Effect of Magnetic Fields on Resonant Solar Acoustic Frequencies

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

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Abstract

Yet to be written

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Chapter 1

Introduction

1.1 What is global helioseismology?

Seismology is the study of vibration of material in a body on the surface to deduce its internal structure.

1.2 Frequency splittings in helioseismology

The standard solar model assumed in this work is that of a spherically symmetric, non-rotating, non-magnetic, adiabatic, isotropic, and static (SNRNMAIS). This SNRNMAIS model admits wave like solutions in density perubations (and as a result, displacement, pressure, etc also) [2]. Normal modes are found to have a discrete spectrum of eigenfrequencies to the operator

The essential idea which drives this work is that one can systematically deduce internal structure parameters, such as convective flow, differential rotation, magnetic fields etc by analysing the frequency splittings of solar acoustic resonant modes.

1.2.1 Quasi Degenerate Perturbation Theory

Quasi-degenerate pertubation theory requires us to account for mixing of modes which lie in a small neighbourhood on the frequency spectrum, and hence satisfy the quasi-degenerate condition.

$$|\omega_k^2 - \omega_{ref}^2| < \tau^2 \tag{1.1}$$

A general magnetohydrodynamic (MHD) description of the Sun using the MHD equations in its full vigour is rather cumbersome. Therefore, it is a standard practice (CITE RL91,92) to implement the degenerate perturbation theory in finding the changes in eigenfrequencies from the standard solar model. This model assume the sun to be spherically symmetric, non-rotating, non-magnetic and non-attenuating. The equation of motion (in fourier space) for such a model is given by [2]

$$\boldsymbol{\mathcal{L}}_0\boldsymbol{\xi} = \rho\omega^2\boldsymbol{\xi} = -\boldsymbol{\nabla}(\rho c^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - \rho g\boldsymbol{\xi}\cdot\hat{\boldsymbol{e}}_r) - g\hat{\boldsymbol{e}}_r\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) \tag{1.2}$$

where ω denotes the temporal frequency of oscillations, $c(r), g(r)$ and $\rho(r)$ are the radial functions of sounds speed, gravity and density. $\boldsymbol{\nabla}$ is the covariant spatial derivative operator. For all ensuing calculations and derivations we write equation 1.2 in the form $\mathcal{L}_0\boldsymbol{\xi} = \rho\omega^2\boldsymbol{\xi}$ where the unperturbed wave operator $\mathcal{L}_0$ is self-adjoint (CITE SOMETHING?). Axi-symmetric or non-axisymmetric flows, rotations, asphericities, anisotropies and non-radial variations in c, g, ρ can be captured as perturbation terms in equation 1.2. Although for the purpose of this study, we restrict ourselves to perturbations induced via presence of global scale magnetic fields only.

Because the solar eigenfunctions lack a toroidal component, we can denote the displacement field $\boldsymbol{\xi}(\boldsymbol{r}, \omega)$ in the basis of spherical harmonics (and thereafter generalized spherical harmonics) as follows:

$$\begin{aligned} \boldsymbol{\xi}(\boldsymbol{r}) &= \sum_k U_k(r)Y_l^m(\theta,\phi)\hat{\boldsymbol{e}}_r + V_k(r)\boldsymbol{\nabla}_1 Y_l^m(\theta,\phi) && (1.3) \\ &= \sum_k \xi_k^0(r)Y_{lm}^0(\theta,\phi)\hat{\boldsymbol{e}}_0 + \xi_k^-(r)Y_{lm}^-(\theta,\phi)\hat{\boldsymbol{e}}_- + \xi_k^+(r)Y_{lm}^+(\theta,\phi)\hat{\boldsymbol{e}}_+ && (1.4) \end{aligned}$$

Here, $\boldsymbol{r} = (r, \theta, \phi)$ in spherical polar coordinate system with basis vectors $(\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi)$, $\boldsymbol{\nabla}_1 \equiv= \left(\hat{\boldsymbol{e}}_\theta\partial_\theta + \hat{\boldsymbol{e}}_\phi\frac{1}{\sin\theta}\partial_\phi\right)$ and $k = (n, l, m)$ where n is the radial order, l is the angular degree and m is the azimuthal order of the particular SNRNMAIS mode. These basis vectors are related to those in generalized spherical harmonics' basis as:

$$\hat{\boldsymbol{e}}_- = \tfrac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_\theta - i\hat{\boldsymbol{e}}_\phi), \quad \hat{\boldsymbol{e}}_0 = \hat{\boldsymbol{e}}_r, \quad \hat{\boldsymbol{e}}_+ = -\tfrac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_\theta + i\hat{\boldsymbol{e}}_\phi) \tag{1.5}$$

We work with normalised eigenfunctions $\boldsymbol{\xi}_k$ with the spherically symmetric background density $\rho(r)$ as the weight factor, which satisfy the orthonormality condition:

$$(\boldsymbol{\xi}_{k'}|\rho\boldsymbol{\xi}_k) = \delta_{n'n}\delta_{l'l}\delta_{m'm} \tag{1.6}$$

where the inner product $(|)$ stands for $(\Phi|\Psi) \equiv \int_\odot d^3\mathbf{r}\Phi^*(\mathbf{r})\Psi(\mathbf{r})$.

1.3 Representation of Splitting Data

Since each multiplet ${}_nS_l$ contains $2l+1$ singlet modes, it is tedious to give the degree of splitting in the mode by exact value of each split frequency ω_{nlm} by itself. Therefore, in helioseismology, splitting data is represented by numbers called splitting coefficients which are the coefficients. Frequencies of a split mode ${}_nS_l$ is given as

$$\omega_{nlm} = \omega_{nl} + \sum_{j=0}^{j_{max}} a_j^{nl}\mathcal{P}_j^{(l)}(m) \tag{1.7}$$

where $\mathcal{P}_j^{(l)}(m)$ with $j \in \{0, 1, 2, \ldots, j_{max}\}$ represents a $j_{max} + 1$ dimensional orthogonal basis function of polynomials on the discrete space of m's which run from $-l$ to l. Since, there are $2l + 1$ data points in a particular mode splitting data set ($2l + 1$ singlet frequencies), j_{max} cannot exceed $2l$. In practice, a-coefficients are recorded till a j_{max} of 10 ([7]). We will use, as a standard, the polynomials prescribed in [6] which are Gram-Schmidt orthogonalised polynomials of increasing degree starting with $\mathcal{P}_0^{(l)}(m) = l$. Given the normalisation condition mentioned above, and this starting condition, the polynomials become well defined. A straight forward recipe for obtaining these can be found in Appendix A of [9]. Some properties to note about these polynomials are

- $\mathcal{P}_j^{(l)}(m)$ is odd/even about $m = 0$ if j is odd/even respectively.
- $\mathcal{P}_j^{(l)}(m)$ has polynomial degree j in m.
- $\mathcal{P}_j^{(l)}(m)$ contains only odd/even powers of m if j is odd/even respectively.
- In limit $l \gg 1$, $\mathcal{P}_j^{(l)}(m) \approx l P_j(m/l)$, where P_j is Legendre polynomial of degree j.

Once we have the splitting data ω_{nlm}, one can easily compute the a coefficients multiplying both sides of Eq.(1.7) by $\mathcal{P}_k^{(l)}(m)$,summing over all m, and finally using using the orthogonality condition

$$\sum_{m=-l}^{l} \mathcal{P}_j^{(l)}(m)\mathcal{P}_k^{(l)}(m) = \delta_{jk} \sum_{m=-l}^{l} \left(\mathcal{P}_j^{(l)}(m)\right)^2$$

as

$$a_j^{nl} = \sum_{m=-l}^{l} \delta\omega_{nlm}\mathcal{P}_j^{(l)}(m) \Big/ \sum_{m=-l}^{l} \left(\mathcal{P}_j^{(l)}(m)\right)^2 \tag{1.8}$$

Note that, even though scaled Legendre polynomials $l P_j(m/l)$ are not perfectly orthogonal on a discretised domain (under the inner product $(A|B) \sum_{m=-l}^{l} A^*(m)B(m)$), they are still linearly independent. However they're still not a great choice for basis functions to represent splitting data, because value of a-coefficients will keep changing depending on maximum value of j being considered during summation over m

Chapter 2

The Rotating Sun

2.1 Differential Rotation

The sun is known to rotate, unlike the earth, differentially. This means that angular rate of rotation of a point in the sun about its spin axis is depends on depth and latitude, $\Omega = \Omega(r, \theta)$. Splitting of p-mode frequencies due to differential rotationis well understood[6] and has a long history of inversion analysis[8]. This Ω is generally taken to be symmetric about the equitorial plane because the antisymmtric part of Ω is found to leave no signature in the acoustic frequency spcectrum in the first order as a result of a selection rule that arises from perturbation theory; this will be shown at the end of this chapter.

As a result of Alfven's freezing theorem, differential rotation is responsible for winding the solar magnetic field around its spin axis in an axis symmtric fashion. (refer something on this); thus for the most part we'll be investigating the effects of an axis symmetric magnetic field on the spectrum in this work.

2.2 Detection from frequency spectrum

Because of axis symmetry of differential rotation, the flow profile is given as,

$$\boldsymbol{v}_{\text{rot}} = \sum_{s=1,3,5,\ldots}^{\infty} -w_s^0(r)\partial_\theta Y_s^0 \hat{\boldsymbol{e}}_\phi \tag{2.1}$$

Note that w_1^0 is responsible for shell like (pure) rotation as it couples with $\partial_\theta Y_1^0 \propto \sin\theta$. Hence w_3^0 onwards components of flow are responsible for the differential part of the rotation. Finding frequency splittings due to differential rotatioin is a problem in degenrate perturbation (dpt) or quasi-degenerate perturbation theory depending on whether we're using the isolated multiplet assumption or not. Below we outline the qdpt approach to the problem because when applied to a single multiplet ${}_nS_l$ it reduces to the dpt approach; [5] contains a more detailed discussion on this topic; it, however, argues via analysis of ${}_nS_1$ and ${}_nS_3$ multiplets that eigenfrequency correction due to mode-mixing between these two is negligible ($\sim 0.1\mu Hz$). I'll show in this work that for higher frequency modes with $l \sim 100$ a frequency correction close to $0.4\mu Hz$ is obtained when qdpt is used, which is close to the correction obtained due presence of realistically strong magnetic fields too, and hence cannot be ignored in an analysis which accounts for both differential rotation and magnetic fields.

2.2.1 QDPT Analysis

The pertubation operator $\delta\boldsymbol{\mathcal{L}}^{\mathrm{dr}}$for a differential rotaional flow is given by

$$\delta\boldsymbol{\mathcal{L}}^{\mathrm{dr}} = -2i\omega\rho\boldsymbol{v}_{\mathrm{rot}} \cdot \boldsymbol{\nabla} \tag{2.2}$$

where ω is the reference frequency in the problem, and ρ is the static background density profile. The supermatrix element $Z_{k'k}$ for a pertubation $\delta\boldsymbol{\mathcal{L}}^{\mathrm{dr}}$is given by

$$Z_{k'k} = \Lambda^{\mathrm{dr}}_{k'k} - \delta_{k'k}(\omega^2_{ref} - \omega^2_k) \tag{2.3}$$

where Λ^{dr} is the coupling matrix element $\Lambda^{\mathrm{dr}} = (\boldsymbol{\xi}_{k'}|\delta\boldsymbol{\mathcal{L}}^{\mathrm{dr}}\boldsymbol{\xi}_k)$. Coupling matrix element is given by

$$\Lambda^{\mathrm{dr}}_{k'k} = 8\pi\omega_{ref}\gamma_{l'}\gamma_l \sum_{s=1,3,5,...} \gamma_s \begin{pmatrix} l' & s & l \\ -m & 0 & m \end{pmatrix} \int_{\odot} dr r^2 w^0_s(r) T_s(r) \tag{2.4}$$

where the sensitivity kernel T_s is given by

$$\begin{aligned} T_s(r) = (1-(-1)^{s+l+l'})\Omega^0_{l'}\Omega^0_l \begin{pmatrix} l' & s & l \\ -1 & 0 & 1 \end{pmatrix} r^{-1} \Big(&U'V + V'U - U'U \\ &- \frac{1}{2}\left(l'(l'+1) + l(l+1) - s(s+1)V'V\right)\Big) \end{aligned} \tag{2.5}$$

where the rounded brackets represent Wigner 3j symbols.

2.2.2 Selecition rules in mode coupling

This matrix element enforces the following selection rules for inter-mode interaction which derive from the properties of Wigner 3j symbols [5].

1. $m' = m$
2. $l' + l + s = \text{odd}$
3. $|l' - l| \leq s \leq l' + l$

It should be noted here that even though only sum over odd s is considered, the expression for T_s is general and holds for all s. This has been verified independently using the Mathematica packaged developed for the sake of this work. This means as far as self coupling is concerned ($l' = l$), the matrix element vanish for all even s.

Chapter 3

Internal Magnetic Fields

3.1 Solar Magnetic Field

It is well known through the observation of much surface solar phenomenon like active regions, solar flares, coronal mass ejections etc, that the sun has in its interior, often highly localised, significant magnetic fields. The source of this magetic field is theorised to be a primordial current which started a dynamo process that is kept going at the expense of continous dissipation of energy from the solar bulk (CITE DYNAMO). It is also widely belived however that mean magnetic field throughout the solar bulk is fairly weak. This chapter is devoted to finding a method to observe signature of this magnetic field in the p-mode frequency spectrum.

Most of high intensity magnetic activity is limited to the solar surface. The tachocline is believed to contain toroidal fields as high as 10^5G (CITE TACH). outside the tachocline the magnetic field is believed to be mostly dipolar such that mean surface magnetic field is about 10G. Very strong local magnetic fields apart from these are also known to exist, but detection of such fields is outside the scope of this work; here we shall only investigate the effect of *global* fields. It will be shown in this chapter why the method of frequency splittings is not the best way to detect strongly localised magentic fields.

3.2 Equation of Motion

In the presence of background magnetic field $\mathbf{B}$, the equation of motion is governed by the new operator $\boldsymbol{\mathcal{L}} \rightarrow \boldsymbol{\mathcal{L}} + \delta\boldsymbol{\mathcal{L}}^B$, where $\delta\boldsymbol{\mathcal{L}}^B$ is estabished in [4] as below.

$$\delta\boldsymbol{\mathcal{L}}^{\mathrm{B}}\boldsymbol{\xi} = \frac{-1}{4\pi}\boldsymbol{\nabla}\cdot[\mathbf{BB}\cdot\boldsymbol{\nabla\xi} + \mathbf{B}\cdot\boldsymbol{\nabla\xi}\mathbf{B} - 2\mathbf{BB}\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\mathbf{B})\mathbf{B} - \mathbf{B}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\mathbf{B}) + B^2\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\boldsymbol{I} - \mathbf{BB}:\boldsymbol{\nabla\xi}\boldsymbol{I} + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}\frac{B^2}{2}\boldsymbol{I}] \tag{3.1}$$

where the : stands for contraction of two second rank tensors ($\mathbf{P}:\mathbf{Q} \equiv P_{ij}Q_{ji}$).

Note that in above expression $\mathbf{B}$ only appears in the second order. This is a consequence of the induction equation where the Lagrangian displacement field $\boldsymbol{\xi}$ creates a current $\mathbf{j} \propto \boldsymbol{\xi} \times \mathbf{B}$ and this current interacts with magnetic field as $\mathbf{j} \times \mathbf{B}$ to give rise to acceleration. [3] contains a detailed derivation of this and a proof of self adjointedness for $\delta\boldsymbol{\mathcal{L}}^B$.

This expression can be put in a more convenient form involving the Lorentz stress tensor $\boldsymbol{\mathcal{H}} \equiv \mathbf{BB}$ as

$$\delta\boldsymbol{\mathcal{L}}^{\mathrm{B}}\boldsymbol{\xi} = \frac{-1}{4\pi}\boldsymbol{\nabla}\cdot[\boldsymbol{\mathcal{H}}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}+(\boldsymbol{\nabla}\boldsymbol{\xi})^T\cdot\boldsymbol{\mathcal{H}}-2\boldsymbol{\mathcal{H}}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{H}}:\boldsymbol{I}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\boldsymbol{I}-\boldsymbol{\mathcal{H}}:\boldsymbol{\nabla}\boldsymbol{\xi}\boldsymbol{I}+\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\left(\frac{\boldsymbol{\mathcal{H}}:\boldsymbol{I}}{2}\right)\boldsymbol{I}] \tag{3.2}$$

The tensor $\boldsymbol{\mathcal{H}}$ is the quantity of interest in this problem and any inversion algorithm must first invert splitting data for its components. It remains unclear if the magnetic field can be recovered from just the knowldge of components of $\boldsymbol{\mathcal{H}}$.

3.3 Coupling Matrix

3.3.1 Lorentz Stress components

The process of taking integrals over a sphere becomes simplified if we're operating in the Generalised Sopherical Harmonics formalism. In this formalism (look at C for basis of this formalism), magnetic field and Lorentz stress are decomposed as

$$\mathbf{B} = \sum_{st}\sum_{\alpha} B_{st}^{\alpha}(r)Y_{st}^{\alpha}(\theta,\phi)\hat{\boldsymbol{e}}_{\alpha}$$

$$\boldsymbol{\mathcal{H}} = \sum_{st}\sum_{\mu\nu} h_{st}^{\mu\nu}(r)Y_{st}^{\mu+\nu}(\theta,\phi)\hat{\boldsymbol{e}}_{\mu}\hat{\boldsymbol{e}}_{\nu}$$

where the generalised spherical harmonic (GSH) coordinate indices given by Greek symbols run from -1 to $+1$. It is not very productive to find general expressions relating $\mathbf{B}$ components to $\boldsymbol{\mathcal{H}}$ components. Instead, we'll find special relations pertaining to the kind of magnetic field at hand when necessary. $\boldsymbol{\mathcal{H}}$ by construction satisfies the symmetry property $h_{st}^{\mu\nu} = h_{st}^{\nu\mu}$ ($\because$ $\boldsymbol{\mathcal{H}} = \mathbf{BB}$), and $(h_{st}^{\mu\nu})^* = (-1)^t h_{s\bar{t}}^{\bar{\mu}\bar{\nu}}$, where overbars represent negatives, follows from its realness condition.

3.3.2 Sensitivity Kernels

Coupling matrix element is given as on integral transform over $\boldsymbol{\mathcal{H}}$ as

$$\Lambda_{k'k}^{\mathrm{B}} = (\boldsymbol{\xi}_{k'}|\delta\boldsymbol{\mathcal{L}}^{\mathrm{B}}\boldsymbol{\xi}_k) = \int_0^{R_\odot} dr r^2 \sum_{\substack{st\\ \mu\nu}} \mathcal{B}_{st}^{\mu\nu}(r)h_{st}^{\mu\nu}(r) \tag{3.3}$$

where $\mathcal{B}_{st}^{\mu\nu}$ are the eigenfunction dependent magnetic sensitivity kernels. Prescription for evaluating these kernels and the explicit expressions can be found in [4]. It should be noted however that the coupling integral $(\boldsymbol{\xi}_{k'}|\delta\boldsymbol{\mathcal{L}}^{\mathrm{B}}\boldsymbol{\xi}_k)$ can be reduced to the radial integral form obtained in (3.3) contains no boundary terms. It is indeed the case that the magnetic field is assumed to vanish at the surface in this analysis. Relaxing this assumption will introduce boundary terms which involve integrals only over the solar surface. Since $h^{\mu\nu}_{st}$ is symmetric in interchange of μ and ν, we may ascribe the same symmetry to $\mathcal{B}^{\mu\nu}_{st}$ too without any loss in generality.

Using the Mathematica package developed for this work [1] which automates manipulation of tensor spherical harmonics via the method of GSHs, the following forms of the kernels were found

$$\begin{aligned}\mathcal{B}^{--}_{st} = \frac{(-1)^{m'+1}}{r^2}\gamma_l\gamma_{l'}\gamma_s\begin{pmatrix} l' & s & l \\ -m' & t & m\end{pmatrix}\Big\{\begin{pmatrix} l' & s & l \\ 1 & -2 & 1\end{pmatrix}\Omega^0_l\Omega^0_{l'}\left[\left(U+\Omega^2_{l'}{}^2V\right)V'-UU'-rV\dot{V}'\right]\\ +\begin{pmatrix} l' & s & l \\ 2 & -2 & 0\end{pmatrix}\Omega^0_{l'}\Omega^2_{l'}\left[(U+r\dot{U})V'-rU\dot{V}'\right]+\begin{pmatrix} l' & s & l \\ 0 & -2 & 2\end{pmatrix}\Omega^0_l\Omega^2_l rV\dot{U}'\\ +\begin{pmatrix} l' & s & l \\ 3 & -2 & -1\end{pmatrix}\Omega^0_l\Omega^0_{l'}\Omega^2_{l'}\Omega^3_{l'}VV'\Big\}\end{aligned} \tag{3.4}$$

$$\begin{aligned}2\mathcal{B}^{0-}_{st} = \frac{(-1)^{m'}}{r^2}\gamma_l\gamma_{l'}\gamma_s\begin{pmatrix} l' & s & l \\ -m' & t & m\end{pmatrix}\Big\{\begin{pmatrix} l' & s & l \\ 0 & -1 & 1\end{pmatrix}\Omega^0_l\Big[\left(2U+\Omega^2_{l'}{}^2V\right)U'\\ +\Omega^0_{l'}{}^2\left(-2UV'-VV'+rV\dot{V}'\right)-r\left(U+V-r\dot{V}'\right)\dot{U}'\Big]\\ -\begin{pmatrix} l' & s & l \\ 1 & -1 & 0\end{pmatrix}\Omega^0_{l'}\Big[(-2U+\Omega^0_l{}^2V)U'+\Omega^0_l{}^2V\left(r\dot{V}'-V'\right)\\ +U\left(2V'+r(\dot{U}'-2\dot{V}'+r\ddot{V}')\right)\Big]+\begin{pmatrix} l' & s & l \\ -1 & -1 & 2\end{pmatrix}\Omega^0_l\Omega^0_{l'}\Omega^2_lV\left[U'-V'+r\dot{V}'\right]\\ +\begin{pmatrix} l' & s & l \\ 2 & -1 & -1\end{pmatrix}\Omega^0_l\Omega^0_{l'}\Omega^2_{l'}\left[V\left(U'-3V'+r\dot{V}'\right)+2r\dot{V}V'\right]\Big\}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{B}^{00}_{st} = \frac{(-1)^{m'}}{2r^2}\gamma_l\gamma_{l'}\gamma_s\begin{pmatrix} l' & s & l \\ -m' & t & m\end{pmatrix}\Big(1\\ +(-1)^{l'+l+s}\Big)\Big\{\frac{1}{2}\begin{pmatrix} l' & s & l \\ 0 & 0 & 0\end{pmatrix}\Big[\left(6U-4\Omega^0_l{}^2V-2r\dot{U}\right)\left(U'-\Omega^0_{l'}{}^2V'\right)+2\Omega^0_{l'}{}^2rU\dot{V}'\\ +r\left(\left(-4U+2\Omega^0_l{}^2V+r\dot{U}\right)\right)\dot{U}'+rU\ddot{U}'\Big]\\ -\begin{pmatrix} l' & s & l \\ -1 & 0 & 1\end{pmatrix}\Omega^0_{l'}\Omega^0_l\Big[V\left(-4U'+2\left(1+\Omega^0_{l'}{}^2\right)V'+r\left(\dot{U}'-2\dot{V}'\right)\right)+2r\dot{V}\left(U'-V'+r\dot{V}'\right)\Big]\Big\}\end{aligned} \tag{3.6}$$

$$\begin{aligned}2\mathcal{B}^{+-}_{st} = \frac{(-1)^{m'}}{r^2}\gamma_l\gamma_{l'}\gamma_s\begin{pmatrix} l' & s & l \\ -m' & t & m\end{pmatrix}\left(1+(-1)^{l'+l+s}\right)\Big\{-2\begin{pmatrix} l' & s & l \\ -2 & 0 & 2\end{pmatrix}\Omega^0_l\Omega^2_l\Omega^0_{l'}\Omega^2_{l'}VV'\\ +\begin{pmatrix} l' & s & l \\ -1 & 0 & 1\end{pmatrix}\Omega^0_{l'}\Omega^0_l\left[-rV\dot{U}'+U\left(U'-V'+r\dot{V}'\right)\right]+\begin{pmatrix} l' & s & l \\ 0 & 0 & 0\end{pmatrix}r^2\left[U\ddot{U}'-\dot{U}\dot{U}'\right]\Big\}\end{aligned} \tag{3.7}$$

Kernel components $\mathcal{B}^{\mu\nu}_{st}$ are found to have these following properties.

1. $\mathcal{B}^{\mu\nu}_{st} = \mathcal{B}^{\nu\mu}_{st}$ (by construction)
2. $\mathcal{B}^{--}_{st} = (-1)^{l+l'+s}\mathcal{B}^{++}_{st}$
3. $\mathcal{B}^{0-}_{st} = (-1)^{l+l'+s}\mathcal{B}^{+0}_{st}$
4. $\mathcal{B}^{00}_{st} = \mathcal{B}^{+-}_{st} = \mathcal{B}^{-+}_{st} = 0$ for odd $(l' + l + s)$

Figure 3.1 shows the four independent components of the sensitivity kernel under self coupling for some modes. It is clear from the plots that for all modes, sensitivity is mostly localised to the solar boundary. This effect is most striking for $\mathcal{B}^{+-}_{st}$ across modes. This is a an indirect consequence of the background density profile of the sun which falls almost exponentially fast with respect to radius towards the outer regions of the sun. The very low density near the boundary makes the eigenfunction peak distinctly near the boundary and as $\mathcal{B}$ consists of the

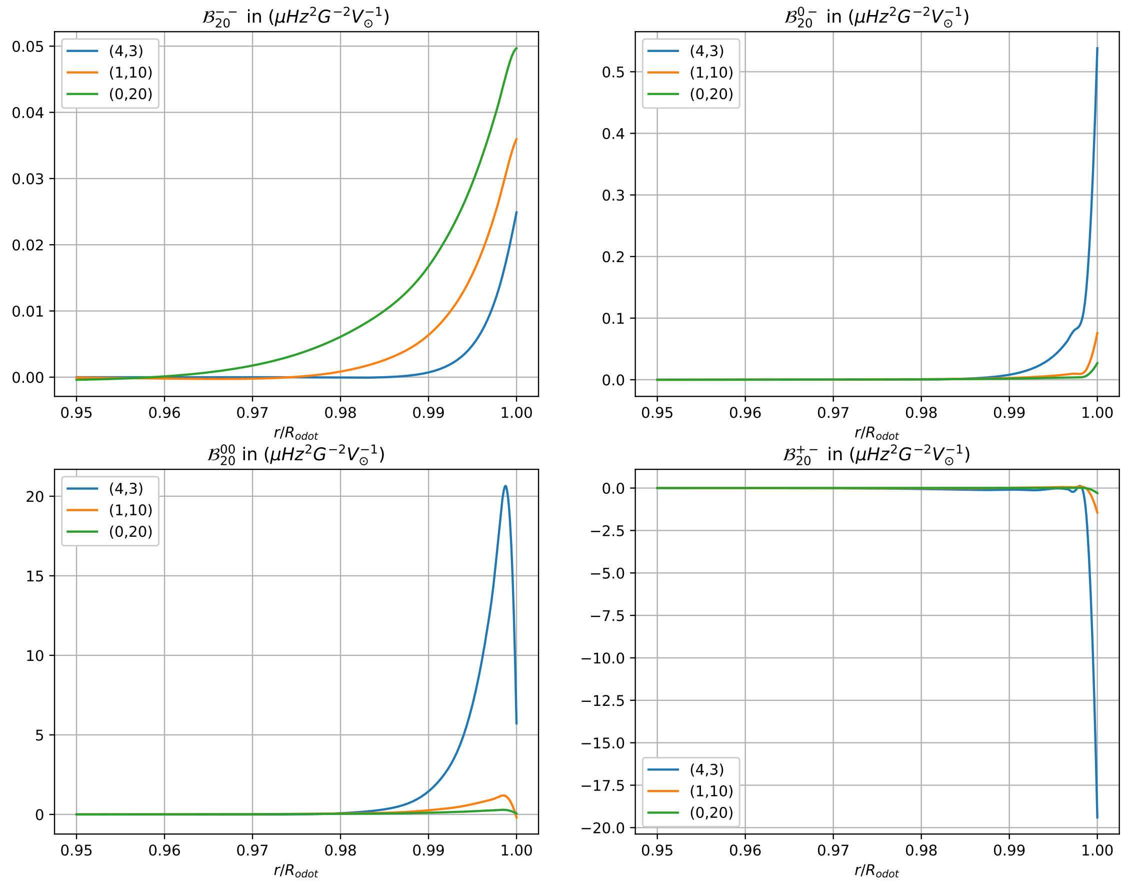


Fig. 3.1 Self coupling Kernels for the modes ${}_4S_3$, ${}_1S_{10}$, and ${}_0S_{20}$.

3.4 Synthetic Magnetic Field

Using some basic pieces of information about mean solar magnetic field as given in 3.1, we can posit the following form of a synthetic magnetic field which will be used for validating our routine of finding frequency splits. We give the following form of the magnetic field which is comoposed of an internal toroidal field concentrated at the tachocline, and a dipolar field which extends from the tachocline to the surface.

3.4.1 Construction of B

Using the identities $\boldsymbol{\nabla}_1 Y_l^m = \Omega_l^0 \left(Y_{lm}^{-1}\hat{\boldsymbol{e}}_- + Y_{lm}^0 \hat{\boldsymbol{e}}_+\right)$, $\hat{\boldsymbol{e}}_r \times \boldsymbol{\nabla}_1 Y_l^m = i\Omega_l^0 \left(Y_{lm}^{-1}\hat{\boldsymbol{e}}_- - Y_{lm}^0 \hat{\boldsymbol{e}}_+\right)$, and $Y_1^0(\theta,\phi) = \gamma_1 \cos\theta$ we see following things: (1) A toroidal field $\mathbf{B} = \alpha(r)\sin\theta\hat{\boldsymbol{e}}_\phi$ can be given as $B_{10} = i\alpha(r)/\gamma_1\,(-1,0,1)$ with all other B_{st} components being 0, and (2) A dipolar field $\mathbf{B} = \beta(r)(2\cos\theta\hat{\boldsymbol{e}}_r + \sin\theta\hat{\boldsymbol{e}}_\theta)$ with $\beta \propto r^{-3}$ can be given as $B_{10} = -\beta(r)/\gamma_1\,(1,-2,1)$ with all other B_{st} components being 0. Note that the row vector refers to the GSH coordinate index μ. This leads to the following final form of $\mathbf{B}$

$$B_{st}(r) = \begin{cases} -i\frac{\alpha(r)}{\gamma_1}\begin{pmatrix}1\\0\\-1\end{pmatrix} - \frac{\beta(r)}{\gamma_1}\begin{pmatrix}b - r\dot{b}\\-2b\\b-r\dot{b}\end{pmatrix}, & \text{for}(s,t) = (1,0)\\ 0, & \text{for}(s,t)\neq(1,0)\end{cases} \tag{3.8}$$

where $b(r) = 1$ where field is perfectly dipolar. The term $r\dot{b}(r)$ appear as a consequence of fixing the diveregence to zero and is only nonzero in the transition region where $b(r)$ goes from 0 to 1. It was can be checked via using $\boldsymbol{\nabla}\cdot\mathbf{B} = g_{\alpha\beta}(\boldsymbol{\nabla}\mathbf{B})^{\alpha\beta}$ ([1] was used) that the two parts in (3.8) (toroidal and dipolar) satisfy the solenoidal condition independently. We plot the forms of the α, β, and b used in our frequency splitting calculations.

3.4.2 Construction of $\mathcal{H}$

After the form of $\mathbf{B}$ has been ascertained, it is straighforward to derive components of $\boldsymbol{\mathcal{H}}$ via taking a tensor product. Decomposing either field in their GSH forms as in 3.3.1, and using orthonormality relation

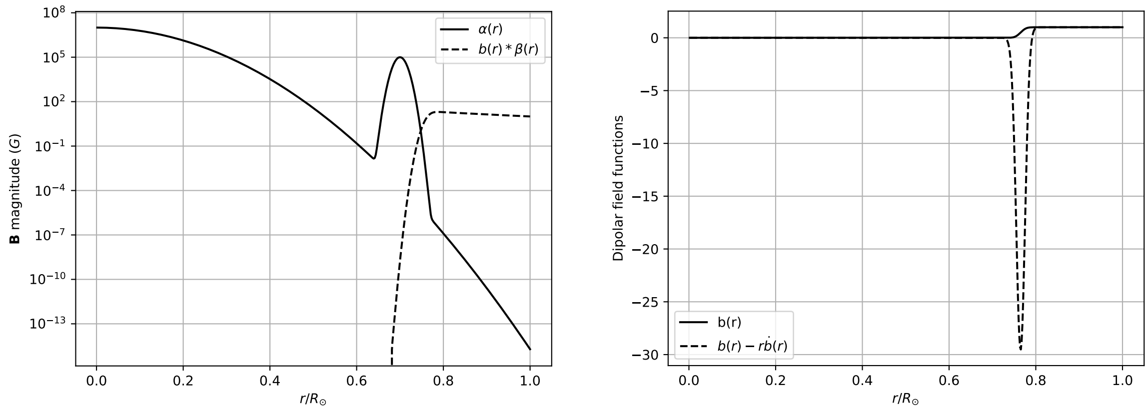
$$\int d\Omega \left(Y_{l_1m_1}^{n_1}\right)^* Y_{l_2m_2}^{n_2} = \delta_{l_1l_2}\delta_{m_1m_2}\delta_{N_1N_2} \tag{3.9}$$

and the triple integral result

$$\int d\Omega \left(Y_{l_1m_1}^{N_1}\right)^* Y_{l_2m_2}^{N_2} Y_{l_3m_3}^{N_3} = (-1)^{m_1+N_1} 4\pi\gamma_{l_1}\gamma_{l_2}\gamma_{l_3} \begin{pmatrix} l_1 & l_2 & l_3\\ -m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3\\ -N_1 & N_2 & N_3\end{pmatrix} \tag{3.10}$$

one can write

$$h_{st}^{\mu\nu} = \sum_{\substack{s_1t_1\\s_2t_2}} \langle Y_{st}^{\mu+\nu}, Y_{s_1t_1}^{\mu} Y_{s_2t_2}^{\nu}\rangle B_{s_1t_1}^{\mu} B_{s_2t_2}^{\nu} \tag{3.11}$$



(a) α and $b * \beta$ in G

(b) b and $a = b - r\dot{b}$

Fig. 3.2 $\alpha(r)$ is addition of two Gaussians centred at $r = 0$ with peak $10^7 G$ and at $r = 0.7R_\odot$ with peak $10^5 G$ respectively. b transitions smoothly from 0 to 1 as a sigmoid around $r = 0.7R_\odot$. $r = 0.7R_\odot$ mark is roughly where the tachocline is placed. Figure 3.2a shows the poloidal (dipolar) field β starting to dominate over the toroidal field by atleast three orders of magnitude as r exceeds $\sim 0.8R_\odot$.

Where $\langle Y^{n_1}_{l_1 m_1}, Y^{N_2}_{l_2 m_2} Y^{N_3}_{l_3 m_3} \rangle$ stands for the integral in eq(3.10). If $\mathbf{B}$ has only $s = s_0$ and $t = t_0$ features, that is $\mathbf{B} = \sum_\alpha B^\alpha_{s_0 t_0} Y^\alpha_{s_0 t_0} \hat{\boldsymbol{e}}_\alpha$, components of $\boldsymbol{\mathcal{H}}$ are given by

$$h^{\mu\nu}_{st} = B^\mu_{s_0 t_0} B^\nu_{s_0 t_0} (-1)^{\mu+\nu+t} (2s_0 + 1)\gamma_s \begin{pmatrix} s_0 & s & s_0 \\ \mu & -(\mu+\nu) & \nu \end{pmatrix} \begin{pmatrix} s_0 & s & s_0 \\ t_0 & -t & t_0 \end{pmatrix} \tag{3.12}$$

For the axis symmetric magnetic field constructed in 3.4.1, we may set $s_0 = 1$ and $t_0 = 0$. Wigner 3j selection rules given in 2.2.2 dicate that $\boldsymbol{\mathcal{H}}$ can only have $s = 0, 1, 2$ and $t = 0$. Then we have the form

$$h^{\mu\nu}_{s0} = 3\gamma_s B^\mu_{10} B^\nu_{10} (-1)^{\mu+\nu} \begin{pmatrix} 1 & s & 1 \\ \mu & -(\mu+\nu) & \nu \end{pmatrix} \begin{pmatrix} 1 & s & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

But we know that $\begin{pmatrix} 1 & s & 1 \\ 0 & 0 & 0 \end{pmatrix}$ vanishes for odd s. Thus we note here that $\boldsymbol{\mathcal{H}}$ has no $s = 1$ and has non-zero $s = 0$ components, which is different from how differential rotation couples modes. The $s = 0$ feature of the Lorentz stress tensor indicates a net shift from the unperturbed mode frequency ω_{nl} for a particular multiplet ${}_nS_l$ as this term couples with $\begin{pmatrix} l' & 0 & l \\ -m & 0 & m \end{pmatrix}$ which is independent of m.

Chapter 4

Results

4.1 Frequency Splittings due to Differential Rotation

4.2 Frequency splittings Due to Lorentz Stresses

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Appendix A

Calculation of Differential Rotation Flow Sensitivity

A.1 Formulation

A.2 Sensitivity Kernels

Appendix B

Lorentz Stress Sensitivity

B.1 Sensitivity Kernels

Appendix C

Generalised Spherical Harmonics Formalism

C.1 Formalism

C.2 Integrals

C.3 Conventions

Appendix D

Fourier Convention

The following convention is employed in this work regarding temporal Fouirer transforms: