Exercise – Understanding convergence speed through Big O notation

1. Aim of the exercise

The aim of this exercise is to illustrate how convergence rates behave across different dimensions using Big O notation. Specifically, we examine the expression $\mathcal{O}(1/n^{1/d})$, which describes how the error of an algorithm decreases as the number of samples n increases, depending on the dimension d. As d grows, convergence slows down significantly, a phenomenon known as the curse of dimensionality. This exercise compares convergence behavior for $d = 1, 2, 3$, and visualizes how the rate of error reduction changes with dimension. Understanding these patterns is essential in numerical analysis and simulation, where algorithm efficiency and sample size requirements are tightly linked to dimensionality. The general form $\mathcal{O}(1/n^{1/d})$ implies:

- $d = 1$, $\mathcal{O}(1/n)$: error decreases rapidly
- $d = 2$, $\mathcal{O}(1/n^{1/2})$: error decreases more slowly
- $d = 3$, $\mathcal{O}(1/n^{1/3})$: error decreases even more slowly

This highlights that higher-dimensional problems require significantly more samples to achieve the same level of accuracy, an important insight in numerical analysis and simulation.

2. Define sample size range

This section defines the range of sample sizes used to explore convergence behavior. By varying n from 1 to 100, we simulate how increasing the number of samples affects error reduction. Starting from 1 avoids division by zero in later calculations.

```
%% 2. Define sample size range

% Sample size range
n = 1:1:100; % Avoid n = 0 to prevent division by zero.
```

3. Model convergence rates

This step computes the expression $\mathcal{O}(1/n^{1/d})$ for dimensions $d = 1, 2, 3$. It models how error decreases as sample size n increases, with slower convergence in higher dimensions. The results illustrate how dimensionality affects the rate of error reduction.

```
%% 3. Model convergence rates

% Convergence rates for d = 1, 2, 3
rate_d1 = 1./(n.^(1/1));
rate_d2 = 1./(n.^(1/2));
rate_d3 = 1./(n.^(1/3));
```

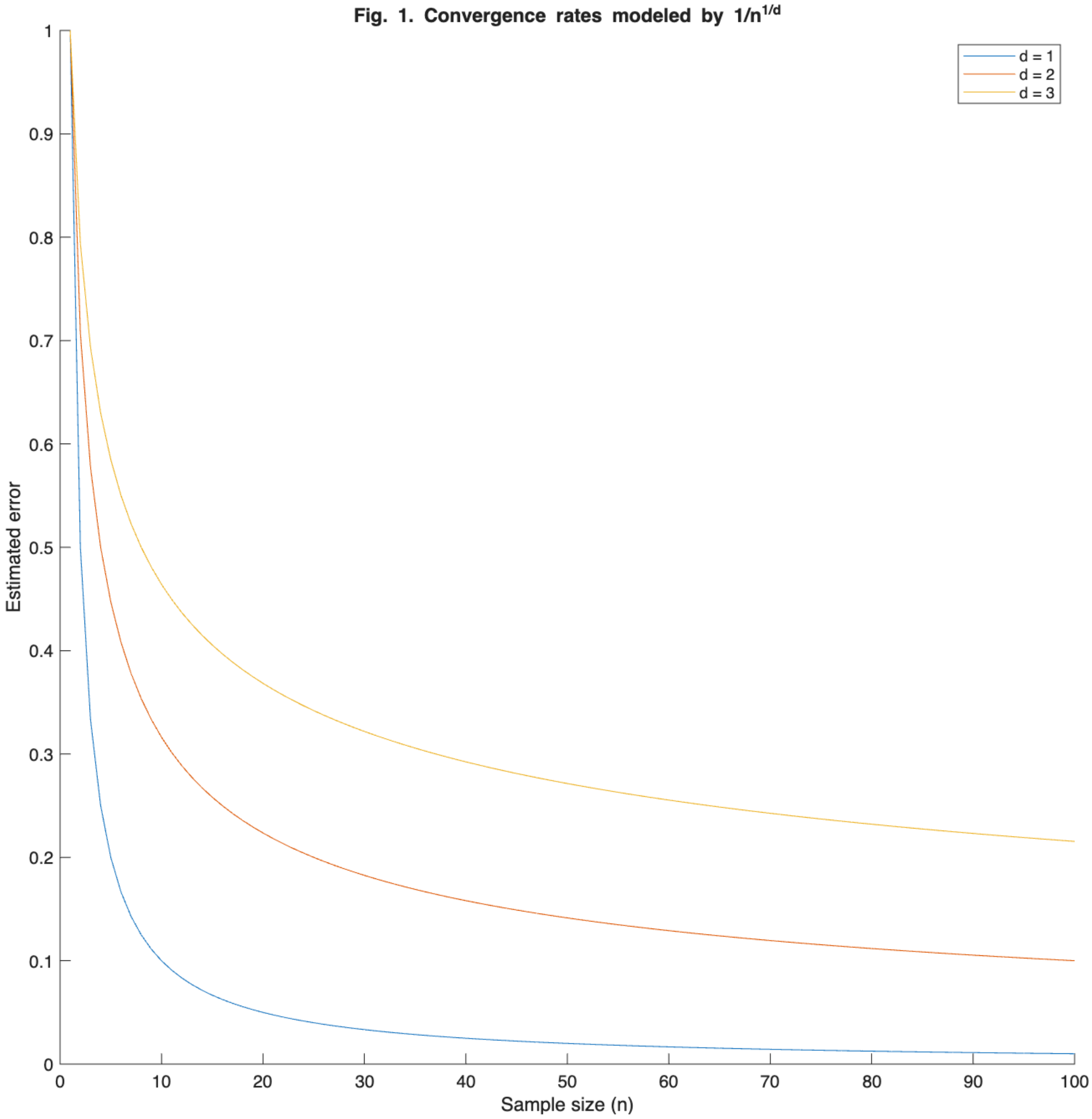
4. Plot convergence behavior across dimensions

This part creates a visual comparison of how quickly error decreases as the number of samples grows, across different dimensions. The curves show that in higher dimensions, improvement happens more slowly, making it harder to achieve accurate results with limited data. The plot helps highlight how dimensionality affects the speed of convergence.

```
%% 4. Plot convergence behavior across dimensions

% Plot
figure
hold on
plot(n,rate_d1,'DisplayName','d = 1');
plot(n,rate_d2,'DisplayName','d = 2');
plot(n,rate_d3,'DisplayName','d = 3');
title('Fig. 1. Convergence rates modeled by 1/n^{1/d}');
xlabel('Sample size (n)');
ylabel('Estimated error');
legend('show');
hold off
```

Fig. 1. Convergence rates modeled by $1/n^{1/d}$



5. Final notes

This file is prepared and copyrighted by Jelmer Wieringa and Tunga Kantarcı. This file and the accompanying MATLAB file are available on GitHub and can be accessed using this link.