Vehicle Models for Ego Motion Estimation

TUM Phoenix Robotics

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1 Constant Turn Rate and Velocity Model (CTRV)

Modelled after Schubert, R.; Richter, E.; Wanielik, G.: *Comparison and evaluation of advanced motion models for vehicle tracking* in 2008 11th International Conference on Information Fusion. To prevent unbounded values we use two models:

- standard model
- simplified (limited) model for $\omega \to 0$

1.1 System Model

$$\vec{x}_{t+T} = \begin{pmatrix} x_{t+T} \\ y_{t+T} \\ \theta_{t+T} \\ v_{t+T} \\ \omega \end{pmatrix} = \vec{x}_t + \begin{pmatrix} \Delta x(T) \\ \Delta y(T) \\ \omega T \\ 0 \\ 0 \end{pmatrix}$$

1.1.1 standard model

$$\Delta x(T) = \frac{v_t}{\omega} \left[\sin(\theta_t + \omega T) - \sin(\theta_t) \right]$$

and

$$\Delta y(T) = \frac{v_t}{\omega} \left[\cos(\theta_t) - \cos(\theta_t + \omega T) \right]$$

1.1.2 simplified model for $\omega \to 0$

$$\begin{aligned} \Delta x(T) &= v_t T \cos(\theta_t) \\ \Delta y(T) &= v_t T \sin(\theta_t) \end{aligned}$$

1.2 System Model Jacobian

$$F = \begin{pmatrix} 1 & 0 & \frac{\partial}{\partial \theta_t} x_{t+T} & \frac{\partial}{\partial v_t} x_{t+T} & \frac{\partial}{\partial \omega} x_{t+T} \\ 0 & 1 & \frac{\partial}{\partial \theta_t} y_{t+T} & \frac{\partial}{\partial v_t} y_{t+T} & \frac{\partial}{\partial \omega} y_{t+T} \\ 0 & 0 & 1 & 0 & T \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1.2.1 x-Derivatives

standard derivative

$$\begin{aligned}
\frac{\partial}{\partial \theta_t} x_{t+T} &= \frac{1}{\omega}\left[v_t(\cos(\theta_t + \omega T) - \cos(\theta_t))\right] \\
\frac{\partial}{\partial v_t} x_{t+T} &= \frac{1}{\omega}\left[\sin(\theta + \omega T) - \sin(\theta)\right] \\
\frac{\partial}{\partial \omega} x_{t+T} &= \frac{1}{\omega^2}\left[v_t \omega T \cos(\theta_t + \omega T) - (v_t(\sin(\theta_t + \omega T) - \sin(\theta_t)))\right]
\end{aligned}$$

simplified derivative for $\omega \to 0$

$$\begin{aligned}
\frac{\partial}{\partial \theta_t} x_{t+T} &= -v_t T \sin(\theta_t) \\
\frac{\partial}{\partial v_t} x_{t+T} &= T \cos(\theta_t) \\
\frac{\partial}{\partial \omega} x_{t+T} &= -\frac{1}{2} v_t T^2 \sin(\theta_t)
\end{aligned}$$

1.2.2 y-Derivatives

standard derivative

$$\begin{aligned}
\frac{\partial}{\partial \theta_t} y_{t+T} &= \frac{1}{\omega}\left[v_t(\sin(\theta_t + \omega T) - \sin(\theta_t))\right] \\
\frac{\partial}{\partial v_t} y_{t+T} &= \frac{1}{\omega}\left[-(\cos(\theta + \omega T) - \cos(\theta))\right] \\
\frac{\partial}{\partial \omega} y_{t+T} &= \frac{1}{\omega^2}\left[v_t(\cos(\theta_t + \omega T) - \cos(\theta_t)) + v_t \omega T \sin(\theta_t + \omega T)\right]
\end{aligned}$$

simplified derivative for $\omega \to 0$

$$\begin{aligned}
\frac{\partial}{\partial \theta_t} y_{t+T} &= v_t T \cos(\theta_t) \\
\frac{\partial}{\partial v_t} y_{t+T} &= T \sin(\theta_t) \\
\frac{\partial}{\partial \omega} y_{t+T} &= \frac{1}{2} v_t T^2 \cos(\theta_t)
\end{aligned}$$

2 Constant Turn Rate and Acceleration Model (CTRA)

Modelled after Schubert, R.; Richter, E.; Wanielik, G.: *Comparison and evaluation of advanced motion models for vehicle tracking* in 2008 11th International Conference on Information Fusion. To prevent unbounded values we use two models:

- standard model
- simplified (limited) model for $\omega \to 0$

2.1 System Model

$$\vec{x}_{t+T} = \begin{pmatrix} x_{t+T} \\ y_{t+T} \\ \theta_{t+T} \\ v_{t+T} \\ a \\ \omega \end{pmatrix} = \vec{x}_t + \begin{pmatrix} \Delta x(T) \\ \Delta y(T) \\ \omega T \\ aT \\ 0 \\ 0 \end{pmatrix}$$

2.1.1 standard model

$$\begin{aligned} \Delta x(T) &= \frac{1}{\omega^2}\left[(v_t\omega + a\omega T)\sin(\theta_t + \omega T) - v_t\omega\sin(\theta_t) + a\cos(\theta_t + \omega T) - a\cos(\theta_t)\right] \\ &= \frac{1}{\omega^2}\left[a\omega T\sin(\theta_t + \omega T) + v_t\omega(\sin(\theta_t + \omega T) - \sin(\theta_t)) + a(\cos(\theta_t + \omega T) - \cos(\theta_t))\right] \end{aligned}$$

and

$$\begin{aligned} \Delta y(T) &= \frac{1}{\omega^2}\left[-(v_t\omega + a\omega T)\cos(\theta_t + \omega T) + v_t\omega\cos(\theta_t) + a\sin(\theta_t + \omega T) - a\sin(\theta_t)\right] \\ &= \frac{1}{\omega^2}\left[-a\omega T\cos(\theta_t + \omega T) - v_t\omega(\cos(\theta_t + \omega T) - \cos(\theta_t)) + a(\sin(\theta_t + \omega T) - \sin(\theta_t))\right] \end{aligned}$$

2.1.2 simplified model for $\omega \to 0$

$$\begin{aligned} \Delta x(T) &= \frac{1}{2}T(2v_t + aT)\cos(\theta_t) \\ \Delta y(T) &= \frac{1}{2}T(2v_t + aT)\sin(\theta_t) \end{aligned}$$

2.2 System Model Jacobian

$$F = \begin{pmatrix} 1 & 0 & \frac{\partial}{\partial\theta_t}x_{t+T} & \frac{\partial}{\partial v_t}x_{t+T} & \frac{\partial}{\partial a}x_{t+T} & \frac{\partial}{\partial\omega}x_{t+T} \\ 0 & 1 & \frac{\partial}{\partial\theta_t}y_{t+T} & \frac{\partial}{\partial v_t}y_{t+T} & \frac{\partial}{\partial a}y_{t+T} & \frac{\partial}{\partial\omega}y_{t+T} \\ 0 & 0 & 1 & 0 & 0 & T \\ 0 & 0 & 0 & 1 & T & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2.2.1 x-Derivatives

standard derivative

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t} x_{t+T} &= \frac{1}{\omega^2}\left[(v_t\omega + a\omega T)\cos(\theta_t+\omega T) - v_t\omega\cos(\theta_t) - a\sin(\theta_t+\omega T) + a\sin(\theta_t)\right] \\
&= \frac{1}{\omega^2}\left[a\omega T\cos(\theta_t+\omega T) + v_t\omega\left(\cos(\theta_t+\omega T)-\cos(\theta_t)\right) - a\left(\sin(\theta_t+\omega T)-\sin(\theta_t)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial v_t} x_{t+T} &= \frac{1}{\omega^2}\left[\omega\sin(\theta_t+\omega T) - \omega\sin(\theta_t)\right] \\
&= \frac{1}{\omega}\left[\sin(\theta_t+\omega T)-\sin(\theta_t)\right]
\end{aligned}
$$

$$
\frac{\partial}{\partial a} x_{t+T} = \frac{1}{\omega^2}\left[\omega T\sin(\theta_t+\omega T) + \cos(\theta_t+\omega T) - \cos(\theta_t)\right]
$$

$$
\begin{aligned}
\frac{\partial}{\partial \omega} x_{t+T} &= \frac{1}{\omega^2}\Big[aT\sin(\theta_t+\omega T) + a\omega T^2\cos(\theta_t+\omega T) + v_t(\sin(\theta_t+\omega T)-\sin(\theta_t)) + v_t\omega T\cos(\theta_t+\omega T) - aT\sin(\theta_t+\omega T)\Big] \\
&- \frac{2}{\omega^3}\left[a\omega T\sin(\theta_t+\omega T) + v_t\omega(\sin(\theta_t+\omega T)-\sin(\theta_t)) + a(\cos(\theta_t+\omega T)-\cos(\theta_t))\right] \\
&= \frac{1}{\omega^2}\Big[(a\omega T^2 + v_t\omega T)\cos(\theta_t+\omega T) + v_t(\sin(\theta_t+\omega T)-\sin(\theta_t))\Big] \\
&- \frac{2}{\omega^3}\left[(a\omega T + v_t\omega)\sin(\theta_t+\omega T) - v_t\omega\sin(\theta_t) + a(\cos(\theta_t+\omega T)-\cos(\theta_t))\right] \\
&= \frac{1}{\omega^2}\left[(aT+v_t)(\omega T)\cos(\theta_t+\omega T) - (2aT+v_t)\sin(\theta_t+\omega T) + v_t\sin(\theta_t)\right] - \frac{2a}{\omega^3}(\cos(\theta_t+\omega T)-\cos(\theta_t))
\end{aligned}
$$

simplified derivative for $\omega \to 0$

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t} x_{t+T} &= -\frac{1}{2}T(2v_t+aT)\sin(\theta_t) \\
\frac{\partial}{\partial v_t} x_{t+T} &= T\cos(\theta_t) \\
\frac{\partial}{\partial a} x_{t+T} &= \frac{1}{2}T^2\cos(\theta_t) \\
\frac{\partial}{\partial \omega} x_{t+T} &= -\frac{1}{6}T^2(3v_t+2aT)\sin(\theta_t)
\end{aligned}
$$

2.2.2 y-Derivatives

standard derivative

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t} y_{t+T} &= \frac{1}{\omega^2}\left[(v_t\omega + a\omega T)\sin(\theta_t+\omega T) - v_t\omega\sin(\theta_t) + a\cos(\theta_t+\omega T) - a\cos(\theta_t)\right] \\
&= \frac{1}{\omega^2}\left[a\omega T\sin(\theta_t+\omega T) + v_t\omega\left(\sin(\theta_t+\omega T) - \sin(\theta_t)\right) + a\left(\cos(\theta_t+\omega T) - \cos(\theta_t)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial v_t} y_{t+T} &= \frac{1}{\omega^2}\left[-\omega\cos(\theta_t+\omega T) + \omega\cos(\theta_t)\right] \\
&= \frac{1}{\omega}\left[-(\cos(\theta_t+\omega T) - \cos(\theta_t))\right]
\end{aligned}
$$

$$
\frac{\partial}{\partial a} y_{t+T} = \frac{1}{\omega^2}\left[-\omega T\cos(\theta_t+\omega T) + \sin(\theta_t+\omega T) - \sin(\theta_t)\right]
$$

$$
\begin{aligned}
\frac{\partial}{\partial \omega} y_{t+T} &= \frac{1}{\omega^2}\left[-aT\cos(\theta_t+\omega T) + a\omega T^2\sin(\theta_t+\omega T) - v_t(\cos(\theta_t+\omega T) - \cos(\theta_t)) + v_t\omega T\sin(\theta_t+\omega T) + aT\cos(\theta_t+\omega T)\right] \\
&- \frac{2}{\omega^3}\left[-a\omega T\cos(\theta_t+\omega T) - v_t\omega(\cos(\theta_t+\omega T) - \cos(\theta_t)) + a(\sin(\theta_t+\omega T) - \sin(\theta_t))\right] \\
&= \frac{1}{\omega^2}\left[(a\omega T^2 + v_t\omega T)\sin(\theta_t+\omega T) - v_t(\cos(\theta_t+\omega T) - \cos(\theta_t))\right] \\
&- \frac{2}{\omega^3}\left[-(a\omega T + v_t\omega)\cos(\theta_t+\omega T) + v_t\omega\cos(\theta_t) + a(\sin(\theta_t+\omega T) - \sin(\theta_t))\right] \\
&= \frac{1}{\omega^2}\left[(aT+v_t)(\omega T)\sin(\theta_t+\omega T) + (2aT+v_t)\cos(\theta_t+\omega T) - v_t\cos(\theta_t)\right] - \frac{2a}{\omega^3}(\sin(\theta_t+\omega T) - \sin(\theta_t))
\end{aligned}
$$

simplified derivative for $\omega \to 0$

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t} y_{t+T} &= \frac{1}{2}T(2v_t + aT)\cos(\theta_t) \\
\frac{\partial}{\partial v_t} y_{t+T} &= T\sin(\theta_t) \\
\frac{\partial}{\partial a} y_{t+T} &= \frac{1}{2}T^2\sin(\theta_t) \\
\frac{\partial}{\partial \omega} y_{t+T} &= \frac{1}{6}T^2(3v_t + 2aT)\cos(\theta_t)
\end{aligned}
$$