Homework 8 Solutions

1 Of 330 male and 270 female employees at the Flagstaff Mall, 210 of the men and 180 of the women are on flex-time (flexible working hours). Given that an employee selected at random from this group is on flex-time, what is the probability that the employee is a woman?

For a randomly selected employee, let W denote the event that the employee is a woman and let F denote the event that the employee is flex-time. We need to find $P(W|F)$.

$$\begin{aligned} P(W|F) &= \frac{P[W \cap F]}{P[F]} \\ P(F) &= \frac{210+180}{330+270} \\ P(F \cap W) &= \frac{180}{330+270} \\ P(W|F) &= \frac{180}{210+180} \\ &= \frac{6}{13} \end{aligned}$$

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2 A new medical test has been designed to detect the presence of the mysterious Brainlesserian disease. Among those who have the disease, the probability that the disease will be detected by the new test is 0.86. However, the probability that the test will erroneously indicate the presence of the disease in those who do not actually have it is 0.08. It is estimated that 16 % of the population who take this test have the disease. If the test administered to an individual is positive, what is the probability that the person actually has the disease?

This, and many other problems involve using Bayes' Theorem. However, we shall see how to solve this problem without using Bayes' Theorem.

Let D denote the event that a particular person has the disease. $P(D) = 0.16$. Let $\bar{D}$ indicate the event that the person doesn't have the disease. Let M denote the event that the medical test returns positive. We need to compute $P(D|M)$.

We know that $P(D) = 0.16, P(\bar{D}) = 0.84$. And, $P(M|D) = 0.86, P(M|\bar{D}) = 0.08$.

We want:

$$P(D|M) = \frac{P(D \cap M)}{P(M)}$$

From

$$P(M|D) = \frac{P(D \cap M)}{P(D)}$$

we get:

$$P(D \cap M) = P(M|D) \times P(D) = 0.86 \times 0.16$$

Similarly,

$$P(\bar{D} \cap M) = P(M|\bar{D}) \times P(\bar{D}) = 0.08 \times 0.84$$

With this, we get,

$$P(M) = P(\bar{D} \cap M) + P(D \cap M) = 0.86 \times 0.16 + 0.08 \times 0.84$$

Hence, combining everything, we get:

$$\begin{aligned} P(D|M) &= \frac{P(D \cap M)}{P(M)} \\ &= \frac{0.86 \times 0.16}{0.86 \times 0.16 + 0.08 \times 0.84} \\ &= 0.67 \end{aligned}$$

■

3 Two marbles are drawn randomly one after the other without replacement from a jar that contains 5 red marbles, 3 white marbles, and 9 yellow marbles. Find the probability of the following events.

1. A red marble is drawn first followed by a white marble.
2. A white marble is drawn first followed by a white marble.
3. A yellow marble is not drawn at all

The answers are:

1. $\frac{5}{17} \times \frac{3}{16}$
2. $\frac{\binom{3}{2}}{\binom{17}{2}}$
3. $\frac{\binom{5+3}{2}}{\binom{17}{2}}$

■

4 Scientific research on popular beverages consisted of 65 studies that were fully sponsored by the food industry, and 35 studies that were conducted with no corporate ties. Of those that were fully sponsored by the food industry, 15 % of the participants found the products unfavorable, 22 % were neutral, and 63 % found the products favorable. Of those that had no industry funding, 35 % found the products unfavorable, 16 % were neutral, and 49 % found the products favorable.

1. What is the probability that a participant selected at random found the products favorable?
2. If a randomly selected participant found the product favorable, what is the probability that the study was sponsored by the food industry?
3. If a randomly selected participant found the product unfavorable, what is the probability that the study had no industry funding?

Let S denote the event that the study was sponsored by industry. For the outcome of the study, let us have the following three events:

- Let F denote the event that the study was Favourable
- Let U denote unfavourable result
- Let N denote a neutral result

Clearly, $P(F) + P(U) + P(N) = 1$. We are given the following values: $P(\bar{S}) = \frac{35}{65}$.

From second statement, we get $P(F|S) = 0.63, P(U|S) = 0.15, P(N|S) = 0.22$. From third statement, we get $P(F|\bar{S}) = 0.49, P(U|\bar{S}) = 0.35, P(N|\bar{S}) = 0.16$. We need to find $P(F)$.

Here are the solutions:

1. $P(F|S) = \frac{P(F \cap S)}{P(S)}$. So, $P(F \cap S) = P(F|S) \times P(S) = 0.63 \times \frac{30}{65}$. Similarly, $P(F \cap \bar{S}) = P(F|\bar{S}) \times P(\bar{S}) = 0.49 \times \frac{35}{65}$.

   Combining, we get, $P(F) = P(F \cap S) + P(F \cap \bar{S}) = 0.63 \times \frac{30}{65} + 0.49 \times \frac{35}{65}$.

2. $P(S|F) = P(F \cap S)/P(F) = (0.63 \times \frac{30}{65})/(0.63 \times \frac{30}{65} + 0.49 \times \frac{35}{65})$.

3. $P(U \cap S) = P(U|S) \times P(S) = 0.15 \times \frac{30}{65}$. Similarly, $P(U \cap \bar{S}) = P(U|\bar{S}) \times P(\bar{S}) = 0.35 \times \frac{35}{65}$. Combining, we get $P(U) = P(U \cap S) + P(U \cap \bar{S}) = 0.15 \times \frac{30}{65} + 0.35 \times \frac{35}{65}$.

   We want to find $P(\bar{S}|U) =$

$$\frac{P(\bar{S} \cap U)}{P(U)} = \frac{0.35 \times \frac{35}{65}}{0.15 \times \frac{30}{65} + 0.35 \times \frac{35}{65}}$$

■

5 If $P(E \cap F) = 0.08, P(E|F) = 0.2, P(F|E) = 0.5$, find:

1. $P(E)$
2. $P(F)$
3. $P(E \cup F)$
4. Are the events E and F independent?

Time spent in resource room	Pass %
None	25
Between 1 and 90 minutes	52
More than 90 minutes	63

Table 1: Table for Problem 6

The answers are:

1. $P(E) = P(F \cap E)/P(F|E) = 0.08/0.5 = 0.16$
2. $P(F) = P(F \cap E)/P(E|F) = 0.08/0.2 = 0.4$
3. $P(F \cup E) = P(F) + P(E) - P(F \cap E) = 0.4 + 0.16 - 0.08 = 0.48$
4. $P(F) \times P(E) = 0.4 \times 0.16 = 0.064 \neq P(E \cap F)$. Hence, E and F are not independent.

■

6 On average 62 % of Finite Mathematics students spend some time in the Mathematics Department's resource room. Half of these students spend more than 90 minutes per week in the resource room. At the end of the semester the students in the class were asked how many minutes per week they spent in the resource room and whether they passed or failed. The passing rates are summarized in table 1.
If a randomly chosen student did not pass the course, what is the probability that he or she did not study in the resource room?

Let S denote the event that a student passed. Let H, M, L denote the events that the student spent "more than 90 mins", "between 1 and 90 mins" and "None" respectively.

We are given $P(H) + P(M) = 0.62$. $P(H) = 0.5 \times 0.62 = 0.31$. So, $P(M) = 0.31, P(L) = 1 - 0.62 = 0.38$. We are also given the following values: $P(S|L) = 0.25, P(S|M) = 0.52, P(S|H) = 0.63$. We are asked to find $P(L|\bar{S})$.

$P(S) = P(S \cap L) + P(S \cap M) + P(S \cap H) = P(S|L) \times P(L) + P(S|M) \times P(M) + P(S|H) \times P(H) = 0.25 \times 0.38 + 0.52 \times 0.31 + 0.63 \times 0.31 = 0.4515.$

$P(\bar{S}) = 1 - P(S) = 0.5485$. And, $P(L \cap \bar{S}) = P(L) - P(L \cap S) = 0.38 - 0.25 \times 0.38 = 0.285$. Combining, we have, $P(L|\bar{S}) = P(L \cap \bar{S})/P(\bar{S}) = \frac{0.285}{0.5485}$

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7 You ask a neighbor to water a sickly plant while you are on vacation. Without water the plant will die with probability 0.9. With water it will die with probability 0.4. You are 82 % certain the neighbor will remember to water the plant.

1. When you are on vacation, find the probability that the plant will die.
2. You come back from the vacation and the plant is dead. What is the probability the neighbor forgot to water it?

Let W be the event that the plant is watered. Let D be the event that the plant would die.

We are given the following values: $P(W) = 0.82, P(D|W) = 0.4, P(D|\bar{W}) = 0.9$. With these, the answers are:

1. $P(D) = P(D \cap W) + P(D \cap \bar{W}) = P(D|W) \times P(W) + P(D|\bar{W}) \times P(\bar{W})$ $P(D) = 0.4 \times 0.82 + 0.9 \times 0.18 = 0.49$
2. $P(\bar{W}|D) = P(D \cap \bar{W})/P(D) = \frac{0.9 \times 0.18}{0.49}$.

■

8 In a survey of 299 people, the data presented in Table 2 were obtained relating gender to political orientation.
A person is randomly selected. What is the probability that the person is:

1. Male?
2. Male and Democrat?
3. Male given that the person is a Democrat?
4. Republican given that the person is Male?
5. Female given that the person is a Independent?
6. Are the events Female and Republican independent?

	Republican (R)	Democrat (D)	Independent (I)	Total
Male (M)	74	39	21	134
Female (F)	85	64	16	165
Total	159	103	37	299

Table 2: Table for Problem 8

The answers are:

1. $\frac{134}{299}$
2. $\frac{39}{299}$
3. $\frac{39}{103}$
4. $\frac{74}{134}$
5. $\frac{16}{37}$
6. P(Female) = $\frac{165}{299}$. P(Republican) = $\frac{159}{299}$. P(Female and Republican) = $\frac{85}{299}$ P(Female)$\times$P(Republican) = $\frac{165}{299} \times \frac{159}{299} \neq$ P(Female and Republican). Hence, they are not independent events.

■

9 In a survey of 347 people, the data presented in Table 3 were obtained relating gender to political orientation.
A person is randomly selected. What is the probability that the person is:

1. Male?
2. Male and Democrat?
3. Male given that the person is a Democrat?
4. Republican given that the person is Male?
5. Female given that the person is a Libertarian?
6. Are the events Male and Republican independent?

	Republican (R)	Democrat (D)	Libertarian (L)	Total
Male (M)	99	80	10	189
Femal (F)	84	58	16	158
Total	183	138	26	347

Table 3: Table for Problem 9

The answers are:

1. $\frac{189}{347}$
2. $\frac{80}{347}$
3. $\frac{80}{138}$
4. $\frac{99}{189}$
5. $\frac{16}{26}$
6. P(Male) = $\frac{189}{347}$. P(Republican) = $\frac{183}{347}$. P(Male and Republican) = $\frac{99}{347}$.
   P(Male)$\times$P(Republican) = $\frac{189}{347} \times \frac{183}{347} \neq$ P(Male and Republican). Hence, they are not independent events.

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10 Factories A, B and C produce computers. Factory A produces 2 times as many computers as factory C. And factory B produces 6 times as many computers as factory C. The probability that a computer produced by factory A is defective is 0.016, the probability that a computer produced by factory B is defective is 0.038, and the probability that a computer produced by factory C is defective is 0.043. A computer is selected at random and it is found to be defective. What is the probability it came from factory C?

Let A, B, C denote the events that any computer is manufactured by factories A, B, and C respectively. Let D denote the event that the computer is defective.

$P(A) = 2 \times P(C), P(B) = 6 \times P(C)$. We get, $P(A) = 2/9, P(B) = 6/9, P(C) = 1/9$.

	Color-Blind (C)	Not Color - Blind ($\bar{C}$)	Total
Male (M)	89	35	124
Femal (F)	56	18	74
Total	145	53	198

Table 4: Table for Problem 11

We are also given $P(D|A) = 0.016, P(D|B) = 0.038, P(D|C) = 0.043$. We need to compute $P(C|D)$.

We can compute

$$\begin{aligned}
P(D) &= P(A \cap D) + P(B \cap D) + P(C \cap D) \\
&= P(D|A) \times P(A) + P(D|B) \times P(B) + P(D|C) \times P(C) \\
&= 0.016 \times 2/9 + 0.038 \times 6/9 + 0.043 \times 1/9 \\
&= 0.303/9
\end{aligned}$$

$$\begin{aligned}
P(C|D) &= P(C \cap D)/P(D) \\
&= \frac{0.043/9}{0.303/9} \\
&= 0.043/0.303
\end{aligned}$$

■

11 In a survey of 198 people, the data presented in Table 4 were obtained relating gender to color-blindness.
A person is randomly selected. What is the probability that the person is:

1. Male?
2. Male and Color-blind?
3. Male given that the person is Color-blind?
4. Color-blind given that the person is Male?
5. Female given that the person is not Color-blind?

The answers are:

1. $\frac{124}{198}$

2. $\frac{89}{198}$

3. $\frac{89}{145}$

4. $\frac{89}{124}$

5. $\frac{18}{53}$

$\blacksquare$

12 If $P(F) = 0.3$ and $P(E|F) = 0.9$, then find $P(E \cap F)$.

$P(E \cap F) = P(E|F) \times P(F) = 0.9 \times 0.3 = 0.27$ $\blacksquare$

13 A fair coin is tossed 12 times. What is the probability that:

1. Exactly 10 heads appear?

2. At least two heads appear?

3. At most 9 heads appear?

Let X be the random variable denoting the number of heads. The answers are:

1. $P(X = 10) = \frac{\binom{12}{10}}{2^{12}}$

2. $P(X \geq 2) = 1 - P(X \leq 1) = 1 - (P(X = 1) + P(X = 0)) = 1 - \frac{\binom{12}{1} + \binom{12}{0}}{2^{12}}$

3. $P(X \leq 9) = 1 - [P(X = 10) + P(X = 11) + P(X = 12)] = 1 - \frac{\binom{12}{10} + \binom{12}{11} + \binom{12}{12}}{2^{12}}$

$\blacksquare$

14 If the letters in the word POKER are rearranged, what is the probability that the word will begin with the letter P and end with the letter E?

Total number of ways of arranging the 5 letters, without restriction is 5!. If we fix the first letter to be P and the last letter to be E, total number of words with such combination is 3!. So, overall probability is $\frac{3!}{5!}$. ■

15 Two cards are drawn from a regular deck of 52 cards, without replacement. What is the probability that the first card is an ace of clubs and the second is black?

$P(\text{First card is ace of clubs}) = \frac{1}{52}$ $P(\text{Second card is black given first card is ace of clubs}) = \frac{25}{51}$

Combining the two, the overall probability is $\frac{25}{51\times 52}$. ■

16 Factories A and B produce computers. Factory A produces 3 times as many computers as factory B. The probability that an item produced by factory A is defective is 0.018 and the probability that an item produced by factory B is defective is 0.043. A computer is selected at random and it is found to be defective. What is the probability it came from factory A?

Let A, B denote the events that a computer is produced by factories A, B respectively. Let D denote the event that the computer is defective.

We know that $P(A) = 3P(B)$ and $P(A) + P(B) = 1$. So, using these, we get, $P(A) = 0.75, P(B) = 0.25$. We are also given $P(D|A) = 0.018, P(D|B) = 0.043$.

Using these, we get $P(D) = P(D \cap A) + P(D \cap B) = P(D|A) \times P(A) + P(D|B) \times P(B) = 0.75 \times 0.018 + 0.25 \times 0.043$.

Now, $P(A|D) = P(A \cap D)/P(D) = \frac{0.75\times 0.018}{0.75\times 0.018+0.25\times 0.043}$

■

17 A box contains 20 yellow, 22 green and 29 red jelly beans. If 8 jelly beans are selected at random, what is the probability that:

1. 2 are yellow?
2. 2 are yellow and 5 are green?
3. At least one is yellow?

Answers are:

1. $\frac{\binom{20}{2}\times\binom{51}{6}}{\binom{71}{8}}$
2. $\frac{\binom{20}{2}\times\binom{22}{5}\times\binom{29}{1}}{\binom{71}{8}}$
3. $1-\frac{\binom{51}{8}}{\binom{71}{8}}$

■

18 Events A_1, A_2 and A_3 form a partiton of the sample space S with probabilities $P(A_1) = 0.3, P(A_2) = 0.2, P(A_3) = 0.5$. If E is an event in S with $P(E|A_1) = 0.3, P(E|A_2) = 0.5, P(E|A_3) = 0.8$, compute

1. $P(E)$
2. $P(A_1|E)$
3. $P(A_2|E)$
4. $P(A_3|E)$

$$\begin{aligned}
P(E\cap A_1) &= P(E|A_1)\times P(A_1) = 0.3\times 0.3 = 0.09\\
P(E\cap A_2) &= P(E|A_2)\times P(A_2) = 0.5\times 0.2 = 0.1\\
P(E\cap A_3) &= P(E|A_3)\times P(A_3) = 0.8\times 0.5 = 0.4\\
P(E) = P(E\cap A_1) &+ P(E\cap A_2) + P(E\cap A_3) = 0.59\\
P(A_1|E) &= \frac{P(A_1\cap E)}{P(E)} = 0.09/0.59\\
P(A_2|E) &= \frac{P(A_2\cap E)}{P(E)} = 0.1/0.59\\
P(A_3|E) &= \frac{P(A_3\cap E)}{P(E)} = 0.4/0.59
\end{aligned}$$

■

19 A card is drawn from a regular deck of 52 cards and is then put back in the deck. A second card is drawn. What is the probability that:

1. The first card is red.
2. The second card is hearts given that the first is red.
3. The first card is red and the second is hearts.

Answers are:

1. $\frac{26}{52} = 0.5$
2. The outcome of second card doesn't depend on first card because outcomes of second and first draws are independent events. So, probability of "the second card is hearts given that the first is red" is same as probability of "the second card is hearts". This is $\frac{13}{52} = 0.25$
3. $P[\text{First card is red}] = \frac{26}{52} = 0.5$. And, $P[\text{Second card is hearts}] = \frac{13}{52} = 0.25$. Overall probability is $0.5 \times 0.25 = 0.125$.

■

20 If $P(E \cap F) = 0.225, P(E|F) = 0.45, P(F|E) = 0.5$, find:

1. $P(E)$
2. $P(F)$
3. $P(E \cup F)$

The answers are:

1. $P(E) = P(F \cap E)/P(F|E) = 0.225/0.5 = 0.45$
2. $P(F) = P(F \cap E)/P(F|E) = 0.225/0.45 = 0.5$
3. $P(F \cup E) = P(F) + P(E) - P(F \cap E) = 0.5 + 0.45 - 0.225 = 0.725$

■