Stat 517 (Decision Theory and Bayesian Inference)

Homework # 1: Bayes Theorem. Solutions.

1. Similarly to our example in class, denote the events...

$$\begin{array}{lll} V &=& \{\text{ donor has the virus }\} \qquad \boldsymbol{P}\{V\} = 0.0001 \\ S &=& \{\text{ test shows a positive result }\} \qquad \boldsymbol{P}\{S \mid V\} = 0.99,\ \boldsymbol{P}\left\{\overline{S} \mid \overline{V}\right\} = 0.995 \end{array}$$

- What proportion of blood that is donated will test positive using the ELISA test?
  By the formula of Total Probability,

$$\begin{aligned} \boldsymbol{P}\{S\} &= \boldsymbol{P}\{S \mid V\}\boldsymbol{P}\{V\} + \boldsymbol{P}\left\{S \mid \overline{V}\right\}\boldsymbol{P}\left\{\overline{V}\right\} \\ &= (0.99)(0.0001) + (1-0.995)(1-0.0001) = \boxed{0.0051} \end{aligned}$$

- Also, what proportion of the blood that tests negative on the ELISA test is actually infected with HIV?
  By the Bayes formula,

$$\boldsymbol{P}\left\{V \mid \overline{S}\right\} = \frac{\boldsymbol{P}\left\{\overline{S} \mid V\right\}\boldsymbol{P}\{V\}}{\boldsymbol{P}\left\{\overline{S}\right\}} = \frac{(1-0.99)(0.0001)}{1-0.0051} = \boxed{0.0000010 \text{ or } 0.0001\%}$$

- Finally, what is the probability that a positive ELISA outcome is truly positive, that is, what proportion of individuals with positive outcomes are actually infected with HIV?
  By the Bayes formula,

$$\boldsymbol{P}\{V \mid S\} = \frac{\boldsymbol{P}\{S \mid V\}\boldsymbol{P}\{V\}}{\boldsymbol{P}\{S\}} = \frac{(0.99)(0.0001)}{0.0051} = \boxed{0.0194}$$

  Very low probability that the patient really has the virus!

2. Let S_3 be the event {three tests gave positive results}. Then $\boldsymbol{P}\{S_3 \mid V\} = 0.95^3$, $\boldsymbol{P}\left\{S_3 \mid \overline{V}\right\} = 0.01^3$, and the conditional probability that a patient has the virus equals

$$\begin{aligned} \boldsymbol{P}\{V \mid S_3\} &= \frac{\boldsymbol{P}\{S_3 \mid V\}\boldsymbol{P}\{V\}}{\boldsymbol{P}\{S_3 \mid V\}\boldsymbol{P}\{V\} + \boldsymbol{P}\left\{S_3 \mid \overline{V}\right\}\boldsymbol{P}\left\{\overline{V}\right\}} \\ &= \frac{(0.95)^3(0.02)}{(0.95)^3(0.02) + (0.01)^3(1-0.02)} = \boxed{0.99994}. \end{aligned}$$

   An overwhelming evidence of a virus, in this case.

3. (a) Denote the events: $H = \{\text{high risk}\}$, $L = \{\text{low risk}\}$, $N = \{\text{no accidents during 1 year}\}$. We have:

$$\boldsymbol{P}\{H\} = 0.2,\ \boldsymbol{P}\{L\} = 0.8,\ \boldsymbol{P}\{N|H\} = 0.368,\ \boldsymbol{P}\{N|L\} = 0.905$$

   (from Poisson distribution, with $\lambda = 1$ and $\lambda = 0.1$).
   By the Bayes' Rule,

$$\begin{aligned} \boldsymbol{P}\{H|N\} &= \frac{\boldsymbol{P}\{N|H\}\boldsymbol{P}\{H\}}{\boldsymbol{P}\{N|H\}\boldsymbol{P}\{H\} + \boldsymbol{P}\{N|L\}\boldsymbol{P}\{L\}} \\ &= \frac{(0.368)(0.2)}{(0.368)(0.2) + (0.905)(0.8)} = \boxed{0.0923} \end{aligned}$$

(b) The prior probabilities of H and L are 0.2 and 0.8. Posterior probabilities are 0.0923 and $1 - 0.0923 = 0.9077$, respectively. The marginal probability of N (the data) is in the denominator,

$$(0.368)(0.2) + (0.905)(0.8) = 0.7976.$$

(c) We have a new event here, $T = \{\text{no accidents in 3 years}\} = \boldsymbol{P}\{X = 0\}$, where X is the number of accidents during 3 years. This variable X has Poisson distribution with parameter 3 for high-risk drivers and with parameter 0.3 for the low-risk group. From Poisson distribution,

$$\boldsymbol{P}\{T|H\} = 0.050 \quad \text{and} \quad \boldsymbol{P}\{T|L\} = 0.741.$$

Then, by the Bayes rule,

$$\begin{aligned} \boldsymbol{P}\{H|T\} &= \frac{\boldsymbol{P}\{T|H\}\,\boldsymbol{P}\{H\}}{\boldsymbol{P}\{T|H\}\,\boldsymbol{P}\{H\} + \boldsymbol{P}\{T|L\}\,\boldsymbol{P}\{L\}} \\ &= \frac{(0.050)(0.2)}{(0.050)(0.2) + (0.741)(0.8)} = \boxed{0.0166} \end{aligned}$$

Of course, this conditional probability must be considerably lower than in (a).

4. Denote good and bad exams by G and B. Also, let GB denote one good and one bad exams, XG denote the event "student X wrote a good exam," etc. We need to find $\boldsymbol{P}\{XG \mid GB\}$ given that $\boldsymbol{P}\{XG\} = 0.8$ and $\boldsymbol{P}\{YG\} = 0.6$.

   By the *Bayes Rule*,

$$\begin{aligned} \boldsymbol{P}\{XG \mid GB\} &= \frac{\boldsymbol{P}\{GB \mid XG\}\,\boldsymbol{P}\{XG\}}{\boldsymbol{P}\{GB\}} = \frac{\boldsymbol{P}\{YB\}\,\boldsymbol{P}\{XG\}}{\boldsymbol{P}\{XG \cap YB\} + \boldsymbol{P}\{XB \cap YG\}} \\ &= \frac{(0.4)(0.8)}{(0.8)(0.4) + (0.2)(0.6)} = \boxed{8/11 \text{ or } 0.727}. \end{aligned}$$