $\xi^{-1} \circ \varphi : \varphi^{-1}(W \cap U) \to \xi^{-1}(W \cap U)$ coincides with $\zeta^{-1} \circ \varphi$, the claim follows. Thus, by (2.11), we see that

$$\int_M \omega = \int_U i_U^*\omega \,,$$

where $i_U : U \hookrightarrow M$ is the inclusion. On the other hand, if $j : U \hookrightarrow A$ is the inclusion map, then $i \circ j = i_U$. Since i is proper, $\operatorname{supp} i^*\omega$ is compact. From (2.7), we have

$$\operatorname{supp} i^*\omega \subseteq i^{-1}(\operatorname{supp}\omega) = \operatorname{supp}\omega \subseteq U \,.$$

Again, from (2.11), it follows that

$$\int_A i^*\omega = \int_U j^* i^*\omega = \int_U (i\circ j)^*\omega = \int_U i_U^*\omega\,.$$

Q.E.D.

Before going into definition of integral, we make a small, but important, remark.

Remark 2.38. Let $(\omega_i)_{i\in I}$ be a family of r-forms of class C^k on a surface M. If $(\operatorname{supp}\omega_i)_{i\in I}$ is locally finite family on M, then $\omega = \sum \omega_i \in C^k$ (cf. 2.5). To see this, let $U \ni x_0$ be a neighborhood of x_0 in M. We have $U \cap \operatorname{supp}\omega_i = \emptyset$ except for a finite set $\{i_1,\dots,i_n\} \subseteq I$. Thus

$$\omega(x) = \sum_{s=1}^{n} \omega_{i_s}(x)$$

for $x \in U$.

Now we extend the definition of integration given above to the case of a compactly supported continuous m-form ω such that $\operatorname{supp}\omega$ is not necessarily contained in the image of a positive parametrization.

Given a cover by images of positive parametrizations $M = \bigcup U_i$ $(\varphi_i : U_{0i} \to U_i)$, we can choose (Theorem 2.11) a partition of unity $\sum \xi_i = 1$ of class C^1 on M which is also strictly subordinated to the open cover $(U_i)_{i\in\mathbf{N}}$, that is, $\operatorname{supp}\xi_i \subseteq U_i$ for every i.

Set $\omega_i = \xi_i\omega$, and let $x \in U_j$ for some $j \in \mathbf{N}$. Since $\operatorname{supp}\omega$ is a compact set and $(\operatorname{supp}\xi_i)_{i\in\mathbf{N}}$ is locally finite, we have $\operatorname{supp}\omega \cap \operatorname{supp}\xi_i = \emptyset$ and $U_j \cap \operatorname{supp}\xi_i = \emptyset$ except for a finite number of indices i. Thus $\omega_i = \xi_i\omega = 0$ on U_j for all but a finitely many i. This shows that $(\operatorname{supp}\omega_i)_{i\in\mathbf{N}}$ is locally finite.

Also, note that

$$\sum_i \omega_i = \sum_i \xi_i\omega = \omega\sum_i \xi_i = \omega\,.$$

For every $i \in \mathbf{N}$, $\operatorname{supp}\omega_i \subseteq U_i$ and $\operatorname{supp}\omega_i \subseteq \operatorname{supp}\omega$ is a closed subset of the compact set $\operatorname{supp}\omega$, therefore $\operatorname{supp}\omega_i$ is also compact. Then we define the integral of ω over M by

$$\int_M \omega = \sum_i \int_M \omega_i\,.$$

This definition independs on the chosen partition of unity. Indeed, take $\sum \zeta_j = 1$ to be another partition of unity on M, strictly subordinated to an open cover by images of positive parametrizations $M = \bigcup V_j$ and define $\omega'_j = \zeta_j\omega$ and $\omega_{ij} = \xi_i\zeta_j\omega$. We have

$$\sum_j \omega_{ij} = \sum_j \xi_i\zeta_j\omega = \xi_i\omega\sum_j \zeta_j = \omega_i\,.$$

Similarly,

$$\omega'_j = \sum_i \omega_{ij}\,.$$

Now, for every $i, j \in \mathbf{N}$, $\operatorname{supp} \omega_{ij} \subseteq U_i$ and $\operatorname{supp} \omega_{ij} \subseteq V_j$. Thus

$$\int_M \omega_i = \sum_j \int_M \omega_{ij} \quad \text{e} \quad \int_M \omega'_j = \sum_i \int_M \omega_{ij}$$

We then conclude that

$$\sum_i \int_M \omega_i = \sum_i \sum_j \int_M \omega_{ij} = \sum_j \sum_i \int_M \omega_{ij} = \sum_j \int_M \omega'_j \,.$$

In the next theorem we sum up some of the properties of surface integrals. The first item tells us that integration of forms is a linear functional on $\Omega_c^r(M)$. This will come in handy during Chapter 3.

Theorem 2.39. *Let M and N be oriented m-dimensional surfaces of class C^1, α and β compactly supported continuous m-forms on M and ω a compactly supported continuous m-form on N.*

1. *If $c \in \mathbf{R}$, then*
$$\int_M (c\alpha + \beta) = c \int_M \alpha + \int_M \beta \,.$$
2. *If $\alpha \geq 0$ and there exists $x \in M$ such that $\alpha(x) > 0$, then*
$$\int_M \alpha > 0 \,.$$
3. *If $f : M \to N$ is an orientation-preserving (resp. reversing) diffeomorphism, then*
$$\int_M f^*\omega = \int_N \omega \qquad \left(\textit{resp.} \int_M f^*\omega = -\int_N \omega\right)$$
4. *If $A \subseteq N$ is an open set,* $\operatorname{supp} \omega \subseteq A$ *and $i : A \hookrightarrow N$ is the inclusion, then*
$$\int_N \omega = \int_A i^*\omega \,.$$
5. *The integral of a differential form is a signed integral:*
$$\int_{-M} \alpha = -\int_M \alpha \,.$$

Proof. We will only prove the fourth item (item 5. follows from (2.10)). For a proof of the first three items, see [**11**]. Let $M = \bigcup_k V_k$ be an open cover of M by the images of positive parametrizations $\varphi_k : V_{0k} \to V_k$, relative to which $\omega = \sum \omega_k$ (finite sum), where each ω_k is a compactly supported continuous m-form on N such that $\operatorname{supp} \omega_k \subseteq V_k \cap \operatorname{supp} \omega \subseteq V_k \cap A$. From $M = \bigcup_k V_k$ we obtain a cover for A by parametrized neighborhoods, namely, $A = \bigcup_k V_k \cap A$. Also, note that $i^*\omega = \sum i^*\omega_k$, where each $i^*\omega_k$ is a compactly supported continuous m-form and $\operatorname{supp} i^*\omega_k \subseteq V_k \cap A$ (see (2.7)). Thus, from Proposition 2.8 applied to ω_k, it follows that

$$\int_N \omega = \sum_k \int_N \omega_k = \sum_k \int_A i^*\omega_k = \int_A i^*\omega_k \, .$$

Q.E.D.

Next we state Stokes' Theorem and a corollary that will be useful in § 3.3. For a proof, see [**11**, p. 391].

Theorem 2.40 (Stokes)**.** *Let ω be a compactly supported m-form of class C^1 on an oriented $(m+1)$-dimensional surface whose boundary ∂M we endow with the orientation induced by that of M. Then*

$$\int_M d\omega = \int_{\partial M} \omega \, .$$

Corollary 2.40.1. *The integral of an exact continuous m-form with compact support on an m-dimensional oriented surface is zero.*

Chapter 3

Poincaré Duality

This chapter puts together the topics developed in the first two, with the main goal being a duality theorem due to H. Poincaré and some of its applications. Main references are [**2**, **3**, **7**, **12**, **13**]. **Exclusively in this chapter, unless otherwise stated, all surfaces and differential forms are smooth** (C^∞).

3.1. de Rham Cohomology

We have already seen that, if M is an m-dimensional surface, exterior differentiation defines a linear map $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ ($r \geq 0$) such that $d_{r+1}d_r = 0$, and $\Omega^r(M) = \{0\}$ whenever $r > m$.

Such facts lead us to definition of the *de Rham complex* of M. By that, we mean the cochain complex (over $\mathbf{R}$) $\Omega^*(M) = (\Omega^r(M), d_r)_{r\in\mathbf{Z}}$, namely,

$$\cdots \longrightarrow 0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{d_m} 0 \longrightarrow \cdots$$

In this case there is a slight change of notation from the one used in Chapter 1, we denote $Z^r(\Omega^*(M))$ by $Z^r(M)$ and $B^r(\Omega^*(M))$ by $B^r(M)$. Also, even though the usual notation for coboundary operators would require the superscript d^r, we will follow [**11**] and keep using the subscript notation d_r for the exterior derivative (this will not cause any problems, since most of the times we will omit the subscript when there is no room for confusion).

The cohomology group of $\Omega^*(M)$ (which is a real vector space) is called *de Rham cohomology group* of M and is denoted by

$$H_{dR}(M) = \left(H^r_{dR}(M) = Z^r(M)/B^r(M)\right)_{r\in\mathbf{Z}}.$$

Note that in this context, elements of $Z^r(M)$ are precisely the closed r-forms on M. Those belonging to $B^r(M)$ are the exact r-forms on M. This means that two closed forms are cohomologous when their difference is exact. Thus $H^r_{dR}(M)$ measures the exactness of closed r-forms on M.

Clearly, $H^r_{dR}(M) = 0$ for $r < 0$ and $r > m$. Thus, during results and manipulations involving the cohomology groups $H^r_{dR}(M)$, the ones that matter the most are those having dimension $0 \le r \le m$.

A more concrete explanation of role played by the de Rham cohomology is as follows. From basic multivariable calculus we know that, on a simply connected (path-connected and without "holes") open set $U \subseteq \mathbf{R}^3$, a smooth vector field $F = (P, Q, R) : U \to \mathbf{R}^3$ is conservative if, and only if, $\operatorname{rot} F = 0$ (irrotational). However, this might not hold if U is not simply connected, as we can see by taking $U = \mathbf{R}^3 - (z\text{-axis})$ and

$$F(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z \right),$$

that is, $\operatorname{rot} F = 0$ and there is no function $f : U \to \mathbf{R}$ such that $F = \operatorname{grad} f$. Equivalently, $\omega_F = Pdx + Qdy + Rdz$ is a closed 1-form on U which is not exact. The de Rham cohomology group $H^1_{dR}(U)$ measures precisely the extent to which this characterization of irrotational fields fails. In other words, one searches for "holes" in the space M by looking for closed forms which are not exact.

Remark 3.1. Henceforth, whenever we write $H^r(M)$, keep in mind we mean the rth de Rham cohomology group $H^r_{dR}(M)$.

Example 3.2. If M is a connected surface, then $H^0(M) \approx \mathbf{R}$. Indeed, we have $B^0(M) = \{0\}$ and Example 2.35 shows that

$$\begin{aligned} Z^0(M) &= \{f : M \to \mathbf{R}\,;\, df = 0\} \\ &= \{f : M \to \mathbf{R}\,;\, f \text{ is constant}\}\,. \end{aligned}$$

Thus $H^0(M) = Z^0(M)/B^0(M) \approx Z^0(M) \approx \mathbf{R}$. In particular, if N is a contractible surface, then $H^0(M) \approx \mathbf{R}$, since M is path-connected.

Since $H^r(M) = \{0\}$ whenever closed forms $\omega \in \Omega^r(M)$ are also exact, Theorem 2.37 implies Poincaré's Lemma.

Theorem 3.3 (Poincaré's Lemma)**.** *For every $r \ge 1$ the rth de Rham cohomology group of a contractible surface is trivial.*

Example 3.4. An immediate consequence of Poincaré's Lemma is that, for $r \ge 1$, the rth de Rham cohomology group of an open star-shaped set in $\mathbf{R}^m$ is trivial.

From Theorem 2.36(4), it follows that there exists a contravariant functor, called *pullback*, from the category of smooth surfaces and smooth maps to the category of cochain complexes and cochain maps which assings to a surface M its de Rham complex and to each smooth map $f : M \to N$ its induced cochain map $f^* : \Omega^*(N) \to \Omega^*(M)$ given by

$$f^* = (f^*_r : \Omega^r(N) \to \Omega^r(M))_{r \in \mathbf{Z}}\,.$$

Again from Theorem 2.36(4) (and Theorem A.5), it follows that the pullback of r-forms f_r^*, where $f : M \to N$, induces a homomorphism at the cohomology level given by $f_r^{**}([\omega]) = [f_r^*\omega]$. Therefore, the morphism of degree 0 between graded $\mathbf{R}$-modules $H(f) = f^{**}$ corresponding to $f^* : \Omega^*(N) \to \Omega^*(M)$ via the cohomology functor is precisely

$$H(f) = (f_r^{**} : H^r_{dR}(N) \to H^r_{dR}(M))_{r\in\mathbf{Z}}\,.$$

The composite of the two functors above is a contravariant functor T from the category of smooth surfaces and smooth maps to the category of graded $\mathbf{R}$-modules which assigns to each surface its de Rham cohomology group and to each smooth map $f : M \to N$ its induced morphism $T(f) = f^{**} : H_{dR}(N) \to H_{dR}(M)$. Thus, diffeomorphic surfaces have isomorphic de Rham cohomology groups. Nevertheless, a stronger result is valid, namely, de Rham cohomology is homotopy invariant. Put precisely:

Theorem 3.5. *There is a contravariant functor from the homotopy category of smooth surfaces and continuous maps to the category of graded* $\mathbf{R}$*-modules and morphisms of degree* 0 *which assigns to each surface its cohomology group and to a homotopy class* $[f]$ *its induced morphism* f^{**}*.*

Proof. Let $f : M \to N$ be a continuous map between smooth surfaces. From the fisrt item in Theorem 2.21, we define a morphism $f^{**} : H_{dR}(N) \to H_{dR}(M)$ by $f^{**} = g^{**}$ for some smooth map $g : M \to N$ such that $f \simeq g$. From the second item in the same theorem and the argument in the proof of Theorem 2.37 we see that, for any two homotopic smooth maps $\xi, \zeta : M \to N$ and $\omega \in Z^r(N)$, the difference $\xi_r^*\omega - \zeta_r^*\omega$ is exact. Thus $\xi^{**} = \zeta^{**}$. Therefore, f^{**} is well-defined, since, by the transitivity of the homotopy relation, it independs on the choice of g. Again by transitivity, one sees that $(g \circ f)^{**} = f^{**} \circ g^{**}$, and $f^{**} = g^{**}$ in case $f \simeq g$.

Q.E.D.

Remark 3.6. From now on, to simplify notation, we will write f^* instead of f^{**}.

This theorem yields another proof of Poincaré's Lemma, since contractible spaces have the same homotopy type as a one-point space ([**14**, p. 26]).

Example 3.7. Let M and N be surfaces, where N is contractible. Since $M \times N$ and M have the same homotopy type (Example 2.25), it follows from Theorem 3.5 that $H_{dR}(M \times N) = H_{dR}(M)$.

Example 3.8. From Example 3.2 we see that $H^0(S^1) \approx \mathbf{R}$. Also, it is obvious that $H^r(S^1) = \{0\}$ for $r > 1$. It can be shown ([**7**, p. 25]) that $H^1(\mathbf{R}^2 - \{0\}) \approx \mathbf{R}$. Thus, Theorem 3.5 and Example 2.22 imply $H^1(S^1) \approx \mathbf{R}$.

Every surface M admits a decomposition $M = U \cup V$, where U and V are open sets of M. Such decomposition yields a short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\ f\ } \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\ g\ } \Omega^*(U \cap V) \longrightarrow 0\,,$$

where

$$f_r(\omega) = (\omega|_U, \omega|_V) \quad \text{and} \quad g_r(\alpha, \beta) = \alpha|_{U\cap V} - \beta|_{U\cap V}\,.$$

Using in Theorem 2.36(4), it is easy to verify that f and g are cochain maps. To see that this sequence is exact, note that it follows immediately from the definitions of f_r and g_r that f_r is injective and $\operatorname{im} f_r = \ker g_r$ for every r. As to the surjectivity of g_r, we take (Theorem 2.11) a partition of unity $\xi_1 + \xi_2$ strictly subordinated to the cover $M = U \cup V$. Thus $\operatorname{supp}\xi_1 \subseteq U$ and $\operatorname{supp}\xi_2 \subseteq V$. We define, for $\omega \in \Omega^r(U\cap V)$, two r-forms $\omega_1 \in \Omega^r(U)$ and $\omega_2 \in \Omega^r(V)$ by

$$\omega_1 = \begin{cases} \xi_2\omega & \text{on } U\cap V \\ 0 & \text{on } U-(U\cap V) \end{cases} \qquad \omega_2 = \begin{cases} -\xi_1\omega & \text{on } U\cap V \\ 0 & \text{on } V-(U\cap V) \end{cases}.$$

From this, it follows that $g(\omega_1, \omega_2) = \omega_1|_{U\cap V} - \omega_2|_{U\cap V} = \xi_2\omega + \xi_1\omega = \omega$, as we wanted to show.

Therefore, the short exact sequence above gives rise, via Theorem 1.17, to a long exact sequence in cohomology

$$\cdots \xrightarrow{\Delta^*_{r-1}} H^r(M) \xrightarrow{f^*_r} H^r(U)\oplus H^r(V) \xrightarrow{g^*_r} H^r(U\cap V) \xrightarrow{\Delta^*_r} H^{r+1}(M) \xrightarrow{f^*_{r+1}} \cdots$$

where

$$f^*_r[\omega] = ([\omega|_U], [\omega|_V]) \quad \text{and} \quad g^*_r([\alpha],[\beta]) = [\alpha|_{U\cap V} - \beta|_{U\cap V}]\,.$$

As to the connecting homomorphism Δ^*_r (Lemma 1.16), for $\omega \in Z^r(U\cap V)$, since g_r is onto, we have $\omega = g_r(\alpha,\beta)$ for some $\alpha \in \Omega^r(U)$ and $\beta \in \Omega^r(V)$. Thus $0 = d_r\omega = (d\alpha)|_{U\cap V} - (d\beta)|_{U\cap V}$. Therefore, $\Delta^*_r[\omega] = [\omega']$, where $\omega' \in Z^{r+1}(M)$ is such that $\omega'|_U = d_r\alpha$ and $\omega'|_V = d_r\beta$.

The long exact sequence above is called the Mayer-Vietoris sequence associated to the decomposition $M = U \cup V$. It allows one to obtain

Proposition 3.1. *If M is a surface of finite type, then $H_{dR}(M)$ is finitely generated. In particular, the same holds for every compact surface M.*

Proof. Let M be a surface of finite type. We argue by induction on n, the cardinality of the finite simple cover, in order to show that each $H^r(M)$ is finite dimensional. The case $n = 1$ follows from Example 3.2 and Poincaré's Lemma (Theorem 3.3). Now suppose the result holds for some $n \in \mathbf{N}$ and let $M = \bigcup_{k=1}^{n+1} U_k$ be a finite simple cover. Set $V = \bigcup_{k=1}^{n} U_k$ so that $M = V \cup U_{n+1}$. The surface $V \cap U_{n+1}$ is of finite type, since $V\cap U_{n+1} = \bigcup_{k=1}^{n}(U_k\cap U_{n+1})$ is a finite simple cover. Thus, by the induction hypothesis, $H_{dR}(V\cap U_{n+1})$ is finitely generated. The Mayer-Vietoris sequence of the decomposition contains an exact three-term sequence

$$H^{r-1}(V\cap U_{n+1}) \xrightarrow{\Delta^*_{r-1}} H^r(M) \xrightarrow{f^*_r} H^r(V)\oplus H^r(U_{n+1}) \quad .$$

The second morphism yields a short exact sequence

$$0 \longrightarrow \ker f^*_r \hookrightarrow H^r(M) \twoheadrightarrow \operatorname{im} f^*_r \longrightarrow 0\,,$$

which splits (Example 1.15). Thus $H^r(M) \approx \ker f_r^* \oplus \operatorname{im} f_r^*$. Since $\ker f_r^* = \operatorname{im} \Delta_{r-1}^*$ and $\dim \operatorname{im} \Delta_{r-1}^* < \infty$, we conclude that $\dim H^r(M)$ is finite, which proves the induction step.

Q.E.D.

In case $H_{dR}(M)$ is finitely generated, we define the *Betti numbers* of M to be the Betti numbers of $H_{dR}(M)$ and the *Euler-Poincaré characteristic*[1] $\chi(M)$ of M to be $\chi(H_{dR}(M))$.

The last result tells us that it is possible to compute $\chi(M)$ whenever M is of finite type, since

$$\chi(M) = \sum_{r=0}^{m}(-1)^r b_r = \sum_{r=0}^{m}(-1)^r \dim H^r(M)\,. \tag{3.1}$$

However, if M is a compact surface, Poincaré duality (Theorem 3.17) yields a relation between the Betti numbers of M, allowing one to compute $\chi(M)$ for certain classes of surfaces, as we shall see in § 3.4.

Example 3.9. Let M be a surface such that $H_{dR}(M)$ is finitely generated. If N is another surface with the same homotopy type as M, then $\chi(M) = \chi(N)$. This follows immediately from Theorem 3.5. Thus, (3.1) and Proposition 3.1 tells us the Euler characteristic is a topological invariant for compact surfaces.

Example 3.10. Let $m \geq 2$. Using the Mayer-Vietoris sequence, Theorem 3.5 and a decomposition $S^m = U \cup V$, where U and V are contractible open such that $U \cap V \approx S^{m-1}$ in the homotopy category[2], one can generalize Example 3.8 by proving that $H^m(S^m) \approx \mathbf{R}$ for every $m > 0$ and $H^r(S^m) \approx \{0\}$ for $0 < r < m$. (See [**7**, p. 32].)

Example 3.11. Upon the identification $\mathbf{C}^{n+1} \approx \mathbf{R}^{2n+2}$, we can write points $z \in S^{2n+1}$ as $z = (z_1, \ldots, z_{n+1})$, where $z_i \in \mathbf{C}$ and $\sum |z_i|^2 = 1$. Consider the map $f : S^{2n+1} \to \mathbf{R}^{2(n+1)^2}$ given by $f(z) = [z_i \bar{z}_j]$ ($(n+1) \times (n+1)$ matrix with complex entries). The n-dimensional complex projective space CP^n is defined to be the image $f(S^{2n+1})$. Proceding along the lines in [**11**, p. 260], one shows that CP^n is a compact, connected, smooth surface of dimension $2n$ in $\mathbf{R}^{2(n+1)^2}$. For $0 \leq r \leq 2n$, one shows that

$$H^r(CP^n) = \begin{cases} \mathbf{R} & ; r \equiv 0 \mod 2 \\ 0 & ; r \equiv 1 \mod 2 \end{cases}.$$

In order to compute these groups, one needs to extend de Rham cohomology to compact sets in Euclidean spaces, which will not be done here. The details can be seen in [**7**, p.73].

[1] Actually, the Euler-Poincaré characteristic is defined in the more general context of topological spaces by means of polyhedra. For further details see [**5**].

[2] For instance $U = S^m - \{\text{north pole}\}$ and $U = S^m - \{\text{south pole}\}$. Then $U \cap V = S^m - \{p, q\}$ and S^{m-1} have the same homotopy type.(See Example 2.24.)

3.2. Compactly Supported de Rham Cohomology

From the de Rham complex $\Omega^*(M)$, we obtain a subcomplex $\Omega_c^*(M)$ by restricting each d_r to $\Omega_c^r(M)$ (§ 2.2), namely,

$$\cdots \longrightarrow 0 \longrightarrow \Omega_c^0(M) \xrightarrow{d_0} \Omega_c^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \Omega_c^m(M) \xrightarrow{d_m} 0 \longrightarrow \cdots$$

The cohomology group of such cochain complex is called *compactly supported de Rham cohomology group*, denoted by

$$H_{dR,c}(M) = (H^r_{dR,c}(M) = Z_c^r(M)/B_c^r(M))_{r\in\mathbf{Z}}\,.$$

To avoid such a heavy notation, we write $H_c(M)$ (resp. $H_c^r(M)$) instead of $H_{dR,c}(M)$ (resp. $H^r_{dR,c}(M)$).

For a compact surface M, the de Rham groups $H_c(M)$ and $H_{dR}(M)$ coincide, since closed sets are compact in compact spaces. In case M is not compact, such groups can be pretty different from each other. For instance, if, in addition, M is connected we see[3] from Example 3.2 that $H^0_{dR}(M) \approx \mathbf{R}$, but $H_c^0(M) = \{0\}$, since $Z_c^0(M) = \{0\}$ (a constant function on M having compact support must be identically zero).

The next result concerns compactly supported cohomology of connected oriented surfaces. The proof is rather lenghty, thus we shall not include it here. (See [**7**, p. 42].)

Proposition 3.2. *Let M be an m-dimensional connected oriented surface. The integration of forms $[\omega] \mapsto \int_M \omega$ defines a linear isomorphism between $H_c^m(M)$ and $\mathbf{R}$.*

From properties of proper maps dicussed in § 2.2, it follows that the algebraic contravariant functor T from last section (discussion preceeding Theorem 3.5) restrics to a functor from the category of smooth surfaces and proper smooth maps which assings to a surface its compactly supported de Rham cohomology group and to a proper smooth map $f : M \to N$ its induced morphism $T(f) = f^{**} : H_{dR,c}(N) \to H_{dR,c}(M)$. In this case, we also write f^* instead of f^{**}.

From the above, it follows that if M and N are (properly) diffeomorphic surfaces, then $H_{dR,c}(N)$ and $H_{dR,c}(M)$ are isomorphic. However, compactly supported cohomology is not homotopy invariant, as shown by the next result.

Proposition 3.3. *If $m \geq 1$, then*

$$H_c^r(\mathbf{R}^m) = \begin{cases} \{0\} & ; 0 \leq r < m \\ \mathbf{R} & ; r = m \end{cases}.$$

Proof. As we have seen above, $H_c^0(M) = \{0\}$ for non-compact connected surfaces. The case $r = m$ comes from Proposition 3.2. We will prove the case $0 < r < m$. Since the stereographic projection $\varphi : S^m - \{p\} \to \mathbf{R}^m$ ($p \in S^m$ is the north pole) is a proper

[3] Compactness does not play a role here.

diffeomorphism, it suffices to prove that $H_c^r(S^m - \{p\}) = \{0\}$. Thus, let $\omega \in Z_c^r(S^m - \{p\})$. Since $\omega_{S^m} \in \Omega_c^r(S^m)$ and $H^r(S^m) = \{0\}$ (Example 3.10), we have

$$\omega_{S^m} = d\alpha$$

for some $\alpha \in \Omega^{r-1}(S^m)$.

We must find $\gamma \in \Omega_c^{r-1}(S^m - \{p\})$ so that $\omega = d\gamma$. First, Proposition 2.5 applied to $A = S^m - \operatorname{supp}\omega$ tells us there exists an open set $p \in V \subseteq S^m$ such that V is diffeomorphic to $\mathbf{R}^m$ and $\omega_{S^m}|_V = 0$. We have two cases:

1. If $r = 1$, then $\alpha \in \Omega^0(S^m)$ is just a C^∞ function on S^m. Since $(d\alpha)|_V = \omega_{S^m}|_V = 0$ and V is connected, it follows that α is constant on U, say $\alpha = c$. Thus, defining $\beta = \alpha - c \in \Omega^0(S^m)$, we see that $\operatorname{supp}\beta \subseteq (S^m - \{p\}) - V \subseteq S^m - V$. Therefore $\beta|_{S^m-\{p\}} \in \Omega_c^0(S^m - \{p\})$ and $d\beta|_{S^m-\{p\}} = (d\alpha)|_{S^m-\{p\}} = \omega$.
2. Since $\omega_{S^m} = d\alpha$, we have $d(\alpha|_V) = \omega_{S^m}|_V = 0$, whence $\alpha|_V \in Z^{r-1}(V)$. From Theorem 3.3 and the fact that V and $\mathbf{R}^m$ are diffeomorphic, it follows that $H^{r-1}(V) \approx H^{r-1}(\mathbf{R}^m) \approx \{0\}$. Thus $\alpha|_V$ is exact, say $\alpha|_V = d\tau$ for some $\tau \in \Omega^{r-2}(V)$. Applying Proposition 2.2 to V and S^m, we see there exists a function $\xi : S^m \to [0, 1]$ such that $\operatorname{supp}\xi \subseteq V$ is compact and $\xi = 1$ on some open set $p \in U \subseteq V$. Thus $\xi\tau \in \Omega_c^{r-2}(V)$. Defining $\sigma = (\xi\tau)_{S^m} \in \Omega_c^{r-2}(S^m)$, we have $\beta = \alpha - d\sigma \in \Omega^{r-1}(S^m)$. Since $U \subseteq V$, $\beta|_U = \alpha|_U - d\sigma|_U = \alpha|_U - d\tau|_U = 0$, whence $\operatorname{supp}\beta \subseteq S^m - U$, that is, $\beta \in \Omega_c^{r-1}(S^m)$. Lastly, $d\beta|_{S^m-\{p\}} = d\alpha|_{S^m-\{p\}} = \omega$.

Q.E.D.

The following result about cohomology will be needed in § 3.3.

Proposition 3.4. *Let M be a surface. If $M = \bigcup_{\lambda\in L} U_\lambda$ is a disjoint union of non-empty open sets $U_\lambda \subseteq M$, then the following hold:*

1. $H_{dR}(M) \approx \prod_{\lambda\in L} H_{dR}(U_\lambda)$.
2. $H_{dR,c}(M) \approx \bigoplus_{\lambda\in L} H_{dR,c}(U_\lambda)$.
3. $\operatorname{Hom}(H_{dR,c}(M), \mathbf{R}) \approx \prod_{\lambda\in L} \operatorname{Hom}(H_{dR,c}(U_\lambda), \mathbf{R})$.

Proof. To prove the first two items, it is enough to see that $\Omega^*(M) \approx \prod \Omega^*(U_\lambda)$ and $\Omega_c^*(M) \approx \bigoplus \Omega_c^*(U_\lambda)$. We define cochain maps $f : \Omega^*(M) \to \prod \Omega^*(U_\lambda)$ and $g : \Omega_c^*(M) \to \bigoplus \Omega_c^*(U_\lambda)$, where

$$\begin{array}{ccc} \Omega^r(M) & \xrightarrow{f_r} & \prod_{\lambda\in L} \Omega^r(U_\lambda) \\ \omega & \longmapsto & (i_r^*\omega)_{\lambda\in L} \end{array} \qquad \begin{array}{ccc} \Omega_c^r(M) & \xrightarrow{g_r} & \bigoplus_{\lambda\in L} \Omega_c^r(U_\lambda) \\ \omega & \longmapsto & (i_r^*\omega)_{\lambda\in L} \end{array}$$

The respective inverse maps are $p : \prod \Omega^*(U_\lambda) \to \Omega^*(M)$ and $q : \bigoplus \Omega^*_c(U_\lambda) \to \Omega^*_c(M)$, given by

$$\begin{array}{ccccccc} \prod_{\lambda\in L} \Omega^r(U_\lambda) & \xrightarrow{p_r} & \Omega^r(M) & \qquad & \bigoplus_{\lambda\in L} \Omega^r_c(U_\lambda) & \xrightarrow{q_r} & \Omega^r_c(M) \\ (\omega_\lambda)_{\lambda\in L} & \longmapsto & \sum \omega_\lambda & & (\omega_\lambda)_{\lambda\in L} & \longmapsto & \sum (\omega_\lambda)_M \end{array}$$

Note that p_r is well-defined, since $M = \bigcup_{\lambda\in L} U_\lambda$ is a disjoint union, which makes $(\operatorname{supp}\omega_\lambda)_{\lambda\in L}$ into a locally finite family (see Remark 2.38). As to the third item, it follows from item 2. and Proposition 1.4. Indeed,

$$\begin{aligned} \big(\operatorname{Hom}(H_{dR,c}(M), \mathbf{R})\big)^r &= \operatorname{Hom}(H^r_c(M), \mathbf{R}) \\ &\approx \operatorname{Hom}\big(\bigoplus H^r_c(U_\lambda), \mathbf{R}\big) \\ &\approx \prod \operatorname{Hom}(H^r_c(U_\lambda), \mathbf{R}) \\ &= \big(\prod \operatorname{Hom}(H_{dR,c}(U_\lambda), \mathbf{R})\big)^r \end{aligned}$$

Q.E.D.

As in last section, a decomposition $M = U \cup V$ yields , in a similar fashion, a short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^*_c(U \cap V) \xrightarrow{f} \Omega^*_c(U) \oplus \Omega^*_c(V) \xrightarrow{g} \Omega^*_c(M) \longrightarrow 0\,,$$

but in this case we use zero extensions of forms intead of restrictions, that is,

$$f_r(\omega) = (\omega_U, \omega_V) \quad \text{and} \quad g_r(\alpha, \beta) = \alpha_M - \beta_M\,.$$

From the definition of zero extension of forms and some easy verifications, it follows that the short sequence above is exact. To see that each g_r is surjective, one uses similar arguments to those used in the last section. Indeed, given $\omega \in \Omega^r_c(M)$, we take a partition of unity $\xi_1 + \xi_2 = 1$ strictly subordidated to $M = U \cup V$ so that $\operatorname{supp}\xi_1 \subseteq U$ and $\operatorname{supp}\xi_2 \subseteq V$. Setting $\alpha = (\xi_1\omega)|_U$ and $\beta = -(\xi_2\omega)|_V$, it follows that $\alpha \in \Omega^r_c(U)$ and $\beta \in \Omega^r_c(V)$. Thefore, $g_r(\alpha, \beta) = (\xi_1\omega)|_U + (\xi_2\omega)|_V = \omega$ (to see this consider points in $U \cap V$, $U - V$ and $V - U$). This shows that g_r is onto.

Again, from Theorem 1.17, we obtain the Mayer-Vietoris sequence with compact supports:

$$\cdots \xrightarrow{\partial^*_{r-1}} H^r_c(U \cap V) \xrightarrow{f^*_r} H^r_c(U) \oplus H^r_c(V) \xrightarrow{g^*_r} H^r_c(M) \xrightarrow{\partial^*_r} H^{r+1}_c(U \cap V) \xrightarrow{f^*_{r+1}} \cdots$$

where

$$f^*_r([\omega]) = ([\omega_U], [\omega_V]) \quad \text{and} \quad g^*_r([\alpha], [\beta]) = [\alpha_M - \beta_M]\,.$$

In order to see how ∂^*_r works, let us consider $[\omega] \in H^r_c(M)$. From the expression for the connecting homomorphism given in Lemma 1.16, it follows that

$$\partial^*_r[\omega] = [(d\alpha)|_{U\cap V}] = [(d\beta)|_{U\cap V}]$$

for some $(\alpha, \beta) \in \Omega^r_c(U) \oplus \Omega^r_c(V)$ such that $\omega = \alpha_M - \beta_M$.

Now, we apply the contravariant $\mathrm{Hom}(\cdot, \mathbf{R})$ (and Proposition 1.4) to the Mayer-Vietoris sequence above and obtain a long sequence[4]

$$\cdots \longrightarrow H_c^{r+1}(U\cap V)^* \stackrel{\partial_{r+1}^{\#}}{\longrightarrow} H_c^r(M)^* \stackrel{g_r^{\#}}{\longrightarrow} H_c^r(U)^* \oplus H_c^r(V)^* \stackrel{f_r^{\#}}{\longrightarrow} H_c^r(U\cap V)^* \longrightarrow \cdots$$

which is exact by Proposition 1.3.

The dual maps above act as follows. For $(\varphi, \psi) \in H_c^r(U)^* \oplus H_c^r(V)^*$ and $[\omega] \in H_c^r(U\cap V)$,

$$f_r^{\#}(\varphi,\psi)\cdot[\omega] = \varphi[\omega_U] - \psi[\omega_V]\,.$$

Given $\varphi \in H_c^r(M)^*$, $g_r^{\#}\varphi = (\xi, \zeta)$, where $(\xi,\zeta) \in H_c^r(U)^* \oplus H_c^r(V)^*$, is such that

$$\xi[\alpha] = \varphi[\alpha_M] \quad \text{and} \quad \zeta[\beta] = \varphi[\beta_M]\,,$$

for $\alpha \in H_c^r(U)$ and $\beta \in H_c^r(V)$.

Lastly, for every $\varphi \in H_c^{r+1}(U\cap V)^*$ and $[\omega] \in H_c^r(M)$,

$$(\partial_{r+1}^{\#}\varphi)\cdot[\omega] = \varphi(\partial^*[\omega]) = \varphi[(d\alpha)|_{U\cap V}] = \varphi[(d\beta)|_{U\cap V}]\,, \tag{3.2}$$

for some $(\alpha,\beta) \in \Omega_c^r(U) \oplus \Omega_c^r(V)$ such that $\omega = \alpha_M - \beta_M$.

3.3. Poincaré Duality

This section is entirely devoted to the proof of the main result, namely, the Poincaré Duality theorem (Theorem 3.17), which states that there is an isomorphism (to be defined in a moment)

$$H_{dR}^r(M) \approx H_{dR,c}^{m-r}(M)^*\,, \tag{3.3}$$

whenever M is an oriented m-dimensional (smooth) surface and $0 \le r \le m$. Clearly, (3.3) holds for $r < 0$ and $r > m$. So we shall focus on the case $0 \le r \le m$.

The proof of (3.3) is rather lengthy so we break it into five lemmas. Also, it is mostly of algebraic nature, since its essence is a relation (Lemma 3.12) between the Mayer-Vietoris sequences of a decomposition $M = U \cup V$ derived in § 3.1 and § 3.2.

Let M be an oriented m-dimensional (smooth) surface and $0 \le r \le m$. The exterior product and integration of forms yield an $\mathbf{R}$-bilinear map

$$P_M = P_M^r : H^r(M) \times H_c^{m-r}(M) \to \mathbf{R} \tag{3.4}$$

given by

$$([\alpha],[\beta]) \mapsto \int_M \alpha\wedge\beta\,,$$

since $\alpha\wedge\beta$ is an m-form on M and $\operatorname{supp}\alpha\wedge\beta \subseteq \operatorname{supp}\alpha \cap \operatorname{supp}\beta$. This bilinear form is well-defined. Indeed, if $([\alpha_1],[\beta_1]), ([\alpha_2],[\beta_2]) \in H^r(M) \times H_c^{m-r}(M)$ are such that

[4] We will keep using the shorter notations $E^* = \mathrm{Hom}(E, R)$ and $f^{\#} = \mathrm{Hom}(f, R)$.

$\alpha_1 = \alpha_2 + d\bar{\alpha}$ and $\beta_1 = \beta_2 + d\bar{\beta}$, then

$$\begin{aligned}\int_M \alpha_1 \wedge \beta_1 &= \int_M (\alpha_2 + d\bar{\alpha}) \wedge (\beta_2 + d\bar{\beta}) \\ &= \int_M \alpha_2 \wedge \beta_2 + \int_M d\bar{\alpha} \wedge \beta_2 + \int_M \alpha_2 \wedge d\bar{\beta} + \int_M d\bar{\alpha} \wedge d\bar{\beta} \\ &= \int_M \alpha_2 \wedge \beta_2 \,,\end{aligned}$$

by Corollary 2.40.1.

The $\mathbf{R}$-bilinear form P_M^r in (3.4) corresponds to a linear map

$$D_M : H^r(M) \to H_c^{m-r}(M)^* \quad (0 \leq r \leq m)\,,$$

called Poincaré duality map, defined by

$$D_M[\alpha] = \int_M \alpha \wedge \cdot \, : H_c^{m-r}(M) \to \mathbf{R}\,,$$

for every $[\alpha] \in H^r(M)$. Thus, Poincaré's duality is stated as

If M is an oriented surface, then D_M is an isomorphism.

This is what we shall prove. Henceforth, whenever we say that Poincaré duality holds for some surface M, we mean that D_M is an isomorphism.

We now proceed to prove our first lemma.

Lemma 3.12. *Let M be an oriented m-dimensional surface. If $M = U \cup V$, where $U, V \subseteq M$ are open, then the diagram below commutes.*

$$\begin{array}{ccc}
H^r(U) \oplus H^r(V) & \xrightarrow{D_U \oplus -D_V} & H_c^{m-r}(U)^* \oplus H_c^{m-r}(V)^* \\
{\scriptstyle g_r^*}\big\downarrow & & \big\downarrow{\scriptstyle f^\#} \\
H^r(U \cap V) & \xrightarrow{D_{U\cap V}} & H_c^{m-r}(U \cap V)^* \\
{\scriptstyle (-1)^{r+1}\Delta_r^*}\big\downarrow & & \big\downarrow{\scriptstyle \partial^\#} \\
H^{r+1}(M) & \xrightarrow{D_M} & H_c^{m-r-1}(M)^* \\
{\scriptstyle f_{r+1}^*}\big\downarrow & & \big\downarrow{\scriptstyle g^\#} \\
H^{r+1}(U) \oplus H^{r+1}(V) & \xrightarrow{D_U \oplus -D_V} & H_c^{m-r-1}(U)^* \oplus H_c^{m-r-1}(V)^* \\
{\scriptstyle g_{r+1}^*}\big\downarrow & & \big\downarrow{\scriptstyle f^\#} \\
H^{r+1}(U \cap V) & \xrightarrow{D_{U\cap V}} & H_c^{m-r-1}(U \cap V)^*
\end{array}$$

Proof. Recalling the definitions in § 3.1 and § 3.2, the first and the third rectangles are easily seen to commute. The one that poses a problem is the second one. In order to prove its commutativity, let $[\omega] \in H^r(U \cap V)$ be such that $\omega = \alpha|_{U\cap V} - \beta|_{U\cap V}$, where $\alpha \in \Omega^r(U)$ and $\beta \in \Omega^r(V)$. As we have already seen in § 3.1,

$$(-1)^{r+1}\Delta_r^*[\omega] = [(-1)^{r+1}\sigma] \in H^{r+1}(M)\,,$$

where $\sigma|_U = d\alpha$, $\sigma|_V = d\beta$ and $\sigma|_{U\cap V} = d\alpha|_{U\cap V} = d\beta|_{U\cap V}$. Thus, for $[\tau] \in H_c^{m-r-1}(M)$, we see that

$$D_M(-1)^{r+1}\Delta_r^*[\omega]\cdot[\tau] = (-1)^{r+1}\int_M \sigma\wedge\tau\,. \tag{3.5}$$

On the other hand, from (3.2), it follows that $\partial^\# D_{U\cap V}[\omega]\cdot[\tau] = D_{U\cap V}[\omega](\partial^*[\tau])$, that is,

$$\partial^\# D_{U\cap V}[\omega]\cdot[\tau] = \int_{U\cap V}\omega\wedge(d\bar{\alpha})|_{U\cap V}\,, \tag{3.6}$$

where $\tau = \bar{\alpha}_M - \bar{\beta}_M$, for some $\bar{\alpha} \in \Omega_c^{m-r-1}(U)$ and $\bar{\beta} \in \Omega_c^{m-r-1}(V)$ such that

$$d\bar{\alpha}|_{U\cap V} = d\bar{\beta}|_{U\cap V}\,.$$

Also, note that $\operatorname{supp} d\bar{\alpha}, \operatorname{supp} d\bar{\beta} \subseteq U\cap V$, since $0 = d\tau = d\bar{\alpha}$ on $U - V$ and $0 = d\tau = d\bar{\beta}$ on $V - U$; this fact will come in handy. Now, we have to show that the integrals in (3.5) and (3.6) are the same. Indeed, from Theorem 2.39(4), we have

$$\begin{aligned}(-1)^{r+1}\int_M \sigma\wedge\tau &= (-1)^{r+1}\int_M \sigma\wedge\bar{\alpha}_M + (-1)^{r+2}\int_M \sigma\wedge\bar{\beta}_M\\ &= (-1)^{r+1}\int_U \sigma\wedge\bar{\alpha} + (-1)^{r+2}\int_V \sigma\wedge\bar{\beta}\\ &= (-1)^{r+1}\int_U d\alpha\wedge\bar{\alpha} + (-1)^{r+2}\int_V d\beta\wedge\bar{\beta}\end{aligned}$$

From Theorem 2.36(3) and Corollary 2.40.1, it follows that

$$(-1)^{r+1}\int_U d\alpha\wedge\bar{\alpha} = \int_U \alpha\wedge d\bar{\alpha} \quad\text{and}\quad (-1)^{r+2}\int_V d\beta\wedge\bar{\beta} = -\int_V \beta\wedge d\bar{\beta}\,.$$

Thus

$$(-1)^{r+1}\int_M \sigma\wedge\tau = \int_U \alpha\wedge d\bar{\alpha} - \int_V \beta\wedge d\bar{\beta}\,,$$

and since $\operatorname{supp} d\bar{\alpha}, \operatorname{supp} d\bar{\beta} \subseteq U\cap V$, Theorem 2.39(4) gives us

$$\begin{aligned}(-1)^{r+1}\int_M \sigma\wedge\tau &= \int_{U\cap V}(\alpha\wedge d\bar{\alpha})|_{U\cap V} - \int_{U\cap V}(\beta\wedge d\bar{\beta})|_{U\cap V}\\ &= \int_{U\cap V}\alpha|_{U\cap V}\wedge d\bar{\alpha}|_{U\cap V} - \int_{U\cap V}\beta|_{U\cap V}\wedge d\bar{\beta}|_{U\cap V}\\ &= \int_{U\cap V}\alpha|_{U\cap V}\wedge d\bar{\alpha}|_{U\cap V} - \int_{U\cap V}\beta|_{U\cap V}\wedge d\bar{\alpha}|_{U\cap V}\\ &= \int_{U\cap V}(\alpha|_{U\cap V} - \beta|_{U\cap V})\wedge d\bar{\alpha}|_{U\cap V}\\ &= \int_{U\cap V}\omega\wedge d\bar{\alpha}|_{U\cap V}\end{aligned}$$

Q.E.D.

Lemma 3.13. *Let M be an m-dimensional surface. If $M = \bigcup_{\lambda \in L} U_\lambda$ is a disjoint union of non-empty open sets $U_\lambda \subseteq M$ and Poincaré duality holds for every U_λ, then it also holds for M, that is, D_M is an isomorphism.*

Proof. If each $D_{U_\lambda} : H^r(U_\lambda) \to H_c^{m-r}(U_\lambda)^*$ is an isomorphism, then the same holds for $D^\times : \prod H^r(U_\lambda) \to \prod H_c^{m-r}(U_\lambda)^*$, where

$$D^\times([\omega_\lambda])_{\lambda \in L} = (D_{U_\lambda}[\omega_\lambda])_{\lambda \in L}\,.$$

From Proposition 3.4(1) and (3) we have a commutative diagram

$$\begin{tikzcd}
H^r(M) \arrow[rr, "f_r"] \arrow[d, "D_M"'] & & \prod_{\lambda \in L} H^r(U_\lambda) \arrow[d, "D^\times"] \\
H_c^{m-r}(M)^* \arrow[r, "q_{m-r}^{\#}"] & \left(\bigoplus_{\lambda \in L} H_c^{m-r}(U_\lambda)\right)^* \arrow[r, "\xi"] & \prod_{\lambda \in L} H_c^{m-r}(U_\lambda)^*
\end{tikzcd}$$

where f_r and $q_{m-r}^{\#}$ are isomorphisms coming from Proposition 3.4 and ξ (also an isomorphism) comes from Proposition 1.4. For $[\omega] \in H^r(M)$ we have

$$D^\times f_r[\omega] = D^\times([i_\lambda^*\omega])_{\lambda \in L} = (D_{U_\lambda}[i_\lambda^*\omega])_{\lambda \in L} \in \prod_{\lambda \in L} H_c^{m-r}(U_\lambda)^*\,,$$

where $i_\lambda : U_\lambda \hookrightarrow M$ is the inclusion map, and for each $\lambda \in L$ and $[\alpha] \in H_c^{m-r}(U_\lambda)$,

$$D_{U_\lambda}[i_\lambda^*\omega] \cdot [\alpha] = \int_{U_\lambda} i_\lambda^*\omega \wedge \alpha \tag{3.7}$$

Denoting by j_λ the natural inclusion $H_c^r(U_\lambda) \hookrightarrow \bigoplus H_c^r(U_\mu)$, we have

$$\xi(q_{m-r}^{\#}(D_M[\omega])) = (\, q_{m-r}^{\#}(D_M[\omega]) \circ j_\lambda \,)_{\lambda \in L} \in \prod_{\lambda \in L} H_c^{m-r}(U_\lambda)^*$$

Since $j_\lambda[\alpha]$ is a family of classes which are null except at the position λ, it follows that, for each $\lambda \in L$ and $[\alpha] \in H_c^{m-r}(U_\lambda)$,

$$\begin{aligned}
(q_{m-r}^{\#}(D_M[\omega]) \circ j_\lambda)[\alpha] &= (q_{m-r}^{\#}(D_M[\omega]) \cdot j_\lambda[\alpha] \\
&= D_M[\omega] \cdot q_{m-r}(j_\lambda[\alpha]) \\
&= D_M[\omega] \cdot [\alpha_M] \\
&= \int_M \omega \wedge \alpha_M\,.
\end{aligned}$$

Note that $\operatorname{supp} \omega \wedge \alpha_M \subseteq U_\lambda$ and $i_\lambda^*(\alpha_M) = \alpha$. Thus, from 3.7 and Proposition 2.39(4) we see that

$$(q_{m-r}^{\#}(D_M[\omega]) \circ j_\lambda)[\alpha] = \int_M \omega \wedge \alpha_M = \int_{U_\lambda} i_\lambda^*(\omega \wedge \alpha_M) = \int_{U_\lambda} i_\lambda^*\omega \wedge \alpha = D_{U_\lambda}[i_\lambda^*\omega] \cdot [\alpha]\,.$$

Q.E.D.

Lemma 3.14. *Let M be a surface and $\mathcal{B}$ a basis for the topology of M which is closed under finite intersections. If Poincaré duality holds for each $U \in \mathcal{B}$, then it also holds for M.*

Proof. By hypothesis we see that D_U, D_V and $D_{U\cap V}$ are isomorphisms. Thus, from Lemma 3.12 and the Five Lemma (Theorem 1.18), it follows that $D_{U\cap V}$ is also an isomorphism. Arguing by induction, one easily verifies that the same holds for arbitrary finite intersections of open sets belonging to $\mathcal{B}$. Now, from Proposition 2.1, we take a decomposition $M = \bigcup_{i\in\mathbf{N}} A_i$, where each A_i is a finite union of open sets belonging to $\mathcal{B}$ such that $A_i \cap A_j = \emptyset$ for every $j \geq i+2$. In particular, D_{A_i} is an isomorphism for every i. Setting

$$U = \bigcup A_{2i} \quad \text{and} \quad V = \bigcup A_{2i-1}\,,$$

we get

$$M = U \cup V \quad \text{and} \quad U \cap V = \bigcup A_i \cap A_{i+1}\,.$$

Note that each $A_i \cap A_{i+1}$ is a finite union of sets belonging to $\mathcal{B}$, which implies that each $D_{A_i\cap A_{i+1}}$ is an isomorphism. Since $A_i \cap A_j = \emptyset$ for every $j \geq i+2$, it follows that U, V and $U \cap V$ are disjoint unions. Then, by Lemma 3.13, the maps D_U, D_V and $D_{U\cap V}$ are isomorphisms. Finally, it follows from Lemma 3.12 and the Five Lemma (Theorem 1.18) that $D_M = D_{U\cup V}$.

Q.E.D.

Lemma 3.15. *Let $0 \leq r \leq m$ and $f : M \to N$ be a proper diffeomorphism between surfaces which is also orientation-preserving. Then the following diagram commutes.*

$$\begin{array}{ccc} H^r(M) & \xleftarrow{\;f^*\;} & H^r(N) \\ {\scriptstyle D_M}\big\downarrow & & \big\downarrow{\scriptstyle D_N} \\ H_c^{m-r}(M)^* & \xrightarrow{\;f^\#\;} & H_c^{m-r}(N)^* \end{array}$$

In particular, if D_N is an isomorphism, then the same holds for D_M.

Proof. Let $[\alpha] \in H^r(N)$. Recall that $f^*[\alpha] = [f^*\alpha]$ and $f^\#\varphi[\omega] = \varphi[f^*\omega]$ for $\varphi \in H_c^{m-r}(M)^*$ and $[\omega] \in H_c^{m-r}(N)$. Thus, for every $[\beta] \in H_c^{m-r}(N)$, we have

$$f^\# D_M[f^*\alpha]\cdot[\beta] = D_M[f^*\alpha]\cdot[f^*\beta] = \int_M f^*\alpha \wedge f^*\beta = \int_M f^*(\alpha\wedge\beta)\,.$$

It follows from Theorem 2.39(3) that

$$f^\# D_M[f^*\alpha]\cdot[\beta] = \int_M f^*(\alpha\wedge\beta) = \int_N \alpha\wedge\beta = D_N[\alpha]\cdot[\beta]\,.$$

Q.E.D.

Lemma 3.16. *Poincaré duality holds for $\mathbf{R}^m$.*

Proof. From Example 3.2, Theorem 3.3 and Proposition 3.3, we have

$$H^r(\mathbf{R}^m) = \begin{cases} \{0\} & ; 0 < r \leq m \\ \mathbf{R} & ; r = 0 \end{cases} \quad \text{and} \quad H_c^{m-r}(\mathbf{R}^m) = \begin{cases} \{0\} & ; 0 < r \leq m \\ \mathbf{R} & ; r = 0 \end{cases}.$$

Thus, we only need to check that $D_{\mathbf{R}^m} : H^0(\mathbf{R}^m) \to H_c^m(\mathbf{R}^m)^*$ is an isomorphism. To do so, it suffices to see that $D_{\mathbf{R}^m}$ is not identically zero, since both spaces have dimension 1 over $\mathbf{R}$ (Proposition 3.2). Let $\alpha : \mathbf{R}^m \to \mathbf{R}$ be the 0-form $\alpha \equiv 1$ and $\beta = f(x)dx_1 \wedge \cdots \wedge dx_m \in \Omega_c^r(\mathbf{R}^m)$, where $f \in C_c^\infty(\mathbf{R}^m)$ comes from Proposition A.1 with $K = B[0; 2]$. Then

$$D_{\mathbf{R}^m}[\alpha] \cdot [\beta] = \int_{\mathbf{R}^m} \alpha \wedge \beta = \int_{\mathbf{R}^m} \beta = \int_K f \neq 0\,.$$

Q.E.D.

Finally, using the results above, we present the proof of the Poincaré duality theorem.

Theorem 3.17 (Poincaré Duality)**.** *If M is an oriented m-dimensional surface, then*

$$H^r(M) \approx H_c^{m-r}(M)^*$$

and, for $0 \leq r \leq m$, this isomorphism is given by the map

$$D_M : H^r(M) \to H_c^{m-r}(M)^*\,,$$

where

$$D_M[\alpha] \cdot [\beta] = \int_M \alpha \wedge \beta$$

for $[\alpha] \in H^r(M)$ and $[\beta] \in H_c^{m-r}(M)$.

Proof. The work is essentially done. We have two cases:

Case #1: Let $M \subseteq \mathbf{R}^m$ be an open set (m-dimensional surface). Denote by $\mathcal{B}$ a basis for the topology of M consisting of open rectangles $C \subseteq \mathbf{R}^m$ with sides parallel to the coordinate axes. Each $C \in \mathcal{B}$ is (properly) diffeomorphic to $\mathbf{R}^m$ (Example 2.18). Thus Lemmas 3.15 and 3.16 tell us that D_C is an isomorphism for every $C \in \mathcal{B}$. Also, note that $C \cap C' \in \mathcal{B}$ for any $C, C' \in \mathcal{B}$. Therefore, by Lemma 3.14, D_M is an isormorphism.

Case #2: Now, for the general case in which $M \subseteq \mathbf{R}^n$ is an oriented m-dimensional surface, we consider $\mathcal{B}$ to be a basis for the topology of M consisting of open sets $U \subseteq M$ which are diffeomorphic to some open subset of $\mathbf{R}^m$ (Example 2.9). By the previous case, we see that D_U is an isomorphism for each $U \subseteq M$. Also, note that $U \cap V \in \mathcal{B}$ for every $U, V \in \mathcal{B}$. Thus, from Lemma 3.14, D_M is an isomorphism.

Q.E.D.

3.4. Applications

The present section was based on [**2, 3, 13**]. We present some applications of Poincaré duality (Theorem 3.17). We begin with some consequences regarding cohomology groups and Betti numbers. Right after, the first subsection deals with applications involving the Euler-Poicaré characteristic and signature of surfaces; we also present an example of irreversible surface, as prosmised at end of subsection 2.1.2. As to the second, the connection between Poincaré duality and the Hodge decomposition theorem is discussed.

First, observe that if M is an oriented surface whose de Rham cohomology group $H_{dR,c}(M)$ is finitely generated, then basic linear algebra tells us that $(H_c^r(M))^* \approx H_c^r(M)$ for every r. Thus, the isomorphism in Theorem 3.17 yields

$$H^r(M) \approx H_c^{m-r}(M)\,.$$

As we know, if M is compact, then $H_c^r(M) = H^r(M)$. Therefore, by Proposition 3.1,

$$H^r(M) \approx H^{m-r}(M)$$

and we have the following

Corollary 3.17.1. *If M is a compact oriented m-dimensional surface, then*

$$b_r = b_{m-r} \quad 0 \leq r \leq m\,,$$

where b_r is the rth Betti number of M.

Poincaré first stated his duality result in his 1895 paper "*Analysis Situs*". The statement given by Poincaré was pretty much the same as in Corollary 3.17.1, but instead of "m-dimensional surface" he considered an m-dimensional topological manifold[5] together with some additional structure, there was no differential structure involved.

More consequences of Poincaré duality are given below.

Corollary 3.17.2. *Let M be an oriented, connected, m-dimensional surface. If M is not compact, then $H^m(M) = \{0\}$.*

Proof. At the begining of § 3.2 we showed that $H_c^0(M) = \{0\}$ if M is non-compact. From Theorem 3.17, it follows that

$$H^m(M) \approx H_c^0(M) = \{0\}\,.$$

Q.E.D.

Corollary 3.17.3. *Let M be an oriented m-dimensional surface. If $H_{dR,c}(M)$ is finitely generated, then*

$$H_c^{r+n}(M \times \mathbf{R}^n) \approx H_c^r(M) \quad (0 \leq r \leq m)\,.$$

[5] A topological n-manifold is a topological space X such that each point has open neighborhood homeomorphic to some open set in $\mathbf{R}^n$.

Proof. Since $\mathbf{R}^n$ is contractible, we see from Example 3.7 that $H_{dR}(M \times \mathbf{R}^n) \approx H_{dR}(M)$. Since $H_{dR,c}(M)$ is finitely generated, we have $(H_c^r(M))^* \approx H_c^r(M)$ for every r. It follows from Theorem 3.17 that

$$\begin{aligned}(H_c^{r+n}(M \times \mathbf{R}^n))^* &\approx H^{m+n-(r+n)}(M \times \mathbf{R}^n) \\ &= H^{m-r}(M \times \mathbf{R}^n) \\ &\approx H^{m-r}(M) \\ &= H_c^r(M)\end{aligned}$$

Thus $\dim(H_c^{r+n}(M \times \mathbf{R}^n))^* < \infty$, which implies $\dim H_c^{r+n}(M \times \mathbf{R}^n) < \infty$[6]. Therefore

$$H_c^{r+n}(M \times \mathbf{R}^n) \approx (H_c^{r+n}(M \times \mathbf{R}^n))^* \approx H_c^r(M)\,.$$

Q.E.D.

Corollary 3.17.4. *Compact oriented surfaces of positive dimension are not contractible.*

Proof. Let M be compact m-dimensional surface with $m > 0$. Write $M = \bigcup_{\lambda \in L} C_\lambda$, where $(C_\lambda)_{\lambda \in L}$ is the family of connected components of M. Example 2.6 tells us that each C_λ is open in M. Thus, Proposition 3.4 together with Example 3.2 yields

$$H^0(M) \approx \prod_{\lambda \in L} H^0(C_\lambda) = \prod_{\lambda \in L} \mathbf{R}\,.$$

Therefore, $H^0(M) \neq \{0\}$. It follows from Poincaré duality and the compactness of M that

$$H^m(M) \approx H_c^0(M) = H^0(M) \neq \{0\}\,.$$

Finally, Poincaré's lemma (Theorem 3.3) implies that M is not contractible.

Q.E.D.

3.4.1. Euler-Poincaré Characteristic and Signature

As to the Euler characteristic, Poincaré duality allows one to compute it modulo 2 for compact oriented surface of even dimension, as shown by the next result.

Corollary 3.17.5. *If M is a compact oriented $2m$-dimensional surface, then*

$$\chi(M) \equiv b_m \mod 2\,.$$

[6]Given a real vector space E, there is always a canonical linear injection $E \hookrightarrow E^{**}$.

Proof. This is a direct application of Corollary 3.17.1:

$$\begin{aligned}\chi(M) &= \sum_{r=0}^{2m}(-1)^r b_r \\ &= (-1)^m b_m + \sum_{r=0}^{m-1}(-1)^r b_r + \sum_{r=m+1}^{2m}(-1)^r b_r \\ &= (-1)^m b_m + \sum_{r=0}^{m-1}(-1)^r b_r + \sum_{r=m+1}^{2m}(-1)^{2m-r} b_{2m-r} \\ &= (-1)^m b_m + \sum_{r=0}^{m-1}(-1)^r b_r + \sum_{q=1}^{m}(-1)^{m-q} b_{m-q} \\ &= (-1)^m b_m + 2\sum_{r=0}^{m-1}(-1)^r b_r \,.\end{aligned}$$

Thus $\chi(M) \equiv (-1)^m b_m \equiv b_m \mod 2$. Q.E.D.

Before presenting the next corollary, we discuss some properties of the bilinear map P_M^r in 3.4. First, if M is a compact oriented m-dimensional surface, then

$$P_M^r([\alpha],[\beta]) = \int_M \alpha\wedge\beta = (-1)^{r(m-r)}\int_M \beta\wedge\alpha = (-1)^{r(m-r)} P_M^{m-r}([\beta],[\alpha])\,,$$

for $[\alpha] \in H^r(M)$ and $[\beta] \in H^{m-r}(M)$. Thus

$$P_M^r([\alpha],[\beta]) = (-1)^r P_M^{m-r}([\beta],[\alpha]) \quad \text{and} \quad P_M^r([\alpha],[\beta]) = P_M^{m-r}([\beta],[\alpha]) \tag{3.8}$$

for $m \equiv 0 \mod 2$ and $m \equiv 1 \mod 2$, respectively.

A bilinear form $b : E \times E \to \mathbf{R}$ on a finite-dimensional real vector space E is said to be *non-degenerate* if the corresponding linear map $E \to E^*$ is an isomorphism. Thus, for a compact oriented $2n$-dimensional surface M, Theorem 3.17 tells us that

$$P_M^n : H^n(M) \times H^n(M) \to \mathbf{R}$$

is a non-degenerate bilinear form on $H^n(M)$. Since M has even dimension, the first equality in (3.8) with $r = n$ tells us that P_M^n is symmetric if $n \equiv 0 \mod 2$ and anti-symmetric otherwise.

Recall from Linear Algebra that the (symmetric) matrix B of a symmetric bilinear form b on an n-dimensional real vector space E can be diagonalized, that is,

$$B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

relative to some basis of E. In this case we define the *signature* of b to be

$$\operatorname{sig}(b) = \operatorname{card}\{i\,;\, \lambda_i > 0\} - \operatorname{card}\{i\,;\, \lambda_i < 0\}\,.$$

The numbers

$$n^+ = \mathrm{card}\{i\,;\,\lambda_i > 0\} \quad \text{and} \quad n^- = \mathrm{card}\{i\,;\,\lambda_i < 0\}$$

are called, respectively, *positive and negative indices of inertia*. The number $\mathrm{sig}(b)$ is well-defined, since Sylverster's Law of Inertia ([**4**, p. 370]) tells us n^+ and n^- are invariant with respect to the choice of bases such that B is diagonal.

In case b is non-degenerate, we have $\det B \neq 0$, whence $\lambda_i \neq 0$ for all $1 \leq i \leq n$. Thus

$$n = n^+ + n^- \,.$$

Now, if M is a compact oriented $2n$-dimensional surface and $n \equiv 0 \mod 2$, then P^n_M is a non-degenerate symmetric bilinear form on $H^n(M)$. In this case we define the *signature* of M to be

$$\mathrm{sig}(M) = \mathrm{sig}(P^n_M)\,.$$

Note that the signature is not homotopy invariant, since $\mathrm{sig}(-M) = -\,\mathrm{sig}(M)$. This follows from Theorem 2.39(5) applied to P^n_M.

From this discussion we have the following result.

Corollary 3.17.6. *The following items hold for a compact oriented m-dimensional surface M.*

1. *If $m \equiv 1 \mod 2$, then $\chi(M) = 0$.*
2. *If $m = 2n$ and $n \equiv 1 \mod 2$, then $\chi(M) \equiv b_n \equiv 0 \mod 2$.*
3. *If $m = 2n$ and $n \equiv 0 \mod 2$, then $\mathrm{sig}(M) \equiv b_n \equiv \chi(M) \mod 2$.*
4. *Let $m \equiv 0 \mod 4$. If $\mathrm{sig}(M) \neq 0$, then M is irreversible.*
5. *If $m = 2n$, $n \equiv 0 \mod 2$ and $b_n \equiv 1 \mod 2$, then $\mathrm{sig}(M), \chi(M) \neq 0$. In particular, M is irreversible.*

Proof. For the first item, if $m \equiv 1 \mod 2$, then Corollary 3.17.1 yields

$$\chi(M) = \sum_{r=0}^{m}(-1)^r b_r = (-1)^m \sum_{r=0}^{m}(-1)^{m-r} b_{m-r} = -\chi(M)\,,$$

whence $\chi(M) = 0$. For the second item, P^n_M non-degenerate and anti-symmetric for $n \equiv 1 \mod 2$. Thus Corollary 3.17.5 and Proposition A.2 give

$$\chi(M) \equiv b_m \equiv 0 \mod 2\,.$$

As to the third item, if $n \equiv 0 \mod 2$, then P^n_M is a non-degenerate symmetric bilinear form.. Thus

$$b_n - \mathrm{sig}(M) = b_n - \mathrm{sig}(P^n_M) = b_n^+ + b_n^- - (b_n^+ - b_n^-) = 2b_n^-\,.$$

Therefore, $b_n \equiv \mathrm{sig}(M) \mod 2$, and Corollary 3.17.5 gives

$$\chi(M) \equiv b_m \equiv 0 \mod 2\,.$$

To prove the fourth item, we argue by contraposition. Let $m = 2n$, where $n \equiv 0 \mod 2$, and $f : M \to M$ an orientation-reversing diffeomorphism. From item 3. in Proposition 2.39 we see that

$$\int_M f^*\omega = -\int_M \omega$$

for $\omega \in \Omega^m(M)$. Thus, given $[\alpha] \in H^n(M)$ and $[\beta] \in H^n(M)$, it follows that

$$P^n_M(f^*[\alpha], f^*[\beta]) = \int_M f^*\alpha \wedge f^*\beta = \int_M f^*(\alpha \wedge \beta) = -\int_M \alpha \wedge \beta = -P^n_M([\alpha], [\beta]) .$$

From Proposition A.3 we conclude that $\operatorname{sig}(P^n_M) = 0$, whence $\operatorname{sig}(M) = 0$.

Lastly, item 5. follows from the previous two.

Q.E.D.

An interesting application of Theorem 3.17 (which will not be proved here) is a suficient condition for the orientability of connected surfaces, namely: an m-dimensional surface M is orientable whenever $H^m(M) \neq \{0\}$. [7]

This result allows one to conclude that the complex projective space CP^n, introduced in Example 3.11, is orientable, since $H^{2n}(CP^n) \approx \mathbf{R}$. Again from Example 3.11, for $n \equiv 0 \mod 2$, $H^n(CP^n) \approx \mathbf{R}$, that is, $b_n = 1$. Thus, Corollary 3.17.6(5) tells us there is no orientation-reversing diffeomorphism from CP^n onto itself, that is, CP^n is irreversible whenever n is even.

3.4.2. Connection to Hodge Theory

Now we briefly discuss the Hodge Decomposition Theorem and how it implies Theorem 3.17 in the case of compact surfaces.[8] Why just the compact case? Well, as we shall see in a moment, compactness allows one to define an inner product on $\Omega^r(M)$.

We begin by observing that, to each point x on an m -dimensional surface $M \subseteq \mathbf{R}^n$, one assings an inner product $g_x = \langle \cdot, \cdot \rangle_x$ on $T_xM \subseteq \mathbf{R}^n$ (in this case, the usual inner product on $\mathbf{R}^n$) so that, given a parametrization $\varphi : U_0 \to U$ of $U \subseteq M$, the maps $g_{ij} : U_0 \to \mathbf{R}$ defined by

$$g_{ij}(a) = \left\langle \frac{\partial \varphi}{\partial x_i}(a), \frac{\partial \varphi}{\partial x_j}(a) \right\rangle_x$$

are smooth. The map $g : x \mapsto g_x = \langle \cdot, \cdot \rangle_x$ is called Riemannian metric.

Let M be an oriented m-dimensional surface. Using the Riemannian metric on M and the Riesz representation theorem (the one from linear algebra), one extends $\langle \cdot, \cdot \rangle_x$ to $\mathcal{A}_r(T_xM)$ by setting

$$\langle u_1 \wedge \cdots \wedge u_r, v_1 \wedge \cdots \wedge v_r \rangle_x = \det(\langle u_i^*, v_j^* \rangle_x) ,$$

[7] The proof of this fact relies on results regarding covering maps. The details can be seen in [**7**, p. 49].

[8] For proofs and further details regarding the subjects discussed here, see [**2**, **13**].

where $u_i^*, v_i^* \in T_xM$ are the vectors correponding to $u, v \in (T_xM)^*$ via the Riesz representation theorem. This leads to the existence of an unique linear isomorphism

$$\star = \star_r : \Omega^r(M) \to \Omega^{m-r}(M)\,,$$

for each $0 \leq r \leq m$, called the *Hodge star operator*, which assigns to each r-form ω an $(m-r)$-form $\star\omega$ such that

$$\alpha(x) \wedge \omega(x) = \langle \alpha(x), \star\omega(x)\rangle_x \, \nu(x)$$

for every $x \in M$ and $\alpha \in \Omega^{m-r}(M)$, where ν denotes the volume form of M (Example 2.27).

Now, let us fix M a compact, oriented m-dimensional surface. Then Hogde's operator yields an inner product on $\Omega^r(M)$, namely,

$$(\alpha|\beta) := \int_M \alpha \wedge \star\, \beta\,. \tag{3.9}$$

Thus, it makes sense to talk about an adjoint $\partial_{r+1} : \Omega^{r+1}(M) \to \Omega^r(M)$, called *codifferential*, for the exterior derivative $d_r : \Omega^r(M) \to \Omega^{r+1}(M)$ with respect to $(\cdot|\cdot)$. Using the fact that $\star\star = (-1)^{m(m-r)}$, one shows that

$$\partial = (-1)^{n(p+1)+1} \star d \star\,.$$

is in fact the adjoint map of d with respect to the inner product $(\cdot|\cdot)$, that is,

$$(d_r\alpha|\beta) = (\alpha|\partial_{r+1}\beta)$$

for every $\alpha \in \Omega^r(M)$ and $\beta \in \Omega^{r+1}(M)$. Moreover, one verifies that $\partial\partial = 0$. Therefore, there is a chain complex $\Omega_*(M) = (\Omega^r, \partial_r)$, namely,

$$\cdots \longrightarrow 0 \longrightarrow \Omega^m(M) \xrightarrow{\partial_m} \Omega^{m-1}(M) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} \Omega^0(M) \xrightarrow{\partial_0} 0 \longrightarrow \cdots$$

In this chain complex, elements of the subspace $Z_r(M) = \ker \partial_r$ are called *co-closed* forms and elements of $B_r(M) = \operatorname{im} \partial_{r+1}$ are called *co-exact* forms.

The Hodge operator relates closed, co-closed, exact and co-exact forms by

$$\star(Z^r(M)) = Z_{m-r}(M) \quad \text{and} \quad \star\,(B^r(M)) = B_{m-r}(M)\,,$$

from which it follows that

$$Z^r(M) \approx Z_{m-r}(M) \quad \text{and} \quad B^r(M) \approx B_{m-r}(M) \quad (0 \leq r \leq m)\,.$$

Thus, Theorem A.5 applied to $\star$ yields isomorphisms between de Rham cohomology groups of M and homology groups $H_r(M)$ of $\Omega_*(M)$, namely,

$$H^r_{dR}(M) \approx H_{m-r}(M)\,. \tag{3.10}$$

for $0 \leq r \leq m$. This is not quite Poincaré duality yet.

In order to state Hodge's theorem we need to define one more object, which is an analogue for differential forms of the usual Laplacian. The *Laplace-de Rham operator*

$$\Delta_r : \Omega^r(M) \to \Omega^r(M)$$

is defined by

$$\Delta_r\omega = \partial_{r+1}d_r(\omega) + d_{r-1}\partial_r(\omega)$$

for every $\omega \in \Omega^r(M)$.

$$\begin{tikzcd} & \Omega^r(M) \arrow[r, "\partial_r"] \arrow[dl, "d_r"'] \arrow[d, "\Delta_r"] & \Omega^{r-1}(M) \arrow[dl, "d_{r-1}"] \\ \Omega^{r+1}(M) \arrow[r, "\partial_{r+1}"] & \Omega^r(M) & \end{tikzcd} \tag{3.11}$$

It can be shown that Δ_r commutes with the Hodge operator, the exterior derivative d and its adjoint ∂. There is then a chain map $\Delta : \Omega_*(M) \to \Omega_*(M)$, given by (3.11) for $0 \le r \le m$. Also, it is not difficult to see that Δ_r is self-adjoint with respect to $(\cdot|\cdot)$.

A differential form $\omega \in \Omega^r(M)$ is said to be *harmonic* if $\Delta_r\omega = 0$. In what follows, we denote by $\mathscr{H}^r(M)$ the vector space of harmonic differential r-forms. Computing the expression $(\Delta\omega|\omega)$ one easily verifies that ω is harmonic if, and only if, it is closed and co-closed.

Since the diagram

$$\begin{tikzcd} \Omega^r(M) \arrow[r, "\star_r"] \arrow[d, "\Delta_r"'] & \Omega^{m-r}(M) \arrow[d, "\Delta_{m-r}"] \\ \Omega^r(M) \arrow[r, "\star_r"] & \Omega^{m-r}(M) \end{tikzcd}$$

is commutative for every r, it follows that $\star_r$ induces an isomorphism

$$\mathscr{H}^r(M) \approx \mathscr{H}^{m-r}(M)\,. \tag{3.12}$$

We are now ready to state Hodge's theorem.

Theorem 3.18 (Hodge Decomposition)**.** *Let M be a compact, oriented m-dimensional surface. For each $0 \le r \le m$, the space $\mathscr{H}^r(M)$ is finite-dimensonal and the inner product space $\Omega^r(M)$ splits into the orthogonal direct sum*

$$\Omega^r(M) = \Delta_r(\Omega^r(M)) \oplus \mathscr{H}^r(M)\,.$$

Moreover,

$$\Omega^r(M) = \operatorname{im} d_{r-1} \oplus \operatorname{im} \partial_{r+1} \oplus \mathscr{H}^r(M) \tag{3.13}$$

is also an orthogonal decomposition.

Some comments regarding the proof of Hodge's theorem are in order. Since the Laplacian is present in Hodge's theorem, it is only fitting to expect the proof of such theorem to be related to PDE's. Indeed, the toughest steps in the proof of Theorem 3.18 are the finiteness of $\dim \mathscr{H}^r(M)$ and the orthogonality $\operatorname{im} \Delta_r = \mathscr{H}^r(M)^\perp$, since they require the Hahn-Banach theorem and two other results. The first of these gives a sufficient condition to the existence of a solution for $\Delta_r\omega = \alpha$, whereas the second one consists of sufficient conditions to obtain a Cauchy sequence from a given sequence of differential forms in $\Omega^r(M)$. In order to prove these two results, one relies on the machinery of elliptic operators and of Sobolev spaces $W^{m,2}$. For further details, we refer the reader to [**13**].

An important consequence of Theorem 3.18 is that it allows us to "find" a harmonic form in each cohomology class in $H^r_{dR}(M)$. To see this, let us fix a surface M as in Theorem 3.18. The orthogonal decomposition (3.13) tells us that each $\omega \in \Omega^r(M)$ can be expressed uniquely as

$$\omega = d_{r-1}\alpha + \partial_{r+1}\beta + \gamma \tag{3.14}$$

for some $\alpha \in \Omega^{r-1}(M)$, $\beta \in \Omega^{r+1}(M)$ and $\gamma \in \mathscr{H}^r(M) = \ker d_r \cap \ker \partial_r$. Note that if $d\omega = 0$, then $d\partial\beta = 0$, whence $(\partial\beta|\partial\beta) = (d\partial\beta|\beta) = 0$. Thus, ω represents a de Rham cohomology class in $H^r_{dR}(M)$ if, and only if, $\partial_{r+1}\beta = 0$.

From this we see that, for $[\omega] \in H^r_{dR}(M)$, the form γ originating from the decomposition (3.14) is a harmonic r-form representing $[\omega]$. Also, this harmonic "component" is unique. Indeed, if $\bar\gamma \in \mathscr{H}^r(M)$ is any other harmonic form which is cohomologous to ω, say $\omega = d_{r-1}\bar\alpha + \bar\gamma$ for some $\bar\alpha \in \Omega^{r-1}(M)$, then $\gamma = \bar\gamma$, by the uniqueness of the orthogonal decomposition. This means that, for $0 \le r \le m$, to each $[\omega] \in H^r_{dR}(M)$ there is an unique harmonic r-form representing $[\omega]$. It then follows that the map

$$\begin{aligned} \mathscr{H}^r(M) &\xrightarrow{\approx} H^r_{dR}(M) \\ \gamma &\longmapsto [\gamma] \end{aligned}$$

is an isomorphism. Therefore, the fact that $\dim \mathscr{H}^r(M) < \infty$ (Theorem 3.18) allows one to obtain the particular case in Proposition 3.1, namely, each $H^r_{dR}(M)$ is a finite-dimensional vector space.

Using (3.12), we summarize the discussion above as follows.

Proposition 3.5. *Let M be a compact, oriented m-dimensional surface. Then*

1. $\mathscr{H}^r(M) \approx H^r_{dR}(M)$ *for every* $0 \le r \le m$*;*
2. $\dim H^r_{dR}(M) < \infty$ *for every* $0 \le r \le m$*;*
3. $H^r_{dR}(M) \approx H^{m-r}_{dR}(M)$ *for every* $0 \le r \le m$*;*

As a consequence of the above result and (3.10), we obtain a duality result for homology groups of $\Omega_*(M)$.

Corollary 3.18.1. *Let M be a compact, oriented m-dimensional surface. For every $0 \le r \le m$, the homology group $H_r(M)$ is finite dimensional and*

$$H_r(M) \approx H_{m-r}(M)\,.$$

Proposition 3.5 together with the existence of the inner product (3.9) implies Theorem 3.17 for compact, oriented surfaces.

Theorem 3.19 (Poincaré Duality, compact case)**.** *If M is a compact, oriented m-dimensional surface, then, for $0 \le r \le m$, the map*

$$D_M : H^r(M) \to H^{m-r}(M)^*\,,$$

given by

$$D_M[\alpha] \cdot [\beta] = \int_M \alpha \wedge \beta$$

for $[\alpha] \in H^r(M)$ *and* $[\beta] \in H^{m-r}(M)$*, is an isomorphism.*

Proof. From Proposition 3.5(2) and (3), it suffices to show that $\ker D_M = \{0\}$. Thus, let $[\alpha] \in H^r(M)$ be such that $D_M[\alpha] = 0$. We have

$$\int_M \alpha \wedge \beta = 0$$

for every $[\beta] \in H^{m-r}(M)$. In particular

$$\int_M \alpha \wedge (\star\alpha) = 0\,,$$

since $\star$ commutes with d. Therefore, $(\alpha|\alpha) = 0$, whence $\alpha = 0$.

Q.E.D.

Appendix A

Some Useful Theorems

This appendix is reserved to state theorems (some of them whitout proof) on integration, differentiablility and modules, which are used throughout the text. For proofs of some these results we refer the reader to [11]. We begin with the classical chain rule.

Theorem A.1 (Chain Rule). *Let $U \subseteq \mathbf{R}^m$, $V \subseteq \mathbf{R}^n$ be open sets, $f : U \to \mathbf{R}^n$ differentiable at $a \in U$, with $f(U) \subset V$, and $g : V \to \mathbf{R}^p$ differentiable at $f(a)$. Then $g \circ f : U \to \mathbf{R}^p$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a) : \mathbf{R}^m \to \mathbf{R}^p \,.$$

In terms of matrices we have

$$[(g \circ f)'(a)]_{p\times m} = [g'(f(a))]_{p\times n} \cdot [f'(a)]_{n\times m} \,.$$

From the definition of matrix multiplication it follows that

$$\frac{\partial (g_i \circ f)}{\partial x_j}(a) = \sum_{k=1}^{n} \frac{\partial g_i}{\partial y_k}(f(a)) \frac{\partial f_k}{\partial x_j}(a) \,,$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, m$, whence

$$\frac{\partial (g \circ f)}{\partial x_j}(a) = \sum_{k=1}^{n} \frac{\partial g}{\partial y_k}(f(a)) \frac{\partial f_k}{\partial x_j}(a) \,.$$

For the next theorem let $U \subseteq \mathbf{R}^m$ be an open set. A map $f : U \to \mathbf{R}^n$ is said to be strongly differentiable at a point $a \in U$ when there exists a linear map $T : \mathbf{R}^m \to \mathbf{R}^n$ such that, for $x, y \in U$,

$$f(x) - f(y) = T(x - y) + r_a(x, y)|x - y| \,, \quad \text{where} \lim_{(x,y)\to(a,a)} r_a(x, y) = 0 \,.$$

It can be shown ([11, p. 221]) that, for a differentiable map $f : U \to \mathbf{R}^n$, strong differentiability at a point $a \in U$ is equivalent to the continuity of the derivate $f' : U \to \mathcal{L}(\mathbf{R}^m; \mathbf{R}^n)$ at a.

Next we have the Inverse Mapping Theorem.

Theorem A.2. *(Inverse Mapping) Let $f : U \to \mathbf{R}^m$ be strongly differentiable at $a \in U$ and $f'(a) : \mathbf{R}^m \to \mathbf{R}^m$ an isomorphism. Then f is a homeomorphism from an open set $V \ni a$ onto an open set $W \ni f(a)$. The inverse homeomorphism $f^{-1} : W \to V$ is strongly differentiable at $f(a)$ and its derivative at $f(a)$ equals $[f'(a)]^{-1}$. If $f \in C^k$ $(k \geq 1)$ then V can be taken so that f maps V diffeomorphically onto W; in particular, $f^{-1} \in C^k$.*

During § 2.3 the following change of variables theorem is used to prove that the integral of a differential form is a signed integral. For a proof see [**11**].

Theorem A.3. *(Change of Variables) Let $\varphi : U \to V$ be a C^1-diffeomorphism between open sets $U, V \subseteq \mathbf{R}^m$, $K \subseteq U$ a Jordan-measurable compact set and $f : \varphi(X) \to \mathbf{R}$ an integrable function. Then $f \circ \varphi : K \to \mathbf{R}$ is integrable and*

$$\int_{\varphi(X)} f = \int_K (f \circ \varphi) |\operatorname{jac} \varphi| \,.$$

During the proof of Theorem 2.37) the following lemma is used in order to extend a homotopy.

Lemma A.4. *There exists a function $\xi : \mathbf{R} \to \mathbf{R}$ such that $\xi \in C^\infty$, $0 \leq \xi \leq 1$, $\xi(t) = 0$ for $t \leq 0$ and $\xi = 1$ for $t \geq 1$.*

Proof. First consider the function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^\infty$, given by

$$\alpha(t) = \begin{cases} e^{-1/t} & ; t > 0 \\ 0 & ; t \leq 0 \end{cases} .$$

Then define $\beta : \mathbf{R} \to \mathbf{R}$ as $\beta(t) = \alpha(t)\alpha(1-t)$. We have $\beta \in C^\infty$ and

$$\beta(t) = \begin{cases} 0 & ; t \leq 0 \\ e^{-1/(t-t^2)} & ; 0 < t < 1 \\ 0 & ; t \geq 1 \end{cases} .$$

Taking $b = \int_{\mathbf{R}} \beta = \int_0^1 \beta$, we define $\gamma : \mathbf{R} \to \mathbf{R}$, $\gamma \in C^\infty$, setting $\gamma(t) = \beta(t)/b$. Thus

$$\int_0^1 \gamma = 1 \,.$$

Lastly, $\xi : \mathbf{R} \to \mathbf{R}$ is defined by

$$\xi(t) = \int_{-\infty}^t \gamma = \int_0^t \gamma \,.$$

We have $0 \leq \xi \leq 1$, $\xi(t) = 0$ for $t \leq 0$ and $\xi = 1$ for $t \geq 1$. Q.E.D.

The following result can be proved in a similar fashion to the previous one. For details, see [**11**, p. 346].

Proposition A.1. *There exists a function $\xi : \mathbf{R}^m \to \mathbf{R}$ such that $\xi \in C_c^\infty(\mathbf{R}^m)$ and*

- $\xi(x) = 1$ *for* $|x| \leq 1$,
- $0 < \xi(x) < 1$ *for* $1 < |x| < 2$,
- $\xi(x) = 0$ *for* $|x| \geq 2$.

In particular,

$$\int_{B[0;2]} \xi \neq 0\,.$$

The following result is useful when dealing with (co)chain complexes in Chapter 1.

Theorem A.5. *Let* $f : M \to N$ *be a homomorphism of* R*-modules. If* $A \subseteq X \subseteq M$ *and* $B \subseteq Y \subseteq N$ *are submodules such that* $f(X) \subseteq Y$ *and* $f(A) \subseteq B$*, then there exists a homomorphism of modules* $f^* : X/A \to Y/B$ *given by* $f^*[x] = [f(x)]$.

Proof. The verification that f^* is a well-defined R-module homomophism is immediate. Q.E.D.

In § 3.4 the following results are needed.

Proposition A.2. *Let* $b : E \times E \to \mathbf{R}$ *be an anti-symmetric bilinear form on a finite-dimensional real vector space* E*. If* b *is non-degenerate, then* $n = \dim E$ *is even.*

Proof. Since b is anti-symmetric, its matrix B relative to a basis of E is anti-symmetric, that is, $B = -B^t$ (here $(\cdot)^t$ means the transpose matrix). Because b is non-degenerate, the corresponding linear map $f : E \to E^*$ is a linear isomorphism, and since $[f]_{n\times n} = B$ it follows that $\det B \neq 0$. Thus

$$1 = (\det B)^{-1} \det(-B^t) = (-1)^n (\det B)^{-1} \det(B^t) = (-1)^n (\det B)^{-1} \det(B) = (-1)^n\,,$$

whence must be n even.

Q.E.D.

Proposition A.3. *Let* $b : E \times E \to \mathbf{R}$ *be a non-degenerate symmetric bilinear form on a finite-dimensional real vector space* E*. If* $A : E \to E$ *is a linear operator such that*

$$b(A(u), A(v)) = -b(u, v) \tag{A.1}$$

for $u, v \in E$*, then* $\operatorname{sig}(b) = 0$*, that is* $n^+ = n^-$.

Proof. Let $n = \dim E$, $B = \{e_1 \ldots, e_n\} \subseteq E$ a basis such that $[b]_E$ is diagonal, $B^+ = \{e \in B\,;\, b(e,e) > 0\}$ and $B^- = \{e \in B\,;\, b(e,e) < 0\}$. First, note that if $u \in E$ is such that $A(u) = 0$, then non-degeneracy and (A.1) imply $u = 0$. This shows that A is an isomorphism. Thus $C = \{A(e_1), \ldots, A(e_n)\} \subseteq E$ is a basis of E. Also, (A.1) tells us that $[b]_C$ is a diagonal matrix. Now let $C^+ = \{A(e)\,;\, e \in B^+\}$ and $C^- = \{A(e)\,;\, e \in B^-\}$. From (A.1) we see that $e \in B^+ \implies A(e) \in C^-$ and $e \in B^- \implies A(e) \in C^+$. Thus $C^+ = C^-$, whence $\operatorname{card} C^+ = \operatorname{card} C^-$. Since A is an isomorphism, it follows that B^+ corresponds bijectively to A^+ and B^- corresponds bijectively to A^-. Therefore

$$n^+ = \operatorname{card} B^+ = \operatorname{card} C^+ = \operatorname{card} C^- = \operatorname{card} B^- = n^-\,.$$

Q.E.D.

List of Symbols

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