Stochastic Overlapping Generation Model Learning Note *

Daniel Tulips Liu † Hans Fehr ‡ Fabian Kindermann §

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Abstract

This paper learns how we can establish a recursive equilibrium. It states that households encounter idiosyncratic shocks in asset market **in which** z-shock, xi-shock and its transition function –a Markov process with invariant transition probability. It is worth noting that this paper had been written by Sevon Hur in 2012.

JEL Classifications: E24, J63, D31

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†Chongqing Technology and Business University,Chongqing, China. Texas Station University, U.S.A. Email address: xudongliu520@vip.qq.com.

‡University of Konstanz, Deutschland.

§Goethe University Frankfurt, Deutschland.

1 General structure and long-run equilibrium

In the following, we extend the life-cycle model with variable labor supply from Section 10.1.2 to a full general equilibrium setup with describes the theoretical structure of the model, how we implement it, and how calibrate parameters. Eventually, we conduct some long-run analysis of policy reforms. Many parts of this section will already sound familiar. Individual decision-making works (almost) exactly like in the previous chapter. The macroeconomic setup, on the other hand, is quite similar to that in Chapter 6 or 9. Hence, we will be brief on things we have discussed before, but as detailed as necessary to paint in a full picture of the model.

1.1 Demographic, Behavior, and Markets

Demographics At each point in time t, the economy is populated by J overlapping generations indexed by $j = 1, \dots, J$. Individuals live up to age J and die with certainty afterwards. Hence, in contrast to Chapter 10, there is no uncertain survival. Cohort sizes grow over time at the constraint rate n_p. Let N_t denote the size of the cohort that enters the labor market at time t, then we have

$$N_t = (1 + n_p)N_{t-1}.$$

As the population size is growing over time at rate n_p, on a balanced grow path all aggregate variables grow at rate n_p. As in chapter 6 and 7, we therefore normalize aggregate variables at time t by the size of the youngest cohort living in this period. On a balanced growth path these normalized aggregates are then constant. Note that since population growth is constant over time, so are the relative population shares

$$m_j = \frac{N_{t-j+1}}{N_t} = (1 + n_p)^{1-j}.$$

Preferences and labor productivity risk Individuals have preferences over streams of consumption $c_{j,t}$ and leisure $\ell_{j,t}$. Like in the earlier chapters, we assume a time endowment of 1. With $l_{j,t}$ denoting the amount of labor hours supplied to the market in period t, we have $\ell_{j,t} + l_{j,t} = 1$. The utility function of the household reads

$$U_0 = E\left[\sum_{j=1}^{J} \beta^{j-1} u(c_{j,t}, 1 - l_{j,t})\right], \tag{1}$$

where the expectation is from an ex ante perspective, i.e. before any information about labor productivity has been revealed to the individual. We again use the specific preferences

$$u(c_{j,t}, 1 - l_{j,t}) = \frac{\left[c_{j,t}^{\nu}(1 - l_{j,t})^{1-\nu}\right]^{1-\frac{1}{\gamma}}}{1 - \frac{1}{\gamma}}.$$

Individuals differ with respect to their labor productivity $h_{j,t}$, which depends on a (deterministic) age profile of earnings e_j, a fixed productivity effect θ that is drawn at the beginning of the life cycle, and an autoregressive component $\eta_{j,t}$ that evolves over time. At the mandatory retirement age j_r, labor productivity falls to zero and households receive a flat pension benefit $pen_{j,t}$ computed as a fraction κ of average labor income in period t. Households maximize the expected utility function (1) subject to the (periodical) budget constraints

$$a_{j+1,t} = (1 + r_t^n)a_{j,t} + w_t^n h_{j,t} l_{j,t} + pen_{j,t} - p_t c_{j,t} \tag{2}$$

where $w_t^n = w_t(1 - \tau_t^w - \tau_t^p), r_t^n = r_t(1 - \tau_t^r)$, and $p_t = 1 + \tau_t^c$ denote the net wage rate, the net interest rate, and the consumer price, respectively. $\tau_t^c, \tau_t^w, \tau_t^p$, and τ_t^r are consumption, labor-income, payroll, and capital-income tax rates. Finally, the household has to respect a non-negativity constraint on savings $a_{j+1,t} \geq 0$ at all ages in every period.

The dynamic programming problem The optimization problem of households reads

$$V_t(z) = \max_{c,l,a^+} u(c, 1-l) + \beta E\left[V_{t+1}(z^+)\middle|\eta\right]$$
$$\text{s.t.}\quad a^+ + p_t c = (1+r_t^n)a + w_t^n h l + pen, \quad a^+ \geq 0, l \geq 0 \tag{3}$$
$$\text{and}\quad \eta^+ = \rho\eta + \epsilon^+ \quad \text{with} \quad \varepsilon^+ \sim N(0, \sigma_\epsilon^2),$$

where $z = (j, a, \theta, \eta)$ again is the vector of individual state variables. Note that we put a time index on the value function and on prices. This will be necessary as soon as we compute transitional dynamic of the model. The terminal condition for the value function is

$$V_t(z) = 0 \quad \text{for} \quad z = (J+1, a, \theta, \eta),$$

which means we assume that the household doesn't value what is happening after death.

Applying the same steps as in Chapter 10, we can formulate the solution to the household problem recognizing that we can write labor hours and consumption as functions of a^+ as

$$l = l(a^+) = \min\left\{\max\left[\nu + \frac{1-\nu}{w_t^n h}(a^+ - (1+r_t^n)a - pen), 0\right], 1\right\} \tag{4}$$

$$c = c(a^+) = \frac{1}{p_t}\left[(1+r_t^n)a + w_t^n h l(a^+) + pen - a^+\right]. \tag{5}$$

The household problem then reduces to solve the first-order condition[1]

$$\frac{\nu\left[c(a^+)^\nu(1-l(a^+))^{1-\nu}\right]^{1-\frac{1}{\gamma}}}{p_t c(a^+)} = \beta(1+r_{t+1}^n)\cdot E\left[\frac{\nu\left[c_{t+1}(z^+)^\nu(1-l_{t+1}(z^+))^{1-\nu}\right]^{1-\frac{1}{\gamma}}}{p_{t+1}c_{t+1}(z^+)}\middle|\eta\right], \tag{6}$$

where a^+ is the unknown.

Aggregate In order to aggregate individual decisions at each element of the state space to economy-wide quantities, we need to determine the distribution of households $\phi_t(z)$ across the state space.[2] For the sake of simplicity, we assume that we have already discretized the state space. Then we can apply exactly the same procedure as in Chapter 10. Specifically, we know that at age $j = 1$ households hold zero assets, experience a permanent productivity shock $\hat{\theta}_i$ with probability π_i^θ, as well as a transitory productivity shock of $\eta_1 = 0$. Hence, we have

$$\phi_t(1, 0, \hat{\theta}_i, \hat{\eta}_g) = \begin{cases} \pi_i^\theta & \text{if } g = \frac{m+1}{2} \quad \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

Knowing the distribution of households over the state space at age 1, we can compute the distribution at any successive age-year combination using the policy function $a_t^+(z)$. Specifically, for each element of the state space z at age j and time t, we compute the left and right interpolation nodes $\hat{a}_l$ and $\hat{a}_r$ as well as the corresponding interpolation weight φ. The nodes and the weight satisfy

$$a_t^+(z) = \varphi \cdot \hat{a}_l + (1-\varphi)\cdot \hat{a}_r.$$

[1] "If you want to do numerical implementation work, you should write equation correctly first", Economist Tao Zha emphasized last year.

[2] Master Sargent had taught me: you should pay more attention to analyze the economy whole widely last year.

Taking into account the transition probabilities for the transitory productivity shock η_{gg^+}, we then distribute the mass of individuals at state z to the state space at the next age $j+1$ and year $t+1$ according to

$$\phi_{t+1}(z^+) = \begin{cases} \phi_{t+1}(z^+) + \varphi \cdot \pi_{gg^+} \cdot \phi_t(z) & \text{if } v = l \\ \phi_{t+1}(z^+) + (1-\varphi) \cdot \pi_{gg^+} \cdot \phi_t(z) & \text{if } v = r \end{cases}$$

with $z^+ = (j+1, \hat{a}_v, \hat{\theta}_i, \hat{\eta}_{g^+})$.[3]

Note that the distributional measure $\phi_t(z)$ satisfies

$$\sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t = 1$$

for any age j and time t. We can hence use it to calculate cohort-specific aggregates

$$\bar{c}_{j,t} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) \cdot c_t(z), \quad \bar{l}_{j,t} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) \cdot h_t(z) l_t(z), \quad \text{and}$$

$$\bar{a}_{j,t} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) \cdot \hat{a}_v.$$

From these cohort values, we can in turn generate economy-wide quantities. We have only to weight the cohort variables with by the respective relative cohort sizes m_j. Consequently, we get

$$C_t = \sum_{j=1}^{J} m_j \cdot \bar{c}_{j,t}, \quad L_t^s = \sum_{j=1}^{J} m_j \cdot \bar{l}_{j,t}, \quad \text{and} \quad A_t = \sum_{j=1}^{J} m_j \cdot \bar{a}_{j,t}. \tag{7}$$

Firms Firms' behaviour is identical to the RBC or the heterogeneous agent model of Chapter 9. Consequently firms hire capital K_t and labor L_t on perfectly competitive factor markets to be transformed into a single output good Y_t according to the Cobb-Douglas production technology

$$Y_t = \Omega K_t^{\alpha} L_t^{1-\alpha},$$

with Ω being the technology level which is constant over time. Capital depreciates at rate δ, so that the capital stock evolves as

$$(1+n_p)K_{t+1} = (1-\delta)K_t + I.$$

Under the assumption of perfect competition, the (inverse) demand functions of the firm for capital and labor are

$$r_t = \alpha\Omega \left[\frac{L_t}{K_t}\right]^{1-\alpha} - \delta \quad \text{and} \quad w_t = (1-\alpha)\Omega \left[\frac{K_t}{L_t}\right]^{\alpha}. \tag{8}$$

The government The last sector in our model is the government. In our model it runs two separate system: the tax system and the pension system, which both operate on a balanced-basis. The government collects taxes on consumption expenditure, labor income, and interest income in order to finance public expenditure G_t and payments related to the shock of debt B_t. In the initial equilibrium, we let government expenditure be a constant share of GDP, that is, $G = g_y Y$. In last periods, the level of public goods is kept constant (per capita), meaning we

[3]See also the appendix about welfare changes in the literature which written by Sevon Hur (2012).

have $G_t = G$. The same applies to public debt, where the initial share is denoted by b_y. At any point in time the budget of the tax system is balanced if

$$\tau_t C_t + \tau_t^w w_t L_t^s + \tau_t^r r_t A_t + (1+n_p)B_{t+1} = G_t + (1+r_t)B_t. \tag{9}$$

In addition to revenue from taxation, the government finances expenditure from issuing new debt $(1+n_p)B_{t+1}$. However it has to repay current debt including interest payments so we have to add $(1+r_t)B_t$ to government consumption on the expenditure side. Therefore, in a steady-state equilibrium, expenditure $(r-n_p)B$ reflects the cost needed to keep the debt level constant. Note that we do not make any a prior restriction about which tax rate has to adjust in order to balance the budget over time.

The pension system operates on a pay-as-you-go basis, meaning that it collects contributions from working-age generation and directly redistributes them to current retirees. There is no capital accumulation process involved. The budget-balance equation of the pension system then reads

$$\tau_t^p w_t L_t^s = \overline{pen}_t \cdot N^R \quad \text{with} \quad N^R = \sum_{j=j_r}^{J} m_j \tag{10}$$

where N^R denotes the (fixed) number of retirees. Since payments are made to all retirees in a lump-sum fashion, one simply has to add up the relative cohort sizes of the retired generations on the expenditure side and multiply this number with the respective benefit. For the evolution of pension payments over time we assume that they are linked to the average labor earnings of the previous period, i.e.

$$\overline{pen}_t = \kappa_t \cdot \frac{w_{t-1} L_{t-1}^s}{N^L} \quad \text{with} \quad N^L = \sum_{j=1}^{j_r-1} m_j,$$

where κ_t is the replacement rate of the pension system and N^L is the (fixed) size of working-age cohorts. We assume that the replacement rate κ_t is given exogenously while the contribution rate τ_t^p adjusts in order to balance the budget.

Markets There are three markets in our economy: the factor markets for capital and labor and the good market. On the factor markets, prices for capital r_t and labor w_t adjust so that market clear, that is

$$K_t + B_t = A_t \quad \text{and} \quad L_t = L_t^s. \tag{11}$$

Note that there are two sectors that demand savings from the households. The firm sector employs savings as capital in the production process, the government uses its public debt in order to finance expenditure. By assumption the government and the firm sector perfectly compete on the capital market. On the goods market, all output produced must be used either as consumption of the private sector or the government or as investment into the future capital stock. The goods market equilibrium therefore reads

$$Y_t = C_t + G_t + I_t. \tag{12}$$

Using walras' law we know that if the factor markets of the economy clear, then the goods market will also be in equilibrium.

A Equilibrium definition

Last but not least, we want formally define an equilibrium path of our OLG economy.

Definition (Equilibrium path) Given a path for government expenditure and debt $\{G_t, B_t\}_{t=0}^{\infty}$, a path for tax rates $\{\tau_t^c, \tau_t^w, \tau_t^r\}_{t=0}^{\infty}$, and a characterization of the pension system, $\{\tau_t^p, \kappa_t\}_{t=0}^{\infty}$, a recursive competitive equilibrium of the economy is a set of policy functions

$$\{c_t(z), l_t(z), a_t(z)\}_{t=0}^{\infty}$$

for the households, a set of input choice $\{K_t, L_t\}_{t=0}^{\infty}$ for the firms, prices $\{r_t, w_t\}_{t=0}^{\infty}$ and a measure of households $\{\phi_t(z)\}_{t=0}^{\infty}$ that are such that

1. *[Household maximization]:* Given prices $\{w_t, r_t\}_{t=0}^{\infty}$, the policy functions solve the household optimization problem as stated in (3).
2. *[Firms maximization]:* Given prices, the firms' factor inputs satisfy their demand equation in (8).
3. *[Government budget constraints]:* The government budget constraints (9) and (10) are satisfied.
4. *[Market clearing]:* The market-clearing equations in (11) and (12) hold with aggregate quantities being computed from (7).
5. *[Consistency of the measure of households]:* The measure of households is consistent with the assumptions about stochastic process and individual decisions.

The above definition applies to any general equilibrium path of the economy. We can furthermore specify a long-run equilibrium of the model or a steady state.

Definition (Long-run equilibrium) A long-run equilibrium of the economy is an equilibrium path on which prices, tax rates, and all individual variables are constant over time and aggregate quantities all grow at the rate of the population n_p.

B Numerical implementation of steady-state equilibrium

To solve our stochastic OLG model numerically, we combine the policy function iteration of Chapter 10 for life-cycle model with the Gauss-Seidel iteration procedure of Chapter 6 for the macroeconomic equilibrium. We have already discussed all the details of the policy function iteration method in Chapter 10, including the discretization of the state space, how to solve the first-order condition, and how to determine the distribution of households across the state space. As there is literally on change to the method, we shall not repeat this. At this stage the important thing is that the policy function iteration method allow us to determine cohort-specific averages, which can be aggregated to economy-wide quantities C, L^s and A, see (7). In a steady state equilibrium, these (normalized) quantities are, of course, constant over time. Note that the solution to the household problem depends on prices and tax rates p, r^n, w^n, as well as pension payment $\overline{pen}$. The same is then obviously true for aggregate quantities.

The macroeconomic iteration procedure We use the Gauss-Seidel procedure discussed in Chapters 2 and 6 to determine the macroeconomic equilibrium of our economy. After initializing the model parameters and providing initial guess for aggregate capital and labor input as well as the pension payment in subroutine *initialize*, the algorithm proceeds according to the following steps:

(1) Given guess for capital and labor input as well as tax rates, compute factor and consumer prices using subroutine *prices*.

(2) Given prices and public pension payments, determine household policy functions using subroutine *solve-household*.

(3) Compute the distribution of households over the state space using subroutine *get-distribution*.

(4) Aggregate household decisions to quantities with subroutine *aggregate*.

(5) Determine new taxes and the pension level with subroutine *government*.

(6) Calculate the absolute value of the relative difference between demand $C+G+I$ and supply Y of goods. If this differences is small enough, we have found the equilibrium and can stop the iteration procedure. If not, start again at point 1.

There are two subroutines involved in this procedure that we haven't discussed so far. The aggregation procedure and the subroutine that determines the parameters of the tax and pension system.

An except of the aggregation procedure is shown in Program 11.1.a. First of all this routine stores the current value for labor supply in a variable *LL-old*. It then calculates cohort average for important variables such as consumption, asset, and labor supply.

In the next step, we aggregate cohort averages to economy-wide quantities using the relative population shares stored in the array *m*. Having fully aggregated household decisions, we can calculate an update level of the capital stock and aggregate labor supply using the factor-market-clearing conditions. Note that the Gauss-Seidel procedure involves using a damping factor *damp*, which ensures that the new levels of capital and labor are some linear combination of the values from the previous iteration step and the newly calculated aggregate quantities. Having derived new values for aggregate capital and labor, we can determine investment and total output. Last but not least, we calculate the difference between supply and demand on the goods market.

C Model parametrization and calibration

One thing we haven't talk about so far–and on which we also were very brief in the last chapter–is how to parameterize this model, that is, how **to map** the model into the data. There are two types of model parameters. **Those that can be directly observed in the data, such as life expectancy, capital shares, etc., and those parameters than influence the model outcome but cannot be observed directly.** To parameterize the latter the literature typically uses a calibration procedure. The idea behind that procedure is to relate each model parameter to a specific target output value of the model, the real-world counterpart of which can be observed in the data. Then the parameter value is adjusted until the respective target output value is sufficiently close to the value taken from the data. There are a couple of things to note when applying such a calibration strategy. First, it is not only one parameter that influences one outcome variable of the model. Instead when we change one parameter usually all variables of the model will change. Therefore the process of calibration involves first getting a feeling for how different parameter shape model outcomes and which parameter we can actually identify from which model outcome. This means one has to run the model many times. Second, the calibration process is not an analytically defined procedure, but a rather vague process in which the researcher decides which fit of the model is good enough to suit her needs. Third, especially for elasticities, it is often useful to get estimates from the literature and directly feed them into the model.

Exogenous parameter values In order to limit the running time of our model, we let one model period cover five real years. The model could easily be expanded so that one period covers one real year by just increasing the maximum age JJ.[4] If we assume that households start their economic life at age $20 (j = 1)$ and face a life expectancy (at birth) of 80 years, the model's life cycle has to cover $JJ = 12$ periods. Assuming a statutory retirement age of 65 years yields a model retirement age of $JR = 10$, so that households spend the last three periods in retirement. The last demographic parameter is the population growth rate. We assume an annuity growth rate of 1 per cent, which needs to be converted into a periodic growth rate $n_p = 1.01^5 - 1 \approx 0.05$. [5]

The capital share in production is set at $\alpha = 0.36$, which is a quite common value and leads the labor-income share to be equal to 0.64. On the household side we choose an intertemporal elasticity of substitution of $\gamma = 0.5$, which implies an individual relative risk-aversion of 2. Note that a risk-aversion of 2 is quite common in the macroeconomic literature, but fairly low compared to values used in finance. The age-productivity profile e_j is taken from the literature. We normalize it to a value of 1 at the first working age $j = 1$. From there on labor productivity roughly doubles until age $45 - 55$. Afterwards it slightly falls again until the data of retirement. Finally, we fix total government expenditure as a fraction of GDP at $g_y = 0.19$, while we specify a government debt to GDP ratio of $b_y = 0.6$.

Calibrated parameter values The remaining parameters of the model need to be pinned down by calibration. We discuss these parameters and their targets by economic sector. On the production side we have to specify the aggregate technology level Ω as well as the capital depreciation rate δ. The former is pinned down by normalizing the wage rate for effective labor to $w = 1$. **This requires setting $\Omega = 1.6$.** The depreciation rate of capital mostly impacts on the size of aggregate investment. In a long-run equilibrium, aggregate investment relative to GDP is defined as

$$\frac{I}{Y} = (n_p + \delta) \cdot \frac{K}{Y} \quad \text{which implies} \quad \delta = \frac{I/Y}{K/Y} - n_p.$$

Assuming an average investment to GDP ratio of 24 per cent, an annual capital-to-output ratio of 3–meaning a five-year ratio of $\frac{3}{5} = 0.6$–and taking into account the (5-year) population growth rate of 5.1 per cent, we arrive at a five-year depreciation rate of 0.349, or an annual value of 8.23 per cent.

We want the capital output ratio to be equal 3 and use this value to pin down the intertemporal discount factor β in the household utility function.[6] The higher is β, the more individuals are willing to save and the higher will be the aggregate capital stock of the economy. The consumption share parameter ν governs the individuals preference for consumption goods purchased on the market in relation to leisure consumption. The larger ν is, the more consumption goods a household buys in the market and the less leisure the household consumes. Therefore ν has a strong influence on the number of hours a household works in the market. We adjust ν to target an average share of working time in the total time endowment of around 33 per cent.[7]

Next we calibrate the wage process of the model by targeting the variance of the log of the household labor income over life cycle. Empirical studies suggest that around age 25 this variance has a value of 0.3, which then increases almost linearly to a value of 0.9 at the age of 60. The variance of log labor earnings in our model is determined by two components: the exogenous processes for idiosyncratic labor productivity θ and η_j, as well as the individual decision about

[4]However, more model periods lead to a longer running time and more computer memory consumption at the size of arrays to store individual decision rules increases.

[5]Note that we report annualized values wherever possible as they are much easier to interpret. The adjustment to model period is done in the code.

[6]Note that estimates of country-specific capital stocks substantially depending on whether housing and other non-productive capital is included or not.

[7]This share is derived from assuming a maximum weekly working-time endowment of 110 hours as well as 50 working weeks per year. We relate to an average annual working hours per employee of around $1,800$.

how many hours of labor to supply in the market. We do not have a closed-form answer to how labor hours react to labor-productivity shocks and therefore share the earnings process over the life cycle as labor hours are also influenced by the amount of wealth and the age of the individual. We do, however, have some information about the structure of the labor-productivity process and how it may influence the variance of log labor earnings. The log of labor earnings of an individual reads(**see also Heejeong Kim (2012), The Empirical model of PSID data do be in line with U.S. data**)

$$\log(w_t h_j l_j) = \log(w_t) + \log(e_j) + \theta + \eta_j + \log(l_j).$$

The first two components are deterministic for each age group, so that their variance is equal to zero. The variance of log labor earnings at age j consequently is

$$\begin{aligned} Var[\log(w_t h_j l_j)] =& Var[\theta] + Var[\eta_j] + Var[\log(l_j)] + 2 \cdot Cov[\theta, \log(l_j)] \\ & 2 \cdot Cov[\eta_j, \log(l_j)]. \end{aligned}$$

Note that term $2 \cdot Cov[\theta, \eta_j]$ is equal to zero since θ and η_j are independent random variables. For our calibration strategy the first two components of the variance are the most interesting, since they can be written more explicitly. First recall that the initial condition for the transitory process is $\eta_1 = 0$, which implies $Var[\eta_1 = 0]$. Knowing this, we can calculate the variances of the stochastic component of log labor productivity for each potential age j as

$$\begin{aligned} Var[\theta] + Var[\eta_1] = Var[\theta] \qquad\qquad\qquad &= \sigma_\theta^2 \\ Var[\theta] + Var[\eta_2] = Var[\theta + \epsilon_2] \qquad\qquad &= \sigma_\theta^2 + \sigma_\epsilon^2 \\ Var[\theta] + Var[\eta_3] = Var[\theta + \rho\epsilon_2 + \epsilon_3] \qquad &= \sigma_\theta^2 + (1+\rho^2)\sigma_\epsilon^2 \\ Var[\theta] + Var[\eta_4] = Var[\theta + \rho^2\epsilon_2 + \rho\epsilon_3 + \epsilon_4] &= \sigma_\theta^2 + (1+\rho^2+\rho^4)\sigma_\epsilon^2 \\ \vdots \qquad\qquad & \end{aligned}$$

The above calculation tells us a lot about the variance of log labor productivity over the life cycle. First of all, since the initial transitory component is normalized to zero, the variance of log labor productivity at the youngest age $j = 1$ is solely due to the variations in the fixed effect σ_θ^2. Second, a strongly increasing variance over life cycle indicates a high autocorrelation parameter ρ. To see this, just assume that persistence was $\rho = 0$. In this case the variance of log labor earnings would simply be $\sigma_\theta^2 + \sigma_\epsilon^2$ in each period $j > 1$. The variance of log labor productivity would therefore jump between age $j = 1$ and $j = 2$ and then stay constant over the remainder of the life cycle. With an extreme autocorrelation of $\rho = 1$, on the other hand, the variance of labor productivity at age j was $\sigma_\theta^2 + (j-1)\sigma_\epsilon^2$, i.e. it would increase linear with age. Since in our model the variance of log labor earnings in influenced by both labor productivity and hours worked, we choose an autocorrelation of $\rho = 0.98$, which make the variance of log labor productivity rise quite substantially but not linearly. We will see that this value is enough to create a strongly rising profile for the variance of log labor earnings. Having choosing a value for ρ, the remaining parameters of the labor productivity process are the variance σ_θ^2 and σ_ϵ^2. We calibrate values for these parameters to target a variance of the log of labor earnings of 0.3 at the age of $25(j = 2)$ and of 0.9 at the age of $65(j = 9)$.

Program 11.1.d shows how to calculate the age-specific variance of log labor earnings. We iterate over the state space at a specific age, add up the log of labor earnings as well as its squire and weight it by the respective distribution of households at a certain grid point. Note that we only select households that work at least 1 per cent of their total time endowment in order to avoid that the program tries to take the log of a number that is close to or equal to zero. Since this selection causes the sum of households to not necessarily add up to a value of 1, we have to normalize the variables *exp-1* and *var-1* by the respective mass of the households we used.

Finally, we have to calibrate the remaining government parameters. We have already defined values for government consumption and public debt as a fraction of GDP. What remains to be specified are the tax rates on consumption and income from capital and labor as well as the parameter of pension system. Assuming that the revenue from the taxation of goods and services is about 4.5 per cent of GDP and aggregate consumption amounts to roughly 60 per cent of GDP, a consumption tax rate of 7.5 per cent is our favorite choice. There are not many more free parameters in the tax system, as the expenditure side is already determined. If we furthermore assume that the government levies a uniform income tax on income from both labor and capital, the income tax rate needs to be the endogenous tax rate in our model. Consequently it is calculated such that the government budget constraint is satisfied. Last but not least, we have

Table 1: Calibrated parameters in OLG model

Parameter	Value	Target	Value
Ω	1.6	w	1.0
δ	0.0823	$\frac{I}{Y}$	0.24
β	0.998	$\frac{K}{Y}$	3.00
ν	0.335	average hours worked	0.33
ρ	0.98	linearly increasing $Var[wh_j l_j]$	
σ^2_θ	0.23	$Var[wh_1 l_1]$	0.30
σ^2_ϵ	0.05	$Var[wh_9 l_9]$	0.90
τ_c	0.075	$\frac{\tau_c C}{Y}$	0.045
κ	0.50	τ^p	0.12

to specify a replacement rate κ for the pension system. We want to target a pension contribution rate of around 12.4 per cent, which leads to a replacement rate of 0.5. Table 1 summarizes our parameter choices and their respective targets.

Parameters for computational purpose The tax system parameters include a variable *tax* that informs the subroutine *government* about which tax rate should be the endogenous one. We set this variable to a value of 2, indicating that a uniform income tax system on labor and capital income should balance the tax system's budget. Next to the model-related parameters we also have to specify the numerical parameter at this stage. These include a damping factor of $damp = 0.3$ for the Gauss-Seidel iteration procedure, a level of tolerance for the relative difference between demand and supply on the goods market $sig = 1d - 4$, and the maximum number of iterations that should be used to calculate the equilibrium values $itermax = 50$. The remainder of the setup procedure happens in subroutine *initialize*, in which we discretize the state space, specify labor productivities e_j, and initialize prices and quantities.

Our parameter choice for the asset grid deserves a more detailed discussion. Obviously the lower end of the state space $a_l = 0$ is defined through the borrowing constraint of households. Specifying the growth rate of asset grid u_a and the maximum asset level a_u remains to be done. We choose a growth rate u_a such that the difference between the first and the second asset grid point is in the order of 10^{-2}. Note that a larger density of points around the lower bound of the asset grid is desirable as **the borrowing constraint** might hit individuals in this range of the asset space and therefore the policy functions might have a **kink**. We suggest the following calibration procedure for the growth rate. Start out with a growth rate of $u_a = 0$ and simulate the model. Then successively increase the growth rate by 0.1 and simulate the model again. This will cause both the macro variable as well as the average life-cycle path of the household to change slightly. As soon as these variables stay constant by increasing the growth rate, you have found a suitable growth rate for the asset grid.[8] Note that the distributional variables

[8]Alternatively, one can try finding a good growth rate by computing Euler equation errors and trying to

will always change slightly when the growth rate is adjusted, especially so at larger ages. So don't pay too much attention to these. Note further that a similar procedure can be used to determine what is a good number of discretization points n for the asset grid space. Finally, we have to come up with a suitable maximum asset level a_u. In order to calibrate this, we identify the maximum asset gridpoint that is used at each age.

D The initial equilibrium

Having calibrated the model, we can now run the program and write some output. The output file in which we write the initial equilibrium variables is called *output-initial-out* and contains all the necessary information. The output file is structured into two parts. The first shows the outcomes of the model on the macroeconomic level, the second part summarizes the average life-cycle profiles of individual variables as well as the variance of their logs. Note that many of the macro variables are shown both in absolute values and as a fraction of GDP in annual terms.[9] The life-cycle averages are expressed as a fraction of working households' average labor earnings, so that they can easily be compared to real-life data.

Table 2 summarizes the macroeconomic data generated by our calibrated OLG model. Most of these data have already been discussed. We found that we can match all our target pretty well. Note that the model generates an interest rate of 4.55 per cent per year, which implies $r > n_p$ and therefore the economy is not falling with **the golden rule**. Rather, **it is on the dynamically efficient side of the golden rule, which all will be import in understanding the welfare effects of policy reforms simulated below.**

Table 2: The initial equilibrium of the OLG model

Variable	Value	Variable	Value
Capital market:		Labor market:	
Private asset	360.82	Average hours worked	33.21
Capital	300.82	(in % time endowment)	
Public debt	60.00	Wage rate (absolute)	1.00
interest rate (in % p.a.)	4.55		
Goods market:		Pension system	
Private consumption	56.93	Pay roll tax rate (in %)	12.27
Public consumption	19.00	Replacement rate (in %)	50.00
Investment	24.07	Total pension payments	7.86
Tax rates:		Tax revenue:	
Consumption (in %)	7.5	Consumption tax	4.27
Labor earnings (in %)	20.87	Labor-earnings tax	13.36
Capital income (in %)	20.87	Capital-income tax	3.75

In % of GDP if not indicated otherwise.

minimize them with the choice of u_a, see the exercise of section Chapter 9.

[9] Where the shock variables as well as the interest rate are adjusted to an annual basis.