Mean Reversion: A New Approach

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Mean reversion - A new approach

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1 Introduction

Mean reversion is an all-time favourite of quantitative traders and has been used in trading strategies for the equities, fixed income, FX and commodities markets. The basic idea is perhaps best expressed by examining stock prices. Stock markets follow a "staircase" like curve that looks like the superposition of a piecewise continuous curve and an oscillatory, quasi-periodic process where the price evolution looks like a “rugged” staircase which sometimes goes up and sometimes goes down as illustrated by the figure 1 below for the Nasdaq index:

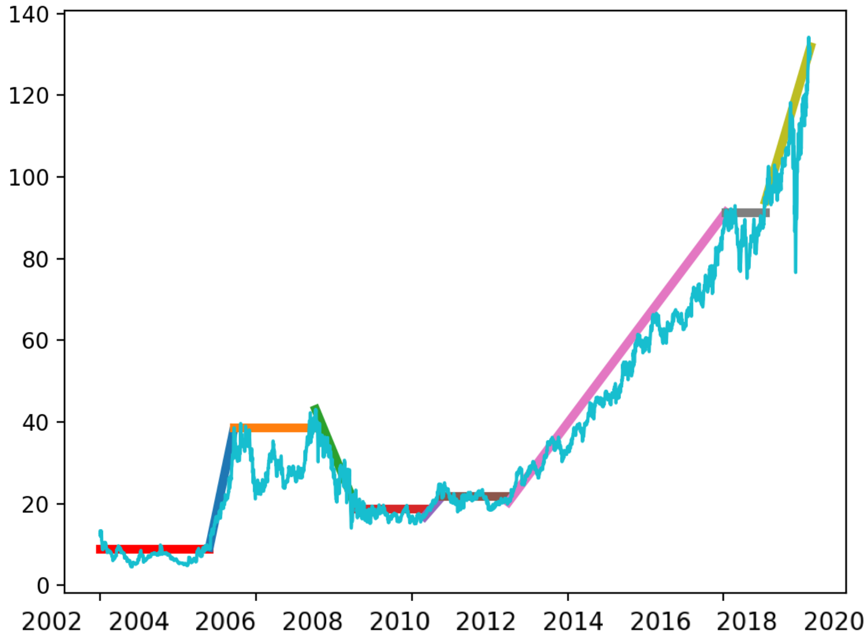


Figure 1: Nasdaq historical prices

In figure 1 above, the "staircase" we are referring to is shown by the straight lines in different colors. Overall the “staircase” points up and this is natural due to inflation and natural growth. If we subtract the “staircase” from the Nasdaq in the graph above we get the following graph:

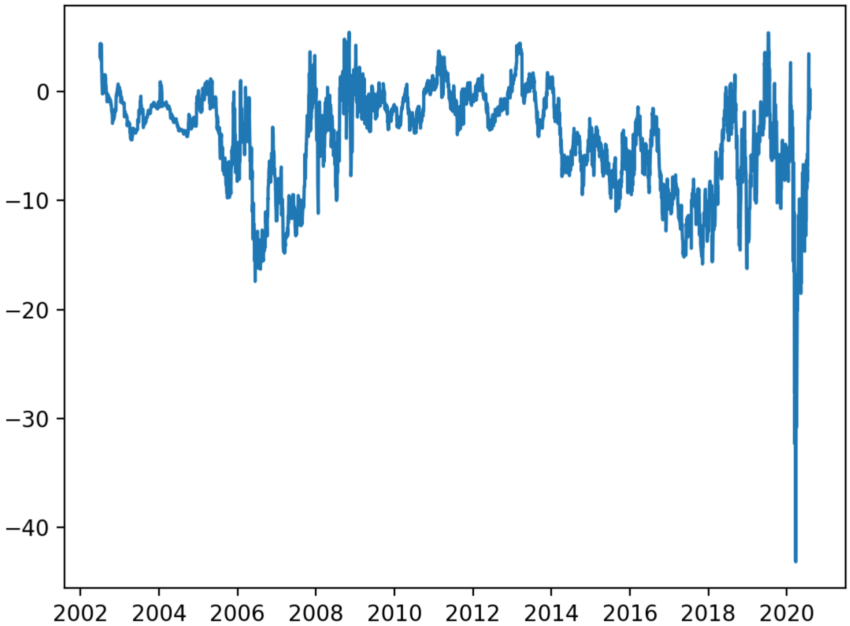


Figure 2: Nasdaq de-trended

We note that this second graph looks somewhat chaotic except for an interesting feature: it looks almost periodic. This almost-periodicity is called “mean reversion” and is typically modeled via the Ornstein-Uhlenbeck process, who first developed the model in physics to account for the motion of a Brownian particle under the effect of friction.

Why do we see this kind of behaviour? There are a thousand theories and although we do not propose to go through these in any detail, a cursory look is instructive. Trends (as illustrated by figure 1) usually occur because of:

- Macro changes across the entire economy, such as the one the world is currently undergoing with the COVID-19 pandemic
- Changes in the specific industry of the company's whose stock is being traded (e.g. the introduction of 3G licences in the early 2000's)
- Herd behaviour (almost any financial crisis in the past 50 years would be a fitting example)
- Exogenous factors such as political turmoil (9/11 terror attacks) or natural events (2004 tsunami)
- Corporate events (earnings forecasts, mergers and acquisitions, share buybacks etc.)
- Substantive changes in monetary or fiscal policy by central banks

The list is not exhaustive but it showcases some of the potential explanatory factors for the behaviour of stock prices. With the exception of herd behaviour, all of the above factors are discrete in nature: we do not have new and impactful news on the economy or on specific companies or industrial sectors every day and, mercifully, we do not have major wars or climatic calamities every other month!

But prices change neverthless: this is reflected by the mean reversion seen in figure 2. Why does it happen? The reason may be deceptively simple: few market agents have long trading horizons and as such, tend to take profit or stop loss on their positions within a short time period. Since most traders have access to substantially similar information, if it so happens that the market is in an uneventful period, there will be as many buyers as sellers. Putting this together gives a behaviour that closely mimics the mean reversion we observe in the market.

During this quasi-periodic part, statistical arbitrage takes place. This basically exploits price mean reversion: when the price is considerably above its expected mean, one sells in the expectation that this price will drop and vice-versa. But the devil is in the detail, namely, where is the mean and when exactly are we "considerably above (or below) the mean"?

Statistical arbitrage is old and widely practised but it is far from being completely understood or mastered. The literature on mean reversion is vast. In references [2, 6, 8, 18], the basics of mean reversion and its applications in various markets are discussed. Extensions to the basic affine framework are discussed in [5,12] where solvable mean reverting models with nonlinear drift are presented while in [14] the nonlinearity is in the diffusion (Brownian motion) term. More recently with the advent of real life use cases of artificial intelligence, the use of neural networks in trading has also become extensive: in [3], reinforcement learning methods are applied to pairs trading strategies, the precursors of most statistical arbitrage trades. In [15], neural networks are used to optimise trading strategies when the underlying return dynamics are given by an Ornstein-Uhlenbeck process while in [16], the impact of transaction costs is explicitly taken into account. In [1], statistical arbitrage based on mean reversion for the US equity markets using PCA factors and sector ETFs is discussed.

In this paper we start by recalling recent approaches to statistical arbitrage and mean reversion and then proceed to present a new approach that combines model-independence (through the use of neural networks) with information theory.

2 Mean reversion - the basics

Typically, practitioners approach mean reversion by modeling stock returns using an Ornstein-Uhlenbeck model (known in the fixed income world as the Vasicek model) or variants thereoff. For instance if we denote by $P_t^{(i)}$ the price of the i^{th} security, a mean reversion model would be:

$$x_{t+1}^{(i)} - x_t^{(i)} = \kappa^{(i)} \left(\mu^{(i)} - x_t^{(i)} \right) + \sigma_i \epsilon_{t+1}^{(i)} \tag{1}$$

where the beta hedged portfolio is defined as:

$$x_t^{(i)} = P_t^{(i)} - \sum_{\alpha} \Lambda_{i\alpha} F_t^{(\alpha)} - \eta_i \tag{2}$$

The factors $F_t^{(\alpha)}$ are usually taken to represent market beta, e.g. the returns on sector ETFs so that equation (1) can be viewed as the cumulative effect of a market wide trend and a stock-specific (idiosyncratic) component. The inclusion of these factors is essential since otherwise, the trends typically seen over sustained periods cannot be captured. The process in (2) is mean reverting and allows us to calculate the idiosyncratic return:

$$R_{t+1}^{(i)} = \kappa^{(i)} \left(\mu^{(i)} - x_t^{(i)} \right) + \sigma_i \epsilon_{t+1}^{(i)} \tag{3}$$

Mean reversion in the context of the process in (2) is usually understood as follows. The variables $\epsilon_t^{(i)}$ for t > 0 are generally assumed to be $\mathcal{N}(0, 1)$ and uncorrelated. This means that, taking the expected value (2) gives:

$$\rho_{t+1}^{(i)} - \rho_t^{(i)} = \kappa^{(i)} \left(\mu^{(i)} - \rho_t^{(i)} \right) \tag{4}$$

where:

$$\rho_t^{(i)} = \mathbb{E}\left[x_t^{(i)}\right] \tag{5}$$

which solves out to:

$$\rho_t^{(i)} = \left(x_0^{(i)} - \mu^{(i)}\right) e^{-\kappa^{(i)} t} + \mu^{(i)} \tag{6}$$

As $t \to \infty, \rho_t \to \mu^{(i)}$, i.e. to a long term equilibrium value: we say that the variable $x_t^{(i)}$ reverts back to it's mean. If $\kappa^{(i)}$ is large enough, we see that, regardless of the value of $x_0^{(i)}$, reversion will happen quickly. Plugging this back into (1) we see that the return on the portfolio consisting of $S_t^{(i)} - \sum_\alpha \Lambda_{i\alpha} F_t^{(\alpha)}$ will revert to $\mu^{(i)}$ quickly. This presents a trading opportunity: if the current value of the signal $x_t^{(i)}$ is below $\mu^{(i)}$, we go long and vice-versa since we know we don't need to wait for long before the return goes back to $\mu^{(i)}$: this is the essence of mean reversion.

Before discussing the estimation of the signal $x_t^{(i)}$, let's recapitulate explicitly the two assumptions needed for mean reversion to work:

1. the errors $\epsilon_t^{(i)}$ are decorrelated with a mean of zero and
2. the mean $\rho_t^{(i)} = \mathbb{E}\left[x_t^{(i)}\right]$ has to tend to some fixed equilibrium value

When these assumptions are met the problem reduces to determining the values of the mean reversion parameters $\Lambda_{i\alpha}$ and b_i. A brute force approach is the basic regression model where the mean reversion parameters are calculated by minimising the overall error over some sample:

$$\epsilon^2 = \sum_{t=1}^{T}\sum_{i=1}^{N}\left(\frac{P_{t+1}^{(i)} - P_t^{(i)}}{P_t^{(i)}} - \sum_\alpha \Lambda_{i\alpha}\frac{F_{t+1}^{(\alpha)} - F_t^{(\alpha)}}{F_t^{(i)}} - \eta_i\right)^2 + \sum_{t=1}^{T}\sum_{i=1}^{N}\left(x_{t+1}^{(i)} - x_t^{(i)} - \kappa^{(i)}\left(\mu^{(i)} - x_t^{(i)}\right)\right)^2 \tag{7}$$

As the factors F may be highly correlated, a Tikhonov regularisation approach (sometimes also known as a "ridge regression" in this context) is typically used where instead of minimising (7) one adds a penalty term:

$$\mathcal{L} = \epsilon^2 + \lambda\left(\|\Lambda\|^2 - 1\right) \tag{8}$$

Alternatively, we can first determine Λ and η through a linear regression between the stock annd the factors and then estimate equation (1) separately. Whatever the method used, the resulting residuals are then used to detect potential trades.

The specific trading rule obviously depends on individual preferences but the typical elements that go into the trade decision are:

1. if the current signal is above or below a certain "threshold" number of standard deviations from the mean (where such thresholds depend on the investor's risk aversion):

$$\kappa^{(i)}\left(\mu^{(i)} - x_t^{(i)}\right) > n_{\text{sell}}\sigma_i \tag{9}$$

$$\kappa^{(i)}\left(\mu^{(i)} - x_t^{(i)}\right) < n_{\text{buy}}\sigma_i \tag{10}$$

2. another parameter of importance in setting mean reversion based trades is the typical time before a deviation from the mean is corrected. This is given by the first passage time of the Ornsetin Uhlenbeck process (see [13]). The exact expression in [13] is complicated and semi-analytical but, to gain some intuition, we can consider the first passage time of the error $\epsilon_t^{(i)}$, i.e. we are asking for the *typical* value of the variable:

$$\hat{\tau}\left(t, \epsilon_t^{(i)}\right) = \text{argmin}_{s>0}\left(\epsilon_{t+s}^{(i)} = 0\right) \tag{11}$$

For Brownian motion (which is typically assumed in the context of stochastic calculus), this can be calculated as:

$$f^{(i)}(t) = \text{Prob}\left(\hat{\tau}\left(t, \epsilon_t^{(i)}\right) = t\right) = \frac{\left|\epsilon_t^{(i)}\right|}{\sigma_i\sqrt{2\pi t^3}} \exp\left(-\frac{\epsilon_t^{(i)2}}{2\sigma_i^2}\right) \tag{12}$$

From the above distribution, the typical first passage time, defined by $f'(t) = 0$ is given by:

$$\tau\left(t, \epsilon_t^{(i)}\right) = \frac{\epsilon_t^{(i)2}}{3\sigma_i^2} \tag{13}$$

Equation (13) shows that, as intuitively expected, the typical mean reversion time increases the further we are from the mean and decreases the more volatile the stock is.

Approaches based on mean reversion discussed previously suffer from two problems. First of all, one assumes that the "error" $\epsilon_t^{(i)}$ in the idiosyncratic return is normally distributed. This assumption, although sometimes valid, has no a priori justification. The reasons why this assumption may fail are many, not least of which the asymmetry in the behaviour of market participants during bull and bear markets and the impact of liquidity. Furthermore, as we will discuss later, the standard approach to calculating the parameters of the mean reversion in (1) is not geared towards insuring that the resulting error is Gaussian as assumed.

The second problem is that these approaches are model-dependent: they assume that the model representing the beta-hedged portfolio returns is affine. This last assumption may sometimes be justified if we interpret $P_{t+1}^{(i)}$ as the price the next minute or maybe the next hour: this is so because, if we look closely enough, any curve looks like a straight line over very short intervals of time. But this starts to fail for longer periods of time and does not adequately capture long term correlations between prices. A less model dependent approach that tries to "learn" the data instead of coercing it to a pre-determined model would, at least conceptually, be more appealing. Model risk has been discussed in the literature (see references [4,7,10,17]), and some approaches to mitigation have been proposed but most of the efforts are around the risk inherent in the pricing of securities and financial instruments. Other statistical models used in finance, such as mean-reversion trading, have received less attention.

3 Ditching the model

In this paper we propose an alternative approach to statistical arbitrage. Our first point of call is to deal with *model risk* by using a neural network as a universal approximator: neural networks with a single hidden layer are universal approximators, i.e. any function can be expressed to an arbitrary degree of accuracy by such a network. Put simply, if we specify a generic form for modeling asset returns:

$$x_{t+1}^{(i)} - x_t^{(i)} = Q\left(x_t^{(1)}, .., x_t^{(N)}\right) + \sigma_i \epsilon_{t+1}^{(i)} \tag{14}$$

where Q is some unknown function, the universal approximation theorem (see [9]) says that this function can be "represented" as

$$Q\left(x_t^{(1)}, .., x_t^{(N)}\right) = \sum_{\alpha=1}^{H} h_{i\alpha} f\left(\sum_{j=1}^{N} W_{\alpha j} x_t^{(j)} + b_\alpha\right) + \sigma_i \epsilon_{t+1}^{(i)} \tag{15}$$

where the matrices W and h are known as weights while the vector b is known as the bias, and the function f is the activation. In this paper, the number of factors used is chosen to correspond to the number of sector ETFs while H, the number of "hidden neurons" represents the architecture of the model:

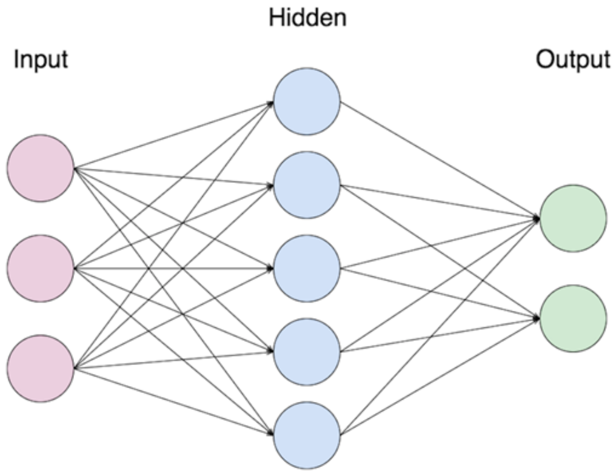


Figure 3: Generic neural network architecture

In figure 3, the architecture, which refers to the specific connections between the different layers of the network; is that of a "plain vanilla" feed forward network. Many other architectures are possible such as convolutional neural networks, LSTM, transformer networks etc. The financial applications of these different architectures are manifold but in this paper, we shall only explore the use of feedforward networks as in equation (15).

Note that in the case of a linear activation, we recover equation (1) which is the basic affine model of mean reversion. Here however, the activation function used will be $f(x) = \tanh(x)$. Remembering that equation (1) is the discretised version of a differential equation we have, under certain conditions, a long term mean since:

$$\rho_t^{(i)} = \mathbb{E}\left[x_t^{(i)}\right] \tag{16}$$

and hence:

$$\sum_{\alpha=1}^{H} h_{i,\alpha} f(\sum_{j=1}^{N} W_{\alpha,j} x_t^{(j)} + b_\alpha) = 0 \tag{17}$$

Since most activation functions used have the property $f(0) = 0$, the long term mean is characterised by:

$$\sum_{j=1}^{N} W_{\alpha j} x_\infty^{(j)} + b_\alpha = 0 \tag{18}$$

The condition of convergence for the differential form of (15) is given by the Picard-Lindelof iteration which basically requires the mapping on the RHS of (15) to be contractive, i.e.:

$$|h(f(Wx+b) - f(Wy+b))| < k|x-y| \tag{19}$$

where $| \, . \, |$ denotes some vector norm.
If the activation itself is a contractive mapping or at least Lipschitz, we will get (using the Cauchy inequality):

$$\begin{aligned} \|h(f(Wx+b) - f(Wy+b))\| &\leq \|h\| \, \|(f(Wx+b) - f(Wy+b))\| \\ &\leq \rho \, \|h\| \, \|W(x-y)\| \\ &\leq \rho \, \|h\| \, \|W\| \, \|(x-y)\| \end{aligned} \tag{20}$$

which implies the constraint:

$$\rho \, \|h\| \, \|W\| < 1 \tag{21}$$

Equation (21) is basically a regularisation condition that must be imposed on the model to ensure it behaves well asymptotically. Such regularisation conditions are familiar in training neural networks and serve to increase generalisation capacity for out-of-sample predictions by "clipping" spurious weights and nodes from the architecture. In addition to this condition, dropout and drop connect methods can be used to get more robust results.

In the neural network literature, determining the weight and bias parameters in equation (15) is usually done by applying some variation on gradient descent method e.g., Stochastic Gradient Decsent (SGD) or adaptive moment minimisation (ADAM [11]) to minimise the in-sample/training error:

$$\epsilon^2 = \sum_{t=1}^{T} \sum_{i=1}^{N} \left(x_{t+1}^{(i)} - x_t^{(i)} - \sum_{\alpha=1}^{H} h_{\mathrm{i}\alpha} f \left(\sum_{j=1}^{N} \Lambda_{\alpha j} x_t^{(j)} + b_\alpha \right) \right)^2 \tag{22}$$

and the "correctness" of the results is then measured by the size of the error on the out-of sample testing set.

4 Built-in mean reversion

As noted earlier, one of the weaknesses of traditional approaches to mean reversion is the assumption that the idiosyncratic residual is basically white noise: a series of uncorrelated, zeero mean random variables. White noise is characterised by a flat power spectrum where the power spectrum is defined as the magnitude of its Fourier transform:

$$\Phi \left[d\hat{s}_t \right] (\omega) = \left| \int_{-\infty}^{\infty} d\hat{s}_t e^{i\omega t} \right|^2 \tag{23}$$

To intuitively understand (16), we recall that the Fourier transform of a signal is a decomposition of the latter into periodic/sinusoidal waves with given amplitudes. When the amplitudes for all possible frequencies are equal, we have have a flat power spectrum: loosely speaking, it means that there is no perceivable regularity in the signal.

Going back to equation (15), the calibration procedure by which the quantities Λ, h and b are obtained from the observed idiosyncratic return $\epsilon_{t+1}^{(i)}$ must have the power spectrum of white noise. This requirement can be fullfilled by minimising the following loss function:

$$p_n^{(i)} = \left| \sum_{t=1}^{T} \left(x_{t+1}^{(i)} - x_t^{(i)} - \sum_{\alpha=1}^{H} h_{\mathrm{i}\alpha} f \left(\sum_{j=1}^{N} \Lambda_{\alpha j} x_t^{(j)} + b_\alpha \right) \right) e^{2\pi i n t/T} \right|^2 \tag{24}$$

with the loss function:

$$\mathcal{L} = -\sum_{i,n} p_n^{(i)} \log p_n^{(i)} \tag{25}$$

and where the parameters are obtained as:

$$\Lambda_*, h_*, b_* = \mathrm{argmin}_{\Lambda,h,b} \mathcal{L} \tag{26}$$

Equation (24) is the power spectrum of the error on the beta-hedged portfolio corresponding to the i^{th} stock. Equation (25) says that we would like to maximize the Shannon information in the power spectrum. Since for any finite trading horizon T, $\sum_{n=1}^{T} p_n^{(i)}$ is bounded we deduce that when the information (25) is maximal, the spectrum is flat, i.e. corresponds to the requirement of mean reversion.

Traditionally, instead of the loss function in (25), one minimises the RMS distance:

$$d^2 = \sum_{t=1}^{T} \left(x_{t+1}^{(i)} - x_t^{(i)} - \sum_{\alpha=1}^{H} h_{\mathrm{i}\alpha} f \left(\sum_{j=1}^{N} \Lambda_{\alpha j} x_t^{(j)} + b_\alpha \right) \right)^2 \tag{27}$$

Minimising (27) however, only guarantees that the first moment (average) of the error will be zero and that the standard deviation will be minimal but it does not guarantee any other properties of the distribution of errors nor does it automatically say anything about their dependence structure.

5 Trading Strategy

Using the results from the preceding section, we would like to generate a trading signal akin to the one in equations (9) and (10). However, we need to note that, in arriving to equations (9) and (10), the parameters have been estimated in a way to make sure that $\mathbb{E}[\epsilon_t] = 0$. The approach outlined in the previous section does not guarantee this last equation since it maximises entropy rather than minimises the mean square error. We therefore have to adapt our decision rule to reflect this as follows:

$$\sum_{\alpha=1}^{H} h_{\mathrm{i}\alpha} f\left(\sum_{j=1}^{N} \Lambda_{\alpha j} x_t^{(j)} + b_\alpha\right) \geq \omega^{(i)} + n_{\mathrm{buy}}\sigma^{(i)} \tag{28}$$

or:

$$\sum_{\alpha=1}^{H} h_{\mathrm{i}\alpha} f\left(\sum_{j=1}^{N} \Lambda_{\alpha j} x_t^{(j)} + b_\alpha\right) \leq \omega^{(i)} - n_{\mathrm{sell}}\sigma^{(i)} \tag{29}$$

where $\omega^{(i)}$ is the (arithmetic) mean of the error. Put simply, equations (28-29) tell us that if the signal is large enough, we should buy, else we should sell !

The specific thresholds, as discussed earlier, are a function of the risk aversion of individual investors and, possibly, of the specific stocks and ETFs being traded. The amount traded can, in principle, be a function of the signal strength and other important variables such as trade volume. In this paper, our focus is really on measuring the impact on statistical arbitrage strategies that comes from ditching pre-specified models (by using neural networks as discussed previously) and changing the way in which parameters are estimated. Therefore, we deem that optimising thresholds or allocations may "muddy the waters" in terms of understanding the efficacy of the outlined approach. As such, we have chosen to use a flat allocation where the same amount is traded regardless of the signal strength and $n_{\mathrm{buy}} = n_{\mathrm{sell}} = 2$.

6 Backtesting

To back test the trading strategy, we have used the following sector ETFs:

XLF	Financial SPDR
XLE	Energy SPDR
VONG, QQQ	Technology SPDR
XBI, IBB	Biotech
VGT	Information technology
VNQ	Realestate
XLV	Healthcare
HHH	Internet
IYR	Realestate
IYT	Transportation
XLP	Consumer staples
XLY	Consumer discretionary
XLI	Industrial
OIH	Oilservices

Table 1. List of ETF factors used

Following [1], we used actual as opposed to synthetic ETFs for backtesting as of 2005. The data was obtained from Yahoo Finance and missing values were linearly interpolated. The dataset was divided 50/50 into training and testing datasets. No attempt was made to evaluate performance on separate historical patches on purpose: we do not want to feed the approach with knowledge gained in hindsight. We also do not take transaction costs, repo rates (for short selling) or borrowing rates into account in this paper.

Unlike linear regression analysis, calibrating the parameters of a neural network has no analytic solution and as such is a numerical gradient-descent based procedure. Such procedures rarely lead to a global optimum and frequently end up in different local minima upon repetition. We deal with this by averaging the signal over multiple runs (we have used 1000 such runs in this paper) which leads to a fairly stable outcome.

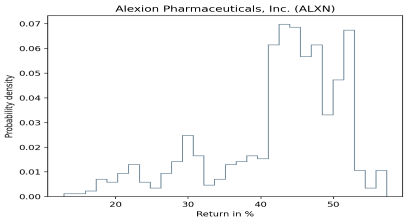
The trading strategy's P&L is summarised by:

$$\Pi_t^{(i)} = s_t^{(i)} \frac{\left(x_{t+\tau}^{(i)} - x_t^{(i)}\right)}{P_t^{(i)}} \tag{30}$$

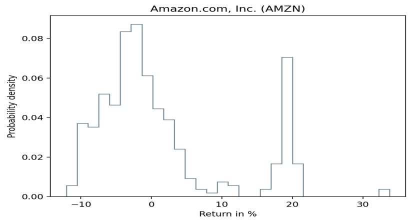
where $x_t^{(i)}$ is given by equation (2), $P_t^{(i)}$ is the price of the specific stock being traded and $s_t^{(i)}$ is the signal. The exit time $t+\tau$ is defined as the first time the signal reverses sign:

$$\tau = \mathrm{argmin}_{\theta=1,\ldots,\infty} \delta\left(s_{t+\theta}^{(i)} = -s_t^{(i)}\right) \tag{31}$$

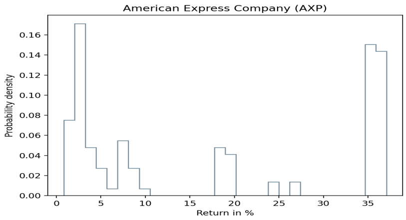
In figure 4 below we show the histogram of returns for a few selected companies in the Biotech, Tech, Financials and Industrial sectors:



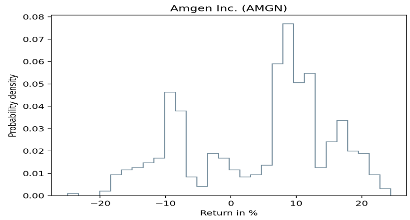
(a) Alexion Pharmaceuticals, Inc.



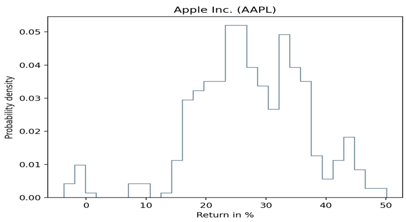
(b) Amazon.com, Inc.



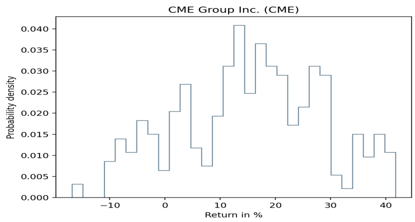
(c) American Express Company



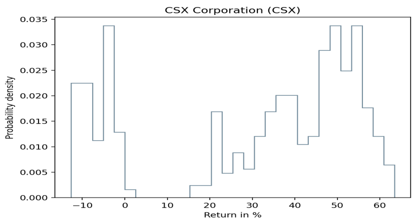
(d) Amgen Inc.



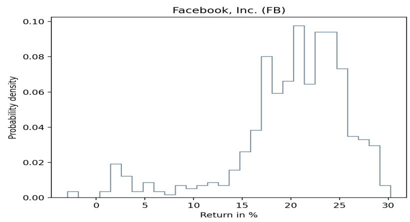
(e) Apple Inc.



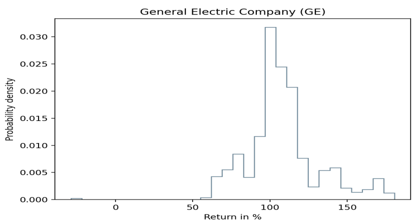
(f) CME Group Inc.



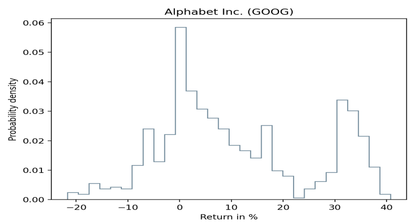
(g) CSX Corporation



(h) Facebook, Inc.



(i) General Electric Company

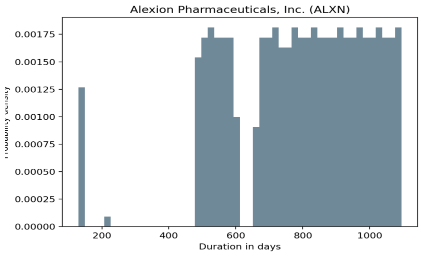


(j) Alphabet Inc.

Figure 4: Plots of mean reversion strategy returns

All of these histograms show returns that are considerably positively skewed. Also, for most companies examined, the percentage of correct trade decisions, i.e. decisions resulting in a positive P&L, 'sell' decisions were more likely to be correct than buy decisions.

However, the holding time betweeen trades is very long as figures 5 below show:



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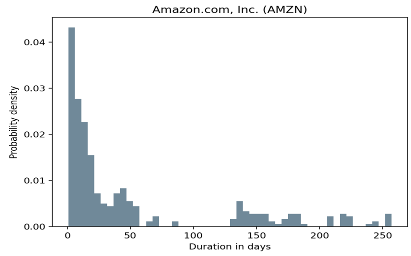

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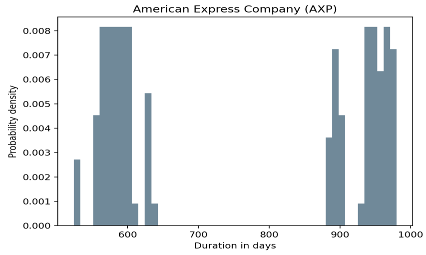
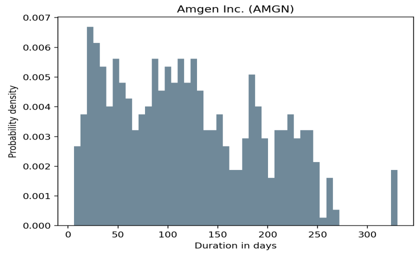
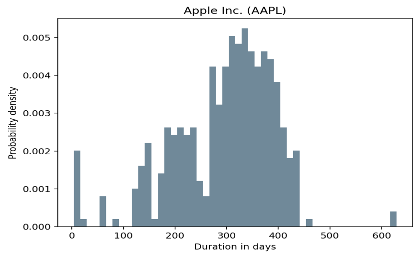


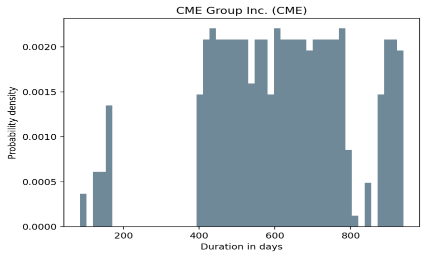
(c) American Express Company



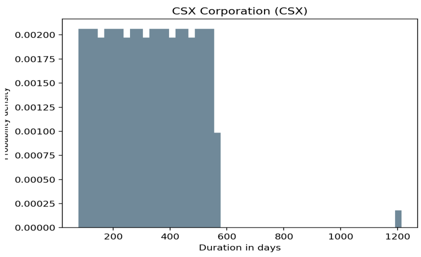
(d) Amgen Inc.



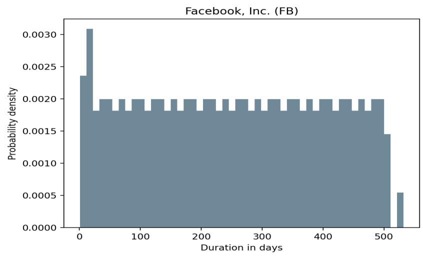
(e) Apple Inc.



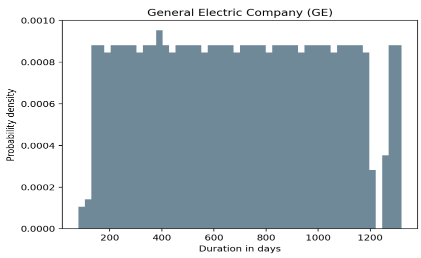
(f) CME Group Inc.



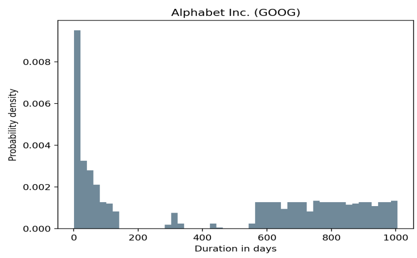
(g) CSX Corporation



(h) Facebook, Inc.



(i) General Electric Company



(j) Alphabet Inc.

Figure 5: Plots of mean reversion strategy durations

The reason for the long holding time is the thresholds in equations (28) and (29): these are quite high and thererfore necessitate long waiting times before unwinding the trades becomes "safe". Finding optimal bounds for these thresholds is in itself the topic of a different study and is quite likely to be specific to the stocks being traded.

7 Conclusions

In this paper, we have explored the impact of using model independent estimators for stock returns and calibrating their parameters in a way that guarantees mean reversion. The results are quite encouraging and suggest future work that focuses on:

- Optimising the architecture of the neural network used
- Finding a more scientific way of determining appropriate threresholds for trading
- Examining the impact of using a non-Markovian approach where current returns depend on longer histories of prices and returns.

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