Violation of the exogeneity assumption, the IV estimator, and the GIV estimator

Empirical Methods, Lecture 7

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SLM, error is exogenous

The SLM assumes that ε_i is strictly exogenous, i.e., $\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_k\right] = 0$.

SLM, error is exogenous

The strict exogeneity assumption states that

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_k\right] = 0.$$

$\boldsymbol{x}_k$ contains n observations for variable k. It says that the mean of ε_i at observation i is independent of the explanatory variable k observed at observation i and also at any other observation j.

SLM, error is exogenous

The weak exogeneity assumption states that

$$\mathrm{E}\left[\varepsilon_i \mid x_{ik}\right] = 0.$$

x_{ik} is the observation i for variable k. Hence, we do not consider all n observations of variable k, denoted by $\boldsymbol{x}_k$, but just the observation i, denoted by x_{ik}.

Skip.

SLM, error is exogenous

Generalising

$$\mathsf{E}\left[\varepsilon_i \mid x_{ik}\right] = 0$$

to K variables, we consider

$$\mathsf{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}.$$

Skip.

SLM, error is exogenous

In this lecture we allow

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] \neq \mathbf{0}.$$

That is, we violate weak exogeneity. But we still assume that

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_j\right] = \mathbf{0}.$$

Skip.

SLM, error is exogenous, implications

$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}$ has a number of implications.

SLM, error is exogenous, implication one

First,

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}$$

implies that

$$\begin{aligned}\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] &= \mathrm{E}_{\boldsymbol{x}_i}\left[\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i \mid \boldsymbol{x}_i\right]\right] \\ &= \mathrm{E}_{\boldsymbol{x}_i}\left[\boldsymbol{x}_i \mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right]\right] \\ &= \mathbf{0}\end{aligned}$$

by the LIE. Keep in mind that when the latter is ever stated, it is because the former holds.

SLM, error is exogenous, implication one

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}$$

implies that

$$\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] = \mathbf{0}.$$

When referring to 'exogeneity', we will use the latter statement instead of the former. There are at least two reasons for doing this. First, we can use the latter when talking about covariance: more on this below. Second, the latter is what we need for showing the consistency of the OLS estimator: see the earlier lecture on this.

SLM, error is exogenous, implication, two

Second,

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}$$

implies that

$$\begin{aligned}\mathrm{E}\left[\varepsilon_i\right] &= \mathrm{E}_{\boldsymbol{x}_i}\left[\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right]\right] \\ &= 0.\end{aligned}$$

by the LIE. It says that if the average of ε_i at all slices of the population determined by the values of $\boldsymbol{x}_i$ equals zero, then the average of these zero conditional means must also be zero.

Skip.

SLM, error is exogenous, implication three

Third,

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = \mathbf{0}$$

implies that

$$\begin{aligned}
\mathrm{Cov}\left[\varepsilon_i, \boldsymbol{x}_i\right] &= \mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] - \mathrm{E}\left[\varepsilon_i\right]\mathrm{E}\left[\boldsymbol{x}_i\right] \\
&= \mathbf{0} - \mathbf{0}\mathrm{E}\left[\boldsymbol{x}_i\right] \\
&= \mathbf{0}
\end{aligned}$$

using the above results. That is, ε_i are $\boldsymbol{x}_i$ are uncorrelated.

SLM, error is endogenous

Violate

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] = 0$$

so that

$$\mathrm{E}\left[\varepsilon_i \mid \boldsymbol{x}_i\right] \neq 0$$

which makes $\boldsymbol{x}_i$ endogenous. When does this happen?

SLM, error is endogenous, case of OVB

Consider the linear model

$$y_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \varepsilon_i.$$

Suppose that

$$\mathrm{E}\left[\varepsilon_i \mid x_{i1}\right] = 0,$$

and

$$\mathrm{E}\left[\varepsilon_i \mid x_{i2}\right] = 0.$$

Hence, the model is correctly specified.

SLM, error is endogenous, case of OVB

Suppose that we do not observe x_{i2} so that it enters the error:

$$y_i = x_{i1}\beta_1 + \varepsilon_i^*$$

where

$$\varepsilon_i^* = x_{i2}\beta_2 + \varepsilon_i.$$

Then,

$$\begin{aligned}
\mathrm{E}\left[\varepsilon_i^* \mid x_{i1}\right] &= \mathrm{E}\left[x_{i2}\boldsymbol{\beta}_2 \mid x_{i1}\right] + \mathrm{E}\left[\varepsilon_i \mid x_{i1}\right] \\
&= \boldsymbol{\beta}_2 \mathrm{E}\left[x_{i2} \mid x_{i1}\right] + 0 \\
&\neq 0
\end{aligned}$$

if

$$\beta_2 \neq 0$$

and

$$\mathrm{E}\left[x_{i2} \mid x_{i1}\right] \neq 0.$$

$\beta_2 \neq 0$ means that x_{i2} should enter the model. $\mathrm{E}\left[x_{i2} \mid x_{i1}\right] \neq 0$ means that x_{i1} and x_{i2} are correlated. A3 is violated for ε_i^*.

SLM, error is endogenous, case of OVB

What is the implication of

$$\mathrm{E}\left[\varepsilon_i^* \mid x_{i1}\right] \neq 0$$

for the OLS estimator $\hat{\beta}_1$? The formula for $\hat{\beta}_1$ when x_{i2} is omitted in the true model, while it should not have been, is

$$\begin{aligned}
\hat{\beta}_1 &= (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{y} \\
&= (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'(\boldsymbol{x}_1\beta_1 + \boldsymbol{x}_2\beta_2 + \boldsymbol{\varepsilon}) \\
&= \beta_1 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2\beta_2 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{\varepsilon}.
\end{aligned}$$

This shows that we regress $\boldsymbol{y}$ (the true model) only on $\boldsymbol{x}_1$, which is the wrong model. Taking the expectation conditional on $\boldsymbol{X}$,

$$\mathrm{E}\left[\hat{\beta}_1 \mid \boldsymbol{X}\right] = \beta_1 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2\beta_2$$

since $\mathrm{E}\left[\boldsymbol{\varepsilon} \mid \boldsymbol{X}\right] = \boldsymbol{0}$ in the true model.

SLM, error is endogenous, case of OVB

$$\mathrm{E}\left[\hat{\beta}_1 \mid \boldsymbol{X}\right] = \beta_1 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2\beta_2.$$

In two cases the estimator is unbiased. First, if

$$\beta_2 = 0,$$

meaning that $\boldsymbol{x}_2$ does not enter the true model. Second, if

$$(\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2 = 0,$$

meaning that there is no correlation between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in the sample. Realize that the stated expression is the OLS estimate of the coefficient of $\boldsymbol{x}_1$ from the regression of $\boldsymbol{x}_2$ on $\boldsymbol{x}_1$! Otherwise the estimator is subject to the omitted variable bias. The equation stated above is the omitted variable bias formula.

SLM, error is endogenous, case of OVB, example

Regress *wage* on *educ* but ignore *exper* because it is, say, unobserved:

```
. regress wage educ

      Source |       SS           df       MS      Number of obs   =       997
-------------+----------------------------------   F(1, 995)       =    251.46
       Model |  7842.35455         1  7842.35455   Prob > F        =    0.0000
    Residual |  31031.0745       995  31.1870095   R-squared       =    0.2017
-------------+----------------------------------   Adj R-squared   =    0.2009
       Total |   38873.429       996  39.0295472   Root MSE        =    5.5845

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   1.135645   .0716154    15.86   0.000     .9951106     1.27618
       _cons |  -4.860424   .9679821    -5.02   0.000    -6.759944   -2.960903
------------------------------------------------------------------------------
```

SLM, error is endogenous, case of OVB, example

Regress *wage* on *educ* and *exper*, and observe that $\hat{\beta}_{educ}$ increases. This suggests that $\hat{\beta}_{educ}$ has downward bias when *exper* is ignored in the previous regression. How do we reach this conclusion?

```
. regress wage educ exper

      Source |       SS           df       MS      Number of obs   =       997
-------------+----------------------------------   F(2, 994)       =    172.32
       Model |  10008.3629         2  5004.18147   Prob > F        =    0.0000
    Residual |  28865.0661       994  29.0393019   R-squared       =    0.2575
-------------+----------------------------------   Adj R-squared   =    0.2560
       Total |   38873.429       996  39.0295472   Root MSE        =    5.3888

------------------------------------------------------------------------------
        wage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   1.246932   .0702966    17.74   0.000     1.108985    1.384879
       exper |   .1327808   .0153744     8.64   0.000     .1026108    .1629509
       _cons |  -8.833768   1.041212    -8.48   0.000    -10.87699   -6.790542
------------------------------------------------------------------------------
```

SLM, error is endogenous, case of OVB, example

In the regression we have ignored *exper*. We suspect that $\hat{\beta}_{educ}$ is biased. That is, we suspect that $\hat{\beta}_{educ}$ would change if we control for *exper* in the regression. Do you expect $\hat{\beta}_{educ}$ to have an upward or downward bias? Use the omitted variable bias formula to form an expectation:

$$\mathrm{E}\left[\hat{\beta}_{educ} \mid \boldsymbol{educ}, \boldsymbol{exper}\right] = \beta_{educ} + (\boldsymbol{educ}'\boldsymbol{educ})^{-1}\boldsymbol{educ}'\boldsymbol{exper}\beta_{exper}.$$

We would expect

$$(\boldsymbol{educ}'\boldsymbol{educ})^{-1}\boldsymbol{educ}'\boldsymbol{exper}$$

to be negative (effect of exper on educ), and

$$\beta_{exper}$$

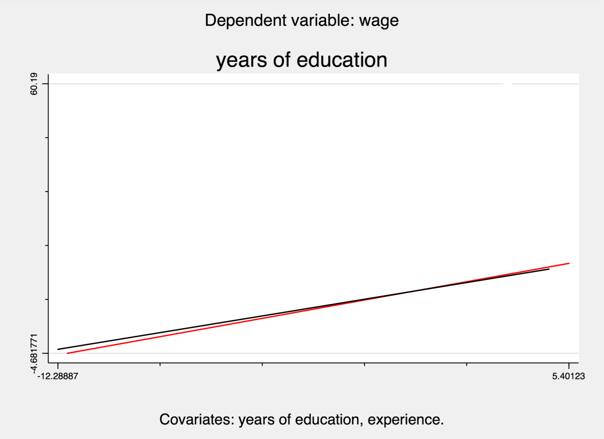
to be positive (effect of exper on wage). Hence, we should expect $\hat{\beta}_{educ}$ to have downward bias when we ignore *exper* in the true regression!

SLM, error is endogenous, case of OVB, example

The fitted line from the regression of *wage* on *educ*. The slope is $\hat{\beta}_{educ}$, and it is biased because we ignore *exper*!

SLM, error is endogenous, case of OVB, example

Adding the fitted line from the regression of *wage* on *educ* after partialling out the effect of *exper* (red line). The slope is $\hat{\beta}_{educ}$, and it is unbiased! The difference in the slopes is the size of the bias due to ignoring *exper* in the regression!



SLM, error is endogenous, case of ME

Consider the linear model

$$y_i = x_i^* \beta + \varepsilon_i.$$

Suppose x_i^* is the true variable we do not observe. Suppose we observe x_i, a noisy version of x_i^* with unobserved measurement error ω_i so that

$$x_i = x_i^* + \omega_i.$$

Since we observe only x_i, replace x_i^* in the model to obtain

$$y_i = x_i \beta \underbrace{-\omega_i \beta + \varepsilon_i}_{\varepsilon_i^*}.$$

x_i is correlated with ε_i^* due to ω_i. OLS estimator of β is subject to the measurement error bias.

Skip.

SLM, error is endogenous, case of SEM

Consider the simultaneous equations model given by

$$y_{i1} = y_{i2}\alpha_1 + z_{i1}\beta_1 + \varepsilon_{i1},$$
$$y_{i2} = y_{i1}\alpha_2 + z_{i2}\beta_2 + \varepsilon_{i2}.$$

In each equation the constant is ignored for simplicity. Assume that

$$\mathrm{E}\left[\varepsilon_{i1} \mid z_{i1}, z_{i2}\right] = 0,$$
$$\mathrm{E}\left[\varepsilon_{i2} \mid z_{i1}, z_{i2}\right] = 0,$$

and that

$$\mathrm{E}\left[\varepsilon_{i1}\right] = 0,$$
$$\mathrm{E}\left[\varepsilon_{i2}\right] = 0.$$

Hence, z_{i1} and z_{i2} are uncorrelated with ε_{i1} and ε_{i2}. Suppose that the interest lies in estimating α_1 in the first equation.

SLM, error is endogenous, case of SEM

Solve the two equations for y_{i2}, in terms of z_{i1}, z_{i2}, ε_{i1}, and ε_{i2}. First, replace y_{i1} in the equation for y_{i2}, and then solve for y_{i2} as

$$
\begin{aligned}
y_{i2} &= y_{i1}\alpha_2 + z_{i2}\beta_2 + \varepsilon_{i2} \\
&= \left(y_{i2}\alpha_1 + z_{i1}\beta_1 + \varepsilon_{i1}\right)\alpha_2 + z_{i2}\beta_2 + \varepsilon_{i2} \\
&= y_{i2}\alpha_1\alpha_2 + z_{i1}\beta_1\alpha_2 + \varepsilon_{i1}\alpha_2 + z_{i2}\beta_2 + \varepsilon_{i2} \\
\left(1 - \alpha_1\alpha_2\right) y_{i2} &= z_{i1}\beta_1\alpha_2 + z_{i2}\beta_2 + \varepsilon_{i1}\alpha_2 + \varepsilon_{i2} \\
y_{i2} &= z_{i1}\frac{\beta_1\alpha_2}{1 - \alpha_1\alpha_2} + z_{i2}\frac{\beta_2}{1 - \alpha_1\alpha_2} + \varepsilon_{i1}\frac{\alpha_2}{1 - \alpha_1\alpha_2} \\
&\qquad + \varepsilon_{i2}\frac{1}{1 - \alpha_1\alpha_2},
\end{aligned}
$$

assuming that $\alpha_1\alpha_2 \neq 1$.

SLM, error is endogenous, case of SEM

The parameter of interest was α_1 in the equation

$$y_{i1} = y_{i2}\alpha_1 + z_{i1}\beta_1 + \varepsilon_{i1},$$

and we have just shown that

$$y_{i2} = z_{i1}\frac{\beta_1\alpha_2}{1-\alpha_1\alpha_2} + z_{i2}\frac{\beta_2}{1-\alpha_1\alpha_2} + \varepsilon_{i1}\frac{\alpha_2}{1-\alpha_1\alpha_2} + \varepsilon_{i2}\frac{1}{1-\alpha_1\alpha_2}.$$

Remember that we need

$$\mathrm{E}\left[y_{i2}\varepsilon_{i1}\right] = 0$$

to hold to consistently estimate α_1! Does it hold?

SLM, error is endogenous, case of SEM

$$y_{i2} = z_{i1}\frac{\beta_1\alpha_2}{1-\alpha_1\alpha_2} + z_{i2}\frac{\beta_2}{1-\alpha_1\alpha_2} + \varepsilon_{i1}\frac{\alpha_2}{1-\alpha_1\alpha_2} + \varepsilon_{i2}\frac{1}{1-\alpha_1\alpha_2}.$$

Multiply both sides with ε_{i1}, take expectations, and use the earlier assumption that $\mathrm{E}\left[z_{i1}\varepsilon_{i1}\right] = 0$ and $\mathrm{E}\left[z_{i2}\varepsilon_{i1}\right] = 0$ to obtain

$$\mathrm{E}\left[y_{i2}\varepsilon_{i1}\right] = \mathrm{E}\left[\varepsilon_{i1}\varepsilon_{i1}\right]\frac{\alpha_2}{1-\alpha_1\alpha_2} + \mathrm{E}\left[\varepsilon_{i2}\varepsilon_{i1}\right]\frac{1}{1-\alpha_1\alpha_2}.$$

If

$$\alpha_2 \neq 0,\ \mathrm{E}\left[\varepsilon_{i2}\varepsilon_{i1}\right] = 0,$$

or

$$\alpha_2 = 0,\ \mathrm{E}\left[\varepsilon_{i2}\varepsilon_{i1}\right] \neq 0,$$

we have

$$\mathrm{E}\left[y_{i2}\varepsilon_{i1}\right] \neq 0,$$

and the OLS estimator of α_1 is subject to the simultaneity bias.

SLM, error is endogenous, implications for OLS estimator

When

$$\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] \neq 0$$

the OLS estimator is biased and inconsistent.

SLM, error is endogenous, OLS estimator is biased

Recall the follow expression we had when proving unbiasedness of $\hat{\boldsymbol{\beta}}$:

$$\begin{aligned}
\mathrm{E}\left[\hat{\boldsymbol{\beta}} \mid \boldsymbol{X}\right] &= \mathrm{E}\left[\boldsymbol{\beta} \mid \boldsymbol{X}\right] + \mathrm{E}\left[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon \mid \boldsymbol{X}\right] \\
&= \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\mathrm{E}\left[\varepsilon \mid \boldsymbol{X}\right] \\
&= \boldsymbol{\beta}.
\end{aligned}$$

If $\mathrm{E}\left[\varepsilon \mid \boldsymbol{X}\right] \neq \mathbf{0}$, $\hat{\boldsymbol{\beta}}$ is biased.

SLM, error is endogenous, OLS estimator is biased

Recall the following expression we had when proving unbiasedness of $\hat{\boldsymbol{\beta}}$:

$$
\begin{aligned}
\mathrm{E}\left[\hat{\beta}_1 \mid \boldsymbol{x}_1, \boldsymbol{x}_2\right] &= \mathrm{E}\left[\boldsymbol{x}_1'\boldsymbol{x}_1^{-1}\boldsymbol{x}_1'\boldsymbol{y} \mid \boldsymbol{x}_1, \boldsymbol{x}_2\right] \\
&= (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'(\boldsymbol{x_1}\beta_1 + \boldsymbol{x}_2\beta_2 + \boldsymbol{\varepsilon}) \\
&= \beta_1 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2\beta_2 + \mathrm{E}\left[(\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{\varepsilon} \mid \boldsymbol{x}_1, \boldsymbol{x}_2\right] \\
&= \beta_1 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\boldsymbol{x}_2\beta_2 + (\boldsymbol{x}_1'\boldsymbol{x}_1)^{-1}\boldsymbol{x}_1'\mathrm{E}\left[\boldsymbol{\varepsilon} \mid \boldsymbol{x}_1, \boldsymbol{x}_2\right].
\end{aligned}
$$

Since $\boldsymbol{x}_2$ is omitted from the regression and left to $\boldsymbol{\varepsilon}$, $\mathrm{E}\left[\boldsymbol{\varepsilon} \mid \boldsymbol{x}_1, \boldsymbol{x}_2\right] \neq \mathbf{0}$, so $\hat{\beta}_1$ is biased.

SLM, error is endogenous, OLS estimator is biased

Suppose that we do not observe x_{i2} so that it enters the error. The model becomes

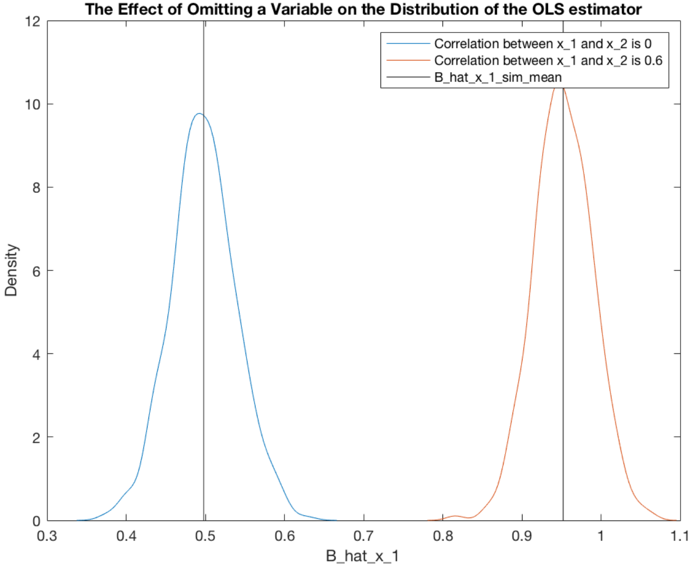
$$y_i = x_{i1}\beta_1 + \varepsilon_i^*$$

where

$$\varepsilon_i^* = x_{i2}\beta_2 + \varepsilon_i.$$

Assume that the true value of β_1 is 0.5. Consider two cases. In the first case, the correlation between the two regressors is 0. In the second case, it is 0.6. Using Monte Carlo simulation, let's check the sampling distribution of $\hat{\beta}_1$ in these two cases.

SLM, error is endogenous, OLS estimator is biased



SLM, error is endogenous, OLS estimator is inconsistent

The OLS estimator, $\hat{\boldsymbol{\beta}}$, is consistent when $\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] = 0$. We proved it as follows:

$$\operatorname{plim} \hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \underbrace{\operatorname{plim} \left[\left(\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i' \right)^{-1} \right]}_{\left(\mathrm{E}\left[\boldsymbol{x}_i \boldsymbol{x}_i'\right]\right)^{-1}} \underbrace{\operatorname{plim} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \varepsilon_i}_{\mathrm{E}\left[\boldsymbol{x}_i \varepsilon_i\right] = \boldsymbol{0}} .$$

Now, $\mathrm{E}\left[\boldsymbol{x}_i \varepsilon_i\right] \neq 0$. Therefore, the second term of the summation does not drop. Hence,

$$\operatorname{plim} \hat{\boldsymbol{\beta}} \neq \boldsymbol{\beta}.$$

SLM, error is endogenous, what to do?

When

$$\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] \neq 0$$

the OLS estimator is biased and inconsistent. We need a new estimator that has at least the desirable large sample properties. For example, a consistent but biased estimator is already better than the OLS estimator.

SLM, error is endogenous, what to do?

There are in fact different estimators that are consistent. The **IV** and LIML estimators estimate a single equation, and hence are called single-equation methods. The 3SLS, GMM, and FIML estimators jointly estimate an entire system of equations, and hence are called system of equations methods.

In this course we study the IV estimator.

IV Model

Consider the linear model

$$y_i = \boldsymbol{x}_i' \boldsymbol{\beta} + \varepsilon_i$$

where $\boldsymbol{x}_i$ is $K \times 1$. Suppose that

$$\mathrm{E}\left[\varepsilon_i \boldsymbol{x}_i\right] \neq 0.$$

IV Model, assumptions, linearity

A1.IV. Linearity. The model is linear in the parameters.

IV Model, assumptions

Suppose $\boldsymbol{z}_i$ is a $L \times 1$ vector of instrumental variables. $\boldsymbol{z}_i$ satisfies two main assumptions.

IV Model, assumptions, rank condition

A2.IV. Relevance. That is,

$$\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]$$

has full column rank. $\boldsymbol{z}_i$ is $L \times 1$. $\boldsymbol{x}_i'$ is $1 \times K$. $\boldsymbol{z}_i \boldsymbol{x}_i'$ is $L \times K$. Hence, the rank of $\boldsymbol{z}_i \boldsymbol{x}_i'$ should be K. Hence, the assumption imposes a rank condition. This condition implies that the variables in $\boldsymbol{z}_i$ are sufficiently linearly related to the variables in $\boldsymbol{x}_i$. What does a rank condition have to do with $\boldsymbol{z}_i$ being related to $\boldsymbol{x}_i$? We do not study this. But it says that $\boldsymbol{z}_i$ and $\boldsymbol{x}_i$ are correlated.

IV Model, assumptions, orthogonality condition

A3.IV. Exogeneity. ε_i is uncorrelated with each variable in $\boldsymbol{z}_i$.

$$\mathrm{E}\left[\boldsymbol{z}_i \varepsilon_i\right] = \mathbf{0}.$$

The assumption imposes an orthogonality condition. There are L such conditions since $\boldsymbol{z}_i$ is $L \times 1$. We do not study what this means. But it says that $\boldsymbol{z}_i$ and ε_i are uncorrelated.

IV Model, assumptions, spherical errors

A4.IV. Errors are homoskedastic and non-autocorrelated. That is,

$$\mathrm{Var}\left[\varepsilon_i \mid \boldsymbol{z}_i\right] = \sigma^2,\ \forall\, i.$$

and

$$\mathrm{Cov}\left[\varepsilon_i, \varepsilon_j \mid \boldsymbol{z}_i\right] = 0,\ \forall\, i \neq j.$$

In the lecture on GMM, we will relax this assumption.

IV Model, assumptions, random sampling

A5.IV. Random sampling. $(\boldsymbol{x}_i, \boldsymbol{z}_i, \varepsilon_i)$, $i = 1, \ldots, n$ are an i.i.d. sequence of random variables.

IV estimator

$\boldsymbol{z}_i$ is $L \times 1$. $\boldsymbol{x}_i$ is $K \times 1$. Suppose that $L = K$. Hence, there are as many instruments as there are endogenous variables. This leads to the $\hat{\boldsymbol{\beta}}_{IV}$ estimator.

IV estimator

$L = K$ has an implication for A2.IV. Since $L = K$, $\boldsymbol{z}_i \boldsymbol{x}_i'$ is a square matrix. This matrix has full rank as A2.IV requires. Square matrices with full rank are invertible. Hence, the inverse of $\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]$ exists. Or, the inverse of $\boldsymbol{Z}'\boldsymbol{X}$ exists. More on this later.

Skip.

IV estimator

$$
\begin{aligned}
y_i &= \boldsymbol{x}_i'\boldsymbol{\beta} + \varepsilon_i \\
\boldsymbol{z}_i y_i &= \boldsymbol{z}_i \boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{z}_i \varepsilon_i \\
\mathrm{E}\left[\boldsymbol{z}_i y_i\right] &= \mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\boldsymbol{\beta}\right] + \mathrm{E}\left[\boldsymbol{z}_i \varepsilon_i\right] \\
\mathrm{E}\left[\boldsymbol{z}_i y_i\right] &= \mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\boldsymbol{\beta} \\
\left(\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right)^{-1}\mathrm{E}\left[\boldsymbol{z}_i y_i\right] &= \left(\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right)^{-1}\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\boldsymbol{\beta} \\
\left(\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right)^{-1}\mathrm{E}\left[\boldsymbol{z}_i y_i\right] &= \boldsymbol{\beta}
\end{aligned}
$$

W used two assumptions. First, we used A3.IV so that $\mathrm{E}\left[\boldsymbol{z}_i \varepsilon\right] = 0$. Second, we used A2.IV so that the inverse of $\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]$ exists.

Skip.

IV estimator

$$\begin{aligned}
\boldsymbol{\beta} &= \left(\mathsf{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right)^{-1} \mathsf{E}\left[\boldsymbol{z}_i y_i\right] \\
&= \left(\operatorname{plim} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{z}_i \boldsymbol{x}_i'\right)^{-1} \operatorname{plim} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{z}_i y_i .
\end{aligned}$$

Expected value terms are unobserved. We can estimate them using sample data, which gives the IV estimator:

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{IV} &= \left(\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{z}_i \boldsymbol{x}_i'\right)^{-1} \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{z}_i y_i \\
&= \left(\sum_{i=1}^{n} \boldsymbol{z}_i \boldsymbol{x}_i'\right)^{-1} \sum_{i=1}^{n} \boldsymbol{z}_i y_i \\
&= \left(\boldsymbol{Z}'\boldsymbol{X}\right)^{-1} \boldsymbol{Z}'\boldsymbol{y} .
\end{aligned}$$

IV estimator, motivation

What motivates the estimator is that $\mathrm{E}[\boldsymbol{z}_i\varepsilon] = 0$ allows us to solve for $\boldsymbol{\beta}$. We obtain K equations in K unknowns in the expression for β. Otherwise we cannot solve for $\boldsymbol{\beta}$, and construct an estimator based on it. See Greene, page 267, for a full treatment of this motivation. We will discuss additional motivation later in this lecture.

Skip.

IV estimator, finite sample properties

$\hat{\boldsymbol{\beta}}_{IV}$

- can be biased if the instruments are only weakly correlated with the endogenous variable,
- can be biased if the instruments are correlated with the error,
- in small samples, it can exhibit bias even if the instruments are uncorrelated with the error. This bias diminishes as the sample size increases, making the IV estimator consistent in large samples.

Therefore, we rely on the large sample properties of $\hat{\boldsymbol{\beta}}_{IV}$.

IV estimator, large sample properties, consistency

$\hat{\boldsymbol{\beta}}_{IV}$ is consistent if A1, A2, A3, and A5 hold. We skip the proof.

IV estimator, large sample properties, asy. normality

$$\hat{\boldsymbol{\beta}}_{IV} \overset{a}{\sim} N\left[\boldsymbol{\beta}, \sigma^2 \frac{1}{n} \left(\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right)^{-1} \mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{z}_i'\right] \left(\mathrm{E}\left[\boldsymbol{x}_i \boldsymbol{z}_i'\right]\right)^{-1}\right].$$

We can estimate Asy. Var $\left[\hat{\boldsymbol{\beta}}_{IV}\right]$ with

$$\text{Est. Asy. Var}\left[\hat{\boldsymbol{\beta}}_{IV}\right] = \hat{\sigma}^2 \left(\boldsymbol{Z}'\boldsymbol{X}\right)^{-1} \boldsymbol{Z}'\boldsymbol{Z} \left(\boldsymbol{X}'\boldsymbol{Z}\right)^{-1},$$

where

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \boldsymbol{x}_i' \hat{\boldsymbol{\beta}}_{IV}\right)^2.$$

GIV estimator

$\boldsymbol{z}_i$ is $L \times 1$. $\boldsymbol{x}_i$ is $K \times 1$. Suppose that $L > K$. Hence, there are more instruments than there are endogenous variables. That is, we have more information than we need to proxy a given endogenous variable. Should we then just use an arbitrary selection of K instruments, and throw away the remaining $L - K$ instruments? No. Throwing away useful information leads to an inefficient estimator: $\hat{\boldsymbol{\beta}}_{IV}$. Linear combinations of the L instruments also satisfy the rank and exogeneity assumptions. This leads to an estimator at least as efficient as the $\hat{\boldsymbol{\beta}}_{IV}$ estimator: $\hat{\boldsymbol{\beta}}_{GIV}$.

GIV estimator

$L > K$ has an implication for A2.IV. Since $L > K$, $\boldsymbol{z}_i \boldsymbol{x}_i'$ is $L \times K$. It is not a square matrix. However, it has full column rank which is K as A2.IV requires. But the inverse of $\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]$ does not exist. Or, the inverse of $\boldsymbol{Z}'\boldsymbol{X}$ does not exist. More on this later.

Skip.

GIV estimator

Consider the linear model

$$y_i = \boldsymbol{x}'_i \boldsymbol{\beta} + \varepsilon_i.$$

We consider that $\boldsymbol{x}'_i$ contains two endogenous variables, instead of only one, to keep the derivation of the $\hat{\boldsymbol{\beta}}_{GIV}$ estimator general.

GIV estimator

The $\hat{\boldsymbol{\beta}}_{GIV}$ estimator is derived, and used, in two stages.

GIV estimator, stage one

For each endogenous regressor, estimate by OLS

$$x_{ik} = \boldsymbol{z}_i' \boldsymbol{\pi}_k + \upsilon_{ik}.$$

$\boldsymbol{z}_i'$ contains the instruments. $1 \times L$. $\boldsymbol{\pi}_k$ contains the parameters for $\boldsymbol{z}_i'$. $L \times 1$. Obtaining the prediction $\hat{x}_{ik}$, and generalising to n observations,

$$\hat{\boldsymbol{x}}_k = \boldsymbol{Z} \underbrace{\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1} \boldsymbol{Z}'\boldsymbol{x}_k}_{\hat{\boldsymbol{\pi}}_k}.$$

$\hat{\boldsymbol{x}}_k$ contains n predictions. $n \times 1$. $\boldsymbol{Z}$ contains L instruments, each with n observations. $n \times L$. $\hat{\boldsymbol{\pi}}_k$ contains L parameter estimates, for variable k. $L \times 1$. Generalising to K endogenous variables,

$$\hat{\boldsymbol{X}} = \boldsymbol{Z} \underbrace{\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1} \boldsymbol{Z}'\boldsymbol{X}}_{\hat{\boldsymbol{\pi}}}.$$

$\hat{\boldsymbol{\pi}}$ contains L parameter estimates, for K endogenous variables. $L \times K$. $\hat{\boldsymbol{X}}$ is $n \times K$.

GIV estimator, stage two

Using the predictions as regressors, estimate by OLS the single equation

$$y_i = \hat{\mathbf{x}}'_i \boldsymbol{\beta} + \varepsilon^*_i$$

where

$$\varepsilon^*_i = \hat{\upsilon}'_i \boldsymbol{\beta} + \varepsilon_i.$$

$\hat{\mathbf{x}}'_i$ is the vector of predicted endogenous variables, for individual i. It is $1 \times K$. Generalising to n observations, the OLS estimator of this model is

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \left(\hat{\mathbf{X}}'\hat{\mathbf{X}}\right)^{-1} \hat{\mathbf{X}}'\mathbf{y} \\ &\equiv \hat{\boldsymbol{\beta}}_{GIV}. \end{aligned}$$

GIV estimator

The estimator is obtained in two stages. Therefore textbooks often call it the two-stage least squares estimator denoted as TSLS.

GIV estimator

How we end up with

$$\varepsilon_i^* = \hat{\upsilon}_i' \boldsymbol{\beta} + \varepsilon_i.$$

Considering that there is only one endogenous variable,

$$x_i = z_i \pi + \upsilon_i.$$

Then,

$$x_i = \hat{x}_i + \hat{\upsilon}_i.$$

Replacing x_i in

$$y_i = x_i \beta + \varepsilon_i,$$

we have

$$y_i = \hat{x}_i \beta + \hat{\upsilon}_i \beta + \varepsilon_i$$

and

$$\varepsilon_i^* \equiv \hat{\upsilon}_i \beta + \varepsilon_i.$$

GIV estimator, small sample properties

$\hat{\boldsymbol{\beta}}_{GIV}$ is biased in a finite sample, like the $\hat{\boldsymbol{\beta}}_{IV}$. Therefore, we rely on the asymptotic properties of the estimator.

GIV estimator, large sample properties, consistency

$\hat{\boldsymbol{\beta}}_{GIV}$ is consistent. The proof is very similar to that of the $\hat{\boldsymbol{\beta}}_{IV}$.

GIV estimator, large sample properties, asy. efficiency

Asymptotic variance of $\hat{\boldsymbol{\beta}}_{GIV}$ is equal to or smaller than that of $\hat{\boldsymbol{\beta}}_{IV}$. That is, $\hat{\boldsymbol{\beta}}_{GIV}$ is at least as efficient as the $\hat{\boldsymbol{\beta}}_{IV}$. We do not prove this.

GIV estimator, large sample properties, asy. normality

Derivation of the asymptotic normality of $\hat{\boldsymbol{\beta}}_{GIV}$ is very similar to that of $\hat{\boldsymbol{\beta}}_{IV}$.

GIV estimator, large sample properties, asy. normality

$$\hat{\boldsymbol{\beta}}_{GIV} \overset{a}{\sim} N\left[\boldsymbol{\beta}, \sigma^2 \frac{1}{n} \left[\mathrm{E}\left[\boldsymbol{x}_i \boldsymbol{z}_i'\right] \left(\mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{z}_i'\right]\right)^{-1} \mathrm{E}\left[\boldsymbol{z}_i \boldsymbol{x}_i'\right]\right]^{-1}\right].$$

We can estimate Asy. Var $\left[\hat{\boldsymbol{\beta}}_{GIV}\right]$ with

$$\text{Est. Asy. Var}\left[\hat{\boldsymbol{\beta}}_{GIV}\right] = \hat{\sigma}^2 \left[\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1}.$$

GIV estimator, note one

Est. Asy. Var $\left[\hat{\beta}_{GIV}\right]$ takes an alternative form. Using $\hat{\boldsymbol{X}} = \boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}$,

$$\begin{aligned}
\text{Est. Asy. Var}\left[\hat{\beta}_{GIV}\right] &= \hat{\sigma}^2\left[\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1} \\
&= \hat{\sigma}^2\left[\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1} \\
&= \hat{\sigma}^2\left[\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\right)'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1} \\
&= \hat{\sigma}^2\left[\left(\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right)'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right]^{-1} \\
&= \hat{\sigma}^2\left[\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}}\right]^{-1}
\end{aligned}$$

Does this look familiar?

GIV estimator, note two

$\boldsymbol{z}_i$ is the $L \times 1$ vector of instruments. $\boldsymbol{x}_i$ is the $K \times 1$ vector of regressors.

GIV estimator, note two

Suppose $L = K$. The number of instruments is equal to the number of endogenous variables. $\boldsymbol{Z}'\boldsymbol{X}$ is a $K \times K$ square matrix. It has full rank. Square matrices are nonsingular and invertible if they have full rank. Hence, $\boldsymbol{Z}'\boldsymbol{X}$ is invertible.

GIV estimator, note two

Using $\hat{\boldsymbol{X}} = \boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}$,

$$
\begin{aligned}
\hat{\beta}_{GIV} &= \left(\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}}\right)^{-1}\hat{\boldsymbol{X}}'\boldsymbol{y} \\
&= \left(\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{y} \\
&= \left(\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{y} \\
&= \left(\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\left(\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\right)^{-1}\boldsymbol{X}'\boldsymbol{Z}\left(\boldsymbol{Z}'\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}'\boldsymbol{y} \\
&= \left(\boldsymbol{Z}'\boldsymbol{X}\right)^{-1}\boldsymbol{Z}'\boldsymbol{y} \\
&\equiv \hat{\beta}_{IV}.
\end{aligned}
$$

GIV estimator, note two

Suppose $L > K$. The number of instruments is larger than the number of endogenous variables. $\boldsymbol{Z}'\boldsymbol{X}$ is $L \times K$ with rank $K < L$. $\boldsymbol{Z}'\boldsymbol{X}$ is not invertible. Then, $\hat{\boldsymbol{\beta}}_{GIV} \neq \hat{\boldsymbol{\beta}}_{IV}$.

GIV estimator, example

```
. reg lwage educ age age2 black

      Source |       SS           df       MS      Number of obs   =     2,220
-------------+----------------------------------   F(4, 2215)      =    143.09
       Model |  88.0908302         4  22.0227076   Prob > F        =    0.0000
    Residual |  340.908673     2,215  .153909108   R-squared       =    0.2053
-------------+----------------------------------   Adj R-squared   =    0.2039
       Total |  428.999503     2,219  .193330105   Root MSE        =    .39231

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |   .0385118   .0032895    11.71   0.000      .032061    .0449627
         age |   .1326507   .0555628     2.39   0.017     .0236901    .2416113
        age2 |  -.0015523   .0009674    -1.60   0.109    -.0034494    .0003448
       black |  -.2127221   .0232691    -9.14   0.000    -.2583537   -.1670906
       _cons |   3.315457   .7883061     4.21   0.000     1.769561    4.861354
------------------------------------------------------------------------------
```

GIV estimator, example

```
. ivregress 2sls lwage (educ = motheduc fatheduc) age age2 black, first

First-stage regressions
-----------------------

                                                Number of obs     =      2,220
                                                F(   5,    2214)  =     157.81
                                                Prob > F          =     0.0000
                                                R-squared         =     0.2628
                                                Adj R-squared     =     0.2611
                                                Root MSE          =     2.2244
```

educ	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
age	.9804534	.314502	3.12	0.002	.3637036	1.597203
age2	-.0160649	.0054764	-2.93	0.003	-.0268043	-.0053256
black	-.1607076	.1376706	-1.17	0.243	-.4306846	.1092694
motheduc	.1975247	.0201066	9.82	0.000	.1580948	.2369545
fatheduc	.2230658	.0167964	13.28	0.000	.1901275	.2560042
_cons	-5.389924	4.472077	-1.21	0.228	-14.15983	3.379979

GIV estimator, example

```
Instrumental variables (2SLS) regression          Number of obs   =      2,220
                                                  Wald chi2(4)    =     503.26
                                                  Prob > chi2     =     0.0000
                                                  R-squared       =     0.1900
                                                  Root MSE        =     .39564
```

lwage	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
educ	.0600324	.0069201	8.68	0.000	.0464692	.0735955
age	.1094726	.0564143	1.94	0.052	-.0010974	.2200426
age2	-.0011585	.0009819	-1.18	0.238	-.003083	.0007659
black	-.1833938	.0248831	-7.37	0.000	-.2321638	-.1346237
_cons	3.354017	.7950635	4.22	0.000	1.795721	4.912313

```
Instrumented:  educ
Instruments:   age age2 black motheduc fatheduc
```

Why the standard normal (z) and not the t distribution (t) is used?