Appendix 2: Proof of Spectral Decomposition

The fundamental theorem of calculus implies that for any fixed F:

$$\begin{aligned}
f(S) &= f(F) + 1_{S>F}\int_F^S f'(u)du - 1_{S<F}\int_S^F f'(u)du \\
&= f(F) + 1_{S>F}\int_F^S \left[f'(F) + \int_F^u f''(v)dv\right]du \\
&\quad -1_{S<F}\int_S^F \left[f'(F) - \int_u^F f''(v)dv\right]du.
\end{aligned}$$

Noting that $f'(F)$ does not depend on u and applying Fubini's theorem:

$$f(S) = f(F) + f'(F)(S-F) + 1_{S>F}\int_F^S\int_v^S f''(v)dudv + 1_{S<F}\int_S^F\int_S^v f''(v)dudv.$$

Performing the integral over u yields:

$$\begin{aligned}
f(S) &= f(F) + f'(F)(S-F) + 1_{S>F}\int_F^S f''(v)(S-v)dv + 1_{S<F}\int_S^F f''(v)(v-S)dv \\
&= f(F) + f'(F)(S-F) + \int_F^\infty f''(v)(S-v)^+dv + \int_0^F f''(v)(v-S)^+dv. \qquad (0.6)
\end{aligned}$$

Setting $F = S_0$, the initial stock price, gives Theorem 1. Note that if $F = 0$, the replication involves only bonds, stocks, and calls:

$$f(S) = f(0) + f'(0)S + \int_0^\infty f''(v)(S-v)^+dv,$$

provided the terms on the right hand side are all finite. Similarly, for claims with $\lim_{F\uparrow\infty} f(F)$ and $\lim_{F\uparrow\infty} f'(F)F$ both finite, we may also replicate using only bonds, stocks, and puts:

$$f(S) = \lim_{F\uparrow\infty} f(F) + \lim_{F\uparrow\infty} f'(F)(S-F) + \int_0^\infty f''(v)(v-S)^+dv.$$