Exercise – Understanding Monte Carlo integration using a profit function

1. Aim of the exercise

Monte Carlo (MC) integration is a technique for estimating integrals. in this exercise we use this technique to estimate a profit function.

2. Theory

A thorough treatment of MC theory is covered in the dedicated exercise on MC integration theory. Here, we illustrate the method through an example.

Consider an energy supplier that provides both electricity and gas. The supplier's goal is to approximate total profit over a representative one-year period using Monte Carlo integration, treating time and daily sales quantities as stochastic inputs. Gas demand is assumed to vary seasonally, while electricity demand remains constant throughout the year. For simplicity, we assume an initial price of 5 euros per day for both products, with 100 units of each available daily.

Let x and y denote the daily quantities sold of gas and electricity, respectively. The profit function $\pi : \mathbb{R}^3 \to \mathbb{R}$, evaluated at a given time t, is defined as

$$\pi(x, y, t) = [P(x, y, t) - C(x, y, t)] \cdot Q(x, y, t),$$

where P, C, and Q represent the price, cost, and demand functions, respectively.

The price function $P : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} P(x, y, t) &= \text{Base price} \cdot \text{Inflation factor} - \text{Price sensitivity} \\ &= 5 \cdot \left(1 + \frac{0.02}{365}\right) - \frac{1}{200}x - \frac{1}{300}y, \end{aligned}$$

where the term $\left(1 + \frac{0.02}{365}\right)$ represents a daily inflation adjustment, referred to as the inflation factor.

The cost function $C : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} C(x, y, t) &= [\text{Fixed costs} + \text{Variable costs}] \cdot \text{Time adjustment factor} \\ &= [2 + 0.015x + 0.01y] \cdot \left(1 + \frac{t}{1000}\right). \end{aligned}$$

The demand function $Q : \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$\begin{aligned} Q(x, y, t) &= [\text{Initial demand} - \text{Projected demand reduction}] \cdot \text{Demand growth factor} \\ &= [80 - 0.05 \cdot x \cdot \rho - 0.08 \cdot y] \cdot e^{\frac{t}{1000}}, \end{aligned}$$

where

$$\rho := \frac{1}{3} \cos\left(\frac{2\pi t}{365} - \frac{\pi}{6}\right) + 1$$

is the seasonal adjustment term. This formulation captures higher demand during winter months relative to summer. Thus, $Q(x, y, t)$ represents the total effective demand across both products, modulated by seasonal and temporal dynamics.

The supplier's goal is to approximate expected annual profit by Monte Carlo integration over time and possible daily sales quantities. Specifically, we define total profit over the year as

$$\Pi := \int_0^{365} \int_0^{100} \int_0^{100} \pi(x, y, t)\, dx\, dy\, dt,$$

and estimate this quantity using random sampling over the domain of time and sales levels.

We begin by defining the random variables $T \sim \text{Unif}(0, 365)$, with PDF

$$f_T(t) = \frac{1}{365} \cdot \mathbb{I}\{t \in [0, 365]\},$$

and $X, Y \sim \text{Unif}(0, 100)$, with PDFs

$$f_X(x) = f_Y(y) = \frac{1}{100} \cdot \mathbb{I}\{x, y \in [0, 100]\}.$$

This formulation allows us to express the total profit Π as the expected value of $\pi(X, Y, T)$, where X, Y, T are random variables uniformly distributed over their respective domains. Since the corresponding PDFs are constant, the integral can be scaled by the total volume. In the representation below, we use lowercase x, y, t to denote integration variables corresponding to realizations of X, Y, T:

$$\Pi = \underbrace{365 \cdot 100 \cdot 100}_{\text{Volume V}} \cdot \int_0^{365} \int_0^{100} \int_0^{100} \pi(x, y, t) \cdot f_X(x) \cdot f_Y(y) \cdot f_T(t)\, dx\, dy\, dt.$$

Note that to compute the expected value of $\pi(X, Y, T)$, we use the law of the unconscious statistician: if we have a function of a random variable, we can compute its expectation by integrating that function times the PDF of the original variable.

Next, generate N i.i.d. samples

$$\theta_i := (X_i, Y_i, T_i) \sim \text{Unif}([0, 100]^2 \times [0, 365]), \quad \text{for } i = 1, \ldots, N.$$

Each sample θ_i represents a random configuration of daily sales quantities and time. The total profit Π can then be approximated via Monte Carlo integration:

$$\Pi \approx \frac{V}{N} \sum_{i=1}^{N} \pi(\theta_i),$$

where $\pi(\theta_i) := \pi(X_i, Y_i, T_i)$ denotes the profit evaluated at sample θ_i. This construction yields an unbiased MC estimator of the total annual profit. Accuracy may be further improved through variance reduction techniques. This approach provides a simulation-based estimate of annual profit under modelled seasonal and pricing dynamics.

3. Simulation setup

Clear the memory. Set the sample size.

```
%% 3. Define the number of samples

% 3.1. Clear workspace and memory
clear;

% 3.2. Set number of random samples
N_samples = 1000;
```

4. Monte Carlo sampling of input variables

We perform MC sampling to generate random input variables for estimating annual profit. We first compute the total volume of the integration domain, 365 days times 100 units of gas and electricity each. Then, we generate random samples for daily gas quantities, electricity quantities, and time (day of the year), each drawn uniformly from their respective ranges and rounded to integer values. These samples serve as input configurations for evaluating the profit function across a representative year.

```
%% 4. Monte Carlo sampling of input variables

% 4.1. Define total volume over time and space
volume_v = 365*100*100;

% 4.2. Generate random x values
x_sample = round(100*random('Uniform',0,1,[N_samples 1]));

% 4.3. Generate random y values
y_sample = round(100*random('Uniform',0,1,[N_samples 1]));

% 4.4. Generate random time values (days)
t_sample = round(365*random('Uniform',0,1,[N_samples 1]));
```

5. Compute profit components

We begin by modeling seasonal variation in gas demand using a cosine function, which gives us a periodic adjustment over time. Then we compute a growth factor to capture exponential demand increase throughout the year. Using these, we define the quantity function, which depends on both the sampled inputs and seasonal effects. Next, we apply a compound inflation adjustment to reflect daily price growth. We calculate the price function by adjusting the base price for inflation and subtracting input-based sensitivities. Similarly, we compute the cost function, which increases with both input quantities and time. Finally, we determine profit per sample by subtracting cost from price and scaling the result by the effective quantity.

```
%% 5. Compute profit components

% 5.1. Seasonal effect: models periodic variation over time
seasonal_effect = (1/3)*cos((2*pi/365).*t_sample-pi/6)+1;

% 5.2. Growth factor: exponential increase over time
growth = exp(t_sample./1000);

% 5.3. Quantity function: depends on inputs and seasonal effects
q_function = (80-0.05.*x_sample.*seasonal_effect-0.08.*y_sample).*growth;

% 5.4. Inflation adjustment: compound interest model
inflation = (1+0.02/365).^t_sample;

% 5.5. Price function: adjusted for inflation and input costs
p_function = 5.*inflation-(1/200).*x_sample-(1/300).*y_sample;
```

```
% 5.6. Cost function: increases with inputs and time
c_function = (2+0.015*x_sample+0.01*y_sample).*(1+t_sample./1000);

% 5.7. Profit per sample: revenue minus cost, scaled by quantity
pi_function = (p_function-c_function).*q_function;
```

6. Estimate total profit

We estimate the total annual profit using Monte Carlo integration by averaging the profit values computed across all sampled configurations. Specifically, we take the mean of the profit values stored in `pi_function` and scale it by the total volume of the integration domain `volumve_v`. This yields an unbiased approximation of the integral representing total profit over the year.

```
%% 6. Estimate total profit

% 6.1. Monte Carlo estimate of total profit
pi = volume_v*mean(pi_function);
```

7. Final notes