MATH 225

Linear Algebra and Differential Equations

Summer 2017 (Bülent Ünal)

Lecture Notes

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Introduction to Differential Equations

Definition: A differential equation is an equation involving derivatives of some unknown functions.

Ex) $x^2 + 3x - 2 = 0$ is not a DE. (no derivative)

$\frac{d}{dx} \sin x = \cos x$ is not a DE. (no unknown function)

$\frac{dy}{dx} = 2x$ is an ordinary differential equation. $y = y(x)$ is the unknown.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$ is a partial differential equation. $u = u(x,y)$ is the unknown.

Definition: A differential equation is ODE if it only involves ordinary derivatives. (The unknown functions are functions of the same variable.

Definition: The order (degree) of an ODE is the highest order of derivatives.

Ex) $y'' + y' = 2y$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$ Second degree ODE

$\left(\frac{dy}{dx}\right)^4 + \frac{d^3y}{dx^3} = y - x$

ODE of order 3

Notation: An n^{th} order ODE is a relation between x (independent variable), y (unknown function), $y', y'', y''' \ldots y^{(n)}$

$(*)$: $F(x, y, y', y'' \ldots y^{(n)}) = 0$, $x \in I \rightarrow$ domain

Ex) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x}}$, $I = (1, \infty)$ since $\sqrt{1-x} \neq 0$ and $\sqrt{1-x}$ is defined.

Ex) $n=1 \Rightarrow (*): F(x, y, y') = 0$ $(*)$ note that n can't be 0.

$n=2 \Rightarrow (*): F(x, y, y', y'') = 0$

Definition: Let (*) be an n^{th} order ODE of the form (*): $F(x, y, y', \dots y^{(n)}) = 0$, $x \in I$. A function f which satisfies (i) $f', f'', f''', \dots f^{(n)}$ are all defined on I. (ii) $F(x, f, f', \dots f^{(n)}) = 0$ is a solution to the DE (*).

Ex) $y'' + y = 0$. Verify that $f(x) = 1984\cos x + 2017\sin x$ is a solution.

$f(x) = 1984\cos x + 2017\sin x$, $x \in I$ ✓
$f'(x) = -1984\sin x + 2017\cos x$, $x \in I$ ✓
$f''(x) = -1984\cos x - 2017\sin x$, $x \in I$ ✓

$f''(x) - f(x) \stackrel{\checkmark}{=} 0$
$\Rightarrow y = f(x)$ is a solution of $y'' + y = 0$.

Definition: A solution of an ODE is explicit if it's given by $y = f(x)$. It's implicit if it's given by $F(x, y) = 0$.

Ex) $\dfrac{dy}{dx} = \dfrac{1 - y\cos(xy)}{x\cos(xy) + 2y}$: (*) | Verify that the following relation is a solution: $[\sin(xy) + y^2 - x = 0]$

$y = f(x)$ can't be expressed algebraicly.

d/dx both sides; $\cos(xy) \cdot \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) - 1 = 0$

$\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y \cdot \frac{dy}{dx} - 1 = 0$

$y\cos(xy) + x\cos(xy) \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$: (*) ✓

Exercise) Verify $e^{y/x} + xy^2 - x = 5$ is a sol for $\dfrac{dy}{dx} = \dfrac{x^2(1 - y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}$.

Verify $y(x) = x^2 - x^2\ln x + \frac{1}{6}x^2(\ln x)^3$ is a sol for

$x^2y'' - 3xy' + 4y = x^2\ln x$, $x > 0$.

Solution to an ODE → General solutions (particular sol.)
→ Singular solutions

Definition: Let (*) be n^{th} order ODE, $F(x, y, y', \ldots y^{(n)}) = 0$, $x \in I$
A solution $y = f(x)$ is a general solution if $y = f(x)$ contains exactly
many ← n-free variables (parameters) say $c_1, c_2, \ldots, c_n$ and $y = f(x)$ is
indeed a solution for any choice of $c_1, c_2, \ldots, c_n$.

Ex (*): $y'' + y = 0$ $(n=2)$ Claim: $f(x) = c_1 \cos x + c_2 \sin x$ is a general solution for (*).

$f'(x) = -c_1 \sin x + c_2 \cos x$
$f''(x) = -c_1 \cos x - c_2 \sin x \Rightarrow f''(x) + f(x) = 0$, i.e, $f(x) = c_1 \cos x + c_2 \sin x$ is a general solution

Definition: A solution $y = f(x)$ of an ODE is a singular solution if it cannot be obtained from the general solution by using some particular values of the free variables.

Ex (*): $(y')^2 + (y-1)^2 = 0$ (If $A^2 + B^2 = 0$, $A = B = 0$)
$y' = 0$ and $(y-1) = 0 \Rightarrow y(x) \equiv c$ and $y(x) \equiv 1 \Rightarrow$ $y(x) \equiv 1$ is the only solution.
Thus (*) fails to have a general solution. The singular solution "$y(x) \equiv 1$" is the unique solution for (*)

Ex $(y')^2 = 4y$ a) Verify $f(x) = (x+c)^2$ is a gen.sol. b) Is there a sing. sol?

a) LHS: $(y')^2 = [2(x+c)]^2 = 4(x+c)^2$
RHS: $4y = 4 \cdot (x+c)^2$
$\Big\}$ LHS = RHS $\Rightarrow$ $f(x) = (x+c)^2$ is a general solution.

b) $y(x) \equiv 0$ is a solution $0^2 = 0$. $y(x) \equiv 0$ cannot be obtained from the general solution, namely $f(x) = (x+c)^2$. Thus $y(x) \equiv 0$ is a singular solution.

Ex $y = xy' + (y')^2$ a) Show that $cx + c^2$ is a gen. sol.
b) Show that $\frac{-x^2}{4}$ is a sing. sol.

a) RHS: $xy' + (y')^2 = x(c) + (c)^2 = cx + c^2$

LHS: $y = cx + c^2$ RHS = LHS $\Rightarrow$ $y(x) = cx + c^2$ is a general solution.

b) RHS: $x \cdot \frac{-x}{2} + \left(\frac{-x}{2}\right)^2 = \frac{-x^2}{2}$

LHS: $y = \frac{-x^2}{2}$ RHS = LHS

$\frac{-x^2}{4}$ cannot be obtained from $cx + c^2$ thus $y(x) = \frac{-x^2}{4}$ is a singular solution.

Exercise $y' = -2y^{3/2}$ a) Verify $y(x) = \frac{1}{(x+c)^2}$ b) Is there a sing. sol?

Definition: A particular solution can be obtained from the general solution for some suitable values of the free variables of gen. sol.

Ex
(IVP)
Initial Value Problem of order 1.

$$\begin{cases} \frac{dy}{dx} = e^x - \sin x \\ y(0) = 2 \end{cases}$$

$\int dy = \int (e^x - \sin x)\, dx \Rightarrow y(x) = e^x + \cos x + C$ is the gen. sol.

$y(0) = e^0 + \cos 0 + c$
$2 = 1 + 1 + C$ $\Rightarrow$ $y(x) = e^x + \cos x$ is part. sol. to DE. is the sol to IVP.

⊛ IVP of order 1:

$$\begin{cases} F(x, y, y') = 0 \\ y(a) = b \end{cases}$$

IVP of order 2:

$$\begin{cases} F(x, y, y', y'') = 0 \\ y(a) = b_1,\ y'(a) = b_2 \end{cases}$$

IVP of order n:

$$\begin{cases} F(x, y, y', \dots, y^{(n)}) = 0 \\ y(a) = b_1,\ y'(a) = b_2 \dots y^{(n)}(a) = b_n \end{cases}$$

Ex

$$\begin{cases} \frac{d^2y}{dx^2} = \frac{1}{x}, \quad x > 0 \\ y(2) = 3 \\ y'(2) = 0 \end{cases}$$

$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x}$

$\int d\left(\frac{dy}{dx}\right) = \int \frac{dx}{x} \Rightarrow \frac{dy}{dx} = \ln x + c_1$ ⊛

$\int dy = \int (\ln x + c_1)\, dx \Rightarrow \underbrace{y = (x\ln x - x) + c_1 x + c_2}_{\text{general solution}}$

⊛: $y'(2) = \ln 2 + c_1 = 0$
$c_1 = -\ln 2$

$y(2) = 2\ln 2 - 2 - 2\ln 2 + c_2 = 3$
$c_2 = 5$

$y(x) = x\ln x - x - \ln 2 \cdot x + 5$ is the sol. for IVP.

(*) The geometry of some ODE of Order 1;

$M(x,y)\,dx + N(x,y)\,dy = 0$ or equivalently $y' = \frac{dy}{dx} = f(x,y)$

<u>The Existance and Uniqueness Theorem:</u> Let (*) be a 1^{st} order IVP of the form: $(*): \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(a) = b \end{cases}$

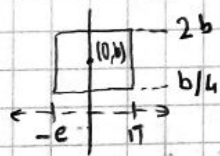
(a) If f is continuous on a rectangular region R of the form $R = [a_1, a_2] \times [b_1, b_2]$ where $a_1 < a < a_2$ and $b_1 < b < b_2$, then the IVP has a solution. (Existance part)

(b) If both f and $\frac{\partial f}{\partial y}$ are continuous on a rectangular region R of the form $R = [a_1, a_2] \times [b_1, b_2]$ where $a_1 < a < a_2$ and $b_1 < b < b_2$, then the IVP has a unique solution. (Uniqueness part)

<u>Ex</u> $\left.\begin{array}{l} \frac{dy}{dx} = 3xy^{1/3} \\ y(0) = b \end{array}\right\}$ Show that the IVP has a unique sol. for all $b > 0$.

$f(x,y) = 3xy^{1/3}$ is continuous everywhere on $\mathbb{R}^2$

$\frac{\partial f}{\partial y} = 3x \cdot \frac{1}{3} \cdot y^{-2/3} = \frac{x}{\sqrt[3]{y^2}}$ is continuous when $y \neq 0$.



$b > 0 \Rightarrow \frac{b}{4} < b < 2b$, Take $R = [-e, \pi] \times [\frac{b}{4}, 2b]$

Both f and $\frac{\partial f}{\partial y}$ are continuous on R, by EU there exist a unique solution for the IVP.

(*) PAGES 29 - 33

<u>Ex</u> (#28 pg29) Verify that $y(x) = kx$ satisfies the ODE $xy' = y$. Then determine (in terms of a and b) how many different solutions of the IVP $\begin{cases} xy' = y \\ y(a) = b \end{cases}$ has one, none or infinitely many.

$y(x)=kx \Rightarrow y'=k$

$xy'=y$

$x\cdot k=kx$

$LHS=RHS \Rightarrow$ verified

$y(a)=b$

$a\cdot y'(a)=y(a)$

$a\cdot y'(a)=b$

$a=0\ ,\ b\neq 0$

$0\cdot y'(0)=b$

$b=0 \Rightarrow$ contradiction

the IVP has no solution if $\begin{matrix} a=0 \\ b\neq 0\end{matrix}$

$a=0\quad b=0$

$0\cdot y'(0)=y(0)$

$0=b$ then,

$y=kx$ is a solution

The IVP has inf. many solutions.

$a\neq 0$

Solutions of Some Special ODE's of order 1

1) Seperable ODE of order 1: An ODE of order 1 is said to be seperable if it is of the form $\dfrac{dy}{dx}=f(x)\cdot g(y)$

Ex) $y'=x+y$ (not seperable) $\quad y'=\tan(xy)$ (not seperable)

Ex) $(x-4)y^4\,dx - x^3(y^2-3)\,dy=0$

$$\frac{dy}{dx}=\frac{(x-4)y^4}{x^3(y^2-3)}=\left(\frac{x-4}{x^3}\right)\cdot\left(\frac{y^4}{y^2-3}\right)$$

so the ODE is seperable

Method to solve seperable ODE of order 1: $\dfrac{dy}{dx}=f(x)\cdot g(y) \Rightarrow$

$$dy=f(x)\cdot g(y)\cdot dx \Rightarrow \int\frac{dy}{g(y)}=\int f(x)\cdot dx \Rightarrow G(x)=F(x)+C$$

$\rightarrow$ Put all x's one side and all y's to the other side

Ex) (The last one continued) $\quad \dfrac{y^2-3}{y^4}\,dy=\dfrac{x-4}{x^3}\,dx \overset{\int}{\Longrightarrow}$

$$\int(y^{-2}-3y^{-4})\,dy=\int(x^{-2}-4x^{-3})\,dx \overset{*}{\Longrightarrow} \boxed{\frac{1}{y^3}-\frac{1}{y}=\frac{2}{x^2}-\frac{1}{x}+C}\ \checkmark$$

The general solution

$$*:\ \frac{y^{-1}}{-1}-3\frac{y^{-3}}{-3}=\frac{x^{-1}}{-1}-4\cdot\frac{x^{-2}}{-2}+C \Rightarrow \frac{-1}{y}+\frac{1}{y^3}=\frac{-1}{x}+\frac{2}{x^2}+C$$

Ex $x \sin y \, dx + (x^2+1)\cos y \, dy = 0$, $y(1) = \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{-x \sin y}{(x^2+1)\cos y} = \left(\frac{-x}{x^2+1}\right) \cdot \frac{\sin y}{\cos y} \Rightarrow -\frac{\cos y}{\sin y} = \frac{x}{x^2+1} dx \Rightarrow ⊛$$

$$-\int \frac{\cos y}{\sin y} dy = \int \frac{x}{x^2+1} dx \quad \begin{matrix} u = x^2+1 \\ du = 2x \end{matrix} \Rightarrow \int \frac{x}{x^2+1} dx = \int \frac{du}{2u} = \frac{1}{2}\ln|u| + c_1 = \frac{1}{2}\ln(x^2+1) + c_1$$

$$\begin{matrix} w = \sin y \\ dw = -\cos y \end{matrix} \Rightarrow \int \frac{-\cos y}{\sin y} = \int \frac{dw}{w} = \ln|w| + C_2 = \ln|\sin y| + C_2$$

$$⊛ \Rightarrow -\ln|\sin y| = \ln\sqrt{x^2+1} + \ln C \Rightarrow \ln\left(\frac{1}{\sin y}\right) = \ln\left(C \cdot \sqrt{x^2+1}\right) \xrightarrow{\ln \text{ is } 1\text{-}1} \underbrace{\frac{1}{\sin y} = C \cdot \sqrt{x^2+1}}_{\text{general solution } (0<y<\pi)}$$

y is near $\frac{\pi}{2}$, $\sin y > 0$, abs(.) is gone.

$$y(1) = \frac{\pi}{2} : \frac{1}{\sin\frac{\pi}{2}} = C \cdot \sqrt{1^2+1} \Rightarrow C = \frac{1}{\sqrt{2}} \Rightarrow y = \arcsin\sqrt{\frac{2}{x^2+1}} \Big\} \text{ soln. to the IVP.}$$

Exercise $8\cos^2 y \, dx + \csc^2 x \, dy = 0$, $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$(3x+8)(y^2+4)\,dx - 4y(x^2+5x+6)\,dy = 0$, $y(1) = 2$

2) Linear ODE's of Order 1: Linear 1^{st} ODE is an ODE of the form ⊛: $\frac{dy}{dx} + p(x) \cdot y = q(x)$ (Remark: If $p(x)$ or $q(x)$ is 0, then its seperable.)

Method to solve: To transform ⊛ into a seperable ODE by multiply both sides of ⊛ by $\mu(x) = e^{\int p(x)\,dx}$ (Remark: $\frac{d}{dx}\mu(x) = p(x) \cdot \mu(x)$)

$$\underbrace{\mu \cdot \frac{dy}{dx} + \mu \cdot p(x) \cdot y}_{} = \mu \cdot q(x) \longrightarrow \text{by Remark}$$

$$\frac{d}{dx}(\mu \cdot y) = \mu \cdot q(x) \Rightarrow \text{Now this became a seperable equation.}$$

$$d(\mu \cdot y) = \mu \cdot q(x) \cdot dx \xRightarrow{\int} \mu \cdot y = \int \mu \cdot q(x) \cdot dx$$

→ Multiply both sides with a term $e^{(\text{integral of } p(x))}$

$$\frac{dy}{dx} + p(x) \cdot y = q(x)$$

Ex] $(x^2+1)\frac{dy}{dx} + 4xy = x$ $\rightarrow p(x) = \frac{4x}{x^2+1} \Rightarrow \int p(x)dx = \int \frac{2\,du}{u} = 2\ln|x^2+1| + c$

$2x\,dx = du$, $x^2+1 = u$

*DOES NOT CONTRIBUTE TO SOL

$$\frac{dy}{dx} + \underbrace{\frac{4x}{x^2+1}}_{p(x)}y = \frac{x}{x^2+1}$$

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$m(x) = e^{\int p(x)dx} = e^{2\ln(x^2+1)} = [e^{\ln(x^2+1)}]^2 = (x^2+1)^2$

So we multiply both sides with $(x^2+1)^2$

$$\underbrace{(x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)y}_{} = x(x^2+1)$$

$$\frac{d}{dx}\left((x^2+1)^2\cdot y\right) = x(x^2+1)$$

$\int d((x^2+1)^2\cdot y) = \int (x^3+x)\,dx$

$(x^2+1)^2\cdot y = \frac{x^4}{4} + \frac{x^2}{2} + C$

$y = \frac{1}{(x^2+1)^2}\cdot\left(\frac{x^4}{4} + \frac{x^2}{2} + C\right)$ → General solution

Ex] $y^2dx + (3xy-1)\,dy = 0$

$\frac{dy}{dx} = \frac{y^2}{1-3xy}$ (not seperable)

$\frac{dy}{dx} - \frac{y^2}{1-3xy} = 0$ (not linear)

* So we think the other way around.

$\frac{dx}{dy} = \frac{1-3xy}{y^2} \Rightarrow \frac{dx}{dy} + \left(\frac{3}{y}\right)\cdot x = \frac{1}{y^2}$

(is linear) $\frac{dx}{dy} + p(y)\cdot x = q(y)$ ✓

$\int p(y)dy = \int \frac{3}{y}dy = 3\ln|y| + C \Rightarrow m = e^{\int p(y)dy} = e^{3\ln y} = y^3$, Multiply both sides

$\underbrace{y^3\cdot\frac{dx}{dy} + 3y^2x}_{} = y \Rightarrow \frac{d}{dy}\left(y^3\cdot x\right) = y$ (is seperable now)

$\int d(y^3x) = \int y\,dy \Rightarrow y^3x = \frac{y^2}{2} + C \Rightarrow x = \frac{1}{2y} + \frac{C}{y^3}$ } → General solution

$\underline{3)}$ * $\frac{dy}{dx} = F(ax+by+c)$, $a,b,c \in \mathbb{R}$ Ex] $\frac{dy}{dx} = \sqrt[3]{\tan(x+2y-1)}$

Method to solve: To substituta $u = ax+by+c$ so that * becomes separable.

$u = ax+by+c$ $\frac{dy}{dx} = F(u)$

$\frac{du}{dx} = a + b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{du}{dx} - a\right)$

$\frac{1}{b}\left(\frac{du}{dx} - a\right) = F(u)$ (seperable)

$du = (b\cdot F(u) + a)\,dx$

Ex) $\frac{dy}{dx} = \sqrt{x+y+1}$, $u = x+y+1 \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \underbrace{\frac{dy}{dx} = \frac{du}{dx} - 1}_{\text{combine}}$

$\frac{dy}{dx} = \frac{du}{dx} - 1 = \sqrt{u} \Rightarrow \int dx = \int \frac{du}{1+\sqrt{u}} \Rightarrow \begin{matrix} u = w^2 \\ du = 2w\,dw \end{matrix}$, $\int \frac{du}{1+\sqrt{u}} = \int \frac{2w}{1+w} dw$

$\int \frac{2w}{1+w} dw = \int \left(2 + \frac{-2}{1+w}\right) dw = 2w - 2\ln|1+w| + C = 2\sqrt{u} - 2\ln(\sqrt{u}+1) + C$

$x = 2\sqrt{u} - 2\ln(\sqrt{u}+1) + \ln C \Longrightarrow 2\sqrt{x+y+1} + 2\ln\left(\frac{C}{(\sqrt{x+y+1}+1)^2}\right) - x = 0$

Exercise) $\frac{dy}{dx} = \tan(x+y+2)$, $\frac{dy}{dx} = \sqrt[3]{2x-y+3}$, $\frac{dy}{dx} = \sec(x+y+1)$

Ex) $\frac{dy}{dx} = \tan(x+y+1)$ $u = x+y+1 \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$ $\frac{dy}{dx} = \frac{du}{dx} - 1 = \tan(u)$

$\frac{du}{dx} = \tan u + 1$ $dx = \frac{du}{\tan u + 1}$ $\int \frac{du}{1+\tan u} = \int \frac{\cos u \, du}{\cos u + \sin u} = \int \frac{\cos u(\cos u - \sin u)\, du}{\cos^2 u - \sin^2 u}$

$= \int \frac{\cos u(\cos u - \sin u)\, du}{\cos 2u} = \int \frac{\cos^2 u}{\cos 2u} du - \int \frac{\cos u \sin u}{\cos 2u} du$ $\left[\begin{matrix} \cos 2u = 2\cos^2 u - 1 \\ \cos^2 u = \frac{\cos 2u + 1}{2} \end{matrix} \right.$

$= \int \frac{\cos 2u + 1}{2\cos 2u} du - \int \frac{\frac{1}{2}\sin 2u}{\cos 2u} du = \frac{1}{2} \cdot \int \left(1 + \frac{1}{\cos 2u}\right) du - \frac{1}{2} \cdot \int \tan 2u \, du =$ $\left[\begin{matrix} \cos 2u = w \\ -2\sin 2u = dw \end{matrix} \right.$

$\frac{1}{2}\left(u + \int \sec 2u \, du + \frac{1}{2} \cdot \ln|\cos 2u|\right) =$ $\left[\int \sec 2u \, du \overset{6}{=} \frac{1}{2} \cdot \int \sec k \, dk = \frac{1}{2} \cdot \int \frac{\sec k(\tan k + \sec k)}{\sec k + \tan k} dk \right.$

$= \frac{1}{2} \ln|\sec k + \tan k| + C$

$\Rightarrow x + C = \frac{1}{4} \ln|\sec 2u + \tan 2u| + \frac{u}{2} + \frac{1}{2} \ln|\cos 2u| =$ (substitute $u = x+y+1$)

4) Homogeneous ODE of order 1: The form of $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Method to solve: To substitute $z = \frac{y}{x}$ so that (*) can be transformed into a seperable ODE.

$z=\frac{y}{x} \Rightarrow y=xz \quad \frac{dy}{dx}=z+x\frac{dz}{dx}$, $(*)$: $\frac{dy}{dx}=F(z)$

$$\frac{dy}{dx}=z+x\frac{dx}{dz}=F(z) \Rightarrow x\frac{dz}{dx}=F(z)-z \Rightarrow \frac{dx}{x}=\frac{dz}{F(z)-z}$$

<u>Ex|</u> $(x^2-3y^2)dx+2xydy=0$: $(*)$

$$\frac{dy}{dx}=\frac{3y^2-x^2}{2xy}=\frac{3}{2}\cdot\frac{y}{x}-\frac{1}{2}\cdot\frac{x}{y} \Rightarrow \frac{dy}{dx}=\overbrace{\frac{3}{2}\cdot z-\frac{1}{2}\cdot\frac{1}{z}}^{F(z)\ \Rightarrow\ \text{so } (*) \text{ is homogeneous.}}$$

$$z=\frac{y}{x} \Rightarrow y=xz,\ \frac{dy}{dx}=z+x\frac{dz}{dx}=\frac{3z}{2}-\frac{1}{2z} \Rightarrow x\frac{dz}{dx}=\frac{z^2-1}{2z}$$

$$\Rightarrow \frac{dx}{x}=\frac{2z}{z^2-1}dz \Rightarrow \int\frac{2z}{z^2-1}dz=\ln|z^2-1|+C_1 \Rightarrow \ln|x|=\ln|z^2-1|+C_1$$

$\Rightarrow \ln|x|+\ln C=\ln|z^2-1|$, If $(y/x)^2>0$, $x>0$ then $\ln(z^2-1)=\ln(cx)$

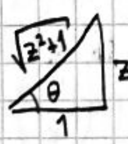
$$\boxed{Cx=\left(\frac{y}{x}\right)^2-1}$$ General solution

<u>Ex|</u> $(y+\sqrt{x^2+y^2})dx-x\,dy=0$

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}=\frac{y}{x}+\sqrt{\frac{x^2+y^2}{x^2}}=\overbrace{\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}}^{F(y/x)}=z+\sqrt{1+z^2}$$

$$z=\frac{y}{x},\ y=zx \Rightarrow \frac{dy}{dx}=z+x\frac{dz}{dx}=z+\sqrt{1+z^2} \Rightarrow \frac{dx}{x}=\frac{dz}{\sqrt{1+z^2}}$$

$\int\frac{dz}{\sqrt{1+z^2}}$ $\quad z=\tan\theta$, $dz=\sec^2\theta\,d\theta$, $1+\tan^2\theta=\sec^2\theta$ $\Rightarrow \int\frac{\sec^2 d\theta}{\sqrt{\sec^2\theta}}=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1$



$\ln|\sqrt{z^2+1}+z|=\ln|x|+\ln C$ If $x,y>0$: $\ln(\sqrt{z^2+1}+z)=\ln(xC)$

$$\sqrt{\left(\frac{y}{x}\right)^2+1}+\frac{y}{x}=xC$$

<u>5) Bernoulli Type ODE's of Order 1:</u> ODE's of given by;

$(*)$: $\frac{dy}{dx}+p(x)y=q(x)y^n$ $\quad$ <u>Remark</u>: $(*)$ is nongeneric if $n\neq 0$

Method to solve Bernoulli: Substitute $v = y^{1-n}$ to transform (*) into a Linear ODE.

$$\left. v = y^{1-n} \Rightarrow \frac{dv}{dx} = (1-n)\cdot y^{-n}\cdot\frac{dy}{dx} \right\} ①$$

This is Linear in "v"

$$(*): \ y^{-n}\frac{dy}{dx} + p(x)\cdot y^{1-n} = q(x)$$

①

$$\left(\frac{1}{1-n}\right)\cdot\frac{dv}{dx} + p(x)\,v = q(x) \Rightarrow \underbrace{\frac{dv}{dx} + (1-n)\,p(x)\,v = (1-n)\,q(x)}$$

Ex $\frac{dy}{dx} + y = x y^3 \ : (*)$ is a Bernoulli type with $n=3$

$$v = y^{1-n} = y^{-2} \Rightarrow v = y^{-2}, \ \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$(*): \frac{dy}{dx} + y = xy^3 \xRightarrow{1/y^3} \frac{1}{y^3}\cdot\frac{dy}{dx} + \frac{1}{y^2} = x \Rightarrow \frac{-1}{2}\cdot\frac{dv}{dx} + v = x$$

$$\Rightarrow \boxed{\frac{dv}{dx} - 2v = -2x} \Big\} \text{Linear in "v"}$$

Recall: $\frac{dy}{dx} + p(x)y = q(x)$, $\mu = e^{\int p(x)dx}$

(Verification is good)

$$m = e^{\int -2dx} = e^{-2x} \longrightarrow e^{-2x}\frac{dv}{dx} - 2ve^{-2x} = -2xe^{-2x} \Longrightarrow \frac{d}{dx}\left(e^{-2x}v\right) = -2xe^{-2x}$$

$$\int d\left(e^{-2x}v\right) = \int -2xe^{-2x}\,dx \quad \Big\{ \int xe^{-2x}dx \quad \begin{matrix} x = a & da = 1 \\ e^{-2x}dx = db & b = \frac{e^{-2x}}{-2} \end{matrix} \Big\} \ x\cdot\frac{e^{-2x}}{-2} - \int\frac{e^{-2x}}{-2}dx = \frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$e^{-2x}v = -2\cdot\frac{-1}{2}\left(xe^{-2x} + \frac{e^{-2x}}{2}\right) + C \xRightarrow{v=\frac{1}{y^2}} \frac{1}{y^2} = \frac{1}{x + ce^{2x} + 1/2} \quad (?)$$

Ex $x\cdot\frac{dy}{dx} + y = (xy)^{3/2}$, $y(1) = 2$

$$\left.\frac{dy}{dx} + \frac{y}{x} = x^{1/2}\cdot y^{3/2}\right\} \text{Bernoulli Type} \qquad v = y^{1-n} = y^{1-\frac{3}{2}} = y^{-1/2}$$

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$$V = y^{-1/2} \Rightarrow \frac{dv}{dx} = -\frac{1}{2} y^{-3/2} \frac{dy}{dx} \Rightarrow -2\frac{dv}{dx} = \frac{1}{y^{3/2}} \cdot \frac{dy}{dx} \quad \Big\} \ 1$$

$$(*) : \frac{dy}{dx} + \frac{y}{x} = x^{1/2} \cdot y^{3/2} \xrightarrow{1/y^{3/2}} \frac{1}{y^{3/2}} \cdot \frac{dy}{dx} + \frac{1}{y^{1/2}} \cdot \frac{1}{x} = x^{1/2} \quad \Big\} \ 1$$

$$-2\frac{dv}{dx} + \frac{1}{x} v = x^{1/2} \Rightarrow \frac{dv}{dx} - \frac{1}{2x} \cdot v = \frac{-x^{1/2}}{2} \quad \text{(Linear in "v")}$$

$$p(x) = \frac{-1}{2x} \quad e^{\int \frac{-1}{2x} dx} = e^{(-1/2)\ln x} = x^{-1/2} \rightarrow x^{-1/2} \frac{dv}{dx} - \frac{1}{2x} \cdot x^{-1/2} = \frac{-1}{2}$$

verif. time

$$\Rightarrow \frac{d}{dx}\left(x^{-1/2} \cdot v\right) = \frac{-1}{2} \xrightarrow{\int} x^{-1/2} v = \frac{-x}{2} + C \Rightarrow \text{(back substitute v)}$$

$$x^{-1/2} y^{-1/2} = \frac{-x}{2} + C \qquad \sqrt{xy} = \frac{2}{\sqrt{2}+1-x}$$

(general solution) (particular solution) → Use $y(1) = 2$ to find C.

6) Ricatti Type of ODE's of order 1:

A 1^{st} order ODE is said to be Ricatti type if it's;

$$\frac{dy}{dx} = A(x) y^2 + B(x) y + C(x) \quad : (*)$$

Method to solve: If $y = f(x)$ is a soln. to $(*)$, then substitute $y = f(x) + \frac{1}{u}$ (where $u = u(x)$), so that $(*)$ can be reduced to a linear equation.

$y = f(x)$ is a soln: $\frac{df}{dx} = A(x)[f(x)]^2 + B(x) f(x) + C(x)$

$$y = f + u^{-1} \Rightarrow \frac{dy}{dx} = \frac{df}{dx} - u^{-2} \cdot \frac{du}{dx},$$

$$\frac{dy}{dx} = A(x) y^2 + B(x) y + C(x) \Rightarrow \frac{df}{dx} - \frac{1}{u^2} \frac{du}{dx} = A(x)\left(f + \frac{1}{u}\right)^2 + B(x)\left(f + \frac{1}{u}\right) + C(x)$$

$$\frac{df}{dx} - \frac{1}{u^2}\cdot\frac{du}{dx} = Af^2 + \frac{2Af}{u} + \frac{A}{u^2} + Bf + \frac{B}{u} + C$$ → All cancels by $y=f(x)$ solution.

$$-\frac{1}{u^2}\cdot\frac{du}{dx} = \frac{2Af}{u} + \frac{A}{u^2} + \frac{B}{u} \Rightarrow \frac{du}{dx} = -2Afu - A - Bu$$

$$\frac{du}{dx} + (2Af + B)u = -A$$ (linear in "u")

Ex) $\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x$, given solution $f(x)=1$

$f(x)\equiv 1$ is a solution of: $LHS = \frac{dy}{dx} = \frac{d(1)}{dx} = 0$

$RHS = (1-x)1^2 + (2x-1)1 - x = 0$, $LHS = RHS \Rightarrow f(x)=1$ is a soln.

$$y = f + \frac{1}{u} \Rightarrow y = 1 + u^{-1} \Rightarrow \frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx} = (1-x)\left(1+\frac{1}{u}\right)^2 + (2x-1)\left(1+\frac{1}{u}\right) - x$$

$$\frac{-1}{u^2}\cdot\frac{du}{dx} = (1-x)\left(1+\frac{2}{u}+\frac{1}{u^2}\right) + (2x-1)\left(1+\frac{1}{u}\right) - x$$

$$\frac{-1}{u^2}\cdot\frac{du}{dx} = 1 + \frac{2}{u} + \frac{1}{u^2} - x - \frac{2x}{u} - \frac{x}{u^2} + 2x + \frac{2x}{u} - 1 - \frac{1}{u} - x$$

$$\frac{-1}{u^2}\cdot\frac{du}{dx} = \frac{1}{u} + \frac{1}{u^2} - \frac{x}{u^2} \Rightarrow \frac{du}{dx} = -u - 1 + x \Rightarrow \overbrace{\frac{du}{dx} + u = x - 1}^{\text{linear in "u"}}$$

$\mu = \int 1\,dx = x + C$, $e^x\frac{du}{dx} + e^x u = xe^x - e^x \Rightarrow \frac{d}{dx}(u\cdot e^x) = xe^x - e^x$

$$\int d(ue^x) = \int (xe^x - e^x)dx \quad \left\{ \begin{array}{l} \int xe^x\,dx = a\cdot b - \int b\,da = xe^x - \int e^x \\ a = x \quad da = dx \qquad\qquad = xe^x - e^x + C \\ db = e^x dx \quad b = e^x \end{array}\right.$$

$$u\cdot e^x = xe^x - e^x - e^x + C$$

$$u = x - 2 + \frac{C}{e^x} \Rightarrow \text{soln: } y = 1 + \frac{1}{u} = 1 + \frac{1}{x - 2 + \frac{C}{e^x}}$$

$\boxed{Ex}$ $\frac{dy}{dx} = -8xy^2 + 4x(4x+1)y - (8x^3+4x^2-1)$, given soln. $f(x)=x$

Verify $f(x)$ is a soln: LHS = RHS so $f(x)=x$ is indeed a solution.

$y = f + \frac{1}{u} = x + u^{-1} \Rightarrow \frac{dy}{dx} = 1 - u^{-2}\frac{du}{dx}$

$\frac{dy}{dx} = 1 - u^{-2}\frac{du}{dx} = -8x\left(x+\frac{1}{u}\right)^2 + (16x^2+4x)\left(x+\frac{1}{u}\right) - (8x^3+4x^2-1)$

$1 - \frac{1}{u^2}\frac{du}{dx} = -8x^3 - \frac{16x^2}{u} - \frac{8x}{u^2} + 16x^3 + \frac{16x^2}{u} + 4x^2 + \frac{4x}{u} - 8x^3 - 4x^2 + 1$

$-\frac{1}{u^2}\frac{du}{dx} = -\frac{8x}{u^2} + \frac{4x}{u} \Rightarrow \frac{du}{dx} = 8x - 4xu \Rightarrow \left.\frac{du}{dx} + 4xu = 8x \right\}$ linear in "u"

Multiply with $e^{\int 4x\,dx} = e^{2x^2}$ $\quad e^{2x^2}\frac{du}{dx} + e^{2x^2}4xu = e^{2x^2}8x$

$\frac{d}{dx}\left(e^{2x^2}\cdot u\right) = e^{2x^2}8x \Rightarrow \int d\left(e^{2x^2}\cdot u\right) = \int e^{2x^2}8x\,dx$ $\quad 2x^2 = t$, $4x\,dx = dt$

$\int e^t \cdot 2dt = 2e^t + C = 2e^{2x^2} + C \Rightarrow e^{2x^2}\cdot u = 2e^{2x^2} + C$

$u = \frac{2e^{2x^2}+C}{2e^{2x^2}} \Rightarrow y = x + \frac{2e^{2x^2}}{2e^{2x^2}+C}$

7) $(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$

$\frac{dy}{dx} = -\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ $\quad$ Remark: (*) is homogeneous if $c_1 = c_2 = 0$

We'll consider two cases: (a) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, (b) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

7a) $(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=k\neq 0$

Method: Substitute $z=a_2x+b_2y$, then $kz=a_1x+b_1y$

$(kz+c_1)dx+(z+c_2)dy=0 \Rightarrow \frac{dy}{dx}=-\frac{kz+c_1}{z+c_2}$

combine $\frac{dy}{dx}$ s

d/dx the substution: $\frac{dz}{dx}=a_2+b_2\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{b_2}\left(\frac{dz}{dx}-a_2\right)$

$\frac{dy}{dx}=\frac{1}{b_2}\left(\frac{dz}{dx}-a_2\right)=-\frac{kz-c_1}{z+c_2} \Rightarrow \frac{dz}{dx}=-\frac{b_2(kz+c_1)}{z+c_2}+a_2$ } seperable equation

Ex $(3x-y+1)dx-(6x-2y-3)dy=0$, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{-1}{2}$

$z=3x-y$

$-2z=-6x+2y$ $\Rightarrow (z+1)dx+(-2z+3)dy=0 \Rightarrow \frac{dy}{dx}=\frac{z+1}{2z-3}$

$\frac{dz}{dx}=3-\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=3-\frac{dz}{dx}$ $\quad 3-\frac{dz}{dx}=\frac{z+1}{2z-3} \Rightarrow \frac{dz}{dx}=3-\frac{z+1}{2z-3}$ } seperable equation

$\frac{dz}{dx}=\frac{6z-9-z-1}{2z-3}=\frac{5z-10}{2z-3} \Rightarrow \frac{2z-3}{5z-10}dz=dx$

$-1+1$

$\ln|5z-10|=\ln 5+\ln|$

$\int\frac{2z-3}{5z-10}dz=\int\left(\frac{2}{5}+\frac{1}{5z-10}\right)dz=\frac{2z}{5}+\frac{1}{5}\ln|5z-10|+C_1=$

$\frac{1}{5}(2z+\ln|z-2|+c)=x \Rightarrow 2(3x-y)+\ln|3x-y-2|+C=5x$

$6x-2y-5x+C=-\ln|3x-y-2| \Rightarrow x-2y+C=-\ln(3x-y-2)$

If $3x-y>2 \Rightarrow \underbrace{e^{-x+2y+C}=3x-y-2}_{\text{general solution}}$

7b) $(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2)\,dy = 0$, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Method: $x = u+h$, $y = v+k$ where $a_1h+b_1k+c_1=0$, $a_2h+b_2k+c_2=0$ to reduce (*) into homogenous equation.

Ex) $(x-2y+1)\,dx + (4x-3y-6)\,dy = 0$, $1/4 \neq -2/-3$

$$\left.\begin{array}{l} x = u+h \\ y = v+k \end{array}\right\} \begin{array}{l} h-2k+1=0 \Rightarrow h=2k-1 \\ 4h-3k-6=0 \Rightarrow 4(2k-1)-3k=6 \end{array} \left.\right\} \begin{array}{l} h=3 \\ k=2 \end{array}$$

$x = u+3$, $dx = du$

$y = v+2$, $dy = dv$ $\Rightarrow ((u+3)-2(v+2)+1)\,du + (4(u+3)-3(v+2)-6)\,dv = 0$

$(u-2v)\,du + (4u-3v)\,dv = 0$ $\qquad \left.\frac{dv}{du} = \frac{2v-u}{4u-3v}\right\}$ now this became homog. eq.

$\frac{dv}{du} = \frac{2\left(\frac{v}{u}\right)-1}{4-3\left(\frac{v}{u}\right)}$ $\quad z = \frac{v}{u}$ $\quad v = z\cdot u \Rightarrow \frac{dv}{du} = z + u\cdot\frac{dz}{du}$ $\qquad \left. z + u\cdot\frac{dz}{du} = \frac{2z-1}{4-3z}\right\}$ now seperable

$u\cdot\frac{dz}{du} = \frac{-2z+1}{3z-4} - z = \frac{-3z^2+2z+1}{3z-4} \Rightarrow \int \frac{3z-4}{-3z^2+2z+1}\,dz = \int \frac{du}{u} \to \ln|u| + C$

$-\int \frac{3z-4}{3z^2-2z-1}\,dz = -\int \frac{3z-4}{(3z+1)(z-1)}\,dz = -\int \left(\frac{A}{3z+1} + \frac{B}{z-1}\right) dz$ $\quad \begin{array}{ll} Az-A+3Bz+B = 3z-4 & \\ A+3B = 3 & B = -1/4 \\ -A+B = -4 & A = 15/4 \end{array}$

$-\frac{1}{4}\cdot\int\left(\frac{15}{3z+1} - \frac{1}{z-1}\right)dz = -\frac{1}{4}\left(15\cdot\frac{\ln|3z+1|}{3} - \ln|z-1|\right) = \frac{-5}{4}\ln|3z+1| + \frac{1}{4}\ln|z-1| + C_1$

$= \ln|z-1|^{1/4} - \ln|3z+1|^{5/4} + C_1$ $\boxed{z>1}$ $\Rightarrow \ln\frac{(z-1)^{1/4}}{(3z+1)^{5/4}} = \ln(uc)$ $\overset{\ln \text{ is } 1-1}{\Rightarrow}$

$\left[\frac{z-1}{(3z+1)^5}\right]^{1/4} = uc \Rightarrow \frac{z-1}{(3z+1)^5} = u^4c^4 \Rightarrow \frac{\frac{v}{u}-1}{\left[3\left(\frac{v}{u}\right)+1\right]^5} = k\cdot u^4$, $\quad u = x-3$, $v = y-2$

$\frac{\frac{y-2}{x-3}-1}{\left[3\left(\frac{y-2}{x-3}\right)+1\right]^5} = k\cdot(x-3)^4 \Rightarrow$ simplify and general solution

* $\int (x^3+1)$ hw 1^{st} (a) exercise from exam

8) Exact ODE's of order 1: A first order ODE of the form;

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $(M_y = N_x)$

Method: If $(*)$: $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, then the soln. is;

$F(x,y) = C$, $(c \in \mathbb{R})$ where $F_x = M$ and $F_y = N$.

Ex) $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$

$M = 3x^2 + 4xy \quad M_y = 4x$
$N = 2x^2 + 2y \quad N_x = 4x$
so the ODE is exact $M_y = N_x$.

$F(x,y) = c$ is the general solution and $F_x = M$, $F_y = N$

$$\frac{\partial F}{\partial x} = 3x^2 + 4xy \Rightarrow F(x,y) = \int (3x^2 + 4xy)\,dx = x^3 + 2x^2y + \phi(y)$$

$$F_y = N \Rightarrow \frac{\partial}{\partial y}\left(x^3 + 2x^2y + \phi(y)\right) = 2x^2 + 2y \Rightarrow 2x^2 + \frac{d\phi}{dy} = 2x^2 + 2y$$

$$\frac{d\phi}{dy} = 2y \Rightarrow \int d\phi = \int 2y\,dy \Rightarrow \phi(y) = y^2 + C_2$$

$$F(x,y) = \underbrace{x^3 + 2x^2y + y^2 = C}_{\text{general solution}}$$

Ex) $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)dy = 0$ Find the constant A to make $(*)$ an exact, then solve the equation.

$$M = \frac{Ay}{x^3} + \frac{y}{x^2} \Rightarrow M_y = \frac{A}{x^3} + \frac{1}{x^2}$$

$$N = \frac{1}{x^2} - \frac{1}{x} \Rightarrow N_x = -2x^{-3} + x^{-2}$$

$M_y = N_x$

$A = -2 \Rightarrow \left(\frac{-2y}{x^3} + \frac{y}{x^2}\right)dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)dy = 0$

The eq. is exact, then $F_x = M$, $F_y = N$ should be satisfied for soln $F(x,y) = c$

$$\frac{\partial F}{\partial x} = \left(\frac{-2y}{x^3} + \frac{y}{x^2}\right) \Rightarrow F(x,y) = \int\left(\frac{-2y}{x^3} + \frac{y}{x^2}\right)dx = -2y\cdot\frac{x^{-2}}{-2} + y\cdot\frac{x^{-1}}{-1} + \phi(y) = \frac{y}{x^2} - \frac{y}{x} + \phi$$

$$F_y = N \Rightarrow \frac{\partial}{\partial y}\left(\frac{y}{x^2} - \frac{y}{x} + \phi(y)\right) = \frac{1}{x^2} - \frac{1}{x} \Rightarrow \frac{1}{x^2} - \frac{1}{x} + \frac{d\phi}{dy} = \frac{1}{x^2} - \frac{1}{x} \Rightarrow \frac{d\phi}{dy} = 0,\ \phi = 0$$

So the gen. sol. $\frac{y}{x^2} - \frac{y}{x} = c$

Exercise $(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$, $y(0) = 2$

Definition: An integrating factor $m = m(x,y)$ for a 1^{st} order ODE of the form "$(*)$: $M(x,y)\,dx + N(x,y)\,dy = 0$" is a function so that $(*)$ is non-exact $(M_y \neq N_x)$ but $(**)$: $mM\,dx + mN\,dy$ is exact $\left((mM)_y = (mN)_x\right)$. [Remark: $(*)$ and $(**)$ have same gen. sol.]

Ex $(*)$: $(3y + 4xy^2)\,dx + (2x + 3x^2 y)\,dy = 0$ } Hint: $m = x^a \cdot y^b$

$\left.\begin{aligned} M_y &= 3 + 8xy \\ N_x &= 2 + 6xy \end{aligned}\right\}$ so the eq is non-exact.

$(**)$: $x^a y^b (3y + 4xy^2)\,dx + x^a y^b (2x + 3x^2 y)\,dy = 0$

$$(3x^a y^{b+1} + 4x^{a+1} y^{b+2})\,dx + (2x^{a+1} y^b + 3x^{a+2} y^{b+1})\,dy = 0$$

$\tilde{M}\,dx + \tilde{N}\,dy = 0$ is exact, $\tilde{M}_y = \tilde{N}_x$ should be true.

$$M_y = 3(b+1)x^a y^b + 4(b+2)x^{a+1} y^{b+1} \qquad 3(b+1) = 2(a+1) \qquad 2a - 3b = 1$$
$$N_x = 2(a+1)x^a y^b + 3(a+2)x^{a+1} y^{b+1} \qquad 4(b+2) = 3(a+2) \Rightarrow 3a - 4b = 2$$
$$a = 2,\ b = 1$$

So the integrating factor is $m = x^a y^b = x^2 y$, so $(**)$ is;

$$(3x^2 y^2 + 4x^3 y^3)\,dx + (2x^3 y + 3x^4 y^2)\,dy = 0$$

$F_x = 3x^2 y^2 + 4x^3 y^3$

$F(x,y) = \int (3x^2 y^2 + 4x^3 y^3)\,dx$

$F(x,y) = x^3 y^2 + x^4 y^3 + \phi(y)$

$\frac{\partial}{\partial y}\left(x^3 y^2 + x^4 y^3 + \phi(y)\right) = 2x^3 y + 3x^4 y^2$

$2x^3 y + 3x^4 y^2 + \frac{d\phi}{dy} = 2x^3 y + 3x^4 y^2$

$\frac{d\phi}{dy} = 0 \Rightarrow \phi = \phi(y) = 0$

So the general solution of $(*)$ and $(**)$ is: $F(x,y) = c$

$$x^3 y^2 + x^4 y^3 = C$$

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Exercise $\left(\frac{1+8xy^{2/3}}{x^{2/3}y^{1/3}}\right)dx + \left(\frac{2x^{4/3}y^{2/3} - x^{1/3}}{y^{4/3}}\right)dy = 0$, $y(1)=8$

$(2y\sin x\cos x + y^2\sin x)\,dx + (\sin^2 x - 2y\cos x)\,dy = 0$, $y(0)=3$

$\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx + \left(\frac{Ax+1}{y^3}\right)dy = 0$ Find A to make exact, then solve.

Theorem: Let $(*)$: $M(x,y)\,dx + N(x,y)\,dy = 0$ be a non-exact ODE.

(i) If $\frac{1}{N}(M_y - N_x)$ depends only on x, then; $m = e^{\int \frac{1}{N}(M_y - N_x)dx}$.

(ii) If $\frac{1}{M}(N_x - M_y)$ depends only on y, then; $m = e^{\int \frac{1}{M}[N_x - M_y]dy}$.

Proof: Suppose $m = m(x)$ is an integrating factor of $(*)$. This means;

$(m\cdot M)_y = (m\cdot N)_x \Rightarrow \frac{\partial}{\partial y}(m(x)\cdot M(x,y)) = \frac{\partial}{\partial x}(m(x)\cdot N(x,y))$

$m\cdot M_y = \frac{dm}{dx}N + m\cdot N_x \Rightarrow \frac{1}{m}\cdot\frac{dm}{dx} = \frac{M_y - N_x}{N}$ → this should depend on x.

$\int \frac{dm}{m} = \int \frac{M_y - N_x}{N}dx \Rightarrow \cdots \Rightarrow m = e^{\int \frac{1}{N}[M_y - N_x]dx}$ // other one is by symmetry.

Ex) $\underbrace{(4x+3y^2)}_{M}dx + \underbrace{2xy}_{N}\,dy = 0$: $(*)$

$M_y = 6y$, $N_x = 2y$ } $M_y \neq N_x$ non-exact , $\frac{1}{N}\cdot(M_y - N_x) = \frac{1}{2xy}[6y - 2y] = \frac{2}{x}$ is a function of x.

$\int \frac{1}{N}(M_y - N_x)dx = \int \frac{2}{x}dx = 2\ln x \Rightarrow m = e^{2\ln x} = x^2$ integrating factor.

$\widetilde{(*)}$: $(4x^3 + 3x^2y^2)\,dx + (2x^3y)\,dy = 0$ $\tilde{M}_y = 6x^2y$, $\tilde{N}_x = 6x^2y$ } $\tilde{M}_y = \tilde{N}_x$ is exact.

$F(x,y)=c$ is the soln. of (*) and (**).

$F_x = \tilde{M} = 4x^3 + 3x^2y^2 \qquad F_y = \tilde{N} = 2x^3y$

$F(x,y) = \int (4x^3 + 3x^2y^2)\,dx = x^4 + x^3y^2 + \phi(y)$

$\frac{\partial}{\partial y}\left(x^4 + x^3y^2 + \phi(y)\right) = 2x^3y \Rightarrow 2x^3y + \frac{d\phi}{dy} = 2x^3y \Rightarrow \begin{array}{l} \frac{d\phi}{dy} = 0 \\ \phi(y) = 0 \end{array}$

So the general solution is; $F(x,y) = x^4 + x^3y^2 = C$

<u>Ex</u> $(\overbrace{y^2 + 2xy}^{M})dx - \overbrace{x^2}^{N}dy = 0$; $M_y = 2y + 2x \neq -2x = N_x$ (non-exact)

$\frac{1}{N}(M_y - N_x) = \frac{1}{-x^2}\cdot(2y + 2x - (-2x)) = \frac{-1}{x^2}[2y + 4x]$ } does not depend only on x.

$\frac{1}{M}(N_x - M_y) = \frac{1}{y^2 + 2xy}(-2x - (2y + 2x)) = \frac{-4x - 2y}{y^2 + 2xy} = \frac{-2(2x+y)}{y(2x+y)} = \frac{-2}{y}$ } depends only on y.

Then, (*) has an integrating factor given by;

$e^{\int \frac{1}{M}(N_x - M_y)\,dy} = e^{\int \frac{-2}{y}dy} = e^{-2\ln y} = \frac{1}{y^2}$ $\quad m = \frac{1}{y^2}$ is the integrating factor.

(**): $\left(1 + 2\frac{x}{y}\right)dx - \frac{x^2}{y^2}dy = 0 \quad M_y = N_x = -2\frac{x}{y^2}$ (is exact)

$F(x,y) = c$ is a solution of (**) and (*);

$F_x = \tilde{M} = 1 + 2\frac{x}{y} \qquad F_y = \tilde{N} = \frac{x^2}{y^2}$

$F(x, y$

⋮

<solve remaining>

(+ Other applications) Lot

Application: Newton's Law of Cooling

The law states that the rate of chang of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (ie, the temperature of its surrounding).

(Temperature of object at time t) = $T(t)$ (Ambient temp.) = S

$$\frac{dT}{dt} = k(T-S) \Rightarrow \int \frac{dT}{T-S} = \int k\,dt$$

$\ln|T-S| = kt + C_1$, Assume $T>S$ (since its cooling)

$$T-S = e^{kt+C_1} \Rightarrow T = S + e^{C_1}\cdot e^{kt} \Rightarrow T(t) = ce^{kt} + S$$

$t=0$: $T(0) = c\cdot e^{k\cdot 0} + S = C+S$ is the initial temp. of object

$$\begin{aligned} T_1 &= T(t_1) = ce^{kt_1} + S \\ T_2 &= T(t_2) = ce^{kt_2} + S \end{aligned} \quad , t_2 > t_1 \Rightarrow \cdots \Rightarrow k = \frac{1}{t_2 - t_1}\cdot \ln\left(\frac{T_2 - S}{T_1 - S}\right)$$

Ex] Suppose a corpse was discovered in a motel room at midnight and its temperature was 80°F. The temp. of the room is kept at 60°F. 2 hours later, the temp of the corpse dropped to 75°F. Find the time of death. (Temp. of a healthy person is 98.6°F)

$$\left.\begin{aligned} t_1\ (00{:}00) \quad T(t_1) &= 80 \\ t_2\ (02{:}00) \quad T(T_2) &= 75 \end{aligned}\right\} T(t) = ce^{kt} + S \Rightarrow \begin{aligned} T(t_1) &= 80 = 60 + ce^{kt_1} \\ T(t_2) &= 75 = 60 + ce^{kt_2} \end{aligned}$$

$$\left.\begin{aligned} ce^{kt_1} &= 20 \\ ce^{kt_2} &= 15 \end{aligned}\right\} e^{k(t_2 - t_1)} = \frac{3}{4} \Rightarrow e^{2k} = \frac{3}{4} , \ k = \frac{1}{2}\ln\left(\frac{3}{4}\right) \simeq 0.1438$$

$$\left.\begin{aligned} t_0\ (\text{death}) \quad T(t_0) &= 98.6 \\ t_1\ (00{:}00) \quad T(t_1) &= 80 \end{aligned}\right] \left.\begin{aligned} 98.6 &= 60 + ce^{kt_0} \\ 80 &= 60 + ce^{kt_1} \end{aligned}\right\} e^{k(t_1 - t_0)} = \frac{20}{38.6}$$

$$k(t_1 - t_0) = \ln\left(\frac{20}{38.6}\right) \qquad (7{:}24 \text{ pm})$$

$$t_1 - t_0 \simeq \frac{\ln(20/38.6)}{0.1438} \simeq 4.57 \text{ hours}$$

Exercise| At midnight, temp inside your house at 70°F, and temp. outside at 20°F, your furnace breaks down. 2 hours later, the temp. in your house fallen to 50°F. Assume that the outside temp remains 20°F. At what time, will the inside temp. of your house reach 40°F ?

Some Special Types of 2^{nd} Degree ODE's:

$F(x, y, y', y'') = 0 \longrightarrow$ Type 1: $F(x, y', y'') = 0$ (y is missing)

$\hookrightarrow$ Type 2: $F(y, y', y'') = 0$ (x is missing)

Method: Reduce the special type to a 1^{st} degree ODE.

Ex| $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}\left(\dfrac{dy}{dx} + x^2\cos x\right)$ $\quad y'' = \dfrac{1}{x}(y' + x^2\cos x)$ (y is missing)

Ex| $\dfrac{d^2y}{dx^2} = \dfrac{-2}{1-y}\cdot\left(\dfrac{dy}{dx}\right)^2$ $\quad y'' = \dfrac{-2}{1-y}\cdot(y')^2$ (x is missing)

Method for type 1: $(*)$: $F(x, y', y'') = 0$, (y is missing)

$y' = p(x)$, $\dfrac{dy}{dx} = p \Rightarrow \dfrac{d^2y}{dx^2} = y'' = p' = \dfrac{dp}{dx}$

$(*)$ now became $F(x, p, p') = 0$ (1^{st} order ODE)

Ex| $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}\left(\dfrac{dy}{dx} + x^2\cos x\right) \Bigg\}\ \begin{matrix} y' = p \\ y'' = p' \end{matrix} \Rightarrow p' = \dfrac{1}{x}(p + x^2\cos x)$

$\dfrac{dp}{dx} - \dfrac{1}{x}p = x\cos x \Big\}$ linear in "p" , $M = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

$\dfrac{1}{x}\dfrac{dp}{dx} - \dfrac{1}{x^2}p = \cos x \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{x}p\right) = \cos x$, $\int d\left(\dfrac{1}{x}p\right) = \int \cos x\,dx$

$$\frac{1}{x}p = \sin x + c_1 \Rightarrow P(x) = x\sin x + c_1 x \qquad \int \overbrace{x \sin x\, dx}^{u \quad dv} = -x\cos x + \sin x + C$$

$$y' = p(x) \Rightarrow y(x) = \int (x\sin x + c_1 x)\,dx \qquad \longrightarrow \int c_1 x\,dx = \frac{c_1 x^2}{2} + C$$

$$y(x) = -x\cos x + \sin x + \frac{c_1}{2}x^2 + C_2 \quad \Big\} \text{ General solution}$$

Ex) $(1+x^2)y'' = -2xy' \quad \Big\{ \ y$ is missing $\Rightarrow \begin{array}{l} y' = p \\ y'' = p' \end{array}$

$(1+x^2)p' = -2xp$ (seperable) $\quad \rightarrow -\int \frac{2x}{1+x^2} = -\int \frac{du}{u} = -\ln|u| + c_1 = -\ln|1+x^2|$

$$\int \frac{dp}{p} = \int \frac{-2x}{1+x^2}\,dx \quad \Rightarrow \ln|p| + \ln C_1 = \ln(1+x^2)$$

ASSUME $p>0 \Rightarrow \ln(pc_1) = -\ln(1+x^2) \Rightarrow p = \frac{1}{c_1(1+x^2)}$

$y' = p = \frac{dy}{dx} = \frac{1}{c_1(1+x^2)}$ (seperable) $\Rightarrow \int dy = \int \frac{dx}{c_1(1+x^2)} \Rightarrow y(x) = \frac{1}{c_1}\arctan x + c_2 \Big\}$ general soln.

Method for Type 2: $F(y, y', y'') = 0$ (x is missing)

$$y' = u(y), \quad \frac{dy}{dx} = u(y) \Rightarrow y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(u(y))$$

$\frac{d}{dx}(u(y)) = \frac{d}{dy}(u(y)) \cdot \frac{dy}{dx} = u'(y) \cdot u \quad \Big\} \ [y' = u, \quad y'' = u' \cdot u]$ substitute to get 1st order

Ex) $\frac{d^2y}{dx^2} = \frac{-2}{1-y}\left(\frac{dy}{dx}\right)^2 \quad \Big\} \begin{array}{l} y' = u \\ y'' = u' \cdot u \end{array}$ (Because x is Missing)

$u'u = \frac{-2}{1-y} \cdot u^2, \quad u \neq 0 \Rightarrow u' = \frac{-2}{1-y}u$ (seperable) $\quad \frac{du}{u} = \frac{-2}{1-y}dy$

$\int \frac{du}{u} = \int \frac{2dy}{y-1} \Rightarrow \ln|u| + \ln c_1 = 2\ln|y-1| \quad \begin{array}{l} y>1 \\ u>0 \end{array}\Big\} \Rightarrow \ln(c_1 u) = \ln(y-1)^2$

$\Rightarrow c_1 u = (y-1)^2, \quad c_1 \cdot y' = (y-1)^2$ (seperable) $\Rightarrow \frac{dy}{(y-1)^2} = \frac{dx}{c_1}$

$\int \frac{dy}{(y-1)^2} = \int w^{-2}dw = \frac{w^{-1}}{-1} + c_2 = -\frac{1}{y-1} + c_2 \quad \Rightarrow \frac{1}{y-1} + c_3 = c_4 x \ \ldots\ y-1 = \frac{k_1}{x+k_2}$

$y-1 = w$
$dy = dw$

$\boxed{\text{Exercise}}$ Determine the displacement at time t of a simple harmonic oscilator that is extended w distance "a" units from its equilibrium position and released from rent at $t=0$.

$$\frac{d^2y}{dt^2} = -w^2 y \quad , \quad y(0)=a \quad , \quad y'(0)=0$$

✓ $\boxed{\text{Exercise}}$ The following IVP arises in the analysis of a cable suspended between two fixed points: $y'' = \frac{1}{a}\left(1+(y')^2\right)^{1/2} \; , \; y(0)=a \; , \; y'(0)=0$

$\boxed{\text{Ex}}$ $y''' = x^{-1}(y''-1) \quad \Big\} \; y''=p \; , \; y'''=p'$

$p' = \frac{1}{x}(p-1)$ (seperable) $\Rightarrow \int \frac{dp}{p-1} = \int \frac{dx}{x} \; , \; \ln|p-1| = \ln|x| + \ln C_1$

$\begin{matrix} p>1 \\ x>0 \end{matrix} \Rightarrow p-1 = C_1 x \; , \; p = y'' = C_1 x + 1 \qquad \frac{d^2y}{dx^2} = C_1 x + 1 \Rightarrow \frac{dy}{dx} = \int (C_1 x + 1)\,dx$

$\frac{dy}{dx} = \frac{C_1}{2}x^2 + x + C_2 \Rightarrow y(x) = \int \left(\frac{C_1}{2}x^2 + x + C_2\right) dx \; \cdots = $ (general solution)

$\boxed{\text{Ex}}$ $yy'' = 2(y')^2 + y^2 \; , \; y(0)=1 \; , \; y'(0)=0$

2^{nd} order, x is missing: $\begin{matrix} y'=u \\ y'=u'u \end{matrix}$

$yu'u = 2u^2 + y^2$

$u' - \frac{2}{y}\cdot u = y\cdot u^{-1} \Big\}$ Bern. $n=-1$

$V = u^{1-n} = u^2 \; , \; \frac{dv}{dy} = 2u\frac{du}{dy}$

$uu' - \frac{2}{y}u^2 = y \Rightarrow \frac{1}{2}\cdot\frac{dv}{dy} - \frac{2}{y}\cdot v = y \Big\}$ Linear in "v"

$\frac{dv}{dy} - \frac{4}{y}v = 2y$

$m = e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = y^{-4}$

$y^{-4}\frac{dv}{dy} - 4y^{-5}v = 2y^{-3}$

$\frac{d}{dy}(y^{-4}v) = 2y^{-3}$

$y^{-4}v = -y^{-2} + C_1$

$V = C_1 y^4 - y^2 = u^2$

$u = \sqrt{C_1 y^4 - y^2}$

$$\frac{dy}{dx} = y' = u = \sqrt{c_1 y^4 - y^2} \Rightarrow \int \frac{dy}{\sqrt{c_1 y^4 - y^2}} = \int dx \Rightarrow$$

$$\int \frac{dy}{y\sqrt{c_1 y^2 - 1}} = \int \frac{dy}{y\sqrt{c_1\left(y^2 - \frac{1}{c_1}\right)}} = \frac{1}{\sqrt{c_1}} \cdot \int \frac{dy}{y\sqrt{y^2 - \frac{1}{c_1}}}$$

$\sec^2\theta - 1 = \tan^2\theta$

$y = \frac{1}{\sqrt{c_1}} \cdot \sec\theta$

$$= \frac{1}{\sqrt{c_1}} \cdot \int \frac{\frac{1}{\sqrt{c_1}} \sec\theta \tan\theta \, d\theta}{\frac{1}{\sqrt{c_1}} \sec\theta \sqrt{\frac{1}{c_1}\sec^2\theta - \frac{1}{c_1}}} = \frac{1}{\sqrt{c_1}} \int \frac{\tan d\theta}{\frac{1}{\sqrt{c_1}} \cdot \sqrt{\sec^2\theta - 1}} = \int d\theta = \theta + C_2 = \operatorname{arcsec}(\sqrt{c_1}\, y) + C_2$$

$\left.\operatorname{arcsec}(\sqrt{c_1}\, y) = x + C_2 \right\}$ General solution, Find particulars with inits.

Exercise | $y'' = (y')^2 \tan y$ | $y'' - 2x^{-1}y' = 6x^4$ | ✓

✓ | $y'' - y'\tan x = 1$ | $y'' = (y')^2 + 2y'$ | ✓

Matrix Algebra

Definition: A matrix A is an array of $m \times n$ numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & a_{mn} \end{bmatrix}_{m \times n} \quad \text{ROW} \times \text{COLUMN}$$

$a_{ij} = (i^{th}$ row, j^{th} column$)$

Notation: $M_{m\times n}(\mathbb{R})$: set of all $m \times n$ matrices.

- $M_{m\times 1}(\mathbb{R})$: column matrices, $M_{1\times n}(\mathbb{R})$: row matrices
- $m = n \Rightarrow M_{n\times n}(\mathbb{R})$: square matrices, a_{ii} are diagonal entries
- $A \in M_{m\times n}$, $B \in M_{p\times q} \Rightarrow A = B$ if and only if
  i) $m = p$ and $n = q$
  ii) $a_{ij} = b_{ij}$ if $A = [a_{ij}]$, $B = [b_{ij}]$
- $A, B \in M_{m\times n} \Rightarrow (A+B) \in M_{m\times n}$, $A + B = [a_{ij} + b_{ij}]$ when $A = [a_{ij}]$, $B = [b_{ij}]$
- $c \in \mathbb{R}$, $cA \in M_{m\times n}$ is defined by $cA = [c\,a_{ij}]_{m\times n}$
- (i) $A + B = B + A$ (commutative rule) (ii) $(A+B)+C = A+(B+C)$ (associative rule) (iii) $A + 0_{m\times n} = A$ ($0_{m\times n}$: additive identity) (iv) $(-1)A = -A$ ($-A$: additive inverse of A)

FABER-CASTELL

Definition: $A \in M_{m\times n}$, $B \in M_{n\times p}$ then $A\cdot B$ is defined by

$$AB \in M_{m\times p}, \quad AB = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]_{m\times p}$$

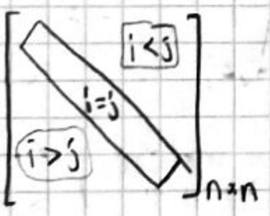
Remark: In general, $AB \neq BA$.

Ex $\underset{2\times 3}{\begin{bmatrix} 1 & 0 & -2 \\ -4 & 3 & 5 \end{bmatrix}} \cdot \underset{3\times 2}{\begin{bmatrix} 7 & -9 \\ 0 & 10 \\ 2 & -1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 3 & -7 \\ -18 & 61 \end{bmatrix}}$

- $A \in M_{m\times n}$, $B \in M_{n\times p}$, $C \in M_{p\times q} \Rightarrow A(BC) = (AB)C$ (assosiative rule)
- $A_{m\times n} \times 0_{n\times p} = 0_{m\times p}$, $0_{n\times p} \times A_{p\times m} = 0_{n\times m}$
- $A(B+C) = AB + AC$, $(B+C)A = BA + BC$ (distrubutive over addition)

Definition: $A \in M_{m\times n}$, $A = [a_{ij}]_{m\times n} \Rightarrow A^T = [a_{ji}]_{n\times m}$

- $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$, $(AB)^T = B^T \cdot A^T$

- $A =$ 

(i) A is an upper triangular matrix if $a_{ij} = 0$ when $i > j$

(ii) A is a lower triangular matrix if $a_{ij} = 0$ when $i < j$.

(iii) A is a diagonal matrix if its both upper and lower diagonal: $(a_{ij} = 0, \; i \neq j)$

Notation: $A = \text{diag}(a_{11}, a_{22}, a_{33}, \ldots, a_{nn})$ is the diagonal matrix notation.

Definition: A square matrix A is said to be symmetric if $A = A^T$

Definition: A square matrix A is said to be skew-symmetric if $A = -A^T$

Ex| Show that any square matrix A can be expressed as a sum of two matrices B and C one of which is symmetric and the other is skew sym.
Furthermore, $A = \underset{(s)}{B} + \underset{(ss)}{C}$ and $A = \underset{(s)}{\tilde{B}} + \underset{(ss)}{\tilde{C}} \Rightarrow B = \tilde{B}$, $C = \tilde{C}$

Hint: $A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right) = (\text{sym}) + (\text{scew-sym})$

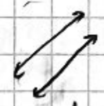
Definition: Trace of matrix $A \in M_{n\times n}$ is; $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

Linear Systems

Definition: A lin. eq. in $x_1, x_2, \dots, x_n$ is an equation of the form; $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

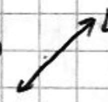
(i) $n=1$: $ax = b$ || $3x = -4$, $S = \{-\frac{4}{3}\}$ | $0x = 4$, $S = \emptyset$ | $0x = 0$, $S = \mathbb{R}$ || $\Rightarrow$ $ax = b$: unique sol. / no sol / inf. sol.

(ii) $n=2$: $\begin{cases} a_1x + b_1y = c_1 & (L_1) \\ a_2x + b_2y = c_2 & (L_2) \end{cases}$ $S = L_1 \cap L_2$



$L_1 // L_2$, $L_1 \cap L_2 = \emptyset$

$P = L_1 \cap L_2$ (unique)



L_1, L_2; $L_1 = L_2$ (inf. many)

(iii) $n=3$: $\begin{cases} a_1x + b_1y + c_1z = d_1 & (D_1) \\ a_2x + b_2y + c_2z = d_2 & (D_2) \\ a_3x + b_3y + c_3z = d_3 & (D_3) \end{cases}$ || $x=2$, $x=3$, $x=4$ parallel (no soln) | $x=0$, $y=0$, $z=0$, $S=(0,0,0)$ (unique) | (i)

$S = (\text{line})$ 1-free param | (ii) $D_1 = D_2 = D_3$, $S = (\text{plane})$ 2-free param

Definition: A linear system in $x_1, x_2, \dots, x_n$ is a collection of m linear equations in $x_1, x_2 \dots x_n$.

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

A solution of $(*)$ is an n-tuple $(x_1, x_2 \cdots, x_n)$ that satisfies all of the m-equations.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

(*) can be expressed as; $\underset{(m \times n)}{A} \underset{(n \times 1)}{X} = \underset{(m \times 1)}{B}$

Ex $\begin{cases} -3x_1 + x_2 - x_3 = 1 \\ 4x_1 + x_3 = -5 \end{cases} \Longrightarrow \begin{bmatrix} -3 & 1 & -1 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Definition: The augmented matrix of (*) is given by $[A \,\vdots\, B]$

Ex Augmented matrix of above example: $\left[\begin{array}{ccc|c} -3 & 1 & -1 & 1 \\ 4 & 0 & -1 & -5 \end{array}\right]$

★ Linear Systems
- Homogeneous
  - unique (trivial) soln.
  - infinitely many solns.
- Non-homogeneous
  - unique soln
  - infinitely solns.
  - Inconsistent (no soln.)

Definition: A Linear system is said to be homogeneous if $b_1 = b_2 = \cdots b_m = 0$

Remark: $AX = B$, $\boxed{B = 0_{m \times 1}} \Longrightarrow AX = B$ is homogeneous.

Definition: A Linear system is said to be non-homogeneous if $b_k \neq 0$ for some k.

★ If (*) is homogeneous then it has the trivial solution: $(0, 0 \ldots, 0)$

That is the solution set contains $(0, 0 \ldots, 0)$ then it's not empty.

→ A homogeneous system is always consistent.

★ Non-homogeneous cannot have a trivial solution. $(0 + 0 + 0 \neq b_k)$

→ Now, our goal is to obtain an equivalent linear system to the given one so that the solution set of the system can be obtained easily.

Ex (*) $\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$ (**) $\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ x_2 + 2x_3 = 4 \\ x_3 = 3 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Definition: Two linear systems are equivalent if they have the same soln. set.

Definition: Elementary Row Operations;

(i) Swap the rows of a matrix. $(R_i \longleftrightarrow R_j)$

(ii) Mutiply a row by a nonzero number. $(R_i \longrightarrow cR_i)$ $(c \neq 0)$

(iii) Add a constant multiple of a row to another. $(R_i \longrightarrow R_i + cR_j)$

Theorem: Let $AX = 0_{m\times1}$ is a homogeneous system. Suppose $\tilde{A}$ is obtained from A by using finitely many elementary row operations. Then, $AX = 0_{m\times1}$ and $\tilde{A}X = 0_{m\times1}$ are equivalent. (have same soln. set.)

Let $AX = B$ is a nonhomogeneous system. Suppose $[\tilde{A}|\tilde{B}]$ is obtained from $[A|B]$ by using finitely many elementary row operations. Then, $AX = B$ and $\tilde{A}X = \tilde{B}$ are equivalent.

→ Our goal is to obtain the RREF of the coefficient / augmented matrix.

Ex $\begin{cases} 3x_1 - 6x_2 - 2x_3 = 1 \\ 2x_1 - 4x_2 + x_3 = 17 \\ x_1 - 2x_2 - 2x_3 = -9 \end{cases}$ $[A|B] = \left[\begin{array}{ccc|c} 3 & -6 & -2 & 1 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & -2 & 9 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_3}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -9 \\ 2 & -4 & 1 & 17 \\ 3 & -6 & -2 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{cccc} 1 & -2 & -2 & -9 \\ 0 & 0 & 5 & 35 \\ 0 & 0 & 4 & 28 \end{array}\right] \xrightarrow[\frac{1}{4}R_3 \to R_3]{\frac{1}{5}R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right] \Rightarrow \left.\begin{array}{l} x_3 = 7 \\ x_1 - 2x_2 - 2x_3 = -9 \end{array}\right\} x_1 - 2x_2 = 5$$

Definition: Matrix A is said to be (Row) Reduced Echelon Form (RREF) if it satisfies the followings:

REF / EF

(i) All rows consisting entirely of 0 are at the bottom of the matrix.

(ii) For each non-zero row, the 1st non-zero entry is 1. (which is called leading 1).

(iii) For two successive non-zero rows, the leading 1 in the upper row appears farther to the left than the leading 1 in the lower row.

(iv) If a column contains a leading 1, then all other entries in that column are 0.

Remark: If a matrix satisfies only first 3 props, then it's (Row) Echelon Form.

Ex $\begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is not in RREF but it's in REF.

$\begin{bmatrix} 0 & 1 & 0 & 5 & -9 \\ 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ is not in REF, the last row violates (ii)

→ If this was 1, then it's REF; if was 0, then it's RREF.

Theorem: Every matrix A is row equivalent to a unique matrix B that is in the RREF.

Explicitly, every matrix has a unique RREF.

Ex $A=\begin{bmatrix}0&0&1&2\\2&3&0&-2\\3&3&6&-9\end{bmatrix}\xrightarrow{\frac{1}{3}R_3\to R_3}\begin{bmatrix}0&0&1&2\\2&3&0&-2\\1&1&2&-3\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_3}\begin{bmatrix}1&1&2&-3\\2&3&0&-2\\0&0&1&2\end{bmatrix}$

$\xrightarrow{-2R_1+R_2\to R_2}\begin{bmatrix}1&1^{①}&2&-3\\0&1&-4&4\\0&0&1&2\end{bmatrix}\xrightarrow{-R_2+R_1\to R_1}\begin{bmatrix}1&0&6^{②}&-7\\0&1&-4&4\\0&0&1&2\end{bmatrix}\xrightarrow[-6R_3+R_1\to R_1]{4R_3+R_2\to R_2}\begin{bmatrix}1&0&0&-19\\0&1&0&12\\0&0&1&2\end{bmatrix}$ ✓

To eliminate ① we can use 2^{nd} row to not ruin, to eliminate ② we use 3^{rd}.

Gaussian Elimination Method (Homogeneous): $(*)$: $AX=0_{m\times 1}$, suppose $\tilde{A}$ is the RREF of A. We know that $(*)$ and $(\tilde{*})$ are equivalent.

Define $r=$ (number of leading 1's).

(i) If $n=r$ then $(*)$, $(\tilde{*})$ have only trivial (unique) solution.

(ii) If $n>r$ then $(*)$, $(\tilde{*})$ have infinitely many solution.

→ If an unknown x_k is in a column that contains a leading 1, its a basic variable

→ If an unknown x_k is not in " " " " its a free variable (parameter)

Ex
$$(*)\begin{cases}x_3+2x_4=0\\2x_1+3x_2-2x_4=0\\3x_1+3x_2+6x_3-9x_4=0\end{cases}\qquad A=\begin{bmatrix}0&0&1&2\\2&3&0&-2\\3&3&6&-9\end{bmatrix},\ \text{we found above},\ \tilde{A}=\begin{bmatrix}1&0&0&-19\\0&1&0&12\\0&0&1&2\end{bmatrix}$$

We write $(\tilde{*})$ using RREF we've found;
$$(\tilde{*})\begin{cases}x_1-19x_4=0\\x_2+12x_4=0\\x_3+2x_4=0\end{cases}\qquad \underbrace{n=4}_{x_1,x_2\cdots x_4}\qquad \underbrace{r=3}_{\text{3 rows with lead}}$$

$n>r\Rightarrow$ infinitely many solutions

$$\begin{array}{cccc}x_1&x_2&x_3&x_4\end{array}\\\begin{bmatrix}1&0&0&-19\\0&1&0&12\\0&0&1&2\end{bmatrix}$$

x_1, x_2, x_3 are ~~free~~ basic variables

x_4 is a parameter.

$x_4=u$ (u: param)

$x_3=-2u$

$x_2=-12u$

$x_1=19u$

$$(\text{Soln. Set})=S=\left\{\begin{bmatrix}-19u\\-12u\\-2u\\u\end{bmatrix}\ \middle|\ u\in\mathbb{R}\right\}.$$

(⊛) If $(*): AX = 0_{m\times 1}$ is a homogeneous system then, exactly one of the following two is true;

(i) It has only the trivial solution (unique).

(ii) It has infinitely many solutions.

→ Recall that a homogeneous system is always consistent.

(⊛) If $(**): AX = B$, is a non-homogeneous system then, exactly one of the following three is true;

(i) It has no solution (the system is inconsistent)

(ii) It has a unique solution (the soln is non-trivial)

(iii) It has infinitely many solutions.

<u>Theorem:</u> Let $(*): AX = B$ be a non-homogeneous system. Then, $(*)$ and $(\tilde{*}): \tilde{A}x = \tilde{B}$ is equivalent when $[\tilde{A} \mid \tilde{B}]$ is the RREF of $[A \mid B]$

<u>Ex</u> $$(*)\begin{cases} 2x_1 + 5x_2 + 12x_3 = 6 \\ 3x_1 + x_2 + 5x_3 = 12 \\ 5x_1 + 8x_2 + 21x_3 = 17 \end{cases} \qquad [A|B] = \left[\begin{array}{ccc|c} 2 & 5 & 12 & 6 \\ 3 & 1 & 5 & 12 \\ 5 & 8 & 21 & 17 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{-R_1+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -7 & 6 \\ 2 & 5 & 12 & 6 \\ 5 & 8 & 21 & 17 \end{array}\right] \xrightarrow[-5R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -4 & -7 & 6 \\ 0 & 13 & 26 & -6 \\ 0 & 28 & 56 & -13 \end{array}\right] \xrightarrow[\frac{1}{28}R_3 \to R_3]{\frac{1}{13}R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -7 & 6 \\ 0 & 1 & 2 & -6/13 \\ 0 & 1 & 2 & -13/28 \end{array}\right] \xrightarrow[-R_2+R_3 \to R_3]{4R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6-24/13 \\ 0 & 1 & 2 & -6/13 \\ 0 & 0 & 0 & \frac{-1}{13\cdot 28} \end{array}\right] \xrightarrow{\text{⊙}R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 54/13 \\ 0 & 1 & 2 & -6/13 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{-54}{13}R_3+R_1 \to R_1]{\frac{6}{13}R_3+R_2 \to R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\text{RREF}} \Longrightarrow (\tilde{*}): \begin{array}{l} x_1 + x_3 = 0 \\ x_2 + 2x_3 = 0 \\ (\text{False})\ 0 = 1 \end{array}$$

$(*)$ and $(\tilde{*})$ are inconsistent

$(\tilde{*})$: $\tilde{A}X = \tilde{B}$

Gaussian Elimination Method (Non-homogeneous): Let $(*)$: $AX = B$ be a non-homogeneous system. Suppose $[\tilde{A}|\tilde{B}]$ is the RREF of $[A|B]$

(i) $(*)$ is consistent if and only if no leading 1 appears in $[\tilde{A}|\tilde{B}]$ in the last column
$(*)$ is inconsistent if and only if a leading 1 appears in " " " "

Suppose $(*)$ is consistent and define r = (number of leading 1's in $\tilde{A}$)

(ii) If $n = r$, then $(*)$ and $(\tilde{*})$ have a unique solution.

(iii) If $n > r$, then $(*)$ and $(\tilde{*})$ have infinitely many solutions.

→ If a variable x_k ---

→ If a variable x_k ---

Ex (The coefficient matrix of a homogeneous system)

$$C = \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \xrightarrow[5R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 8 & -20 & 4 & 59 \end{bmatrix} \xrightarrow{2R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & -4 & 10 & -2 & -29 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2 \to R_2} \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 0 & 1 & -5/2 & 1/2 & 29/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-10R_3+R_1 \to R_1]{\frac{-29}{4}R_3+R_2 \to R_2}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 3/2 & 3/2 & 0 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ RREF✓}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3/2 & 3/2 & 0 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

x_1, x_2, x_5: basic variables
x_4, x_3: free variables (parameters)

$n=5$, $r=3$, $n>r$, inf. many

$x_3 = u$

$x_4 = v \Rightarrow x_1 = -\frac{3}{2}u - \frac{3}{2}v$

$x_2 = \frac{5}{2}u - \frac{1}{2}v$

$$S = \left\{ \begin{bmatrix} -\frac{3}{2}u - \frac{3}{2}v \\ \frac{5}{2}u - \frac{1}{2}v \\ u \\ v \\ 0 \end{bmatrix} \middle| u, v \in \mathbb{R} \right\}$$

Ex (*) $\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 10 \\ 3x_1 - x_2 + 7x_3 + 4x_4 = 1 \\ -5x_1 + 3x_2 - 15x_3 - 6x_4 = 9 \end{cases}$

The augmented matrix of (*) is the same with the coefficient matrix of the last example.

$$C = \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{array}\right]$$

RREF of C: $\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} & 11/4 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ → The leading 1 appears in the last column (bad row) So (*) has no solution.

Consider doing with swapping.

Algorithm to find RREF:

→ Make the left-most entry in the top row 1 by multiplication.
→ Then use that 1 as a pivot to eliminate everything below it.
→ Go to the next row and make the left-most entry 1.
→ Then use that 1 as a pivot to eliminate everything below and above it.
→ Repeat the last two steps until the last row.

Ex

$$C = \begin{bmatrix} 2 & 4 & -1 & -2 & 2 & 6 \\ 1 & 3 & 2 & -7 & 3 & 9 \\ 5 & 8 & -7 & 6 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 2 & -7 & 3 & 9 \\ 2 & 4 & -1 & -2 & 2 & 6 \\ 5 & 8 & -7 & 6 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[-5R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 2 & -7 & 3 & 9 \\ 0 & -2 & -5 & 12 & -4 & -12 \\ 0 & -7 & -17 & 41 & -14 & -41 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 3 & 2 & -7 & 3 & 9 \\ 0 & 1 & 5/2 & -6 & 2 & 6 \\ 0 & -7 & -17 & 41 & -14 & -41 \end{bmatrix}$$

$$\xrightarrow[+7R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -11/2 & 11 & -3 & -9 \\ 0 & 1 & 5/2 & -6 & 2 & 6 \\ 0 & 0 & 1/2 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{2R_3 \to R_3} \begin{bmatrix} 1 & 0 & -11/2 & 11 & -3 & -9 \\ 0 & 1 & 5/2 & -6 & 2 & 6 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[+\frac{11}{2}R_3+R_1 \to R_1]{-\frac{5}{2}R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{bmatrix} \text{ RREF } \checkmark$$

Solve the homogeneous system (*) which have the coefficient Matrix C.

Basic Var. Free Var.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{bmatrix}$$

$\left.\begin{array}{l} x_4 = u \\ x_5 = v \\ x_6 = w \end{array}\right\}$ parameters

Translate

$\left.\begin{array}{l} x_1 - 3x_5 + 2x_6 = 0 \\ x_2 - x_4 + 2x_5 + x_6 = 0 \\ x_3 - 2x_4 + 2x_6 = 0 \end{array}\right\}$ $\begin{array}{l} x_1 = 3v - 2w \\ x_2 = u - 2v - w \\ x_3 = 2u - 2w \end{array}$

So the solution set; (inf. many sols)

$$S = \left\{ \begin{bmatrix} 3v-2w \\ u-2v-w \\ 2u-2w \\ u \\ v \\ w \end{bmatrix} \middle| \; u, v, w \in \mathbb{R} \right\}$$

Now, solve the non-homo system (*) which has the augmented matrix C.

Basic Free

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{array}\right]$$

$\left.\begin{array}{l} x_4 = u \\ x_5 = v \end{array}\right\}$ parameters

Translate

$\left.\begin{array}{l} x_1 - 3x_5 = 2 \\ x_2 - x_4 + 2x_5 = 1 \\ x_3 - 2x_4 = 2 \end{array}\right\}$ $\begin{array}{l} x_1 = 3v + 2 \\ x_2 = u - 2v + 1 \\ x_3 = 2u + 2 \end{array}$

So the solution set; (inf. many sols)

$$S = \left\{ \begin{bmatrix} 2+3v \\ u-2v+1 \\ 2u+2 \\ u \\ v \end{bmatrix} \middle| \; u, v \in \mathbb{R} \right\}$$

$\boxed{\text{* Mt2 question Harder Ver.}}$

$\boxed{\text{Ex}}$ Find the value(s) of k such that (*) has no soln, unique soln, inf soln.

$$(*) \begin{cases} 3x + 2y = 1 \\ 6x + 4y = k \end{cases} \qquad \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & k \end{bmatrix} \xrightarrow[-2R_1 + R_2 \to R_2]{} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & k-2 \end{bmatrix} \xrightarrow[\frac{1}{3} R_1 \to R_1]{}$$

$$\begin{array}{ccc} x & y & \\ \end{array}$$
$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & k-2 \end{bmatrix}$$

If $k - 2 \neq 0 \Rightarrow$ (*) is inconsistent

If $k - 2 = 0 \Rightarrow$ (*) has inf. many solns.

(*) Cannot have a unique solution.

$y = u \}$ param

$x = \frac{1}{3} - \frac{2}{3}u$

$$S = \left\{ \begin{bmatrix} \frac{1}{3} - \frac{2}{3}u \\ u \end{bmatrix} \middle| \; u \in \mathbb{R} \right\}$$

$\boxed{\text{Ex}}$ Find the value(s) of k such that (*) has no soln, unique soln, inf. soln

$$(*) \begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{cases} \qquad \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & (k^2-5) & k \end{bmatrix} \xrightarrow[\substack{-R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3}]{} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{bmatrix} \to (k^2-4)z = k-2 \Rightarrow (k-2)(k+2)z = (k-2)$$

To have a bad row (leading 1 on last column) we need to have: $(k^2-4)=0$ but $(k-2)\neq 0$ so;

If $k=-2 \Rightarrow$ the system is inconsistent.

If $k=2 \Rightarrow$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left.\begin{array}{l} z = u \text{ } \} \text{ param} \\ x = 2+u-(1-2u) \\ y = 1-2u \end{array}\right\} S = \left\{ \begin{bmatrix} 1+3u \\ 1-2u \\ u \end{bmatrix} \middle| u \in \mathbb{R} \right\}$$

(inf. many solutions)

If $k\neq 2$, $k \neq -2 \Rightarrow \cancel{(k-2)}(k+2)z = \cancel{(k-2)}$ (we can do)

$$z = \frac{1}{k+2} \Rightarrow \left.\begin{array}{l} y = 1 - \frac{2}{k+2} \\ x = 1 + \frac{3}{k+2} \end{array}\right\} \text{(unique soln)} \quad S = \left\{ \begin{bmatrix} (k+5)/(k+2) \\ k/(k+2) \\ 1/(k+2) \end{bmatrix} \right\}$$

<u>Exercise</u>

$$\begin{cases} x+y+z=2 \\ 2x+3y+2z=5 \\ 2x+3y+(k^2-1)z=k+1 \end{cases} \qquad \begin{cases} x+y+z=2 \\ x+2y+z=3 \\ x+y+(k^2-5)z=k \end{cases}$$

$$\begin{cases} x+y+kz=2 \\ 3x+4y+2z=k \\ 2x+3y-z=1 \end{cases} \qquad \begin{matrix} \text{2 inf.} \\ \text{-5 no.} \\ \text{others, uniq} \end{matrix} \begin{cases} x-3z=-3 \\ 2x+ky-z=-2 \\ x+2y+kz=1 \end{cases}$$

<u>Ex</u>

$$\begin{cases} kx+y+z=1 \\ x+ky+z=1 \\ x+y+kz=1 \end{cases} \quad \begin{bmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 1 & k & 1 \\ k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \end{bmatrix} \xrightarrow[-R_1+R_2\to R_2]{-kR_1+R_2\to R_2}$$

$$\begin{bmatrix} 1 & 1 & k & 1 \\ 0 & 1-k & 1-k^2 & 1-k \\ 0 & k-1 & 1-k & 0 \end{bmatrix}$$

If $k=1$:

$$\begin{array}{cccc} x & y & z & \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{array}{l} y = u \\ z = v \\ x = 1-u-v \end{array} \Rightarrow S = \left\{ \begin{bmatrix} 1-u-v \\ u \\ v \end{bmatrix} \middle| u, v \in \mathbb{R} \right\} \text{ (inf. many)}$$

If $k \neq 1$:

$$\xrightarrow[\frac{1}{1+k}R_3 \to R_3]{\frac{1}{1-k}R_2 \to R_2} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & 1 & 1+k & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[-R_2\to R_1]{-R_2 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1+k & 1 \\ 0 & 0 & -2-k & -1 \end{bmatrix}$$

→ If $-2-k=0 \Rightarrow$ no soln. ($k=-2$)

→ If $-2-k \neq 0 \Rightarrow$ unique ($k\neq -2$ and $k \neq 1$)

$$S = \left\{ \begin{bmatrix} 1/(k+2) \\ 1/(k+2) \\ 1/(k+2) \end{bmatrix} \right\}$$

Ex] Find a relation between a, b, c such that the system is consistent

$$(*)\begin{cases} x+2y-3z=a \\ 3x-y+2z=b \\ x-5y+8z=c \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-3R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{array}\right] \xrightarrow{-R_2+R_3\to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & c-b+2a \end{array}\right] \to (*) \text{ is consistent if } c-b+2a=0$$

(In this case, we have inf. many solutions)

Exercise

$$(*)\begin{cases} x-2y+4z=0 \\ 2x+3y-z=b \\ 3x+y+2z=c \end{cases}$$

always

Find a relation between a, b, c such that the system is consistent.

Exercise

Find a, b vals to make system inc. cons.

$$\begin{cases} 2x-y+2az+t=b \\ -2x+ay-3z=4 \\ 2x-y+(2a+1)z+(a+1)t=0 \\ -2x+y+(1-2a)z-2t=-2b-2 \end{cases}$$

$$\begin{cases} x_1+kx_2-3x_4=0 \\ (k-1)x_1-(k+1)x_2+3x_4=0 \\ x_1+kx_2+(k+2)x_3-3x_4=0 \\ (k+1)x_1+kx_2+(k+5)x_3-3x_4=0 \end{cases}$$

Inverces of Matrices

Definition: Let $n \in \mathbb{N}$ (no-zero). The identity matrix $I_n = \text{diag}(1,1,\dots,1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*) $n=1 \Rightarrow M_{1\times1}$ can be identified with $\mathbb{R}$, so $I_1 = 1$ → Real Number

Definition: $n\times n$ matrix A is said to be invertible (non-singular) if there is an $n\times n$ matrix B such that, $A\cdot B = B\cdot A = I_n$

A is called as singular if there is no such B satisfies the condition.

B is called the inverse of A.

⊛ Suppose A is non-singular. Then its inverse A^{-1} is unique.

Proof: Suppose A has two inverses, say B and C. That means;

$AB = BA = I_n \Rightarrow C(AB) = C(I_n) \Rightarrow \underbrace{(CA)}_{I_n} B = C \Rightarrow \underbrace{B = C}_{\square}$

$AC = CA = I_n$

⊛ Suppose both A and B are $n \times n$, non-singular matrices.

(i) A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

(ii) $A \cdot B$ is also non-singular and $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

(iii) A^n is non-singular and $(A^n)^{-1} = (A^{-1})^n = A^{-n}$

Notation: $A^0 = I_n$ when A is non-singular.

Proof: $(AB)(AB)^{-1} = (AB).(B^{-1}A^{-1}) \Rightarrow I_n = A \cdot (BB^{-1}) \cdot A^{-1} \Rightarrow I_n = I_n$ ✓

⊛ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ (i) A is non-singular if and only if $ad - bc \neq 0$

(ii) If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (Proof (ii): $A \cdot A^{-1} = I_2$)

Proof (i): Suppose $A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$

A^{-1} since $xyzt$ should be unique

$$(*) \begin{cases} ax + bz = 1 \\ ay + bt = 0 \\ cx + dz = 0 \\ cy + dt = 1 \end{cases} \quad \begin{array}{cccc|c} x & y & z & t & \\ \hline a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 1 \\ 0 & c & 0 & d & 0 \end{array}$$

→ A is non-singular iff $(*)$ has a unique soln

Hint: (i) $a = 0$ (ii) $a \neq 0$ { ⊛ show that $(*)$ has a unique soln iff $ad - bc \neq 0$ }

Goal (i) A is nonsingular iff RREF of A is I_n. (will prove)

(ii) $[A \mid I_n]_{n \times 2n} \longrightarrow$ (Elementary Row Ops) $\longrightarrow [I_n \mid B]$, $B = A^{-1}$

Recall: Elementary row operations are;

(i) $R_i \longleftrightarrow R_j$ (ii) $cR_i \rightarrow R_i$, $c \neq 0$ (iii) $cR_i + R_j \rightarrow R_j$

Definition: An $n \times n$ matrix E is said to be an <u>elementary matrix</u> if E is obtained from I_n by using only one el. row op.

Ex $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are elementary matrices.

⊛ Any elementary matrix is non-singular.

Proof: Suppose E is an elementary matrix.

(i) $R_i \leftrightarrow R_j$ (E_1)

(ii) $cR_i \rightarrow R_j$ (E_2)

(iii) $cR_i + R_j \rightarrow R_j$ (E_3)

(ii) E_2 is non-singular: $E_2 : I_n \xrightarrow{\frac{1}{c}R_1} E_2^{-1}$ have inverse

(i) E_1 is non-singular: $E_1^{-1} = E_1$ inverse is itself

(iii) ⟨Exercise⟩ E_3^{-1} is obtained by $I_n \xrightarrow{\boxed{-cR_i + R_j \rightarrow R_j}} E_3^{-1}$

Notation: Let $\mathcal{E}$ be an elementary row operation. $\mathcal{E}(A)$ is the matrix $A \xrightarrow{\mathcal{E}} \mathcal{E}(A)$.

⊛ Suppose A is an $n \times n$ matrix. If $\mathcal{E}$ is an el. row. op. then

$$\mathcal{E}(A) = E \cdot A \quad \text{where} \quad E = \mathcal{E}(I_n). \qquad \boxed{\mathcal{E}(A) = \mathcal{E}(I_n) \cdot A}$$

Ex $\mathcal{E} = 2R_1 + R_3 \rightarrow R_3$

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 4 & 7 & -9 \\ 2 & 0 & 5 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 4 & -7 & 9 \\ 8 & -4 & 5 \end{bmatrix}}_{\mathcal{E}(A)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{\mathcal{E}(I_n)} \cdot \underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 4 & 7 & -9 \\ 2 & 0 & 5 \end{bmatrix}}_{A}$$

⊛ Suppose B is the RREF of $n \times n$ matrix A.

$$A \xrightarrow{\mathcal{E}_1} \underbrace{E_1 A}_{\mathcal{E}(A)} \xrightarrow{\mathcal{E}_2} \underbrace{E_2 E_1 A}_{\mathcal{E}(E_1 A)} \xrightarrow{\mathcal{E}_3} \cdots \xrightarrow{\mathcal{E}_k} B = \underbrace{E_k E_{k-1} E_{k-2} \cdots E_2 E_1}_{\text{nonsingular}} A$$

A is nonsingular iff $B = I_n$ ⇐ nonsingular

Theorem: $n \times n$ matrix A is nonsingular if and only if its RREF is I_n.

↳ In this case; $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$

Ex $A=\begin{bmatrix}-2 & 3\\ 5 & -7\end{bmatrix}$, $[A\,|\,I_2]=\left[\begin{array}{cc|cc}-2 & 3 & 1 & 0\\ 5 & -7 & 0 & 1\end{array}\right] \xrightarrow{-\frac{1}{2}R_1\to R_1} \left[\begin{array}{cc|cc}1 & -3/2 & -1/2 & 0\\ 5 & -7 & 0 & 1\end{array}\right]$

$\xrightarrow[-5R_1+R_2\to R_2]{} \left[\begin{array}{cc|cc}1 & -3/2 & -1/2 & 0\\ 0 & 1/2 & 5/2 & 1\end{array}\right] \xrightarrow[2R_2\to R_2]{} \left[\begin{array}{cc|cc}1 & -3/2 & -1/2 & 0\\ 0 & 1 & 5 & 2\end{array}\right] \xrightarrow[\frac{3}{2}R_2+R_1\to R_1]{}$

$\left[\begin{array}{cc|cc}1 & 0 & 7 & 3\\ 0 & 1 & 5 & 2\end{array}\right] \Rightarrow$ The RREF of A is I_2, thus A is nonsingular and $A^{-1}=\begin{bmatrix}7 & 3\\ 5 & 2\end{bmatrix}$

Ex $A=\begin{bmatrix}2 & 7 & 3\\ 1 & 3 & 2\\ 3 & 7 & 9\end{bmatrix}$, $[A\,|\,I_3]=\left[\begin{array}{ccc|ccc}2 & 7 & 3 & 1 & 0 & 0\\ 1 & 3 & 2 & 0 & 1 & 0\\ 3 & 7 & 9 & 0 & 0 & 1\end{array}\right] \xrightarrow[\substack{-2R_2+R_1\to R_1\\ -3R_2+R_3\to R_3\\ R_2\leftrightarrow R_1}]{}$

$\left[\begin{array}{ccc|ccc}1 & 3 & 2 & 0 & 1 & 0\\ 0 & 1 & -1 & 1 & -2 & 0\\ 0 & -2 & 3 & 0 & -3 & 1\end{array}\right] \xrightarrow[\substack{-3R_2+R_1\to R_1\\ 2R_2+R_3\to R_3}]{} \left[\begin{array}{ccc|ccc}1 & 0 & 5 & -3 & 7 & 0\\ 0 & 1 & -1 & 1 & -2 & 0\\ 0 & 0 & 1 & 2 & -7 & 1\end{array}\right] \xrightarrow[\substack{+R_3+R_2\to R_2\\ -5R_3+R_1\to R_1}]{}$

$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -13 & 42 & -5\\ 0 & 1 & 0 & 3 & -9 & 1\\ 0 & 0 & 1 & 2 & -7 & 1\end{array}\right] \Rightarrow (\text{RREF of } A)=I_3 \Rightarrow$ A is non-singular, $A^{-1}=\begin{bmatrix}-13 & 42 & -5\\ 3 & -9 & 1\\ 2 & -7 & 1\end{bmatrix}$

Ex $A=\begin{bmatrix}-6 & 2\\ 9 & -3\end{bmatrix}$, $[A\,|\,I_2]=\left[\begin{array}{cc|cc}-6 & 2 & 1 & 0\\ 9 & -3 & 0 & 1\end{array}\right] \to \cdots \to \left[\begin{array}{cc|cc}1 & -1/3 & -1/6 & 0\\ 0 & 0 & 1/6 & 1/9\end{array}\right]$

(RREF of A) $\neq I_2$, then A is singular. (Easily; $ad-bc=0$)

(*) Suppose A is a square matrix where $A\cdot X=B$

$$(*)\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\ \quad\vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n\end{cases}$$

Preposition: (*) has a unique solution iff A is non-singular.

→ If $B=0_{n\times 1}$, then the system is homogeneous and the solution is the trivial soln.

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Determinants

det: $M_{n\times n} \to \mathbb{R}$, $A \mapsto \det(A) \in \mathbb{R}$

Definition: Let $A = [a_{ij}]_{n\times n}$ be square matrix.

(i) $A(i|j)$ is an $(n-1)\times(n-1)$ matrix that is obtained from A by deleting the i^{th} row and j^{th} column.

(ii) The $(i,j)^{th}$ minor of A is defined by $M_{ij} = \det(A(i|j))$

(iii) The $(i,j)^{th}$ cofactor of A is defined by $A_{ij} = (-1)^{i+j} \cdot M_{ij}$

Definition: $A \in M_{1\times 1}(\mathbb{R})$, $A = [a] \Rightarrow \det[a] = a$

⊛ Definition: Suppose $A \in M_{n\times n}(\mathbb{R})$

(i) $\det A = \sum_{j=1}^{n} a_{1j} A_{1j} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \dots + a_{1n}A_{1n}$ (expansion with respect to 1^{st} row)

Remark:
$n=2$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$\det A = a_{11}A_{11} + a_{12}A_{12}$
$= a_{11}a_{22} - a_{12}a_{21}$

$A_{11} = (-1)^{1+1} a_{22}$
$A_{12} = (-1)^{1+2} a_{21}$

(ii) $\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$ (expansion with respect to i^{th} row)

→ Since the det A and det A^T should give the same result, ...

(iii) $\det A = \sum_{i=1}^{n} a_{i1} A_{i1} = a_{11}A_{11} + a_{21}A_{21} + \dots + a_{n1}A_{n1}$ (expansion with respect to 1^{st} column)

(iv) $\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$ (expansion with respect to j^{th} column)

Ex $A=\begin{bmatrix}3 & -2 & 7\\ 5 & 1 & 0\\ -4 & -8 & 6\end{bmatrix}$ Expansion with respect to 2^{nd} row;

$\det A = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$$= 5\cdot(-1)^{2+1}\cdot\det\begin{bmatrix}-2 & 7\\ -8 & 6\end{bmatrix} + 1\cdot(-1)^{2+2}\cdot\det\begin{bmatrix}3 & 7\\ -4 & 6\end{bmatrix} + 0\cdot(-1)^{2+3}\cdot\det\begin{bmatrix}3 & -2\\ -4 & -8\end{bmatrix}$$

$$= (-5)\cdot(-12+56) + 1\cdot(18+28) + 0 = -5\cdot 44 + 46 = -174$$

Ex $A=\begin{bmatrix}-4 & 0^{\ominus} & 2 & -1\\ 3 & 9^{\oplus} & -7 & 6\\ 10 & 0^{\ominus} & 5 & 11\\ 1 & -8^{\oplus} & 12 & -12\end{bmatrix}$ Expansion with respect to 2^{nd} column:

$\det A = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + a_{42}A_{42}$

$$= 9\det\begin{bmatrix}-4 & 2 & -1\\ 10 & 5 & 11\\ 1 & 12 & -12\end{bmatrix} - 8\det\begin{bmatrix}-4 & 2 & -1\\ 3 & -7 & 6\\ 10 & 5 & 11\end{bmatrix} = \cdots$$ (rest is exercise)

<u>Notation:</u> $A \in M_{n\times n}(\mathbb{R}) \Rightarrow A=\begin{bmatrix}R_1\\ R_2\\ \vdots\\ R_n\end{bmatrix}$ Rows, $A=[C_1 \ C_2 \ \cdots \ C_n]$ Columns

<u>Properties of Determinant Function</u>

(i) $\det\begin{bmatrix}R_1\\ R_2\\ cR_3\\ R_n\end{bmatrix} = c\cdot\det\begin{bmatrix}R_1\\ R_2\\ R_3\\ R_n\end{bmatrix}$, $\det[C_1 \ k\cdot C_2 \cdots C_n] = k\cdot\det[C_1 \ C_2 \cdots C_n]$

Ex $$\det\begin{bmatrix}3 & -6 & 9\\ 5 & -10 & 15\\ 7 & 49 & -63\end{bmatrix} = 3\cdot\det\begin{bmatrix}1 & -2 & 3\\ 5 & -10 & 15\\ 7 & 49 & -63\end{bmatrix} = 7\det\begin{bmatrix}3 & -6 & 9\\ 5 & -10 & 15\\ 1 & 7 & -9\end{bmatrix} = 51\cdot\det\begin{bmatrix}3 & -6/51 & 9\\ 5 & -10/51 & 15\\ 1 & 7/51 & -9\end{bmatrix}$$

(ii) $\det(kA) = k^n\cdot\det(A)$ where A is $n\times n$ matrix.

Ex
$$\det\begin{bmatrix}-64 & 36 & 18\\ 50 & -10 & 32\\ 6 & -12 & -24\end{bmatrix}_{3\times 3} = (-2)^3\cdot\det\begin{bmatrix}32 & -18 & -9\\ -25 & 5 & -16\\ -3 & 6 & 12\end{bmatrix}$$

negative comes when swapped row/column

(iii) $\det\begin{bmatrix} R_i \\ R_j \\ R_n \end{bmatrix} = -\det\begin{bmatrix} R_j \\ R_i \\ R_n \end{bmatrix}$, $\det[C_i\ C_j\ C_n] = -\det[C_j\ C_i\ C_n]$

(iv) $\det\begin{bmatrix} R_i \\ R_i \\ R_n \end{bmatrix} = 0$ det is 0 when two rows/columns are the same , $\det[C_i\ C_i\ C_n] = 0$

(v) $\det\begin{bmatrix} R_1 \\ \vdots \\ A+B \\ \vdots \\ R_n \end{bmatrix} = \det\begin{bmatrix} R_1 \\ A \\ R_n \end{bmatrix} + \det\begin{bmatrix} R_1 \\ B \\ R_n \end{bmatrix}$, $\det[C_1\ X+Y\ C_n] = \det[C_1\ X\ C_n] + \det[C_1\ Y\ C_n]$

Ex

$$\det\begin{bmatrix} 3 & -6 & -7 \\ 4 & 2 & 0 \\ 11 & 13 & -9 \end{bmatrix} = \det\begin{bmatrix} 3 & -6 & -7 \\ 4 & 2 & 0 \\ 5 & 6 & -10 \end{bmatrix} + \det\begin{bmatrix} 3 & -6 & -7 \\ 4 & 2 & 0 \\ 6 & 7 & 2 \end{bmatrix}$$

split

Multiplying and adding it to another row does not change the det.

(vi) $\det\begin{bmatrix} R_1 \\ A \\ B \\ R_n \end{bmatrix} = \det\begin{bmatrix} R_1 \\ A+cB \\ B \\ R_n \end{bmatrix}$ $\det[C_1\ A\ B\ C_n] = \det[C_1\ A+Bk\ B\ C_n]$

(vii) If A is a lower triangular, upper triangular or diagonal matrix, then its determinant is $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$

Ex $\det\begin{bmatrix} 3 & 0 & 0 \\ 5 & -3 & 0 \\ 7 & 13 & 4 \end{bmatrix} = -36$

(viii) $\det A = \det A^T$ <u>Remark</u>: $\det I_n = 1$

(ix) $\det(AB) = (\det A)\cdot(\det B)$

$AA^{-1} = I_n \Rightarrow \det(AA^{-1}) = \det I_n \Rightarrow \underbrace{(\det A)\cdot(\det A^{-1})}_{\det A \neq 0} = 1$

<u>Theorem:</u> Matrix A is non-singular iff $\det A \neq 0$

→ Corollary: If A is nonsingular then, $\det A^{-1} = \dfrac{1}{\det A} = (\det A)^{-1}$

Ex Find $\det A$ if $A^2 = A$. $\det(AA) = \det A$

$(\det A)(\det A) = \det A \Rightarrow \det A = 0$ or $\det A = 1$

Ex) A and B are said to be similar if there is a nonsingular if there is a nonsingular matrix P such that $B = P^{-1}AP$.
Show that the similar matrices have the same determinant.

$\det B = \cancel{\det(P^{-1})} \cdot \det A \cdot \cancel{\det P} \implies \det B = \det A$

Definition: Suppose A is a n×n-matrix. The adjoint matrix of A (denoted by adj(A)) is an n×n matrix given by;

$$\operatorname{adj}(A) = [A_{ij}]^T \quad \text{where} \quad A_{ij} = (-1)^{i+j} \cdot \det A(i|j)$$

Goal) $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A)$

In exam, $A_{11} = (-1)^{1+1} \det \begin{bmatrix} 4 & 5 \\ 5 & 1 \end{bmatrix} = -21$

Ex)

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{bmatrix} \quad \operatorname{adj} A = [A_{ij}]^T = \begin{bmatrix} -21 & 9 & -3 \\ 11 & -5 & 3 \\ 8 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} -21 & 11 & 8 \\ 9 & -5 & -2 \\ -3 & 3 & 0 \end{bmatrix}$$

$$\det A = A_{11}a_{11} + A_{12}a_{12} + A_{13}a_{13} = (-21)\cdot 1 + 9 \cdot 4 + (-3)\cdot 3 = 6$$

Let's compute $(\operatorname{adj} A)\cdot A = \begin{bmatrix} -21 & 11 & 8 \\ 9 & -5 & -2 \\ -3 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

So we got; $(\operatorname{adj} A) A = 6 \cdot I_n$ 6 was actually $\det A$.

(x) $(\operatorname{adj} A) \cdot A = (\det A) \cdot I_n$

Suppose A is nonsingular. Multiply both sides with A^{-1}

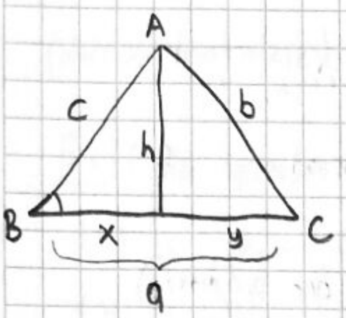
$$(\operatorname{adj} A) \underbrace{A A^{-1}} = (\det A) I_n A^{-1} \implies (\operatorname{adj} A) = \det A \cdot A^{-1}$$

(xi) If A is nonsingular, $\boxed{A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A}$

(sec3 question?)

256 exam type questions

Ex (pg 217, pr 58)



(same operations)

$$a = x + y = \overbrace{c\cos B + b\cos C}^{①} \longrightarrow b = a\cos C + c\cos A \} ②$$

$$\cos B = \frac{x}{c} \Rightarrow x = c\cos B \qquad \longrightarrow c = a\cos B + b\cos A \} ③$$

$$\cos C = \frac{y}{b} \Rightarrow y = b\cos B$$

We got a linear system ①, ②, ③.

Let $x_1 = \cos A$, $x_2 = \cos B$, $x_3 = \cos C$ $\Rightarrow$

$$(*) \begin{cases} c x_2 + b x_3 = a \\ c x_1 + a x_3 = b \\ a x_2 + b x_1 = c \end{cases}$$

Write the matrices,

$$\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So we can compute the $(^{-1})$ and write the cosine laws for.

— End of Section 3

General Vector Spaces

Goal To generalize the algebraic structure of $\mathbb{R}^n$ to define gen. vec spaces.

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots x_n) \mid x_k \in \mathbb{R}\} : \text{(set of all ordered n-tuples)}$$

$a = (x_1, x_2 \dots x_n) \in \mathbb{R}^n$ $\qquad$ $a + b = (x_1 + y_1, \dots, x_n + y_n)$ (vector addition)

$b = (y_1, y_2 \dots y_n) \in \mathbb{R}^n$ $\qquad$ $k \cdot a = (kx_1, \dots kx_n)$ (scalar multiplication)

$\hookrightarrow k \in \mathbb{R}$

(i) $a + b = b + a$ (commutative)

(ii) $ka = ak$ (associative)

(iii) $0_{\mathbb{R}^n} = (0, \dots 0)$ (additive identity)

(iv) $-a = (-1)a$ (additive inverse)

(v) $(k + L)a = ka + La$

(vi) $k(a + b) = ka + kb$

(vii) $k(La) = (kL)a$

$\rightarrow 0_{\mathbb{R}} a = 0_{\mathbb{R}^n}$

Note: Multiplication does not have to be commutative. / Actually it can't be since param. types are not the same. (vector, real)

Definition: Let V be a non-empty set. Suppose;

$+: V \times V \to V$ (addition) and $\cdot : \mathbb{R} \times V \to V$ (scalar multiplication)

V is said to be a <u>vector space</u> under the given operations if;

(i) $(a+b) \in V$ for all $a, b \in V$ (V is closed under addition)

(ii) $a + b = b + a$ (addition is commutative)

(iii) $(a+b)+c = a+(b+c)$ (addition is associative)

(iv) There exists a Zero Vector, $0_V \in V$ such that $0_V + a = a$ (0_V is additive identity)

(v) For any a, $-a = (-1)a$ satisfies $a + (-a) = 0_V$

(vi) $ka \in V$ for any $a \in V$, $k \in \mathbb{R}$ (V is closed under scalar multiplication)

(vii) $(k+l)a = ka + la$ (viii) $k(la) = (kl)a$ (ix) $k(a+b) = ka + kb$

Ex $\mathbb{R}^n$ is itself a vector space.

Ex $V = M_{m \times n}(\mathbb{R})$: collection of all $m \times n$ matrices

V is a vector space under; $A + B = [a_{ij} + b_{ij}]$, $kA = [ka_{ij}]$, $0_V = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

Ex $V = P_2[x]$: collection of all polynomials of degree 2 or less.

$V = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$. V is a vector space under;

$\alpha + \beta = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)$

$k\alpha = (ka_1)x^2 + (kb)x + (kc)$

$0_V = 0$

$V = P_n[x]$ is also a vector space.

Ex) $W = \{ax^2+bx+c \mid a \neq 0\}$: collection of polynomials of degree exactly 2.

W is not a vector space since $(x^2)+(-x^2)=0 \notin W$

Ex) We know $V = M_{2\times2}(\mathbb{R})$ is a vector space.

(a) Is $W_1 = \{A \in V \mid \det A = 0\}$ a vector space?

(b) Is $W_2 = \{A \in V \mid \det A \neq 0\}$ a vector space?

(a) No, since $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \notin W$ since $\det(A+B) \neq 0$

(b) $0_{2\times2} = 0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2$ since $\det 0_{2\times2} = 0$.

Ex) $V = C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$: all continuous fncs. on $[0,1]$

V is a vector space under $(f+g)(x) = f(x)+g(x)$, $(kf)(x) = k \cdot f(x)$, $0_V = 0$.

Ex) $P[x]$: set of all polynomials is a vector space. (inf-dimensional)

Ex) Is $W = \{(a,b,c) \in \mathbb{R}^3 \mid a+b=c+1\}$ a vector space?

No, since $0_{\mathbb{R}^3} = (0,0,0) \notin W$, $0+0 \neq 1$.

Ex) $W = \{ax^2+bx+c \mid a, b, c \in \mathbb{Z}\}$

$\left.\begin{matrix} \alpha = x^2 \in W \\ k = \frac{1}{2} \in \mathbb{R} \end{matrix}\right\} k\alpha = \frac{1}{2}x^2 \notin W$, then W is not closed under scalar mult. So W is not a vector space.

Exercise) Show that $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a+2d = 3b-c \right\}$ is a vector space.

Suppose V is a vector space. Then V is equipped with two operations, namely, vector addition and scalar multiplication. Thus,

$a+b \in V$, $ka \in V$ when $a, b \in V$, $k \in \mathbb{R}$ are already defined.

Let W be a non empty subset of V. W is said to be a subspace of V if W is itself a vector space under the operations defined on V.

① 0_V must be in W.

② W must be closed under vector addition

③ W must be closed under scalar multiplication.

Other conditions are already satisfied since W is a subset of V.

$\boxed{\text{Ex}}$ $V=\mathbb{R}^3$. Is $W=\{(a,b,c) \mid c \geq 0\}$ a subspace of V?

✓ (i) $0_V=(0,0,0) \in W$, since $c=0 \geq 0$

✓ (ii) $\left.\begin{array}{l} \alpha=(a_1, b_1, c_1) \in W \ (\text{i.e. } c_1 \geq 0) \\ \beta=(a_2, b_2, c_2) \in W \ (\text{i.e. } c_2 \geq 0) \end{array}\right\}$ $\left.\begin{array}{l} \alpha+\beta=(a_1+a_2, b_1+b_2, c_1+c_2) \in W \\ \text{since } c_1+c_2 \geq 0 \end{array}\right]$ is closed under addition

✗ (iii) $\alpha=(a,b,c) \in W$, $k\alpha=(ka,kb,kc) \notin W$ if $\boxed{\begin{array}{l} c \neq 0 \\ k<0 \end{array}}$

Take $-2\cdot(1,2,3)=(-2,-4,-6) \notin W$

is not closed under multiplication,

So we conclude that W is not a subspace of V.

$\begin{array}{cc} 1 & 2 \\ -3 & 2 \end{array}$

$\boxed{\text{Ex}}$ $V=M_{2\times2}(\mathbb{R})$ (a) $W_1=\{A \in V \mid \det A=0\}$

(b) $W_2=\{A \in V \mid \det A \neq 0\}$ (c) $W_3=\{A \in V \mid A=A^T\}$ (symmetric)

(e) $W_5=\{A \in V \mid tr(A)=0\}$ (d) $W_4=\{A \in V \mid A=-A^T\}$ (skew sym)

(b) $0_V=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2$ so W_2 is not a subspace.

(a) (i) $0_V=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_1$ ✓ ✗ (ii) $\underset{\substack{\in W_1 \\ \det=0}}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} + \underset{\substack{\in W_1 \\ \det=0}}{\begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}}_{\substack{\notin W_1 \\ \det \neq 0}}$

(iii) $A \in W_1$, $\det A=0$, $k \in \mathbb{R}$

$\det(kA)=k^2\cdot\det A=0$ ✓

W_1 is not closed under addition, so its not a subspace.

(d) (i) $0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_4$ (iii) $A = -A^T,\ A \in W_4$, $k \in \mathbb{R}$ $\quad kA = k(-A^T) = -kA^T = -(kA)^T \Rightarrow kA \in W_4$ / W_4 is closed under..

(ii) $A = -A^T,\ A \in W_4$, $B = -B^T,\ B \in W_4$ $\Rightarrow A+B = -A^T - B^T = -(A^T + B^T) = -(A+B)^T \Rightarrow (A+B) \in W_4$ W_4 is closed un..

Thus, W_4 is a subspace. (c) and (e) are also subspaces.

Actually these are true for $V = M_{n\times n}(\mathbb{R})$ vector space.

Ex] $V = P_2[x]$: space of all polynomials with degree 2 or less.

(a) $W_1 = \{ax^2 + bx + c \mid a \neq 0\}$: all polys. with degree exactly 2

(b) $W_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{Q}\}$ = all polys with rational coefficients

(c) $W_3 = \{ax^2 + bx + c \mid 2a - b = 3c\}$

Exercise: Prove $x^2\sqrt{2} \notin W_2$, $\sqrt{2} \notin \mathbb{Q}$

(a) $0_V = 0x^2 + 0x + 0 \notin W_1$ so W_1 is not a subspace

(b) (i) $0_V \in W_2$ (ii) W_2 is closed since two rationals add up to another rational.

(iii) Take $k = \sqrt{2} \in \mathbb{R}$, $\alpha = x^2 + 2x + 1 \Rightarrow$ W_2 is not closed under scalar mult. Thus W_2 is not a subspace

(c) (i) $0_V \in W_3$ (ii) $\alpha = a_1x^2 + b_1x + c_1 \in W_3$, $2a_1 - b_1 = 3c_1$

$\beta = a_2x^2 + b_2x + c_2 \in W_3$, $2a_2 - b_2 = 3c_2$

W_3 is a subspace

$(\alpha+\beta) = (a_1+a_2)x^2 + (b_1+b_2)x + c_1 + c_2$, $2(a_1+a_2) - (b_1+b_2) = 3(c_1+c_2)$ ✓

Ex] $W_4 = \{ax^2 + bx + c \mid 2a - b = 3c + 1\}$ is not a subspace since $0_V \notin W_4$

Remark: $\mathbb{R}^n \longleftrightarrow M_{n\times 1} \longleftrightarrow M_{1\times n} \longleftrightarrow P_{n-1}[x]$ these spaces are actually identical.

Exercise $V = \mathbb{R}^3$

(a) a Line L is a subspace of $\mathbb{R}$ iff L passes through the origin

(b) a plane P is a subspace of $\mathbb{R}$ iff P passes through the origin.

Definition: Let $A \in M_{m\times n}$. The solution space generated by A is the solution set of the homogeneous system given by

$$(*)\ AX = 0_{m\times 1} \quad \text{where} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\text{Soln}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in M_{n\times 1}(\mathbb{R}) \;\middle|\; AX = 0_{m\times 1} \right\}$$

(*) $\text{Soln}(A)$ is a subspace of $M_{n\times 1}(\mathbb{R})$.

Proof: (i) $0_V = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \text{Soln}(A)$ since $A \cdot 0_V = 0_{m\times 1}$

(ii) $\left.\begin{matrix} X \in \text{Soln}(A) & AX = 0_{m\times 1} \\ Y \in \text{Soln}(A) & AY = 0_{m\times 1} \end{matrix}\right\} A(X+Y) = AX + AY = 0_{m\times 1} + 0_{m\times 1} = 0_{m\times 1}$ | $(X+Y) \in \text{Soln}(A)$

(iii) $X \in \text{Soln}(A),\ k \in \mathbb{R} \Rightarrow A\cdot(kX) = k(AX) = k \cdot 0_{m\times 1} = 0_{m\times 1}$ | $kX \in \text{Soln}(A)$

→ Soln. set of a non-homogeneous system is not a subspace since it does not contain the 0_V, and also does not satisfy other two properties.

Ex | Is $V = \mathbb{R}^3$ a vector space under given operations?

→ $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1+a_2,\ b_1+b_2,\ c_1+c_2)$

→ $k \odot (a, b, c) = (a, b, kc)$

If V is a vector space, then $0_{\mathbb{R}} \cdot (a, b, c) = 0_V$ should be satisfied

$0 \cdot (1, 1, 1) = (1, 1, 0) \neq (0, 0, 0)$ so V is not a vector space

(*) Answer ✗ Exercise | $V = \mathbb{R}^+ = (0, \infty)$ Prove/disprove that V is a vector space under,

$x \boxplus y = xy$ where $x, y \in V$

$k \boxdot x = k^x$ where $k \in \mathbb{R}$

Hint: Show $1_{\mathbb{R}}$ is additive identity

$0_V = 1_{\mathbb{R}}$

$1 \boxplus x = x$

Answer ✓ Exercise 2 | $k \boxdot x = x^k$ [is vector]

FABER-CASTELL

<u>Exercise</u> $A \in M_{n\times n}(\mathbb{R}) = V$ Show that, $W = \{B \in V \mid AB = BA\}$ is a subspace.

<u>Exercise</u> Suppose W_1 and W_2 are subspaces of V. Prove / disprove that;

Answers

✓ (a) $W_1 + W_2 = \{\alpha + \beta \mid \alpha \in W_1 \text{ and } \beta \in W_2\}$ is a subspace of V

✓ (b) $W_1 \cap W_2$ is a subspace of V.

✗ (c) $W_1 \cup W_2$ is a subspace of V.

<u>Linear Combination:</u> Suppose that V is a vector space. If $\alpha_1, \alpha_2 \dots \alpha_k$ are finitely many vectors in V, then the any expression of the form $c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k$ for some real $c_1, c_2 \dots c_k$'s is called as a linear combination of $\alpha_1, \alpha_2 \dots \alpha_k$.

⊛ The set of all linear combinations of $\alpha_1, \alpha_2 \dots \alpha_k$ is denoted by;

$$\text{span}\{\alpha_1, \alpha_2, \dots, \alpha_k\} = \{c_1\alpha_1, c_2\alpha_2, \dots, c_k\alpha_k \mid c_1, c_2 \dots c_k \in \mathbb{R}\}$$

<u>Pop:</u> $W = \text{span}\{\alpha_1, \alpha_2 \dots \alpha_k\}$ is a subspace of V.

In this case, $\{\alpha_1, \alpha_2 \dots \alpha_k\}$ is said to be a spanning set for W and W is called spanned by $\alpha_1, \alpha_2 \dots \alpha_k$.

<u>Remark:</u> W does not necessarily have a unique spanning set.

<u>Proof:</u> (i) $0_V \in W$ since $0_V = 0_{\mathbb{R}}\alpha_1 + 0_{\mathbb{R}}\alpha_2 + 0_{\mathbb{R}}\alpha_3 \cdots 0_{\mathbb{R}}\alpha_k$

(ii) $\left.\begin{aligned}\alpha &= c_1\alpha_1 + \dots + c_k\alpha_k \\ \beta &= d_1\alpha_1 + \dots + d_k\alpha_k\end{aligned}\right\}$ $\alpha + \beta = (c_1 + d_1)\alpha_1 + \dots + (c_k + d_k)\alpha_k \in W$ since its another linear combination.

(iii) $\left.\begin{aligned}&\alpha = c_1\alpha_1 + \dots + c_k\alpha_k \\ &k \in \mathbb{R}\end{aligned}\right\}$ $k\alpha = (kc_1)\alpha_1 + \dots (kc_k)\alpha_k \in W$ since its another linear combination

Ex) $V=\mathbb{R}^3$ $\quad (a,b,c) = a\underbrace{(1,0,0)}_{e_1} + b\underbrace{(0,1,0)}_{e_2} + c\underbrace{(0,0,1)}_{e_3}$

$= ae_1 + be_2 + ce_3$

$e_1=(1,0,0)$

$e_2=(0,1,0)$ → any vector (a,b,c) is a linear comb. of e_1, e_2, e_3.

$e_3=(0,0,1)$ → $\{e_1, e_2, e_3\}$ is a spanning set for $\mathbb{R}^3$.

$\text{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$

Exercise) $V=\mathbb{R}^n$

Ex) $V = P_2[x] = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$ $\qquad$ Exercise) $P_n[x]$

$ax^2+bx+c = a(x^2)+b(x)+c(1)$ i.e $ax^2+bx+c \in V$ is a linear combination of $x^2, x, 1$.

$\text{Span}\{x^2, x, 1\} = P_2[x]$

Exercise) Define $W = \{ax^2+bx+c \in V \mid 2a+b=3c\}$ $\quad$ show that W is a subspace of V.

→ $2a+b=3c$
$b=3c-2a$ $\Rightarrow$ $ax^2+bx+c = ax^2+(3c-2a)x+c$

$= (ax^2-2ax)+(3cx+c)$

$= a(x^2-2x)+c(3x+1)$

any $\alpha = ax^2+bx+c \in W$ is a lin. comb of (x^2-2x) and $(3x+1)$.

$\text{span}\{(x^2-2x), (3x+1)\} = W$ $\quad$ (Remark: There can be different spanning as well.

Ex) $V = M_{2\times2}(\mathbb{R})$

(a) $W_1 = \{A \in V \mid A = A^T\}$

(b) $W_2 = \{A \in V \mid A = -A^T\}$

(c) $W_3 = \{A \in V \mid \text{tr}(A) = 0\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\cdot\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3} + d\ldots_{E_4}$

Then any 2×2 matrix is a lin comb of E_1, E_2, E_3, E_4.

So, $\text{span}\{E_1, E_2, E_3, E_4\} = M_{2\times2} = V$

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_1$ iff $A = A^T$, ie, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\begin{matrix} a=a \\ b=c \\ c=b \\ d=d \end{matrix}$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_1$ iff $b=c \Rightarrow W_1 = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{F_1} + b \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F_2} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{F_3}$ Any 2x2 matrix is a lin comb of F_1, F_2, F_3.

span$\{F_1, F_2, F_3\} = W_1$

(b) The process is the same, Answer: $W_2 = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

(c) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_1$ iff $\left. \begin{matrix} a+d=0 \\ d=-a \end{matrix} \right\}$ <solve like above in exam> $W_3 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Exercise: Solve the above question with; Find a spanning set for $V = M_{3\times3}(\mathbb{R})$ and for W_1, W_2, W_3.

Remark: (i) span$\{0_V\} = \{0_V\}$

(ii) span$\{\alpha_1, \alpha_2, \dots \alpha_k\}$ = span$\{\alpha_1, \alpha_2, \dots, \alpha_k, 0_V\}$

(iii) span$\{\alpha\} = \{c\alpha \mid c \in \mathbb{R}\}$

Linear Independence: Let V be a vector space. A set of finitely many vectors $\{\alpha_1, \alpha_2, \dots \alpha_k\}$ is said to be linearly independent if the homogenious system $c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k = 0$ has only the trivial soln, ie, $c_1 = c_2 = c_3 \dots = c_k = 0$.

$\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is said to be linearly dependent if the homogeneous equation above has some non trivial solutions (i.e inf. many)

Remark: $\{\alpha\}$ is linearly independent iff $\alpha \neq 0_V$

Ex) $V = \mathbb{R}^3$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Claim: $\{e_1, e_2, e_3\}$ is linearly independent.

$(*)$ $c_1 e_1 + c_2 e_2 + c_3 e_3 = 0_{\mathbb{R}^3} = (0,0,0) \Rightarrow \left.\begin{array}{l} c_1=0 \\ c_2=0 \\ c_3=0 \end{array}\right\}$ Only has the trivial. Thus $\{e_1, e_2, e_3\}$ is lin. ind.

Ex) $V = P_2[x]$, Claim: $\{x^2, x, 1\}$ is lin. ind.

$(*)$ $c_1 x^2 + c_2 x + c_3 1 = 0x^2 + 0x + 0 \cdot 1 \Rightarrow \left.\begin{array}{l} c_1=0 \\ c_2=0 \\ c_3=0 \end{array}\right\}$ $\{x^2, x, 1\}$ is lin. ind.

Ex) $W = \{ax^2 + bx + c \mid 2a + b = 3c\}$, Claim: $\{(x^2-2x), (3x+1)\}$ is lin. ind.

$(*)$ $\left.\begin{array}{l} c_1(x^2-2x) + c_2(3x+1) = 0x^2 + 0x + 0 \\ c_1x^2 + (-2c_1 + 3c_2)x + c_2 = 0x^2 + 0x + 0 \end{array}\right\}$ $\left.\begin{array}{l} c_1=0 \\ c_2=0 \end{array}\right\}$ Only has the trivial soln. So the set is lin. ind.

Ex) Is $\{\underbrace{(2,3)}_{\alpha_1}, \underbrace{(-4,7)}_{\alpha_2}, \underbrace{(1,-2)}_{\alpha_3}\}$ lin ind. or dep.?

$(*)$: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0)$

$c_1(2,3) + c_2(-4,7) + c_3(1,-2) = (0,0)$

$(*)$: $\begin{array}{l} 2c_1 - 4c_2 + c_3 = 0 \\ 3c_1 + 7c_2 - 2c_3 = 0 \end{array}$

$$\begin{bmatrix} 2 & -4 & 1 \\ 3 & 7 & -2 \end{bmatrix} \xrightarrow{\text{(Exercise)}} \begin{array}{ccc} c_1 & c_2 & c_3 \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} -2 \\ 1 \end{array} & \left.\begin{array}{c} 1/2 \\ -7/26 \end{array}\right] \end{array}$$

c_1, c_2 basic variables, c_3 free var.

$\left.\begin{array}{l} c_3 = 26u \\ c_2 = 7u \\ c_1 = u \end{array}\right\}$ $S = \left\{ \begin{bmatrix} u \\ 7u \\ 26u \end{bmatrix} \middle| u \in \mathbb{R} \right\}$

Since $(*)$ has inf. many solns, $\{(2,3), (-4,7), (1,-2)\}$ is dependent.

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0)$

$u\alpha_1 + 7u\alpha_2 + 26u\alpha_3 = (0,0)$ → Take $u=1$ $\alpha_1 + 7\alpha_2 + 26\alpha_3 = (0,0)$

$\alpha_1 = -7\alpha_2 - 26\alpha_3$, α_1 is lin. comb of α_2,

$\alpha_1 \in \text{span}\{\alpha_2, \alpha_3\}$, $\alpha_2 \in \text{span}\{\alpha_1, \alpha_3\}$, $\alpha_3 \in \text{span}\{\alpha_1, \alpha_2\}$

$\Rightarrow \text{Sol}(A) = u\begin{bmatrix} 1 \\ 7 \\ 26 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 7 \\ 26 \end{bmatrix}\right\}$

<u>Remarks:</u> (i) $\alpha_1 \neq 0_V$, $\alpha_2 \neq 0_V$

$\{\alpha_1, \alpha_2\}$ is lin. dep. iff $\alpha_1 = k\alpha_2$ for some k.

$\{\alpha_1, \alpha_2\}$ is lin. ind. iff $\alpha_1 \neq k\alpha_2$ for any k.

(ii) $\{\alpha_1, \alpha_2, \dots \alpha_k, 0_V\}$ is lin. dep.

(iii) if $\{\alpha_1, \dots \alpha_k\}$ is lin ind. then, none of the vectors α_i is a linear combination of the others, ie,

$\alpha_i \notin \text{span}\{\alpha_1, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_k\}$

(iv) if $\{\alpha_1, \dots \alpha_k\}$ is lin dep then, any vector α_i is a linear combination of the others, ie, → some?

$\alpha_i \in \text{span}\{\alpha_1, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_k\}$

<u>Ex)</u> Prove or disprove that $\{2x^2-3, 4x^2-5x-6, -2x^2+x+3\}$ lin. dep.

$V = P_2[x]$. Construct, (*): $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0_V$

$c_1(2x^2-3) + c_2(4x^2-5x-6) + c_3(-2x^2+x+3) = 0$

$x^2(2c_1 + 4c_2 - 2c_3) + x(-5c_2 + c_3) + 1\cdot(-3c_1 - 6c_2 + 3c_3) = 0$

$$(*)\begin{cases} 2c_1+4c_2-2c_3=0 \\ -5c_2+c_3=0 \\ -3c_1-6c_2+c_3=0 \end{cases} \qquad \begin{bmatrix} 2 & 4 & -2 \\ 0 & -5 & 1 \\ -3 & -6 & 3 \end{bmatrix} \xrightarrow{\langle \text{Exercise} \rangle} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -3/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

c_1, c_2 basic var. $\quad c_3 = 5u$

c_3 free var. $\quad c_2 = u$

$c_1 = 3u$

$$S = \left\{ \begin{bmatrix} 3u \\ u \\ 5u \end{bmatrix} \middle| u \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$$

$(*)$ has inf. many sols, then $\{\alpha_1, \alpha_2, \alpha_3\}$ is lin. dep.

$$3u\,\alpha_1 + u\,\alpha_2 + 5u\,\alpha_3 = 0 \overset{u=1}{\Longrightarrow} \alpha_2 = -5\alpha_3 - 3\alpha_1, \quad \alpha_2 \in \text{span}\{\alpha_1, \alpha_3\}$$

Ex (pr 26, pg 253) Given that $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly ind., show that $\{\alpha_2+\alpha_3, \alpha_1+\alpha_3, \alpha_1+\alpha_2\}$ is linearly ind.

Let $\{\beta_1, \beta_2, \beta_3\}$. Construct, $(*)$: $c_1\beta_1 + c_2\beta_2 + c_3\beta_3 = 0_V$

$$c_1(\alpha_2+\alpha_3) + c_2(\alpha_1+\alpha_3) + c_3(\alpha_1+\alpha_2) = 0_V$$

$(c_2+c_3)\alpha_1 + (c_1+c_3)\alpha_2 + (c_1+c_2)\alpha = 0_V$, given $\{\alpha_1, \alpha_2, \alpha_3\}$ is ind.

which means,
$$\left. \begin{array}{l} c_2+c_3=0 \\ c_1+c_3=0 \\ c_1+c_2=0 \end{array} \right\} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\langle ex \rangle} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} c_1=0 \\ c_2=0 \\ c_3=0 \end{array}$$

This implies, $(*)$ only has the trivial solution, so $\{\beta_1, \beta_2, \beta_3\}$ is ind.

Exercise Find a spanning set for the soln. space generated by

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix}$$

$\{\alpha_1, \alpha_2, \alpha_3\}$ is lin. dep. So, $\alpha_1 \in \operatorname{span}\{\alpha_2, \alpha_3\}$ which leads to $\operatorname{span}\{\alpha_1, \alpha_2, \alpha_3\} = \operatorname{span}\{\alpha_2, \alpha_3\}$

Finding a basis for a subspace

Definition: Let V be a vector space. Assume that B is a set of vectors in V. B is a basis for V if;

(plural is bases)

exam question

(i) B is linearly independent

(ii) $\operatorname{span} B = V$ (B is a spanning for V)

+1 lin ind question.

Explicitly, a lin. ind. spanning set of V is a basis.

Ex We showed $\operatorname{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is lin. ind. Thus $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{R}^3$.

Which means any vector $\alpha \in V$ can be written as linear combination of $\alpha_1, \alpha_2 \cdots \alpha_n$ (B's elements.)

In this case, V is said to be a n-dimensional vector space.

$\dim V = n$

Remark: The dimension of V remains unchanged from basis to basis. If B' is basis for a n-dimensional vector space then B' has exactly n vectors.

Ex $V = \mathbb{R}^n$ $e_1 = (1, 0 \dots 0)$ $e_2 = (0, 1, 0 \dots 0)$ $\dots$ $e_n = (0, 0 \dots 1)$

Claim: $B = \{e_1, e_2 \dots e_n\}$ is a basis for $\mathbb{R}^n$

(i) B is lin. indd: $c_1e_1 + c_2e_2 + \dots c_ne_n = 0_{\mathbb{R}^n} \Rightarrow (c_1, c_2 \dots c_n) = (0, 0 \dots 0)$

(ii) $\operatorname{span} B = \mathbb{R}^n$: $\alpha = (x_1, x_2 \dots x_n) \in \mathbb{R}^n \Rightarrow \alpha = x_1e_1 + x_2e_2 + \cdots$ ✓

α is a lin. comb of

Ex $V = P_2[x] = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$

$ax^2+bx+c \in V$, $a(x^2)+b(x)+c(1)$, ie, any $\alpha \in V$ is a lin. comb of $x^2, x, 1$.

$\text{span}\{x^2, x, 1\} = P_2[x]$ and $\{x^2, x, 1\}$ is lin ind: ----

So $\{x^2, x+1\}$ is a basis for $P_2[x]$, $\dim P_2[x] = 3$

Exercise Show that $\{x^n, x^{n-1}, \dots 1\}$ is a basis for $P_n[x]$, $\dim P_n[x] = n+1$

Ex $V = M_{2\times3}(\mathbb{R})$: space of all 2×3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times3} = a\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_1} + b\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_2} \cdots f\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_6}$$

$\text{span}\{E_1, \dots E_6\} = M_{2\times3}(\mathbb{R})$, $B = \{E_1, \dots E_6\}$ is a spanning set for V.

B is a basis for V if B is lin ind:

$c_1E_1 + c_2E_2 + \cdots c_6E_6$ (*) $\nearrow$ $c_1 = c_2 = \dots = 0$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (*) has only the trivial solution. $\Rightarrow B$ is lin. ind.

Thus B is a basis for V. $\dim M_{2\times3}(\mathbb{R}) = 2\cdot3 = 6$

Exercise Show that $\dim M_{m\times n}(\mathbb{R}) = m\cdot n$ by providing a basis for $M_{m\times n}$

Ex $P[x]$: space of all polynomials.

$B = \{1, x^2, x^3 \dots\}$ is a basis for $P[x]$

$P[x]$ is a infinite dim. vector space.

<u>Preposition:</u> Suppose V is a n-dimensional vector space. ($\dim V = n < \infty$)
If B has k vectors in it.

(i) When $k \leq n-1$ then B can't be a basis (since $\operatorname{span} B \neq V$)

(ii) When $k \geq n+1$ then B can't be a basis (since B is lin. dep.)

<u>Ex)</u> $V = P_3[x]$. Is $B = \{x^3 + 4x^2 - 1,\ x^2 + x + 5,\ 3x^3 - x\}$ a basis for $P_3[x]$?

$\dim P_3[x] = 4 > 3 = |B| \Rightarrow \operatorname{span} B \neq V \Rightarrow B$ is not a basis.

<u>Ex)</u> $V = \mathbb{R}^3$ Is $B = \{(3,-2,0), (0,1,0), (4,-2,7), (5,9,-11)\}$ a basis for $\mathbb{R}^3$?

$\dim \mathbb{R}^3 = 3 < 4 = |B| \Rightarrow B$ is lin. dep $\Rightarrow B$ is not a basis.

<u>Ex)</u> $V = \mathbb{R}^2$ Is $B = \{\overbrace{(-2,3)}^{\alpha_1}, \overbrace{(1,-4)}^{\alpha_2}\}$ a basis for $\mathbb{R}^2$?

$\dim \mathbb{R}^2 = 2 = 2 = |B| \Rightarrow$ We should show (i) B is lin. ind. (ii) $\operatorname{span}(B) = \mathbb{R}^2$

(i) $c_1\alpha_1 + c_2\alpha_2 = 0$

$c_1(-2,3) + c_2(1,-4) = 0$

$$(*)\begin{cases} -2c_1 + c_2 = 0 \\ 3c_1 - 4c_2 = 0 \end{cases} \qquad \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix} \xrightarrow{\ \ } \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}$$

$(*)$ has only trivial solution.
Then B is linearly independent

(ii) Any vector $\alpha = (a,b)$ should be able to written as a linear combination of α_1 and α_2. i.e $(a,b) = x_1\alpha_1 + x_2$ $(**)$

Show that $(**)$ is consistent for any a, b values.

$$(**): (a,b) = (-2,3)x_1 + (1,-4)x_2 \left.\right\} \underbrace{\begin{matrix} -2x_1 + x_2 = a \\ 3x_1 - 4x_2 = b \end{matrix}}_{(**)} \qquad \left[\begin{array}{cc|c} -2 & 1 & a \\ 3 & -4 & b \end{array}\right] \longrightarrow \begin{bmatrix} 1 & -3 & a+b \\ 0 & 5 & -3a-2b \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \to \begin{bmatrix} 1 & -3 & a+b \\ 0 & 1 & \frac{-3a-2b}{5} \end{bmatrix} \end{array}$$

No leading entry can appear in the last column $\Rightarrow$ (**) has always a solution.

x_1, x_2 basic vars, no free vars

$$x_2 = \frac{-3a-2b}{5} \qquad x_1 = a+b+3\left(\frac{-3a-2b}{5}\right) = \frac{-4a-b}{5}$$

So we get a way to get a linear combination of any (a,b);

$$(**) = (a,b) = x_1\alpha_1 + x_2\alpha_2 \Rightarrow (a,b) = \left[\frac{-4a-b}{5}\right]\alpha_1 + \left[\frac{-3a-2b}{5}\right]\alpha_2$$

So any (a,b) is a lin. comb. of α_1, α_2.

(i), (ii) $\Rightarrow$ B is a basis for $\mathbb{R}^2$.

Ex: Prove/disprove that $\{x^3+2x^2-2x+3,\ 2x^3+5x^2-3x+4,\ x^3+5x^2-2,\ x^2+3x-2\}$ is a basis for $P_3[x]$.

Theorem: Let V be an n-dimensional vector space. ($\dim V = n < \infty$) Suppose that B has exactly n-vectors. The following are equivalent:

(i) B is basis
(ii) $\operatorname{span}(B) = V$
(iii) B is lin ind.

B has 4 vectors and $\dim P_3[x] = 4$ so the theorem applies. So it's enough to show that B is lin. ind to say B is a basis.

Construct $(*)$: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0_V$

$$c_1(x^3+2x^2-2x+3) + c_2(2x^3+5x^2-3x+4) + c_3(x^3+5x^2-2) + c_4(x^2+3x-2)$$

$(*): \; x^3(c_1+2c_2+c_3) + x^2(2c_1+5c_2+5c_3+c_4) + x(-2c_1-3c_2+3c_4) + (3c_1+4c_2-2c_3-3c_4) = 0$

$$(*)\begin{cases} c_1+2c_2+c_3=0 \\ 2c_1+5c_2+5c_3+c_4=0 \\ -2c_1-3c_2+3c_4=0 \\ 3c_1+4c_2-2c_3-3c_4=0 \end{cases} \qquad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{bmatrix} \xrightarrow{\text{exercise}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e, $(*)$ has only the trivial soln. $c_1=c_2=\dots=c_4=0$

Alternatively, $(*)$ has only the trivial soln, since $\det A \neq 0$

→ If the last row of rref was 0000 or $\det A = 0$ we would say this $(*)$ is linearly dependent and B is not a basis

B is lin. ind, thus B is a basis for $P_3[x]$.

Preposition: Let V be a n-dimensional vector space ($\dim V = n < \infty$)

Suppose W is a subspace of V.

(i) $W = \{0_V\}$ iff $\dim W = 0$

(ii) $W = V$ iff $\dim W = n$

(iii) $W \neq \{0_V\}$ and $W \neq V$ iff $1 \leq \dim W \leq n-1$

Ex) $V = \mathbb{R}^3$, $W = \{(a,b,c) \mid a+2b=3c\}$

(a) Show that W is a subspace of V. <exercise>

(b) Find a basis for W and determine its dimension.

$(a,b,c) \in W$ iff $a+2b=3c$ ⇓ $a=3c-2b$ $\Rightarrow$ $(a,b,c) = (3c-2b, b, c) =$

$(3c, 0, c) + (-2b, b, 0) = c(3,0,1) + b(-2,1,0)$

$B = \{\underbrace{(3,0,1)}_{\alpha_1}, \underbrace{(-2,1,0)}_{\alpha_2}\}$ span$(B) = W$ ① since any $(a,b,c) \in W$ is a lin. com. of α_1 and α_2. B is lin. ind. ② since α_1 is not a constant multiple of α_2. ($\alpha_1 \neq \alpha_2 \cdot k$ for some k)

①,② ⇒ Thus B is a basis for W. $\dim W = 2$

Ex Find a basis for the soln. space generated by,

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 10 \\ 3 & -1 & 7 & 4 & 1 \\ -5 & 3 & -15 & -6 & 9 \end{bmatrix} \quad S = \text{soln}(A) \text{ of } (*) \begin{cases} x_1 + x_2 - x_3 + 2x_4 + 10x_5 = 0 \\ - - - - \quad = 0 \\ - - - - \quad = 0 \end{cases}$$

$$\text{rref of } A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 3/2 & 3/2 & 11/4 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

x_1, x_2, x_5 basic variables, x_3, x_4 free variables

$x_3 = 2u$
$x_4 = 2v$
$x_5 = 0$

$x_1 = -3u - 3v$
$x_2 = 5u - v$

$$S = \left\{ \begin{bmatrix} -3u-3v \\ 5u-v \\ 2u \\ 2v \\ 0 \end{bmatrix} ; u, v \in \mathbb{R} \right\}$$

$$\begin{bmatrix} -3u-3v \\ 5u-v \\ 2u \\ 2v \\ 0 \end{bmatrix} = u \overbrace{\begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \\ 0 \end{bmatrix}}^{\alpha_1} + v \overbrace{\begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}^{\alpha_2}$$

$B = \{\alpha_1, \alpha_2\}$ is a spanning set for soln$(A) = S$.

Soln(A) is a subspace of $M_{5\times 1}(\mathbb{R})$. We need to show that $\{\alpha_1, \alpha_2\}$ is a basis for Soln$(A) = S$. $\dim \text{Soln}(A) = 2$

Exercise Find the value(s) of k for which

$B = \{(1,2,-1,1), (-1,-1,-1,-1), (0, 2k-1, 1, k), (2,5,-3,4)\}$

is a basis for $\mathbb{R}^4$. ⟨All except $k = \frac{-2}{7}$⟩

Exercise) Find a basis for W.

(a) $W = \{A \in M_{2\times2} \mid \overbrace{A = A^T}^{*}\}$ (b) $*$: $A = -A^T$ (c) $*$ $tr(A) = 0$

Exercise) Solve the above question for $M_{3\times3}$.

Exercise) Find a basis for W when $V = P_n[x]$ and

$W = \{P \in V \mid x \cdot p'(x) = p(x)\}$ Ans: $\{x\}$

Exercise) Show that $\{1, 1+x, 1+x+x^2, \dots 1+x+x^2+\dots x^n\}$ is a basis for $P_n[x]$.

Recall: $A = M_{m\times n}(\mathbb{R})$, $Soln(A) = S$ where S is the soln. set of

(*) $AX = 0_{m\times1}$, $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\rightarrow$ $Soln(A)$ is a subspace of $M_{n\times1}(\mathbb{R})$

(*) Suppose RREF of A has r-many nonzero rows, ie, there are exactly r-many leading ones in the RREF of A.

Thus, we have r-many basic variables, then (*) has (n−r)-many free variables (parameters).

Proposition: $\dim Soln(A) = n - r$ (number of parameters in S)

Term: r is so called the rank of A.

$\approx{W} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \cdots \right.$ $\quad \widetilde{W} = \{ ax^3+bx^2+cx+d \mid \cdots$ $\quad \widetilde{\widetilde{W}} = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \cdots \right\}$ (all equivalent)

Ex| Find a basis for $W = \{ (a,b,c,d) \in \mathbb{R}^4 \mid a-3b = c+2d = 0 \}$

$$(*) \begin{cases} a-3b=0 \\ c+2d=0 \end{cases} \Rightarrow \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array} \quad \begin{array}{l} a,c: \text{basic} \\ b,d: \text{free} \end{array} \quad \begin{array}{l} b=u \Rightarrow a=3u \\ d=v \Rightarrow c=-2v \end{array}$$

$$(a,b,c,d) = (3u, u, -2v, v) = u \underbrace{(3,1,0,0)}_{\alpha_1} + v \underbrace{(0,0,-2,1)}_{\alpha_2}$$

So any $(a,b,c,d) \in W$ can be written as a linear comb of α_1 and α_2.

Then $B = \{ (3,1,0,0), (0,0,-2,1) \}$ is a spanning set for W.

B is also linearly independent since one can't be written as a multiple of other.

$(\alpha_1 \neq k\alpha_2$ for any $k \in \mathbb{R}) \Rightarrow (B$ is lin. ind$)$

Thus, B is a basis for W. and $\dim W = |B| = 2$

For the equivalent $\widetilde{W}$, $\widetilde{B} = \{ (3x^3+x^2), (-2x+1) \}$

Ex| Find span $\{ 1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4 \} = W$

Since the biggest dim is 4, W is a subspace of $P_4[x]$

$\dim P_4[x] = 5$ since $P_4[x]$ has a basis $\{ x^4, x^3, x^2, x^1, 1 \}$

$B = \{ 1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4 \}$

B is a set of lin. ind. $\langle$Verify$\rangle$: $c_1\alpha_1 + c_2\alpha_2 \cdots = 0 \; (*)$

✓ Is B a basis for $P_4[x]$? yes, since $|B| = \dim P_4 = 5$ and B is lin. ind.

Then we can also conclude span$\{B\} = P_4[x]$

$\to W$ is the commutator of $A=\begin{bmatrix}5 & 0\\ 0 & -7\end{bmatrix}$

Ex) Find a basis for $W=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in M_{2\times 2} \;\middle|\; \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & -7\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & -7\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right\}$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & -7\end{bmatrix}=\underbrace{\begin{bmatrix}5a & -7b\\ 5c & -7d\end{bmatrix}}_{1},\quad \begin{bmatrix}5 & 0\\ 0 & -7\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\underbrace{\begin{bmatrix}5a & 5b\\ -7c & -7d\end{bmatrix}}_{2}$$

$$(1=2)\begin{cases}5a=5a\ (\checkmark)\\ -7b=5b \Rightarrow b=0\\ 5c=-7c\Rightarrow c=0\\ -7d=-7d\ (\checkmark)\end{cases}\Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix}\in W \text{ if } \begin{matrix}b=0\\ c=0\end{matrix}.$$

$$\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}=a\cdot\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+d\cdot\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\Rightarrow B \text{ is a spanning set for } W,$$
and B is lin. ind.

Thus, B is a basis for W and $\dim W=2$.

Exercise) $V=P_2[x]$. Find a basis for W_1 and W_2 if they are a subspace.

(a) $W_1=\{p\in W \mid p(0)+p(1)=1\}$ ⟨no subspace⟩

(b) $W_2=\{p\in W \mid p(1)=-p(-1)\}$ $\langle\{x^2-1,\ x\}\rangle$

Exercise) For which values of a and b it is possible to express $\begin{bmatrix}4a\\ a-3\\ a+2\end{bmatrix}$ and $\begin{bmatrix}4\\ -b\\ 2b+1\end{bmatrix}$ as a linear comb of $\begin{bmatrix}4\\ 0\\ 1\end{bmatrix}$ and $\begin{bmatrix}0\\ -1\\ 2\end{bmatrix}$. ? ⟨$a=2$, $b=$ any value⟩

Exercise) Find all possible values for a which $\{(0,a,3),(a,a,0),(-3,0,a)\}$ is a basis of $\mathbb{R}^3$. ⟨$k\neq 0$, $k\neq 3$, $k\neq -3$⟩

Exercise) Find a basis for $W=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix} \;\middle|\; a+b-c=a+b+2d=0\right\}$

$$\left\langle\left\{\begin{bmatrix}-1\\ 1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 2\\ -1\end{bmatrix}\right\}\right\rangle$$

Exercise $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\det A = 2$

(a) Find $\det\begin{bmatrix} a+2d-3g & b+2e-3h & c+2f-3i \\ 3g & 3h & 3i \\ 2d & 2e & 2f \end{bmatrix} =$ (some numeric answer) $\langle -12 \rangle$

(b) Find $\det(\text{adj}(A))$ and $\det(2\text{adj}(A))$ $\langle 4; 32 \rangle$

Exercise Find a relation between a, b, c such that (a,b,c) is a linear combination of $(1,-1,1), (2,1,-1), (4,-1,1)$ $\langle b=-c \rangle$?

Row and Column Spaces:

Let $A \in M_{m\times n}(\mathbb{R})$, $A=\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} = [C_1 \; C_2 \cdots C_n]$

Ex $A=\begin{bmatrix} 3 & 0 & -1 & 21 \\ -2 & 15 & 6 & -9 \\ 0 & 4 & -11 & 19 \end{bmatrix}_{3\times 4}$ $R_1=(3,0,-1,21) \in \mathbb{R}^4$ $\quad C_1=(3,-2,0) \in \mathbb{R}^3$

$R_2= \cdots \in \mathbb{R}^4 \quad C_2 = \cdots \in \mathbb{R}^3$

So at the above, $R_i \in \mathbb{R}^n$ and $C_j \in \mathbb{R}^m$.

(i) The row space of A is a subspace of $\mathbb{R}^n$ that is spanned by the rows of A, ie,

$\text{Row}(A) = \text{span}\{R_1, R_2 \cdots R_m\}$. $\text{Row}(A)$ is a subspace of $\mathbb{R}^n$.

(ii) The column space of A is a subspace of $\mathbb{R}^m$ that is spanned by the columns of A, ie,

$\text{Col}(A) = \text{span}\{C_1, C_2 \cdots C_n\}$. $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Ex| For the above example the row and column spaces are;

Row(A) = span{(3, 0, -1, 21) --} Row(A) is a subspace of $\mathbb{R}^4$

Col(A) = span{(3, -2, 0), --} $\mathbb{R}^3$

(✱) Let $A \in M_{m\times n}(\mathbb{R})$. If the nonzero rows of RREF of A are given by $\widetilde{R}_1, \widetilde{R}_2 \cdots \widetilde{R}_r$, then $\{\widetilde{R}_1, \widetilde{R}_2 \cdots \widetilde{R}_r\}$ forms a basis for Row(A).

A basis of Row(A) contains the nonzero rows of RREF of A.

Definition: The dimension of the row space is said to be the row rank of A.

Ex| $A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 11 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$ Find a basis for row space of A.

Row(A) = span{(3, 21, 0, 9, 0), (1, 7. } Row(A) is a subspace of $\mathbb{R}^5$

RREF of A: $\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow B = \{(1,7,0,3,0), (0,0,1,5,0), (0,0,0,0,1)\}$

is a basis for Row(A). dimRow(A) = 3 = (row rank of A)

Ex| An equivalent question for above example: Find a basis for

$W = \{3x^4 + 21x^3 + 9x,\ x^4 + 7x^3 - x^2 - 2x - 1,\ 2x^4 + 14x^3 + 6x + 11,\ 6x^4 \cdots\}$

W is a subspace of $P_4[x]$. Construct a matrix A such W = Row(A)

From the previous example, $\{x^4 + 7x^3 + 3x,\ x^2 + 5x,\ 1\}$ is a basis for W.

★ Let $A \in M_{(m \times n)}(\mathbb{R})$. The collection of all columns of A that correspond a pivot column (a column containing a leading 1) in the RREF of A

Definition: The dimension of the column space is called the column rank of A.

Ex
$$A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & 1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 8 \end{bmatrix} \quad \text{, the RREF of } A = \begin{array}{c} \begin{array}{ccccc} 1 & & 3 & & 5 \end{array} \\ \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

1 3 5

$Col(A) = span\{(3,1,2,6), (21,7,14,42) \dots\}$
$Col(A)$ is a subspace of $\mathbb{R}^4$.

$B = \{(3,1,2,6), (0,-1,0,-1), (0,1,1,0)$ is a basis for $Col(A)$.

$\dim Col(A) = 3 =$ (column rank of A)

Preposition: $\dim Row(A) = \dim Col(A)$ (row rank = col rank)

Definition: The rank of A is the number of pivot columns (non-zero rows, leading ones).

$rank(A) = r$ (number of basic variables in the $Soln(A)$)

$\dim Soln(A) = n - r$ (number of free variables in A)

$nullity(A) = \dim Soln(A) = n - r$

<u>Rank-Nullity Theorem</u>: If $A \in M_{m\times n}(\mathbb{R})$ then,

$$\text{rank}(A) + \text{nullity}(A) = n$$

$\boxed{\text{Exercise}}$ $W = \text{span}\{3x^3+x^2+2x+6,\ 21x^3+x^2+14x+42,\ -x^2-1,\ 9x^3-2x^2+6x+13,\ -x^2+x\}$

Find a basis for W that is a subset of the spanning set.

Construct an A such $Col(A)=W$ because in col(A) we take the original columns, but in row(A) we take from the RREF.

<u>Definition</u>: Let $A \in M_{m\times n}(\mathbb{R})$. The null space of A is the collection of all $(x_1, x_2, \dots, x_n)$ such that $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in Soln(A)$

$\boxed{\text{Ex}}$ $A = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$, RREF of A: $\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

So the answer for the above exercise: $B = \{3x^3+x^2+2x+6,\ -x^2-1,\ -x^2+x\}$

Soln(A) is the solution for: x_1, x_3, x_5: basic variables x_2, x_4: free vars.

$(*)\begin{cases} 3x_1 + 21x_2 + 9x_4 = 0 \\ x_1 + 7x_2 - x_3 - 2x_4 - x_5 = 0 \\ \vdots \end{cases}$ | $x_2 = u \quad x_4 = v$, $x_1 = -7u - 3v$, $x_3 = -5v \quad x_5 = 0$ | $Soln(A) = \left\{ \begin{bmatrix} -7u-3v \\ u \\ -5v \\ v \\ 0 \end{bmatrix} \middle|\ u, v \in \mathbb{R} \right\}$

just write horizontal

$\left\{ \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for soln(A), $\dim Soln(A) = 2$, $\text{rank}(A) = 3$,

$Null(A) = \{(-7u-3v,\ u,\ -5v,\ v,\ 0) \mid u, v \in \mathbb{R}\}$

$n = 5 = \text{nullity}(A) + \text{rank}(A)$
$= 2 + 3$

$\{(-7,1,0,0,0), (-3,0,-5,1,0)\}$ is a basis for Null(A)

Ex $W = \text{span}\{(1,1,1,1), (3,2,1,1), (1,2,3,1), (0,2,4,1)\}$

Find a basis for W (a) B_1 that is not a subset of W

(b) B_2 that is a subset of W.

(a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix}$, $\text{Row}(A) = W$. RREF of A: $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

W has a basis $B_1 = \overbrace{\{(1,0,-1,1), (0,1,2,0)\}}^{\text{(nonzero rows of RREF)}}$ which is not a subset of the spanning set.

(b) $B = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ \begin{matrix} 1 & 2 & & \end{matrix} \end{array}$, $\text{Col}(B) = W$. RREF of A: $\begin{array}{c} \begin{matrix} 1 & 2 & & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow$

(1st and 2nd columns of A)

W has a basis $B_2 = \{(1,1,1,1), (3,2,1,1)\}$ which is a subset of the spanning set.

Ex $W = \{(a,b,c) \in \mathbb{R}^3 \mid 3a - b = 2c\}$

Find a basis for W and extend this basis for $\mathbb{R}^3$.

$(a,b,c) \in W$ if $b = 3a - 2c \Rightarrow (a, 3a-2c, c) = a(1,3,0) + c(0,-2,1)$

$B = \{(1,3,0), (0,-2,1)\}$ is a basis for W.

$\dim W = 2$, $\dim \mathbb{R}^3 = 3 \Rightarrow 3 - 2 = \underline{+1}$ vector should be added for extension.

$\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$.

Construct $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 3 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$, $\text{Col}(A) = \{(1,3,0), (0,-2,1), e_1, e_2, e_3\}$
$\text{Col}(A) = \mathbb{R}^3$

RREF of A: $\begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \\ \begin{matrix} 1 & 2 & & 4 & \end{matrix} \end{array} \Rightarrow$ $\text{Col}(A)$ has the basis given by

$\{\underset{1}{(1,3,0)}, \underset{2}{(0,-2,1)}, \underset{4}{(0,1,0)}\}$. ✓

1 Question in Final in this section.

(Q2, MT2, SS2016)

Ex) $V = P_3[x]$ $W = \text{span}\{x^3+2x^2+x-1,\ x^3+x^2+x,\ 3x^3+2x^2+3x-2,\ 4x^3+3x^2+4x-2,\ 2x^3+3x^2+2x-1\}$

(a) Find a basis B for W.

(b) Extend B to a basis for V. $ax^3+bx^2+cx+d \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longleftrightarrow (a,b,c,d)$

(a) $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 3 & 2 & 3 & -2 \\ 4 & 3 & 4 & -2 \\ 2 & 3 & 2 & -1 \end{bmatrix}$ RREF of A: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

row(A) = W.

$B' = \{(1,0,1,0), (0,1,0,0), (0,0,0,1)\} \longleftrightarrow B = \{x^3+x,\ x^2,\ 1\}$ is a basis for W.

(b) dim W = 3, dim V = 4 (+1 vector to B we need.)

Construct
$$K = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccccccc} \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{B}} & x^3 & x^2 & x & 1 \end{array} \end{array}$$

rref of K:
$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \\ \begin{array}{ccccccc} 1 & 2 & 3 & 4 & & & \end{array} \end{array}$$

So we take 1^{st}, 2^{nd}, 3^{rd}, 4^{th} columns. $\Rightarrow$ $\tilde{B} = \{\underbrace{x^3+x,\ x^2,\ 1}_{B},\ x^3\}$ is a basis for $P_3[x]$

Inner Product

Recall: The dot product was like; $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$ $\Rightarrow \underbrace{\alpha \cdot \beta}_{\text{vectors}} = \overbrace{x_1y_1 + x_2y_2 + x_3y_3}^{\text{Real Number}}$

(i) $\alpha \cdot \beta = \beta \cdot \alpha$

(ii) $(k\alpha)\cdot\beta = \alpha\cdot(k\beta) = k(\alpha\cdot\beta)$

(iii) $\alpha(\beta+\theta) = \alpha\beta + \alpha\theta$

(iv) $\alpha\cdot\alpha \geq 0$, $\alpha\cdot\alpha = 0_{\mathbb{R}}$ iff $\alpha = 0_{\mathbb{R}^n}$

Definition: Let V be a vector space and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be a map (i.e $\langle \alpha, \beta \rangle \in \mathbb{R}$ for any $\alpha, \beta \in V$). Then,

$\langle , \rangle$ is said to be an inner product on V if;

(i) $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$ for any $\alpha, \beta \in V$.

(ii) $k\langle \alpha, \beta \rangle = \langle k\alpha, \beta \rangle = \langle \alpha, k\beta \rangle$ for any $k \in \mathbb{R}$

(iii) $\langle \alpha, \beta + \theta \rangle = \langle \alpha, \beta \rangle + \langle \alpha, \theta \rangle$

(iv) $\langle \alpha, \alpha \rangle \geq 0$ and $\langle \alpha, \alpha \rangle = 0$ iff $\alpha = 0_V$

$\boxed{\text{Ex}}$ The dot product in $\mathbb{R}^n$ is an inner product.

$\boxed{\text{Ex}}$ $V = C[0,1]$: space of all continuous functions on $[0,1]$.

Define $\langle f, g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$

f, g are continuous on $[0,1] \Rightarrow f \cdot g$ is continuous on $[0,1]$

$\Rightarrow \int_0^1 f(t) g(t)\, dt$ exists and is a real number.

$\langle f, g \rangle \in \mathbb{R}$ for any $f, g \in C[0,1]$

(i) $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt = \int_0^1 g(t) f(t)\, dt = \langle g, f \rangle$ for any $f, g \in C[0,1]$

(ii) $k \langle f, g \rangle = \int_0^1 k f(t) g(t)\, dt = \int_0^1 f(t)\, k g(t)\, dt$

$= \langle kf, g \rangle = \langle f, kg \rangle$ for any $k \in \mathbb{R}$

(iii) $\langle f, g+h \rangle = \int_0^1 f(t)[g(t) + h(t)]\, dt = \int_0^1 [f(t)g(t) + f(t)h(t)]\, dt$

$= \int_0^1 f(t) g(t) + \int_0^1 f(t) h(t) = \langle f, g \rangle + \langle f, h \rangle$

(iv) $\langle f, f \rangle = \int_0^1 [f(t)]^2\, dt \geq 0$ since $[f(t)]^2 \geq 0$

Also, $\int_0^1 [f(t)]^2\, dt = 0$ iff $f(t) \equiv 0$ on $[0,1]$ since f is continuous.

Ex) $V = \mathbb{R}^3$, $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$

Define $\cdot_L$ by $\alpha \cdot_L \beta = -x_1 y_1 + x_2 y_2 + x_3 y_3$

Is $\cdot_L$ product an inner product?

$(1,1,0) \cdot_L (1,1,0) = 0$ but $(1,1,0) \neq (0,0,0) = 0_{\mathbb{R}^3}$

Then, $\cdot_L$ is not an inner product.

Exercise Prove or disprove:

(a) $V = M_{n\times n}(\mathbb{R})$, $\langle A, B\rangle = tr(A^T \cdot B)$ is an inner product on V.

(b) $V = \mathbb{R}^n$, $\langle \alpha, \beta\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 + \dots + nx_ny_n$ is an inner product on V.

Definition: Let $(V, \langle , \rangle)$ be an inner product space. (a space V with property of inner product.)
If $\alpha \in V$ then the norm (magnitude, length) of α is defined;

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$$

Remark: $V = \mathbb{R}^n$ furnished by the usual dot product; $\|\alpha\| = \sqrt{x_1^2 + x_2^2 + x_3^2 \cdots x_n^2}$

Preposition: Let $(V, \langle , \rangle)$ be an inner product space.

(i) $\|\alpha\| \geq 0$ and $\|\alpha\| = 0$ iff $\alpha = 0_V$

(ii) $\|k\alpha\| = |k| \cdot \|\alpha\|$

Cauchy-Schwarz Inequaty: $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \cdot \|\beta\|$

Remark: $|\alpha \cdot \beta| \leq \|\alpha\| \cdot \|\beta\|$

$$\frac{|\langle \alpha, \beta \rangle|}{\|\alpha\| \cdot \|\beta\|} \leqslant 1 \quad \text{if} \quad \begin{matrix} \alpha \neq 0_V \\ \beta \neq 0_V \end{matrix} \implies -1 \leqslant \frac{\langle \alpha, \beta \rangle}{\|\alpha\| \cdot \|\beta\|} \leqslant 1$$

<u>Definition:</u> If $\alpha \neq 0_V$ and $\beta \neq 0_V$ then the angle θ $(0 \leqslant \theta \leqslant \pi)$ between α and β is defined by $\cos\theta = \dfrac{\langle \alpha, \beta \rangle}{\|\alpha\| \cdot \|\beta\|}$

<u>Definition:</u> α and β are said to be orthogonal $(\alpha \perp \beta)$ if $\theta = \pi/2$ ie, $\langle \alpha, \beta \rangle = 0$.

<u>Ex</u> Find the angle between $\alpha = (1,1,1)$ and $\beta = (2,3,0)$

$\|\alpha\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ $\quad \langle \alpha, \beta \rangle = \alpha \cdot \beta = 1\cdot 2 + 1 \cdot 3 = 5$

$\|\beta\| = \sqrt{2^2+3^2} = \sqrt{13}$ $\quad \cos\theta = \frac{5}{\sqrt{39}} \implies \theta = \arccos\left(\frac{5}{\sqrt{39}}\right)$

<u>Ex</u> $V = C[0,1]$, $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$.

$f(t) = \sin t \implies \langle f, f \rangle = \int_0^1 \sin^2 t\, dt = \int_0^1 \frac{1-\cos 2t}{2} dt$ $\qquad \cos 2t = 1 - 2\sin^2 t$, $\sin^2 t = \frac{1-\cos 2t}{2}$

$= \frac{1}{2}\left[t \Big|_0^1 - \frac{1}{2}\sin 2t \Big|_0^1 \right] = \frac{1}{2} - \frac{1}{4}\sin 2 \implies \|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\frac{1}{2} - \frac{1}{4}\sin 2}$

<u>Preposition</u> (Triangle Inequality): $\|\alpha + \beta\| \leqslant \|\alpha\| + \|\beta\|$

<u>Preposition:</u> If $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is a set of k-mutuonally orthogonal non-zero vectors ($\alpha_i \perp \alpha_j$ for any $\alpha_i \neq \alpha_j$) then $\{\alpha_1, \dots, \alpha_k\}$ is linearly independent.

Proof: Construct $(*)\ c_1\alpha_1 + c_2\alpha_2 + c_k\alpha_k = 0_V$.

$\langle c_1\alpha_1 + c_2\alpha_2 + \dots + c_k\alpha_k , \alpha_1 \rangle = \langle 0_V, \alpha_1\rangle$ since $\langle \alpha, 0_V\rangle = 0_{\mathbb{R}}$

$c_1\langle \alpha_1, \alpha_1\rangle + \underbrace{c_2\langle \alpha_2, \alpha_1\rangle}_{0} + \dots \underbrace{c_k\langle \alpha_k, \alpha_1\rangle}_{0} = 0$

$c_1\langle \alpha_1, \alpha_1\rangle = 0$

$\alpha_1 \neq 0_V \Rightarrow \langle \alpha_1, \alpha_1\rangle \neq 0 \Rightarrow c_1 = 0$

Similarly one can show $c_2=0$, $c_3=0 \dots c_k=0$. Thus, $(*)$ has only the trivial solution.

Definition: Let $(V, \langle , \rangle)$ is an inner product space. Suppose that W is a subspace of V. Then the orthogonal complement of W is def,

$$W^\perp = \{\alpha \in V \mid \langle \alpha, \beta\rangle = 0 \text{ for any } \beta \in W\}$$

Preposition: $W^\perp$ is a subspace of V where W is a subspace of the inner product space $(V, \langle , \rangle)$.

Proof: (i) $0_V \in W^\perp$ since $\langle 0_V, \beta\rangle = 0$ for any $\beta \in W$.

(ii) $\alpha_1, \alpha_2 \in W^\perp$, i.e, $\langle \alpha_1, \beta\rangle = 0$ for any $\beta \in W$
$\langle \alpha_2, \beta\rangle = 0$ for any $\beta \in W$.

Consider $\langle \alpha_1 + \alpha_2, \beta\rangle = \langle \alpha_1, \beta\rangle + \langle \alpha_2, \beta\rangle = 0 + 0 = 0$ for any $\beta \in W$

Thus $\alpha_1 + \alpha_2 \in W^\perp$ for any $\alpha_1, \alpha_2 \in W^\perp$.

(iii) $\alpha_1 \cdot k \in W^\perp$.

Preposition: Let $(V, \langle , \rangle)$ be an inner product space. W is a subspace of V,

(i) $\{0_V\}^{\perp} = V$, $V^{\perp} = \{0_V\}$

(ii) $(W^{\perp})^{\perp} = W$

(iii) If $\dim V = n < \infty$, $\dim W = k < \infty$ and $\{\alpha_1, \alpha_2 \dots \alpha_k\}$ is a basis for W, then $W^{\perp}$ admits a basis $\{\beta_1, \beta_2 \dots \beta_{n-k}\}$ so that $\dim W^{\perp} = n-k$, $\{\alpha_1, \alpha_2 \dots \alpha_k, \beta_1, \beta_2, \dots \beta_{n-k}\}$ is a basis for V $(\langle \alpha_i, \beta_j \rangle = 0)$.

(iv) Any vector θ can be written as a sum of two vectors in a <u>unique way</u> as $\theta = \alpha + \beta$ where $\alpha \in W$, $\beta \in W^{\perp}$,

Preposition: Suppose $W = \text{span}\{R_1, R_2, \dots, R_m\}$ where $R_i \in \mathbb{R}^n$. (Notice W is a subset of $\mathbb{R}^n$) Then, $W^{\perp} = \text{Null}(A)$ where $W = \text{Row}(A)$ and

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m \times n} \in M_{m \times n}(\mathbb{R})$$

Proof: $A = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}_{m \times n} \Rightarrow W = \text{Row}(A)$.

$(x_1, x_2 \dots x_n) \in W^{\perp}$ iff $(x_1, \dots x_n) \perp R_i$ for all i.

dot product

$R_i \cdot (x_1, x_2 \dots x_n) = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \text{Soln}(A) \text{ where } (*) \begin{cases} R_1 \cdot (x_1, x_2 \dots x_n) = 0 \\ R_2 \cdot (x_1, x_2 \dots x_n) = 0 \\ \vdots \\ R_m \cdot (x_1, x_2 \dots x_n) = 0 \end{cases}$$

3-2=1

Ex) $V=\mathbb{R}^3$, $W=\text{span}\{(1,2,0),(0,1,-1)\}$. Find $W^\perp$.

Note that W is a subspace of V. $\dim V=3$, $\dim W=2 \Rightarrow \dim W^\perp = 1$

$(x_1,x_2,x_3)\in W^\perp$ iff $\begin{cases}(x_1,x_2,x_3)\cdot(1,2,0)=0\\(x_1,x_2,x_3)\cdot(0,1,-1)=0\end{cases} \Rightarrow (*)\begin{cases}x_1+2x_2=0\\x_2-x_3=0\end{cases}$

$A=\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix}$

$W=\text{Row}(A)$

$W^\perp=\text{Null}(A)$

$$\begin{array}{ccc}\text{basic}&\text{basic}&\text{free}\\x_1&x_2&x_3\end{array}$$
$$\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix}$$

$x_3=u$

$x_1=-2u$

$x_2=u$

$W^\perp=\text{Null}(A)=\{(-2u,u,u)\mid u\in\mathbb{R}\}=\text{span}\{(-2,1,1)\}=$

And, $\{(-2,1,1)\}$ is a basis for $W^\perp$ and $\dim W^\perp=1$ as expected.

And, $\{\underset{\alpha_1}{(1,2,0)},\underset{\alpha_2}{(0,1,-1)},\underset{\beta}{(-2,1,1)}\}$ is a basis for $V=\mathbb{R}^3$

Also, two bases are orthogonal to each other, $\alpha_1\cdot\beta=0$, $\alpha_2\cdot\beta=0$.

Ex) Find $W^\perp$ where $W=\text{span}\{(3,21,0,9,0),(1,7,-1,6,1),(2,14,0,6,1),(6,4,2,-1,13)\}$.

W is a subspace of $\mathbb{R}^5$.

Construct a matrix A so that $\text{Row}(A)=W$.

$$A=\begin{bmatrix}3&21&0&9&0\\1&7&-1&6&1\\2&14&0&6&1\\6&4&2&-1&13\end{bmatrix}_{4\times5}\longrightarrow \text{RREF}: \begin{bmatrix}1&7&0&3&0\\0&0&1&5&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}_{4\times5}$$

Then, $B=\{\underset{\alpha_1}{(1,7,0,3,0)},\underset{\alpha_2}{(0,0,1,5,0)},\underset{\alpha_3}{(0,0,0,0,1)}\}$ is a basis for W.

$\dim W=3 \Rightarrow \dim W^\perp=\dim\mathbb{R}^5-\dim W=2$

If $W = Row(A)$, then $W^\perp = Null(A)$. So we do the soln:

x_2, x_4 : free
x_1, x_3, x_5 : basic
$x_5 = 0$, $x_2 = u$, $x_4 = v$, $x_1 = -7u - 3v$, $x_3 = -5v$ $\Rightarrow W^\perp = Null(A) = \{(-7u-3v, u, -5v, v, 0) \mid u,v \in \mathbb{R}\}$

And, $\{\underbrace{(-7,1,0,0,0)}_{\beta_1}, \underbrace{(-3,0,-5,1,0)}_{\beta_2}\}$ is a basis for $W^\perp = Null(A)$.

And, $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}$ is a basis for $\mathbb{R}^5$ and $\alpha_i \cdot \beta_j = 0$ for any i, j.

$\boxed{Ex}$ $V = P_2[x]$, $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$

$W = span\{x^2+1\}$, $\dim W = 1$, $\dim W^\perp = 2$.

The previous method does not work here since we don't have the dot product as our inner product. Instead;

$p \in W^\perp$ iff $p \perp (x^2+1)$, ie, $\int_0^1 (ax^2+bx+c)(x^2+1)\,dx = 0$.
$p(x) = ax^2+bx+c$

Eigenvalues and Eigenvectors

Goal: to obtain conditions for a matrix A to guarantee A is diagonizable.

Definition: Suppose A is an $n \times n$ matrix. A real number c is an eigenvalue of A (characteristic value of A) if there is a nonzero X in $M_{n\times1}(\mathbb{R})$ such that; $AX = cX$

Definition: Suppose c is an eigen value of $n \times n$ matrix A. Then a vector X is said to be an eigenvector associated to c if $AX = cX$ $(X \in M_{n\times1}(\mathbb{R}))$

<u>Notation:</u> $W_c = \{X \in M_{n\times 1}(\mathbb{R}) \mid AX = cX\}$ is the collection of all eigenvectors associated to c.

<u>Preposition:</u> If c is an eigenvalue of $n\times n$ matrix A, then W_c is a subset of $M_{n\times 1}(\mathbb{R})$.

<u>Proof:</u> (i) $0_{n\times 1} \in W_c$ since $A\cdot 0 = c\cdot 0$

(ii) $X_1, X_2 \in W_c$ then $\left.\begin{array}{l} AX_1 = cX_1 \\ AX_2 = cX_2 \end{array}\right\}$ $A(X_1+X_2) = c(X_1+X_2)$
$X_1 + X_2 \in W_c$

(iii) $X \in W_c$, $k \in \mathbb{R}$ then $\left.\begin{array}{l} AX = cX \\ kAX = ckX \end{array}\right\}$ $kX \in W_c$

<u>Terminology:</u> W_c is the eigenspace associated to the eigenvalue c.

<u>Remark:</u>

<u>Recall:</u> $B \in M_{n\times n}(\mathbb{R})$ $(*)$ $BX = 0_{n\times 1}$

$$(*)\begin{cases} b_{11}x_1 + \cdots b_{1n}x_n = 0 \\ \vdots \qquad\qquad \vdots \\ b_{n1}x_1 + \cdots b_{nn}x_n = 0 \end{cases} \Rightarrow B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

The system $(*)$ has a non-trivial (inf. many) solution iff $\det B = 0$

$AX = cX$ $(X \in M_{n\times 1}(\mathbb{R}) - \{0_{n\times 1}\}$

$AX - cX = 0_{n\times 1}$

$(*)$ $(A - cI)X = 0_{n\times 1}$ $\Rightarrow$ c is an eigenvalue iff $(*)$ has non-trivial soln.
$\det(A - cI) = 0$ by the last recall

Preposition: c is an eigenvalue of A iff $\det(A - cI_n) = 0$

Ex $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \Rightarrow A - cI_n = \begin{bmatrix} 4-c & -2 \\ 1 & 1-c \end{bmatrix}$

$\det(A - cI_n) = (4-c)(1-c) - (-2) = c^2 - 5c + 6 = (c-3)(c-2)$

$(c-3)(c-2) = 0 \Rightarrow \left.\begin{matrix} c_1 = 2 \\ c_2 = 3 \end{matrix}\right\}$ These are eigenvalues of A.

Definition: If A is an $n \times n$ matrix, then its characteristic polynomial is defined by; $\quad char_A(x) = \det(A - cI_n)$

Preposition: c is an eigenvalue of A iff c is a root of the characteristic polynomial of A.

Ex For above example, $char_A(x) = \det(A - xI_n) = (x-3)(x-2)$ $\quad \begin{matrix} x_1 = 2 \\ x_2 = 3 \end{matrix}$

Recall that $W_c = \{ X \in M_{n \times 1}(\mathbb{R}) \mid AX = cX \}$

$AX = cX$ is equivalent to $(A - cI_n)X = 0_{n \times n}$, so we can write

$W_c = \{ X \in M_{n \times 1}(\mathbb{R}) \mid (A - cI_n)X = 0_{n \times 1} \}$

$c_1 = 2$: $W_{c_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| (A - c_1 I_n) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$A - 2I_n = \begin{bmatrix} 4-2 & -2 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} x_2 = u \\ x_1 = -u \end{matrix} \Rightarrow \; = \left\{ \begin{bmatrix} u \\ u \end{bmatrix} \middle| u \in \mathbb{R} \right\}$

$u \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B_{c_1} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for W_{c_1}.

$c_2=3: \; W_{c_2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| (A - c_2 I_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$A - c_2 I_2 = \begin{bmatrix} 4-3 & -2 \\ 1 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $\quad x_2 = u$ (par) $\quad x_1 = 2u$

$W_{c_2} = \left\{ \begin{bmatrix} 2u \\ u \end{bmatrix} \middle| u \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $B_{c_2} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for W_{c_2}.

Definition: Suppose that c is an eigenvalue of $n \times n$ matrix A.

(i) The geometric multiplicity of c is the dimension of W_c.

$$(\text{Geometric Multiplicity}) = \dim W_c$$

(ii) The algebraic multiplicity of c is the highest power of $(x-c)^k$ which divides $\text{char}_A(x)$. ie,

(The alg. mult of c) $= k$ if $(x-c)^k \mid \text{char}_A(x)$ but $(x-c)^{k+1} \nmid \text{char}_A(x)$

Ex) For the above example,

The geometric multiplicity of $c_1=2$ is; $\dim W_{c_1} = 1$

The geometric multiplicity of $c_2=3$ is; $\dim W_{c_2} = 1$

The algebraic multiplicity of $c_1=2$ is; $(x-3)^1 (x-2)^{1} \Rightarrow 1$

The algebraic multiplicity of $c_2=3$ is; $(x-3)^{1} (x-2) \Rightarrow 1$

$40 \to 37.5$

Preposition: Suppose A is an $n\times n$ matrix.

$p(0)=a_0$

(i) $\text{char}_A(x) = \det(A - cI_n) = (-1)^n \cdot x^n + \overset{(-1)^{n-1}}{tr(A)} \cdot x^{n-1} + \dots + \det(A)$

(ii) 0 is an eigenvalue iff $\det A = 0$. ($p(0)=0$ iff $a_0=0$)

(Any eigenvalue of A is nonzero iff $\det A \neq 0$)

$\boxed{\text{Ex}}$ $A = \begin{bmatrix} 5 & -6 & 3 \\ 6 & -7 & 3 \\ 6 & -6 & 2 \end{bmatrix}$ $\text{char}_A(x) = \det(A - xI_3) = \det \begin{bmatrix} 5-x & -6 & 3 \\ 6 & -7-x & 3 \\ 6 & -6 & 2-x \end{bmatrix}$

$\text{char}_A(x) = (5-x)\begin{vmatrix} -7-x & 3 \\ -6 & 2-x \end{vmatrix} - (-6)\begin{vmatrix} 6 & 3 \\ 6 & 2-x \end{vmatrix} + 3\begin{vmatrix} 6 & -7-x \\ 6 & -6 \end{vmatrix} =$

$= (5-x)(-14+7x-2x+x^2+18) + 6(12-6x-18) + 3(-18+42+6x)$

$= (5-x)(x^2+5x+4) + 6(-6x-6) + 3(6x+6)$

$= \cancel{5x^2} + 25x + 20 - x^3 - \cancel{5x^2} - 4x - 18x - 18 = \underline{-x^3 + 3x + 2}$

$\text{char}_A(x) = -x^3 + 3x + 2 = (-1)^n x^n + (-1)^{n-1} \cdot tr(A) x^{n-1} + \dots + \det A$

$= -x^3 + tr(A)x^2 + ax + \det A$

2'nin çarpanlarını bul.

$2 \to -1, 1 \;;\; -2, 2$ $\Rightarrow$ $x=-1$: $\text{char}_A(-1) = 0$ ✓

Thus, $x-(-1)$ divides $\text{char}_A(x) = -x^2 + 3x + 2$

$-x^3+3x+2 \,\big|\, x+1$, quotient $-x^2+x+2$, remainder 0 $\Rightarrow$ $\text{char}_A(x) = (x+1)(-x^2+x+2)$

$= -(x+1)(x-2)(x+1) = -(x+1)^2(x-2)$

$c_1 = -1$, alg.mult $= 2$

$c_2 = 2$, alg.mult $= 1$

$$W_{c_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ (A - c_1 I_3) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (c_1 = -1)$$

$$A - (-1) I_3 = \begin{bmatrix} 5-(-1) & -6 & 3 \\ 6 & -7-(-1) & 3 \\ 6 & -6 & 2-(-1) \end{bmatrix} = \begin{bmatrix} 6 & -6 & 3 \\ 6 & -6 & 3 \\ 6 & -6 & 3 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} \text{basic} & \overbrace{\quad\quad\quad\quad}^{\text{free}} & \\ x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\left.\begin{array}{l} x_2 = u \\ x_3 = 2v \\ x_1 = u - v \end{array}\right\} \quad W_{c_1} = \left\{ \begin{bmatrix} u-v \\ u \\ 2v \end{bmatrix} \middle| \ u, v \in \mathbb{R} \right\}, \quad B_{c_1} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

is a basis for W_{c_1}.

$$\dim W_{c_1} = |B_{c_1}| = 2 = (\text{Geometric multiplicity of } c_1 = -1)$$

$$W_{c_2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \ (A - c_2 I_3) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (c_2 = 2)$$

$$A - 2 I_3 = \begin{bmatrix} 5-2 & -6 & 3 \\ 6 & -7-2 & 3 \\ 6 & -6 & 2-2 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \xrightarrow{\text{(some steps)}} \begin{array}{c} \begin{array}{ccc} \overbrace{x_1 \quad x_2}^{\text{basic}} & & \overbrace{x_3}^{\text{free}} \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\left.\begin{array}{l} x_3 = u \\ x_2 = u \\ x_1 = u \end{array}\right\} \quad W_{c_2} = \left\{ \begin{bmatrix} u \\ u \\ u \end{bmatrix} \middle| \ u \in \mathbb{R} \right\}, \quad B_{c_2} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } W_{c_2}.$$

$$\dim W_{c_2} = |B_{c_2}| = 1 = (\text{Geometric multiplicity of } c_2 = 2)$$

<u>Definition:</u> An $n \times n$ matrix A is said to be diagonalizable if there is a nonsingular matrix P such that,

$$P^{-1} A P = D \text{ is a diagonal matrix.}$$

<u>Ex</u> Any diagonal matrix D is diagonalizable; Take $P = I_n$. Thus, $P^{-1} D P = I_n \cdot D \cdot I_n = D$ is diagonal.

Corollary: Sum of algebraic mults (e_i) $e_1+e_2+\cdots+e_k = n$ (degree of $char_A(x)$)

Preposition: If c is an eigenvalue of an $n \times n$ matrix A, then,

(geometric mult. of c) $\leq$ (algebraic mult. of c)

Theorem: Let $A \in M_{n\times n}(\mathbb{R})$. Suppose that $c_1, c_2 \ldots c_k$ are distinct eigenvalues of A $(k \leq n)$. If $\dim W_{c_i} = d_i$ (geo. mult of c_i) For all i, the alg. mult is denoted e_i, the following are equivalent:

(i) A is diagonalizable.

(ii) $d_i = e_i$ for all i

(iii) $d_1 + d_2 + \cdots + d_k = n$

(iv) $B_{c_1} \cup B_{c_2} \cup \cdots \cup B_{c_k}$ forms a basis for $M_{n\times n}(\mathbb{R})$ when B_{c_i} is the basis of W_{c_i}.

(v) $P = [B_{c_1} \; B_{c_2} \cdots B_{c_k}]$ is a diagonalizing matrix, ie,

$$P^{-1}AP = \operatorname{diag}(\underbrace{c_1, c_1 \cdots c_1}_{\# d_1}, \underbrace{c_2, c_2 \cdots c_2}_{\# d_2}, \cdots \underbrace{c_k, c_k, \cdots c_k}_{\# d_k})$$

(diagonalized form of A)

Ex/
$A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$. Is A diagonalizable? If so, find a diagonalizing matrix.

$char_A(x) = \det(A - xI_2) = \begin{vmatrix} -1-x & 4 \\ -1 & 3-x \end{vmatrix} = x^2 - 2x + 1 = (x-1)^2$

$c_1 = 1$ with algebraic multiplicity $\underline{2}$. ✓

$W_{c_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| (A - c_1 I_2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $A - 1I_2 = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$x_2 = u$
$x_1 = 2u$ $\Rightarrow B_{c_1} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for W_{c_1}. Geo. mult. for c_1 is $\underline{1}$

(Alg. Mult. of c_1) $= 2 \neq 1 =$ (Geo. Mult. of c_1) $\Rightarrow$ A is not diagonalizable.

Ex) $A = \begin{bmatrix} 5 & 0 & 4 \\ -6 & 1 & -1 \\ 6 & 0 & -5 \end{bmatrix}_{3\times 3}$ Is A diagonalizable?

$$Char_A = \det(A - xI_3) = \begin{vmatrix} 5-x & 0 & 4 \\ -6 & 1-x & -1 \\ 6 & 0 & -5-x \end{vmatrix} = (5-x)\begin{vmatrix} 1-x & -1 \\ 0 & -5-x \end{vmatrix} - 0 + 4\begin{vmatrix} -6 & 1-x \\ 6 & 0 \end{vmatrix}$$

$$= (5-x)(1-x)(-5-x) + 4(-6+6x) = -(x-5)(x-1)(x+5) + 24(x-1) =$$

$$= (1-x)[x^2-25-24] = (1-x)(x^2-49) = (1-x)(x-7)(x+7)$$

$\left.\begin{array}{l} c_1 = 1 \\ c_2 = 7 \\ c_3 = -7 \end{array}\right\}$ A has 3 distinct eigenvalues where $n=3$
Thus, A is diagonalizable.

$$W_{c_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| (A - c_1 I_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad A - 1I_3 = \begin{bmatrix} 4 & 0 & 4 \\ 6 & 0 & -1 \\ 6 & 0 & -6 \end{bmatrix} \to \begin{array}{ccc} x_1 & & x_2 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$

$\left.\begin{array}{l} x_2 = u \\ x_1 = 0 \\ x_3 = 0 \end{array}\right\}$ $W_{c_1} = \left\{ \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} \middle| u \in \mathbb{R} \right\}$, $B_{c_1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$W_{c_2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| (A - c_2 I_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad A - 7I_3 = \begin{bmatrix} -2 & 0 & 4 \\ -6 & -6 & -1 \\ 6 & 0 & -12 \end{bmatrix} \to \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline 1 & 0 & -2 \\ 0 & 6 & 13 \\ 0 & 0 & 0 \end{array}$$

$\left.\begin{array}{l} x_3 = 6u \\ x_1 = 12u \\ x_2 = -13u \end{array}\right\}$ $W_{c_1} = \left\{ \begin{bmatrix} 12u \\ -13u \\ 6u \end{bmatrix} \middle| u \in \mathbb{R} \right\}$, $B_{c_2} = \left\{ \begin{bmatrix} 12 \\ -13 \\ 6 \end{bmatrix} \right\}$

$$-- A + 7I_3 = \begin{bmatrix} 12 & 0 & 4 \\ -6 & 8 & -1 \\ 6 & 0 & 2 \end{bmatrix} \to \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline 3 & 0 & 1 \\ 0 & 8 & 1 \\ 0 & 0 & 0 \end{array} \quad \left.\begin{array}{l} x_3 = 24u \\ x_1 = -8u \\ x_2 = -3u \end{array}\right\} B_{c_3} = \left\{ \begin{bmatrix} -8 \\ -3 \\ 24 \end{bmatrix} \right\}$$

$D = P^{-1}AP$ | $D = \mathrm{diag}(1, 7, -7)$, $P = \begin{bmatrix} 0 & 12 & -8 \\ 1 & -13 & -3 \\ 0 & 6 & 24 \end{bmatrix}$ (?)

Ex $A = \begin{bmatrix} 5 & 0 & 4 \\ -6 & 1 & -1 \\ -6 & 0 & -5 \end{bmatrix}$ Is A diagonalizable?

$char_A(x) = \det(A - xI_n) = -(x-1)^2(x+1)$

$c_1 = -1$ with alg. mult $= 1$ ✓ equal to geo.
$c_2 = 1$ with alg. mult $= 2$ ✗ not equal to geo.

$A - c_1 I_n = \begin{bmatrix} 6 & 0 & 4 \\ -6 & 2 & -1 \\ -6 & 0 & 4 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \end{array}$

$W_{c_1} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| (A - c_1 I_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$x_3 = 6u$ (free)
$x_1 = -4u$
$x_2 = 9u$

$W_{c_1} = \left\{ \begin{bmatrix} -4u \\ 9u \\ 6u \end{bmatrix} \middle| u \in \mathbb{R} \right\}$

$B_{c_1} = \left\{ \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} \right\} \Rightarrow \dim W_{c_1} = 1 = (\text{geo. mult of } c_1)$

$W_{c_2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| (A - c_2 I_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ $A - c_2 I_3 = \begin{bmatrix} 4 & 0 & 4 \\ -6 & 0 & -1 \\ -6 & 0 & -6 \end{bmatrix} \longrightarrow$

$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

$x_2 = u$ (free)
$x_1 = 0$
$x_3 = 0$

$W_{c_2} = \left\{ \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} \middle| u \in \mathbb{R} \right\}$

$B_{c_2} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \dim W_{c_2} = 1 = (\text{geo. mult. of } c_2)$

Thus, A is not diagonalizable.

Corollary: If $n \times n$ matrix A has n distinct eigenvalues, say, $c_1, c_2 \dots c_n$, then; $char_A(x) = (-1)^n (x - c_1)(x - c_2) \cdots (x - c_n)$

$\Rightarrow$ each alg. mult. is $1 \geq$ geo. mult. ≥ 1

$\Rightarrow$ each geo. mult has to be $1 \Rightarrow$ A is diagonalizable.

★ Thus, if an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

FABER-CASTELL

Ex) $A=\begin{bmatrix}3&-2&0\\0&1&0\\-4&4&1\end{bmatrix}$ Is A diagonalizable?

$char_A(x) = \det(A - xI_3) = \det\begin{bmatrix}3-x&-2&0\\0&1-x&0\\-4&4&1-x\end{bmatrix}$

$= -0 + (1-x)\begin{vmatrix}3-x&0\\-4&1-x\end{vmatrix} - 0 = (1-x)(3-x)(1-x) = -(x-3)(x-1)^2$

$c_1 = 3$ with alg. mult 1 ✓

$c_2 = 1$ with alg. mult 2. ✓ Now, we find eigen spaces;

$W_{c_1} = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| (A-c_1I_3)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$ $(A-c_1I_3) = \begin{bmatrix}0&-2&0\\0&-2&0\\-4&4&-2\end{bmatrix} \longrightarrow$

$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$
$\begin{bmatrix}1&-1&1/2\\0&1&0\\0&0&0\end{bmatrix}$ $\begin{matrix}x_3 = 2u\\ x_2 = 0\\ x_1 = -u\end{matrix}$ | $W_{c_1} = \left\{\begin{bmatrix}-u\\0\\2u\end{bmatrix} \middle| u \in \mathbb{R}\right\}$

$B_{c_1} = \left\{\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\} \Rightarrow \dim W_{c_1} = 1 = $ (geo. mult. for c_1)

$W_{c_2} = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| (A-c_2I_3)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$ $(A-c_2I_3) = \begin{bmatrix}2&-2&0\\0&0&0\\-4&4&0\end{bmatrix} \longrightarrow$

$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$
$\begin{bmatrix}1&-1&0\\0&0&0\\0&0&0\end{bmatrix}$ $\begin{matrix}x_3 = v\\ x_2 = u\\ x_1 = u\end{matrix}$ | $W_{c_1} = \left\{\begin{bmatrix}u\\v\\u\end{bmatrix} \middle| u \in \mathbb{R}\right\}$

$B_{c_1} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \Rightarrow \dim W_{c_2} = 2 = $ (geo. mult for c_2)

Since $1=1$, $2=2$, we can apply the theorem. $B = B_{c_1} \cup B_{c_2}$

A is diagonalizable.

Also, $B = \left\{\begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $M_{3\times1}(\mathbb{R})$

$B_{c_1} = \left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$, $c_1 = 3$

$B_{c_2} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $c_2 = 1$

$P = \begin{bmatrix}-1&1&0\\0&1&0\\1&0&1\end{bmatrix}$ with columns $\underbrace{\text{col } 1}_{B_{c_1}}$, $\underbrace{\text{cols } 2,3}_{B_{c_2}}$

$P^{-1}AP = \text{diag}(\underbrace{3}_{\substack{d_1 \text{ many}\\(1 \text{ many})}}, \underbrace{1, 1}_{\substack{d_2 \text{ many}\\(2 \text{ many})}})$

<u>Remark:</u> ~~Diagonalizing~~ The matrix $P^{-1}AP$ is not unique. (B_{c_1}-B_{c_2} order can change)

Ex) $A=\begin{bmatrix}5 & -6 & 3\\ 6 & -7 & 3\\ 6 & -6 & 2\end{bmatrix}$ $\quad char_A(x) = -(x+1)^2(x+2)$ (we found these before)

$c_1=-1$ with alg. mul. 2 , $B_{c_1}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}$

$c_2=-2$ with alg. mul. 1 , $B_{c_2}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

Since $2=|B_{c_1}|$, $1=|B_{c_2}|$, A is diagonalizable.

$$P=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 2 & 1\end{bmatrix} \quad (\text{columns: } \underbrace{1,2}_{B_{c_1}},\ \underbrace{3}_{B_{c_2}}), \qquad P^{-1}AP = diag(\underbrace{-1,\ -1}_{|B_{c_1}|\text{ many}},\ \underbrace{2}_{|B_{c_2}|\text{ many}})$$

Theorem: If A is an $n\times n$ matrix, then

$char_A(A) = 0_{n\times n}$ (cayley-hamilton theorem)

Idea: $char_A(x) = det(A - xI_n)$, $x=A$

Recall: A is nonsingular iff 0 is not an eigenvalue of A.

Ex) $A=\begin{bmatrix}5 & 0 & 4\\ -6 & 1 & -1\\ 6 & 0 & -5\end{bmatrix}$ $\quad char_A(x) = -x^3+x^2+x-1 = -(x-1)^2(x+1)$

0 is not an eigenvalue of A $\Rightarrow$ A is nonsingular.

By cayley-hamilton theorem, $-A^3+A^2+A-I_n = 0_{n\times n}$, $n=3$

Multiply A^{-1} both sides; $A^{-1}(-A^3+A^2+A-I_n) = A^{-1}0_{n\times n}$

$-A^2+A+I_n-A^{-1} = 0_{n\times n} \Rightarrow A^{-1} = \underline{-A^2+A+I_n}$

* Another method to compute A^{-1}.

Preposition: If c is an eigenvalue of a nonsingular matrix A, then $(c \neq 0)$ and $\frac{1}{c} = c^{-1}$ is ~~the~~ an eigenvalue of A^{-1}.

Proof: $AX = cX$ for some matrix X.

A is nonsingular $\Rightarrow$ $A^{-1}AX = A^{-1}cX$ $\quad X = c \cdot A^{-1}X$ $\Rightarrow$ $c^{-1}X = A^{-1}X$

c^{-1} is an eigenvalue of A^{-1}

Ex) Suppose that a 5×5-diagonalizable matrix A has the eigenvalues as $-2, \frac{4}{7}, \frac{5}{9}, 1, 3$. Then find;

a) Char. poly of A.

b) Char. poly of A^{-1} if A is nonsingular.

c) Is A^{-1} diagonalizable?

0 is not eigenvalue $\Rightarrow$ $\det(A) \neq 0 \Rightarrow$ A is nonsingular. ✓

(a) A has 5 distinct eigenvalues;

$$\text{char}_A(x) = (-1)^5 \cdot (x+2)\left(x-\frac{4}{7}\right)\left(x-\frac{5}{9}\right)(x-1)(x-3)$$

(b) A is nonsingular since 0 is not an eigenvalue of A.

The eigenvalues of A^{-1} are, $(-2)^{-1}, \left(\frac{4}{7}\right)^{-1}, \left(\frac{5}{9}\right)^{-1}, (1)^{-1}, (3)^{-1}$

$$\text{char}_{A^{-1}}(x) = -\left(x+\frac{1}{2}\right)\left(x-\frac{7}{4}\right)\left(x-\frac{9}{5}\right)(x-1)\left(x-\frac{1}{3}\right)$$

(c) A^{-1} is diagonalizable since A^{-1} has 5 distinct eigenvalues.

Remark: If the last eg.val. 3 was not given, Then we couldn't write char poly since we have no data to say which one is repeating.

Applications

Suppose A is a diagonalizable matrix, that is $P^{-1}AP=D$ for some diagonal matrix D.

⊛ $D=\text{diag}(d_1, d_2, \dots, d_n) \Rightarrow D^k=\text{diag}(d_1^k, d_2^k, \dots, d_n^k)$

$P^{-1}AP=D \Rightarrow A=PDP^{-1}$

$$A^k = A\cdots A = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$$
$$= PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1}$$
$$= PDD\cdots DP^{-1} \Longrightarrow \underline{A^k = PD^kP^{-1}}$$

Ex] $A=\begin{bmatrix}4 & -2 & 1\\ 2 & 0 & 1\\ 2 & -2 & 3\end{bmatrix}$ Compute A^{65}.

(Exercise): Show that A is diagonalizable with these; $P^{-1}AP=D=\text{diag}(3,2,2)$ and $P=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}$

$$A^{65}=PD^{65}P^{-1}=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}\cdot\begin{bmatrix}3^{65} & 0 & 0\\ 0 & 2^{65} & 0\\ 0 & 0 & 2^{65}\end{bmatrix}\cdot\begin{bmatrix}2 & -2 & 1\\ 2 & 3 & -1\\ -1 & 1 & 0\end{bmatrix} \to \text{(exercise)}$$

Systems of Lin. ODEs with Constant Coefficients:

Ex] $\left.\begin{aligned}\frac{dx_1}{dt} &= 4x_1-3x_2\\ \frac{dx_2}{dt} &= -x_1+2x_2\end{aligned}\right\}$ where x_1 and x_2 are functions of t.

$$\frac{dX}{dt}=AX$$

⊛ $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $\frac{dX}{dt}=\begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{bmatrix}$, $A=\begin{bmatrix}4 & -3\\ -1 & 2\end{bmatrix}$

A is an $n \times n$ matrix over $\mathbb{R}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ (where $x_k = x_k(t)$)

$$\frac{dX}{dt} = AX \Rightarrow \begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n \end{cases} \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$(n=1)$ $\frac{dx}{dt} = kx \Rightarrow$ (seperable) $\overset{\langle\text{exercise}\rangle}{\Longrightarrow}$ $x(t) = c \cdot e^{kt}$ if $x > 0$.

Ex] $\left.\begin{aligned} \frac{dx_1}{dt} &= 2x_1 \\ \frac{dx_2}{dt} &= -3x_2 \end{aligned}\right\}$ $\frac{dX}{dt} = AX$, $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ the coefficient matrix is diagonal.

$\left.\begin{aligned} x_1(t) &= c_1 e^{2t} \\ x_2(t) &= c_2 e^{-3t} \end{aligned}\right\}$ from the $(n=1)$ case above.

★ $\frac{dX}{dt} = DX$ where $D = \operatorname{diag}(d_1, d_2 \cdots d_n) \Longrightarrow X = \begin{bmatrix} c_1 e^{d_1 t} \\ c_2 e^{d_2 t} \\ \vdots \\ c_n e^{d_n t} \end{bmatrix}$

Suppose A is a diagonalizable matrix, (ie, $P^{-1}AP = D$ is diagonal.)

(*) $\frac{dX}{dt} = AX$

$$P^{-1}AP = D$$
$$P^{-1}A = DP^{-1}$$
$$P^{-1}AX = DP^{-1}X$$
$$P^{-1}\left(\frac{dX}{dt}\right) = DP^{-1}X$$
$$\frac{d}{dt}\left(P^{-1}X\right) = D \cdot P^{-1}X$$

$Y = P^{-1}X$

$\frac{dY}{dt} = DY$, $X = PY$

can be written directly

(Finalde çıkabilir.)

Ex (*)
$$\begin{cases} \dfrac{dx_1}{dt} = 4x_1 - 2x_2 + x_3 \\ \dfrac{dx_2}{dt} = 2x_1 + x_3 \\ \dfrac{dx_3}{dt} = 2x_1 - 2x_2 + 3x_3 \end{cases}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$(*): \frac{dX}{dt} = AX$$

<Exercise>: Show that a is diagonalizable with $P^{-1}AP = D = \mathrm{diag}(3,2,2)$ and $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$P^{-1}AP = D$$
$$P^{-1}A = DP^{-1}$$
$$P^{-1}AX = DP^{-1}X$$

$$P^{-1}\frac{dX}{dt} = DP^{-1}X$$
$$\frac{d}{dt}(P^{-1}X) = D\underbrace{P^{-1}X}_{=Y}$$
$$\frac{dY}{dt} = DY$$

$$\frac{dY}{dt} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\frac{dy_1}{dt} = 3y_1 \Rightarrow y_1(t) = c_1 e^{3t}$$
$$\frac{dy_2}{dt} = 2y_2 \Rightarrow y_2(t) = c_2 e^{2t}$$
$$\frac{dy_3}{dt} = 2y_3 \Rightarrow y_3(t) = c_3 e^{2t}$$

$$X = P \cdot Y$$

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix}$$

$$\left.\begin{aligned} x_1(t) &= c_1 e^{3t} + c_2 e^{2t} - c_3 e^{2t} \\ x_2(t) &= \\ x_3(t) &= \end{aligned}\right\} \text{answer}$$

Exercise

$$\frac{dx_1}{dt} = -11x_1 - 5x_2 - 3x_3$$
$$\frac{dx_2}{dt} = 12x_1 + 7x_2 + 2x_3$$
$$\frac{dx_3}{dt} = 12x_1 + 5x_2 + 4x_3$$

Note: The coefficient matrix A is always diagonalizable.

Ex $V = P_2[x]$, $W = \text{span}\{x+1, x^2\}$. Find a basis for $W^\perp$ if V is furnished by; $\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$.

$\left.\begin{array}{l}\dim V = 3\\ \dim W = 2\end{array}\right. \Rightarrow \dim W^\perp = 1$ | $ax^2+bx+c \in W^\perp$ iff $\begin{array}{l} ax^2+bx+c \perp x+1 \\ ax^2+bx+c \perp x^2\end{array}$ and

$$\begin{array}{l}\langle ax^2+bx+c,\ x+1\rangle = 0\\ \langle ax^2+bx+c,\ x^2\rangle = 0\end{array} \Longrightarrow \begin{array}{l}\int_0^1 (ax^2+bx+c)(x+1)\, dx = 0\\ \int_0^1 (ax^2+bx+c)\, x^2\, dx = 0\end{array}$$

$\int_0^1 (ax^3+bx^2+cx+ax^2+bx+c)\,dx = 0$ and $\int_0^1 (ax^4+bx^3+cx^2)\,dx = 0$

$\frac{a}{4}+\frac{b}{3}+\frac{c}{2}+\frac{a}{3}+\frac{b}{2}+c = 0$ $\qquad$ $\frac{a}{5}+\frac{b}{4}+\frac{c}{3} = 0$

(1) $7a+10b+18c = 0$ $\qquad$ (2) $12a+15b+20c = 0$

$$(*): \begin{cases}(1)\\(2)\end{cases} \Rightarrow \begin{bmatrix} 7 & 10 & 18 \\ 12 & 15 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 10/7 & 18/7 \\ 0 & -5/28 & -13/21 \end{bmatrix} \Bigg\} \begin{array}{l} c = 21u \\ b = \frac{-532}{5}u \end{array} \quad a = 206u$$

$$ax^2+bx+c = \left(206x^2 - \frac{532}{5}x + 21\right)u \Longrightarrow W^\perp = \text{span}\left\{206x^2 - \frac{532}{5}x + 21\right\}$$