SCHOOL OF ENGINEERING AND DESIGN

TECHNISCHE UNIVERSITÄT MÜNCHEN

Riemann solver status report

Jonas Erbesdobler

Riemann solver - dissipation limiter

The following document exists for the sole purpose of presenting the current state of my semester thesis in order to have all vital information in one joint document and it is no official document whatsoever. That being said, this document will introduce the equations used for the dissipation limiter, show plots of the calculated kinetic energy as well as the maximum velocity, and present open questions.

1 SPH Discretisation

The approach used in the reference paper [1]. is to model the viscous term Navier Stokes equations implicit through numeric dissipation introduced by the Riemann problem. Therefore, the NS equations are simplified to the inviscid Euler equations [2]. These can be formulated as follows for a weakly compressible fluid [1].

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} \tag{1}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \nabla P \tag{2}$$

Where ρ is the density, $\mathbf{v}$ is the velocity, P is the pressure, and t is the time. Here, $\frac{d}{dt} = \frac{\delta}{\delta t} + \mathbf{v} \cdot \nabla$ refers to the material derivative [1]. Taking the weakly compressible assumption into account, the pressure is calculated from the artificial equation of state, eq. (3) [1].

$$P = c_0^2 \left(\rho - \rho_0\right) \tag{3}$$

Here, c_0 denotes the artificial speed of sound and is calculated via $c_0 = 10 V_{max}$ [1]. Applying the SPH operator to the mass conservation eq. (1), leads to the discretised formulation in eq. (4) [1][3].

$$\frac{d\rho_i}{dt} = \rho_i \sum_j \frac{m_j}{\rho_j} \mathbf{v}_{ij} \cdot \nabla_i W_{ij} \tag{4}$$

The mass of particle j is denoted by m_j, $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$, and $\nabla_i W_{ij}$ is the gradient of the Kernel function [1]. This can be reformulated again into the following equation,

$$\frac{d\rho_i}{dt} = 2\rho_i \sum_j \frac{m_j}{\rho_j} \left(\mathbf{v}_i - \overline{\mathbf{v}}_{ij}\right) \cdot \nabla_i W_{ij} \tag{5}$$

where $\overline{\mathbf{v}}_{ij} = \left(\mathbf{v}_i + \mathbf{v}_j\right)/2$ is the average velocity between the particles i and j [1]. In a similar fashion, we get the SPH formulation for the momentum equation [1][3].

$$\frac{d\mathbf{v}}{dt} = -2 \sum_j m_j \frac{\overline{P}_{ij}}{\rho_i \rho_j} \nabla_i W_{ij} \tag{6}$$

2 Riemann Solver

Based on the abovementioned equations, we can now introduce the Riemann solver into these equations. First, the inter-particle Riemann problem is constructed along a unit vector, which is defined as the following [1].

$$\mathbf{e}_{ij} = -\frac{\mathbf{r}_{ij}}{\|\mathbf{r}_{ij}\|_{L_2}} \tag{7}$$

In this Riemann problem, the initial left and right states are on particles i and j, respectively. The L and R states are described in the following equation [1].

$$\begin{aligned} (\rho_L, U_L, P_L) &= \left(\rho_i, \mathbf{v}_i \cdot \mathbf{e}_{ij}, P_i\right) \\ (\rho_R, U_R, P_R) &= \left(\rho_j, \mathbf{v}_j \cdot \mathbf{e}_{ij}, P_j\right) \end{aligned} \tag{8}$$

Using the left and right states, a linearised Riemann solver for smooth flows with only moderately strong shocks can be written as

$$\begin{aligned} U^* &= \overline{U} + \frac{1}{2}\frac{(P_L - P_R)}{\overline{\rho} c_0} \\ P^* &= \overline{P} + \frac{1}{2}\overline{\rho} c_0 \left(U_L - U_R\right) \\ \mathbf{v}^* &= U^* \mathbf{e}_{ij} + \left(\overline{\mathbf{v}}_{ij} - \overline{U}\mathbf{e}_{ij}\right) \end{aligned} \tag{9}$$

where $\overline{U} = (U_L + U_R)/2$ and $\overline{P} = (P_L + P_R)/2$ are inter-particle averages [1]. With the solution of the Riemann problem, eq. (5) and (6) are discretized as shown below [1].

$$\frac{d\rho_i}{dt} = 2\rho_i \sum_j \frac{m_j}{\rho_j} \left(\mathbf{v}_i - \mathbf{v}^*\right) \cdot \nabla_i W_{ij} \tag{10}$$

$$\frac{d\mathbf{v}}{dt} = -2 \sum_j m_j \frac{P^*}{\rho_i \rho_j} \nabla_i W_{ij} \tag{11}$$

3 Dissipation Limiter

This section will introduce the approach used for the dissipation limiter and afterwards present open questions regarding its formulation.

3.1 Equations

Since the above discretisation is very dissipative, a straightforward modification is to apply a limiter to decrease the implicit numerical dissipations introduced in eq. (9). In the reference paper [1], it is proposed to limit the intermediate pressure as

$$P^* = \overline{P} + \frac{1}{2}\beta\overline{\rho}\left(U_L - U_R\right) \tag{12}$$

where the limiter is defined as

$$\beta = \min\left(\eta \max\left(U_L - U_R, 0\right), \bar{c}\right). \tag{13}$$

It is suggested to use $\eta = 3$ according to numerical tests [1].

3.2 Problems - What to do?

$\bar{c}$ is not defined

The definition of $\bar{c}$ is, unfortunately, missing in [1]. The author suggests, following previous conventions in [1], $\bar{c}$ might be defined as

$$\bar{c} = 10\overline{U}. \tag{14}$$

However, this leads to unphysical results, which is why we assume $\bar{c} = c_0$ for further calculations.

Kernel is not clearly defined

Furthermore, it is mentioned in [1] that the 5th-order Wendland kernel from [4] is used. However, as one can see, the 5th-order Wendland kernel has different possible formulations [4][5], and it is not specified which one is used. Therefore, the already implemented quintic spline kernel is used for now.

4 Taylor Green Vortex Plots

4.1 Reference Plot

The figure below shows the results obtained in [1]. The "Present" data refers to a solution obtained by including the dissipation limiter, and "Without limiter" refers to data obtained from directly solving eq. (10) and (11).

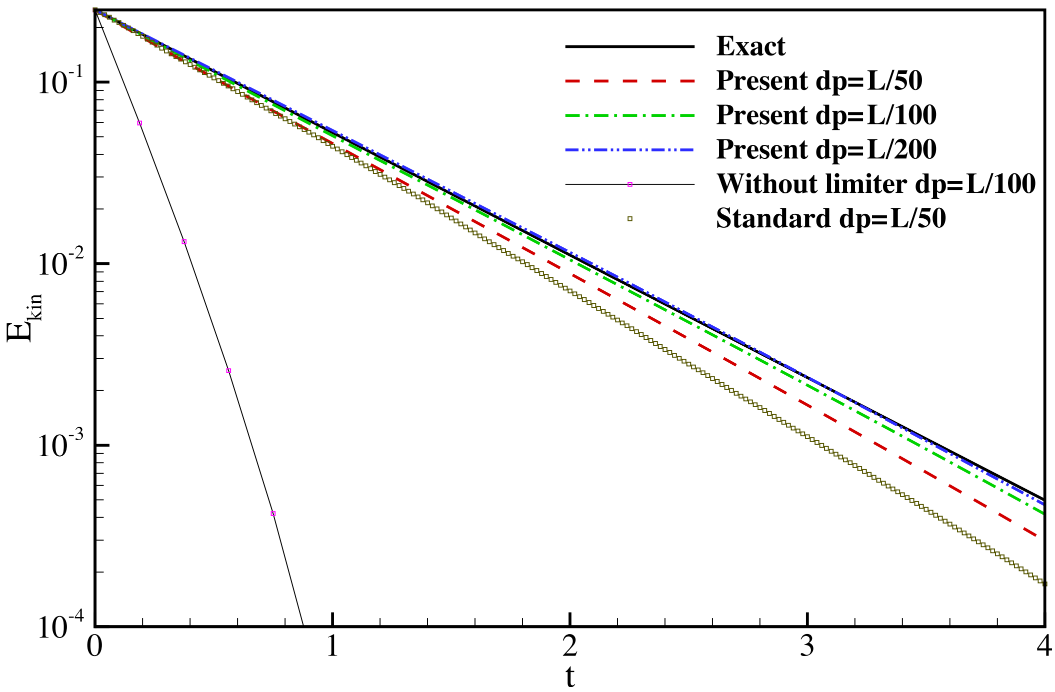

Figure 1: Taylor Green Vortex kinetic energy plot from [1]

The input parameters used in the reference paper are shown in table 1 below. The smoothing length is also used for the quitic spline kernel later.

Property	Value
Smoothing length	$h = 1.3dx$
Cutoff radius	$r_c = 2h$
Re	100
η	3

Table 1: input parameters from [1]

4.2 Calculated Plots

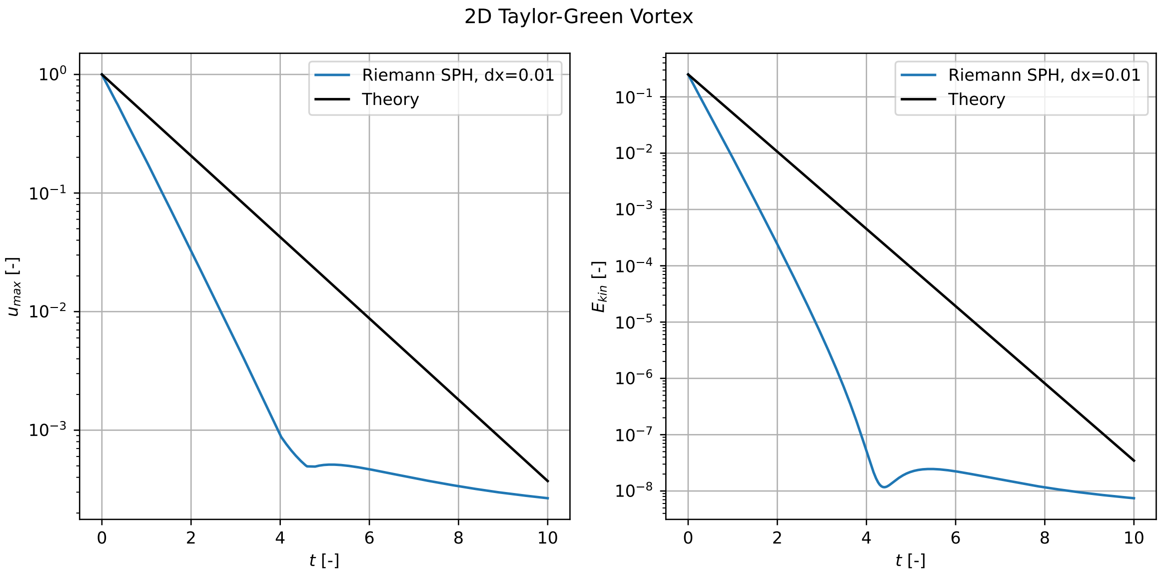

Figure 2: Taylor Green Vortex kinetic energy and maximum velocity plot without dissipation limiter

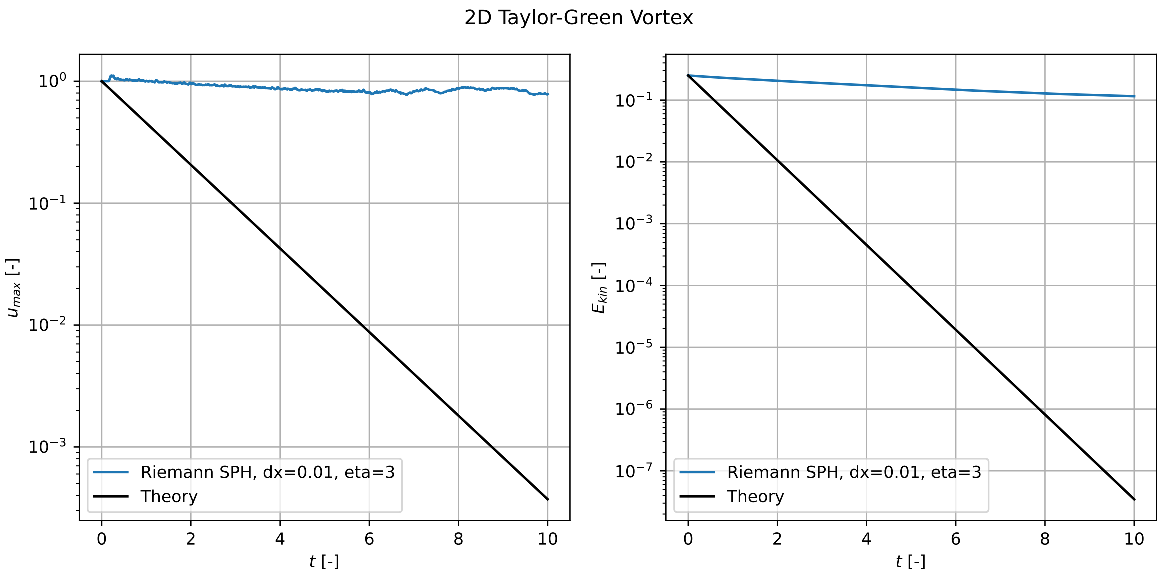

Figure 3: Taylor Green Vortex kinetic energy and maximum velocity plot with dissipation limiter $\eta = 3$

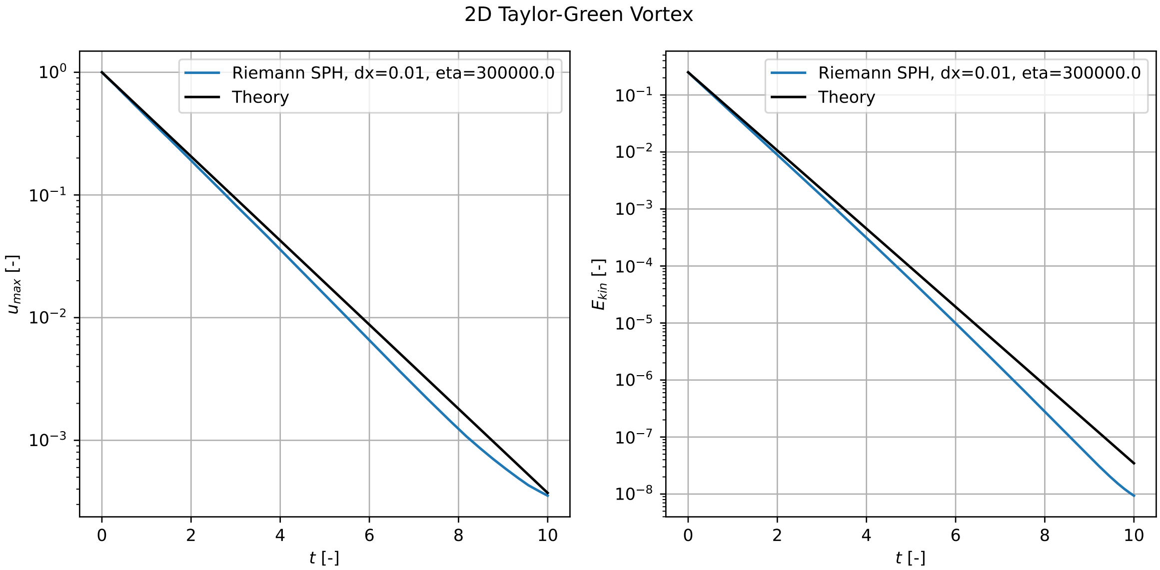

Figure 4: Taylor Green Vortex kinetic energy and maximum velocity plot with dissipation limiter $\eta = 300000$

Probably due to the usage of another kernel function, the implemented Riemann solver is less dissipative compared to the implementation in [1], which you can see in Fig. 1. This leads to the limiter being too effective for the proposed η. However, since the difference between the proposed η and the used one is large, it is not clear whether the usage of the quintic spline kernel is the only error source. However, I am confident that the rest is correctly implemented, apart from $\bar{c}$ being unknown.

4.3 What to do from here?

Implement different versions of 5th-order Wendland kernel?

optimize η and try Hydrostatic test / 2D dam break?

Bibliography

[1] C. Zhang, X. Hu, and N. Adams. "A weakly compressible SPH method based on a low-dissipation Riemann solver". In: *Journal of Computational Physics* 335 (2017), pp. 605–620.

[2] L. Pan, P. Padoan, and A. Nordlund. "Inaccuracy of Spatial Derivatives in Riemann Solver Simulations of Supersonic Turbulence". In: *The Astrophysical Journal* 876 (2019), pp. 90–102.

[3] M. Liu and G. Liu. "Smoothed Particle Hydrodynamics (SPH): an Overview and Recent Developments". In: *Arch Comput Methods Eng* 17 (2010), pp. 25–76.

[4] H. Wendland. "Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree". In: *Advances in Computational Mathematics* 4 (1995), pp. 389–396.

[5] *PySPH documentation*. URL: `https://pysph.readthedocs.io/en/latest/reference/kernels.html`. (accessed: 08.01.2024).