Fundamentals of Learning and Information Processing

Session 14: Statistical Machine Learning (IV)

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The Fundamental Theorem of PAC Learning

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1. $\mathscr{H}$ has the uniform convergence property
2. Any ERM rule is a successful agnostic PAC learner for $\mathscr{H}$
3. $\mathscr{H}$ is agnostic PAC learnable
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  - If $VCDim(\mathcal{H}) = d$, then even though $\mathcal{H}$ might be infinite, when restricting to a finite set $C \subseteq \mathcal{X}$, the size of $|\mathcal{H}_C|$ grows polynomially with $|C|$
  - Uniform convergence holds for the "small effective size" class $\mathcal{H}$.

Let $\mathcal{H}$ be a hypothesis class of functions from a domain $\mathcal{X}$ to $\{0,1\}$, and let the loss function be the $0-1$ loss. Assume that $VCDim(\mathcal{H}) = d < \infty$. Then, there are absolute constants C_1 and C_2 such that

1. $\mathcal{H}$ has the uniform convergence property with sample complexity

$$C_1 \frac{d + \log(1/\delta)}{\epsilon^2} \le m_{\mathcal{H}}^{UC}(\epsilon,\delta) \le C_2 \frac{d + \log(1/\delta)}{\epsilon^2}$$

2. $\mathcal{H}$ is agnostic PAC learnable with sample complexity

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The Fundamental Theorem of PAC Learning (Quantitative)

Let $\mathscr{H}$ be a hypothesis class of functions from a domain $\mathscr{X}$ to $\{0,1\}$, and let the loss function be the $0-1$ loss. Assume that $VCDim(\mathscr{H}) = d < \infty$. Then, there are absolute constants C_1 and C_2 such that

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Intuition.

- *NFL* is on the lower bound; while *Hoffding's Inequality* is on the upper bound.

Proof of the Lower Bounds

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- Pick a set $S = \{x_1, \cdots, x_d\}$ of size d that is shattered by $\mathcal{H}$. Let $\mathcal{D}$ be:

$$p(x_i) = \begin{cases} 1-\epsilon & \text{if } i = 1 \\ \frac{\epsilon}{d-1} & i \in [2, d] \\ 0 & \text{otherwise} \end{cases}$$

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- So the overall expected error will be $\frac{\epsilon}{4}$.
- Only roughly $m\epsilon$ of a sample S of size m will hit $\{x_2, \cdots, x_d\}$.
- So in order to get the expected error below $\frac{\epsilon}{4}$, we need the $m\epsilon > \frac{d-1}{2}$, so $m > \frac{d-1}{2\epsilon}$

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- So in order to get the expected error below $\frac{\epsilon}{4}$, we need the $m\epsilon > \frac{d-1}{2}$, so $m > \frac{d-1}{2\epsilon}$
- Now we have $\forall \epsilon$, $E_{S\sim\mathscr{D}^m}[L_{\mathscr{D}}(A(S))] < \frac{\epsilon}{4}$, then $m > \frac{d-1}{2\epsilon}$

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- Consider one sample with two points $S = \{x, y\}$ both points with probability $\frac{1}{2}$ to be selected. $P(1|X) = P(0|Y) = \frac{1}{2} + \epsilon$ and $P(0|X) = P(1|Y) = \frac{1}{2} - \epsilon$

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- In order to detect if a coin has $\frac{1}{2} + \epsilon$ towards tails or heads, one needs $\frac{1}{\epsilon^2}$ coin tosses, as ϵ gets small

$\square$

The Growth Function

Let $\mathscr{H}$ be a hypothesis class, then the growth function of $\mathscr{H}$, denoted $\tau_{\mathscr{H}} : \mathscr{N} \to \mathscr{N}$, is defined as:

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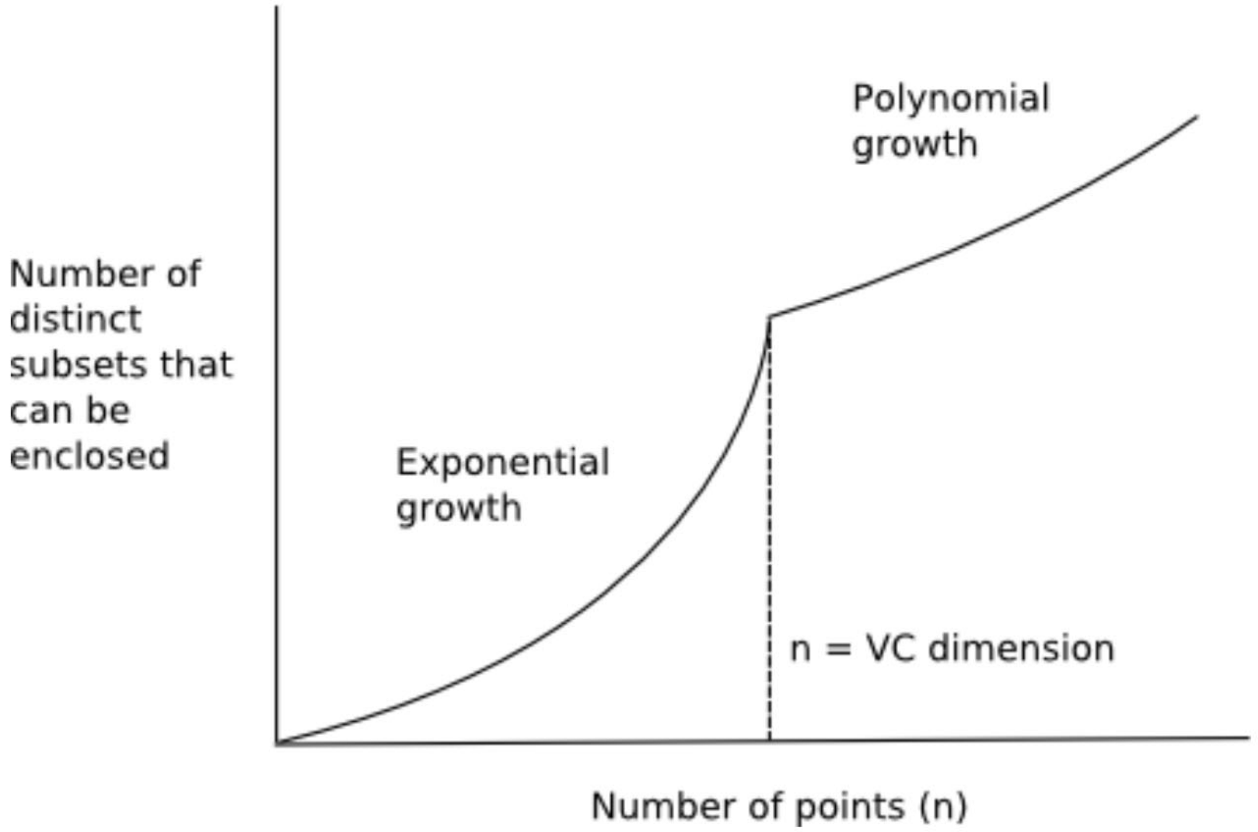
Intuition

$$\begin{aligned}\tau_{\mathcal{H}}(m) &= \max_{|A| \le m} |h \cap A : h \in \mathcal{H}| \\ &= |\{g : A \to \{0,1\} : \\ &\quad \exists h \in \mathcal{H}, \forall x \in A, g(x) = h(x)\}|\end{aligned}$$

Exponential $2^m, 3^m, \cdots$
... $m^{\log m}$, etc.
Polynomial $m^2, m^3, \cdots$



Sauer's Lemma

Let $\mathscr{H}$ be a hypothesis class with $VCDim(\mathscr{H}) \leq d \leq \infty$, then $\forall m$, $\tau_{\mathscr{H}}(m) \leq \sum_{i=0}^{d} \binom{m}{i} \leq m^d$. In particular, if $m > d+1$ then $\tau_{\mathscr{H}}(m) \leq (em/d)^d$

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  - $VCDim(\mathscr{H}_{HS^2}) = 3$

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  - $1000^3 << 2^{1000}$

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1. We prove a stronger claim: $\forall C = \{c_1, \cdots, c_m\}$, we have $\forall \mathscr{H}$, $|\mathscr{H}_C| \le |\{B \subseteq C : \mathscr{H} \text{ shatters } B\}|$.

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Let $\mathcal{H}$ be a hypothesis class with $VCDim(\mathcal{H}) \leq d \leq \infty$, then $\forall m$, $\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{d} \binom{m}{i} \leq m^d$. In particular, if $m > d+1$ then $\tau_{\mathcal{H}}(m) \leq (em/d)^d$

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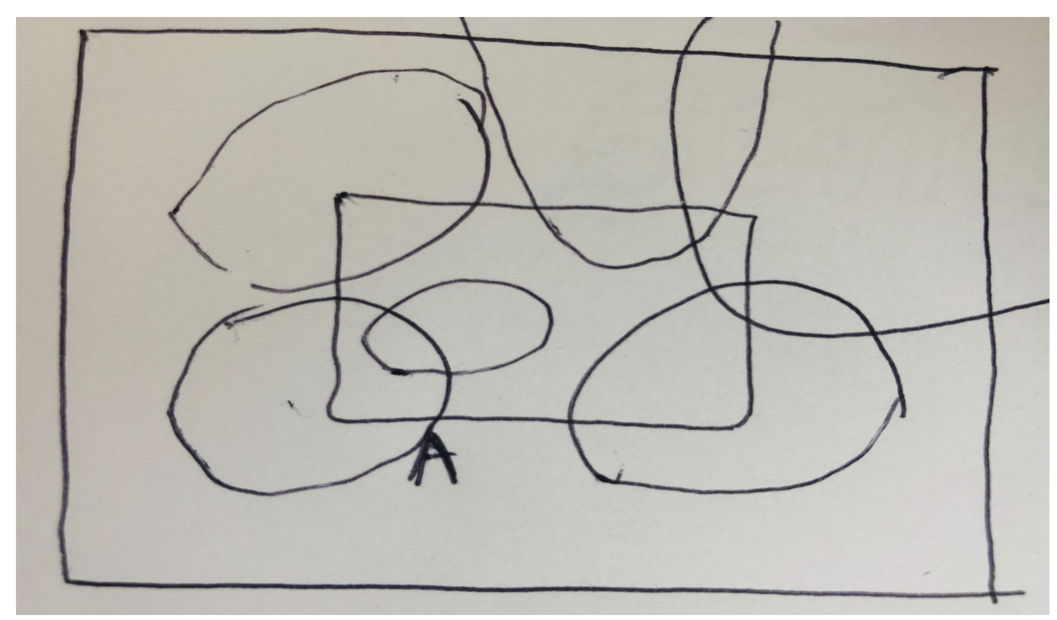
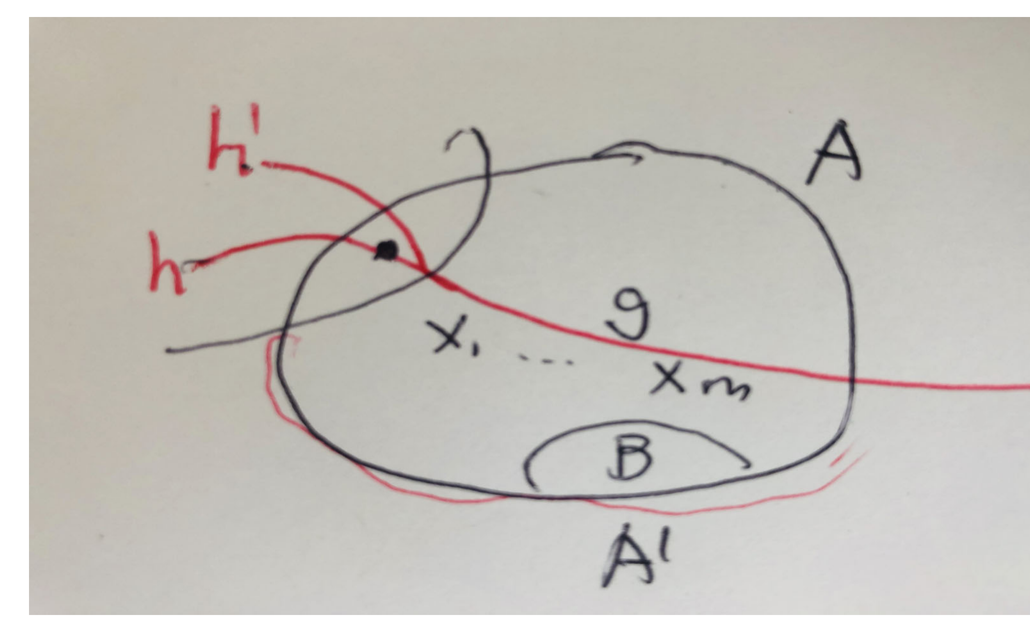
2. If $VCDim(\mathcal{H}) \leq d$, then no set with size larger than d is shattered by $\mathcal{H}$: $|\{B \subseteq C : \mathcal{H} \text{ shatters } B\}| \leq \sum_{i=0}^{d} \binom{m}{i}$





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$\forall C = \{c_1, \cdots, c_m\}$, we have $\forall \mathscr{H}$, $|\mathscr{H}_C| \leq |\{B \subseteq C : \mathscr{H} \text{ shatters } B\}|$.

Inductive Argument.

1. The empty set ϕ is always shattered by $\mathscr{H}$.

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Inductive Argument.

1. The empty set ϕ is always shattered by $\mathscr{H}$.
2. Assume it holds for any size $m-1$, let us prove it for the size m. Denote $C' = \{c_2, \cdots, c_m\}$ and define:

$$Y_0 = \{(y_2, \cdots, y_m) : (0, y_2, \cdots, y_m) \in \mathscr{H}_C \vee (1, y_2, \cdots, y_m) \in \mathscr{H}_C\}$$

and

$$Y_1 = \{(y_2, \cdots, y_m) : (0, y_2, \cdots, y_m) \in \mathscr{H}_C \wedge (1, y_2, \cdots, y_m) \in \mathscr{H}_C\}$$

We have $|\mathscr{H}_C| = |Y_0| + |Y_1|$.

Let $\mathcal{H}$ be a hypothesis class with $VCDim(\mathcal{H}) \leq d \leq \infty$, then $\forall m$, $\tau_{\mathcal{H}}(m) \leq \sum_{i=0}^{d} \binom{m}{i} \leq m^d$. In particular, if $m > d+1$ then $\tau_{\mathcal{H}}(m) \leq (em/d)^d$

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2. Assume it holds for any size $m-1$, let us prove it for the size m. Denote $C' = \{c_2, \cdots, c_m\}$ and define:

   $$Y_0 = \{(y_2, \cdots, y_m) : (0, y_2, \cdots, y_m) \in \mathcal{H}_C \vee (1, y_2, \cdots, y_m) \in \mathcal{H}_C\}$$

   and

   $$Y_1 = \{(y_2, \cdots, y_m) : (0, y_2, \cdots, y_m) \in \mathcal{H}_C \wedge (1, y_2, \cdots, y_m) \in \mathcal{H}_C\}$$

   We have $|\mathcal{H}_C| = |Y_0| + |Y_1|$.
3. Since $Y_0 = \mathcal{H}_{C'}$, using induction assumption, we have $|Y_0| = |\mathcal{H}_{C'}| \leq |\{B \subseteq C' : \mathcal{H} \text{ shatters } B\}| = |\{B \subseteq C : c_1 \notin B \wedge \mathcal{H} \text{ shatters } B\}|$

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Sauer's Lemma

Let $\mathscr{H}$ be a hypothesis class with $VCDim(\mathscr{H}) \le d \le \infty$, then $\forall m$, $\tau_{\mathscr{H}}(m) \le \sum_{i=0}^{d} \binom{m}{i} \le m^d$. In particular, if $m > d+1$ then $\tau_{\mathscr{H}}(m) \le (em/d)^d$

$\forall C = \{c_1, \cdots, c_m\}$, we have $\forall \mathscr{H}$, $|\mathscr{H}_C| \le |\{B \subseteq C : \mathscr{H}$ shatters $B\}|$.

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4. Let $\mathscr{H}' \subseteq \mathscr{H}$,
   $\mathscr{H}' = \{h \in \mathscr{H} : \exists h' \in \mathscr{H}(1-h'(c_1), h'(c_2), \cdots, h'(c_m)) = (h(c_1), h(c_2), \cdots, h(c_m))\}$
   - $\mathscr{H}'$ contains pairs of hypotheses that agree on C' and differ on c_1.
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6. Overall we have shown that:

$$|\mathscr{H}_C| = |Y_0| + |Y_1| \le |\{B \subseteq C : c_1 \notin B \wedge \mathscr{H} \text{ shatters } B\}| + |\{B \subseteq C : c_1 \in B \wedge \mathscr{H} \text{ shatters } B\}| \\ = |\{B \subseteq C : \mathscr{H} \text{ shatters } B\}|$$

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  - m can not be too big...

$\square$

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- If $\mathcal{H}$ is infinite or large, the union bound yields a meaningless bound

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Main Idea.

Two Samples Trick We evaluate the error of any hypothesis $h \in \mathcal{H}_B$ on a fresh sample T, a.k.a. the ghost sample.

- Since S and T are i.i.d., we can think of first sampling $2m$ samples, and then splitting them to S and T at random
- On this evaluation, it suffices to consider $\mathcal{H}_{S \cup T}$, the set of possible behaviors of $S \cup T$
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Combination When m is sufficiently large, the probability of bad hypothesis will $< \epsilon$:

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4. As h_S is independent of the ghost sample, we can use Chebyshev's Inequality:

$$P(|L_{\mathcal{D}}(h_S) - L_T(h_S)| \geq \epsilon/2) \leq \frac{4var(h_S)}{m\epsilon^2} \leq \frac{1}{m\epsilon^2}, \quad \text{because } var(h_S) \leq 1/4$$
$$P(|L_{\mathcal{D}}(h_S) - L_T(h_S)| < \epsilon/2) \geq 1 - \frac{1}{m\epsilon^2} \geq 1/2, \quad \text{when } 1 - \frac{1}{m\epsilon^2} \geq 1/2$$

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□

- With Symmetrization, hypothesis classes with finite VC dimension can be PAC learnable. Hence ⑥ → ① in the The Fundamental Theorem of PAC Learning.

The fundamental theorem of statistic learning

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What is Learnable and *How to Learn*?

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(Left to right) J. Rissanen, V. Vapnik, A. Gammerman, A. Chervonenkis, C. Wallace and R. Solomonoff

Learning Beyond Classes with Finite VC Dimension

Nonuniform Learnability

A hypothesis class $\mathcal{H}$ is nonuniformly learnable (NUL) if there exist a learning algorithm A, and a function $m_{\mathcal{H}}^{NUL} : (0,1)^2 \times \mathcal{H} \to \mathcal{N}$ such that:

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- A class that is agnostic PAC learnable is also nonuniformly learnable

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Finite functions Let $\mathscr{X} = \mathscr{R}$, $\mathscr{Y} = \{0,1\}$, for each k, let $\mathscr{H}_k = \{h : h(x) = 0 \text{ for at most } k \text{ many } x\text{s}\}$

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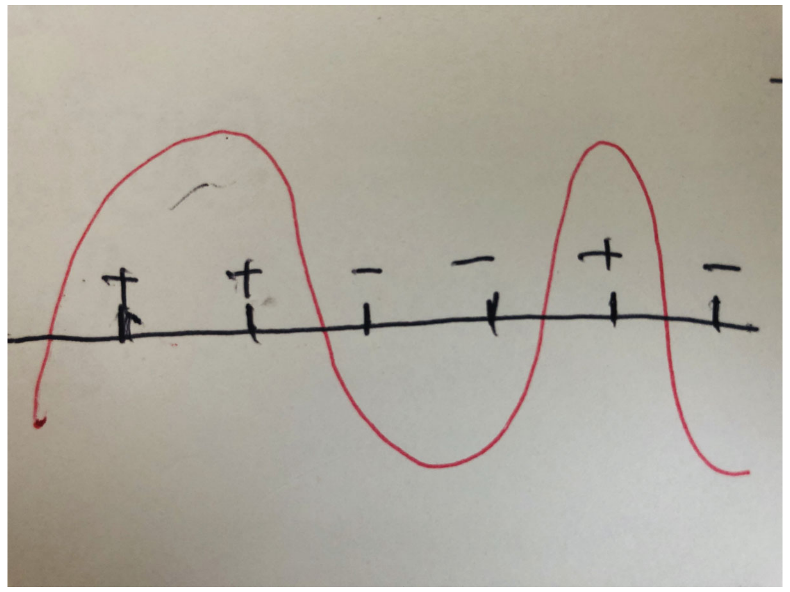
- $\mathscr{H}$ is not PAC learnable, because it has infinite VC dimension



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- This implies that there exist some classes that are not NUL

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  - Contradiction!

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- Apply the union bound over n, we have at most $\sum_n \delta_n = \delta$,

$$\forall h \in \mathscr{H}, |L_{\mathscr{D}}(h) - L_S(h)| \geq \min_{n: h\in\mathscr{H}_n} \epsilon_n(m, \delta_n)$$

NUL versus A-PAC

Table 1: Comparison of NUL and Agnostic PAC (I)

Learnability	Conditions
Agnostic PAC	$\forall h$, $\forall \mathscr{D}$, $\forall \delta$ and $\forall \epsilon$, if $m \geq \ldots$, we have $$P_{S\sim\mathscr{D}^m}[\exists h : \lvert L_{\mathscr{D}}(h) - L_S(h)\rvert > \epsilon] < \delta$$
NUL	$\mathscr{H} = \cup_{n\in\mathscr{N}} \mathscr{H}_n$, each $\mathscr{H}_n$ with $m_n^{UC}(\epsilon,\delta)$, $\omega : \mathscr{N} \mapsto [0,1]$ and $\sum \omega(n) \leq 1$. $\forall \mathscr{D}$, $\forall \delta$ and $\forall \epsilon$, $\forall h$, we have $$P_{S\sim\mathscr{D}^m}[\exists h : \lvert L_{\mathscr{D}}(h) - L_S(h)\rvert > \epsilon_{n(h)}(m,\omega(n(h))\delta)] < \delta$$ This implies that $$L_{\mathscr{D}}(h) \leq L_S(h) + \epsilon_{n(h)}(m,\omega(n(h))\delta)$$

Convergence bounds on S: $\lvert L_S(h) - L_{\mathscr{D}}(h)\rvert$

NUL versus A-PAC

Table 2: Comparison of NUL and Agnostic PAC (II)

Learnability	Learner
Agnostic PAC	ERM learner: $\forall h$, $\forall \mathscr{D}$, $\forall \delta$ and $\forall \epsilon$, if $m \geq \ldots$, we have $$L_{\mathscr{D}}(A(S)) \leq L_S(h) + \epsilon$$ for $m \geq m_{\mathscr{H}}^{UC}(\epsilon/2, \delta)$ with probability at least $1-\delta$.
NUL	SRM learner: let $A(S)$ be any hypothesis $h \in \mathscr{H}$ that minimizes $$L_{\mathscr{D}}(A(S)) \leq L_S(h) + \epsilon_{n(h)}(m, \omega(n(h))\delta)$$

Learner bounds its error on S: $|L_{\mathscr{D}}(A(S)) - L_{\mathscr{D}}(h)|$

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- and prior weight $\omega(n)$ for each $\mathscr{H}_n$, with $n(h) = \min\{n : h \in \mathscr{H}_n\}$

$\square$

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Given $\mathscr{H} = \cup_{n \in \mathscr{N}} \mathscr{H}_n$, where each $\mathscr{H}_n$ satisfies the uniform convergence property with a sample complexity function $m^{UC}_{\mathscr{H}_n}$. Given any weight function $\omega : \mathscr{N} \mapsto [0,1]$ and $\sum \omega(n) \leq 1$. Any SRM algorithm is a successful Non-Uniform Learner with sample

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- Combining together we obtain that at most δ we have

$$L_{\mathcal{D}}(A(S)) > L_{\mathcal{D}}(h) + \epsilon$$

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Bias to Shorter Description:

- Let $\mathscr{H}$ be a countable hypothesis class
- Let $\omega : \mathscr{H} \mapsto \mathscr{R}$ be such that $\sum_{h \in \mathscr{H}} \omega(h) \leq 1$
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- Let $|h|$ be the length of $d(h)$, then set $\omega(h) = \frac{1}{2^{|h|}}$.
- From Kraft's inequality, we have $\sum_h \omega(h) \le 1$
  - define probability over words in $d(H)$ as follows: repeatedly toss an unbiased coin, until the sequence of outcomes is a member of $d(H)$, and then stop.
  - Since $d(H)$ is prefix-free, this is a valid probability over $d(H)$, and the probability to get $d(h)$ is $\omega(h)$.

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Let $\omega : \mathcal{H} \mapsto \mathcal{R}$ be such that $\sum_{h \in \mathcal{H}} \omega(h) \leq 1$. Then, with probability of at least $1 - \delta$ over $S \sim \mathcal{D}^m$, we have $\forall h \in \mathcal{H}$,

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- Applying the union bound,

$$\mathcal{D}^m(\{S : \exists h \in \mathcal{H}, L_{\mathcal{D}}(h) > L_S(h) + \sqrt{\frac{\log(2/\delta_h)}{2m}}\})$$

$$= \mathcal{D}^m(\cup_{h\in\mathcal{H}}\{S : L_{\mathcal{D}}(h) > L_S(h) + \sqrt{\frac{\log(2/\delta_h)}{2m}}\})$$

$$\leq \sum_{h\in\mathcal{H}} \delta_h \leq \delta$$

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Algorithms

ERM minimizes the VC bound
MDL minimizes the MDL bound

□

Consistency Learnability

Let $\mathcal{Z}$ be a domain set, let $\mathcal{P}$ be a set of probability over $\mathcal{Z}$, and let $\mathcal{H}$ be a hypothesis class. A learning rule A is consistent with respect to $\mathcal{H}$ and $\mathcal{P}$ if there exists a function $m_{\mathcal{H}}^{CON} : (0,1)^2 \times \mathcal{H} \times \mathcal{P} \to \mathcal{N}$ such that, for every $\epsilon, \delta \in (0,1)$, for every $h \in \mathcal{H}$, and every $\mathcal{D} \in \mathcal{P}$, if $m \geq m_{\mathcal{H}}^{CON}(\epsilon, \delta, h, \mathcal{D})$, then for every distribution $\mathcal{D}$ over $\mathcal{X}$, with probability of at least $(1-\delta)$ over the choice of $S \sim \mathcal{D}^m$, it holds that:

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  - What to do if the returned hypothesis with a large risk?

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A `Memorize` algorithm is universally consistent for every countable domain $\mathscr{X}$ and a finite label set $\mathscr{Y}$, with respect to the zero-one loss.

Intuition. $\square$

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- `Memorize` is a consistent learner for every class of binary valued functions over any countable domain.

□

Quiz

The VC Dimension

- Let $\mathscr{H}_1, \ldots, \mathscr{H}_r$ be hypothesis classes over some fixed domain set $\mathscr{X}$. Let $d = max_i VCDim(\mathscr{H}_i)$ and assume for simplicity that $d \geq 3$.

- Let $\mathcal{H}_1, \ldots, \mathcal{H}_r$ be hypothesis classes over some fixed domain set $\mathcal{X}$. Let $d = max_i VCDim(\mathcal{H}_i)$ and assume for simplicity that $d \geq 3$.

  1. Prove that for $r = 2$, it holds that

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  2. Prove that

  $$VCDim(\cup_{i=1}^{r}\mathscr{H}_i \leq 4d\log(2d) + 2\log(r)$$

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  2. The weights could be monotonically nondecreaasing, namely, if $i < j$, then $\omega(\mathscr{H}_i) \leq \omega(\mathscr{H}_j)$.

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Questions?

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