MATH 101
Calculus I

Fall 2016 (Mefharet Kocatepe)

Lecture Notes

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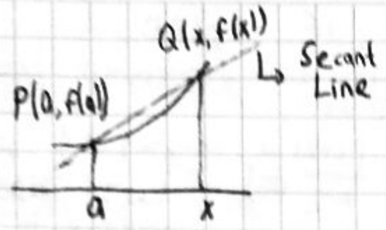
Subject : Date



$m_{secant} = \dfrac{f(x) - f(a)}{x - a}$
(slope)

Tangent Line: is the limiting position of the secant line PQ as Q approaches P along the curve.

$$m_{tangent} = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex) A dynamite blows up a rock straight up with a launch velocity of 50 m/s. It reaches a height $s(t) = 50t - 4.9t^2$ m after t seconds.

a) What is the average velocity during first 2 seconds?

b) What is the (instantaneous) velocity at $t = 2$?

a) $V_{avg} = \dfrac{\text{change in position}}{\text{time elapsed}} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{(50)(2) - (4.9)(2^2)}{2} = 40.2$ m/s

b) find the $\lim_{t \to 2} V_{[t,2]}$ or $\lim_{t \to 2} V_{[2,t]}$

$$\lim_{t \to 2} V_{[2,t]} = \lim_{t \to 2} \frac{s(2) - s(t)}{2 - t} = \frac{4.9t^2 - 50t + 80.4}{2 - t} = \frac{-(2-t)(4.9t - 40.2)}{2 - t} = 30.4 \text{ m/s}$$

Rate of Change: $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ $\quad \lim_{x_2 \to x_1} \dfrac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the rate of change of y with respect to x at the point x_1.

Limit Notation: $\lim_{x \to a} f(x) = L$ or "$f(x) \to L$ as $x \to a$"

Ex) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ This function's plot makes infinite oscilation between $y = -1$ and $y = 1$. So limit at 0 doesn't exist. (Notation: DNE)

Ex) Heaviside function; $H(t) \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

As $t \to 0$ from the right hand side, $H(t) \to 1$ (Right handed limit) (Right limit)

As $t \to 0$ from the left hand side, $H(t) \to 0$ (Left handed limit) (Left limit)

So limit DNE at the point $x = 0$.

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<u>Ex Solution</u>) $\lim_{x\to 2} \frac{|x-2|}{x-2} = ?$ $|x-2| \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases}$

$\lim_{x\to 2^-} \frac{|x-2|}{(x-2)} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$ $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{(x-2)}{x-2} = 1$ $-1 \neq 1$ so limit DNA at $x=2$.

<u>Ex</u>| $\lim_{x\to 3} \frac{|5-2x|-|x-2|}{|x-5|-|3x-7|} = ?$

$\lim_{x\to 3} |5-2x| = 2x-5$ $\lim_{x\to 3} |x-2| = x-2$ $\lim_{x\to 3} \frac{(2x-5)-(x-2)}{(5-x)-(3x-7)} = \frac{x-3}{12-4x} = \frac{-1}{4}$

$\lim_{x\to 3} |x-5| = 5-x$ $\lim_{x\to 3} |3x-7| = 3x-7$

(★) $\lim_{x\to 0} \frac{1}{x^2} = \infty$ but, limit does not exist.

Limits must be numbers, ∞ and $-\infty$ are not numbers.

So, $\lim_{x\to a} f(x) = \pm\infty$ just shows the behaivor. Limit still does not exist.

<u>Asymptotes:</u> The line $x=a$ is called a vertical asymptote of $y=f(x)$ if at least one of the following true: $\lim_{x\to a^{\pm}} f(x) = \pm\infty$ → 4 conditions

→ Candidates for rational functions are the roots of denominator.

<u>Ex</u>| Vertical asymptots of $f(x)=\frac{x}{x-1}$ and $g(x)=\frac{x^3+x}{x^2-x}$

$x-1=0$, $x=1$ $\lim_{x\to 1^+} \frac{x \to 1}{x-1 \to 0^+} = \infty$, so "$x=1$" is an asymptote for $f(x)$.

$x^2-x=0$, $x=0$ $x=1$ $\lim_{x\to 0^+} \frac{x^3+x}{x^2-x} = \lim_{x\to 0^+} \frac{x(x^2+1) \to 1}{x(x-1) \to -1} = -1$ ⇒ $-1 \neq \pm\infty$, so "$x=0$" is not an asymptote for $g(x)$

$\lim_{x\to 1^+} \frac{x^3+x}{x^2-x} = \lim_{x\to 1^+} \frac{(x^3+x) \to 2}{x(x-1) \to 1 \cdot 0^+} = \infty$, so "$x=1$" is an asymptote for $g(x)$

<u>Limit Laws:</u> Let c be a constant and suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists.

1- Sum Rule
2- Difference Rule
3- Constant Multiple Rule
4- Product Rule
5- Quotient Rule $\left(\frac{f(x)}{g(x)}\right)$ $g(x)\neq 0$
6- Power Rule $(f(x))^n$ positive integer
7- $\lim_{x\to a} x = a$
8- $\lim_{x\to a} x^n = a^n$
9- $\lim_{x\to a} c = c$
10- $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

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Ex] $\lim_{x\to -2} \frac{x^3+8}{x^2-x-6} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}(x-3)} = \frac{(-2)^2-2(-2)+4}{-2-3} = \frac{-12}{5}$

Ex] $\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2}$, we multiply by conjugate $\left(\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}\right)$

$\lim_{u\to 2} \frac{(\sqrt{4u+1}-3)(\sqrt{4u+1}+3)}{(u-2)(\sqrt{4u+1}+3)} = \lim_{u\to 2} \frac{4u+1-9}{(u-2)(\sqrt{4u+1}+3)} = \lim_{u\to 2} \frac{4\cancel{(u-2)}}{\cancel{(u-2)}(\sqrt{4u+1}+3)} = \frac{4}{\sqrt{9}+3} = \frac{2}{3}$

Ex] $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x^2-1} = \lim_{x\to 1} \frac{\sqrt[3]{x}-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{\cancel{(\sqrt[3]{x}-1)}}{\cancel{(\sqrt[3]{x}-1)}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(x+1)} = \frac{1}{(1+1+1)(2)} = \frac{1}{6}$

→ We also could've multiply by $\left(\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}\right)$ which from; $a^3-b^3 = \underbrace{(a-b)}_{\sqrt[3]{x}-1}(a^2+ab+b^2)$

Ex] $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$, Let $x = t^6$, We have that $t\to 1$ as $x\to 1$

$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{\sqrt[3]{t^6}-1}{\sqrt{t^6}-1} = \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(t^2+t+1)} = \frac{2}{3}$

→ We also could've multiply by $\left(\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}\right)$.

Ex] $\lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{3\cancel{h}(3+h)} = \frac{-1}{9}$

Geometrically; $f(x) = \frac{1}{x}$, $a=3 \Rightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, $x = a+h$, $x\to a$, as $h\to 0$, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = m_T$

★ $\lim_{x\to 0} x\cdot \sin\left(\frac{\pi}{x}\right)$ $\quad \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE, so we can't use product rule. So we use;

? $\frac{\sin\left(\frac{\pi}{\frac{1}{t}}\right)}{\frac{1}{t}} = \frac{\sin \pi t}{t}$

Squeeze (Sandwitch) Theorem: Assume $f(x) \leq g(x) \leq h(x)$ for all x near a, but not necessarily for $x=a$. Assume $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$

Remarks: 1) Theorem is true if L is a number or $L = \pm\infty$.

2) Theorem is true if all $\lim_{x\to a}$ are replaced by $\lim_{x\to a^-}$ OR $\lim_{x\to a^+}$.

For all $x \neq 0$; $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$. We multiply by x.

If $x>0$; $-x \leq x\sin\left(\frac{\pi}{x}\right) \leq x$. Let $x\to 0^+$. $\lim_{x\to 0^+}(-x) = 0$ $\quad \lim_{x\to 0^+} x = 0$ $\quad$ same → $\lim_{x\to 0^+} x\sin\frac{\pi}{x} = 0$

If $x<0$; $-x \geq x\sin\left(\frac{\pi}{x}\right) \geq x$. Let $x\to 0^-$. $\lim_{x\to 0^-}(-x) = 0$ $\quad \lim_{x\to 0^-} = 0$ $\quad$ same → $\lim_{x\to 0^-} x\sin\frac{\pi}{x} = 0$

Since $\lim_{x\to 0^+} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to 0^-} x\sin\left(\frac{\pi}{x}\right) = 0$; $\lim_{x\to 0} x\sin\left(\frac{\pi}{x}\right) = 0$

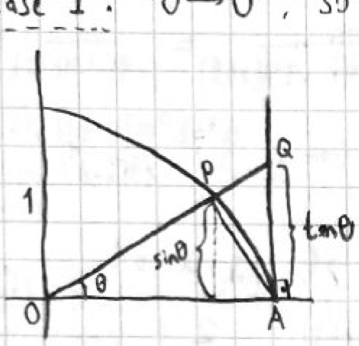
Remark: If $\lim_{x \to a} |f(x)| = 0$ then $\lim_{x \to a} f(x) = 0$

Proof: For any number y; $-|y| \leq y \leq |y|$ Then $\lim_{x\to a} f(x) = 0$ by the squeeze theorem.

$\underbrace{-|f(x)|}_{\to 0} \leq f(x) \leq \underbrace{|f(x)|}_{\to 0}$

(★) Assume θ measured in radians. Then $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. Proof;

Case 1: $\theta \to 0^+$, so we may take $0 < \theta < \frac{\pi}{2}$.



Area(triangle OPA) $\leq$ Area(sector OPA) $\leq$ Area(triangle OQA)

$$\frac{1}{2}\cdot 1 \cdot \sin\theta \leq \frac{1^2 \cdot \theta}{2} \leq \frac{1}{2}\cdot 1 \cdot \tan\theta$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \Rightarrow A = \frac{\pi r^2 \cdot \theta}{2\pi} = \frac{r^2 \cdot \theta}{2}$$

$\sin\theta \leq \theta \leq \tan\theta$ We're trying to get $\left(\frac{\sin\theta}{\theta}\right)$ so divide by $\sin\theta$

$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$ Take reciprocals. (Means $f(x) = \frac{1}{x}$)

$1 \leq \frac{\sin\theta}{\theta} \leq \cos\theta$ Let $\theta \to 0^+$; (A) $\lim_{\theta \to 0^+} 1 = 1$ (B) $\lim_{\theta \to 0^+} \cos\theta = 1$

So with (A) and (B) by Squeeze Theorem; $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

Case 2: $\theta \to 0^-$ $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta}$ Let $u = -\theta$, so $\theta = -u$

$u \to 0^+$ as $\theta \to 0^-$

$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{u \to 0^+} \frac{\sin(-u)}{-u} = \lim_{u \to 0^+} \frac{-\sin u}{-u} = 1 \text{ (From the Case 1)}$$

Conclusion: $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1 \Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Ex) $$\lim_{x \to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x \to 0} \frac{\sin 5x}{\sin 3x} \cdot \frac{5x}{5x} \cdot \frac{3x}{3x} = \lim_{x \to 0} \frac{\sin 5x}{5x} \cdot \frac{3x}{\sin 3x} \cdot \frac{5x}{3x}$$

$$= \frac{5}{3} \cdot \lim_{x \to 0} \underbrace{\frac{\sin 5x}{5x}}_{u=5x} \cdot \frac{1}{\underbrace{\frac{\sin 3x}{3x}}_{v=3x}} = \frac{5}{3} \cdot \lim_{u \to 0} \frac{\sin u}{u} \cdot \frac{1}{\lim_{v \to 0} \frac{\sin v}{v}} = \frac{5}{3} \cdot 1 \cdot \frac{1}{1} = \frac{5}{3}$$

(★) General Rule: Let a, b non-zero constants; $\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$

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Ex) $\lim\limits_{x\to 1} \dfrac{\sin(x^2-1)}{x^2+x-2} = \lim\limits_{x\to 1} \dfrac{\sin(x^2-1)}{(x+2)(x-1)} \cdot \dfrac{(x+1)}{(x+1)} = \lim\limits_{x\to 1} \dfrac{\sin(x^2-1)}{(x^2-1)} \cdot \lim\limits_{x\to 1} \dfrac{x+1}{x+2}$ $u = x^2-1$, $u \to 0$ as $x \to 1$

$\lim\limits_{u\to 0} \dfrac{\sin u}{u} \cdot \lim\limits_{x\to 1} \dfrac{x+1}{x+2} = 1 \cdot \dfrac{2}{3} = \dfrac{2}{3}$

(★) General Rule: Assume $\lim\limits_{x\to a} f(x) = 0$ Then; $\lim\limits_{x\to a} \dfrac{\sin(f(x))}{f(x)} = 1$

Ex) $\lim\limits_{x\to 0} \dfrac{\sin x^2 + \sin^2 x}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2} + \left(\dfrac{\sin x}{x}\right)^2 = 1 + 1^2 = 2$

Ex) $\lim\limits_{x\to 0} \dfrac{\sin x}{x+\tan x} = \dfrac{\frac{\sin x}{x}}{\frac{x}{x} + \frac{\tan x}{x}} = \dfrac{1}{1+1} = \dfrac{1}{2}$ $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = 1$

Ex) $\lim\limits_{x\to \pi} \dfrac{(x-\pi)^3}{(x-\pi)^2 + \sin^2\left(\frac{1}{(x-\pi)^4}\right)} = \lim\limits_{u\to 0} \dfrac{u^3}{u^2 + \sin^2\left(\frac{1}{u^4}\right)}$ $0 \leqslant \sin^2\left(\frac{1}{u^4}\right) \leqslant 1$ Add u^2

$u = (x-\pi)$ $u \to 0$ As $x \to \pi$ $u^2 \leqslant u^2 + \sin^2\left(\frac{1}{u^4}\right) \leqslant u^2 + 1$

$\dfrac{1}{u^2} \leqslant \dfrac{1}{u^2 + \sin^2\left(\frac{1}{u^4}\right)} \leqslant \dfrac{1}{u^2+1}$ Multiply by u^3

$u > 0 \Rightarrow u^3 > 0$ $\dfrac{u^3}{u^2} \leqslant \dfrac{u^3}{u^2+\sin^2\left(\frac{1}{u^4}\right)} \leqslant \dfrac{u^3}{u^2+1}$; 0 ... same ... 0

$u < 0 \Rightarrow u^3 < 0$ $\dfrac{u^3}{u^2} \geqslant \dfrac{u^3}{u^2+\sin^2\left(\frac{1}{u^4}\right)} \geqslant \dfrac{u^3}{u^2+1}$; 0 ... same ... 0

So $\lim\limits_{u\to 0^+} \dfrac{u^3}{u^2+\sin^2\left(\frac{1}{u^4}\right)} = \lim\limits_{u\to 0^-} \dfrac{u^3}{u^2+\sin^2\left(\frac{1}{u^4}\right)}$ Then; $\lim\limits_{u\to 0} \dfrac{u^3}{u^2+\sin^2\left(\frac{1}{u^4}\right)} = 0$

Ex) $\lim\limits_{x\to 3} \dfrac{\sin(\pi x)}{x-3}$ Let $u = x-3$ Then $u \to 0$ As $x \to 3$, $\pi x = \pi u + 3\pi$

$\lim\limits_{u\to 0} \dfrac{\sin(\pi u + 3\pi)}{u}$

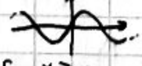
$\sin(\pi u + 3\pi) = -\sin(\pi u)$

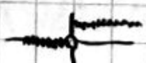
$\lim\limits_{u\to 0} \dfrac{-\sin(\pi u)}{\pi u} \cdot \pi = -\pi \cdot \lim\limits_{u\to} \dfrac{\sin u}{u} = -\pi$

Continuity: We say f is continuous at point a ; $\lim\limits_{x\to a} f(x) = f(a)$

$f(a)$ should be defined and $\lim\limits_{x\to a} f(x) = L$ should exist. ($f(x)$ should be defined near a)

Ex) At what points are the following functions discontinuous?

a) $f(x) = \sin x$  So f is coninuous at every point x.

b) $H(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$  There is a jump discontinuity at $x=0$ infinite disc. infinite jump

c) $f(x) \begin{cases} \dfrac{x^2+x-2}{x^2+2x} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$ → $f(x)$ is undefined at $x=0$ so it's discontinous at $x=0$.

→ $f(-2) = 1$ $\lim\limits_{x\to -2} \dfrac{x^2+x-2}{x^2+2x} = \lim\limits_{x\to -2} \dfrac{(x+2)(x-1)}{(x+2)\,x} = \dfrac{3}{2}$

$1 \neq \dfrac{3}{2}$ so f is discontinuous at $x=-2$ → removable discontinuity

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★ We say f has a removable discontinuity at $x=a$ if $\lim_{x\to a} f(x)=L$ exists. We can remove the discontinuity by defining $f(a)=L$

★ We say f is continuous on an interval I if f is continuous at every point x of interval I.

If I includes an end point, then at the point we need one-sided continuity.

Ex) $g(x)=\sqrt{3-x}$ is continuous on $I=(-\infty, 3]$. $\lim_{x\to 3^-} \sqrt{3-x}=0$ is enough.

★ When we say f is continuous on an interval I, that means the graph of f above the interval I is one piece.

Theorem: The following functions are continuous at every point of their domains; polynomials, rational functions, root functions, trigonometric functions.

Theorem: If g is continuous at c, and if f is continuous $g(c)$; then the composition $f\circ g$ is continuous at c.

Ex) $h(x)=\sqrt{\tan x}$ $c=\frac{\pi}{3}$ Is h continuous at c?

$h(x)=f(g(x))$ where $g(x)=\tan x$, $f(x)=\sqrt{x}$

$g(\frac{\pi}{3})=\tan(\frac{\pi}{3})=\sqrt{3}$ and $f(x)=\sqrt{x}$ is continuous at $g(\frac{\pi}{3})=\sqrt{3}$. So $h=f\circ g$ is continuous at $c=\frac{\pi}{3}$.

Intermediate Value Theorem (IVT): Assume f is continuous on the closed bounded interval $[a,b]$. Assume also $f(a)\neq f(b)$. Let N be any number between $f(a)$ and $f(b)$. Then there is at least one point c in (a,b) such that $f(c)=N$.

Special Case: Suppose f is continuous on $[a,b]$ and $f(a)\cdot f(b)<0$ then there is at least one point c such that $f(c)=0$

Ex) $f(x)=x^2+10\sin x$ Show that there is a number c such that $f(c)=1000$.

f is continuous on $(-\infty,\infty)$. So
$a=0$ $f(0)=0^2+10\sin 0=0<1000$
$b=100$ $f(100)=100^2+\underbrace{10\sin 100}_{\text{at least } -10} \geq 9990>1000$

(at least -1 for $\sin 100$)

Then by IVT, there is a point c in $(0,100)$ such that $f(c)=1000$.

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Ex) Show that the equation $x^4+9=7x^3$ has at least one root.

Let $f(x)=x^4-7x^3+9$. So we should find a c such that $f(c)=0$.

f is continuous since it's a polynomial.

$f(2)=2^4-7\cdot2^3+9=-31<0$ $a=0,\ b=2$, f is continuous on $[0,2]$. So
$f(0)=0-0+9=9>0$ by IVT, there is a point c such that $f(c)=0$

Since $f(c)=0$, $c^4-7c^3+9=0$, so there is a real root of this equation.

Derivatives: The slope of tangent line to the curve $y=f(x)$ at the point $P(a,f(a))$ is $f'(a)=m_{Tangent}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, if this limit exists.

f': read "f prime". $f(a)$ is velocity at a, $f'(a)$ is rate of change of $y=f(x)$ with respect to x at a.

Ex) $f(x)=\frac{2}{x}$ $f'(1)=\lim_{x\to1}\frac{\frac{2}{x}-\frac{2}{1}}{x-1}=\lim_{x\to1}\frac{2\overset{-1}{\cancel{(1-x)}}}{x\cancel{(x-1)}}=\frac{-2}{1}=-2$

Ex) $f(x)=\sqrt{x}$ a) Find $f'(a)$ b) Find an equation of a line that is tangent to the curve $y=\sqrt{x}$ and has slope $=1/4$.

a) $f'(a)=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{x-a\,(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$

Note that $f'(a)=\frac{1}{2\sqrt{a}}$ is defined for $a>0$ (Tangent line is vertical at 0.)

b) Let $P(a,\sqrt{a})$ be the point of tangency. Then $m_{Tangent}=\frac{1}{4}$ and $m_{Tangent}=f'(a)=\frac{1}{2\sqrt{a}}$

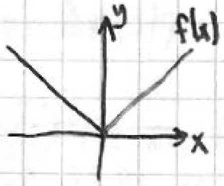
$\frac{1}{2\sqrt{a}}=\frac{1}{4}\Rightarrow a=4$. Then the point is $P(4,2)$. Tangent line; $y-2=\frac{1}{4}(x-4)$

Another notation for derivative: $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$

We may consider derivative itself as a function $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

Ex) for $f(x)=\sqrt{x}$, we have the derivative; $f'(x)=\frac{1}{2\sqrt{x}}$

Ex) $f(x)=|x| \Rightarrow f(x) \begin{cases} -x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ x & \text{if } x>0 \end{cases}$ and $f'(x)=\lim\limits_{h\to 0} \dfrac{|x+h|-|x|}{h}$:

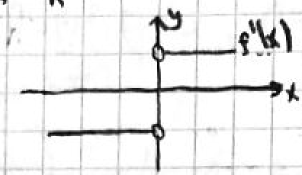


$x<0$, tangent line is $y=-x$ and its slope is -1.

$x>0$, tangent line is $y=x$ and its slope is 1.

$x=0$ $\quad f'(0)=\lim\limits_{h\to 0} \dfrac{|0+h|+|0|}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ $\quad \lim\limits_{h\to 0^-} \dfrac{|h|}{h} = -1$, $\lim\limits_{h\to 0^+} \dfrac{|h|}{h} = 1$ $\Rightarrow -1 \neq 1$ so $f'(0)=\lim\limits_{h\to 0} \dfrac{|h|}{h}$ does not exists.

So, for $f(x)=|x|$, $\quad f'(x) = \begin{cases} -1 & \text{if } x<0 \\ \text{DNE} & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$



★ $f'_+(a)=\lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ and the same way $f'_-(a)$.

If $f'_+(a) \neq f'_-(a)$ then the graph has a corner at $x=a$. Then $f'(a)$ DNE.

If $f'(a)$ exists, we can say f is differentiable at the point a.

Theorem: If f is differentiable at a, then f is continuous at a.

(Differentiability implies continuity.) Proof;

Hypothesis: f is differentiable at a that is $f'(a)=\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists.

Prove that f is continuous at a that is $\lim\limits_{x\to a} f(x)=f(a)$.

Let $g(x)=\dfrac{f(x)-f(a)}{x-a}$, then $\lim\limits_{x\to a} g(x) = f'(a)$.

$g(x)=\dfrac{f(x)-f(a)}{x-a}$ so, $f(x)=(x-a)\cdot g(x)+f(a)$

Then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a}\left((x-a)\cdot g(x)+f(a)\right) = \underbrace{\lim\limits_{x\to a}(x-a)}_{0} \cdot \underbrace{\lim\limits_{x\to a} g(x)}_{f'(a)} + \underbrace{\lim\limits_{x\to a} f(a)}_{f(a)}$

$\lim\limits_{x\to a} f(x) = 0\cdot f'(a)+f(a) \Rightarrow \lim\limits_{x\to a} f(x)=f(a)$. $\square$

Converse of this theorem is not true. There are functions which are continuous but not differentiable. Ex: $f(x)=|x|$ at $x=0$

(Not differentiable: nondifferentiable)

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(*) How can a function be non differentiable?

→ a discontinuity → a corner → a vertical tangent

We say f has a vertical tangent at a if, $\lim_{x\to a} |f'(x)| = \infty$

Ex) $f(x)=\sqrt[3]{x}$, $x=0$, $f(x)=\sqrt[3]{|x|}$, $x=0$

Derivative Notations: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D_x f(x)$ (function)

$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \frac{dy}{dx}\Big]_{x=a}$ (at the point a)

Higher Order Derivatives: $f'''(x) = \frac{d}{dx}f''(x) = \frac{d^3y}{dx^3} = f^{(3)}(x)$

Differantiation Formulas: Assume f an c differentiable and c is a constant.

1) $\frac{d}{dx}c = 0$, Derivative of a constant is zero.

2) $\frac{d}{dx}x^n = n\cdot x^{n-1}$, n is a positive integer. [Positive power rule]

3) $\frac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$, [Constant multiple rule

4) $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$ [Sum rule]

5) $\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$ [Difference rule]

6) $\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ [Product rule]

7) $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{[g(x)]^2}$ [Reciprocal rule]

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$ [Quotient rule]

at the points $g(x) \neq 0$

8) $\frac{d}{dx}x^{-n} = -n\cdot x^{-n-1}$, n is a positive integer. [Negative power rule]

9) $\frac{d}{dx}x^n = n\cdot x^{n-1}$, n is any real constant [General power rule]

Proof of 2: $\frac{d}{dx}x^n = \lim_{u\to x}\frac{u^n-x^n}{u-x} = \lim_{u\to x}\frac{(u-x)(u^{n-1}+u^{n-2}\cdot x+u^{n-3}\cdot x^2+\dots x^{n-1})}{(u-x)} =$

$\lim_{u\to x} u^{n-1}+u^{n-2}\cdot x+\dots+x^{n-1} = \underbrace{x^{n-1}+x^{n-2}\cdot x+x^{n-3}\cdot x^2+\dots+x^{n-1}}_{n \text{ terms}} = n\cdot x^{n-1}$

Proof of 6: $\frac{d}{dx}(f(x)\cdot g(x)) = \lim\limits_{u\to x} \frac{f(u)g(u) - f(x)g(x)}{u-x} = \lim\limits_{u\to x} \frac{f(u)g(u) - f(x)g(u) + f(x)g(u) - f(x)g(x)}{u-x}$

$\lim\limits_{u\to x}\left(\frac{f(u)-f(x)}{u-x}\cdot g(u) + f(x)\cdot\frac{g(u)-g(x)}{u-x}\right) = \underbrace{\lim\limits_{u\to x}\frac{f(u)-f(x)}{u-x}}_{f'(x)} \cdot \underbrace{\lim\limits_{u\to x} g(u)}_{\substack{g(u)\text{ is continuous because}\\ \text{differentiable } g(x)}} + \underbrace{\lim\limits_{u\to x} f(x)}_{f(x)} \cdot \underbrace{\lim\limits_{u\to x}\frac{g(u)-g(x)}{u-x}}_{g'(x)}$

$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Ex) The line normal to the parabola $y = 4x - x^2$ at the point $P(1,3)$ intersects the parabola at a second point A. Find point A.

$m_{Normal} = \frac{-1}{m_{Tangent}} = \frac{-1}{\frac{d}{dx}(4x-x^2)\big|_{x=1}} = \frac{-1}{(4-2x)\big|_{x=1}} = \frac{-1}{2}$ Normal; $y - y_0 = m(x - x_0)$

$y - 3 = \frac{-1}{2}(x-1)$

$y = \frac{-x}{2} + \frac{7}{2}$

Intersection Points; $4x - x^2 = \frac{-x}{2} + \frac{7}{2}$; $2x^2 - 9x + 7 = 0$

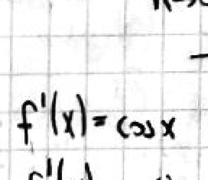
$8x - 2x^2 = 7 - x$; $(x-1)(2x-7) = 0 \Rightarrow$ $P(1,3)$, $A\left(\frac{7}{2}, \frac{7}{4}\right)$

Ex) $\lim\limits_{x\to 1}\frac{x^{1000}-1}{x-1} = \lim\limits_{x\to 1}\frac{x^{1000}-1^{1000}}{x-1} = \frac{d}{dx}\left(x^{1000}\right)\Big|_{x=1} = (1000\cdot x^{999})\big|_{x=1} = 1000$

Ex) P.143 # 106

Derivatives of Trigonometric Functions:

Review:



$|\sin\theta| < |\theta| \Rightarrow -|\theta| < \sin\theta < |\theta|$ By squeeze theorem; $\lim\limits_{x\to 0}\sin x = 0$

$|1-\cos\theta| \leq |\theta| \Rightarrow -|\theta| < 1-\cos\theta < |\theta|$ " $\lim\limits_{x\to 0}\cos x = 1$

Ex) $\lim\limits_{h\to 0}\frac{1-\cos h}{h} = \lim\limits_{h\to 0}\frac{(1-\cos h)(1+\cos h)}{h(1+\cos h)} = \lim\limits_{h\to 0}\frac{\sin h\cdot\sin h}{h\cdot(1+\cos h)} = \lim\limits_{h\to 0}\frac{\sin h}{h}\cdot\lim\limits_{h\to 0}\frac{\sin h}{1+\cos h} = 1\cdot 0 = 0$

——— o ———

$f(x) = \sin x$ $f'(x) = \cos x$

$f(x) = \cos x$ $f'(x) = -\sin x$

$f(x) = \tan x$ $f'(x) = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}$

$f(x) = \cot x$ $f'(x) = -1 - \cot^2 x = -\csc^2 x = \frac{-1}{\sin^2 x}$

$f(x) = \sec x$ $f'(x) = \sec x\cdot\tan x$

$f(x) = \csc x$ $f'(x) = -\csc x\cdot\cot x$

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Proof for $\frac{d}{dx}(\sin x) = \cos x$: $f(x) = \sin x$ $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cdot \cosh + \sinh \cdot \cos x - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cosh - 1)}{h} + \frac{\cos x \cdot \sinh}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cosh - 1}{h}}_{0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sinh}{h}}_{1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$

Ex) Find constants A, B such that $y = A\sin x + B\cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

$y' = A\cos x - B\sin x$

$y'' = -A\sin x - B\cos x$

$-A\sin x - B\cos x + A\cos x - B\sin x - 2A\sin x - 2B\cos x = \sin x$

$(-3A - B)\sin x + (A - 3B)\cos x = \sin x$ (true for every x),

$-3A - B = 1$

$A - 3B = 0 \Rightarrow A = 3B$

$-9B - B = 1$

$B = -1/10 \Rightarrow A = -3/10$ so $(A, B) = (-0.3, -0.1)$.

The Chain Rule: Assume f is differentiable at x (with $u = f(x)$ and g is differentiable at $u = f(x)$. Then composition $F = g\circ f$ is differentiable;

$$F'(x) = (g\circ f)'(x) = g'(u) \cdot f'(x) = g'(f(x)) \cdot f'(x)$$

By Leibnitz Notation; $y = g(u)$, $u = f(x)$ $F'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex) $y = (x^3 + x^2)^2$ Let $u = x^3 + x^2$, $y = u^2$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dy}{d_1} = 2u \cdot (3x^2 + 2x) = 2(x^3 + x^2)(3x^2 + 2x)$

Ex) $y = \sin(x^3 + 5x^2 - 7x^{-1/2}) = \cos(x^3 + 5x^2 - 7x^{-1/2}) \cdot (3x^2 + 10x + \frac{7}{2}x^{-3/2})$

Ex) $y = \cos(\sin(\sec x))$ $y' = \underline{-\sin(\sin(\sec x))} \cdot \underline{\cos(\sec x)} \cdot \underline{\sec x \cdot \tan x}$ # From outer to inner.

Ex) $y = \sec^2(x^5 + \frac{100}{x})$ $y' = 2\sec(x^5 + \frac{100}{x}) \cdot \sec(x^5 + \frac{100}{x}) \cdot \tan(x^5 + \frac{100}{x}) \cdot (5x^4 - \frac{100}{x^2})$

Ex) $f(x) \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ $f'(x) = ?$

$x \neq 0$ $f'(x) = \sin\left(\frac{1}{x}\right) + x \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} = \sin\frac{1}{x} - \frac{1}{x} \cdot \cos\frac{1}{x}$

$x = 0$ $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \cdot \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} \sin\frac{1}{x}$ DNE.

So derivative function; $f'(x) = \begin{cases} \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}$

Ex) $f(x) \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ $f'(x) = ?$

$x \neq 0$ $f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\frac{1}{x} \cdot \frac{-1}{x^2} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

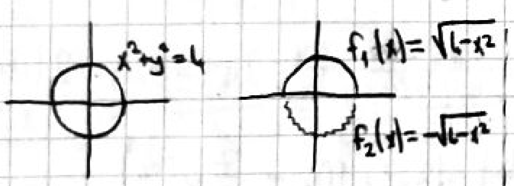
$x = 0$ $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \cdot \sin\frac{1}{x} = 0$ (by squeeze theorem)

So derivative function; $f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not continuous

Implicit Differentiation: Functions can be expressed;

implicitly; $x^2 + y^2 = 4$ or explicitly; $y = \sqrt{4 - x^2}$, $y = -\sqrt{4 - x^2}$

So there is two different functions for y at equation $x^2 + y^2 = 4$.

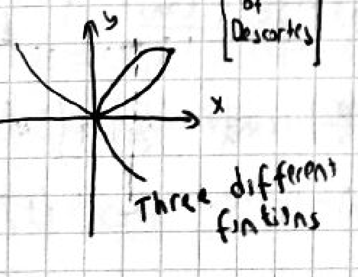


$\rightarrow x^2 + y^2 = 4$ find $\frac{dy}{dx}$ and $\frac{dy}{dx}\Big|_{(x,y) = (1, -\sqrt{3})}$

Solution 1: $y = f(x)$, $x^2 + (f(x))^2 = 4$ | Take $\frac{d}{dx}$

$2x + 2f(x) \cdot f'(x) = 0 \Rightarrow f'(x) = -\frac{x}{f(x)}$

$x^3 + y^3 = 6xy$ [Folium of Descartes]



or; $x^2 + y^2 = 4$ | Take $\frac{d}{dx}$ | $2x + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 0$

$2x + 2y \cdot \frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{x}{y}$ $\frac{dy}{dx}\Big|_{(x,y) = (1, -\sqrt{3})} = \frac{1}{\sqrt{3}}$

This solution is the Implicit Differentiation.

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Solution 2: From $x^2+y^2=4$, we get two functions; $y=f_1(x)=\sqrt{4-x^2}$ $y=f_2(x)=-\sqrt{4-x^2}$

The point $(1,-\sqrt{3})$ is on the f_2 function. So we use $y=f_2(x)=-\sqrt{4-x^2}$

Then $\frac{dy}{dx}=-\frac{1}{2\sqrt{4-x^2}}\cdot(-2x)=\frac{x}{\sqrt{4-x^2}}$ $\quad \frac{dy}{dx}\Big|_{x=1}=\frac{1}{\sqrt{4-1}}=\frac{1}{\sqrt{3}}$

Ex) $x^3+2y^3=5xy$, $\frac{d^2y}{dx^2}\Big|_{(x,y)=(2,1)}=?$ Take $\frac{d}{dx}$, Substitute $(2,1)$, Take $\frac{d}{dx}$ again.

Or take $\frac{d}{dx}$ of the $\frac{dy}{dx}$. [very Long]

$3x^2+6y^2\cdot\frac{dy}{dx}=5y+5x\cdot\frac{dy}{dx}$ Substitute $x=2$, $y=1$: $12+6\frac{dy}{dx}=5+10\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\Big|_{(x,y)=(2,1)}=\frac{7}{4}$

$6x+12y\cdot\frac{dy}{dx}\cdot\frac{dy}{dx}+6y^2\cdot\frac{d^2y}{dx^2}=5\frac{dy}{dx}+5\frac{dy}{dx}+5x\cdot\frac{d^2y}{dx^2}$ Substitute $x=2$, $y=1$, $\frac{dy}{dx}=\frac{7}{4}$

$12+24\cdot\frac{7}{4}\cdot\frac{7}{4}+6\cdot y''=5\cdot\frac{7}{4}+5\cdot\frac{7}{4}+10\cdot y'' \Rightarrow y''=\frac{125}{16}$, $\frac{d^2y}{dx^2}\Big|_{(x,y)=(2,1)}=125/16$

Proof of $\frac{d}{dx}(x^n)=nx^{n-1}$: $n=\frac{p}{q}$ where p and q are integers. (Rational Constant)

$y=x^{\frac{p}{q}}$

$y^q=x^p \xrightarrow{\frac{d}{dx}} q\cdot y^{q-1}\cdot\frac{dy}{dx}=p\cdot x^{p-1}$

$\frac{dy}{dx}=\frac{p\cdot x^{p-1}}{q\cdot y^{q-1}}=\frac{p}{q}\cdot\frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}}=\frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-\frac{p}{q}}}=\frac{p}{q}\cdot x^{\frac{p}{q}-1}$

$\frac{dy}{dx}=n\cdot x^{n-1}$

We can combine this with chain rule: $\frac{d}{dx}(y)^n=n\cdot y^{n-1}\cdot\frac{dy}{dx}$

(★) Read the section 2.7 from your book.

Related Rates: If a quantity y is changing with time t, then the rate of change of y with respect to time t is; $\frac{dy}{dt}$.

Ex) The lengths of the sides of a rectangle are changing as a function of time. Find the rate of change of the area of the rectangle with respect to time at the moment when the length of one side is 2m and decreasing at the rate of 3m/s, and the length of the other side is 5m and increasing at the rate of 4m/s.

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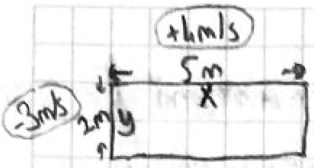


Expres what is wanted in terms of these variables and their derivatives

Solve the calculus problem and translate the result back to the language

Given: $\frac{dx}{dt} = +4$ m/s when $x=5$m

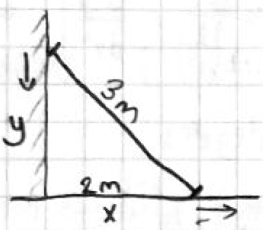
$\frac{dy}{dt} = -3$m/s when $y=2$m

Wanted: $\frac{dA}{dt}$ when $x=5$, $y=2$

$$\frac{d(xy)}{dt} = \frac{dx}{dt}\cdot y + \frac{dy}{dt}\cdot x = 4\cdot 2 + (-3)\cdot 5 = -7$$

The area is decreasing at the rate of $7m^2/s$.

Ex| A 3m long ladder which is leaning against a wall starts to slide away from the wall. If the bottom end is moving with a speed $\frac{5}{4}$ m/s at the moment when it is 2m away from the wall, how fast is the top end moving at this same moment?



Given: $\frac{dx}{dt} = \frac{5}{4}$ m/s when $x=2$m.

Wanted: $\frac{dy}{dt}$ when $x=2$ m.

$y^2 = 9-x^2$

$y = \sqrt{9-4} = \sqrt{5}$

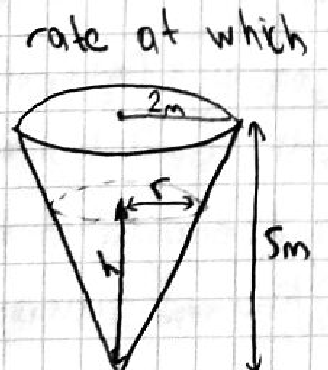
$x^2+y^2=3^2$ Take derivative with respect to t.

$$2x\cdot\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2\cdot 2\cdot\frac{5}{4} + 2\cdot\sqrt{5}\cdot\frac{dy}{dt} = 0$$

$$5 + 2\sqrt{5}\cdot\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-\sqrt{5}}{2}$$

The top of the ladder is falling at the rate of $\frac{\sqrt{5}}{2}$ m/s.

Ex| A water tank has the shape of an upside-down cone with radius 2m and height 5m. If the water is running out of tank at the rate of $3m^3/min$ when the depth of water in the tank is 4m. Find the rate at which the water level is changing at this moment.



Given: $\frac{dV}{dt} = -3m^3/min$ when $h=4$m

By similar triangles; $\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2h}{5}$

Wanted: $\frac{dh}{dt}$ when $h=4$m.

$$\frac{dV}{dt} = \frac{d\left(\frac{\pi\cdot r^2\cdot h}{3}\right)}{dt} = \frac{d\left(\frac{\pi\cdot 4h^2\cdot h}{3\cdot 25}\right)}{dt} = \frac{\pi\cdot 12h^2}{3\cdot 25}\cdot\frac{dh}{dt} = -3 \quad \pi\cdot\frac{64}{25}\cdot\frac{dh}{dt} = -3$$

$$\frac{dh}{dt} = \frac{-75}{64\pi}$$

Water level is falling at the rate of $\frac{75}{64\pi}$ m/min.

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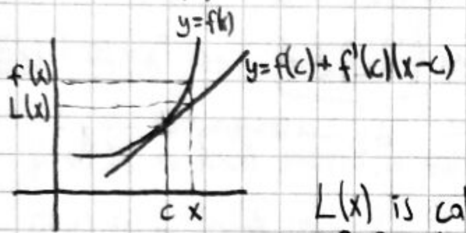
Linear approximations and differentials: Suppose f is differentiable at c. Then when x is near c, we have $\frac{f(x)-f(c)}{x-c}$ is near to $f'(c)$ that is;

$$x \approx c, \quad \frac{f(x)-f(c)}{x-c} \approx f'(c) \Rightarrow f(x)-f(c) \approx f'(c)\cdot(x-c)$$

So the approximation formula: $f(x) = f(c) + f'(c)(x-c)$ when x is near c



Tangent line: $y - f(c) = f'(c)(x-c)$

$$y = \underbrace{f(c) + f'(c)(x-c)}_{L(x)}$$

$L(x)$ is called linearization of f at c.

Ex a) Find the linearization of the function $f(x)=\sqrt{x+7}$ at $c=9$

b) By using (a), find $\sqrt{16.05}$ and $\sqrt{15.98}$ approximately.

(a) $f(x)=\sqrt{x+7}$, $c=9$

$L(x) = f(c) + f'(c)(x-c)$

$L(x) = \sqrt{16} + \frac{1}{2\sqrt{16}}\cdot(x-9)$

$L(x) = 4 + \frac{(x-9)}{8} = \boxed{\frac{x+23}{8}}$

(b) $f(x) \approx L(x)$ when $x \approx c$

So $\sqrt{x+7} \approx \frac{23}{8} + \frac{x}{8}$ when $x \approx c$

$\sqrt{16.05} = \sqrt{\underbrace{9.05}_{\substack{x=9.05 \\ \text{is near } 9}}+7} \approx \frac{23}{8} + \frac{9.05}{8} = 4.00625$

Calculator: 4.00624512...

$\sqrt{15.98} = \sqrt{8.98+7} \approx \frac{23}{8} + \frac{8.98}{8} = 3.9975$

Calculator: 3.9974992...

Ex Find $\sqrt{15.98}$ approximately; $f(x)=\sqrt{x}$, $c=16 \rightarrow L(x) = 2 + \frac{x}{8} = 2 + \frac{15.98}{8} = 3.9...$

$4 + \frac{-0.2}{8}$

★ Some useful linearizations at $c=0$ and linear approximations:

1) $f(x)=\sin x$, $c=0$ $\quad L(x)=x$ $\quad \sin x \approx x$ for $x \approx 0$

2) $f(x)=\cos x$, $c=0$ $\quad L(x)=1$ $\quad \cos x \approx 1$ for $x \approx 0$

3) $f(x)=\tan x$, $c=0$ $\quad L(x)=x$ $\quad \tan x \approx x$ for $x \approx 0$

4) $f(x)=(1+x)^k$ $c=0$ $\quad L(x)=1+kx$ $\quad (1+x)^k \approx 1+kx$ for $x \approx 0$

Ex $\sqrt{1.2} = (1+0.2)^{\frac{1}{2}} \approx 1 + \frac{1}{2}\cdot 0.2 = 1.1$

$x=0.2 \quad k=\frac{1}{2}$

Differentials: Let $y=f(x)$ be a differentiable function of x.

dx: differential of x.

dy: differential of y. $\Rightarrow dy = f'(x)\cdot dx$

Ex) $y = x^3$

$dy = 3x^2\,dx$

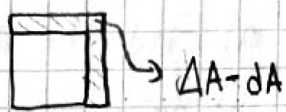
So $\frac{dy}{dx}$ has two meanings: derivative of y and differential of y / differential of x

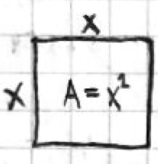
→ If we take $dx = \Delta x$: change in x, then:

$$f(x) \approx f(c) + f'(c)\underbrace{(x-c)}_{\Delta x = dx} \Rightarrow \underbrace{f(x)-f(c)}_{\text{True change} \leftarrow \Delta y} \approx \underbrace{f'(c)\,dx}_{dy \rightarrow \text{Differential}}$$

So dy is the approximate change in y.

Ex) Each side of a square is increased from 1m to 1.01m. Find the true change and approximate change in the area.





True change: $\Delta A = (1.01)^2 - 1^2 = 0.0201$

Approximate: $dA = \left.\frac{dA}{dx}\right|_{x=1} dx = (2x|_{x=1})\cdot(0.01) = 2\cdot(0.1) = 0.02$

↳ $dx = 1.01 - 1 = 0.01$

Ex) Assume that radius of a sphere increased by 1%. Estimate the percentage increase in its volume. We take $\Delta r = dr \Rightarrow \frac{dr}{r}\cdot 100 = 1$

$V = \frac{4}{3}\pi r^3$ $\quad \Delta V \approx dV = 4\pi r^2 dr$

$= \frac{4\pi r^2 \cdot r}{100} = \frac{\pi r^3}{25} \Rightarrow \frac{dV}{V}\cdot 100 = \frac{\frac{\pi r^3}{25}}{\frac{4}{3}\cdot\pi r^3}\cdot 100 = \frac{3}{4}\cdot\frac{1}{25}\cdot 100 = 3$ //

So the volume will be approximately increased by 3%.

(★) Sometimes we consider $\Delta x = dx$ as the error in the measurement of x as in an experiment. Then $dy = f'(x)dx$ will be corresponding approximate error in the outcome.

In the previous example, if we consider 1% as the error in the radius measurement; $\left|\frac{dr}{r}\right| \leq \frac{1}{100} \Rightarrow \left|\frac{dV}{V}\right| \leq \frac{3}{100}$

If the error in the radius measurement at most 1% then The error in the volume calculation will be at most 3%

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Maximum and Minimum Values: Let f be a function with domain D and let c be a point in D. So we say:

$f(c)$ is the absolute maximum value of f in D if $f(c) \geq f(x)$ for all x in D.

$f(c)$ is the absolute minimum value of f in D if $f(c) \leq f(x)$ for all x in D.

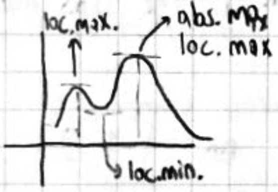
Ex) $f(x)=x^2$ on $[-1,2]$. Abs. min. value: $f(0)=0$
Abs. max. value: $f(2)=4$

Ex) $f(x)=x^2$ on $(-1,2)$. Abs. min. value: $f(0)=0$
Abs. max. value: DNE.

$f(c)$ is a local maximum value of $f(x)$ if $f(c) \geq f(x)$ for all x in some open interval containing c. (for all x near c.)

$f(c)$ is a local minimum value of $f(x)$ if $f(c) \leq f(x)$ for all x in some open interval containing c.

Ex)

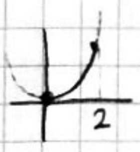


Ex) $f(x)=x^3$ on $[0,2]$

$f(0)=0$ is abs. min. but not loc. min.

$f(2)=8$ is abs. max. but not loc. max.

Ex) $f(x)=x^2$ on $[0,2]$



$f(0)$ is abs. min and loc. min.

$f(2)$ is abs. max but not loc. max.

★ maximum / minimum = extremum

singular	→ plural
maximum	maxima, maximums
minimum	minima, minimums
extremum	extrema, extremums

Extreme Value Theorem: If f is continuous on a closed and bounded interval $[a,b]$, then f has an abs. max $f(c)$ and an abs. min $f(d)$ in $[a,b]$.

Fermat's Theorem: Assume $f(x)$ has a local extremum at a point c and $f'(c)$ exists. Then, $f'(c)=0$.

Proof: WLOG, assume $f(x)$ has a minimum at c. We have $f(c) < f(x)$ for all x near c.

$f'(c)=\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. So $f'_+(c)=f'_-(c)$;

$$f'_+(c)=\lim_{x\to c^+} \frac{\overbrace{f(x)-f(c)}^{\geq 0}}{\underbrace{x-c}_{>0}} \geq 0 \qquad f'_-(c)=\lim_{x\to c^-} \frac{\overbrace{f(x)-f(c)}^{\leq 0}}{\underbrace{x-c}_{<0}} \leq 0$$

So $f'_+(c)=f'_-(c) \Rightarrow f'_+(c)=f'_-(c)=0 \to f'(c)=0$

★ So at a local extremum $f'(c)$ DNE or 0.

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Critical Number: $f'(c)=0$ or $f'(c)=DNE$, then c is critical number.

(★) If we want to find local extrema, we should look for critical points.

Ex) $f(x)=5x^{\frac{2}{3}}-2x^{\frac{5}{3}}$ Find the critical points.

$$f'(x)=\frac{10}{3}x^{\frac{-1}{3}}-\frac{10}{3}x^{\frac{2}{3}}=\frac{10}{3x^{\frac{1}{3}}}(1-x)=\frac{10(1-x)}{3\cdot\sqrt[3]{x}} \Rightarrow \text{Critical Points; } \begin{matrix} x=1 & (f'(1)=0) \\ x=0 & (f'(0)\ DNE) \end{matrix}$$

Closed Interval Method: Suppose f is continuous on the closed and bounded interval $[a,b]$. To find the abs. max and min of f on $[a,b]$;

1- Find all critical points of f on the open interval (a,b) and find the values of f there.

2- Find the values of f at the end points $x=a$ and $x=b$.

3- The largest of these values from the steps 1 and 2 is the abs. max. value, on the smallest one is the abs. min. value.

Ex) $f(x)=5x^{\frac{2}{3}}-2x^{\frac{5}{3}}$ Find the abs. max. and min. on $[-1,2]$.

Critical points; $x=1$ $f(1)=5-2=3$
$x=0$ $f(0)=0$

End points; $x=-1$ $f(-1)=5-(-2)=7$
$x=2$ $f(2)=\cdots=\sqrt[3]{4}$

$\Rightarrow$ $\underset{\text{abs. min.}}{0} < \sqrt[3]{4} < 3 < \underset{\text{abs. max.}}{7}$

Ex) $f(x)=x^{\frac{2}{3}}(x-4)^2$ Find the abs. max. and min. on $[-1,2]$.

$$f'(x)=\frac{2}{3}\cdot x^{\frac{-1}{3}}\cdot(x-4)^2+x^{\frac{2}{3}}\cdot 2(x-4)$$
$$=\frac{2}{x^{\frac{1}{3}}}\cdot(x-4)\left[\frac{x-4}{3}+x\right]=\frac{8}{3}\cdot\frac{(x-4)(x-1)}{x^{\frac{1}{3}}} \Rightarrow \text{Critical Points; } \begin{matrix} x=0 & (DNE) \\ x=1 & (f'(1)=0) \\ \cancel{x=4} & (f'(4)=0) \end{matrix}$$

Not in domain

Critical Points; $x=0$ $f(0)=0$
$x=1$ $f(1)=9$

End Points; $x=-1$ $f(-1)=25$
$x=2$ $f(2)=\sqrt[3]{4}$

$\underset{\text{abs. min}}{0} < \sqrt[3]{4} < 9 < \underset{\text{abs. max.}}{25}$

(★) If the interval doesn't have end points (not bounded), replace step 2 by: $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

Ex) $f(x)=\frac{x}{x^2+1}$ on $(-\infty,\infty)$

$$f'(x)=\frac{-x^2+1}{(x^2+1)^2}=0 \Rightarrow \underbrace{\begin{matrix} x=-1 \\ x=1 \end{matrix}}_{\text{Critical points}}$$

$f(-1)=\frac{-1}{2}$
$f(1)=\frac{1}{2}$
$\lim_{x\to\infty} f(x)=0$
$\lim_{x\to-\infty} f(x)=0$

$\frac{1}{2}$ is abs. max.
$-\frac{1}{2}$ is abs. min.

These won't be included to abs values. Just to check if they are bigger/smaller. (see next example)

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Ex) $f(x) = x + \frac{1}{x}$ on $(0, \infty)$. Check the abs. max. and min values.

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$ Critical Points; $x = 1$, $\cancel{x = -1}$ → Not in interval.

$f(1) = 1 + \frac{1}{1} = 2$, $\lim_{x \to 0^+} f(x) = \infty$ $\lim_{x \to \infty} f(x) = \infty$

So the abs. min is 2 but there is no abs. max value.

Mean Value Theorem:

Rolle's Theorem: Let f be a function that satisfies the following hypotheses:

(i) f is continuous on the closed and bounded interval $[a, b]$.

(ii) f is differentiable in the open interval (a, b).

(iii) f has same values at the endpoints: $f(a) = f(b)$

Then, there is at least one point c in (a, b) such that $f'(c) = 0$

Proof: Case 1: $f(x) = k$, a constant function. Then for every c in (a,b), $f'(c) = 0$.

Case 2: $f(x)$ is not a constant function. Let $f(c_1)$ be the abs. max. and $f(c_2)$ be the abs. min. value of f on $[a, b]$. So we can't have $c_1 = a$ and $c_2 = b$ since $f(x)$ is not a constant function ~~.~~ and from hyp (iii), $f(c_1) = f(a) = f(b) = f(c_2)$. We can't have $c_1 = b$ and $c_2 = a$ same way.

So one of c_1 and c_2 must be an inside point;

Say c_1 is an inside point, c_1 in (a,b). Then $f(c_1)$ is local max. By hyp (ii) $f'(c_1)$ exists then by Fermat's Theorem; $f'(c_1) = 0$.

Ex) Show that the equation $\frac{x^3}{3} - \frac{x^2}{2} + x - 1 = 0$ has $\underbrace{\text{exactly one}}_{\text{at least one and at most one}}$ real root.

1) At least one: $f(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 1$, f is continuous at every point.

$f(0) = -1$, $f(2) = \frac{8}{3} - 2 + 2 - 1 = \frac{+5}{3}$ $\Rightarrow$ by IVT $f(0) < 0$, $f(2) > 0$. There is a $f(c) = 0$ so there is at least one root c, $0 < c < 2$

2) At most one: Assume the equation has more than one root, say $f(a) = 0$ and $f(b) = 0$ and $a < b$.

f is $\underset{(i)}{\underline{\text{continuous}}}$ at $[a,b]$ and $\underset{(ii)}{\underline{\text{differentiable}}}$ on (a,b) since $f(x)$ is a polynomial

$\underset{(iii)}{\underline{f(a) = f(b) = 0}}$

Three hypotheses of Rolle's Theorem is true, then there is a k in (a,b), $f'(k) = 0$

$f'(x) = \underset{\Delta = -3}{\underline{x^2 - x + 1}}$, Since $f'(x)$ can't be 0, so the assumption "...has more than one root." is wrong. It has at most one root.

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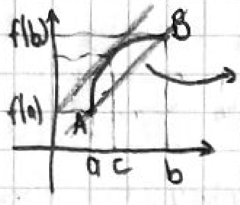
Mean Value Theorem (MVT): Assume

(i) f is continuous on the closed and bounded interval $[a, b]$.

(ii) f is differentiable in the open interval (a, b).

Then there is at least one point c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Geometric meaning:  $\longrightarrow m = \frac{f(b)-f(a)}{b-a} = f'(c)$ $\quad f(b)-f(a) = f'(c)(b-a)$

Proof: Secant line AB has equation: $y = \frac{f(b)-f(a)}{b-a}(x-b) + f(b)$

Let $g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}(x-b) + f(b)\right)$ ($g(x)$ is vertical distance between $f(x)$ and (AB))

Apply Rolle's Theorem to $g(x)$ on $[a, b]$;

(i) $g(x)$ is continuous. (ii) $g(x)$ is differentiable

(iii) $g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-b) - f(b) = 0$

$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-b) - f(b) = 0$ $\Rightarrow g(a) = g(b)$

Then there is a point c in (a,b) such that $g'(c)=0$

$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, $g'(c)=0 \Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

P.220#31 Prove that (a) for all real numbers a, b $|\sin a - \sin b| \leq |a-b|$ ... (*)

(b) for all real numbers x $|\sin x| \leq x$

(a) if $a=b$ then both sides of (*) is zero

if $a \neq b$ Assume $a<b$. Apply MVT to $f(x) = \sin x$ on interval $[a, b]$

There is a c in (a,b); $f(b)-f(a) = f'(c)(b-a)$

$\sin b - \sin a = \cos c \cdot (b-a)$

Take absolute value: $|\sin a - \sin b| = |a-b| \cdot \underbrace{|\cos c|}_{\leq 1}$

$|\sin a - \sin b| \leq |a-b|$

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(b) In part (a) choose $a=x$, $b=0$. Then, $|\sin x - \sin 0| \leq |x-0|$
$|\sin x| \leq |x|$ (x in radius)

P.220 #25) $f(1)=10$, $f'(x) \geq 2$ for $1 \leq x \leq 4$. Find the smallest value possible of $f(4)$.

f is differentiable and continuous on $[1,4]$. Apply MVT.
(Because $f'(x)$ exists.)

$f(4)-f(1) = f'(c)\cdot(4-1)$

$f(4) = 3\underbrace{f'(c)}_{\geq 2}+10 \implies f(4) = 3\cdot 2 + 10 = 16 \implies f(4) \geq 16$

Is 16 actually possible? Yes.

Let $f(x)=2x+8$, Then $f(1)=10$, $f'(x)=2 \geq 2$, For this function $f(4)=16$

Theorem: If $f'(x)=0$ for all x in (a,b), then $f(x)$ is a constant function.

→ Fix a point k in (a,b). Let x be an arbitrary point (a,b). Apply MVT to f on the interval with end points k and x.

There is a point c between k and x such that $f(k)-f(x) = f'(c)(k-x)$

c is in (a,b) so $f'(c)=0 \implies f(k)-f(x)=0$ $f(k)-f(x)$

True for every x in (a,b), so $f(x)$ is a constant function.

Corollary: If $f'(x)=g'(x)$ for all x in the interval (a,b), then $f(x)-g(x)$ is a constant function, that is $f(x)=g(x)+C$.

→ Let $h(x)=f(x)-g(x)$ for x in (a,b), $h'(x)=f'(x)-g'(x)=0$ for all x in (a,b)

So $h(x)=C$, a constant function on (a,b). That is $f(x)=g(x)+C$

Ex) $f'(x)=\sin 2x + 3x^2$, $f(0)=1$, find $f(x)$.

$f(x) = \frac{-1}{2}\cos 2x + x^3 + C$ $f(0) = \frac{-1}{2}+C=1$ $C=\frac{3}{2} \implies f(x) = \frac{-1}{2}\cos 2x + x^3 + \frac{3}{2}$

Ex) Let $f(x)=\frac{1}{x}$, $g(x) = \begin{cases} \frac{1}{x} & \text{if } x>0 \\ 1+\frac{1}{x} & \text{if } x<0 \end{cases}$

(a) $f'(x)=g'(x)$ for all x in their domains. $f'(x) = \frac{-1}{x^2}$ $g'(x)\begin{cases} -\frac{1}{x^2} & \text{if } x>0 \\ -\frac{1}{x^2} & \text{if } x<0 \end{cases}$

Corollary is not true, because domain of f, g is not an interval. Theorem and corollary are true in intervals.

$(f(x)-g(x) \neq C)$

1 or 0

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Derivatives and Shape of a Graph:

We say f is increasing on an interval I, if for all x_1, x_2 in I such that $x_1 < x_2$ we have $f(x_1) < f(x_2)$. ($\nearrow$)

We say f is decreasing on an interval I, if for all x_1, x_2 in I such that $x_1 < x_2$ we have $f(x_1) > f(x_2)$. ($\searrow$)

Increasing / Decreasing (I/D) Test:

(i) If $f'(x) > 0$ for all x in interval I, then f is $\nearrow$ on I.

(ii) If $f'(x) < 0$ for all x in interval I, then f is $\searrow$ on I.

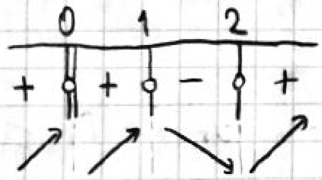
Proof for (ii): Let x_1, x_2 be two arbitrary points in I such that $x_1 < x_2$. Apply MVT to f on interval $[x_1, x_2]$. There is a point c in (x_1, x_2) such that $f(x_2) - f(x_1) = \underbrace{f'(c)}_{-}\underbrace{(x_2 - x_1)}_{+} \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow f(x_2) < f(x_1)$

Ex) $f(x) = 12x^5 - 45x^4 + 40x^3$

a) Find all intervals on which f is $\nearrow$, $\searrow$.

b) Find all local min. and max. points.

$$f'(x) = 60x^4 - 180x^3 - 120x^2 = 60x^2(x^2 - 3x + 2) = 60x^2(x-1)(x-2)$$



a) Increasing interval $(-\infty, 0)$, $(0, 1)$, $(2, \infty)$

Decreasing interval $(1, 2)$ — [,] are also acceptable.

b) Local max. $x = 1$, Local min. $x = 2$

First Derivative Test for Local Extrema: Suppose f is continuous and c is a critical point of f.

(i) If $f'(x)$ changes sign $+$ to $-$ at c, then c is a local maximum

(ii) If $f'(x)$ changes sign $-$ to $+$ at c, then c is a local minimum.

(iii) If $f'(x)$ does not change sign at c, then there is no local extrema at c.

Concavity: Let f be differentiable on an interval I. We say;

$\rightarrow$ f is concave upward on I if graph of f is above the graph of the tangent line at every point x on I. We say;

f is CU on I. (f is ∪ on I).

→ f is concave downward on I if graph of f is below the graph of the tangent line at every point x on I. We say;

f is CD on I. (f is ∩ on I).

Concavity test: Assume $f''(x)$ exist at every point of I.

(i) If $f''(x)>0$ at every point x in I, then f is CU on I. ∪

(ii) If $f''(x)<0$ at every point x in I, then f is CD on I. ∩

Ex) $f(x)=x^3-6x^2+9x+5$ Find intervals on which f is ∪ , ∩.

$f'(x)=3x^2-12x+9$

$f''(x)=6x-12$

2: − ∩ | + ∪

f is CD on $(-\infty,2)$

f is CU on $(2,\infty)$

Inflection Point: Assume f is continuous at c and concavity changes at x=c. Then the point $P(c,f(c))$ is called an inflection point.

Ex) Find inflection points of $f(x)=x^{\frac{1}{3}}$ and $g(x)=x^4$

$f'(x)=\frac{1}{3}x^{\frac{-2}{3}}$

$f''(x)=\frac{-2}{9}x^{\frac{-5}{3}}=\frac{-2}{9x^{\frac{5}{3}}}$

0: + | −

→ $f''(0)$ undefined but we only cosider f.

f is continuous on 0.

So f has an inf. point at x=0.

$g'(x)=4x^3$

$g''(x)=12x^2$

0: + ∪ | + ∪

Concavity doesn't change at x=0 so there isn't an inflection point for g.

Ex) Draw the graph of $f(x)=x^3-6x^2+9x+5$

$f'(x)=3x^2-12x+9=3(x-3)(x-1)$

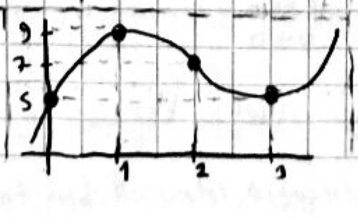
1, 3: + | − | +

f(x): ↗ ↘ ↗

$f''(x)=6x-12$

2: − | +

f(x): ∩ ∪



x		1		2		3	
f'(x)	+	0	−		−	0	+
f''(x)	−		−	0	+		+
f(x)	↗	9	↘	7	↘	5	↗

Ex) Sketch the graph $f(x)=5x^{\frac{2}{3}}-2x^{\frac{5}{3}}$. Domain is $(-\infty,\infty)$.

$f'(x)=\frac{10}{3}x^{\frac{-1}{3}}+\frac{10}{3}x^{\frac{2}{3}}=\frac{10}{3}x^{-1/3}(1-x)=\frac{10(1-x)}{3x^{1/3}}$

	0 (undefined)	1	
$f'(x)$: −	$-\infty$ / ∞	+	−

$\lim_{x\to0^-}\frac{10}{3}\frac{(1-x)}{x^{1/3}}=-\infty$

$\lim_{x\to0^+}\frac{10}{3}\frac{(1-0)}{x^{1/3}}=\infty$ $\Rightarrow$ cusp at $x=0$

$f''(x)=\frac{-10}{9}x^{\frac{-4}{3}}+\frac{-20}{9}x^{-1/3}=\frac{-10x^{\frac{-4}{3}}}{9}(1+2x)=\frac{-10}{9}\cdot\frac{(2x+1)}{x^{4/3}}$

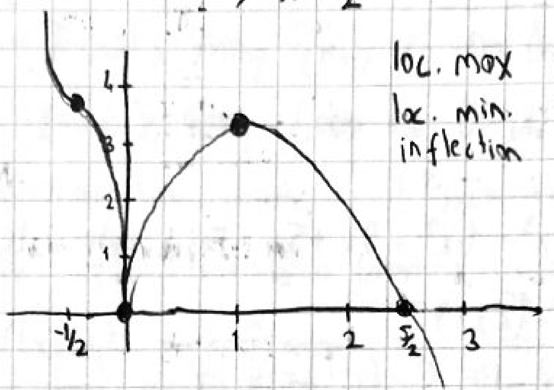
	$-1/2$		0 (undefined)	
$f''(x)$:	−	+		+
$f(x)$:	∩	∪		∩

y intercept: $f(0)=0-0=0$

x intercepts: $f(x)=0 \Rightarrow x^{\frac{2}{3}}(5-2x)=0$

$x=0$, $x=\frac{5}{2}$

x		$-1/2$		0		1	
$f'(x)$	−		−	0 ($\pm\infty$)	+	0	−
$f''(x)$	+	0	−		−		−
$f(x)$	↘	3,75	↘	0	↗	3	↘



loc. max $x=1$
loc. min. $x=0$
inflection $x=-1/2$

Second Derivative Test: Suppose $f''(0)$ is continuous on an open interval that contains $x=c$ and also suppose $f'(c)=0$;

if $f''(c)>0$, then f has a local min at c. ∪

if $f''(c)<0$, then f has a local max at c. ∩

Ex) $f(x)=\frac{x^2}{1+x^2}$ Find all local max and min values of f.

$f'(x)=\frac{1-x^2}{(1+x^2)^2}=0 \Rightarrow 1-x^2=0$, $x=1$, $x=-1$

$f''(x)=\frac{-6x+2x^3}{(1+x^2)^3}$

$f''(1)=\frac{-4}{8}<0$, loc. max at $x=1$

$f''(-1)=\frac{4}{8}>0$, loc. min at $x=-1$

Limits of Infinity and Horizontal Asymptotes: Let f be defined on some interval (a,∞). We write $\lim_{x\to\infty}f(x)=L$ if $f(x)$ values can be made arbitrarily close to L by taking x sufficently large.

Ex] $\lim_{x\to\pm\infty}x\cdot\sin\frac{1}{x}=\lim_{x\to\pm\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{u\to0}\frac{\sin u}{u}=0$

$u=\frac{1}{x}$, $x\to\pm\infty$, $u\to0$

Ex] $\lim_{x\to\pm\infty}\frac{\sin x}{x}=0$ by Squeeze theorem

$-1\le\sin x\le1$

$\underbrace{\frac{-1}{x}}_{0}\le\frac{\sin x}{x}\le\underbrace{\frac{1}{x}}_{0}\Rightarrow\lim_{x\to\pm\infty}\frac{\sin x}{x}=0$ by squeeze theorem.

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Remark: All limit rules are valid for $\lim\limits_{x\to\pm\infty}$ except: $\lim\limits_{x\to a} x^n = a^n$

Ex $\lim\limits_{t\to\infty} \dfrac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$ Divide the numerator and denominator by the highest power in the denominator, that is $t^{3/2}$.

$$\lim_{t\to\infty}\frac{\frac{t-t\sqrt{t}}{t^{3/2}}}{\frac{2t^{3/2}+3t-5}{t^{3/2}}} = \frac{\frac{1}{t^{1/2}}-1}{2+\frac{3}{t^{1/2}}-\frac{5}{t^{3/2}}} = \frac{0-1}{2+0-0} = \frac{-1}{2}$$

Remark: $\lim\limits_{x\to\infty}(x^2-x) = \lim\limits_{x\to\infty} \underbrace{x}_{\infty}\underbrace{(x-1)}_{\infty} = \infty$

Ex $\lim\limits_{x\to\infty}\sqrt{x^2+x}-x = \dfrac{(\sqrt{x^2-x}-x)(\sqrt{x^2-x}+x)}{\sqrt{x^2-x}+x} = \dfrac{x^2-x-x^2}{\underbrace{\sqrt{x^2(1-\frac{1}{x})}}_{|x|\cdot\sqrt{1-1/x}}+x} = \dfrac{-x}{x\left(\sqrt{1-\frac{1}{x}}+1\right)} = \dfrac{-1}{\sqrt{1-\frac{1}{x}}+1} = \dfrac{-1}{2}$

$\infty-\infty$ undefined $\Rightarrow$ multiply by conj.

Ex $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right) = \dfrac{(x+\sqrt{x^2+2x})(x-\sqrt{x^2+2x})}{(x-\sqrt{x^2+2x})} = \dfrac{x^2-x^2-2x}{x-|x|\sqrt{1+\frac{2}{x}}} = \dfrac{-2x}{x+x\sqrt{1+\frac{2}{x}}} = \dfrac{-2}{1+\sqrt{1+\frac{2}{x}}} = -1$

$-\infty+\infty$ = undefined $\Rightarrow$ multiply by conj.

Ex $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = \dfrac{x}{x\sqrt{1+\frac{1}{x^2}}} = \dfrac{1}{\sqrt{1+\frac{1}{x^2}}} = 1$

Ex $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = \dfrac{x}{-x\sqrt{1+\frac{1}{x^2}}} = \dfrac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1$

Horizontal Asymptote: The (horizontal) line $y=b$ is called a horizontal asymptote of f if $\lim\limits_{x\to\infty} f(x)=b$ or $\lim\limits_{x\to-\infty} f(x)=b$ or both.

Slant Asymptote: If as $x\to\infty$ or $x\to-\infty$, the graph of f comes close to the line $y=mx+b$, then we say the line $y=mx+b$ is a slant asymptote (or inclined asymptote) of f.

$\lim\limits_{x\to\infty}\left(f(x)-(mx+b)\right)=0 \Longrightarrow (mx+b)$ is an inclined asymptote.

Ex $f(x)=\dfrac{x^2-x}{x-2} \xrightarrow{\text{polynomial division}} \dfrac{x^2-x}{x-2} = x+1+\dfrac{2}{x-2} \Rightarrow f(x)-(x+1) = \dfrac{2}{x-2} \Rightarrow$

limit at infinity $\lim\limits_{x\to\pm\infty}\left(f(x)-(x+1)\right)=0$, then $y=x+1$ is a slant asymptote.

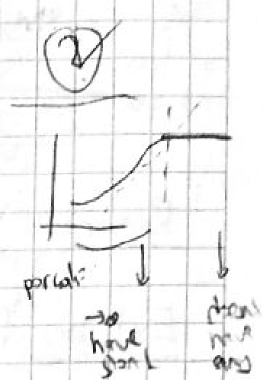
Remark: We use this method at rational functions like; $\dfrac{P(x)}{Q(x)}$, degree $[P(x)]$ = degree $[Q(x)]+1$

Remark: A function cant have slant and horizontal asymptote at the same time, same side.

Ex) $\lim_{x\to-\infty}(x+\sqrt{x^2+2x})=-1$ so find a slant asymptote at $-\infty$ of $g(x)=\sqrt{x^2+2x}$

$\lim_{x\to-\infty}(x+\sqrt{x^2+2x}+1)=0$ $\lim_{x\to-\infty}\left(\sqrt{x^2+2x}-(-x-1)\right)=0$ → $y=-x-1$ is the slant asymptote

Curve Sketching: To sketch the graph of $y=f(x)$;



1- Find domain of f.

2- Find $f'(x)$ and $f''(x)$, then examine their signs.

3- Find all asymptotes (vertical, horizontal, slant) if there are any.

4- Find intercepts as possible.

5- Make a table with all information, then sketch it.

Ex) $f(x)=\dfrac{2x^3}{x^3+1}$ sketch the graph of $y=f(x)$.

Vertical Asymptote: $\lim_{x\to-1^+}f(x)=-\infty$ $\lim_{x\to-1^-}f(x)=\infty$ $x=-1$

Domain: All $x\neq-1$

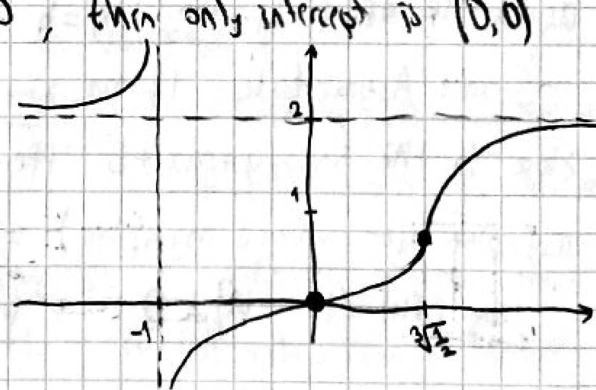
Horizontal Asymptote: $\lim_{x\to\pm\infty}f(x)=2$ $y=2$

$f'(x)=\dfrac{6x^2}{(x^3+1)^2}$ sign: $+$ ‖ -1 ‖ $+$ | 0 | $+$

$f''(x)=6\cdot\dfrac{2x(x^3+1)^2-x^2\cdot2(x^3+1)\cdot3x^2}{(x^3+1)^{\cancel{4}\,3}}=\dfrac{2x^4+2x-6x^4}{(x^3+1)^3}=12\,\dfrac{x(1-2x^3)}{(x^3+1)^3}$ sign: $+$ ‖ -1 ‖ $-$ | 0 | $+$ | $1/\sqrt[3]{2}$ | $-$

Intercepts: $x=0\Rightarrow y=0$, $y=0\Rightarrow x=0$, then only intercept is $(0,0)$

	$-\infty$		-1		0		$\sqrt[3]{1/2}$		∞
$f'(x)$		$+$		$+$	0	$+$		$+$	
$f''(x)$		$+$		$-$	0	$+$	0	$-$	
$f(x)$	2	↗ $+\infty$	‖	$-\infty$ ↗	0	↗	$\frac{2}{3}$	↗	≈2



How to find abs. ext. values: The interval I (maybe closed-bounded or not)

(I) Modified Closed Interval) 1- Compute $f'(x)$. → Critical: $f'(c)=0$, $f'(c)$-DNE ✱

2- Find the critical points of f in the interval I.

3- Add end points to the step 2. (End points are open or close values of I including $\pm\infty$)

4- For all these points in the I, compute the value of f.
For all these points not in the I, compute the limit of f.

5- If the largest "number" occurs in I, then the number is abs. max. value.
If the smallest "number" occurs in I, then the number is abs. min. value.

(II) Graph the curve of the function

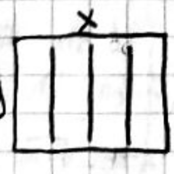
(III) The first derivative for abs. extrema) Suppose c is the only critical point of f on I.

If $f'(x)>0$ for all $x<c$ and $f'(x)<0$ for all $x>c$, then $f(c)$ is abs. max.

(IV) The second derivative for abs. extrema) Suppose c is the only critical point of f on I and $f''(c)<0$.
Then $f(c)$ is abs. max of f.
$\rightarrow$ It's similar with abs. min. at methods III and IV.

<u>Optimization Problems:</u> Deal with the abs. extrema. Translate to normal lang at last.

Ex) A farmer with 300 m fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the possible largest area of the four pens?



$A = x \cdot y$ $\qquad 2x+5y=300 \qquad x=\frac{300-5y}{2}$

$A=\frac{y(300-5y)}{2}=\frac{5}{2}(60y-y^2)$

We should have; $x \geq 0$, $y \geq 0$ $\quad 2x=300-5y \geq 0 \Rightarrow 60 \geq y$

Domain: $[0, 60]$

Find the abs. max value of $A=\frac{5}{2}(60y-y^2)$ on the $[0, 60]$

$\frac{dA}{dy}=\frac{5}{2}(60-2y)=0$

$y=30$

<u>Method I</u>

$y=0 \Rightarrow A=0$
$y=30 \Rightarrow A=2250$
$y=60 \Rightarrow A=0$

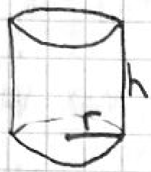
Endpoints, critical point
So by closed interval method, ...

<u>Method III</u>

y		30	
$\frac{dA}{dy}$	+	0	−

By the first derivative test for abs. extrema, A has abs. max at $y=30$

Ex A right circular cylinder is to be made to hold 1 lt of oil. Find the dimensions of the can that will minimize the total surface area.



$S = \underbrace{2\pi r h + 2\pi r^2}_{\text{Find abs. min}}$ | $\pi r^2 h = 1000 \text{ cm}^3$, $h = \dfrac{1000}{\pi r^2}$ | $S = \dfrac{2\pi r \cdot 1000}{\pi r^2} + 2\pi r^2$, $S = \dfrac{2000}{r} + 2\pi r^2$

We should have: $r>0$, $h>0$, so the domain is $(0,\infty)$.

Find abs. min of $S = \frac{2000}{r} + 2\pi r^2$ in the domain $(0,\infty)$.

$$\frac{dS}{dr} = \frac{-2000}{r^2} + 4\pi r = 0 \qquad \pi r^3 - 500 = 0$$
$$\frac{-2000 + 4\pi r^3}{r^2} = 0 \qquad r = \sqrt[3]{\frac{500}{\pi}}$$

Method III

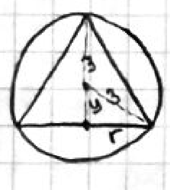
r		$\sqrt[3]{\frac{500}{\pi}}$	
$\frac{dS}{dr}$	$-$	0	$+$
	$\searrow$		$\nearrow$

By the first derivative test for abs. extrema, S has abs. min at $r = \sqrt[3]{\frac{500}{\pi}}$

So the dimensions for the min. surface area, $r = \sqrt[3]{\frac{500}{\pi}}$ cm , $h = \dfrac{1000}{\pi \cdot \sqrt[3]{\left(\frac{500}{\pi}\right)^2}}$ cm

$h = 2 \cdot \sqrt[3]{\frac{500}{\pi}}$ cm

Ex Find the dimensions of the right cone of the largest volume that can be inscribed in a sphere of radius 3 dm.



$h_c = y + 3$ $\qquad V_c = \dfrac{\pi r^2 h}{3}$ $\qquad$ Domain: $[0,3]$

$r^2 = 9 - y^2$ $\qquad V_c = \dfrac{\pi}{3}(9-y^2)(3+y) = \dfrac{\pi}{3}(27 + 9y - 3y^2 - y^3)$

Find abs. min of V_c when $0 \leq y \leq 3$.

$$\frac{dV}{dy} = \frac{\pi}{3}(9 - 6y - 3y^2) = -\pi(y+3)(y-1) = 0$$
$$y = -3 \quad y = 1$$

Method I

$y=0 \Rightarrow V_c = 9\pi$

$y=1 \Rightarrow V_c = \frac{32\pi}{3}$) Largest

$y=3 \Rightarrow V_c = 0$

So the dimensions for the largest;

$r = \sqrt{9-1}$

$r = 2\sqrt{2}$ $\quad h = 4$

Newton's Method: This method is used to find the roots of equations approximately. $f(x) = 0$

1- Start with a point x_0.

2- Draw the tangent line at x_0. ($y - f(x_0) = f'(x_0)(x - x_0)$

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3- Find the point x_1 where the tangent line crosses the x-axis. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

4- Draw a tangent line at x_1 and... $\lim_{n\to\infty} x_n = c$ is the root for $f(x)=0$

Antiderivatives: A function $F(x)$ is called an antiderivative of a function $f(x)$ on an interval I if $F'(x) = f(x)$ at all points x in I.

Ex) $f(x) = \cos x$, Then $F_1(x) = \sin x$, $F_2(x) = \sin x - 3$, $F_3(x) = \sin x + \pi$ are all antiderivatives of $f(x) = \cos x$, on $(-\infty, \infty)$

If $F'(x) = G'(x)$ then, $F(x) = G(x) + C$.

If $F(x)$ is an antiderivative of $f(x)$ then, "$F(x)+C$" is the most general antiderivative.

Ex) Find the antiderivative of $f(x) = 2\sec^2 x \cdot \tan x$.

$f(x) = 2\sec^2 x \tan x = \frac{d}{dx}(\tan^2 x)$ so $F(x) = \tan^2(x)$ is an antiderivative

$f(x) = 2\sec x \cdot \tan x \sec x = \frac{d}{dx}(\sec^2 x)$ so $G(x) = \sec^2(x)$ is an antiderivative

Note that $\tan^2 x + 1 = \sec^2 x$ so writing $+C$ in the most general antiderivative is important

Function	Antiderivative
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$\cos(kx) \quad (k \neq 0)$	$\frac{1}{k}\sin(kx) + c$
$\sin(kx) \quad (k = 0)$	$-\frac{1}{k}\cos(kx) + c$
$\sec^2 x$	$\tan x + c$
$\csc^2 x$	$-\cot x + c$
$\sec x \cdot \tan x$	$\sec x + c$
$\csc x \cdot \cot x$	$-\csc x + c$

Ex) $f'(x) = 1 + 3\sqrt{x}$, $f(4) = 25$, $f(x) = ?$

$f'(x) = 1 + 3 \cdot x^{1/2} \Rightarrow f(x) = x + \frac{3 \cdot x^{3/2}}{\frac{3}{2}} = x + 2x^{3/2} + c$

for all $x \geq 0$

$f(4) = 4 + 2 \cdot \sqrt{4^3} + c$

$25 = 4 + 16 + c \quad c = 5$

$f(x) = x + 2x^{3/2} + 5$

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Ex) $f'(x) = x^{-1/3}$ a) Find the general antiderivative of f' on its domain.
b) Find $f(x)$ if we have: $f(1)=1$, $f(-1)=-1$

$f'(x) = \dfrac{1}{x^{1/3}}$ $f'(x) = x^{\frac{-1}{3}} = \dfrac{d}{dx}\left(\frac{3}{2}x^{2/3} + c\right)$

domain: $(-\infty, 0) \cup (0, \infty)$

a) So the general a.d: $f(x) = \begin{cases} \dfrac{3x^{2/3}}{2} + C_1, & x<0 \\ \dfrac{3x^{2/3}}{2} + C_2, & x>0 \end{cases}$

And for (b), $\frac{3}{2} + C_2 = 1$, $C_2 = \frac{-1}{2}$ $\frac{3}{2} + C_1 = -1$, $C_1 = \frac{-5}{2}$ $\Rightarrow$ $f(x) = \begin{cases} \dfrac{3x^{2/3}}{2} - \dfrac{5}{2}, & x<0 \\ \dfrac{3x^{2/3}}{2} - \dfrac{1}{2}, & x>0 \end{cases}$

Ex) $f''(x) = 2 + \cos x$, $f(0) = -1$, $f\left(\frac{\pi}{2}\right) = 0$, find $f(x)$.

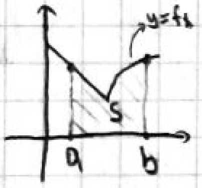
$f'(x) = 2x + \sin x + A$
$f(x) = x^2 - \cos x + Ax + B$

$f(0) = -1 \Rightarrow -1 + B = -1$ $B = 0$
$f\left(\frac{\pi}{2}\right) = 0 \Rightarrow \frac{\pi^2}{4} + A\cdot\frac{\pi}{2} = 0$ $A = \frac{-\pi}{2}$

$f(x) = x^2 - \cos x - \frac{\pi}{2}x$

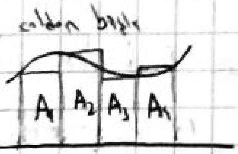
Areas and Distances:

Area Problem: Suppose f is continuous and $f(x) \geq 0$ on the $[a,b]$

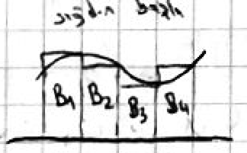


$S = \{(x,y) : a \leq x \leq b,\ 0 < y < f(x)\}$, How to find $A(S) = ?$

To find the area of S, we divide the interval $[a,b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$, where n is a arbitrary positive int.



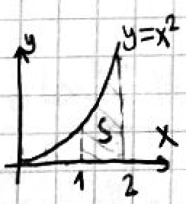
$L_4 = A_1 + A_2 + A_3 + A_4$



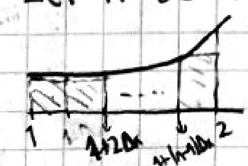
$R_4 = B_1 + B_2 + B_3 + B_4$

Then we sum the rectangles made by right rectangles (L_n or R_n). And finally we take limit $n \to \infty$ to find $A(s)$.

Ex)



Let n be an arbitrary positive integer. $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$



$L_n = \Delta x \cdot 1^2 + \Delta x(1+\Delta x)^2 + \Delta x(1+2\Delta x)^2 + \dots + \Delta x(1+(n-1)\Delta x)^2$

$L_n = \Delta x \cdot \left[(1^2 + 1^2 + 1^2 \dots) + (2\Delta x + 4\Delta x + 6\Delta x \dots) + (\Delta x^2 + (2\Delta x)^2 \dots)\right]$

$L_n = \Delta x\left[n + 2\Delta x(1+2+3\dots(n-1)) + \Delta x^2(1^2+2^2+\dots(n-1)^2)\right]$

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$L_n = \Delta x \cdot \left[n + 2\Delta x \cdot \frac{(n-1)n}{2} + (\Delta x)^2 \cdot \frac{(n-1)\cdot n\,(2n-1)}{6}\right]$, $\Delta x = \frac{1}{n}$

$L_n = \frac{1}{n}\left[n + \frac{\cancel{2}(n-1)\cancel{n}}{\cancel{n}\cdot\cancel{2}} + \frac{1}{n^{\cancel{2}}}\cdot\frac{(n-1)\cancel{n}(2n-1)}{6}\right] = \frac{1}{n}\cdot\left[2n-1 + \frac{(n-1)(2n-1)}{6}\right]$

So, $L_n = 2 - \frac{1}{n} + \frac{(n-1)(2n-1)}{6n^2}$ In this case we have $L_n < A(S)$

$R_n = \Delta x(1+\Delta x)^2 + \Delta x(1+2\Delta x)^2 \cdots + \Delta x(1+n\Delta x)^2 = \cdots$

So, $R_n = 2 + \frac{1}{n} + \frac{(n-1)(2n-1)}{6n^2}$ In this case we have $R_n > A(S)$

So, $\underbrace{2 - \frac{1}{n} + \frac{(n-1)(2n-1)}{6n^2}}_{2+\frac{2}{6} = 7/3} < A(S) < \underbrace{2 + \frac{1}{n} + \frac{(n-1)(2n-1)}{6n^2}}_{2+\frac{2}{6} = 7/3}$ True for every natural number n.

Let $n \to \infty$

By Squeeze Theorem; $A(S) = 7/3$

<u>The Definite Integral</u>: Assume f is defined on $[a,b]$. Given any natural number n, we divide the $[a,b]$ into n subintervals of length $\Delta = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2 \cdots x_n = b$ be the endpoints of these subintervals. From every subinterval, we choose sample points $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ so that x_i^* is in the $[x_{i-1}, x_i]$. We form the sum

$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\cdot\Delta x = \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$ (Riemann sum)

Take limit of the Reimann sum S_n. If $\left(\lim_{n\to\infty} S_n\right)$ exists, and has the same value for all choices of the sample points in the corresponding intervals, we say that "f is integrable on $[a,b]$" and we call;

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot\Delta x = \int_a^b f(x)\,dx$$

$$\int_{a\,(\text{lower limit})}^{b\,(\text{upper limit})} \underbrace{f(x)}_{\text{integrand}}\,\overbrace{dx}^{\text{Means that the integral is with respect to the variable } x.}$$

Note

$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \cdots$

Definite integral does not depend on the variable.

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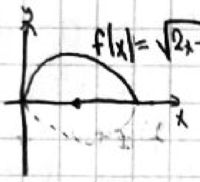
Remark: Although we divided $[a,b]$ into subintervals of equal length Δx. But they might have different lengths $\Delta x_1, \Delta x_2, \dots \Delta x_n$, where the $\max(\Delta x_1, \Delta x_2 \dots \Delta x_n) \to 0$ as $n \to \infty$. But if f is continuous on $[a,b]$, this doesn't make any difference since continuous functions are always integrable.

Remark: If $f(x)$ is integrable on $[a,b]$ and $f(x) \geq 0$, then $\int_a^b f(x)\,dx = A(S)$, area of the region under the graph of f.

Ex) $\int_0^2 \sqrt{2x-x^2}\,dx = ?$ $\quad f(x) \geq 0$ so the answer is an area A.

$f(x) = y = \sqrt{2x-x^2}$

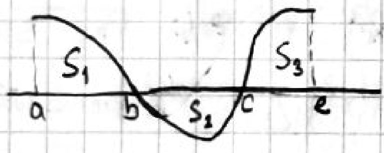
$y^2 = 2x - x^2 \Rightarrow (x-1)^2 + y^2 = 1$



so $A = \frac{\pi \cdot 1^2}{2} = \frac{\pi}{2}$

Remark: If $f(x)$ is integrable on $[a,b]$ and $f(x) \leq 0$, then $-\int_a^b f(x)\,dx = A(S)$, area of the region above the graph of f.

Ex)



$A(S_1) = 2$
$A(S_2) = 1$
$A(S_3) = 5$

$\Rightarrow \int_a^e f(x)\,dx = S_1 - S_2 + S_3 = 2 - 1 + 5 = 6$

Remark: If the sample points $x_1^*, x_2^*, \dots x_n^*$ are chosen

(i) as the left end points of the corresponding subintervals, then the Riemann Sum is called the left sum. $\quad x_0 — x_1 — x_2 \dots$ (x_1^* at x_0, x_2^* at x_1)

(ii) as the right end points of the corresponding subintervals, then the Riemann Sum is called the right sum. $\quad x_0 — x_1 — x_2 \dots$ (x_1^* at x_1, x_2^* at x_2)

(iii) as the midpoints of the corresponding subintervals, then the Riemann Sum is called the mid sum. $\quad x_0 — x_1 — x_2 \dots$ (x_1^*, x_2^* at midpoints)

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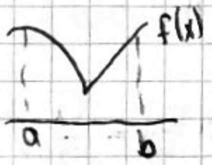
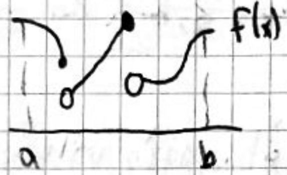
(iv) as the points where $f(x)$ takes abs. max. in the corresponding subintervals, then the Riemann sum is called the upper sum.

(v) as the points where $f(x)$ takes abs. min. in the corresponding subintervals, then the Riemann sum is called the lower sum.

<u>Theorem</u>: If f is continuous on $[a,b]$ or if f has only a finite number of jump discontinuities on $[a,b]$ then f is integrable on $[a,b]$.

<u>Ex</u>





These functions are both integrable on $[a,b]$

<u>Fact</u>: Density of rational and irrational numbers: Given any two real numbers a, b such that $a<b$. Then there is a rational number r and irrational number s such that $a<r<b$ and $a<s<b$.

<u>Ex</u>

$$f(x)=\begin{cases} 0 & \text{if } x \text{ is a rational number.} \\ 1 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Is it integrable on $[0,1]$?

Given any n, choose sample points all rational; $S_n=0$
choose sample points all irrational; $S_n=1$, $\lim_{n\to\infty} S_n = DNE$
It's not integrable.

<u>Remark</u>: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ and $\int_a^a f(x)\,dx = 0$

<u>Properties</u>: 1) $\int_a^b c\,dx = c\cdot(b-a)$

2) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

3) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$ Note: It is NOT necessary to $a<c<b$

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5) $\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$

6) If $f(x)\geq 0$ for all x in $[a,b]$, $\int_a^b f(x)\,dx \geq 0$

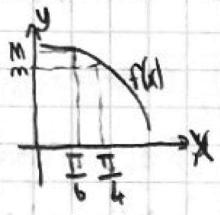
7) If $f(x)\geq g(x)$ for all x in $[a,b]$, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8) If $m\leq f(x)\leq M$ for all x in $[a,b]$, $m(b-a)\leq \int_a^b f(x) \leq M(b-a)$

Ex) Show that $\frac{\pi\sqrt{2}}{24} \leq \underbrace{\int_{\pi/6}^{\pi/4}\cos x\,dx}_{I} \leq \frac{\pi\sqrt{3}}{24}$

$f(x)=\cos x$ on $\left[\frac{\pi}{6},\frac{\pi}{4}\right]$ has abs. max. value $M=f\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

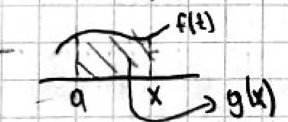
has abs. min. value $m=f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$



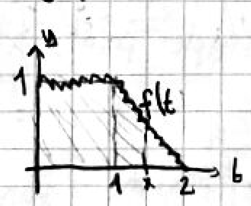
By property 8, $\underbrace{\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}\leq I\leq \underbrace{\frac{\sqrt{3}}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}$

The Fundamental Theorem of Calculus: Assume f is continuous on $[a,b]$. We define a new function $g:[a,b]\to\mathbb{R}$ as follows:

$g(x)=\int_a^x f(t)\,dt$



Ex) $f(t)\begin{cases}1 & \text{if } 0\leq t\leq 1\\ 2-t & \text{if } 1<t\leq 2\end{cases}$



$g(x)=\begin{cases}x & \text{if } 0\leq x\leq 1\\ \frac{-x^2+4x-1}{2} & \text{if } 1<x\leq 2\end{cases}$

Part 1 (FTC1): Let f be continuous on $[a,b]$. Let $g(x)=\int_a^x f(t)\,dt$, $a\leq x\leq b$

Then g is continuous on $[a,b]$, differentiable on (a,b) and, $g'(x)=f(x)$

By using Leibniz' notation; $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Ex) $\frac{d}{dx}\int_1^x \sqrt{t^7+1}\,dt=\sqrt{x^7+1}$ Ex) $\frac{d}{dx}\int_x^7 \sin^2 t\,dt = -\sin^2 x$

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Ex] $\frac{d}{dx}\int_2^{x^2}\frac{1}{t+t^3}\,dt = \frac{dx^2}{dx}\cdot\frac{d}{dx^2}\int_2^{x^2}\frac{1}{t+t^3} = 2x\left(\frac{1}{x^2+x^6}\right) = \frac{2x}{x^2+x^6}$

$$\frac{d}{dx}\int_a^{v(x)} f(t)\,dt = f(v(x))\cdot v'(x) \quad \text{[By chain rule]}$$

$$\left.\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)\right\}\ \underline{\text{Leibniz Rule}}$$

Ex] $\frac{d}{dx}\int_{\tan x}^{x^2}\frac{1}{\sqrt{2+t^4}}\,dt = \frac{1}{\sqrt{2+x^8}}\,2x - \frac{1}{\sqrt{2+\tan^4 x}}\cdot\sec^2 x$

<u>Average Value of a Function:</u> If $y_1, y_2, y_3 \cdots, y_n$ are n numbers, then their average is, $y_{ave} = \frac{y_1+y_2+\ldots+y_n}{n}$.

How do we define the average of a continuous function f on $[a,b]$?

Assume $y=f(x)$ for $a<x<b$ and f is continuous on $[a,b]$. We divide $[a,b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. From each subinterval we choose sample points $x_1^*, x_2^* \ldots x_n^*$. We find average of $f(x_1^*)$, $f(x_2^*), \ldots, f(x_n^*)$ which is $\frac{f(x_1^*)+f(x_2^*)+\ldots+f(x_n^*)}{n} = \quad n = \frac{b-a}{\Delta x}$

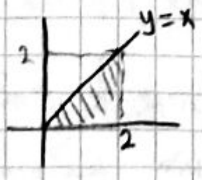
$= \frac{1}{b-a}\left(\Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \cdots + \Delta x\, f(x_n^*)\right)$ $\frac{1}{b-a}\cdot$(Reimann Sum)

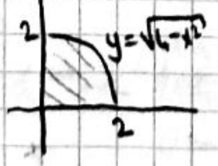
Take limit $n\to\infty$

$$f_{ave} = \frac{1}{b-a}\cdot\int_a^b f(x)\,dx.$$

Ex] $f(x) = x+\sqrt{4-x^2}$ on $[0,2]$. Find the fave.

$$f_{ave} = \frac{1}{2-0}\cdot\int_0^2 (x+\sqrt{4-x^2})\,dx = \frac{1}{2}\left(\int_0^2 x\,dx + \int_0^2\sqrt{4-x^2}\,dx\right)$$





$$\Rightarrow f_{ave} = \frac{1}{2}(2+\pi)$$
$$= 1+\frac{\pi}{2}$$

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Mean Value Theorem for Integrals: Assume f is continuous on $[a,b]$. Then there is a number c in $[a,b]$ such that; $\int_a^b f(x)\,dx = (b-a)\cdot f(c)$

That is, $f(c) = \frac{1}{b-a}\cdot \int_a^b f(x)\,dx$. $(f(c) = f_{ave})$

Proof: Since f is continuous on $[a,b]$, m = abs. min and M = abs. max values exists on $[a,b]$ by Extreme Value Theorem.

$$m \leq f(x) \leq M$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \Rightarrow m \leq \underbrace{\frac{\int_a^b f(x)dx}{(b-a)}}_{N} \leq M$$

Since f is continuous on $[a,b]$ there are points α, β in $[a,b]$ such that $f(\alpha)=m$, $f(\beta)=M$. Then $f(\alpha) \leq N \leq f(\beta)$.

By IVT, there is a point c between α and β such that $f(c)=N$

Ex) Assume f is continuous on $[1,3]$ and $\int_1^3 f(x)dx = 8$. Show that f takes on the value 4 at least once on interval $[1,3]$.

$\int_1^3 f(x)dx = (3-1)\,f(c)$ for some c in $[1,3]$ by MVT for integrals.

$8 = 2\cdot f(c) \Rightarrow f(c)=4$ for some c in $[1,3]$.

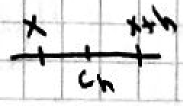
FTC 1: Let f be continuous on $[a,b]$ and, $g(x) = \int_a^x f(t)dt$, $a \leq x \leq b$. Then $g(x)$ is continuous on $[a,b]$, differentiable in (a,b) and $g'(x) = f(x)$, $a<x<b$.

Proof of differentiability: Fix a point x in (a,b).

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h} =$$

$$= \frac{(\cancel{x}+h-\cancel{x})\cdot f(c_h)}{\cancel{h}} \quad \text{for some point } c_h \text{ between } x \text{ and } x+h$$

$= f(c_h)$ As $h\to 0$, we have $x+h \to x$ and $c_h \to x$. Since f is continuous at x, $f(c_h) \to f(x)$ as $h\to 0$.



$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} f(c_h) = f(x)$$

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<u>FTC 2:</u> Let f be continuous on $[a,b]$ and let F be antiderivative of f on $[a,b]$ that is $F'(x)=f(x)$ for all x in $[a,b]$. Then,

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

<u>Proof:</u> Let $g(x)=\int_a^x f(t)\,dt$, $a \leq x \leq b$. Then $g'(x)=f(x)$ by FTC1.

Also $F'(x)=f(x)$ so $F'(x)=g'(x)$ and $F(x)=g(x)+C$

Let $x=a$ $\quad F(a)=\underbrace{g(a)}_{\int_a^a f(t)dt=0}+C \Rightarrow F(a)=C \Rightarrow F(x)=g(x)+F(a)$

Let $x=b$ $\quad F(b)=g(b)+F(a) \Rightarrow F(b)-F(a)=\int_a^b f(x)\,dx$

Ex) $\int_1^2 x^2\,dx$, $\frac{d}{dx}\left(\frac{x^3}{3}\right)=x^2$ so $F(x)=\frac{x^3}{3}$, $F(2)-F(1)=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}$

<u>Notation</u> = $F(b)-F(a)=F(x)\Big|_a^b=F(x)\Big]_a^b$

<u>Remark</u>: In definite integral, there is no need to write C in the antiderivative.

Ex) $\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \sqrt{x}-\frac{1}{\sqrt{x}}\,dx = \int_1^9 x^{1/2}-x^{-1/2} = \frac{x^{3/2}}{3/2}-\frac{x^{1/2}}{1/2}\Big|_1^9 = \frac{40}{3}$

Ex) $L=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\cdot\sqrt{n+1}}+\frac{1}{\sqrt{n}\cdot\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\cdot\sqrt{n+n}}\right)=?$

Try to recognize S_n as a Riemann sum of some function $f(x)$ on $[a,b]$.
S_n has n terms so $[a,b]$ divided into n subintervals of length $\Delta x=\frac{b-a}{n}$
So we must have $\frac{1}{n}$ at each term. (b-a is a constant so not important)
Multiply each term of S_n with $\frac{\sqrt{n}}{\sqrt{n}}$ to get $\frac{1}{n}$.

$$S_n=\frac{\sqrt{n}}{n(\sqrt{n+1})}+\frac{\sqrt{n}}{n(\sqrt{n+2})}+\cdots+\frac{\sqrt{n}}{n(\sqrt{n+n})}=\frac{\Delta x}{b-a}\cdot\frac{1}{\sqrt{1+\frac{1}{n}}}+\frac{\Delta x}{b-a}\cdot\frac{1}{\sqrt{1+\frac{2}{n}}}+\cdots\frac{\Delta x}{b-a}\cdot\frac{1}{\sqrt{1+\frac{n}{n}}}$$

$\frac{1}{n}=\frac{\Delta x}{b-a}$

$$=\frac{1}{b-a}\cdot\left(\frac{\Delta x}{\sqrt{1+\frac{1}{n}}}+\frac{\Delta x}{\sqrt{1+\frac{2}{n}}}+\cdots\frac{\Delta x}{\sqrt{1+\frac{n}{n}}}\right)$$

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$$S_n = \frac{1}{b-a} \cdot \left(\Delta x \frac{1}{\sqrt{1+\frac{1}{n}}} + \Delta x \frac{1}{\sqrt{1+\frac{2}{n}}} + \cdots \Delta x \frac{1}{\sqrt{1+\frac{n}{n}}} \right)$$

$x_1^* = 1 + \frac{1}{n}$

$x_2^* = 1 + \frac{2}{n}$

$x_n = 1 + \frac{1}{n}$

Reimann Sum: $\Delta x \cdot f(x_1^*) + \Delta x\, f(x_2^*) + \cdots + \Delta x \cdot f(x_n^*) \Rightarrow f(x) = \frac{1}{\sqrt{x}}$

As $n \to \infty$, $x_1^* \to a$ and $x_n^* \to b$. So $a = \lim_{n\to\infty} (1 + \frac{1}{n}) = 1$ $b = \lim_{n\to\infty} (1 + \frac{n}{n}) = 2$

Then $S_n = \frac{1}{2-1}$ Riemann sum for $f(x) = \frac{1}{\sqrt{x}}$ on $[1,2]$

Antiderivative for $x^{-\frac{1}{2}} \Rightarrow \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_1^2 = 2\sqrt{2} - 2$

Indefinite Integral and Net Change Theorem: The most general antiderivative of $f(x)$ on an interval (a,b) is called the indefinite Integral of f on (a,b). It's denoted by $\int f(x)\,dx$

Ex] $\int x^2\,dx = \frac{x^3}{3} + C$. | By FTC2; $\int_a^b f(x)\,dx = \left. \int f(x)\,dx \right|_a^b$

Table of Integrals: That we should all know:

$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$ $\int k\,dx = kx + C$

$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

$\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \tan x\,dx = \sec x + C$ $\int \csc x \cot x\,dx = -\csc x + C$

Remark: When a formula for indefinite integral is written, we should know that it's valid on an interval.

Ex] $\int \sec^2 x\,dx = \tan x + C$ is valid on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or $(\frac{\pi}{2}, \frac{3\pi}{2})$ or ...

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Ex) $\int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + C_1 + \frac{x^{-1}}{-1} + C_2 = \frac{x^3}{3} - \frac{1}{x} + C$ valid on $(-\infty,0)$ or $(0,\infty)$

Ex) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\cdot\tan^2\theta}{\sec^2\theta} = \int_0^{\pi/3} \frac{\sin\theta\,\cancel{(1+\tan^2\theta)}}{\cancel{\sec^2\theta}} = -\cos\theta \Big|_0^{\pi/3} = -\cos\frac{\pi}{3} - (-\cos 0) = \frac{1}{2}$

Ex) $\int_0^2 |2x-1|\,dx = ?$ $\quad |2x-1| = \begin{cases} -(2x-1) & \text{if } x < \frac{1}{2} \\ 2x-1 & \text{if } x \geq \frac{1}{2} \end{cases}$

$\int_0^{1/2} -(2x-1)\,dx + \int_{1/2}^{2} (2x-1)\,dx = -(x^2-x)\Big|_0^{1/2} + (x^2-x)\Big|_{1/2}^{2} = -\frac{1}{4} + \frac{1}{2} + 4 - 2 - \frac{1}{4} + \frac{1}{2} = \frac{5}{2}$

Net Change Theorem: $\int_a^b F'(x)\,dx = \underbrace{F(b) - F(a)}_{\text{Net change in } F(x).}$

→ Definite integral of a rate of change is net change.

The Substitution Rule: We know $\frac{d}{dx}(F(g(x))) = F'(g(x))\cdot g'(x)$ by chain rule.

Then, $\int F'(g(x))\cdot g'(x)\,dx = F(g(x)) + C$. Let $u = g(x) \Rightarrow du = g'(x)dx$

$\int F'(u)\cdot du = F(u) + C = F(g(x)) + C$

→ If $u = g(x)$ is a differentiable function of x whose range is interval I, and f is continuous on I, then; $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

Ex) $\int (3x^2+1)^{15}\cdot 6x\,dx = ?$ $\quad \begin{matrix} u = 3x^2+1 \\ du = 6x\,dx \end{matrix} \Rightarrow \int u^{15}\,du = \frac{u^{16}}{16} + C = \frac{(3x^2+1)^{16}}{16} + C$

Ex) $\int (3x^2+1)^{15}\cdot x\,dx = ?$ $\quad \begin{matrix} u = 3x^2+1 \\ du = 6x\,dx \\ du/6 = x\,dx \end{matrix} \Rightarrow \int \frac{u^{15}}{6}\,du = \frac{u^{16}}{6\cdot 16} + C = \frac{(3x^2+1)^{16}}{16\cdot 6} + C$

Ex) $\int \frac{\cos(\pi/x)}{x^2}\,dx = ?$ $\quad \begin{matrix} u = \frac{\pi}{x} \\ du = \frac{-\pi}{x^2}\,dx \\ \frac{du}{-\pi} = \frac{dx}{x^2} \end{matrix} \Rightarrow \int \frac{\cos u}{-\pi}\cdot du = \frac{\sin u}{-\pi} + C = \frac{-\sin\left(\frac{\pi}{x}\right)}{\pi} + C$

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Ex) $\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx$ $\quad u = 1-\frac{1}{x}$, $du = \frac{1}{x^2}dx$

$= \int \sqrt{u}\,du = \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2u^{3/2}}{3} + C = \frac{2\cdot\left(1-\frac{1}{x}\right)^{3/2}}{3} + C$

Substitution for Definite Integrals: Assume we have $I = \int_a^b f(g(x))g'(x)\,dx$

Suppose we have $u = g(x)$. $\int_\alpha^\beta f(u)\,du$ Then, $\alpha = g(a)$, $\beta = g(b)$

Ex) $I = \int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\,dx$ $\quad u = x^2+1$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$ $\quad x=0 \Rightarrow u = 0+1 = 1$, $x=\sqrt{3} \Rightarrow u = 3+1 = 4$

$\int_1^4 \frac{du}{2\sqrt{u}} = \frac{1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \Big|_1^4 = \sqrt{4} - \sqrt{1} = 2-1 = 1$ //

Ex) $\int_1^2 x\sqrt{x-1}\,dx$ $\quad u = x-1$, $du = dx$, $u+1 = x$ $\quad x=2 \Rightarrow u=1$, $x=1 \Rightarrow u=0$

$\int_0^1 (u+1)\cdot\sqrt{u}\,du = \int_0^1 (u^{3/2}+u^{1/2})\,du = \frac{u^{5/2}}{\frac{5}{2}} + \frac{u^{3/2}}{\frac{3}{2}} \Big|_0^1 = \frac{2}{5}+\frac{2}{3} = \frac{6+10}{15} = \frac{16}{15}$ /

Ex) $\int x^3\cdot\sqrt{x^2+1}\,dx$ $\quad u = x^2+1$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$ $\quad \int (u-1)\cdot\sqrt{u}\cdot\frac{du}{2} = \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\cdot\frac{1}{2} + C$

$\frac{u^{\frac{5}{2}}}{5} - \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + C$ /

Separable Equations: A first order differential equation is an equation of the form $\frac{dy}{dx} = h(x,y)$. (Ex: $\frac{dy}{dx} = x^2+y^3$). Solving the equation is finding a function y of x (explicitly or implicitly) whose derivative with respect to x is $h(x,y)$. If $h(x,y) = f(y)\cdot g(x)$ that is depending on x and y. Then, the differential equation $\frac{dy}{dx} = f(y)\cdot g(x)$ is called a seperable equation.

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We can write the equation as $\frac{1}{f(y)}\,dy = g(x)\,dx$. Now we can integrate.

$$\int \frac{1}{f(y)}\,dy \overset{(*)}{=} \int g(x)\,dx$$ (integration with respect to x and y seperately.)

The equation $(*)$ defines y implicitly as a function of x.

This equation $(*)$ actually satisfies the differential equation. To see, take $\frac{d}{dx}$.

$$\frac{d}{dx}\int \frac{1}{f(y)}\,dy = \underbrace{\frac{d}{dx}\int g(x)\,dx}$$

$$\frac{d}{dy}\left(\int \frac{1}{f(y)}\,dy\right)\cdot\frac{dy}{dx} = g(x) \Rightarrow \frac{1}{f(y)}\cdot\frac{dy}{dx} = g(x) \Rightarrow \frac{dy}{dx} = f(y)\cdot g(x)$$

__Ex]__ $\frac{dy}{dx} = xy^2$, $\overbrace{y(0)=1}^{x=0 \Rightarrow y=1}$, Find $y = y(x)$.

$$\frac{dy}{y^2} = x\,dx \Rightarrow \int \frac{1}{y^2}\,dy = \int x\,dx \Rightarrow \frac{y^{-1}}{-1} + C_1 = \frac{x^2}{2} + C_2$$

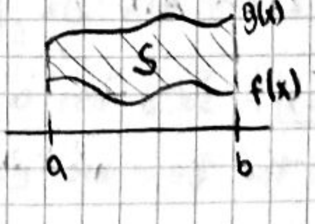
$$-\frac{1}{y} = \frac{x^2}{2} + C \qquad -\frac{1}{1} = \frac{0}{2} + C \Rightarrow \frac{-1}{y} = \frac{x^2}{2} - 1 \qquad y = \frac{1}{1-\frac{x^2}{2}} = \frac{2}{2-x^2}$$

$$C = -1$$

__Applications of Integration:__ We'll use definite integrals to compute areas between curves, volumes of solids etc. Let Q be the quantity that we want to compute. We break up Q into a large number of small parts. We approximate each small part in the form $f(x_i^*)\cdot\Delta x$, and so

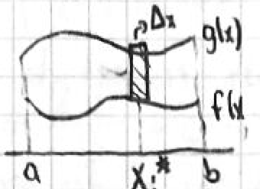
$$\lim_{n\to\infty} \sum_{i=a}^{b} f(x_i^*)\cdot\Delta x = \int_a^b f(x)\,dx$$

__Areas Between Curves:__ Let f, g be continuous on the interval $[a,b]$ and assume $f(x) \leq g(x)$ for all x in $[a,b]$.



$$S = \{(x,y) : a \leq x \leq b,\ f(x) \leq y \leq g(x)\}$$

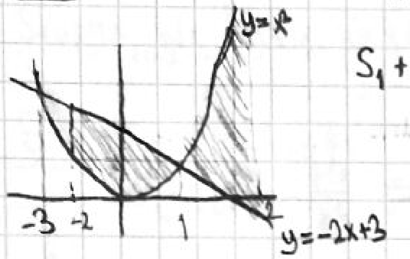
To find the area of S, we divide $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$.



$A(\text{strip}) = \Delta x \cdot (g(x_i^*) - f(x_i^*))$

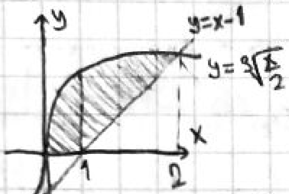
$$A(S) = \lim_{n\to\infty} \sum_{i=a}^{b} (g(x_i^*) - f(x_i^*)) \cdot \Delta x = \int_a^b (g(x) - f(x))\, dx$$

Ex Find area between $y = x^2$ and $y = -2x+3$ for $-2 \leq x \leq 2$.



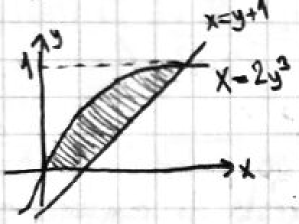
$$S_1 + S_2 = \int_{-2}^{1} (-2x+3) - x^2\, dx + \int_1^2 x^2 - (-2x+3)\, dx$$
$$= \int_{-2}^{1} -x^2 - 2x + 3\, dx + \int_1^2 x^2 + 2x - 3\, dx = \frac{-x^3}{3} - x^2 + 3x \Big|_{-2}^{1} + \frac{x^3}{3} + x^2 - 3x \Big|_1^2$$
$$= 9 + \frac{7}{3} = \frac{34}{3}$$

Ex Area of the region in the first quadrant bounded by $y = \sqrt[3]{\frac{x}{2}}$ and $y = x - 1$.



$$S_1 = \int_0^1 \sqrt[3]{\frac{x}{2}}\, dx = \frac{1}{\sqrt[3]{2}} \cdot \frac{x^{4/3}}{\frac{4}{3}} \Big|_0^1 = \frac{1}{\sqrt[3]{2}} \cdot \frac{3}{4}$$
$$S_2 = \int_1^2 \sqrt[3]{\frac{x}{2}} - (x-1)\, dx = \frac{1}{\sqrt[3]{2}} \cdot \frac{x^{4/3}}{\frac{4}{3}} - \frac{x^2}{2} + x \Big|_1^2 = \ldots$$

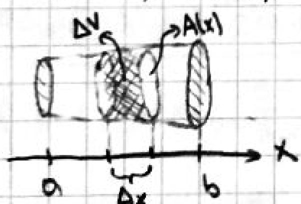
$S_1 + S_2 = 1$



$$S = \int_0^1 y + 1 - 2y^3\, dy = \frac{y^2}{2} + y - \frac{2y^4}{4} \Big|_0^1 = \frac{1}{2} + 1 - \frac{1}{2} = 1$$

$\Delta A \approx [(y+1) - (2y^3)] \cdot \Delta y$

Volumes: Suppose a solid S in space is bounded by two planes perpendicular to the x-axis, say $x = a$ and $x = b$.



To find the volume $V(S)$, we divide S into thin slices of thickness Δx by planes perpendicular to the x axis. If we know the $A(x)$ area for every x in $[a, b]$,

$$\Delta V \approx A(x) \cdot \Delta x \implies V = \int_a^b A(x)\, dx.$$

Ex A solid has a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

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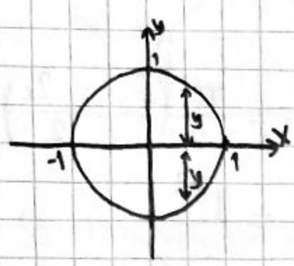
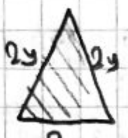
Assume base is the disk $x^2+y^2 \le 1$ in the xy-plane.

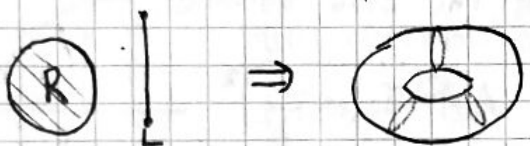




Area $= \sqrt{3}\,y^2$
where $x^2+y^2=1$

$$A(x)=\sqrt{3}\cdot(1-x^2) \Rightarrow V = 2\cdot\int_0^1 A(x)\,dx = 2\int_0^1 \sqrt{3}(1-x^2)\,dx = \frac{4}{\sqrt{3}}$$

<u>Solids of Revolution</u>: Assume a plane region R is revolved about a line L by 360°. The solid obtained this way is called solid of revolution.

→ 

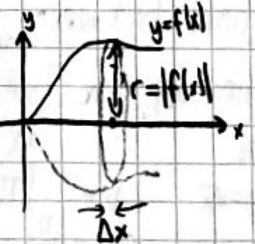
↳ A torus
(also known as donut)

Assume we have a continuous function f on $[a,b]$ and the region R is revolved about the x-axis. To find its volume, we cut it by planes perpendicular to the x-axis (the axis of revolution) into thin slices of Δx thick.

The cross section is a circle of radius $|f(x)|$

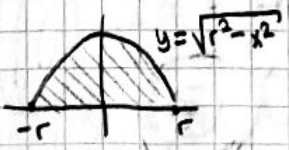
So $A(x) = \pi r^2 = \pi\cdot(f(x))^2$

$$V = \int_a^b \pi\cdot(f(x))^2\,dx$$



This method is called the disk method.

<u>Ex</u> Find the volume of a sphere of radius r.

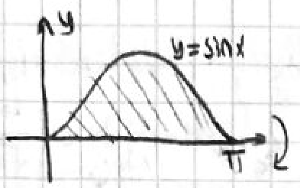


The sphere is obtained by revolving this area.

$$\int_{-r}^{r} \pi\cdot(\sqrt{r^2-x^2})^2\,dx = 2\cdot\int_0^r \pi\cdot(r^2-x^2)\,dx = 2\pi\cdot\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r =$$

$$= 2\pi\left(r^3-\frac{r^3}{3}-0\right) = 2\pi\cdot\frac{2r^3}{3} = \frac{4}{3}\pi r^3$$

Ex| The region between $y=\sin x$ and $y=0$, $0\leq x\leq\pi$, is rotated about the x-axis. Find the volume of the solid obtained.



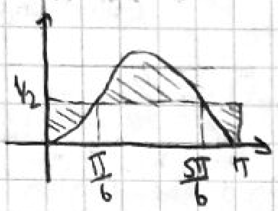
$$V=\int_0^{\pi}\pi\cdot\sin^2x\,dx=\pi\cdot\int_0^{\pi}\frac{1}{2}-\frac{\cos 2x}{2}\,dx=\frac{\pi}{2}\left(x-\frac{\sin 2x}{2}\right)\Big|_0^{\pi}$$

$\cos 2x=1-2\sin^2x$

$\sin^2x=\frac{1-\cos 2x}{2}$

$$=\frac{\pi^2}{2}$$

Ex| The region between $y=\sin x$ and $y=\frac{1}{2}$, $0\leq x\leq\pi$, is rotated about the line $y=\frac{1}{2}$. Find the volume of the solid obtained.



For $\frac{\pi}{6}\leq x\leq\frac{5\pi}{6}$ $A(x)=\pi\cdot\left(\sin x-\frac{1}{2}\right)^2$

For $0\leq x\leq\frac{\pi}{6}$, $\frac{5\pi}{6}\leq x\leq\pi$ $A(x)=\pi\cdot\left(\frac{1}{2}-\sin x\right)^2$

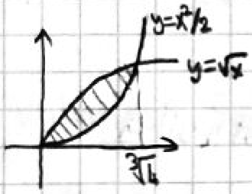
same

$$\int_0^{\pi}\pi\cdot\left(\sin x-\frac{1}{2}\right)^2dx=\pi\cdot\int_0^{\pi}\sin^2x-\sin x+\frac{1}{4}\,dx=\pi\left(\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+\cos x+\frac{x}{4}\right)\Big|_0^{\pi}=$$

↳ Double angle form.

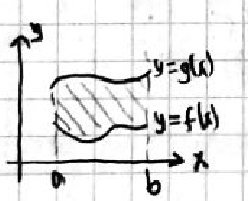
$$\pi\left(\frac{\pi}{2}-1+\frac{\pi}{4}\right)-\pi=$$

Ex| The region between $y=\frac{x^2}{2}$ and $y=\sqrt{x}$ is revolved about the x-axis. Find the volume of the solid so generated.



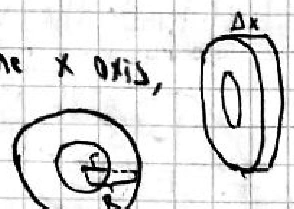
$$V=\int_0^{\sqrt[3]{4}}\pi\left(\sqrt{x}^2-\left(\frac{x^2}{2}\right)^2\right)dx$$ by washer method.

Remark:



If this region is rotated about the x axis,

$$A(x)=\pi R^2-\pi r^2=\pi\left(g^2(x)-f^2(x)\right)$$



This method is called washer method: $V=\int_a^b\pi\cdot\left[(g(x))^2-(f(x))^2\right]dx$

Volumes by Cylindrical Shells: Consider about the example region given below:

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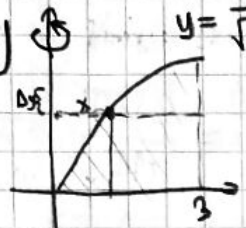
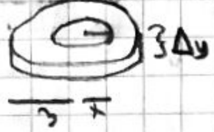
Ex) $y=\frac{x}{\sqrt{x^3+1}}$ The region is revolved about the y-axis.



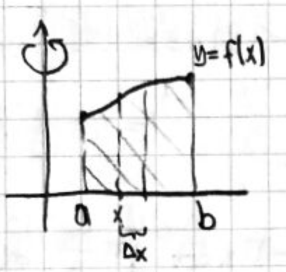


$\Delta V \approx (\pi\cdot 3^2-\pi x^2)\cdot \Delta y$

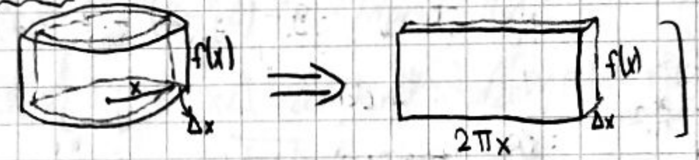
$V=\int_0^{3/\sqrt{28}} 9\pi-\pi\cdot(f(y))^2\, dy$ Problem: $f(y)=?$

So the washer method is useless since we cant find $f(y)$ easily.

<u>Cylindrical Shells Method:</u> Suppose we have a region R and revolve about y axis.



To find volume, take thin strips of area parallel to the axis of revolution. So there will be a thin cylindrical shell;

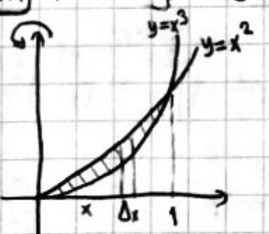


$\Delta v \approx 2\pi x\cdot f(x)\cdot \Delta x$ then find V

$$\boxed{V=\int_a^b 2\pi x\, f(x)\, dx}$$

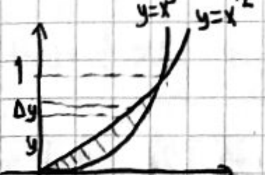
x: shell radius
$2\pi x$: circumference
$f(x)$: shell height
Δx: thickness

Ex) The region bounded by $y=x^2$ and $y=x^3$ is revolved about y-axis. Find V.



Shell thickness: Δx
Shell radius: x
Shell height: x^2-x^3

$V=\int_0^1 2\pi x\cdot(x^2-x^3)\cdot dx = 2\pi\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{\pi}{10}$

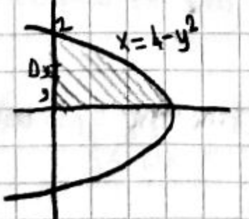


Washer thickness: Δy
Big radius: $y^{1/3}$
Small radius: $y^{1/2}$

$\Delta v \approx \left(\pi(y^{2/3})-\pi(y^{2/2})\right)\cdot \Delta y$

$V=\int_0^1 \pi\left(y^{\frac{2}{3}}-y\right)dy = \pi\cdot\left(\frac{3}{5}-\frac{1}{2}\right)=\frac{\pi}{10}$

Ex) Region from the first quadrant by $x=4-y^2$ is revolved about the y-axis. Find V



Washer thickness: Δy
Radius: $4-y^2$

$\Delta v \approx \pi(4-y^2)^2\cdot \Delta y$

$\int_0^2 \pi(16-8y^2+y^4)\,dy =$

$\pi\cdot\left(16y-\frac{8y^3}{3}+\frac{y^5}{5}\right)\Big|_0^2 =$

$\pi\left(32-\frac{64}{3}+\frac{32}{5}\right)=\frac{256\pi}{15}$

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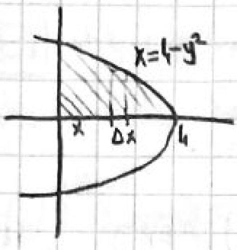


Shell thickness: Δx

Shell radius: x

Shell height: $\sqrt{4-x}$

$\Delta V \approx 2\pi x \cdot \sqrt{4-x} \cdot \Delta x$

$V = \int_0^4 2\pi x \cdot \underset{u=4-x}{\sqrt{4-x}}\, dx = \dfrac{256\pi}{15}$

Ex) Region in the upper plane bounded by $x=3y^2-2$, $x=y^2$ and $\underset{(x\text{-axis})}{y=0}$ is revolved about the x-axis. Find the volume.

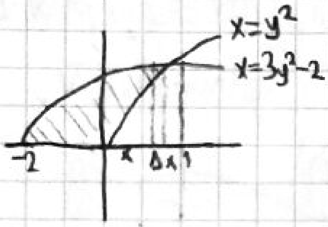
Shell thickness: Δy

Shell radius: y

Shell height: $y^2-(3y^2-2) = 2-2y^2$

$V = \int_0^1 2\pi y \cdot (2-2y^2)\, dy = 4\pi \int_0^1 y - y^3\, dy$

$= 4\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \pi$



Washer: thickness: Δx

Big radius: $\sqrt{\frac{x+2}{3}}$

Small radius: $\sqrt{x}$ (0, 1)

$V = \int_0^1 \pi \frac{x+2}{3} - \pi x\, dx + \int_{-2}^{0} \pi \frac{x+2}{3}\, dx = \pi$

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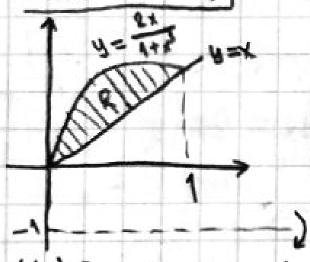


(a) Revolve about $y=-1$

$r = x-(-1) = x+1$

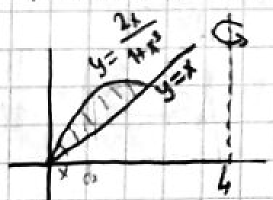
$R = \dfrac{2x}{1+x^3} - (-1) = \dfrac{2x}{1+x^3}+1$

$\Delta V \approx \left(\pi\left(\frac{2x}{1+x^3}+1\right)^2 - \pi (x+1)^2\right)\cdot \Delta x$

$V = \int_0^1 \pi\left(\frac{2x}{1+x^3}+1\right)^2 - \pi (x+1)^2\, dx$

(b) Revolve about $x=4$

$C = \int_0^1 2\pi (4-x)\left(\frac{2x}{1+x^3}-x\right) dx$



Shell radius: $4-x$

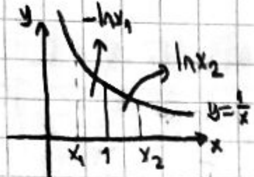
Shell height: $\dfrac{2x}{1+x^3} - x$ $\Rightarrow V = \int_0^1 2\pi (4-x)\cdot\left(\frac{2x}{1+x^3}-x\right) dx$

Shell thickness: dx

Section 6.1. Read Inverse functions from the book.

The Natural Logarithmic Function: For $x>0$ we define the natural logarithm of x;

$$\ln x = \int_1^x \frac{1}{t}\, dt$$



$x>1 \Rightarrow \ln x>0$

$0<x<1 \Rightarrow \ln x<0$

$$\Rightarrow \ln x = \begin{cases} \text{positive} & \text{if } 1<x \\ 0 & \text{if} \\ \text{negative} & 0<x<1 \end{cases}$$

By FTC1, $\frac{d}{dx}\ln x = \frac{1}{x}$, so $(\ln x)'>0$ for $x>0$ thus $y=\ln x$ is increasing in $(0,\infty)$.

Laws of Logarithm: Let $x>0$, $y>0$ and r be a rational number.

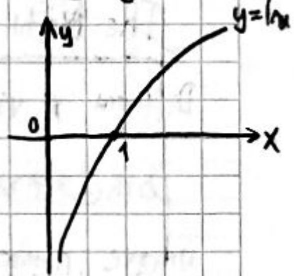
(i) $\ln(xy)=\ln x+\ln y$ (ii) $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$ (iii) $\ln(x^r)=r\cdot\ln x$

Proof of (iii): Let $f(x)=\ln(x^r)$, $g(x)=r\cdot\ln x$, $x>0$.

$f'(x)=r\cdot x^{r-1}\cdot\frac{1}{x^r}=\frac{r}{x}$ $g'(x)=r\cdot\frac{1}{x}=\frac{r}{x}$. So $f'(x)=g'(x)$ in $(0,\infty)$.

$f(x)=g(x)+C$ $\ln(x^r)=r\cdot\ln x+C$ Let $x=1$, $\ln 1^r=r\cdot\ln 1+C \Rightarrow C=0$

Ex] Expand $\ln\left(\sqrt[3]{\frac{x-1}{x+1}}\right)=\frac{1}{3}\left(\ln(x-1)-\ln(x+1)\right)$



Graph: ① $\lim_{x\to\infty}\ln x=?$ For $x\to\infty$ choose x as $2, 2^2, 2^3 \cdots 2^n$

$\lim_{x\to\infty}\ln 2^n=\lim_{x\to\infty} n\cdot\underbrace{\ln 2}_{\text{constant}}=\infty$. Since $y=\ln x$ is increasing on $(0,\infty)$, $\lim_{x\to\infty}\ln x=\infty$

② $\lim_{x\to 0^+}\ln x=?$ $x=\frac{1}{u}$, $u\to\infty$ $\lim_{u\to\infty}\ln\left(\frac{1}{u}\right)=\lim_{u\to\infty}-\underbrace{\ln u}_{\infty}=-\infty$, $\lim_{x\to 0^+}\ln x=-\infty$

③ $\frac{d}{dx}(\ln x)=\frac{1}{x}>0$, so $y=\ln x$ is $\nearrow$ on $(0,\infty)$ ④ $\frac{d^2}{dx}(\ln x)=\frac{-1}{x^2}$, so $y=\ln x$ is $\cap$ on $(0,\infty)$

Derivatives and Integrals: $\frac{d}{dx}\ln x=\frac{1}{x}$ so $\frac{d}{dx}\ln(g(x))=\frac{g'(x)}{g(x)}$

$\underbrace{\int\frac{1}{x}dx}_{\text{defined for } x<0 \text{ as well}}=\ln|x|+C$ $\ln|x|\begin{cases}\ln x & \text{if } x>0\\ \ln(-x) & \text{if } x<0\end{cases}$ $(\ln|x|)'=\begin{cases}\frac{1}{x} & \text{if } x>0\\ \frac{1}{-x}\cdot(-1) & \text{if } x<0\end{cases}=\frac{1}{x}$

different constants for $(-\infty,0)$ and $(0,\infty)$.

So $\int x^n dx=\begin{cases}\frac{x^{n+1}}{n+1}+C & \text{if } n\neq 1\\ \int\frac{1}{x}dx=\ln|x|+C & \text{if } n=0\end{cases}$

Ex) * $\int \frac{dx}{2x+2\sqrt{x}} = \int \frac{dx}{2\sqrt{x}(\sqrt{x}+1)}$ $\quad \sqrt{x}+1=u,\ \frac{1}{2\sqrt{x}}dx = du$ $\quad \int \frac{du}{u} = \ln|u| + C = \ln|\sqrt{x}+1| + C$

$= \ln(\sqrt{x}+1) + C \quad (\sqrt{x}>0)$

* $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$ $\quad u = \cos x,\ du = -\sin x\, dx$ $\quad \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln \frac{1}{|\cos x|} + C$

* $\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$ $\quad = -\ln|\sec x| + C$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)dx \Rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$

Ex) Find $\frac{dy}{dx}$. $\quad y = \frac{(x+1)^4 \cdot (x-5)^5}{(x-3)^8}$ $\quad \ln y = 4\ln(x+1) + 5\ln(x-5) - 8\ln(x-3)$

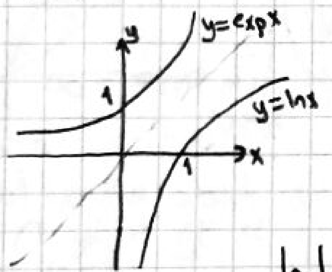
Now find $\frac{dy}{dx}$ implicitly. $\quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{4}{x+1} + \frac{5}{x-5} - \frac{8}{x-3} \Rightarrow \frac{dy}{dx} = \frac{(x+1)^4(x-5)^5}{(x-3)^8}\cdot\left(\frac{4}{x+1} + \frac{5}{x-5} - \frac{8}{x-3}\right)$

<u>The Natural Exponential Function:</u> $f(x) = \ln x$ is 1-1 (one to one function)

Different x values are taken to different y values. So $\ln a = \ln b \Rightarrow a = b$

Since range of f in $(-\infty, \infty)$ (that is all y), we have that there is a unique number (denoted by e) whose $\ln$ is 1. $\quad \ln e = 1$, $\quad e \approx 2.718$

$f(x) = \ln x$ has an inverse function $f^{-1}(x) = \exp x$



$y = \exp x \Leftrightarrow x = \ln y \quad (f(x)=y,\ f^{-1}(y)=x)$

$\exp(\ln x) = x, \quad x>0$

$\ln(\exp x) = x, \quad$ all x.

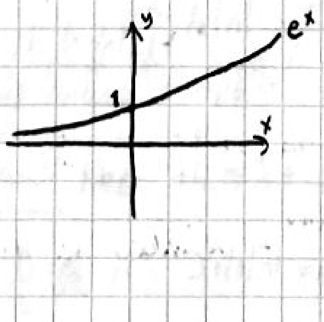
$$\left.\begin{aligned} \ln(e^r) &= r\cdot \ln e = r \\ \ln(\exp r) &= r \end{aligned}\right\} \exp r = e^r \text{ for rational numbers } r.$$

Then for all real numbers x, we define $\exp x = e^x$

So $\quad y = e^x \Leftrightarrow x = \ln y$

$e^{\ln x} = x, \quad x>0$

$\ln e^x = x, \quad$ all x



$e^x > 0$ for all x.

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Laws of Exponents: Let x, y be real numbers, r be a rational number.

(i) $e^{x+y} = e^x \cdot e^y$ (ii) $e^{x-y} = \frac{e^x}{e^y}$ (iii) $(e^x)^r = e^{rx}$

Theorem: $\frac{d}{dx} e^x = e^x$ for all x.

Proof: Let $y = e^x$, then $x = \ln y$. Take $\frac{d}{dx}$ $1 = \frac{1}{y}\frac{dy}{dx} \Rightarrow y = \frac{dy}{dx} = e^x$

Remark: $f(x) = e^x$ is the only function, $f'(x) = f(x)$ and $f(0) = 1$

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x dx = e^x + C \qquad \frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$$

Ex) $y = \sqrt{1 + x \cdot e^{-2x}}$ $\quad \frac{dy}{dx} = \frac{e^{-2x} + (-2) \cdot x e^{-2x}}{2\sqrt{1 + x \cdot e^{-2x}}}$

Ex) $y = \sin(e^x) + e^{\sin x}$ $\quad \frac{dy}{dx} = e^x \cdot \cos e^x + e^{\sin x} \cdot \cos x$

Ex) $\int \frac{(1+e^x)^2}{e^x} dx = \int \frac{1 + 2e^x + e^{2x}}{e^x} dx = \int e^{-x} + 2 + e^x \, dx = -\frac{1}{e^x} + 2x + e^x + C$

Ex) $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x} dx$ $\quad 1 + e^{-x} = u$, $\frac{-1}{e^x} = du$ $\Rightarrow \int_2^{1+\frac{1}{e}} -\sqrt{u}\, du = -\left(\frac{\sqrt{u^3} \cdot 2}{3}\right)\Big|_2^{1+\frac{1}{e}}$

General Logarithmic and Exponential Functions: If $a > 0$ and r is rational,

$a = e^{\ln a}$ and $(e^x)^r = e^{xr}$ $\Rightarrow a^r = (e^{\ln a})^r = e^{r \ln a}$

So if $a > 0$ and r is rational, then $a^r = e^{r \ln a}$

For every real number x, we define $a^x = e^{x \ln a}$ where $a > 0$ real number.

Ex) $2^\pi = e^{\pi \cdot \ln 2}$, $(\sqrt{3})^{\sqrt{2}} = e^{\sqrt{2} \cdot \ln\sqrt{3}}$, but $(-3)^{\sqrt{2}}$ is undefined since $a < 0$.

$a > 0$, $y = a^x$ is called a general function with base a.

$\rightarrow \ln(a^x) = \ln(e^{x \ln a}) = x \ln a$ is true for every real powers x.

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Derivative of $y=a^x$ when $a>0$ constant;

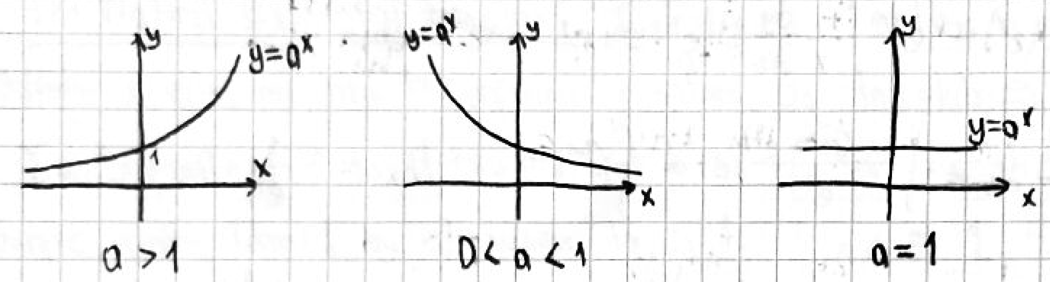
$y=a^x=e^{x\ln a} \Rightarrow \frac{dy}{dx}=e^{x\ln a}\cdot \ln a = a^x\cdot \ln a$. So we have;

$$\frac{d}{dx}(a^x)=a^x\cdot \ln a \quad , \quad \int a^x dx = \frac{a^x}{\ln a}+C$$

Graph of $y=a^x$, $a>0$: Domain: $(-\infty,\infty)$, $y=a^x=e^{x\ln a}>0$ for all x.

$$\frac{dy}{dx}=a^x\ln a \begin{cases} \text{positive} & \text{if } a>0 & \rightarrow \text{Its increasing } (\nearrow) \text{ at } (-\infty,\infty) \\ 0 & \text{if } a=1 & \rightarrow \text{Its constant } (\rightarrow) \text{ at } (-\infty,\infty) \\ \text{negitive} & \text{if } 0<a<1 & \rightarrow \text{Its decreasing } (\searrow) \text{ at } (-\infty,\infty) \end{cases}$$

$$\frac{d^2y}{dx^2}=a^x\cdot(\ln a)^2 \Big\}$$ Allways positive, so its concave up $(\cup)$ at $(-\infty,\infty)$.



Ex) $\int x\cdot 2^{(x^2)}dx$ $\quad x^2=u$, $2x\,dx=du \Rightarrow \int \frac{2^u}{2}du=\frac{1}{2}\cdot\frac{2^u}{\ln 2}+C=\frac{2^{x^2-1}}{\ln 2}+C$

Ex) $y=x^4+4^x \quad \frac{dy}{dx}=4x^3+4^x\cdot\ln 4$

Logarithmic Differentiation: We use to find $\frac{d}{dx}\left((f(x))^{g(x)}\right)=?$

Ex) $y=x^{\cos x} \xRightarrow{\text{Take ln}} \ln y=\cos x\cdot \ln x \Rightarrow$ Take $\frac{d}{dx}$ both sides.

$$\frac{1}{y}\cdot\frac{dy}{dx}=-\sin x\cdot\ln x+\cos x\cdot\frac{1}{x} \Rightarrow \frac{dy}{dx}=x^{\cos x}\cdot\left(-\sin x\cdot\ln x+\frac{\cos x}{x}\right)$$

General Logarithmic Function: Consider $f(x)=a^x$ where $a>0$, $a\neq 1$

Then f is either $(\nearrow)$ or $(\searrow)$ so f has an inverse function.

$f^{-1}(x)=\log_a x$ (logarithm with base a.)

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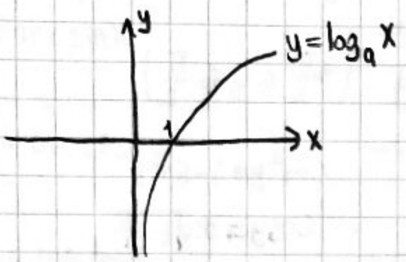
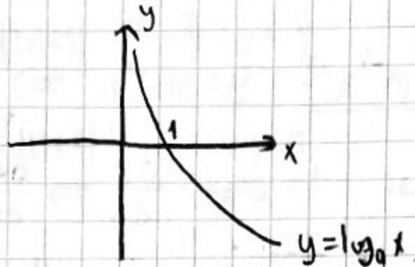
In particular if $a=e$, then $\log_e x = \ln x$. $(y=\log_e x \Rightarrow x=e^y \Rightarrow y=\ln x)$



$a>1$



$0<a<1$

These graphs are symetrical to the $y=a^x$ graph on the line $y=x$ (because they're inverse)

$$y=\log_a x \iff x=a^y \xrightarrow{\text{take } \ln} \ln x = y\cdot \ln a \Rightarrow y=\frac{\ln x}{\ln a}$$ } Change of the base formula

Then, $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)=\frac{1}{x\cdot \ln a}$ $(a>0,\ a\neq 1,\ a \text{ constant})$

Ex) $y=\log_3(x^2+5x)$ $\quad \frac{dy}{dx}=\frac{1}{(x^2+5x)}\cdot\frac{1}{\ln 3}\cdot(2x+5)=\frac{2x+5}{\ln 3(x^2+5x)}$

Theorem: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{x\to 0}(1+x)^{1/x}=e$

Proof: If $f(x)=\ln x$, then $f'(x)=\frac{1}{x}$, so $f'(1)=1$

$$1=f'(1)=\lim_{x\to 0}\frac{f(1+x)-f(1)}{x}=\frac{\ln(1+x)-\overbrace{\ln 1}^{0}}{x}=\frac{\ln(1+x)}{x}=\frac{1}{x}\cdot\ln(1+x)=\ln(1+x)^{\frac{1}{x}}\Rightarrow$$

$$\lim_{x\to 0}\ln\underbrace{(1+x)^{\frac{1}{x}}}=1$$

Since e^x is a continuous function.

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0}e^{\ln(1+x)^{\frac{1}{x}}}=e^{\overbrace{\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}}^{1}}\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e^1=e$$

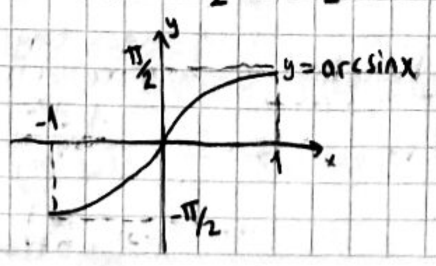
Section 6.5 omitted.

Inverse Trigonometric Functions: $f(x)=\sin x$ is not 1-1 if $-\infty<x<\infty$.

But if we consider $f(x)=\sin x$ for $\frac{-\pi}{2}<x<\frac{\pi}{2}$, then it has an inverse function

$f^{-1}(x)=\sin^{-1}(x)$

$y=\arcsin x$



$y=\arcsin x \iff x=\sin y$ and $\frac{-\pi}{2}<y<\frac{\pi}{2}$

Ex) $\sin^{-1}1=\frac{\pi}{2}$, $\sin^{-1}2$ = undefined, $\sin^{-1}\left(\sin\frac{\pi}{4}\right)=\frac{\pi}{4}$, $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)=\frac{-\pi}{4}$

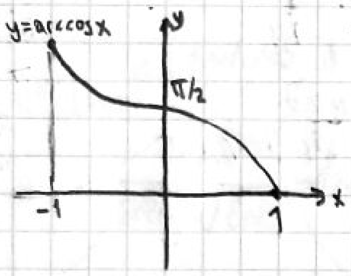
Derivative of $\sin^{-1}x$: $y=\sin^{-1}x \Rightarrow x=\sin y \quad \left(-\frac{\pi}{2}\le y\le\frac{\pi}{2}\right)$

Take $\frac{d}{dx}$ both sides. $1=\cos y\cdot\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{\cos y}$ $\qquad \cos^2 y=1-\sin^2 y$

$\cos y=\pm\sqrt{1-x^2}$

Since $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, $\cos y=+\sqrt{1-x^2}\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}} \to -1<x<1$

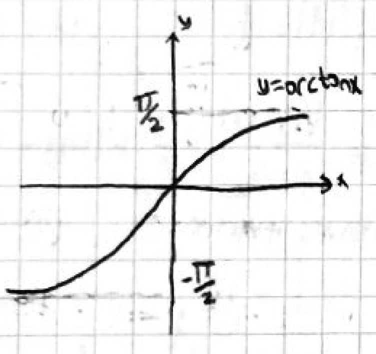
$f(x)=\cos x$, Take $0\le x\le\pi$, $f^{-1}(x)=\cos^{-1}x=\arccos x$



$y=\arccos x$ domain: $[-1,1]$, range: $[0,\pi]$

$\frac{dy}{dx}=\frac{-1}{\sin y}=\frac{-1}{\sqrt{1-x^2}}$

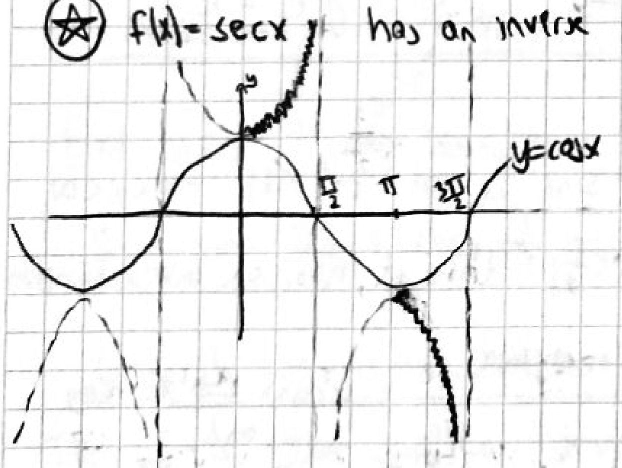
$f(x)=\tan x$, has an inverse on $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ $f^{-1}(x)=\tan^{-1}x=\arctan x$



$y=\arctan x$ domain: $(-\infty,\infty)$, range: $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}$

$f(x)=\sec x$, has an inverse on $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ $f^{-1}(x)=\sec^{-1}x=\operatorname{arcsec}x$



$y=\operatorname{arcsec}x$ domain: $(-\infty,-1]\cup[1,\infty)\Rightarrow|x|\ge1$

range: $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$

$x=\sec y$, $0\le y<\frac{\pi}{2}$ · $\pi\le y<\frac{3\pi}{2}$

$1=\sec y\cdot\tan y\cdot\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{1}{\sec y\cdot\tan y}=\frac{1}{x\sqrt{x^2-1}}$

$\sec^2 y=1+\tan^2 y$

$\tan y=\pm\sqrt{\sec^2 y-1}$ always (+) on domain of y

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Function	Domain	Range	Derivative
$y = \arcsin x$	$-1 \leq x \leq 1 \quad (\lvert x\rvert \leq 1)$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$\frac{1}{\sqrt{1-x^2}}$
$y = \arccos x$	$-1 \leq x \leq 1 \quad (\lvert x\rvert \leq 1)$	$0 \leq y \leq \pi$	$\frac{-1}{\sqrt{1-x^2}}$
$y = \arctan x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\frac{1}{1+x^2}$
$y = \operatorname{arccot} x$	$-\infty < x < \infty$	$0 < y < \pi$	$\frac{-1}{1+x^2}$
$y = \operatorname{arcsec} x$	$\lvert x\rvert \geq 1$	$0 \leq y < \frac{\pi}{2}$ $\pi \leq y < \frac{3\pi}{2}$	$\frac{1}{x\sqrt{x^2-1}}$
$y = \operatorname{arccsc} x$	$\lvert x\rvert \geq 1$	$0 < y \leq \frac{\pi}{2}$ $\pi < y \leq \frac{3\pi}{2}$	$\frac{-1}{x\sqrt{x^2-1}}$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$

Ex Simplify $A = \sin\left(\operatorname{arcsec}\frac{\sqrt{x^2+4}}{x}\right)$, $x > 0$

$u = \operatorname{arcsec}\frac{\sqrt{x^2+4}}{x}$, then $\sec u = \frac{\sqrt{x^2+4}}{x}$ and $\overbrace{0 \leq u < \frac{\pi}{2}}^{①}, \overbrace{\pi \leq u < \frac{3\pi}{2}}^{②}$

If ② is true, then $\frac{\sqrt{x^2+4}}{x} < 0$ so $x < 0$. But we have $x > 0$ so ② is false.

Then we have ① which is $0 \leq u < \frac{\pi}{2}$. So we should move to the $\sin u$.

$\cos u = \frac{1}{\sec u} = \frac{x}{\sqrt{x^2+4}} \qquad \sin^2 u = 1 - \cos^2 u = 1 - \frac{x^2}{x^2+4} = \frac{4}{x^2+4} \qquad \sin u = \pm\frac{2}{\sqrt{x^2+4}}$

Since we have $0 \leq u < \frac{\pi}{2}$, $\sin u = \frac{+2}{\sqrt{x^2+4}} \Rightarrow A = \sin u = \frac{2}{\sqrt{x^2+4}}$

Ex $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for all x in $[-1, 1]$. Proove by derivative.

$f(x) = \sin^{-1}x + \cos^{-1}x$

$f'(x) = \frac{1}{\sqrt{x^2-1}} + \frac{-1}{\sqrt{x^2-1}} = 0$

Since $f'(x) = 0$, $f(x)$ should be a constant C.

$f(x) = C$. Let $x = 0$ $\quad f(0) = \sin^{-1}0 + \cos^{-1}0 = \frac{\pi}{2} \to C$

$f(x) = \frac{\pi}{2}$

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Ex $\int \frac{x}{1+x^4}dx$ $\quad u=x^2,\ du=2x\,dx \Rightarrow \int \frac{1}{1+u^2}\cdot\frac{du}{2} = \frac{\arctan u}{2}+C = \frac{1}{2}\arctan x^2 + C$

Ex $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$ $\quad u=e^{2x},\ du=2\cdot e^{2x}dx \Rightarrow \int \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{2} = \frac{\arcsin u}{2}+C = \frac{1}{2}\arcsin e^{2x}+C$

L'Hôpital's Rule: Suppose f and g are differentiable on an open interval I which contains the point a, and $g'(x)\neq 0$ for all $x\neq a$ in I. Suppose $\left(\lim_{x\to a} f(x)=0,\ \lim_{x\to a} g(x)=0\right)$ or $\left(\lim_{x\to a} f(x)=\pm\infty,\ \lim_{x\to a} f(x)=\pm\infty\right)$

Suppose also $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$ exists where L is a number or $\pm\infty$. Then, $\lim_{x\to a}\frac{f(x)}{g(x)}=L$

Remarks:
- It's important to check the conditions before using the rule.
- The rule is true if all limits are approaching a^+ or all approaching a^-.
- This rule can be used only for quotients like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Ex $\lim_{x\to 0}\frac{\sin x - x}{x^3} \cancel{=} \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{-\cos x}{6} = \frac{-1}{6}$

shouldn't put (=)

Ex $\lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x}\right)}{\sin x} \cancel{=} \left(\frac{0}{0}\right) \Rightarrow \lim_{x\to 0}\frac{2x\sin\left(\frac{1}{x}\right)+\cancel{x^2}\cos\frac{1}{x}\cdot\left(\frac{-1}{\cancel{x^2}}\right)}{\cos x} = \lim_{x\to 0}\underbrace{\frac{\overbrace{2x\cdot\sin\frac{1}{x}}^{0}-\overbrace{\cos\frac{1}{x}}^{\text{No limit}}}{\underbrace{\cos x}_{1}}}_{\text{No Limit}}$

by squeeze theorem

Since $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$ doesn't exists, we can't use the L'Hôpital's.

$\lim_{x\to 0}\underbrace{\frac{x}{\sin x}}_{1}\cdot\underbrace{x\cdot\sin\left(\frac{1}{x}\right)}_{0 \text{ by squeeze}} = 1\cdot 0 = 0$

(A) Don't use L'Hôpitals directly and don't put (=) directly.

Ex $\lim_{x\to 0^+} x\cdot\ln x$ We have $0\cdot(-\infty) \Rightarrow \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}$ Now we have $\frac{-\infty}{\infty}$.

$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$ So we got $\lim_{x\to 0^+} x\ln x = 0$

Ex $\lim_{x\to 0}\frac{1}{x^2}-\frac{1}{x\tan x} = \lim_{x\to 0}\frac{\tan x - x}{x^2\cdot\tan x} = \frac{\frac{\sin x}{\cos x}-x}{x^2\cdot\frac{\sin x}{\cos x}} = \frac{\cancel{\cos x}(\sin x - x\cos x)}{\cancel{\cos x}\,x^2\sin x} = \frac{\sin x - x\cos x}{x^2\sin x}$

$\lim_{x\to 0}\frac{\cos x - \cos x + x\sin x}{2x\sin x + x^2\cos x} = \frac{x\sin x}{2x\sin x + x^2\cos x} \Rightarrow \frac{\cos x}{2\cos x+\cos x - x\sin x} = \frac{\cos x}{3\cos x - x\sin x} = \frac{1}{3-0} = \frac{1}{3}$

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Ex] $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$, we have 1^∞. We should take ln of $y=(\cos x)^{1/x^2}$

$\ln(\cos x)^{1/x^2} = \frac{1}{x^2}\cdot\ln(\cos x) \Rightarrow \lim_{x\to 0}\frac{\ln(\cos x)}{x^2} \Rightarrow \frac{\frac{-\sin x}{\cos x}}{2x} = \frac{-\tan x}{2x} \Rightarrow \frac{-\sec^2 x}{2} = \frac{-1}{2}$

$\ln y = \frac{-1}{2} \Rightarrow y = e^{-1/2} \Rightarrow \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$

Integration By Parts: $\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\,g(x) + f(x)\cdot g'(x)$

$\underbrace{\int \frac{d}{dx}(f(x)\cdot g(x))\,dx}_{f(x)\cdot g(x)} = \int f(x) g'(x)\,dx + \int f'(x) g(x)\,dx$

$f(x)\cdot g(x) = \int f(x)\cdot g'(x)\,dx + \int f'(x)\,g(x)\,dx$ } include the C.

$\int f(x)\,g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\,g(x)\,dx$, Let $u=f(x)$, $v=g(x)$

$\int u\cdot dv = u\cdot v - \int v\cdot du$ } Integration by parts formula.

Ex] $\int x\cdot\cos x\cdot dx$, Let $u=x$ and $dv=\cos x\,dx \Rightarrow \begin{matrix} du=dx \\ v=\sin x\end{matrix}$

$\int u\cdot dv = u\cdot v - \int v\cdot du = x\cdot\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

Ex] $\int x^2 e^x\,dx$, Let $u=x^2$ and $dv=e^x dx \Rightarrow \begin{matrix} du=2x\,dx \\ v=e^x\end{matrix}$

$= \int u\cdot dv = u\cdot v - \int v\cdot du = x^2\cdot e^x - \underbrace{\int e^x\cdot 2x\,dx}_{2\left(x\cdot e^x - \int e^x dx\right)}$ } $\begin{matrix} u=x \\ dv=e^x dx\end{matrix} \Rightarrow \begin{matrix} du=dx \\ v=e^x\end{matrix}$

$= x^2 e^x - 2\left(xe^x - e^x\right) + C$

Ex] $A = \int e^x\cdot\cos x\,dx$ $\quad u=e^x \quad dv=\cos x\,dx \Rightarrow \begin{matrix} du=e^x dx \\ v=\sin x\end{matrix}$

$= e^x\cdot\sin x - \underbrace{\int e^x\cdot\sin x\,dx}_{\begin{matrix} u=e^x & dv=\sin x\,dx \\ du=e^x dx & v=-\cos x\end{matrix}} = e^x\cdot\sin x - \left(-e^x\cos x - \int -\cos x\,e^x dx\right) =$

$e^x\sin x + e^x\cos x - \underbrace{\int \cos x\,e^x dx}_{A} = A \Rightarrow A = \frac{e^x\sin x + e^x\cos x}{2} + C$

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Ex) $\int \ln x \, dx$ $\quad u = \ln x \Rightarrow du = \frac{1}{x}dx$, $dv = dx \Rightarrow v = x$

$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x\ln x - x + C$

Ex) $\int e^{\sqrt{x}} dx$ $\quad t = \sqrt{x}$, $t^2 = x \Rightarrow dx = 2t\,dt \Rightarrow \int e^t \cdot 2t \cdot dt$

$u = t \Rightarrow du = dt$
$dv = e^t dt \Rightarrow v = e^t$ $\quad = 2\left(t \cdot e^t - \int e^t dt\right) = 2\sqrt{x} \cdot e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

$$\int_a^b f(x) \cdot g'(x)\, dx = f(x) \cdot g(x)\Big|_a^b - \int_a^b f'(x) \cdot g(x) \cdot dx$$

Ex) $\int_0^2 (x-2) \cdot \left(\int_0^x e^{(t-2)^3} dt\right) dx$ $\quad u = \int_0^x e^{(t-2)^3} dt \Rightarrow du = e^{(x-2)^3}$

$dv = (x-2)dx \Rightarrow v = \frac{(x-2)^2}{2}$

$= \left(\int_0^x e^{(t-2)^3} dt\right) \cdot \frac{(x-2)^2}{2}\Big|_0^2 - \int_0^2 e^{(x-2)^3} \cdot \frac{(x-2)^2}{2} dx =$

$= \underbrace{\left(\int_0^2 e^{(t-2)^3} dt\right) \cdot \frac{(2-2)^2}{2}}_{0} - \underbrace{\left(\int_0^0 e^{(t-2)^3} dt\right) \cdot \frac{(0-2)^2}{2}}_{0} - \int_0^2 e^{(x-2)^3} \cdot \frac{(x-2)^2}{2} dx$ $\quad y = (x-2)^3$, $dy = (x-2)^2 \cdot 3\,dx$

$= \int_{-8}^{0} e^y \frac{dy}{6} = \frac{1}{6}\left(e^0 - e^{-8}\right) = \frac{1 - \frac{1}{e^8}}{6} + C$

Trigonometric Integrals:

Ex) $I = \int \sin^5 x \cdot \cos^2 x \cdot dx = \int \sin^4 x \cdot \cos^2 x \cdot \sin x \, dx$ $\quad \cos x = u$, $-\sin x\, dx = du$

$= \int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx = -\int (1-u^2)^2 \cdot u^2 \cdot du = -\int (1 - 2u^2 + u^4) \cdot u^2 \, du$

$= -\int (u^2 - 2u^4 + u^6) du = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C = -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + C$

$= -\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$

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(*) We can use the previous method, if one of the sin/cos power is odd.

Ex) $\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4} \cdot \int 1 - 2\cos 2x + \cos^2 2x\, dx$

$= \frac{1}{4} \int 1 - 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right) dx = \frac{1}{4}\left(x - 2 \cdot \frac{\sin 2x}{2} + \frac{1}{2}\left(x + \frac{1}{4} \cdot \sin 4x\right)\right) + C$

$\boxed{(*)\ \sin^2 x = \frac{1-\cos 2x}{2} \quad \cos^2 x = \frac{1+\cos 2x}{2}}$ $= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C$

(*) We use double angle formulas if the sin/cos powers are even.

Ex) $I = \int \tan^6 x \cdot \sec^4 x \cdot dx = \int \tan^6 x \cdot (1 + \tan^2 x) \cdot \sec^2 x\, dx$ $\quad u = \tan x$, $du = \sec^2 x\, dx$

$\int u^6 (1 + u^2) \cdot du = \int u^6 + u^8\, du = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$

(*) We use the previous method, if the sec power is even.

Ex) $\int \tan^3 x \cdot \sec^5 x \cdot dx = \int \underbrace{\tan^2 x}_{(1-\sec^2 x)} \cdot \sec^4 x \cdot \tan x \sec x\, dx$ $\quad \sec x = u$, $\tan x \sec x = du$

$= \int (1-u^2) \cdot u^4 \cdot du = \int u^4 - u^6\, du = \frac{\sec^5 x}{5} - \frac{\sec^6 x}{6} + C$

(*) We use the previous method, if the tan power is odd.

Ex) $A = \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$ $\quad u = \sec x$, $dv = \sec^2 x\, dx \Rightarrow du = \sec x \tan x\, dx$, $v = \tan x$

$\sec x \cdot \tan x - \int \underbrace{\tan x \cdot \sec x \tan x}_{\substack{\tan^2 x \cdot \sec x \\ (\sec^2 x - 1)}}\, dx = \sec x \cdot \tan x - \int \sec^3 x - \sec x\, dx$

$\sec x \tan x - \underbrace{\int \sec^3 x}_{A} + \int \sec x = A$

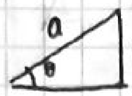
$\Rightarrow A = \frac{1}{2}\left(\sec x \tan x + \int \sec x\right)$

$= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

Trigonometric Substitutions: Suppose we have an integral which involves $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ where $a>0$ and constant. Ex: $\int \frac{dx}{\sqrt{x^2-3}}$

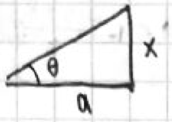
Remember Pythagorean theorem for a proper substitution;

1) For $\sqrt{a^2-x^2}$



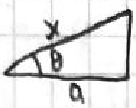
Let $x = a\cdot\sin\theta \Rightarrow \sqrt{a^2-x^2} = a\cos\theta$

2) For $\sqrt{a^2+x^2}$



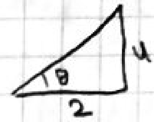
Let $x = a\cdot\tan\theta \Rightarrow \sqrt{a^2+x^2} = a\sec\theta$

3) For $\sqrt{x^2-a^2}$



Let $x = a\cdot\sec\theta \Rightarrow \sqrt{x^2-a^2} = a\tan\theta$

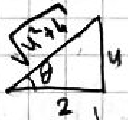
Ex $\int \frac{dx}{(x^2+6x+13)^2} = \int \frac{dx}{((x+3)^2+2^2)^2}$ $u=(x+3)$, $du=dx$ $\int \frac{du}{(u^2+2^2)^2}$



$u = 2\tan\theta$
$du = 2\sec^2\theta\, d\theta$

$\int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \int \frac{1}{8}\cdot\cos^2\theta\, d\theta = \frac{1}{8}\cdot\int\frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

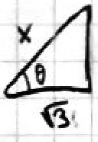
$= \frac{1}{16}\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C \Longrightarrow$ $\theta = \arctan\left(\frac{u}{2}\right)$, $u = x+3$



$\Rightarrow \frac{1}{16}\left(\arctan\frac{x+3}{2} + \frac{(x+3)\cdot 2}{(x+3)^2+4}\right) + C$

$\sin\theta = \frac{u}{\sqrt{u^2+4}}$, $\cos\theta = \frac{2}{\sqrt{u^2+4}}$

Ex $\int \frac{dx}{\sqrt{x^2-3}}$



$\cos\theta = \frac{\sqrt{3}}{x}$
$x = \sqrt{3}\cdot\sec\theta$
$dx = \sqrt{3}\cdot\sec\theta\tan\theta\, d\theta$

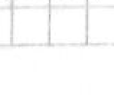
$\int \frac{\sqrt{3}\sec\theta\tan\theta\, d\theta}{\sqrt{3}\tan\theta} = \int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\, d\theta$

$= \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{x}{\sqrt{3}} + \frac{\sqrt{x^2-3}}{\sqrt{3}}\right| + C$

Ex $\int \frac{x\,dx}{\sqrt{5-6x-x^2}} = \int \frac{x\,dx}{\sqrt{-((x+2)^2-9)}} = \int \frac{x\,dx}{\sqrt{3^2-(x+2)^2}}$ $u = x+2$, $du = dx$ $\int \frac{(u-2)\,du}{\sqrt{3^2-u^2}} =$

$= \int \frac{u\,du}{\sqrt{9-u^2}} - 2\cdot\int\frac{du}{\sqrt{9-u^2}} = \int \frac{-1}{2}\cdot\frac{dy}{\sqrt{y}} - 2\int\frac{3\cos\theta\, d\theta}{3\cos\theta} = \frac{-1}{2}\cdot\frac{y^{1/2}}{\frac{1}{2}} - 2\theta + C$

$y = 9-u^2$
$dy = -2u\,du$



$u = 3\sin\theta$
$du = 3\cos\theta\, d\theta$

$= -\sqrt{y} - 2\theta + C = -\sqrt{9-u^2} - 2\arcsin\left(\frac{u}{3}\right) + C$

$= -\sqrt{9-(x+2)^2} - 2\arcsin\left(\frac{x+2}{3}\right) + C$

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Integration of Rational Functions by Partial Fractions Method: $\int \frac{P(x)}{Q(x)}dx = ?$

Suppose we have $I = \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$ We can evaluate easily.

The same integral can become $I = \int \frac{-2x+4}{(x^2+1)(x-1)} dx$. If we're asked to evaluate this form, we should try to bring the easy form back by the partial fractions method.

Suppose we have $f(x) = \frac{P(x)}{Q(x)}$, $P(x)$ and $Q(x)$ are polynomials.

(1) This method works only if $\deg[P(x)] < \deg[Q(x)]$. If we have $\deg[P(x)] \geq \deg[Q(x)]$ then first do polynomial division. So assume now, that $\deg[P(x)] < \deg[Q(x)]$.

(2) Factor $Q(x)$ as much as possible. The Fundamental Theorem of Algebra says; $Q(x)$ can be written in a unique way as a product of linear factors $(x-r)$ and a product of irreducible quadratic factors x^2+bx+c (where $b^2-4c<0$) and a constant

(3) To each linear factor $x-r$, if $(x-r)^n$ is its highest power, then for this factor we write $\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} \cdots + \frac{A_n}{(x-r)^n}$, where $A_1, A_2 \ldots A_n$ are constants

(4) For each irreducible factor x^2+px+q, if $(x^2+px+q)^m$ is its highest power then we write $\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots \frac{B_mx+C_m}{(x^2+px+q)^m}$ where B and C are constants.

(5) Set $f(x) = \frac{P(x)}{Q(x)}$ equal to the sum of all sums coming from (3) and (4).

(6) Determine the values of constants A, B, C.

Ex) $I = \int \frac{4x+1}{x^2+x-6} dx \Rightarrow \frac{4x+1}{(x+3)(x-2)} = \frac{A}{x+3}_{(x-2)} + \frac{B}{x-2}_{(x+3)} \Rightarrow 4x+1 = Ax - 2A + Bx + 3B$

Method 1) $4x+1 = (A+B)x + (-2A+3B)$ Always works

Method 2) $4x+1 = A(x-2) + B(x+3)$ sometimes works

$x=2 \Rightarrow B = \frac{9}{5}$ $x=-3 \Rightarrow A = \frac{11}{5}$

$I = \int \frac{11/5}{x+3} + \frac{9/5}{x-2} dx = \frac{11}{5}\ln|x+3| + \frac{9}{5}\ln|x-2| + C$

Ex) $I = \int \frac{3x+1}{(x-1)^2(x^2+1)} dx \Rightarrow \frac{3x+1}{(x-1)^2\cdot(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \Rightarrow$

$(x-1)(x^2+1) \quad (x^2+1) \quad (x-1)^2$

$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2\cdot(x^2+1)} = \frac{3x+1}{(x-1)^2(x^2+1)}$ Since denominators are equal,

$x=1 \Rightarrow B=2 \qquad A(x^3-x^2+x-1) + 2(x^2+1) + (Cx^3-2Cx^2+C+Dx^2+2Dx+1) = 3x+1$

$(A+C)x^3 + (-A+2-2C+D)x^2 + (A+C-2D)x + (-A+B+D) = 3x+1$

$A+C=0$

$-A-2C+D=-2$

$A+C-2D=3$

$-A+D=-1$

$D=\frac{-3}{2}$

$C=\frac{1}{2}$

$A=-C=-\frac{1}{2}$

$I = \int \frac{-1/2}{x-1} + \frac{2}{(x-1)^2} + \frac{\frac{1}{2}x - 3/2}{x^2+1} dx$

$I = \frac{-1}{2}\ln|x-1| + \frac{-2}{x-1} + \frac{1}{2}\int \frac{x}{x^2+1}dx - \frac{3}{2}\int \frac{1}{x^2+1}dx$

$I = \frac{-1}{2}\ln(x-1) - \frac{2}{x-1} + \frac{1}{4}\ln|x^2+1| - \frac{3}{2}\arctan x + C$

⟹ Solve all exercises in the section 7.5

Ex) $\int e^{\sqrt{x}} dx \qquad y^2=x,\ 2y\,dy=dx \qquad \int 2ye^y dy \qquad u=y,\ du=dy \qquad dv=e^y dy,\ v=e^y$

$2\left(y\cdot e^y - \int e^y dy\right) = 2\cdot(ye^y - e^y) = 2\cdot(\sqrt{x}\cdot e^{\sqrt{x}} - e^{\sqrt{x}}) + C$

Ex) $\int \frac{dx}{1+e^x} \qquad e^x=u,\ e^x dx=du,\ dx=\frac{du}{u} \qquad \int \frac{du}{u(1+u)} = \int\left(\frac{1}{u} - \frac{1}{u+1}\right)du = \ln|u| - \ln|u+1|$

$= \ln|e^x| - \ln|e^x+1| = x - \ln(e^x+1) + C$

Improper Integrals: An integral $\int_a^b f(x)dx$ is an improper integral if;

(i) The interval is unbounded, like $a=\pm\infty$ or/and $b=\pm\infty$

(ii) The integrand becomes $\pm\infty$ at some point in the interval of integration.

One or two of these conditions are satisfied for improper integrals.

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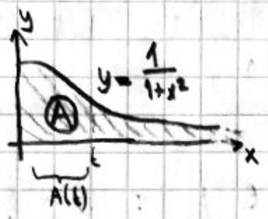
Ex $\int_0^\infty \frac{1}{1+x^2}\,dx \quad \int_0^1 \frac{1}{1-x}\,dx \quad \int_0^2 \frac{1}{1-x}\,dx \quad \int_0^\infty \frac{1}{x}\,dx$ are all improper integrals

Ex $\int_0^\infty \frac{1}{1+x^2}\,dx = A$



To find the unbounded shaded area, We take $0<t$ and compute the area 0 to t which is $A(t)$

$$A(t) = \int_0^t \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^t = \arctan t - \arctan 0 = \arctan t$$

Then take limit as $t \to \infty$

$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}$$

Definitions

Type 1: Infinite Intervals

(a) Assume f is continuous on $[a,\infty)$. We define $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

(b) Assume f is continuous on $(-\infty, b]$. We define $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

Type 2: Infinite Discontinuity

(a) Assume f is continuous on $[a,b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$ then,
We define $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

(b) Assume f is continuous on $(a,b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$ then,
We define $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

These are called basic type improper integrals. These integrals have exactly one "bad point" which means $\pm\infty$ for x or $f(x)$. At all basic type improper integrals, the bad point is an endpoint.

If the limit in the definition exists and is a number then we say the improper integral convergent, and if the limit is DNE or $\pm\infty$ we say the improper integral is divergent.

Ex) $\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} \left(\frac{-1}{x}\right)\Big|_1^t = 0 - \left(\frac{-1}{1}\right) = 1$ (convergent)

Ex) $\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} (\ln x)\Big|_1^t = \lim_{t\to\infty} \ln t = \infty$ (divergent)

Ex) $\int_0^\infty e^{-x}dx = \lim_{t\to\infty} \int_0^t e^{-x}dx = \lim_{t\to\infty} (-e^{-x})\Big|_0^t = 0 - (-1) = 1$ (convergent)

Ex) $\int_0^1 \ln x = \lim_{t\to 0^+} \int_t^1 \ln x = \lim_{t\to 0^+} (x\ln x - x)\Big|_t^1 = -1 - 0 = -1$ (convergent)

$\lim_{t\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{t\to 0^+} -x = 0$

Ex) $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} (2\sqrt{x})\Big|_t^1 = 2$ (convergent)

★ Let $p>0$ constant, and $I = \int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx$

If $p \neq 1$;

$I = \lim_{t\to\infty} \left(\frac{x^{1-p}}{1-p}\right)\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}-1}{1-p}$ $\quad \lim_{t\to\infty} t^{1-p} = \begin{cases} 0 & \text{if } p>1 \\ \infty & \text{if } 0<p<1 \end{cases}$

So; $\int_a^\infty \frac{1}{x^p}dx$ is $\begin{cases} \text{convergent} & \text{if } p>1 \\ \text{divergent} & \text{if } 0<p\leq 1 \end{cases}$ $\quad a>0$

★ Let $p>0$ costant, and $I = \int_0^1 \frac{1}{x^p}dx = \lim_{t\to\infty} \int_t^1 \frac{1}{x^p}dx$

If $p \neq 1$

$I = \lim_{t\to 0^+} \left(\frac{x^{1-p}}{1-p}\right)\Big|_t^1 = \lim_{t\to 0^+} \left(\frac{1-t^{1-p}}{1-p}\right)$ $\quad \lim_{t\to 0^+} t^{1-p} = \begin{cases} \infty & \text{if } p>1 \\ 0 & \text{if } 0<p<1 \end{cases}$

So; $\int_0^b \frac{1}{x^p}dx$ is $\begin{cases} \text{convergent} & \text{if } 0<p<1 \\ \text{divergent} & \text{if } p\geq 1 \end{cases}$ $\quad b>0$

Ex) $\int_1^\infty \frac{1}{x^3}$ is convergent $(p=3>1)$ $\qquad \int_0^1 \frac{1}{x^3}$ is divergent $(p=3\geq 1)$

Ex) $\int_1^\infty \frac{1}{\sqrt[3]{x}}$ is divergent $(p=\frac{1}{3}\leq 1)$ $\qquad \int_0^1 \frac{1}{\sqrt[3]{x}}$ is convergent $(p=\frac{1}{3}<1)$

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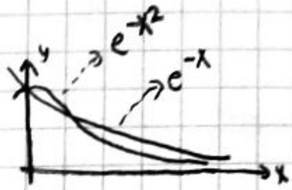
__Ex]__ Consider $I = \int_0^\infty e^{-x^2}\,dx$. We cannot evaluate the integral I but we can say something about convergence / divergence.



For $x \geq 1$, we have $x^2 \geq x$, $-x^2 \leq x$ $\Rightarrow e^{-x^2} \leq e^{-x} \Rightarrow$

$\int_1^\infty e^{-x} = \lim_{t \to \infty} (-e^{-x})\Big|_1^t = \frac{1}{e}$ Since the graph above (e^{-x}) is bigger than (e^{-x^2}) and its area is a number $(\frac{1}{e})$ then $\int_1^\infty e^{-x^2}dx$ is finite.

And since $\int_0^1 e^{-x^2}dx$ is not an improper integral; I is convergent

→ We'll see that $\frac{\sqrt{\pi}}{2}$ in the course MATH 102...

__Theorem (Comparasion Test):__ Let $\int_a^b f(x)$, $\int_a^b g(x)$ are improper integrals of the same basic type. Also suppose $0 \leq f(x) \leq g(x)$ on (a, b)

Then, if $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent.