Cooperative Manipulators

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1 Introduction

1.1 What is a Robot? What is a Manipulator?

At first sight a **robot** is very similar to a crane for example. Both are made from several **links** attached serialy with either revolute or prismatic **joints**, at both the end effector can be placed in any place in the **workspace** by defining the **joint parameters** of each individual joint. The only difference between a robot and a **manipulator** (the crane) is that the manipulator ist controlled by a human and the robot is guided by an program running on a computer.

2 Flexible Manipulators

As the focus of modern manufactoring is more and more on light-weight tecnology, as for example space schiences, the structures will lose their rigid character and become more and more flexible. For this reason it is necessary not only to calculate the rigid-body movements, but also simulate the deformations and use the results in the simulation of the movement of the robot. One, probably the most common, method of calculating this deformations I will show in the next subsection.

2.1 A short trip to Finite Elements Analysis (FEA)

To calculate deformations and stresses of structures, **finite element analysis** is an important and often used method. It divides the structure in several points, calles **nodes**, and **elements** between those nodes.

As the finite element analysis divides a part in very many elements and there are a lot of equations to solve for every node, the finite element analysis was not very popular until the 1950s. With the rise of the computer it was possible to solve thousands of equations in a few minutes. The more the computer was developed, the more programs for finite element analysis were developed.

In the following subsection I will give a short review about finite element analysis showing on the example of an one dimensional bar leading to methods of solving more complex problems of two- and three-dimensional problems.

Let us assume that the rod is on simple support and under a constant axial load of q_0, length L, Young's Module E and cross-section area A:
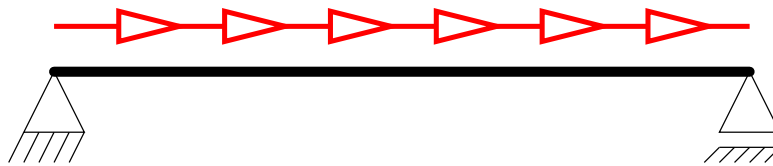
$$EA\frac{du}{dx} + q_0 = 0 \tag{1}$$

From the elementary courses of mechanics we know that

As we now proceed we can assume any trial function $\hat{u}(x)$ that satisfies the boundary conditions that $\hat{u} = 0$ at $x = 0$ and $\frac{d\hat{u}}{dx} = 0$ at $x = L$.

Let $\hat{u}$ be a polynomial function of x with

$$\hat{u}(x) = c_0 \ + \ c_1 \, x \ + \ c_2 \, x$$

with the unknown constants c_0, c_1 and c_2. According to the boundary conditions we have $c_0 = 0$ and $c_1 = -c_2 \, L$. Now our trial function looks like

$$\hat{u}(x) = -2c_2 Lx \ + \ c_2 x = c_2(x \ - \ 2Lx)$$

As we substitute $\hat{u}$ in eq.1 we come to the following equation:

$$2AEc_2 + q_0 = R_d$$

with R_d as the **residual**. We see that for $c_2 = \frac{-q_0}{2AE}$ the residual is equal to zero on every point of the rod. We will see later, that this will not happen for every trial function. As the residual is equal to zero on every point our solution is equal to the exact solution:

$$\hat{u}(x) = (2xL - x\check{s})\frac{q_0}{2AE} = u(x)$$

Assuming a different trial solution, e.g. $\hat{u}(x) = c_0 \, sin(\frac{\pi x}{2L})$, we see that the boundary conditions $\hat{u} = 0$ at $x = 0$ and $\frac{d\hat{u}}{dx} = 0$ at $x = L$ are satisfied. As the equation has only one free parameter, c_0, it is called a **one-parameter solution**.

As we substitute $\frac{d^2\hat{u}}{dx^2}$ again in eq.1 we come to:

$$-EA \, c_0 \, sin\left(\frac{\pi \, x}{2L}\right) \frac{\pi^2}{4L^2} + q_0 = R_d$$

We see that there is no constant c_0 that makes R_d equal to zero on every point of the rod. We can try to make R_d equal to zero on one point (for example $x = 0.5L$), but the errors would be quite huge.

An opportunity to make the mistakes smaller is to add more parameters to our trial solution like

$$\hat{u}(x) = c_0 \, sin\left(\frac{\pi \, x}{2L}\right) \ + \ c_1 \, sin\left(\frac{3 \, \pi \, x}{2l}\right) \ + \ c_2 \, sin\left(\frac{5 \, \pi \, x}{2L}\right) \ + ... \tag{2}$$

which decreases the mistake the more parameters we invent how fig.1 shows. This technique is called the **point collocation technique**.

An other method is the **weighted residual technique**, where $W(x)$ is the **weighted function**. We will multiplicate the residual of our trial function with this weighted function and integrate it over the whole area.

$$\int W(x) \, R_d(x) \, dx = 0$$

Theoretically we can chose every function as weighted function as long as they are integrable, but GALERKIN introduced the idea of letting $W(x)$ be $\hat{u}(x)$ itself in 1915, which works pretty well with most problems (see fig.2)

So for our problem with the one dimensional rod $W(x)$ will be

$$W(x) = sin\left(\frac{\pi\,x}{2L}\right)$$

And the integral would be

$$\int sin\left(\frac{\pi\,x}{2L}\right)\left(-EA\,c_0\,sin\left(\frac{\pi\,x}{2L}\right)\frac{\pi^2}{4L^2} + q_0\right)dx = 0$$

which will lead to

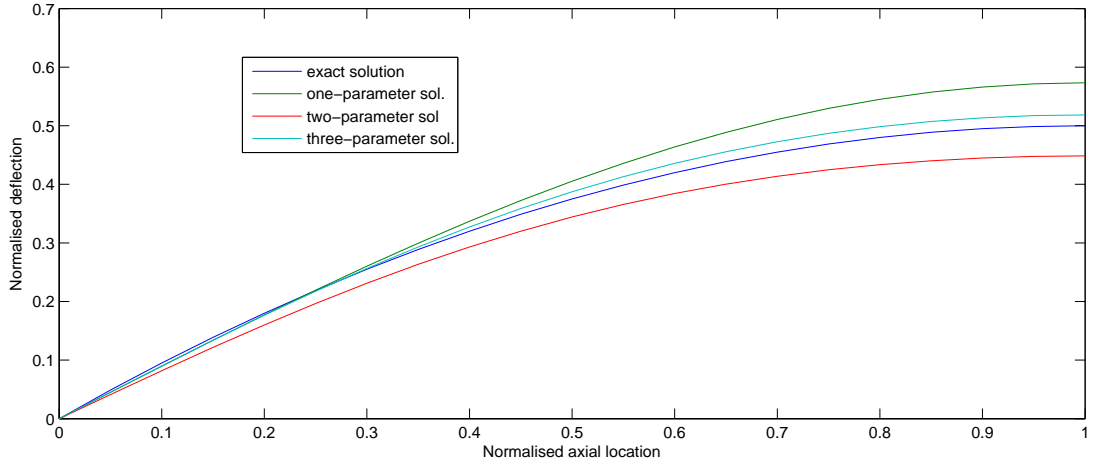$$c_0 = \frac{16\,L^2 q_0}{\pi^3 EA}$$

Figure 1: The exact solution compared to the one-, two-, and three-parameter solution

Note: The GALERKIN-solution here is made for a one-parameter trial function. For a more-parameter trial function with $\hat{u}(x) = \sum c_i\,\hat{u}_i(x)$ there would be weighted functions W_i that make the approximative solution even more exact

2.2 Basic methods of my FEA-program

In the following I will derive the equations used in my one-dimensional finite element program, first for the static case and then, much more complicated, for the dynamic case. I will only give a very short overview of the theoretical basics of FEA, the interested reader is referred to [1] and [2].
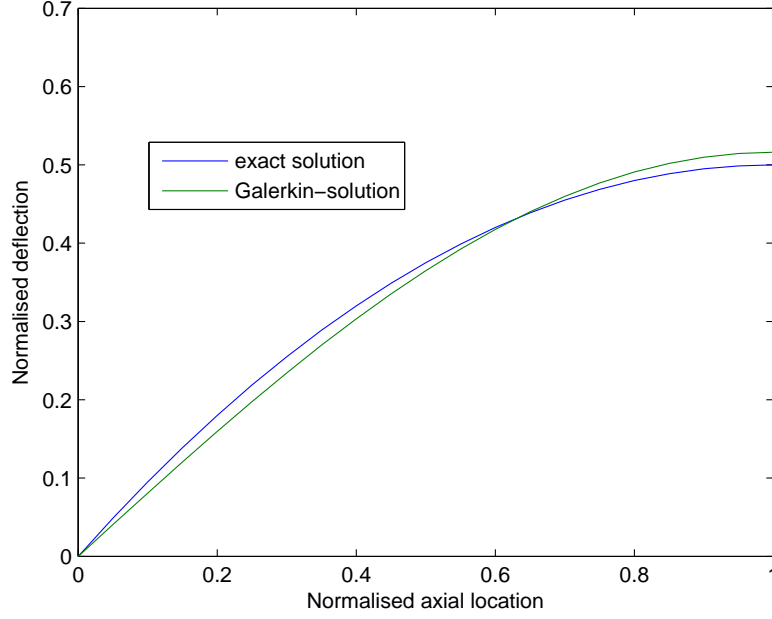
Figure 2: The exact solution compared to the solution with a Galerkin-weighted function

The Beam Element   Our one-dimensional beam element consists of two nodes at each end. Every node has two degrees of freedom as it can move normally to the beams axis and rotate around an axis normally to the grid. We define a vector $\boldsymbol{q}$ describing the deformations of one element and a vector $\boldsymbol{f}$ describing the forces and torques attached to the nodes:
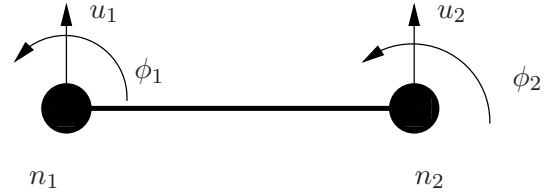

$$\boldsymbol{q} = \begin{pmatrix} u_1 \\ \phi_1 \\ u_2 \\ \phi_2 \end{pmatrix} \qquad\qquad \boldsymbol{f} = \begin{pmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{pmatrix}$$

where u_1 and ϕ_1 are the displacements and rotations of the first node, respectively, and u_2 and ϕ_2 for the second node in the same manner. F_i and M_i are for the force and torques at node i.

The k-Matrix   We now want to derive a matrix $\boldsymbol{k}$, so that $\boldsymbol{f} = \boldsymbol{k} * \boldsymbol{q}$. For that we will look at a beam, deformed in only one degree of freedom, based on the **Beam Theory** of EULER and BERNOULLI. There are a whole bunch of books about the Beam Theory, the interested reader can read for example [3]. For deriving the k-matrix we will deform our beam element four times, so it will just deform in one of the four degrees of freedom.

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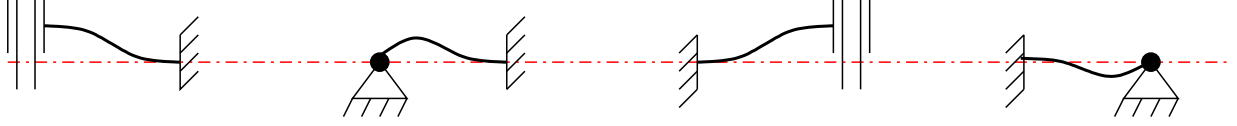
For a beam of lenght l I will demonstrate the derivation of the k-matrix on an example of a beam which has only a displacement in its first degree of freedom. I will use the Beam Theory of EULER and BERNOULLI to calculate the needed forces and torques at each node.
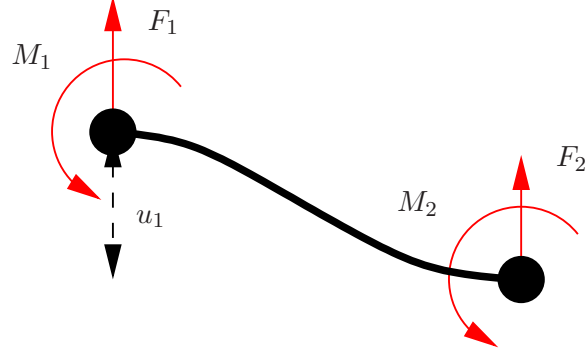
The moment in the beam $M(x)$ for a x running from left to right over the beam's axis is:

$$M(x) = F_1\, x - M_1 \tag{3}$$

Integrating $EI\, w'' = M(x)$ twice with 3 gives:

$$w' = \frac{1}{EI}\left(\frac{1}{2}F_1\, x^2 - M_1\, x + C_1\right) \tag{4}$$

$$w = \frac{1}{EI}\left(\frac{1}{6}F_1\, x^3 - \frac{1}{2}M_1\, x^2 + c_1\, x + C_2\right) \tag{5}$$

With the boundary conditions

$$w(0) = u_1; \qquad w'(0) = 0; \qquad w(l) = 0; \qquad w'(l) = 0$$

inserted in 5 and 4 we get the force-vector $\boldsymbol{f} = k_1 * u_1$:

$$\boldsymbol{f} = \frac{EI}{l^3}\begin{pmatrix} 12 \\ 6\,l \\ -12 \\ 6\,l \end{pmatrix} u_1 \tag{6}$$

Analogously we find k_2, k_3 and k_4 for deformation of an other degree of freedom. Because of Superposition we can write for $\boldsymbol{f}$ for deformation of all degrees of freedom just the sum of all deformations:

$$\boldsymbol{f} = \sum \boldsymbol{k_i}\, \boldsymbol{q_i}$$
$$\boldsymbol{f} = \boldsymbol{k_1}\, \boldsymbol{q_1} + \boldsymbol{k_2}\, \boldsymbol{q_2} + \boldsymbol{k_3}\, \boldsymbol{q_3} + \boldsymbol{k_4}\, \boldsymbol{q_4}$$
$$\boldsymbol{f} = \boldsymbol{k}\, \boldsymbol{q} \tag{7}$$

with

$$k = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \end{pmatrix} \quad (8)$$

For our one dimensional beam element we can derive k as:

$$k = \frac{EI}{l^3} \begin{bmatrix} 12 & 6\,l & -12 & 6\,l \\ 6\,l & 4\,l^2 & -6\,l & 2\,l^2 \\ -12 & -6\,l & 12 & -6\,l^2 \\ 6\,l & 2\,l^2 & -6\,l^2 & 4\,l^2 \end{bmatrix} \quad (9)$$

Shape Functions   In our model forces and torques can only be applied at the nodes, but unluckily in many cases forces and torques are acting somewhere between the nodes. For this reason you need shape functions defining which part of the force is acting on which node of the element. A shape functions of a node has the property that it will be 1 at that specific node and 0 on all other nodes, which is trivial, because forces acting on that node will act with 100% on that node and forces acting on other nodes will not affect that node. I will show two different types of famous shape functions

References

[1] P. Seshu, *Textbook of Finite Element Analysis.* Prentice-Hall of India Private Limited, New Delhi, Fourth Printing, 2006

[2] G. Ramamurty, *Applied Finite Element Analysis.* I.K. International Publishing House Pvt. Ltd., New Delhi, Second Edition, 2010

[3] S. Timoschenko, *History of strength of materials.* McGraw-Hill, New-York, 1953