FUNDAMENTALS OF LEARNING AND INFORMATION PROCESSING

SESSION 15: STATISTICAL MACHINE LEARNING (V)

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Table of Content

Paradox

Learning for Decipher

Let F_n be a class of trapdoor functions over $\{0,1\}^n$: $F_n = \{f_{s_n} : s_n \text{ is the key}\}$. Now given a set of encrypted messages:

$$(x_1, f(x_1)), \cdots, (x_n, f(x_n))$$

Learn the class $F_n^{-1} = \{f^{-1} : f \in F_n\}$, whether it is feasible?

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Proof.

- $|F_n| \leq 2^{Poly(n)}$, so $VCDim(F_n) = Poly(n)$
- So the number of examples we need will be polynomial in n.
- There is no efficient learner for this class. Otherwise, ...?

□

Computational Complexity of Learning

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How much computation is involved in carrying out a learning task?

Sample Complexity as in *the Fundamental Theorem of PAC Learning*

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Output **of the learning algorithm** $\mathscr{A}$ must be efficient
Runtime **of the learning algorithm** $\mathscr{A}$
max(time for $\mathscr{A}$ to output some h, time for h to output a label for any given X)

□

Computational Complexity of Learning

Given a function $f : (0,1)^2 \to \mathcal{N}$, a learning task $(Z, \mathcal{H}, l)$, and a learning algorithm $\mathcal{A}$ solves the learning task in time $O(f)$ if there exists a constant number c, such that for every probability distribution $\mathcal{D}$ over $\mathcal{A}$, and input $\epsilon, \delta \in (0,1)$, when $\mathcal{A}$ has access to samples generated i.i.d. by $\mathcal{D}$.

1. $\mathcal{A}$ terminates after performing at most $cf(\epsilon,\delta)$ operations
2. The output of $\mathcal{A}$, h_A, can be applied to predict the label of a new example while performing at most $cf(\epsilon,\delta)$ operations
3. The output of $\mathcal{A}$ is probably approximately correct: with probability of at least $1-\delta$, $L_{\mathcal{D}} \leq \min_{h^* \in \mathcal{H}} L_{\mathcal{D}(h^*)} + \epsilon$

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Consider a sequence of learning problems, $(Z_n, \mathcal{H}_n, l_n)_{n=1}^{\infty}$, where n is defined by a domain Z_n, a hypothesis class $\mathcal{H}_n$ and a loss function l_n. Let $\mathcal{A}$ be a learning algorithm designed for solving learning problems of this form. Given a function $g : \mathcal{N} \times (0,1)^2 \to \mathcal{N}$, we say that the runtime of $\mathcal{A}$ with respect to the preceding sequence is $O(g)$, if for all n, $\mathcal{A}$ solves the problem in time $O(f_n)$, where $f_n : (0,1)^2 \to \mathcal{N}$ is defined by $f_n(\epsilon,\delta) = g(n,\epsilon,\delta)$.

- $\mathcal{A}$ is an efficient algorithm with respect to a sequence $(Z_n, \mathcal{H}_n, l_n)$ if its runtime is $O(p(\frac{1}{\epsilon}, \frac{1}{\delta}, n))$ for some polynomial p.

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A Sequence of Problem

- If we define a sequence of problems in which $|\mathcal{H}_n| = n$, then the exhaustive search is still efficient;
- If we define a sequence of problems in which $|\mathcal{H}_n| = 2^n$, then the exhaustive search is not efficient;

□

Implementing the ERM Rule

On a finite input sample $S \in \mathcal{Z}^m$, the ERM rule outputs some $h \in \mathcal{H}$ that minimizes the empirical loss $L_S(h) = \frac{1}{|S|} \sum_{z \in S} l(h, z)$

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Axis aligned rectangles $\mathcal{X} = \mathcal{R}^d$, let $\mathcal{H}_{a_1,\cdots,a_d,b_1,\cdots,b_d} = \{h_{a_1,\cdots,a_d,b_1,\cdots,b_d}(x) : a_i < b_i\}$, where $h_{a_1,\cdots,a_d,b_1,\cdots,b_d}(\vec{x}, y) = 1$ iff $x_1 \in \forall i, x_i \in [a_i, b_i]$.

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- Every such rectangle is determined by at most $2d$ points from S; there are $\leq m^{2d}$ such tuples, so the run time $\sim [C \frac{2d + \ln(2/\delta)}{\epsilon^2}]^{2d}$

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- So for a variable d, the exhaustive search runs in *exponential* time.

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Boolean Conjunctions is a mapping from $\mathcal{X} = \{0,1\}^d$ to $\mathcal{Y} = \{0,1\}$ that can be expressed as $x_{i_1} \wedge \cdots \wedge x_{i_k} \wedge \neg x_{j_1} \wedge \cdots \wedge \neg x_{j_r}$. The function is defined as:

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$$h(\vec{x}) = \begin{cases} 1 & \text{if } x_{i_1} = \cdots = x_{i_k} = 1 \text{ and } x_{j_1} = \cdots = x_{j_r} = 0 \\ 0 & \text{otherwise} \end{cases}$$

Step 1: Upper bound the sample size -

- $VCdim(\mathcal{H}) \leq \log(|\mathcal{H}|)$, and $|\mathcal{H}| \leq 2^{2d}$ or $\leq 3^d$. Therefore, $m_{\mathcal{H}}(\epsilon, \delta) \sim C \frac{2d + \ln(2/\delta)}{\epsilon^2}$

Step 2: Algorithm - *Realizable* cases

- Let $h_0 = x_{i_1} \wedge \cdots \wedge x_{i_k} \wedge \neg x_{i_1} \wedge \cdots \wedge \neg x_{i_k}$ and $L_S(h_0) = \frac{|h(\vec{x}_i, y_i) : y_i = 1|}{m}$.
- $h_{t+1} = \begin{cases} h_t & h_t(x_t) = y_t \\ \text{delete any literal that fails in } h_t & \text{otherwise} \end{cases}$
- h_{t+1} is the minimal (the most demanding) conjunction that accepts all t labelled examples among $(x_1, y_1), \cdots, (x_t, y_t)$.

Step 3: Upper bound the time -

- $m \times 2d = C \frac{2d + \ln(2/\delta)}{\epsilon^2} \times 2d$, so it is polynomial.

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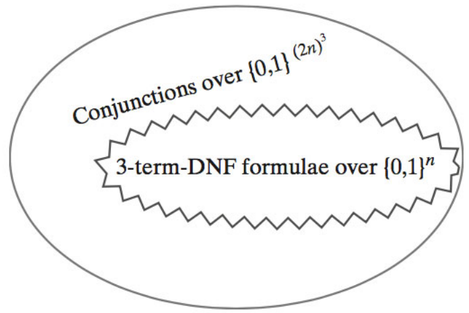

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- Such conjunction can be learned in time $poly(\frac{1}{\epsilon}, \frac{1}{\delta}, 8d^3)$

Hardness of Learning

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Table 1: Hardness of Learning

	Realizable	**Agnostic**
Unrestricted Proper	3-term DNF `Conjunctions` over d variables with complexity $O(\frac{d}{\epsilon})$	$\mathscr{H}_{rec}^{d}$ with complexity $O(\frac{d}{\epsilon^2})^d$

It is *hard* from left to right, or from top to bottom.

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Learning Half-Spaces $\mathscr{H}_{linear}^{d}$

- It is easy for *realizable* case.
- But it is NP Hard in the *agnostic* case, for *proper* learning and *unrestricted* learning.

□

Cryptographic-Hardness

Cryptographic-Hardness Many cryptographic systems assume that there exists a one way function $f : \{0,1\}^n \to \{0,1\}^n$, that is easy to compute but is hard to invert: f can be computed in time polynomial $p(n)$, but no polynomial time algorithm can invert it: for any randomized polynomial time algorithm $\mathscr{A}$, for any every polynomial $p(\cdot)$,

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- Discrete log function

□

Validation and Model Selection

Model Validation

We have already learned some hypothesis h, now we want to estimate how good is h. When we can randomly partition the set of examples into two parts, one as training set S, and one as validation set V, also known as hold-out set.

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- This is in general more accurate than the estimate on the *training* set S, which is from fundamental theorem of learning:

$$L_{\mathscr{D}}(h) \leq L_S(h) + \sqrt{C\frac{d + \log(1/\delta)}{m}}$$

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Let $\mathscr{H} = \{h_1, \cdots, h_r\}$ be the set of all output predictors of the different algorithms. Assume a validation set V of size m_V is sampled independent of $\mathscr{H}$. Then, with probability of at least $1-\delta$ over the choice of V, we have $\forall h \in \mathscr{H}$,

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- This theorem tells us that the error on the validation set approximates the true error as long as $\mathscr{H}$ is not too large.
- Suppose the chosen one is $h_S \in ERM_{\mathscr{H}}(V)$, then from the uniform convergence property, we have:

$$L_{\mathscr{D}}(h_S) \leq \min_{h^* \in \mathscr{H}} L_{\mathscr{D}}(h^*) + \sqrt{\frac{2\log(2|\mathscr{H}|/\delta)}{m_V}}$$

Model Selection in Practise

When data is plentiful, the best model selection approach is the training-validation-test split:

Training Set apply the learning algorithm with different parameters on the training set to produce $\mathscr{H} = \{h_1, \cdots, h_r\}$

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Implications.

- When data is scarce, we use k-fold cross validation.

□

Machine Learning Diagnosis

Consider a scenario:

> You were given a learning task and have approached it with a choice of hypothesis class, a learning algorithm, and parameters. You used a validation set to tune the parameters and tested the learned predictor on the test set. The test result turns out to be unsatisfactory.

What went wrong? and what should you do next?

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- Change the feature representation of the data

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Let h_S be an $ERM_{\mathcal{H}}$ hypothesis: $h_S \in \text{argmin}_{h \in \mathcal{H}} L_S(h)$. Assume $f \in \mathcal{Y}^{\mathcal{X}}$ be the true hypothesis, and h^* is the best hypothesis in $\mathcal{H}$: $h^* = \min_{h^* \in \mathcal{H}} L_{\mathcal{D}}(h^*)$. Then we can have:

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- The estimation error ϵ_{est} depends on the sample size. If it is large, we should obtain more training examples; or reduce the hypothesis class.

□

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$$L_{\mathscr{D}}(h_S) = L_{\mathscr{D}}(h_S) - L_V(h_S) + L_V(h_S) - L_S(h_S) + L_S(h_S)$$

How to Fix?

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$$L_S(h_S) = (L_S(h_S) - L_S(h^*)) + (L_S(h^*) - L_{\mathcal{D}}(h^*)) + L_{\mathcal{D}}(h^*)$$

  - When h_S is an ERM result, we have $L_S(h_S) - L_S(h^*) \le 0$
  - Since h^* is not dependent on S, $(L_S(h^*) - L_{\mathcal{D}}(h^*))$ can be bounded tightly.
  - $L_{\mathcal{D}}(h^*)$ is the approximation error. So if $L_S(h_S)$ is large, the approximation error $L_D(h^*)$ is generally large, though not always (why?)

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- When $L_S(h_S)$ is *small*, it does not necessarily imply that $L_D(h^*)$ is small.

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  - When the $VCDim(h_S)$ is d, but the training sample size $m < d$, how is the approximation error?
  - When the $VCDim(h_S)$ is d, but the training sample size $m > 2d$, how is the approximation error?

□

Boosting

Weak Learner

An algorithm $\mathscr{A}$ is a γ-weak learner for a class $\mathscr{H}$ if for some function $m_{\mathscr{H}}(\delta)$ for every probability distribution D over $\mathscr{X}$ and every $f \in \mathscr{H}$, for every $m > m_{\mathscr{H}}(\delta)$:

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- For every sample that is labelled by some hypothesis from $\mathscr{H}$, any ERM_B hypothesis will have an error of at most $\frac{1}{2} - \gamma$.

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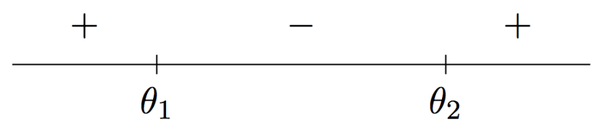

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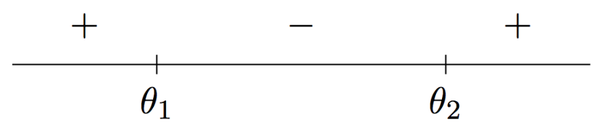

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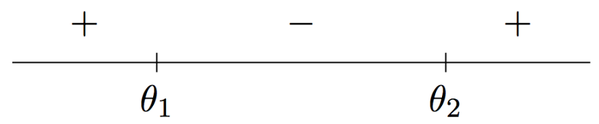

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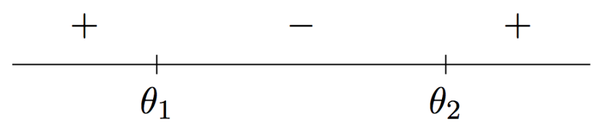

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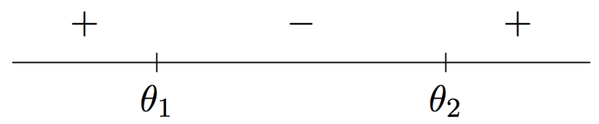

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  - Given $\gamma < \frac{1}{6}$, let ϵ be such that $\gamma + \epsilon < \frac{1}{6}$, and let $m(\gamma) = m_{\mathscr{B}}(\epsilon, \delta)$, it will guarantee that $L_{(D,f)}(ERM_B(S)) \leq L_S(ERM_B(S)) + \epsilon \leq \frac{1}{6}$

One Popular Weak Learner $\mathscr{H}_{DS}^{d}$

In practise, the most popular weak learner class is $\mathscr{H}_{DS}^{d}$, the class of Decision Stumps over $\mathscr{R}^{d}$.

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  - Compute $L_S(h)$ for each such h, output the one with minimum loss. The time cost will be $O(2(m+1) \times d \times m)$, which is polynomial, hence efficient.

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Boosting the Confidence

Suppose an algorithm $\mathscr{A}$ is a γ_0-weak learner for some class $\mathscr{H}$ that requires $m_{\mathscr{H}}(\delta_0)$ examples, namely in the PAC learning scenario, it is $m_{\mathscr{H}}(\epsilon_0, \delta_0)$ with $\epsilon_0 = \frac{1}{2} - \gamma_0$ is fixed. For any $\epsilon, \delta \in (0,1)$, we show how to learn $\mathscr{H}$ to accuracy $\epsilon_0 + \epsilon$ with confidence δ.

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2. Take additional validation sample of size m_V and output $\hat{h} \in \arg min_{h_i} L_V(h_i)$.

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  This is valid when $k \geq \lceil \frac{\log(2/\delta)}{\log(1/\delta_0)} \rceil$.
- Apply the union bound, for at least $1-\delta$, we have $L_{\mathscr{D}}(\hat{h}) \leq \epsilon_0 + \epsilon$.

□

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Suppose an algorithm $\mathscr{A}$ is a γ_0-weak learner for some class $\mathscr{H}$ that requires $m_{\mathscr{H}}(\delta_0)$ examples, namely in the PAC learning scenario:

Input $S = \{(x_1, y_1), \cdots, (x_m, y_m)\}$, where for each i, $y_i = f(x_i)$

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- Problem raised by *Kearns* and *Valiant* in 1988
- Solved in 1990 by then MIT student, Robert *Schapire*; In 1995 improved into AdaBoost by *Schapire* and *Freund*

AdaBoost Algorithm

Input Training set $S = \{(x_1, y_1), \cdots, (x_m, y_m)\}$, a weaker learner WL, number of rounds T

Output $h_S(x) = sign(\sum_{t=1}^{T} \omega_t h_t(x))$

AdaBoost Algorithm

Input Training set $S = \{(x_1, y_1), \cdots, (x_m, y_m)\}$, a weaker learner WL, number of rounds T

Output $h_S(x) = sign(\sum_{t=1}^{T} \omega_t h_t(x))$

Algorithm.

- Initialize $D^{(1)} = (\frac{1}{m}, \cdots, \frac{1}{m})$

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  - invoke weak learner $h_t = WL(D^{(t)}, S)$
  - compute $\epsilon_t = L_{D^{(t)}}(h_t) = \sum_{i=1}^{m} D_i^{(t)} 1_{[y_i \neq h_t(x_i)]}$
  - let $\omega_t = \frac{1}{2} \log(\frac{1}{\epsilon_t} - 1)$
  - update $D_i^{(t+1)} = \frac{D_i^{(t)} e^{-\omega_t y_i h_t(x_i)}}{\sum_{j=1}^{m} D_j^{(t)} e^{-\omega_t y_j h_t(x_j)}}$ for all $i = 1, \ldots, m$

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□

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Intuition. AdaBoost forces WL to focus on problematic examples

- The error of h_t with respect to $D^{(t+1)}$ is exactly 1/2.

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Intuition. AdaBoost forces WL to focus on problematic examples

- The error of h_t with respect to $D^{(t+1)}$ is exactly 1/2.
  - Considering the fact that:

$$\sum_{i=1}^{m} D_i^{(t+1)} 1_{[y_i \neq h_t(x_i)]} = \frac{\sum_{i=1}^{m} D_i^{(t)} e^{-\omega_t y_i h_t(x_i)} 1_{[y_i \neq h_t(x_i)]}}{\sum_{j=1}^{m} D_j^{(t)} e^{-\omega_t y_j h_t(x_j)}} = \frac{e^{\omega_t} \epsilon_t}{e^{\omega_t} \epsilon_t + e^{-\omega_t}(1 - \epsilon_t)}$$

$$= \frac{\epsilon_t}{\epsilon_t + e^{-2\omega_t}(1 - \epsilon_t)} = \frac{\epsilon_t}{\epsilon_t + \frac{\epsilon_t}{1-\epsilon_t}(1 - \epsilon_t)} = \frac{1}{2}$$

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Let S be a training set and assume that at each iteration of AdaBoost, the γ-weak learner $\mathscr{A}$ returns a hypothesis for which $\epsilon_t \leq 1/2 - \gamma$, then the training error of the output hypothesis of AdaBoost is at most

$$L_S(h_S) = \frac{1}{m}\sum_{i=1}^{m} 1_{[h_s(x_i)\neq y_i]} \leq e^{-2\gamma^2 T}$$

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- For any $\epsilon > 0$ and $\gamma \in (0, 1/2)$, if $T \geq \frac{\log(1/\epsilon)}{2\gamma^2}$, then AdaBoost will output a hypothesis h_S with $L_S(h_S) \leq \epsilon$.

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- Since the weak learner is involved on a distribution over S, in many cases δ can be 0. In any case, by "Boosting the confidence", we can assume without loss of generality that δ is very small.
- How about $L_D(h_S)$? Need to check VC Dimension.

□

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- Denote $\psi(x) = (g_1(x), \cdots, g_d(x))$, we have

  $$L(B,T) = \{x \mapsto sign(< \omega, \psi(x) >) : \omega \in \mathscr{R}^d, ||\omega||_0 \leq T\}$$

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- That is, AdaBoost learns a composition of the class of halfspaces with sparse coefficients over the mapping $x \mapsto \psi(x)$.

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Example. Suppose $X = \mathscr{R}$ and B is Decision Stumps:

$$B = \{x \mapsto sign(x - \theta) \cdot \omega : \theta \in B, \omega \in \{\pm 1\}\}$$

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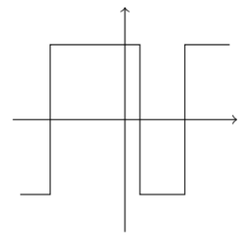

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Let B be a base class with VC dimension $VCDim(B) = d$, and let $L(B,T)$ be: $L(B,T) = \{x \mapsto sign(\sum_{t=1}^{T} \omega_t h_t(x)) : \omega \in \mathscr{R}^T, \forall t, h_t \in B\}$. Assume that both T and d are at least 3, then

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- Therefore, the parameter T of AdaBoost enables us to control the bias-complexity tradeoff.

□

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Proof.

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  Namely, $(em/d)^{dT} \times (em/T)^T \leq m^{(d+1)T}$

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- Since we assume C is shattered, so we have $2^m \leq (em/d)^d T \times (em/T)^T \leq m^{(d+1)T}$
- m can not be too big: $m \leq \log(m)\frac{(d+1)T}{\log(2)}$

□

Quiz

The VC Dimension

- Let $\mathcal{H}_1,\ldots,\mathcal{H}_r$ be hypothesis classes over some fixed domain set $\mathcal{X}$. Let $d = max_i VCDim(\mathcal{H}_i)$ and assume for simplicity that $d \geq 3$.

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Questions?

Contact Information

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