MATH 102

Calculus II

Spring 2017 (Mehmet Okan Tekman)

Lecture Notes

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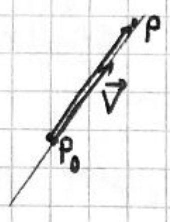
Subject :   Date ....../....../......

Lines in the Plane:

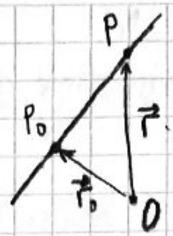


$\overrightarrow{P_0P} \parallel \vec{v}$

$\Updownarrow$ $-\infty < t < \infty$

$\boxed{\overrightarrow{P_0P} = t \cdot \vec{v}}$ parameter

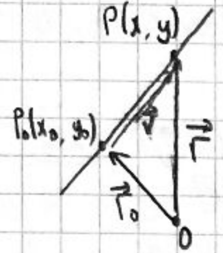
(vector equation of line)



$\vec{r}$ → position vector

$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0} = r - r_0$

$\boxed{\vec{r} = \vec{r}_0 + t \cdot \vec{v}}$



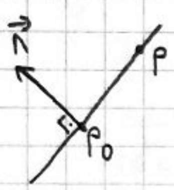
$\vec{v} = ai + bj$

$\vec{r} = xi + yj$

$\vec{r}_0 = x_0 i + y_0 j$

$\longrightarrow$ $\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}}$ } (parametric equation of the line)

Normal Equations



$P_0P \perp \vec{n}$

$\boxed{\vec{n} \cdot P_0P = 0}$ → $(P_0P = r - r_0)$ → $\boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0}$ → $\vec{n} = ai + bj$

(vector equation of line)

$\boxed{ax + by = ax_0 + by_0}$

or

$\boxed{a(x - x_0) + b(y - y_0) = 0}$ (scalar equation of the line)

Ex Write an equation for the line passing through $P_0(1, 4)$ and parallel to $\vec{v} = 3i - 2j$.

$\vec{r} = \vec{r}_0 + t \cdot \vec{v}$

$xi + yj = (i + 4j) + t \cdot (3i - 2j)$ or $\begin{aligned} x &= 1 + 3t \\ y &= 4 - 2t \end{aligned}$

Ex Write an equation for the line passing through $P_0(3, -4)$ and normal to $7i + 2j$

$7(x - 3) + 2(y - (-4)) = 0 \implies 7x + 2y = 13$ → Can you find the param. from this?

Lines in the Space: We just add 3rd element to the coordinates.

$P(x, y, z)$
$P_0(x_0, y_0, z_0)$ $\implies$ $\begin{aligned} \vec{v} &= ai + bj + ck \\ \vec{r} &= xi + yj + zk \\ \vec{r}_0 &= x_0 i + y_0 j + z_0 k \end{aligned}$ $\implies$ $\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$

Planes in the Space: When we write the normal equations for the space, it represents a plane instead of a line.

$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \implies a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Ex| Find an equation for the line passing through $P_0(1, 2, -1)$ and parallel to $\vec{v} = 4\vec{i} - \vec{j} + 2\vec{k}$

$$\left.\begin{aligned} x &= 4t+1 \\ y &= -t+2 \\ z &= 2t-1 \end{aligned}\right\} \text{parametric equation} \xrightarrow{\text{solve for } t} \underbrace{\frac{x-1}{4} = \frac{y-2}{-1} = \frac{z-(-1)}{2}}_{\text{symmetric equation}}$$

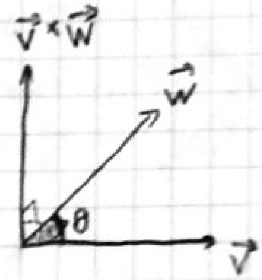
Cross Product: Only happens in 3-dimensions.

$\vec{v} \times \vec{w}$ is the vector in space such that:

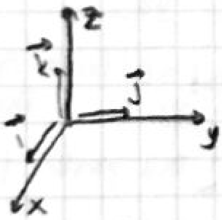
1- $\vec{v} \times \vec{w} \perp \vec{v}$ and $\vec{v} \times \vec{w} \times \vec{w}$

2- $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin\theta$

3- $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ form a right handed system.



Right Handed Coordinate System:



$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

$\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$

$\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$

$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$

Ex| $v = 3\vec{i} + 2\vec{j} - 4\vec{k}$

$w = 5\vec{i} + \vec{j} + 2\vec{k}$

$$v \times w = \begin{vmatrix} i & j & k \\ 3 & 2 & -4 \\ 5 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 1 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & -4 \\ 5 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix}\vec{k} = 8\vec{i} - 26\vec{j} - 7\vec{k}$$

★ $\vec{u} \cdot \vec{w} = 0 \Leftrightarrow \vec{u} \perp \vec{w}$, $\vec{u} \times \vec{w} = \vec{0} \Leftrightarrow \vec{u} \parallel \vec{w}$

Ex| Find an equation for the plane passing through $P(1, -1, 2)$ and $Q(3, 1, -1)$ and parallel to the line $L: x = t-5,\ y = 3t,\ z = 4t+1$

$\vec{v} \perp \vec{n}$ $\quad \vec{v} = \vec{i} + 3\vec{j} + 4\vec{k}$

$\vec{PQ} \perp \vec{n}$ $\quad \vec{PQ} = 2\vec{i} + 2\vec{j} - 3\vec{k}$

$$n = \vec{v} \times \vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 3 & 4 \\ 2 & 2 & -3 \end{vmatrix} =$$

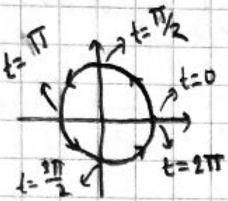
$\vec{n} = -17\vec{i} + 11\vec{j} - 4\vec{k} \Rightarrow -17(x-1) + 11(y-(-1)) - 4(z-2) = 0$

$17x - 11y + 4z = 36$

Parametric Curves: (Sections 10.1, 10.2 & 13.1, 13.3)

Definition: If $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ for t in some interval I, then the set of points $(x, y) = (f(t), g(t))$ traced on as t varies in I is a parametric curve and t is called the parameter.

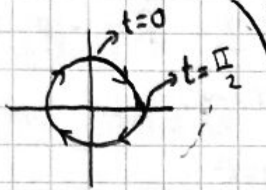
Ex) $x = \cos t$
$y = \sin t$
$0 < t < 2\pi$



$x^2 + y^2 = 1$

$x = t^2$
$y = t^3$
$-\infty < t < \infty$

$x = \sin t$
$y = \cos t$
$-\infty < t < \infty$



$x^2 + y^2 = 1$

$x = t$
$y = t^2$
$-\infty < t < \infty$

$y = x^2$

Ex) $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j}$

$\vec{V} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$|\vec{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

$\vec{r} = t^2\vec{i} + t^3\vec{j}$

$\vec{V} = 2t\vec{i} + 3t^2\vec{j}$

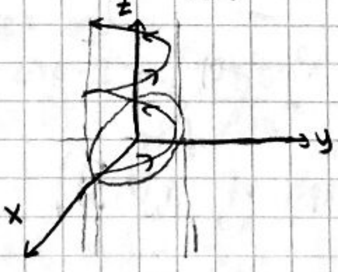
$|\vec{v}| = \sqrt{4t^2 + 9t^4}$

★ $\vec{V}$ is called velocity vector or tangent vector. $\dfrac{\vec{V}}{|\vec{V}|} = \vec{T}$, is the unit tangent vector.

Ex) $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

$\vec{V} = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

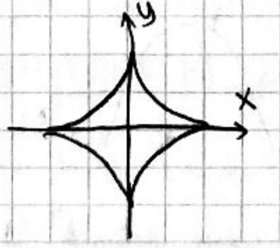


Arclength: $\displaystyle\int_{t=0}^{t=b} |\vec{v}|\,dt$ (integral of speed) = (distance travelled)

Ex) $\vec{r} = \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j}$

$\Downarrow$

$x^{2/3} + y^{2/3} = 1$



$\displaystyle 4 \cdot \int_{\pi/2}^{0} |\vec{v}|\,dt = 4\int_{\pi/2}^{0} \frac{3}{2}\sin 2t\,dt = 6$

$\vec{V} = 3\cos^2 t(-\sin t)\,\vec{i} + 3\sin^2 t\cos t\,\vec{j}$

$|\vec{v}| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = 3\cos t\sin t\sqrt{\cos^2 t + \sin^2 t} = 3\cos t\sin t = \frac{3}{2}\sin 2t$

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<u>Functions with Several Variables:</u> A function f of n-variables is a rule that assigns a unique real number $w = f(x_1, x_2 \dots, x_n)$ to each ordered n-tuple $(x_1, x_2, \dots, x_n)$ of real number in some subset D of $\underbrace{\mathbb{R}^n}$

↳ The set of all ordered n-tuples of $\mathbb{R}$.

$x_1, x_2, \dots x_n$ are independent variables.

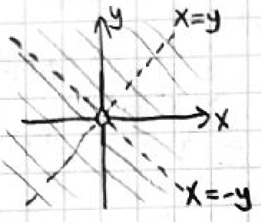
w is the dependent variable, D is the domain of f.

$R = \{f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots x_n) \text{ in } D\}$ is the range of f.

Ex| $n = 2 \Rightarrow w = f(x_1, x_2)$, more often: $z = f(x, y)$

$f(x,y) = \dfrac{1}{x^2 - y^2}$

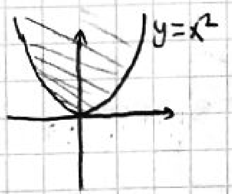
$D = \{(x,y) : x^2 - y^2 \neq 0\}$



$R = (-\infty, 0) \cup (0, \infty)$

$f(x,y) = \sqrt{y - x^2}$

$D = \{(x,y) : y - x^2 \geq 0\}$



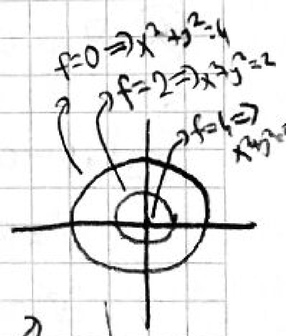
$R = [0, \infty)$

Ex| $n = 3 \Rightarrow w = f(x_1, x_2, x_3)$, more often: $w = f(x, y, z)$

$f(x,y,z) = \dfrac{\ln z}{x^2 + y^2}$

$D = \{(x,y,z) : z > 0,\ x^2 + y^2 \neq 0\}$

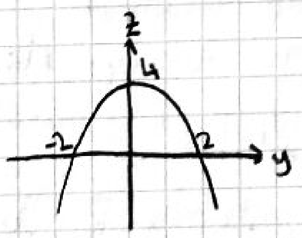
The half space above the xy-plane but the z-axis is excluded.

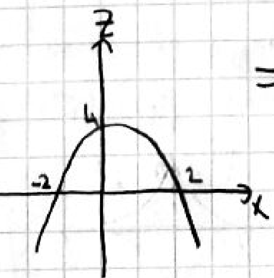


[level curves]

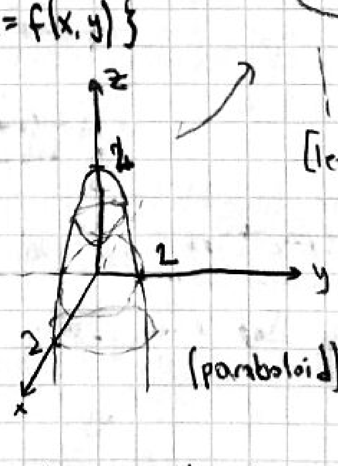
Ex| $n = 2 \Rightarrow$ (Graph of $f(x,y)$) $= \{(x,y,z) : z = f(x,y)\}$

$f(x,y) = 4 - x^2 - y^2$





$\Rightarrow$



[paraboloid]

Let k be a constant. The curve defined by $f(x,y) = k$ in the domain of f is called a <u>level curve</u> of f.

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(☆) Read Section 12.6 from the book. (Cylinders and

Limits and Continuity:

Definition: Let $f(x, y)$ be a function with domain D. We say that the number L is the limit of $f(x, y)$ as (x, y) approaches (a, b) and we write $\lim_{(x,y)\to(a,b)} f(x, y) = L$. If for every $\varepsilon > 0$, there is $d > 0$ such that for all (x, y) in D, we have;

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < d \Rightarrow |f(x, y) - L| < \varepsilon$$

Definition: $f(x,y)$ is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a, b)$.

→ If $f(x,y)$ and $g(x, y)$ are continuous, then $f \pm g$, $f \cdot g$, $k \cdot f$, $\frac{f}{g}$, f^k are also continuous in their domains.

→ If $f(x, y)$ and $\ln(z)$ are continuous, then $\ln(f(x, y))$ is continuous.

(☆) Squeeze (Sandwitch) theorem holds for 2-variable functions too.

Ex) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ $\quad 0 \leq \left|\frac{xy^2}{x^2+y^2}\right| = \underbrace{\frac{y^2}{x^2+y^2}}_{} \cdot |x| \leq 1 \cdot |x| = |x|$ → 0 as $(x,y)\to(0,0)$

$0 < x^2$, $0 \leq y^2 \Rightarrow 0 \leq y^2 \leq x^2+y^2 \Rightarrow \frac{y^2}{x^2+y^2} \leq 1$

By squeeze therem, $\lim_{(x,y)} \left|\frac{xy^2}{x^2+y^2}\right| = 0$

Ex) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ } $f(x,y)$

Lets try $y=0$: $\lim_{\substack{(x,y)\to(0,0) \\ \text{on } x\text{-axis}}} f(x,y) = \lim_{x\to 0} f(x, 0) = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$ → ☆ Limit is 0 or DNE

Lets try $y=x$: $\lim_{\substack{(x,y)\to(0,0) \\ \text{on } x=y \text{ line}}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x \cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$ → ☆ Limit is $\frac{1}{2}$ or DNE

Limit DNE by 2-Path Test

Two-Path Test: If $f(x,y)$ opprâches to different numbers as $(x,y) \to (a,b)$, then the limit DNE on the point (a, b).

Note that $f(x, mx) = \frac{m}{1+m^2}$ so f is constant on any line $y = mx$ through the $(0,0)$. So, every $y=mx$ is a level curve. $|f(x, y)| \leq \frac{1}{2}$ for all (x,y) except $(0,0)$.

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Remark: We must use Squeeze Theorem to show that the limit exists.
We must use 2-Path Test to show the limit does not exist.
These two rules can not be used to show vice-versa.

Ex] $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = ?$ Lets' try to look at what happens if we approach the origin through all lines $y=mx$ ($\Rightarrow f(x,mx)$) — does not include this.

$\lim_{(x,y)\to(0,0)} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2} = 0$ And we approach $y=0$ lastly;

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \frac{0}{0+y^4} = 0$; $f(x,y)$ approaches 0 from all the lines through the origin. Is it possible that $\lim_{(x,y)\to(0,0)} f(x,y) \neq 0$? Yes.

What if we approach the origin from the parabola $x=y^2$

$\lim_{(x,y)\to(0,0)} f(\overset{y^2}{x},y) = \lim_{y\to 0} \frac{y^2 \cdot y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$ So lets approach from $x=my^2$

$\left.\lim_{y\to 0} \frac{(my^2)\cdot y^2}{(my^2)^2+y^4} = \frac{m}{m^2+1} \right\}$ So the value of the function is constant for all these.

Partial Derivatives:

- The partial derivative of f with respect to y at (a,b):

$$\frac{\partial f}{\partial y}(a,b) = \frac{d}{dy} f(a,y)\Big|_{y=b} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

- The partial derivative of f with respect to x at (a,b):

$$\frac{\partial f}{\partial x}(a,b) = \frac{d}{dx} f(x,b)\Big|_{x=a} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

Observation: Our candidate for the equation of the tangent plane to the graph of $z=f(x,y)$ at the point $(a,b,f(a,b))$ is:

$$z = \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b) + f(a,b)$$

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Notation: $\frac{\partial f}{\partial x} = f_x = f_1 = \frac{\partial z}{\partial x} = z_x = D_x f = D_1 f \quad (z = f(x, y))$

Ex) $f(x, y) = e^{xy} + x^2 \sin y \Rightarrow f_x = e^{xy} \cdot y + 2x \sin y \quad f_y = e^{xy} \cdot x + x^2 \cos y$

Ex) $f(x, y, z) = \frac{xy^2}{z^3+1} \Rightarrow f_x = \frac{y^2}{z^3+1} \quad f_y = \frac{2xy}{z^3+1} \quad f_z = -\frac{3xy^2z^2}{(z^3+1)^2}$

Ex) Suppose z is a differentiable function of x, y satisfying the equation, $yz^3 - x^3z + xy^2 = -9$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, -1, 2)$.

Take $\frac{\partial}{\partial x} \xRightarrow{y \text{ is constant}} y \cdot 3z^2 \cdot \frac{\partial z}{\partial x} - \left(3x^2z + x^3 \cdot \frac{\partial z}{\partial x}\right) + y^2 = 0 \xrightarrow{\text{substitute}} (x, y, z) = (1, -1, 2)$

$-12\frac{\partial z}{\partial x} - \left(6 + \frac{\partial z}{\partial x}\right) + 1 = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-5}{13}$ at $(1, -1, 2)$.

Take $\frac{\partial}{\partial y} \xRightarrow{x \text{ is constant}} z^3 + y3z^2\frac{\partial z}{\partial y} - x^3\frac{\partial z}{\partial y} - 2xy = 0 \xrightarrow{\text{substitute}} (x, y, z) = (1, -1, 2)$

$8 - 12\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} - 2 = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{6}{13}$ at $(1, -1, 2)$.

Exercise: For the same equation, consider x as a function of y and z and compute $\frac{\partial x}{\partial y}, \frac{\partial x}{\partial z}$. Then do the same for y. Are these 6 numbers related in some way?

Exercise:
$$xw^3 - z^3 + xy^2 = 1$$
$$yw^2 - xz^2 + x^3 = 7$$
Suppose w and z are differentiable functions of x and y satisfying these.

Find w_x, w_y, z_x, z_y at $(x, y, z, w) = (2, 1, -1, -1)$ | A: $\frac{-11}{6}, \frac{-17}{18}, \frac{-11}{3}, \frac{-7}{9}$

Higher Order Derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = \frac{\partial^2 z}{\partial x^2} = z_{xx} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = z_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = \frac{\partial^2 z}{\partial y^2} = z_{yy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = z_{yx}$$

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Theorem: Suppose f_{xy} and f_{yx} are continuous on some disk centered at (a, b), then: $f_{xy}(a,b) = f_{yx}(a,b)$ [Clairaut's / Euler's Theorem]

Tangent Plane and Differentiability:

Definition: $f(x, y)$ is differentiable at (a, b) if $f_x(a, b)$ and $f_y(a, b)$ exist and $f(x, y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \varepsilon_1(x, y)(x-a) + \varepsilon_2(x,y)(y-b)$ where ε_1 and ε_2 are functions such that $\varepsilon_1(x,y) \to 0$ and $\varepsilon_2(x,y) \to 0$ as $(x,y) \to (a,b)$.

Remark: This condition is the geometric condition for our candidate tangent plane to be the real tangent plane.

Theorem: If f_x and f_y are continuous at (a,b), then f is differentiable at (a,b).

(*) Graph of $z = f(x, y)$ has a tangent plane at $(a, b, f(a,b))$ $\overset{def}{\equiv}$ f is differentiable at (a,b)

f_x and f_y are continuous at (a,b) $\Downarrow$ f is differentiable at (a,b) $\Rightarrow$ $f_x(a,b)$ and $f_y(a,b)$ exists; $\Rightarrow$ f is continuous at (a,b)

Definition: Suppose $f(x, y)$ is differentiable at (a, b). Then:

An equation of the tangent plane at the point $(a, b, f(a, b))$ is:

$$z = f(a, b) + f_x(a,b)(x-a) + f_y(a, b)(y-b)$$

The Linearization of $f(x,y)$ centered on (a,b) is:

$$L(x, y) = f(a, b) + f_x(a,b)(x-a) + f_y(a, b)(y-b)$$

The Linear Approximation of $f(x,y)$ centered on (a,b) is:

$$f(x, y) \approx L(x, y) \quad \text{for} \quad (x, y) \approx (a, b)$$

Ex $f(x, y) = y^3 + xy^2 - x^2$ at the point $(2, 1)$.

$f_x = y^2 - 2x \Rightarrow f_x(2,1) = -3$

$f_y = 3y^2 + 2xy \Rightarrow f_y(2,1) = 7$

$f(2,1) = -1$

$\Rightarrow z = -3(x-2) + 7(y-1) + (-1)$

$\rightarrow$ Compute $f(2.01, 1.02)$ approximately: $L(2.01, 1.02) = -3(0.01) + 7(0.02) - 1$
$= -0.03 + 0.14 - 1 = -0.89$

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$\Delta x = dx = x-a$
$\Delta y = dy = y-b$

Definition: If $f(x,y)$ is differentiable at (a,b) then,

$$dz = df = f_x(a,b)\,dx + f_y(a,b)\,dy$$ is the differential of f. $\Rightarrow$ Approximate change in f.

Ex If we measure the height of a cone with 1% error and its radius with 2% error, and use this data to compute its volume, then estimate the error in the volume.

$$V = \frac{\pi}{3}\cdot r^2 h \Rightarrow dV = \frac{\partial V}{\partial r}\cdot dr + \frac{\partial V}{\partial h}dh = \frac{2\pi}{3} rh\,dr + \frac{\pi}{3} r^2\,dh \qquad \left|\frac{dh}{h}\right| \leq 1\%$$

$$\frac{dV}{V} = \frac{\frac{2\pi}{3} rh\,dr + \frac{\pi}{3} r^2\,dh}{\frac{\pi}{3} r^2 h} = 2\frac{dr}{r} + \frac{dh}{h} \Longrightarrow \left|\frac{dV}{V}\right| = \left|2\frac{dr}{r} + \frac{dh}{h}\right| \qquad \left|\frac{dr}{r}\right| \leq 2\%$$

$$\left|2\frac{dr}{r} + \frac{dh}{h}\right| \overset{\text{triangle inequality}}{\leq} 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \Longrightarrow \left|\frac{dV}{V}\right| \leq 2\cdot 2\% + 1\% = 5\%$$

Chain Rules: **Theorem:** If $z = f(x,y)$ is a differentiable function and $x = g(t)$, $y = h(t)$ are differentiable functions of t, then $z = f(g(t), h(t))$ is a differentiable function of t, and;

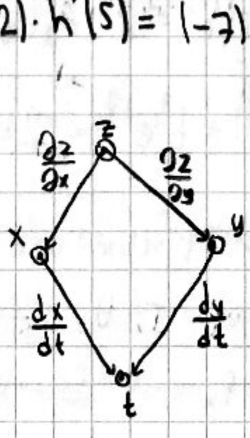
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Ex Suppose $f(x,y)$, $g(t)$, $h(t)$ are differentiable functions such that $f(1,2)=3$, $g(5)=1$, $h(5)=2$, $f_x(1,2)=-7$, $f_y(1,2)=-4$, $g'(5)=11$, $h'(5)=-9$. Then let $f(g(t), h(t)) = z$ find $\left.\frac{dz}{dt}\right|_{t=5}$. $f(g(5), h(5)) = f(1,2)$

$$\frac{dz}{dt} = f_x(1,2)\cdot g'(5) + f_y(1,2)\cdot h'(5) = (-7)\cdot 11 + (-4)\cdot(-9) = -41$$

General Chain Rule:

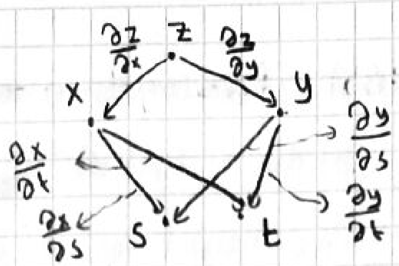
$$\left.\begin{aligned} z &= f(x,y) \\ x &= g(t) \\ y &= h(t) \end{aligned}\right\} z = f(g(t), h(t))$$



For each road going from z to t;

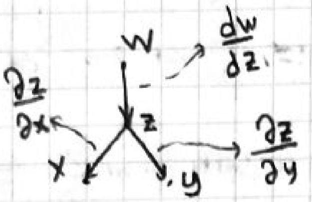
$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}}_{\text{Road 1}} + \underbrace{\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}_{\text{Road 2}}$$

$$\left.\begin{aligned} z &= f(x, y) \\ x &= g(s, t) \\ y &= h(s, t) \end{aligned}\right\} z = f(g(s,t), h(s,t))$$



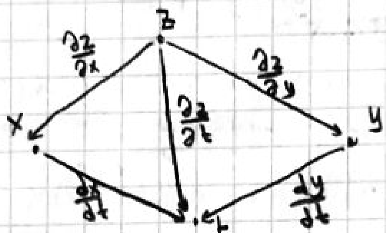
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\left.\begin{aligned} w &= f(z) \\ z &= g(x, y) \end{aligned}\right\} w = f(g(x, y))$$



$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\cdot\frac{\partial z}{\partial x} \qquad \frac{\partial w}{\partial y} = \frac{dw}{dz}\cdot\frac{\partial z}{\partial y}$$

$$\left.\begin{aligned} z &= f(x, y, t) \\ x &= g(t) \\ y &= h(t) \end{aligned}\right\} z = f(g(t), h(t), t)$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial y}{\partial t}$$

Exercise: $z = x\cdot e^{x^2 y t}$, $x = t^2$, $y = t^3$. Find $\frac{dz}{dt}$ by substituting and doing chain rule.

1) $z = t^2\cdot e^{t^4\cdot t^3\cdot t} = t^2\cdot e^{t^8} \Rightarrow \frac{dz}{dt} = 2te^{t^8} + t^2\cdot 8t^7\cdot e^{t^8} = 2te^{t^8} + 8t^9\cdot e^{t^8}$

2) $\frac{\partial z}{\partial x} = e^{x^2yt} + x\cdot 2xyt\cdot e^{x^2yt}$ $\quad \frac{dx}{dt} = 2t$ $\quad \frac{\partial z}{\partial y} = x\cdot x^2 t\cdot e^{x^2yt}$ $\quad \frac{dy}{dt} = 3t^2$

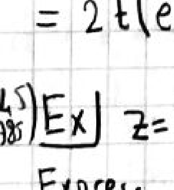
$\frac{\partial y}{\partial t} = x\cdot x^2 y\cdot e^{x^2yt} \Rightarrow \frac{dz}{dt} = 2t(e^{x^2yt} + 2x^2yt\,e^{x^2yt}) + 3t^2(x^3te^{x^2yt}) + x^3ye^{x^2yt}$

$= 2t(e^{t^8} + 2t^8e^{t^8}) + 3t^2(t^7\cdot e^{t^8}) + t^9 e^{t^8} = 2tk + 4t^9k + 3t^9k + t^9k = 2tk + 8t^9k$

$e^{t^8} \to k$

(ex 4.5 pg 985) Ex] $z = f(x, y)$ is a differentiable function and $x = r\cos\theta$, $y = r\sin\theta$.

Express $(f_x)^2 + (f_y)^2$ in terms of r, θ, $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$.

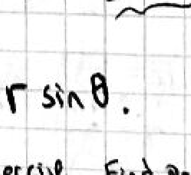


$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cdot\cos\theta + f_y\cdot\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x(-r\cos\theta) + f_y(r\sin\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \dots = (f_x)^2 + (f_y)^2$$

Exercise Find answer by



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Exercise: Ex53, Pg 985. $f_{xx} = \frac{\partial (f_x)}{\partial x} = \dots$ use the $\frac{\partial z}{\partial x}$'s table, substitute f_x to $\boxed{z}$.

⊛ Suppose $y = f(x)$ is a differentiable function which satisfies $F(x, y) = \text{constant}$, where F is a differentiable function too. Then consider $z = F(x, y) = F(x, f(x))$

$$\begin{tikzcd} & z \arrow[dl, "\frac{\partial z}{\partial x}"'] \arrow[dr, "\frac{\partial z}{\partial y}"] & \\ x & & y \arrow[ll, "\frac{dy}{dx}"] \end{tikzcd}$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$F(x, f(x)) = \text{const} \Rightarrow \frac{dz}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercise ⊛ Express $\frac{d^2y}{dx^2}$ with $F_x, F_y, F_{xx}, F_{yy}, F_{xy}$

Gradient Vector: $\frac{d}{dt} f(g(t), h(t)) = \frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \Big\}$ We write this as a dot product.

$$\frac{df}{dt} = \underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)}_{\vec{\nabla} f(x)} \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right) \quad \vec{V}\Big|_{t=t_0}$$

Definition: If $f(x, y)$ is differentiable at $P_0(a, b)$ then;

$$\vec{\nabla} f(P_0) = \frac{\partial f}{\partial x}\Big|_{P_0} \vec{i} + \frac{\partial f}{\partial y}\Big|_{P_0} \vec{j} \quad \Big\} \text{ Gradient of } f \text{ at } P_0.$$

⊛ $$\frac{d}{dt} f(g(t), h(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)}_{\vec{\nabla} f} \cdot \underbrace{\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right)}_{\vec{V}}$$

$$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} = \vec{\nabla} f(P_0) \cdot \vec{V}\Big|_{t_0} \quad \left[\vec{r}(t_0) = \overrightarrow{OP_0}\right]$$

Definition: The directional derivative of $f(x, y)$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = a\vec{i} + b\vec{j}$ is;

$$D_{\vec{u}} f(P_0) = \frac{d}{dt} f(x_0 + at, y_0 + bt)\Big|_{t=0} \qquad |\vec{u}| = 1$$

Theorem: If f is differentiable at P_0: $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$

Remark: $D_{\vec{u}} f(P_0)$ is the rate of change of f as we move from P_0 in the direction $\vec{u}$ with unit speed with respect to time.

In other words: $D_{\vec{u}} f(P_0)$ is the rate of change of f as we move from P_0 in the direction $\vec{u}$ with respect to distance.

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Ex] $f(x,y) = y^3 + xy^2 - x^2$, $P_0 = (2,1)$, $\vec{u} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$

Find the directional derivative of f at P_0 in direction $\vec{u}$.

$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$ $\quad \vec{\nabla} f = f_x \vec{i} + f_y \vec{j} = (y^2 - 2x)\vec{i} + (3y^2 + 2xy)\vec{j}$

$= (-3\vec{i} + 7\vec{j}) \cdot (\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}) = \frac{7\sqrt{3} - 3}{2}$ $\quad \vec{\nabla} f(P_0) = (1-4)\vec{i} + (3+4)\vec{j} = -3\vec{i} + 7\vec{j}$

→ Fix f and P_0. How does $D_{\vec{u}} f(P_0)$ change as we change $\vec{u}$.

$D_{\vec{u}} f(P_0) = |\vec{\nabla} f(P_0)| \cdot |\vec{u}| \cdot \cos\theta$ ① We get the fastest increase of f in the direction of $\vec{\nabla} f(P_0)$ $(\theta = 0)$ which is $|\nabla f(P_0)|$

② We get the fastest decrease of f in the direction of $-\vec{\nabla} f(P_0)$ $(\theta = 180)$ which is $-|\nabla f(P_0)|$

③ If we move perpendicularly to $\vec{\nabla} f(P_0)$ $(\theta = 90)$ then the rate of change is 0.

Ex] Find the direction in which f increases fastest. (Direction means unit vector)

$\frac{\vec{\nabla} f(P_0)}{|\vec{\nabla} f(P_0)|} = \frac{-3\vec{i} + 7\vec{j}}{\sqrt{(-3)^2 + 7^2}} = \frac{-3}{\sqrt{58}}\vec{i} + \frac{7}{\sqrt{58}}\vec{j}$ is the direction. And the r.o.c. is $\sqrt{58}$ in this direction.

Observation: Consider the level curve of $f(x,y)$ passing through the point $P_0(x_0, y_0)$. If we parametrize the level curve $x = g(t)$, $y = h(t)$ with $x_0 = g(0)$, $y_0 = h(0)$. Then,

$$0 = \frac{d}{dt} f(g(t), h(t)) \Big|_{t=0} = \vec{\nabla} f(P_0) \cdot \vec{v} \Big|_{t=0} \Rightarrow \vec{\nabla} f(P_0) \perp \vec{v} \Big|_{t=0}$$

So $\vec{\nabla} f(P_0)$ is normal to the level curve passing through P_0, at P_0.

Since we know $\vec{\nabla} f(P_0)$ is normal to the tangent line at that point, we can write the equation for the tangent line: $f_x(P_0) \cdot (x - x_0) + f_y(P_0) \cdot (y - y_0) = 0$

Since we know $\vec{\nabla} f(P_0)$ is parallel to the normal line at that point, we can write the parametric equation for the normal line: $\left.\begin{matrix} x = f_x(P_0)t + x_0 \\ y = f_y(P_0)t + y_0 \end{matrix}\right\}$ ✓

Ex] Find the normal and tangent equations to the curve $y^3 + xy^2 - x^2 = 0$ at $P_0(2,1)$.

Take $f(x,y) = y^3 + xy^2 - x^2$ and consider the level curve $f(x,y) = 0$.

$\vec{\nabla} f(P_0) = -3\vec{i} + 7\vec{j}$

Tangent: $-3(x-2) + 7(y-1) = 0$

Normal: $x = -3t + 2$
$y = 7t + 1$

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Trap: its not a unit vector.

Ex) $f(x,y,z) = \ln(x^2+y^2) + xz^2$, $P_0(1,-1,2)$, $\vec{A} = 2\vec{i} + 2\vec{j} - k$

a) Find the directional derivative of f at P_0 in the direction of $\vec{A}$.

$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u}$ $\qquad \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \left(\frac{2x}{x^2+y^2} + z^2\right)\vec{i} + \left(\frac{2y}{x^2+y^2}\right)\vec{j} + (2xz)\vec{k}$

$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$ $\qquad \vec{\nabla} f(1,-1,2) = 5\vec{i} - \vec{j} + 4\vec{k}$

$$D_{\vec{u}} f(P_0) = (5\vec{i} - \vec{j} + 4\vec{k}) \cdot \left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}\right) = \frac{10}{3} - \frac{2}{3} - \frac{4}{3} = \frac{4}{3}$$

b) Find the direction which f increases fastest at P_0. $\frac{\vec{\nabla} f(P_0)}{|\vec{\nabla} f(P_0)|} = \frac{5\vec{i} - \vec{j} + 4\vec{k}}{\sqrt{42}}$ → direction → rate

⊖ $F(x,y,z)$ is a 3-variable function. $\vec{\nabla} F(P_0)$ is normal to the level surface of F through P_0 at P_0. So the tangent plane is:

$$F_x(P_0)(x-x_0) + F_y(P_0)(y-y_0) + F_z(P_0)(z-z_0) = 0$$

And the parametric equations to normal line are: $\begin{cases} x = F_x(P_0)t + x_0 \\ y = F_y(P_0)t + y_0 \\ z = F_z(P_0)t + z_0 \end{cases}$

Ex) Find equations of the tangent plane and the normal line to the surface $\underbrace{xy^2 + z^3 - xz}_{F(x,y,z)} = 7$ at $P_0(1,-1,2)$.

Level Surface

$\vec{\nabla} F = (y^2 - z)\vec{i} + (2xy)\vec{j} + (3z^2 - x)\vec{k}$

$\vec{\nabla} F(P_0) = -\vec{i} - 2\vec{j} + 11\vec{k}$

→ Tangent plane: $(-1)(x-1) + (-2)(y+1) + 11(z-2) = 0$

→ Normal line: $(x = -t+1)$, $(y = -2t-1)$, $(z = 11t+2)$

Remark: Consider $z = f(x,y)$. Then an equation for the tangent plane to the graph of $z = f(x,y)$ at the point $(x,y) = (x_0,y_0)$ is:

$$z = f_x(P_0)\cdot(x-x_0) + f_y(P_0)(y-y_0) + f(P_0)$$

On the other hand, consider the surface defined by the equation $\overbrace{f(x,y) - z}^{F(x,y,z)} = 0$ at the point $(x,y,z) = (x_0,y_0,z_0)$, $z_0 = f(x,y)$

$$\underbrace{f_x(P_0)}_{F_x(x_0,y_0,z_0)}(x-x_0) + \underbrace{f_y(P_0)}_{F_y(x_0,y_0,z_0)}(y-y_0) + \underbrace{(-1)}_{F_z(x_0,y_0,z_0)}(z-z_0) = 0$$

→ So we can write the tangent plane equation in two ways.

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Maximum and Minimum Values: Definition: Suppose $f(x,y)$ is a function with domain D. $f(a,b)$ is the absolute maximum value of f on D if $f(a,b) \geq f(x,y)$ for all (x,y) in domain D. The definition is similar with abs. min. value.

Definition: $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all (x,y) in domain D in some disk centered at (a,b). The difinition is similar with...

Theorem (1st Derivative Test): If $f(x,y)$ has a local extreme value at an interior point $P_0(a,b)$, then $f_x(P_0)$ and $f_y(P_0)$ are 0 or undefined.

↳ (a,b) is an interior point of D if there's a disk centred at (a,b) and contained in D.

↳ (a,b) is a boundary point of D if every disk centered at (a,b) contains points both in D and not in D.

Proof: Fix $y=b$ and apply Fermat's Theorem for 1-variable function $f(x,b)$, to get $f_x(a,b)$ is 0 or undefined. Do the same for $f_y(a,b)$.

Definition: An interior point of the domain of $f(x,y)$ where both f_x and f_y are 0 or undefined is called a critical point of f.

A critical point which is not a local max or min, is called saddle point of f.

Theorem (2nd Derivative Test): Suppose 2nd partial derivatives of $f(x,y)$ are continuous on a disk centered at (a,b) and also suppose that $f_x(a,b) = f_y(a,b) = 0$ then;

Definition: The discriminant of $f(x,y)$: $\Delta(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$

1) If $\Delta(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.
2) If $\Delta(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.
3) If $\Delta(a,b) < 0$ then (a,b) is a saddle point.
4) If $\Delta(a,b) = 0$ then the test does not give an answer.

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Ex $f(x,y)=x^2+y^2$

$$\left.\begin{array}{l} f_x=2x=0 \\ f_y=2y=0 \end{array}\right\} (0,0) \text{ is critical}$$

$$\Delta=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}=4$$

$$\Delta(0,0)=4>0,\quad f_{xx}(0,0)=2>0 \Rightarrow$$ f has a local min. on $(0,0)$.

Ex $f(x,y)=-x^2+y^2$

$$\left.\begin{array}{l} f_x=-2x=0 \\ f_y=2y=0 \end{array}\right\} (0,0) \text{ is critical}$$

$$\Delta=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix}=-4$$

$\Delta(0,0)=-4<0 \Rightarrow$ f has a saddle point on $(0,0)$.

Ex Find and classify the critical points of following functions.

1) $f(x,y)=x^3+5y^2-5xy^2+x^2$

$$\left.\begin{array}{l} f_x=3x^2-5y^2+2x=0 \\ f_y=10y-10xy=0 \end{array}\right\} \begin{array}{l} y-xy=0 \\ y(1-x)=0 \end{array}$$

$y=0 \Rightarrow 3x^2+2x=0;\ x=0,\ x=\frac{-2}{3}$ — $(0,0)$, $(\frac{-2}{3},0)$

$x=1 \Rightarrow 3-5y^2+2=0;\ y=1,\ y=-1$ — $(1,1)$, $(1,-1)$

So there are four critical points for 2^{nd} derivative test → Writing order is not correct!

$$\Delta=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} 6x+2 & -10y \\ -10y & 10-10x \end{vmatrix}$$

$\Delta(0,0)=\begin{vmatrix} 2 & 0 \\ 0 & 10 \end{vmatrix}=20>0$ $\quad \Delta(1,1)=\begin{vmatrix} 8 & -10 \\ -10 & 0 \end{vmatrix}=-100<0$

$\Delta(\frac{-2}{3},0)=\begin{vmatrix} -2 & 0 \\ 0 & \frac{50}{3} \end{vmatrix}=-\frac{100}{3}<0$ $\quad \Delta(1,-1)=\begin{vmatrix} 8 & 10 \\ 10 & 0 \end{vmatrix}=100>0$

$f_{xx}=6x+2$

$$\left.\begin{array}{ll} f_{xx}(0,0)=2 & \cancel{f_{xx}(1,1)=8} \\ \cancel{f_{xx}(\frac{-2}{3},0)=-2} & f_{xx}(1,-1)=8 \end{array}\right\}$$ $(0,0)$ and $(1,-1)$ has local min, others are saddle points.

2) $f(x,y)=x^4-x^2y+y^2$

$$\left.\begin{array}{l} f_x=4x^3-2xy=0 \\ f_y=-x^2+2y=0 \end{array}\right\} \begin{array}{l} x^2=2y \\ \downarrow \\ 4x^3-x^3=0 \end{array} \quad \left.\begin{array}{l} x^3=0 \\ x=0 \\ y=0 \end{array}\right\} (0,0) \text{ is only critical point}$$

$$\Delta=\begin{vmatrix} 12x^2-2y & -2x \\ -2x & 2 \end{vmatrix}$$

$\Delta(0,0)=\begin{vmatrix} 0 & -0 \\ -0 & 2 \end{vmatrix}=0$ → 2^{nd} derivative test fails!

$$f(x,y)=y^2-x^2y+x^4=\underbrace{\left(y-\frac{x^2}{2}\right)^2}_{\geq 0}+\underbrace{\frac{3}{4}x^4}_{\geq 0} \geq 0=f(0,0) \text{ for all } (x,y).$$

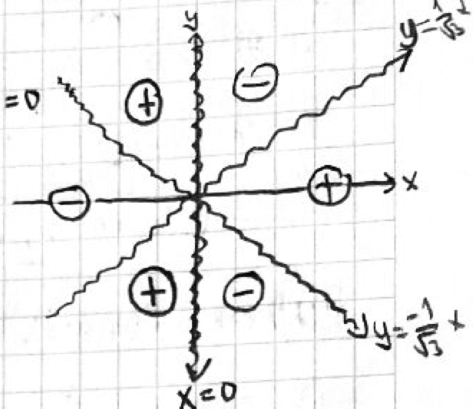
So f has an abs. min value at $(0,0)$.

3) $f(x,y)=x^3-3xy^2$

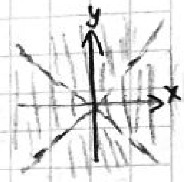
Exercise: show that 2^{nd} fails and only crit is $(0,0)$

$f(x,y) = x^3 - 3xy^2$. Consider the restriction of $f(x,y)$ to x-axis: $f(x,0) = x^3$
This 1-variable has a critical point at $x=0$ which is not a local max/min.
Therefore 2-variable function $f(x,y)$ cannot have a local max or min so we conclude that $(0,0)$ is a saddle point of f.

Another idea: $f(x,y) = x^3 - 3xy^2 = \underbrace{x}_{0}\underbrace{(x-\sqrt{3}y)}_{\text{or } 0}\underbrace{(x+\sqrt{3}y)}_{\text{or } 0} = 0$
So we conclude $(0,0)$ is a saddle point.



Some Topology: Let D be a region in the plane.
→ Interior of D is the set of all interior points of D.
→ Boundary of D is the set of all boundary points of D.
→ D is open if every point of D is an interior point. — Example Def. of $z = \frac{1}{x^2-y^2}$
→ D is closed if every point of D is a boundary point.
*→ D is bounded if there is a disk containing it, otherwise it's unbounded.



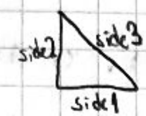
Extreme Value Theorem: A continuous function on a closed and bounded set in the plane has an abs. max and an abs. min.

Finding Abs. Min/Max: $f(x,y)$ is a continuous function on a closed and bounded set D.
1. Compute f_x and f_y.
2. Find critical points of f in the interior of D.
3. Find critical points of the restriction of f to the boundry of D.
4. Find the min/max values of the function at those points.

Ex) Find abs. max-min of $f(x,y) = x^3 - xy + y^2 - x$ on $D = \left\{(x,y) : \begin{matrix} x \geq 0 \\ y \geq 0 \\ x+y \leq 2 \end{matrix}\right\}$

Interior of D: $\left.\begin{matrix} f_x = 3x^2 - y - 1 = 0 \\ f_y = -x + 2y = 0 \end{matrix}\right\}$ $(x,y) = \left(\frac{2}{3}, \frac{1}{3}\right), \underbrace{\cancel{\left(-\frac{1}{2}, -\frac{1}{4}\right)}}_{\text{not in D}}$

Subject : Date : Not in D.

Boundary of D: 

Side 1: $\begin{matrix}0\le x\le 2\\ y=0\end{matrix}$ $\left.\begin{matrix} f(x,0)=x^3-x \\ f'(x,0)=3x^2-1=0\end{matrix}\right\}$ $\begin{matrix}x=\frac{1}{\sqrt{3}}\\ \Downarrow \\ y=0\end{matrix}$ $\cancel{\begin{matrix}x=\frac{-1}{\sqrt{3}}\\ \Downarrow \\ y=0\end{matrix}}$

Criticals we found: $(\frac{1}{\sqrt{3}},0),(0,0),(2,0)$

Endpoints: $x=0\Rightarrow y=0$, $x=2\Rightarrow y=0$

Side 2: $\begin{matrix}0\le y\le 2\\ x=0\end{matrix}$ $\left.\begin{matrix}f(0,y)=y^2\\ f'(0,y)=2y=0\end{matrix}\right\}$ $\begin{matrix}y=0\\x=0\end{matrix}$

Criticals: $(0,0),(2,0)$

Endpoints: $y=0\Rightarrow x=0$, $y=2\Rightarrow x=0$.

Side 3: $\begin{matrix}y=2-x\\ 0\le x\le 2\end{matrix}$ $\left.\begin{matrix}f(x,2-x)=x^3+2x^2-7x+4\\ f'(x,2-x)=3x^2+4x-7=0\end{matrix}\right\}$ $\begin{matrix}x=1\\ \Downarrow\\ y=1\end{matrix}$ Not in D $\cancel{\begin{matrix}x=\frac{-7}{3}\\ \Downarrow \\ y=\end{matrix}}$ Endpoints: $x=0\Rightarrow y=2$, $x=2\Rightarrow y=0$

Criticals: $(1,1),(0,2),(2,0)$ Now we calculate the values on all criticals.

$f(0,0)=0\quad f(0,2)=4\quad f(2,0)=6\quad f(1,1)=0\quad f(\frac{2}{3},\frac{1}{3})=\frac{-13}{27}\quad f(\frac{1}{\sqrt{3}},0)=\frac{-2}{3\sqrt{3}}$

Min value is $\frac{-13}{27}$ which occurs at $(\frac{2}{3},\frac{1}{3})$, Max value is 6 which occurs at $(2,0)$.

Lagrange Multipliers Method: Question: Find abs. max-min of the restriction of $f(x,y)$ to the curve $g(x,y)=\text{constant}$.

In other words: Find abs. max-min of $f(x,y)$ subject to the constraint $g(x,y)=\text{constant}$

Say our curve is the unit circle, $x^2+y^2=1$

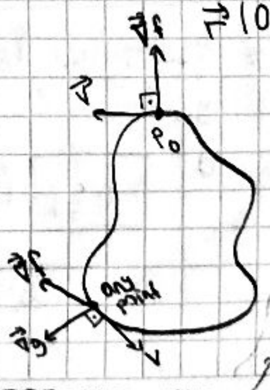
Idea 1: Solve for y in terms of x, then substitute.

$y=\pm\sqrt{1-x^2}$ → $f(x,\sqrt{1-x^2})=\dots$ for $-1\le x\le 1$ → $f(x,-\sqrt{1-x^2})$ for $-1\le x\le 1$

Idea 2: Parametrize the curve. $\left.\begin{matrix}x=\cos t\\ y=\sin t\end{matrix}\right\}$ $0\le t\le 2\pi \Rightarrow f(\cos t,\sin t)$ for $0\le t\le 2\pi$

* Idea 3: Suppose the restriction of $f(x,y)$ to the curve $g(x,y)=c$ has a local extreme at P_0. Take a parametrization $\vec{r}(t)$ of the curve with $\vec{r}(0)=\vec{OP_0}$

$$0=\frac{d}{dt}f(\vec{r}(t))\Big|_{t=0}=\vec{\nabla}f(P_0)\cdot\vec{v}\Big|_{t=0}\Rightarrow\vec{\nabla}f(P_0)\perp\vec{v}\big|_{t=0}$$



The curve should be bounded. and $\vec{\nabla}g\neq 0$

Remember $\vec{\nabla}g$ is perpendicular to the curve at every point.

$\underbrace{\vec{\nabla}g(P)\perp\vec{v}\big|_{t=a}}_{\text{at any point}}\Rightarrow\vec{\nabla}g(P_0)\parallel\vec{\nabla}f(P_0)$ f: function, g: constraint

$$\Rightarrow \left.\begin{matrix}\vec{\nabla}f=\lambda\cdot\vec{\nabla}g\\ g=c\end{matrix}\right\}$$ (★)

Solve for (x,y,λ) — Get rid of λ. — Find (x,y) → List of critical points.

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Ex) Find the closest and farthest points to the origin on the ellipse $x^2+xy+y^2=1$

$f(x,y) = (\text{distance from } (x,y) \text{ to } (0,0))^2$ → To make the computation easier.

$g(x,y) = x^2+xy+y^2$ → This is our constraint.

$$\left.\begin{array}{l}\vec{\nabla} f = \lambda \cdot \vec{\nabla} g \\ g = c\end{array}\right\} \Rightarrow \left.\begin{array}{l} f_x = \lambda \cdot g_x \\ f_y = \lambda \cdot g_y \\ g = c\end{array}\right\} \Rightarrow \begin{array}{ll} 2x = \lambda\cdot(2x+y) & (1) \\ 2y = \lambda\cdot(x+2y) & (2) \\ x^2+xy+y^2=1 & (3)\end{array}$$

(1) × (2) $\quad 2x\cdot\lambda(x+2y) = 2y\cdot\lambda(2x+y)$ → $\lambda = 0$ (***) ; → $2x^2+4xy = 4xy+4y^2$ → $x=-y$ (*) ; $x=y$ (**)

(3) and (**) $\quad x^2+x\cdot x+x^2=1 \quad 3x^2=1$ → $x=\frac{1}{\sqrt3}$; $x=\frac{-1}{\sqrt3}$

(3) and (*) $\quad x^2-x\cdot x+x^2=1 \quad x^2=1$ → $x=1$; $x=-1$

(x, y) solutions: $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right), \left(\frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}\right), (1,-1), (-1,1)$

(***) $\lambda = 0$ → $x=0$, $y=0$ ⇒ This creates contradiction

Since the ellipse is bounded; $(1,-1)$ and $(-1,1)$ are farthest, other two are the closest points.

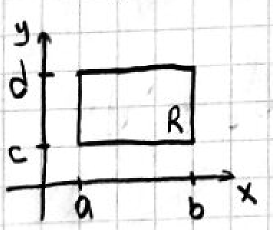
Ex) Find abs. max-min of $f(x,y,z) = 2x-y-2z$ on the sphere $x^2+y^2+z^2=4$

$$\left.\begin{array}{l}\vec{\nabla} f = \vec{\nabla} g\cdot\lambda \\ g = c\end{array}\right\} \Rightarrow \left.\begin{array}{l} f_x = \lambda\cdot g_x \\ f_y = \lambda\cdot g_y \\ f_z = \lambda\, g_z \\ g = c\end{array}\right\} \Rightarrow \left.\begin{array}{l} 2 = \lambda\cdot 2x \\ -1 = \lambda\, 2y \\ -2 = \lambda\, 2z \\ x^2+y^2+z^2=4\end{array}\right\} \begin{array}{l} \frac{1}{\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{\lambda^2}=4 \quad \lambda = \pm\frac{3}{4} \\ (x,y,z) = \left(\frac{4}{3}, \frac{-2}{3}, \frac{-4}{3}\right) \\ (x,y,z) = \left(\frac{-4}{3}, \frac{2}{3}, \frac{+4}{3}\right)\end{array}$$

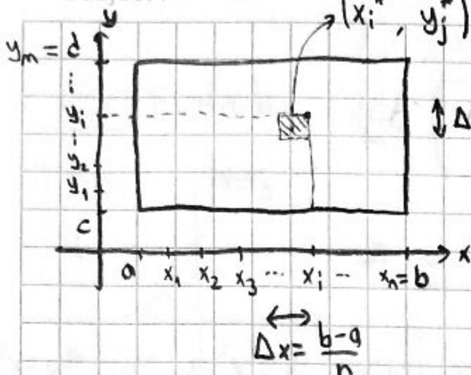
Since the sphere is bounded abs max is 6 and abs min is -6.

Multiple Integrals:

Double Integrals in Cartesian Coordinates: Suppose $f(x,y)$ is defined on a rectangle $R = \{(x,y): a \le x \le b \text{ and } c \le y \le d\}$. Then;



double integral of f over R: $\displaystyle\iint\limits_{\underbrace{R}_{\text{Region of integration}}} \underbrace{f(x,y)}_{\text{integrand}}\, \underbrace{dA}_{\text{Area Element}}$



$\Delta A = \Delta_x \cdot \Delta_y = $ (area of subrectangle)

$$\iint_R f(x,y)\, da = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta A$$

→ If the limit exists, we say that $f(x,y)$ is integrable over R.

1. __Fubini's Theorem:__ Suppose $f(x,y)$ is continuous on $R = \left\{(x,y): \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}\right\}$ Then;

$$\underbrace{\iint_R f(x,y)\, dA}_{\text{double integral}} = \underbrace{\int_c^d \left(\int_a^b f(x,y)\, dx \right) dy}_{\text{iterated integral}} = \underbrace{\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx}_{\text{iterated integral}}$$

⊛ $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA \quad (R = R_1 \cup R_2)$ ⊛ $\iint_R dA = \iint_R 1\, dA = $ (Area of R.)

__Ex]__ Evaluate $\iint_R (xy - y^2)\, dA$ where $R = \{(x,y): 1 \le x \le 2,\ 0 \le y \le 2\}$

$$= \int_0^2 \int_1^2 (xy - y^2)\, dx\, dy = \int_0^2 \left[\frac{x^2 y}{2} - x y^2 \right]_{x=1}^{x=2} dy = \int_0^2 \left(-y^2 + \frac{3y}{2} \right) dy = \left[\frac{-y^3}{3} + \frac{3y^2}{4} \right]_{y=0}^{y=2} = \frac{1}{3}$$

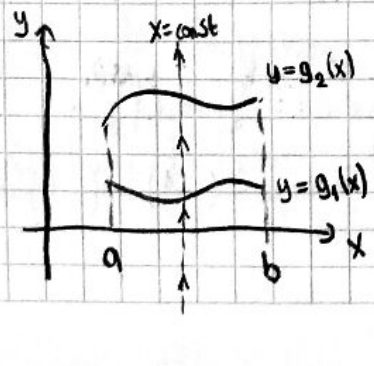
__Definition:__ Let R be a bounded region in the plane and $f(x,y)$ be defined on R. Take a rectangle R_0 containing R. Define;

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } R \\ 0 & \text{if } (x,y) \text{ not in } R \end{cases} \implies \iint_R f(x,y)\, dA = \iint_{R_0} F(x,y)\, dA$$

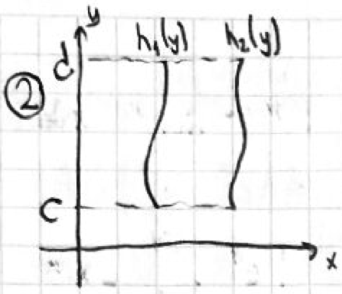
2. __Fubini's Theorem:__ ⇉ If you're integrating with y ($\int \ldots dy$), then you move along $x = $ const.

① Assume $g_1(x) \le g_2(x)$ on $[a,b]$, and $R = \{(x,y): a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Also assume that g_1, g_2 are continuous on $[a,b]$ and $f(x,y)$ is continuous on R.

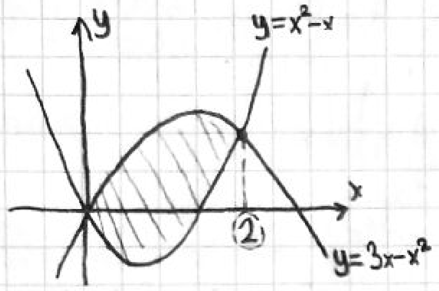


$$\iint_R f(x,y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$

② 

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$$

Ex | Evaluate $\iint_D x^2y\,dA$ where D is between the region $y=x^2-x$, $y=3x-x^2$

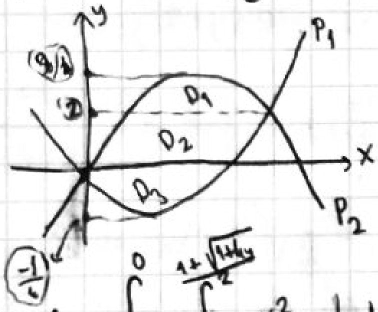


$$\int_0^2 \int_{x^2-x}^{3x-x^2} x^2y\,dy\,dx =$$

$$\int_0^2 \left[\frac{x^2y^2}{2}\right]_{y=x^2-x}^{y=3x-x^2} dx = \int_0^2 \frac{x^2}{2}\left[(x^4-6x^3+9x^2)-(x^4-2x^3+x^2)\right]dx$$

$$= \int_0^2 \left(\frac{x^2}{2}(-4x^3+8x^2)\right)dx = \int_0^2 -2x^5+4x^4 = \frac{-\cancel{2}x^6}{\cancel{6}3} + \frac{4x^5}{5}\Big|_0^2 = \frac{128}{5}-\frac{64}{3} = \frac{64}{15}$$

The other way of computation: $D = D_1 \cup D_2 \cup D_3$

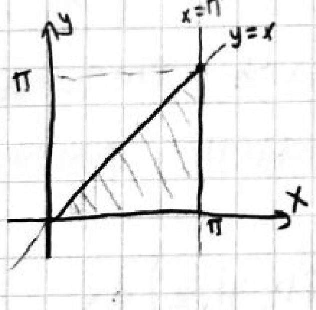


P_1: $x^2-x-y=0 \Rightarrow x=\frac{1\pm\sqrt{1+4y}}{2}$ → $x=\frac{1-\sqrt{1+4y}}{2}$ (Left side), → $x=\frac{1+\sqrt{1+4y}}{2}$ (Right side)

P_2: $x^2-3x+y=0 \Rightarrow x=\frac{3\pm\sqrt{9-4y}}{2}$ → $x = \dots$ (−) (Left), → $x = \dots$ (+) (Right)

$$A = \int_{-1/4}^{0} \int_{\frac{1-\sqrt{1+4y}}{2}}^{\frac{1+\sqrt{1+4y}}{2}} x^2y\,dx\,dy + \int_0^2 \int_{\frac{3-\sqrt{9-4y}}{2}}^{\frac{1+\sqrt{1+4y}}{2}} x^2y\,dx\,dy + \int_2^{9/4} \int_{\frac{3-\sqrt{9-4y}}{2}}^{\frac{3+\sqrt{9-4y}}{2}} x^2y\,dx\,dy$$

Ex) Evaluate $\iint_R \frac{\sin x}{x}\,dA$ where R is triangular region bounded by $y=x$, $x=\pi$, x-axis.



$$\iint_R \frac{\sin x}{x}\,dA = \begin{cases} \int_0^\pi \left(\int_y^\pi \frac{\sin x}{x}\,dx\right) dy & \text{Cannot be evaluated!} \\ \int_0^\pi \int_0^x \frac{\sin x}{x}\,dy\,dx = \end{cases}$$

$$= \int_0^\pi \left[\frac{\sin x}{x}y\right]_{y=0}^{y=x} dx = \int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi$$

$$= -(-1)-(-1) = 2$$

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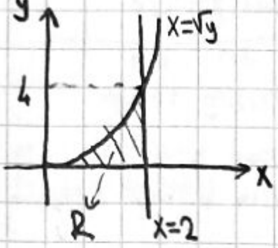
Ex| Evaluate $\int_0^4\int_{\sqrt{y}}^2 \frac{e^{y/x}}{x}\,dx\,dy$ → Cannot be expressed, Can be expressed

iterated $dx\,dy$ → double over R → iterated $dy\,dx$
* Changing the order of integration



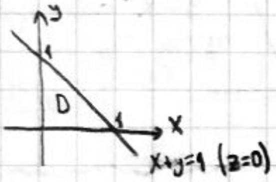
$$\Rightarrow = \iint_R \frac{e^{y/x}}{x}\,dA = \int_0^2\int_0^{x^2} \frac{e^{y/x}}{x}\,dy\,dx = \int_0^2 \left[e^{y/x}\right]_{y=0}^{y=x^2} dx = \int_0^2 e^x-1\,dx$$

$$= \left[e^x - x\right]_0^2 = e^2-2-1 = e^2-3$$

Exercise| Reverse the order of integration and compute if possible:

1) $\int_0^3\int_{\sqrt{y}}^2 \frac{e^{y/x}}{x}\,dx\,dy$ 2) $\int_0^5\int_{\sqrt{y}}^2 \frac{e^{y/x}}{x}\,dx\,dy = \iint_{R_1} - \iint_{R_2}$

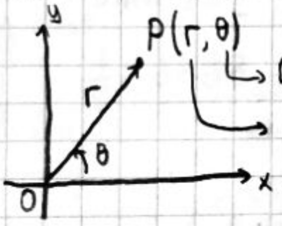
Ex| Find the volume of the solid cut from 1^{st} octand by the plane $x+y+z=1$



$$\iint_D (1-x-y)\,dA = \int_0^1\int_0^{1-y} (1-x-y)\,dx\,dy = \int_0^1 \left[(1-y)x - \frac{x^2}{2}\right]_{x=0}^{x=1-y} dy$$

$$= \int_0^1 (1-y)^2\cdot\frac{1}{2}\,dy = -\frac{1}{6}(1-y)^3\Big]_0^1 = 1/6$$

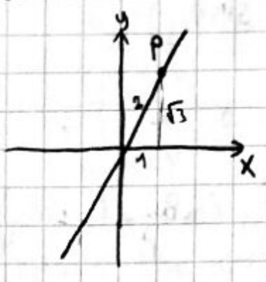
Polar Coordinates in the Plane:



θ → Directed angle between the positive x-axis and $\overrightarrow{OP}$

r → Directed distance from the origin to P.

Ex|



$P(2,\frac{\pi}{3}),\ P(2,\frac{\pi}{3}+2\pi),\ P(2,\frac{\pi}{3}+4\pi)\ \ldots$

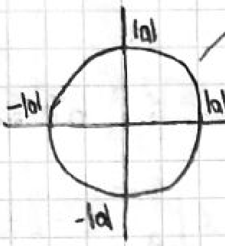
$P(2,\frac{-5\pi}{3}),\ P(2,\frac{-5\pi}{3}-2\pi),\ \ldots$

$P(-2,\frac{\pi}{3}+\pi),\ (-2,\frac{\pi}{3}+3\pi),\ \ldots \quad P(-2,\frac{\pi}{3}-\pi),\ \ldots$

Polar to Cartesian: $P(r,\theta) \longrightarrow \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}$

Cartesian to Polar: $P(x,y) \longrightarrow r^2 = x^2+y^2,\quad \tan\theta = \frac{y}{x}$

Coordinate Curves:

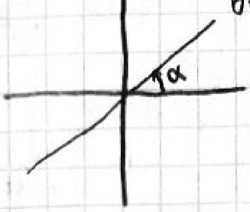


$r = a$

$\Downarrow$

$x^2+y^2=a^2$

* A circle with center origin and radius $|a|$



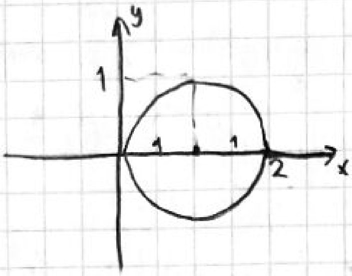
$\theta = \alpha \Rightarrow y = (\tan\alpha)\cdot x$

* A line passing through the origin and have an α angle with positive x-axis.

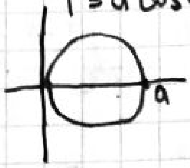
Ex $r = 2\cos\theta$

$r^2 = r \cdot 2\cos\theta$

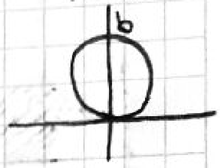
$x^2+y^2 = x\cdot 2 \Rightarrow (x-1)^2+y^2=1$

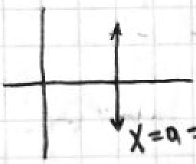


$r = a\cos\theta$

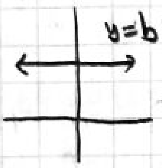


$r = b\sin\theta$

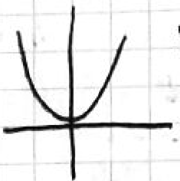




$x = a \Rightarrow r = a\sec\theta$

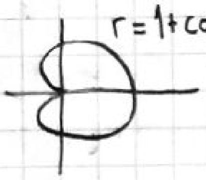


$y = b \Rightarrow r = b\csc\theta$

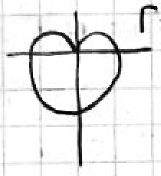


$y = x^2 \Rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$

$r = \dfrac{\sin\theta}{\cos^2\theta}$



$r = 1+\cos\theta$ (cardioid)



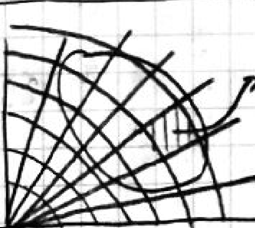
$r = 1-\sin\theta$

Remember the transformation

$x = r\cos\theta$

$y = r\sin\theta$

Double Integrals in Polar Coordinates:



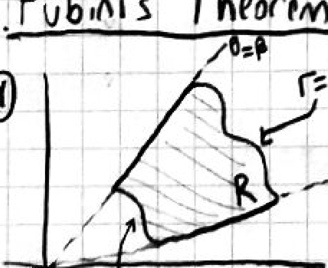
$(x,y) = (r_{ij}\cdot\cos\theta_{ij},\ r_{ij}\cdot\sin\theta_{ij})$

$\Delta A \approx \Delta r\,(r_{ij}\cdot\Delta\theta)$

$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to\infty\\ m\to\infty}} \sum_j \sum_i f(r_{ij}\cos\theta_{ij},\ r_{ij}\sin\theta_{ij})\cdot \overbrace{r_{ij}\cdot\Delta r\,\Delta\theta}^{\Delta A}$$

3. Fubini's Theorem:

1



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\overbrace{\underbrace{r}_{\text{Don't Forget}}\cdot\,dr\,d\theta}^{dA}$$

Note: $r \geq 0$ for integration.

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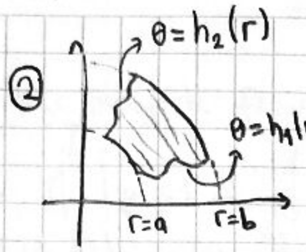
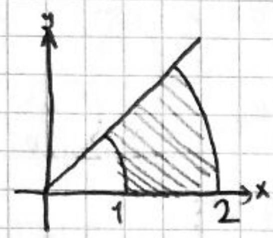
② 

$$\iint_D f(x,y)\,dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

Exercise: Try for dxdy, dydx

Ex] Evaluate $\iint_D \frac{xy}{x^2+y^2}$ where D is the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$, and the lines $y=x$ and the x axis in the 1st quadrant



$x = r\cos\theta$
$y = r\sin\theta$

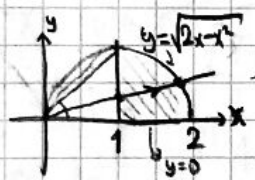
$$\int_0^{\pi/4}\int_1^2 \frac{r^2\sin\theta\cos\theta}{r^2}\cdot r\; dr\, d\theta =$$

$$= \int_0^{\pi/4}\left[\frac{\sin 2\theta}{2}\cdot\frac{r^2}{2}\right]_{r=1}^{r=2} d\theta = \int_0^{\pi/4} \frac{3\sin 2\theta}{4}\, d\theta = \frac{3}{4}\cdot\frac{(-\cos 2\theta)}{2}\Bigg]_0^{\pi/4} = \frac{3}{8}$$

Ex] $\displaystyle\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\, dy\, dx$ $\qquad \displaystyle\iint_A \frac{1}{(x^2+y^2)^2}\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} \frac{1}{(r^2)^2}\cdot r\, dr\, d\theta$

$\hookrightarrow x = r\cos\theta,\quad y = r\sin\theta$

$y=\sqrt{2x-x^2}$ $\downarrow$ $y=0$ $\qquad$ $y=2$ $\downarrow$ $y=1$



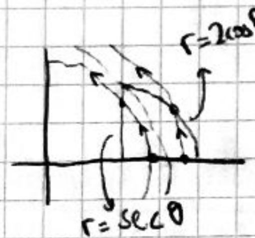
ray enters: $\boxed{r_1 = \sec\theta \quad (x=1)}$

ray exits: $\boxed{r_2 = 2\cos\theta \quad (y=\sqrt{2x-x^2})}$

$\boxed{\theta_1 = 0}$ → region begins

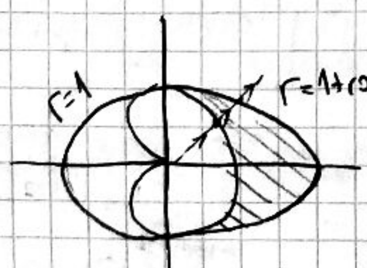
$\boxed{\theta_2 = \pi/4}$ → region ends

$$\int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^3}\, dr\, d\theta = \int_0^{\pi/4}\left[\frac{r^{-2}}{-2}\right]_{\sec\theta}^{2\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{-1}{4}\sec\theta + \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta = \frac{1}{2}\left[\frac{-1}{4}\tan\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} = \frac{\pi}{16}$$

$\boxed{*}$ 

$$\iint_A \frac{1}{r^4}\cdot r\, d\theta\, dr = \int_1^{\sqrt{2}}\int_0^{\operatorname{arcsec}(r)} \frac{1}{r^3}\, d\theta\, dr + \int_{\sqrt{2}}^{2}\int_0^{\arccos(\frac{r}{2})} \frac{1}{r^3}\, d\theta\, dr$$

Ex] Compute the area of the region lying inside cardioid $r = 1+\cos\theta$, and outside the unit circle $r=1$.



$$(\text{Area}) = \iint_A dA = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\, dr = \ldots = 2 + \frac{\pi}{4}$$

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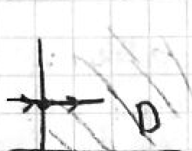
Subject :

<u>Ex</u> Let's compute the improper integral $I = \int_0^\infty e^{-x^2}\,dx$

$= I^2$

$= I \cdot I$



Consider D is the first quadrant.

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^\infty \left(\int_0^\infty e^{-x^2}\cdot e^{-y^2}\,dx\right)dy = \int_0^\infty e^{-y^2}\cdot\left(\int_0^\infty e^{-x^2}dx\right)dy = \left(\int_0^\infty e^{-x^2}dx\right)\cdot\left(\int_0^\infty e^{-y^2}dy\right)$$

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta \overset{*}{=} \int_0^{\pi/2} \frac{1}{2}\,d\theta = \boxed{\frac{\pi}{4}}$$

$*$: $\int_0^\infty e^{-r^2}\,r\,dr = \frac{1}{2}\cdot\int_0^\infty e^{-u}\,du = \frac{1}{2}$

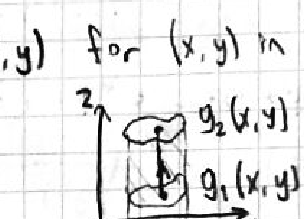
$u = r^2, \quad du = 2r\,dr$

$$I^2 = \frac{\pi}{4} \Rightarrow I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

<u>Triple Integrals In Cartesian Coordinates:</u>

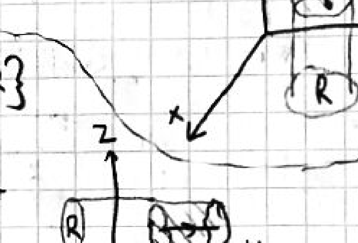
<u>Fubini's Theorem:</u> (1) $D = \{(x,y,z) : g_1(x,y) \leq z \leq g_2(x,y) \text{ for } (x,y) \text{ in } R\}$

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right) dA$$



(2) $D = \{(x,y,z) : h_1(x,z) \leq y \leq h_2(x,z) \text{ for } (x,z) \text{ in } R\}$

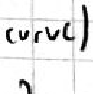
$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\right) dA$$



(3) Is the one with dx taken first...

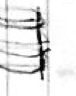
— Summary for Section 12.6 —

plane: $3x + 2y + z = 6$ sphere: $x^2 + y^2 + z^2 = 4$

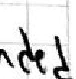


paraboloid: $z = x^2 + y^2$ (circlic level curve), $z = 4x^2 + y^2$ (eliptic level curve)

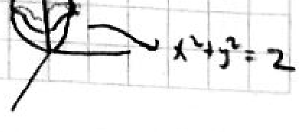
cone: $z^2 = x^2 + y^2$ (circlic level curve), $z^2 = 4x^2 + y^2$ (eliptic level curve)



cylinder: $x^2 + y^2 = 1$ (infinite circlic cylinder), $4x^2 + y^2 = 1$ (eliptic cylinder)



<u>Ex</u> Evaluate $\iiint_D z\,dV$ where D is the region in the space bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3x^2 - y^2$



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We integrate from the lower $(z=x^2+y^2)$ to upper $(z=4-3x^2-y^2)$ limit.

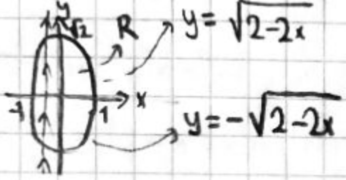
$$\iiint_D z\,dV = \iint_R \left(\int_{x^2+y^2}^{4-3x^2-y^2} z\,dz \right) dA$$

$\hookrightarrow$ dxdy or dydx

R = (projection of the D to the xy plane) (widest part of the D)

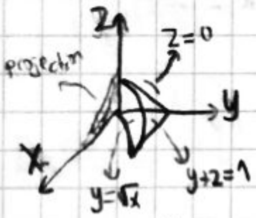
R's boundary is the projection of the intersection of $z=x^2+y^2$ and $z=4-3x^2-y^2$.

$\hookrightarrow$ $z=x^2+y^2=4-3x^2-y^2 \Rightarrow 4x^2+2y^2=4 \Rightarrow 2x^2+y^2=2$ is the boundary of R.



$$\int_{-1}^{1} \int_{-\sqrt{2-2x}}^{\sqrt{2-2x}} \int_{x^2+y^2}^{4-3x^2-y^2} z\,dz\,dy\,dx = \cdots = \frac{11\sqrt{2}}{3}\pi$$

Ex] Express $\iiint_D f(x,y,z)\,dV$ in terms of iterated integrals in all 6 orders of integration, where D is the region bounded at the top by the plane $y+z=1$, on the sides by the parabolic cylinder $y=\sqrt{x}$ and at the bottom by the xy plane. Top: $y+z=1$ Bottom: $z=0$ Sides: $y=\sqrt{x}$



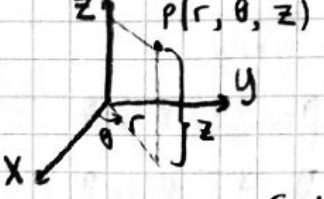
$$\iint_R \left(\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy \right) dA$$

$y=\sqrt{x}$
$y+z=1$
$\sqrt{x}+z=1$ → R's bound

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

Think on this graph after this point. $\boxed{\text{Exercise Above}}$

Triple Integrals in Cylindrical Coordinates:



Polar coordinates (r,θ) on xy-plane
Cartesian coordinates for z-dimension } (r,θ,z)

Cylindrical to Cartesian: $x=r\cos\theta$
$y=r\sin\theta$
$z=z$

Cartesian to cylindrical: $r^2=x^2+y^2$
$\tan\theta=\frac{x}{y}$
$z=z$

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Examples of Surfaces: $r=2$ (cylinder) $\theta=\frac{\pi}{3}$ (plane) $z=2$ (plane)

$r=2\sin\theta$ (cylinder on $(0,1,z)$ center) $x^2+y^2+z^2=4 \Rightarrow r^2+z^2=4$ (sphere)

$z=x^2+y^2 \Rightarrow z=r^2$ (paraboloid) $z^2=x^2+y^2 \Rightarrow z^2=r^2 \Rightarrow z=r$ or $z=-r$ (cone)

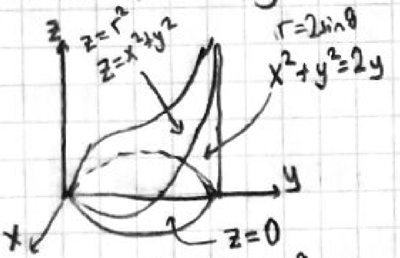
Fubini's Theorem: $\iiint_D f(x,y,z)\,dV = \int_*^*\int_*^*\int_*^* f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,dz\,dr\,d\theta}_{dV}$ $(r \geq 0)$

order changes

Ex| Evaluate $\iiint_D y\,dV$ where D is the region in space bounded on top by the paraboloid $z=x^2+y^2$ and at the bottom by xy-plane, and on the sides by the cylinder $x^2+y^2=2y$.

Exercise Try Cartesian

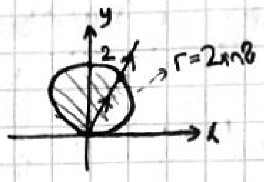
r,θ fixed $\Rightarrow x,y$ fixed



$$\iiint_D y\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} r\sin\theta\cdot r\,dz\,dr\,d\theta = 2\pi$$

switch to projection

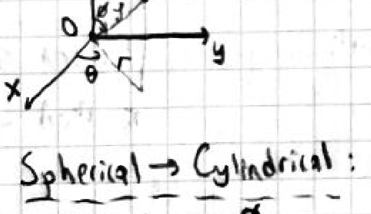
exercise



$$\langle\text{Extra}\rangle = \int_0^{\pi}\int_0^{4\sin^2\theta}\int_{\sqrt{z}}^{2\sin\theta} r\sin\theta\; r\,dr\,dz\,d\theta$$

$dr\,dz\,d\theta$

Triple Integrals in Spherical Coordinates:



ρ = distance between O and P $(\rho \geq 0)$

ϕ = angle between OP and z-axis. $(0 \leq \phi \leq \pi)$

θ = same with cylindirical coordinates $(-\infty < \theta < \infty)$

Spherical → Cylindrical:

$r=\rho\cdot\sin\phi$

$\theta=\theta$

$z=\rho\cdot\cos\phi$

Cylindrical → Spherical:

$\rho=\sqrt{r^2+z^2}$

$\tan\phi=\frac{r}{z}$ or $\cos\phi=\frac{z}{\sqrt{r^2+z^2}}$

$\theta=\theta$

Spherical → Cartesian

$x=r\cos\theta=\rho\sin\phi\cos\theta$

$y=r\sin\theta=\rho\sin\phi\sin\theta$

$z=z=\rho\cos\phi$

Cartesian → Spherical

$\rho=\sqrt{x^2+y^2+z^2}$

$\cos\phi=\frac{z}{\sqrt{x^2+y^2+z^2}}$ or $\tan\phi=\frac{\sqrt{x^2+y^2}}{z}$

$\tan\theta=\frac{x}{y}$

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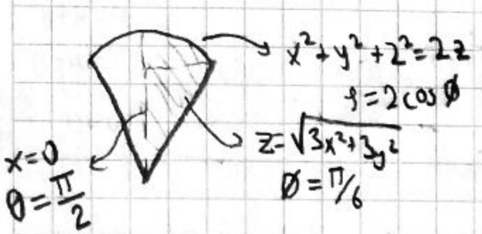
Examples of Surfaces: $\rho = 2$ (sphere)

$\phi = \frac{\pi}{6}$ half (cone)

$\theta = \frac{\pi}{3}$ (half plane)

Fubini's Theorem: $\iiint_D f(x,y,z) = \int_{*}^{*}\int_{*}^{*}\int_{*}^{*} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\ \overbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}^{dV}$

Ex Evaluate $\iiint_D (x^2+y^2+z^2)^2\, dV$ where D is the region above the cone $z=\sqrt{3x^2+3y^2}$ and inside the sphere $x^2+y^2+z^2=2z$, in the 1^{st} octand.



$$\int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\cos\phi} (\rho^2)^2\ (\rho^2\sin\phi\, d\rho\, d\phi\, d\theta) \overset{(ex)}{=} \frac{2\pi}{92}$$

Change of Variables: (Remember the 1-variable case,)

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du \quad \begin{pmatrix} x = g(u) \\ dx = g'(u)\,du \end{pmatrix}$$

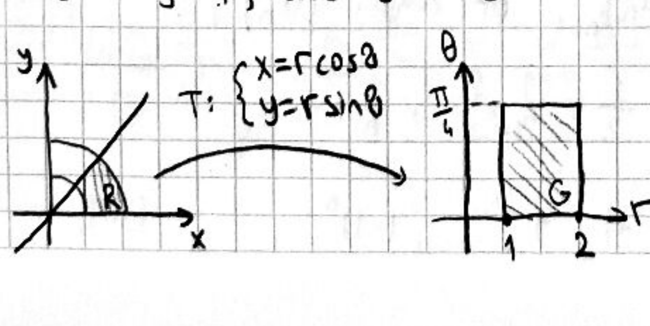
Definition: The Jacobian of the transformation T: $x=g(u,v)$, $y=h(u,v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u \cdot y_v - x_v \cdot y_u$$

(*) $$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Provided that $R=T(G)$ and T is 1-1 on interior of G.

Ex Evaluate $\iint_R \frac{xy}{x^2+y^2}\,dA$ where R is the region bounded by circles $x^2+y^2=1$ and $x^2+y^2=4$, and $y=x$ and x-axis on the 1^{st} quadrant.



$$\iint_G \frac{(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2} \cdot \left|\underbrace{\frac{\partial(x,y)}{\partial(r,\theta)}}_{|r|}\right|^{(*)} \cdot dr\,d\theta$$

$$(*) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cdots = r$$

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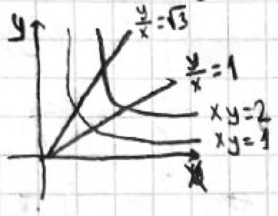
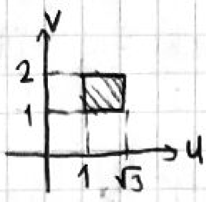
Ex| Evaluate $\iint_R \frac{1}{x^2+y^2}\,dA$ where R is bounded by hyperbolas $xy=1$ and $xy=2$ the lines $y=x$, $y=\sqrt{3}x$ in the 1st quadrant



$(*)$: $\begin{cases} u=\frac{y}{x} \\ v=xy \end{cases}$

T: $\begin{cases} x=\sqrt{\frac{v}{u}} \\ y=\sqrt{u\cdot v} \end{cases}$ $\dashrightarrow$ $f(x,y)=\frac{1}{\frac{v}{u}+uv}$



$$\int_1^2\int_1^{\sqrt{3}} \frac{1}{\frac{v}{u}+uv}\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv =$$

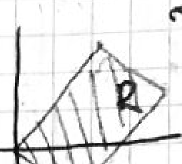
$\boxed{\text{Theorem}}^*$

$$=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{-1}{2u}$$

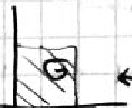
$$=\int_1^2\int_1^{\sqrt{3}} \frac{u}{v+u^2v}\cdot\frac{+1}{2u}\cdot du\,dv=\dots=\frac{\pi}{24}\cdot\ln 2$$

Exercise: Ex. 24 on page 1100. $\left(\iint_R (x+y)e^{x^2-y^2}\,dA\right)$

$u=x+y$
$v=x-y$



Answer: $\frac{e^4-7}{4}$



Theorem*: $\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

Remark: Triple integrals has a similar change of variable formula.

The Jacobian for triple integrals; $\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$

Exercise: Show the Jacobian for spherical transformation.

——— o ———

Sequences: A sequence is a function whose domain is the set of all integers, greater than or equal to some integer n_0.

Most of the time, $n_0=1$, so the domain is positive integers.

frequent notations: $f(x),\ g(t) \longrightarrow a_n,\ b_m$

n is called index, a_n is called the n'th term of the sequence.

Denoting the entire sequence $\{a_n\}_{n=1}^{\infty}$ or simply (a_n)

			Limit
Ex|	$a_n=\frac{1}{n}$, $n\geq 1$	$a_n=1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots$	0
	$b_n=(-1)^n$, $n\geq 0$	$b_n=1,-1,1,\dots,(-1)^n,\dots$	DNE

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		Limit
$c_n = \sqrt{n}$, $n \geq 1$	$c_n: 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots$	∞ (DNE)
$d_n = 5$, $n \geq 2$	$d_n: 5, 5, 5, \dots, 5, \dots$	5
$e_n = (-1)^n \frac{n+1}{n}$, $n \geq 1$	$e_n: -2, \frac{3}{2}, \frac{-4}{3}, \dots, (-1)^n \frac{n+1}{n}, \dots$	DNE
$f_n = \frac{3n-1}{2n}$, $n \geq 1$	$f_n: 1, \frac{5}{4}, \frac{4}{3}, \frac{11}{8}, \dots, \frac{3n-1}{2n}, \dots$	$\frac{3}{2}$
$g_n =$, $n \geq 1$	$g_n: 1, 2, 2, 3, 3, 3 \dots$	∞
$h_n = \frac{(-1)^{n-1}}{n}$, $n \geq 1$	$h_n: 1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \dots$	0

$x = \frac{n(n+1)}{2}$

$n = \frac{-1 \pm \sqrt{1+8x}}{2}$

$a_n = \left\lceil \frac{\sqrt{1+8n}-1}{2} \right\rceil$

not necessary: **Definition:** We say L is the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$, there is an integer N such; $n > N \Rightarrow |a_n - L| < \varepsilon$

If $\{a_n\}_{n=1}^{\infty}$ has a limit L, then we say that the sequence converges to L,

If $\{a_n\}_{n=1}^{\infty}$ hasn't got a limit, then we say that the sequence diverges.

Alternative notations; $\lim a_n$ or $a_n \to L$ as $n \to \infty$

Squeeze Theorem: Suppose $a_n \leq b_n \leq c_n$ for all sufficiently large n, and also suppose that $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = L$.

Ex Lets prove $\lim h_n = \lim \frac{(-1)^{n-1}}{n} = 0$ above. $-1 \leq (-1)^{n-1} \leq 1 \Rightarrow \underbrace{\frac{-1}{n}}_{\to 0} \leq \overbrace{\frac{(-1)^{n-1}}{n}}^{\text{By squeeze theorem, } 0} \leq \underbrace{\frac{1}{n}}_{\to 0}$

Ex $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ $\quad 0 < \frac{n!}{n^n} = \frac{1}{n} \cdot \underbrace{\frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n}}_{<1} < \overbrace{\frac{1}{n}}^{\to 0}$

Theorem (L'Hopital): Let $f(x)$ be defined on $[n_0, \infty)$ and $a_n = f_n$ for $n \geq n_0$. Then, if $\lim_{x\to\infty} f(x) = L \Longrightarrow \lim_{n\to\infty} a_n = L$.

Warning: The opposite is not true;

$f(x) = \sin(\pi x)$, $a_n = \sin(\pi n)$, $n \geq 1$

$\lim_{x\to\infty} f(x) = \text{DNE}$, $\lim_{n\to\infty} a_n = 0$

Ex $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

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<u>Theorem:</u> (Limits apply on pieces of cont. functions): Suppose $f(x)$ is continuous and $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} f(a_n) = f(L)$. Ex) $\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1$.

<u>Recursively Defined Sequences:</u> Ex) $\left.\underbrace{a_0=1}_{\text{initial condition}},\ \underbrace{a_n = n\cdot a_{n-1},\ \text{for } n\geq 1}_{\text{recursion relation}}\right\} a_n = n!$

Ex) $a_1=1$, $a_2=1$, $a_n = a_{n-1}+a_{n-2}$ is Fibonacci sequence.
This can be shown by testing three conditions (a_1, a_2, a_n) $\Bigg\}\ a_n = \dfrac{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n}{\sqrt5}$

Ex) $x_0 = 2$, $x_n = \dfrac{x_{n-1}}{2} + \dfrac{1}{x_{n-1}}$, $n \geq 1$ $\quad x_1 = \frac{3}{2} = 1.5$ $\quad x_2 = \frac{17}{12} = 1.417$ $\quad x_3 = 1.414215686\ldots$ $\quad x_4 = 1.414213562\ldots$ (Getting closer to $\sqrt2$)

Lets assume $L = \lim_{n\to\infty} x_n$ exists. Then:

$L = \frac{L}{2} + \frac{1}{L} \Rightarrow L^2 = 2 \Rightarrow L = \sqrt2$ or $\cancel{L=-\sqrt2}$ → Can't happen because $x_0 > 0$ and $x_n > 0$ for all $n \geq 1$. So $L \geq 0$ (Can be shown by Induction)

So we showed that if L is exists, then it should be $\sqrt2$.

<u>Def:</u> A sequene $\{a_n\}$ is increasing if $a_{n+1} > a_n$ for all n.

A sequence $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all n.

A sequence $\{a_n\}$ is Monotonic if its either increasing or decreasing.

A sequence $\{a_n\}$ is bounded above if there is a real number n such that $a_n \leq M$ for all n. M is called an upper bound. The smallest upper bound is called the least upper bound of $\{a_n\}$

A sequence $\{a_n\}$ is bounded below if there is a real number n such that $a_n \geq K$ for all n. K is called an lower bound The largest lower bound is called the greatest lower bound of $\{a_n\}$

$\{a_n\}$ is called bounded if its bounded both from above and below.

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Monotonic Sequence Theorem (MST):

→ If $\{a_n\}$ is increasing and bounded above, $\{a_n\}$ coverges to least upper bound.

→ If $\{a_n\}$ is decreasing and bounded below, $\{a_n\}$ converges to greatest lower bound.

Remark: Proof of MST uses "Completeness Axiom" which says: every non-empty subset of real numbers which is bounded above has a least upper bound.

⟨Exercise⟩ For the previous example $x_0 = 2$, $x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}$ for $n \geq 1$

① Show that $\{x_n\}$ is bounded below. → Induction shows > 0.

② Show that $\{x_n\}$ is decreasing. → Hard one. Solution on Sample Calc. Problems.

} use MST to conclude that $\{x_n\}$ converges.

Ex) $a_1 = 7$ $\quad a_n = \frac{a_{n-1}+1}{4}$ for $n \geq 2$.

We are going to show that $\{a_n\}$ is bounded below by 0 and decreasing.

Claim $a_n > 0$ for all $n \geq 1$.

If $n=1$, $a_1 = 7 > 0$

Assume "$a_k > 0$" for some k:

"$a_k + 1 > 1$" $\Rightarrow \underbrace{\frac{a_k + 1}{4}}_{} > \frac{1}{4}$

$a_{k+1} > \frac{1}{4} > 0$

We conclude that by induction, $a_n > 0$ for all $n \geq 1$

Claim $a_{n+1} < a_n$ for all $n \geq 1$

If $n=1$, $a_2 = 2 < 7 = a_1$

Assume $a_{k+1} < a_k$ for some k: (Then we want to show $a_{k+2} < a_{k+1}$)

$\frac{a_{k+1}+1}{4} < \frac{a_k + 1}{4} \Rightarrow a_{k+2} < a_{k+1}$

We conclude that by induction, $a_{n+1} < a_n$ for all $n \geq 1$

Since $\{a_n\}$ is bounded below by 0 and decreasing, by monotonic sequence theorem, $\{a_n\}$ converges.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L \Rightarrow L = \frac{L+1}{4} \Rightarrow L = 1/3$

⟨Exercise⟩ What would happen if $a_1 = 1$ was the case.

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Infinite Series: Definition: Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, the expression $a_1+a_2+\dots+a_n+\dots = \sum_{n=1}^{\infty} a_n$ is called a series. (infinite series)

The sequence of partial sums of the series are defined by:

$S_1 = a_1$

$S_2 = a_1 + a_2$ .... $S_n = \sum_{i=1}^{n} a_i$, $S_1 = a_1$

$S_n = S_{n-1} + a_n$ for $n \geq 2$

S_n is called n^{th} partial sum. If the sequence $\{S_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} S_n = S$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converge and its sum is S. If it diverge, then the series diverges as well.

Geometric Series: $a \neq 0$, r is real. $a + ar + ar^2 \dots ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$ is called the geometric series with common ratio r.

Suppose $r \neq 1$. Then,

$$S_n = a + ar + ar^2 + \dots ar^{n-2} + ar^{n-1}$$

$$- \quad r \cdot S_n = \quad ar + ar^2 + \dots ar^{n-2} + ar^{n-1} + ar^n$$

$$S_n - r\cdot S_n = a - ar^n \implies S_n = a \cdot \frac{1-r^n}{1-r} \quad \text{for } n \geq 1$$

$$\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \begin{cases} \dfrac{a}{1-r} & , |r|<1 \\ \text{diverges} & , |r| \geq 1 \end{cases}}$$

Then we take $\lim_{n\to\infty}$ for S_n to find sum.

Ex $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{2}{1-\frac{1}{2}} = 2$ $\boxed{a=1,\ r=\frac{1}{2}}$ $|r|<1$

Ex $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \dots \frac{1}{3}\cdot\left(\frac{-2}{3}\right)^{n-1} + \dots = \sum_{n=1}^{\infty} \frac{1}{3}\cdot\left(\frac{-2}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\left(\frac{-2}{3}\right)} = \frac{1}{5}$ $a = \frac{1}{3}$, $r = \frac{-2}{3}$ $|r|<1$

Ex $1 - 2 + 4 - 8 + 16 - 32 \dots = \sum_{n=1}^{\infty} (-2)^{n-1} =$ $a=1$, $r=(-2)$ $|r| \geq 1 \implies$ diverges

Ex $1 - 1 + 1 - 1 + 1 \dots = \sum_{n=1}^{\infty} (-1)^{n-1} =$ $a=1$, $r=-1$ $|r| \geq 1 \implies$ diverges by definition.

S_n: 1, 0, 1, 0 .... $\lim_{n\to\infty} S_n = \text{DNE} \implies$ diverge by definition.

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Remark: We can't do infinitely many cancelations in an infinite series:

$$\cancel{1}-\cancel{1}+\cancel{1}-\cancel{1}+\cancel{1}\cdots=0 \quad , \quad 1-\cancel{1}+\cancel{1}-\cancel{1}+\cancel{1}\cdots=1 \Rightarrow 0=1$$

$a=\frac{12}{100}, r=\frac{1}{100}$

Ex] $0.121212\ldots = 0.\overline{12} = \frac{12}{100}+\frac{12}{(100)^2}+\frac{12}{(100)^3}\cdots = \sum_{n=1}^{\infty}\frac{12}{100}\cdot\left(\frac{1}{100}\right)^{n-1} = \frac{\frac{12}{100}}{1-\frac{1}{100}} = \frac{12}{99}$

Telescoping Series: **Ex]** $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}\cdots$

$$S_n=\sum_{i=1}^{n}\frac{1}{i(i+1)}=\sum_{i=1}^{n}\frac{1}{i}-\frac{1}{i+1}=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$S_n=1-\frac{1}{n+1}, \ n\geq 1 \Rightarrow \lim_{n\to\infty} 1-\frac{1}{n+1}=1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

Remark: We still can't do infinitely many cancelations. We're considering partial sums.

⟨Exercises⟩ $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ $\quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

Harmonic Series: $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$

Observation: If $a_n\geq 0$ for all $n\geq 1$, then $\sum_{n=1}^{\infty}$ either converges or ∞.

$$1+\underbrace{\frac{1}{2}}_{1/2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{1/2}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{1/2}+\frac{1}{9}\cdots \qquad S_{2^n}\geq 1+\frac{n}{2}\Rightarrow \lim_{n\to\infty}S_{2^n}=\infty$$

(because all $S_n>0$) $\Rightarrow \lim_{n\to\infty}S_n=\infty$

— o —

Ⓣ **n^{th} Term Test for Divergence:** If $\sum_{n=1}^{\infty}a_n$ converges, then $\lim_{n\to\infty}a_n=0$.

Equivalently; If $\lim_{n\to\infty}a_n\neq 0$, then $\sum_{n=1}^{\infty}a_n$ diverges.

Note that the opposite is not true (harmonic series for example)

Ex] $\sum_{n=1}^{\infty}\frac{n}{2n+1}$ diverges by nTT because $\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}\neq 0$

Ex] $\sum_{n=1}^{\infty}(-1)^{n-1}$ diverges by nTT because $\lim_{n\to\infty}(-1)^{n-1}\neq 0$

★ A few remarks: $\sum_{n=1}^{\infty}a_n=\sum_{n=1+k}^{\infty}a_{n-k}$, $\sum_{n=1}^{\infty}2^n=\sum_{n=2}^{\infty}2^{n-1}=\sum_{n=0}^{\infty}2^{n+1}$

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(T) Integral Test (IT): Let N be an integer and suppose $f(x)$ is continuous, positive and decreasing on $[N, \infty)$. Let $a_n = f(n)$ for $n \geq N$,

$\sum_{n=N}^{\infty} a_n$ converges $\iff \int_N^{\infty} f(x)dx$ converges | $\sum_{n=N}^{\infty} a_n$ diverges $\iff \int_N^{\infty} f(x)dx$ diverges

By geometry: $\int_1^{n+1} f(x)dx < S_n$, $S_n < a_1 + \int_1^n f(x)dx$

Ex | Determine the following series converges or diverges.

1) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ $f(x) = \frac{\ln x}{x}$ $\begin{cases} \text{continuous? is continuous for } x \geq 0 \\ \text{positive? } \frac{\ln x}{x} > 0 \text{ for } x > 1 \checkmark \\ \text{decreasing? } f'(x) = \frac{1 - \ln x}{x^2} < 0 \text{ for } x \geq e \checkmark \text{ (good enough)} \end{cases}$

$u = \ln x$, $du = \frac{dx}{x}$

$\int_2^{\infty} \frac{\ln x}{x} dx = \int_{\ln 2}^{\infty} u\, du = \infty \implies \sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ is divergent by IT.

2) $\sum_{n=1}^{\infty} \frac{1}{n(1 + (\ln n)^2)}$ $f(x) = \begin{cases} \text{continuous? continuous for } x \geq 0 \checkmark \\ \text{positive? positive for } x \geq 0 \checkmark \\ \text{decreasing? decreasing for } x \geq 1 \checkmark \end{cases}$ $\frac{1}{\underbrace{x}_{\text{inc}}\underbrace{(1 + (\ln x)^2)}_{\text{inc}}}$ dec.

$\int_1^{\infty} \frac{1}{x(1 + (\ln x)^2)} dx = \int_{\ln 1}^{\infty} \frac{1}{1 + u^2} = \frac{\pi}{2} \implies \sum_{n=1}^{\infty} a_n$ is convergent by IT.

(★) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p is a positive constant)

$f(x) = \frac{1}{x^p}$ $\begin{cases} \text{continuous for } x > 0 \\ \text{positive for } x > 0 \\ \text{decreasing for } x > 0 \text{ (because } p > 0) \end{cases}$

$$\boxed{\int_1^{\infty} \frac{dx}{x^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}}$$

$\Downarrow$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}}$$

Note that $p = 1$ gives the harmonic series.

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$$\int_1^{n+1} f(x)\,dx < a_1+a_2+\cdots+a_n < a_1+\int_1^n f(x)\,dx$$

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\int_1^n \frac{1}{x}\,dx = 1+\ln(n)$$

So $s_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln(n)$. To get $s_n \approx 10$, we need $\ln(n)=10 \Rightarrow n=e^{10} \approx 22000$!

Ⓣ **Direct Comparasion Test (DCT)**: Suppose $0 \leqslant a_n \leqslant b_n$ for all large n;

If $\sum b_n$ converges, then $\sum a_n$ converges.

If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Use these: $\sum_{n=1}^{\infty} ar^{n-1}$ Geometric series; $\sum_{n=1}^{\infty} \frac{1}{n^p}$ p-series

Ex| Determine the following series are convergent or divergent.

1) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ $\quad \ln(n) \geqslant 1$ for $n \geqslant 3 \Rightarrow \frac{\ln(n)}{n} \geqslant \frac{1}{n} \geqslant 0$ for $n \geqslant 3$.
$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$\Rightarrow \sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges by DCT.

2) $\sum_{n=0}^{\infty} \frac{1}{n!}$ $\quad \underbrace{n! \geqslant 2^n}_{\text{Show by Induction.}}$ for $n \geqslant 4$ $\Rightarrow 0 \leqslant \frac{1}{n!} \leqslant \frac{1}{2^n}$ for $n \geqslant 4$

2^n → n^2 would also work.

$\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges by $r=\frac{1}{2}$, $|r|=\frac{1}{2}<1$

$\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by DCT.

Ⓣ **Limit Comparasion Test (LCT)**: Suppose that $\sum a_n$ and $\sum b_n$ have positive terms for large n. Suppose the limit $c = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists or ∞.

① If $0<c<\infty$, $\sum a_n$ and $\sum b_n$ both converge or both diverge.

② If $c=0$, If $\sum b_n$ converges, $\sum a_n$ converges.
If $\sum a_n$ diverges, $\sum b_n$ diverges

③ If $c=\infty$ If $\sum a_n$ converges, $\sum b_n$ converges.
If $\sum b_n$ diverges, $\sum a_n$ diverges.

Use lopital to show $\frac{n^2}{2^n} \to 0$ as $n\to\infty$

Ex) 1) $\sum_{n=5}^{\infty} \frac{1}{2^n-n^2}$ $\quad C=\lim_{n\to\infty}\dfrac{\frac{1}{2^n-n^2}}{\frac{1}{2^n}} = \lim_{n\to\infty}\dfrac{1}{1-\frac{n^2}{2^n}} \overset{!}{=} \frac{1}{1}=1$

and $\sum_{n=5}^{\infty} \frac{1}{2^n}$ converges ($\overset{1/2}{|r|}<1$) $\Rightarrow \sum_{n=5}^{\infty}\frac{1}{2^n-n^2}$ converge by LCT.

2) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{3/2}}$ $\quad C=\lim_{n\to\infty}\dfrac{\frac{\ln(n)}{n^{3/2}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty}\frac{\ln(n)}{n} \overset{LH}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0$ $\quad \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges, $p=\frac{1}{2}<0$
Does not say anything !!

$C=\lim_{n\to\infty}\dfrac{\frac{\ln(n)}{n^{3/2}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\ln(n)=\infty$ $\quad \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, Does not say anything !!

We should put $\frac{1}{n^p}$ with some p, bigger than $\overset{(\text{actually } 1)}{\frac{1}{2}}$, smaller than $\frac{3}{2}$. $1<p<\frac{3}{2}$,

$C=\lim_{n\to\infty}\dfrac{\frac{\ln(n)}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty}\frac{\ln(n)}{n^{1/4}}=0$, $\quad \sum_{n=2}^{\infty}\frac{1}{n^{5/4}}$ (converges) $p=\frac{5}{4}>1$ $\Rightarrow \sum_{n=2}^{\infty}\frac{\ln(n)}{n^{3/2}}$ converges as well.

nTT → Apply any series
IT, DCT, LCT → Apply positive series

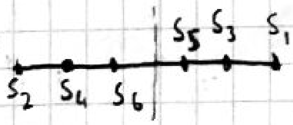
Alternating Series: Terms are alternating positive and negative.

Ex) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots$ (Alternating harmonic series)

(T) **Alternating Series Test (AST):** Suppose the series $\sum_{n=1}^{\infty}\underbrace{(-1)^{n-1}b_n}_{a_n} = b_1-b_2+b_3-b_4+\cdots$

$$\left.\begin{array}{l} ⓪\ |a_n|=b_n>0 \quad ①\ b_{n+1}\le b_n \text{ for all } n \\ \qquad\qquad\qquad\quad ②\ \lim_{n\to\infty} b_n=0 \end{array}\right\} \Rightarrow \text{The series } \sum_{n=1}^{\infty}(-1)^{n-1}b_n \text{ converges.}$$

Proof:



$S_2\le S_4\le S_6\le\cdots\le S_5\le S_3\le S_1$

By MST, $\lim_{n\to\infty}S_{2n}=L_2$, $\lim_{n\to\infty}S_{2n-1}=L_1$ exists.

$\lim_{n\to\infty}(S_{2n}-S_{2n-1})=\lim_{n\to\infty}(a_{2n})=\lim_{n\to\infty}(-b_{2n})=0 \Rightarrow L_1=L_2=L \Rightarrow \lim_{n\to\infty}S_n=L$

Alternating Series Estimate (ASE): Suppose we have the above series,

$S=\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, then: $|S-s_n|<b_{n+1}$ $\qquad$ $(S-s_n)$ has same sign with b_{n+1}

first unused term $\qquad$ first unused term

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⊛ **Ex** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ ($p>0$, constant) (Alternating p-series)

Aply AST: 0) $b_n = \frac{1}{n^p} > 0$ for $n \geq 1$

1) $n+1 < n \Rightarrow 0 < \frac{1}{n+1} < \frac{1}{n} \xRightarrow{(p>0)} 0 < \frac{1}{(n+1)^p} < \frac{1}{n^p} \Rightarrow b_{n+1} < b_n$ for $n \geq 1$

2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = 0 \dashrightarrow (p>0)$

So by AST, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges for all $p>0$.

Ex In particular $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (alternating harmonic series) converges. (Actually to $\ln 2$)

We can use ASE here; $\left.\begin{aligned} s_{10} &= 1 - \frac{1}{2} + \frac{1}{3} \cdots - \frac{1}{10} \approx 0.646 \\ s_{11} &= 1 - \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{11} \approx 0.737 \end{aligned}\right\} 0.64 < \ln 2 < 0.74$

Ex ① $\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \cdot \underbrace{(\sqrt{n^2+n} - n)}_{b_n}}_{a_n}$ $\emptyset$. $n>0 \Rightarrow n^2+n > n^2 \Rightarrow \sqrt{n^2+n} > \sqrt{n^2} \Rightarrow b_n > 0$ ✓

2. (Check 2 first because if $\lim_{n\to\infty} b_n \neq 0$ then $\lim_{n\to\infty} a_n \neq 0$ so series diverges by nTT.

2. $\lim_{n\to\infty} \frac{\sqrt{n^2+n} - n}{(\sqrt{n^2+n}+n)} = \lim_{n\to\infty} \frac{(n^2+n)-(n^2)}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1} = 1/2 \neq 0$. So series diverges by nTT.

② $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot (\sqrt{n^2+1} - n)$ $\emptyset$. $n>0 \Rightarrow n^2+1 > n^2 \Rightarrow \sqrt{n^2+1} > \sqrt{n^2} \Rightarrow b_n > 0$ ✓

2. $\lim_{n\to\infty} \frac{\sqrt{n^2+1} - n}{(\sqrt{n^2+1}+n)} = \lim_{n\to\infty} \frac{\cancel{(n^2+1)-(n^2)}}{\underbrace{\sqrt{n^2+1}+n}_{\infty}} = 0$ ✓ 1. Show; $b_{n+1} \leq b_n$ or $f(x)$ is decreasing:

Let $f(x) = \sqrt{x^2+1} - x$ $f'(x) = \frac{x}{\sqrt{x^2+1}} - 1 = \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}} < 0$ so 2 is satisfied.

So the series converge by AST.

Ⓣ <u>Absolute Convergence Test (ACT):</u> If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well. Proof: $0 \leq a_n + |a_n| \leq 2 \cdot |a_n|$ and use DCT.

Proof: $\left.\begin{array}{l} 0 \leqslant a_n + |a_n| \leqslant 2\cdot|a_n| \quad \text{for all } n \geqslant 0 \\ \sum_{n=1}^{\infty} 2\cdot|a_n| \text{ converges.} \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty}(a_n+|a_n|)$ converges by DCT.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(\underbrace{(a_n+|a_n|)}_{\text{converges}} - \underbrace{|a_n|}_{\text{converges}}) \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

Ex $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ consider, $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| \Rightarrow 0 \leqslant |\sin n| \leqslant 1 \Rightarrow 0 \leqslant \frac{|\sin n|}{n^2} \leqslant \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, p serie $p>1$ $\Rightarrow \sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$ converges by DCT $\Rightarrow \sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges by ACT.

Definition: $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty}|a_n|$ is convergent.

If $\sum_{n=1}^{\infty}|a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Ex $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ absolutely converges, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ conditionally converges.

Ex $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ $\begin{cases} \text{converges absolutely if } p>1 \\ \text{converges conditionally if } p \leqslant 1 \end{cases}$

Definition: A sequence $\{b_n\}_{n=1}^{\infty}$ is a rearrangement of $\{a_n\}_{n=1}^{\infty}$ if it contains exactly the same terms but possibly in a different order.

Rearrangement Theorem: ① If $\sum_{n=1}^{\infty} a_n$ converge absolutely and $\{b_n\}$ is a rearrangement of $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

② Suppose $\sum_{n=1}^{\infty} a_n$ converges conditionally and let M be any real number or $\pm\infty$, then there is a rearrangement $\{b_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} b_n = M$.

↳ $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$ is conditionally convergent. We want to get π;

add until $>\pi$: $S' = 1 + \frac{1}{3} + \frac{1}{5} \cdots = \infty$

add until $<\pi$: $S'' = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} \cdots = -\infty$

$\underbrace{1 + \frac{1}{3} + \frac{1}{5} \cdots \frac{1}{x}}_{>\pi} \underbrace{- \frac{1}{2}}_{<\pi} \underbrace{+ \frac{1}{x+2} + \cdots}_{>\pi}$

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Ⓣ Ratio Test (RT):

If $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists or ∞,

Ⓣ nth Root Test (nRT):

If $L = \lim_{n\to\infty} |a_n|^{1/n}$ exists or ∞,

nTT } applies all series
IT, DCT, LCT } applies series with positive terms
AST } applies alternating series
ACT, RT, nRT } applies all series

1) If $L<1$, then the series $\sum^{\infty} a_n$ converges absolutely.

2) If $L>1$, then the series $\sum^{\infty} a_n$ diverges.

3) If $L=1$, then this test doesn't give an answer.

Ex ① $\sum_{n=0}^{\infty} \frac{1}{n!}$ $\quad \frac{\left|\frac{1}{(n+1)!}\right|}{\left|\frac{1}{n!}\right|} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$ $\quad L = \lim_{n\to\infty} \frac{1}{n+1} = 0$ $\quad \boxed{L=0<1}$ then $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges absolutely by RT.

$\hookrightarrow (n+1)\cdot n!$

② $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ $\quad \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \left(\frac{n+1}{n}\right)^2$ $\quad L = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}$ $\quad L = \frac{1}{2} < 1$ series converges by RT

⇨ In exam solution, write $\lim_{n\to\infty}$ for everything. Write $a_n = \cdots$ and write this marked ones.

✪ Solve all exercises in Section 11.7. ✓

③ $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}$ $\quad = \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \frac{\cancel{(n!)^2}}{(n+1)^2 \cdot \cancel{(n!)^2}} = \frac{2(2n+1)}{n+1}$ $\quad L = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$ then the series diverges by RT.

④ $\sum_{n=0}^{\infty} \frac{n!}{n^n}$ $\quad = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{\cancel{(n+1)\cdot n!}}{\cancel{n!}} \cdot \frac{n^n}{(n+1)^{\cancel{n+1}}} = \left(\frac{n}{n+1}\right)^n$ $\quad L = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n+1}\right)} = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{e}$

Useful limits: $\left(1-\frac{1}{n+1}\right)^{n+1} \to e^{-1}$, $\left(1-\frac{1}{n+1}\right) \to 1$

converges by RT $\leftarrow L = \frac{1}{e} < 1$

Remark: RT does not work for $\frac{1}{n}$ or $\frac{1}{n^2}$ $(L=1)$

Ex ① $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ $\quad L = \lim_{n\to\infty} |a_n|^{1/n} = \frac{n^{2/n}}{2^{n/n}} = \frac{n^{\frac{2}{n}}}{2} = \frac{1}{2} < 1 \Rightarrow$ converges by nRT

$n^{\frac{2}{n}} \to 1^2$ → Useful limits

Remark: If L_{RT} and L_{nRT} both exist or both infinity, then $L_{RT} = L_{nRT}$

$\hookrightarrow$ So if $L_{RT} = 1$, then don't even try L_{nRT} since it will be 1.

② $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ $L = e > 1$ Series diverges by nRT.

<u>Power Series:</u> A power series centered at a is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots c_n(x-a)^n \dots$

x is a variable and a; $c_0, c_1, \dots$ are constants.
a is the center of the power series and c_n is the n^{th} coefficient.

<u>Remark:</u> By convention, $(x-a)^0 = 1$ for all x in a power series.

<u>Remark:</u> If $a=0$, then $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 \dots c_n x^n$

<u>Examples</u> ① $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$ is a geometric series $a=1$, $r=x$ $\begin{cases} \text{converges absolutely, } |x|<1 \\ \text{diverges } |x| \geq 1 \end{cases}$

Moreover, $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$ -1 0 1 coefficient with abs() constant

② $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} \dots$ Apply ratio test $a_n = \frac{x^n}{n!}$, $L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$

$L = 0 < 1$, $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges absolutely for all x, by RT. 0

③ $\sum_{n=0}^{\infty} n!\cdot x^n = 1 + 2!x^2 + 3!x^3 \dots$ $a_n = n!x^n$ $L = \lim_{n\to\infty}\frac{|(n+1)!\cdot x^n|}{|n!\cdot x^n|} = \lim_{n\to\infty}(n+1)\cdot|x| = \infty$

$L = \infty > 1$, $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ diverges for $x \neq 0$, by RT. 0

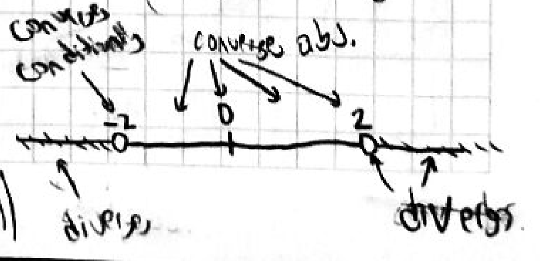
④ $\sum_{n=1}^{\infty}\frac{x^n}{2^n\cdot n}$ $a_n = \frac{x^n}{2^n\cdot n}$ $L = \lim_{n\to\infty}\frac{\left|\frac{x^{n+1}}{2^{n+1}(n+1)}\right|}{\left|\frac{x^n}{2^n\cdot n}\right|} = \lim_{n\to\infty}\frac{|x|}{2}\cdot\frac{n}{n+1} = \frac{|x|}{2}$

If $L = \frac{|x|}{2} < 1$, $|x|<2$, the series converges absolutely for all $|x|<2$ by RT

If $L = \frac{|x|}{2} > 1$, $|x|>2$, the series diverges for all $|x|>2$ by RT

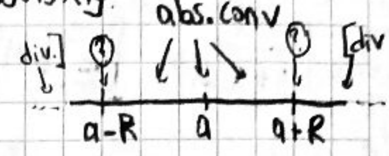
If $x=2$, the series diverges since its harmonic.

If $x=-2$ the series converge by AST. conditionally $(I = [-2, 2))$



⊖ Don't try ratio test on endpoints below since it gives $L=1$ and $\sqrt[n]{}$RT

Theorem: Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there is a R, $0 \leq R \leq \infty$,
- → If $|x-a| < R$, then the series converges absolutely.
- → If $|x-a| > R$, then the series diverges.



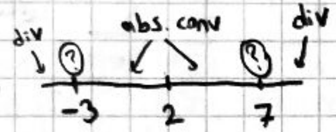
(★) $\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$, $\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}$ [radius of convergence]

if either of those limits exists. [The interval I of x which the series converges is called interval of convergence]

Ex) Find the interval of convergence I of the power series and determine the type of convergence at each x in I.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{5^n \cdot \sqrt{n^2+1}} \qquad \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} \quad \left(c_n = \frac{(-1)^n}{5^n \cdot \sqrt{n^2+1}} \right) \quad \frac{1}{R} = \lim_{n\to\infty} \frac{|\cdots|}{|\cdots|} = \frac{1}{5}$$

$R = 5 \Rightarrow$



< Suggestion: First look at the endpoint which makes might all terms positive (or all negative.) > If it converges you can use ACT for the other.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-3-2)^n}{5^n \sqrt{n^2+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}} \xrightarrow{\text{use LCT}} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^2}{n^2+1}\right)^{1/2} = 1$$

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges

So the series diverges for $x=-3$ by LCT.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(7-2)^n}{5^n \sqrt{n^2+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$$

$b_n = \frac{1}{\sqrt{n^2+1}}$: $b_n > 0$, b_n is decreasing for $n > 0$, $\lim_{n\to\infty} b_n = 0$

The series for $x=7$ converges by AST. (conditionally)

$I = (-3, 7]$, abs. conv on $(-3, 7)$ and cond. conv on $\{7\}$.

Differentiation / Integration Theorem: Suppose $\sum_{n=0}^{\infty} c_n (x-a)^n$ has interval of convergence I and radius of convergence $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$.

$$f'(x) = \sum_{n=1}^{\infty} c_n \, n \cdot (x-a)^{n-1} \quad \text{and} \quad \int f(x)\,dx = \sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1} + C$$

for all x in I.

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Ex ① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ $\xRightarrow{x \to -x}$ $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$ for $|x|<1$ $\xRightarrow{\int}$

$\int \frac{dx}{1+x} = \int \sum_{n=0}^{\infty} (-1)^n \cdot x^n \Rightarrow \underbrace{\ln|1+x|}_{x=0 \Rightarrow \ln 1 = 0 + C \Rightarrow C=0} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$ for $|x|<1$

$|x|<1 \Rightarrow \ln|1+x| = \ln(1+x)$

$n+1 \to n$

$\to \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for $-1 < x \leq 1$ → It can be shown, ("useful Power Series")

② $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ $\xRightarrow{x \to -x^2}$ $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|-x^2|<1$, $|x|<1$ $\xRightarrow{\int}$

$\int \frac{dx}{1+x^2} = \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$ $\xrightarrow{x=0}$ $\arctan 0 = 0 + C$, $C = 0$ for $|x|<1$

$\to \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| \leq 1$ → It can be shown separately.

③ Let $f(x) = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$ for all x. $\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$ for all x

$y = f(x)$ $\quad \frac{dy}{dx} = x \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + C \Rightarrow |y| = e^x \cdot \overset{e^c}{A}$ for all x.

Take $x=0 \Rightarrow f(0) = 1 = e^0 \cdot A \Rightarrow A = 1$ $\xRightarrow[\text{(How?)}]{\text{remove abs(|)}}$ $y = f(x) = e^x$

$\to e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. , $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

④ $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = ?$ $\quad = \sum_{n=1}^{\infty} n^2 \cdot \left(\frac{1}{2}\right)^n$ $\quad$ $x = 1/2$ $\quad f(x) = \sum_{n=1}^{\infty} n^2 \cdot x^n$, $f\left(\frac{1}{2}\right) = ?$

$f(x) = \sum_{n=1}^{\infty} n^2 x^n = x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 \dots$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \dots$

d/dx

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \dots$

$\cdot x$

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 \dots$

d/dx

$\left(\frac{x}{(1-x)^2}\right)' = 1 + 2^2 x + 3^2 x^2 + 4^2 x^3$

$\cdot x$

$\underbrace{x\left(\frac{x}{(1-x)^2}\right)' = x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4}_{f(x)}$

$\Downarrow$

$f(x) = \sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for all $|x|<1$

$f\left(\frac{1}{2}\right) = 6 = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

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Taylor Series: Question: Given a function f and a point a in the domain of f, is it possible to find a power series $P(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a such that $f(x)=P(x)$ for some x in some open interval I contains a.
The answer is ~~no~~ not for all f and a. (For example. $f(x)=|x|$, $a=0$)

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ for x in some interval I contains a.

$$f^{(n)}(a)=n!\cdot c_n \quad \text{for } n\geq 0. \quad \Rightarrow c_n=\frac{f^{(n)}(a)}{n!} \text{ for } n\geq 0$$

Definition: Suppose $f^{(n)}(a)$ are defined for all $n\geq 0$. Then,

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)\cdot(x-a)+f''(a)\cdot\frac{(x-a)^2}{2}+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the Taylor Series generated by f and centred at a.

If $a=0$, $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n = f(0)+f'(0)\cdot x+\frac{f''(0)}{2}\cdot x^2+\dots+\frac{f^{(n)}(0)}{n!}\cdot x^n$ is called Maclurin Series.

(*) (*) (*) If $f(x)$ can be expresed as a power series centered at a on an open interval containing a, then that series must be its Taylor series centered at a. So the Taylor Series is the only candidate.

Given a function for which $f^{(n)}(a)$ are defined for all $n\geq 0$,
Q1) Does the Taylor Series converge with some $R>0$?
Q2) If the answer is yes to Q1, then does the Taylor series converge to $f(x)$ on some open interval I contains a.

Ex) ① $f(x)=\frac{1}{x}$ and $a=2$

$f'(x)=\frac{-1}{x^2}$ $\quad f^{(3)}(x)=\frac{-6}{x^4}$ $\xrightarrow[\text{Induction}]{\text{shown by}}$ $f^{(n)}(x)=(-1)^n\frac{n!}{x^{n+1}}$

$f''(x)=\frac{2}{x^3}$ $\quad f^{(4)}(x)=\frac{24}{x^5}$

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So at the center $a=2$, $f^{(n)}(2) = (-1)^n \cdot \frac{n!}{2^{n+1}}$. Thus, Taylor Series of $\frac{1}{x}$ centered at 2 is $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} \cdot (x-2)^n = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot (x-2)^n}_{\text{this is a geometric series}}$

$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(-\frac{x-2}{2}\right)^n, \quad a=\frac{1}{2}, \quad r=-\frac{x-2}{2}}$

Converges when $|r| = \left|\frac{x-2}{2}\right| < 1$ $\quad |x-2|<2$, $0<x<4$ $\Rightarrow = \frac{\frac{1}{2}}{1-\left(-\frac{x-2}{2}\right)} = \frac{1}{x}$

So Taylor series of the function $\frac{1}{x}$ converges to itself on interval $|x-2|<2$

Ex $f(x) = e^x$, $a=0$ → Maclurin series of e^x is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$f'(x) = e^x$
$f''(x) = e^x$
$\vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$ for all $n \geq 1$

So, series of e^x converges to itself for all x.

Ex $f(x) = \sin x \quad a=0$

$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f^{(4)}(x) = \sin x$
$\vdots$

$f(0)=0$
$f'(0)=1$
$f''(0)=0$
$f'''(0)=-1$
$f^{(4)}(0)=0$
$\vdots$

Maclurin series of $\sin x$;

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

Exercise: show that this converges for all x.

We will show that this actually converges to $\sin x$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \xrightarrow{d/dx} \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Ex

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad a=0$$

(exercise)
$f^{(n)}(0) = 0$ for all $n \geq 0$

Maclurin of $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = 0$ for all x.

So maclurin series of $f(x)$ does not converge to itself other than $x=0$.

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Taylor's Theorem: Suppose f is $(n+1)$-times differentiable on I and a is a point in I. Then for any x in I, there is a c between a and x, such that

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}}_{}$$

$T_n(x) =$ n^{th} partial sum S_n of Taylor Series. (n^{th} order Taylor Polynomial) $R_n(x) =$ n^{th} order remainder term error.

Remark: $n=0$ gives Mean Value Theorem, $n=1$ gives Linearization formula.

Observation: The Taylor Series generated by f centered at a converges to $f(x)$ for all x in I exactly when $\lim_{n\to\infty} R_n = 0$ for all x in I.

Ex] $f(x) = \sin x$, $a=0$. Taylor therem says:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

$$R_{2n+1}(x) = \frac{f^{(2n+2)}(c)}{(2n+2)!}\cdot x^{2n+2}$$

$f^{(k)}(c) = \pm\sin c$ or $\pm\cos c$

$|f^{(2n+2)}| \leq 1$ $\Rightarrow$

$$0 \leq |R_{2n+1}(x)| \leq |f^{(2n+2)}(c)| \cdot \frac{|x^{2n+2}|}{(2n+2)!} \leq \frac{|x^{2n+2}|}{(2n+2)!}$$

Take limit $n\to\infty$, $\to 0$

$\lim_{n\to\infty} R_{2n+1}(x) = 0$ by Squeeze Theorem for all x.

So, maclurin series of $\sin x$ converges to $\sin x$ for all x.