Hypothesis testing in finite and large samples

Econometrics for minor Finance, Lecture 5

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Hypothesis testing in finite and large samples

Consider the population model

$$y_i = \beta x_i + u_i$$

So far we have been interested in estimating the parameter

$$\beta$$

the true effect in the population.

Hypothesis testing in finite and large samples

We studied that we can estimate the population parameter

$$\beta$$

by the OLS estimator

$$\hat{\beta}$$

This estimator is itself a random variable: it has a sampling distribution with a mean and a variance. Thus, the particular value of

$$\hat{\beta}$$

obtained from our sample reflects the randomness of the sampling process.

Hypothesis testing in finite and large samples

Hypothesis testing is about the true parameter

$$\beta$$

in particular, about testing whether it is equal to a certain hypothesized value

$$\beta^0$$

We do not observe the true parameter but we can use its estimate

$$\hat{\beta}$$

to conduct the test. But because this estimate reflects randomness of the sampling distribution process, we have to take account for this uncertainty while conducting the test.

Hypothesis testing in finite and large samples

Hypothesize that the true value

$$\beta$$

is equal to the particular value

$$\beta^0$$

Then, the null hypothesis is

$$H_0 : \beta = \beta^0$$

and, say, the alternative is

$$H_1 : \beta \neq \beta^0$$

The alternative could also be that true value is larger or smaller.

Hypothesis testing in finite and large samples

That is, we want to check whether

$$\beta = \beta^0$$

But we do not observe β. So we cannot make this check. But we can estimate it using the OLS method. Then we can check whether

$$\hat{\beta} = \beta^0$$

is true.

Suppose that this is true. Then, our test is complete, and we can conclude that

$$H_0$$

is true. But this conclusion has a problem.

Hypothesis testing in finite and large samples

Until we estimate it, $\hat{\beta}$ is a random variable. Hence, there is a probability associated with the condition

$$\hat{\beta} = \beta^0$$

Therefore, we need to check the equality in a statistical sense. Therefore, we need a bit of a different form of

$$\hat{\beta} = \beta^0$$

That form is a test statistic. But since the form still depends on the random $\hat{\beta}$, the test statistic is a random variable.

Hypothesis testing in finite and large samples

The test statistic is a random variable, so it has a distribution. Which distribution? The statistic we will derive is a function of

$$\hat{\beta}$$

and if we take

$$x$$

as given, that is a function of

$$u$$

Hence, the distribution of u will determine the distribution of the statistic.

Hypothesis testing in finite and large samples

If we assume that

$$u$$

is normal, the test statistic has an exact distribution. An exact distribution means that the distribution is valid in any finite sample size

$$n$$

For example, if u is normal, the t statistic has a t distribution.

Hypothesis testing in finite and large samples

If we do not assume that

$$u$$

is normal, the test statistic does not have an exact distribution. However, if

$$n$$

is large, then the test statistic has an asymptotic distribution that approximates an exact distribution. For example, the t statistic approximates the normal distribution.

Hypothesis testing in finite and large samples

Therefore we need to make a distinction between a finite and large sample.

Hypothesis testing in finite samples: Single restriction

The condition

$$\hat{\beta} = \beta^0$$

represents a single restriction. We could also wish to test conditions for multiple slope coefficients. We will study a multiple restrictions test later.

Hypothesis testing in finite samples: Single restriction

We know from an earlier lecture that if

$$u \mid x \sim N\left(0, \sigma^2\right)$$

then

$$\hat{\beta} \mid x \sim N\left[\beta,\ \frac{\sigma^2}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

Hypothesis testing in finite samples: Single restriction

$$\hat{\beta} \mid x \sim N\left[\beta,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

means

$$\hat{\beta} - \beta \mid X \sim N\left[0,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

Hypothesis testing in finite samples: Single restriction

We were interested in testing the null hypothesis

$$H_0 : \beta = \beta^0$$

against the alternative

$$H_1 : \beta \neq \beta^0$$

Hypothesis testing in finite samples: Single restriction

We have

$$\hat{\beta} - \beta \mid X \sim N\left[0,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

Under the null hypothesis, we have

$$\beta = \beta^0$$

Hence, under the null we have

$$\hat{\beta} - \beta^0 \mid X \sim N\left[0,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

This implies that $\mathrm{E}\left[\hat{\beta} \;\middle|\; x\right] = \beta^0$.

Hypothesis testing in finite samples: Single restriction

We have

$$\hat{\beta} - \beta^0 \mid X \sim N\left[0, \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

Standardize this variable to get

$$z \left| x := \frac{\hat{\beta} - \beta^0}{\sqrt{\frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \right| x \sim N[0, 1]$$

The distribution does not depend on $\hat{\beta}$, β^0, σ, or x. Hence,

$$z \sim N[0, 1]$$

This is a convenient simplification. We do not need to condition on x while using the test statistic.

Hypothesis testing in finite samples: Single restriction

$$z = \frac{\hat{\beta} - \beta^0}{\sqrt{\dfrac{\sigma^2}{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}}} \sim N[0, 1]$$

is not usable because σ is unknown. Replace σ with its unbiased estimator

$$\hat{\sigma} = \sqrt{\frac{\sum\limits_{i=1}^{n} \hat{u}_i^2}{n - K}}$$

where $\hat{u}_i$ is the residual for i.

Hypothesis testing in finite samples: Single restriction

We obtain

$$t = \frac{\hat{\beta} - \beta^0}{\sqrt{\frac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}}$$

But we just changed from z to t. Then how is t distributed?

Hypothesis testing in finite samples: Single restriction

By replacing σ in

$$z = \frac{\hat{\beta} - \beta^0}{\sqrt{\dfrac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \sim N[0, 1]$$

with $\hat{\sigma}$ in

$$t = \frac{\hat{\beta} - \beta^0}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \sim t[n - K]$$

we move from a standard normal distribution to a t distribution which has slightly thicker tails.

Hypothesis testing in finite samples: Single restriction

Note that

$$\text{SEE}\left[\hat{\beta}\right] = \sqrt{\frac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

Hypothesis testing in finite samples: Single restriction

$$t = \frac{\hat{\beta} - \beta^0}{\mathsf{SEE}\left[\hat{\beta}\right]}$$

What is the intuition of this test statistic?

Is the distance between $\hat{\beta}$ and β^0 sufficiently large, with the distance measured in terms of the sampling variance of $\hat{\beta}$? Is t sufficiently large? If it is, reject the null for the true β:

$$\beta = \beta^0$$

This is the decision rule of the test.

Hypothesis testing in finite samples: Single restriction

How large t should be depends on a threshold t value we want to set. We call this threshold value, the critical t value and denote it as

$$t^c_{\frac{\alpha}{2},n-K}$$

Hypothesis testing in finite samples: Single restriction

$$t^c_{\frac{\alpha}{2},n-K}$$

is a value from the t distribution and depends on

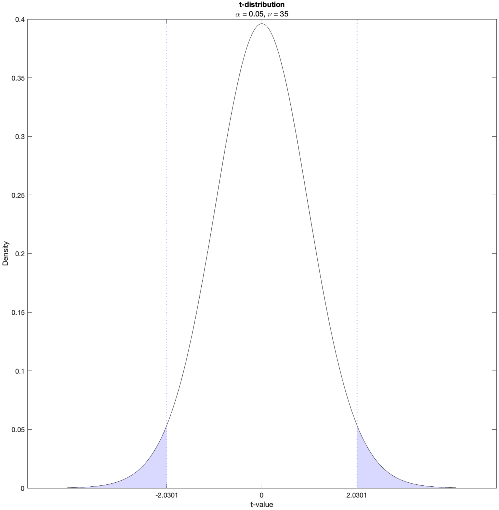
$\frac{\alpha}{2}$: area under the t distribution covering up to where we want t^c to rest. Hence, $\frac{\alpha}{2}$ determines the t^c. α is often taken as 5%.

$n-K$: degrees of freedom which determines the shape of the t distribution.

Hypothesis testing in finite samples: Single restriction

Let's demonstrate the t distribution for $\alpha = 0.05$ and $n - K = 35$.

Hypothesis testing in finite samples: Single restriction



Hypothesis testing in finite samples: Single restriction

$$t = \frac{\hat{\beta} - \beta^0}{\mathsf{SEE}\left[\hat{\beta}\right]}$$

One point left unclear is that the distance $\hat{\beta} - \beta^0$ can be positive or negative. Hence, t can be positive or negative.

Hypothesis testing in finite samples: Single restriction

If t is positive, we reject the null if

$$t > t^c_{1-\frac{\alpha}{2},n-K}$$

If t is negative, we reject the null if

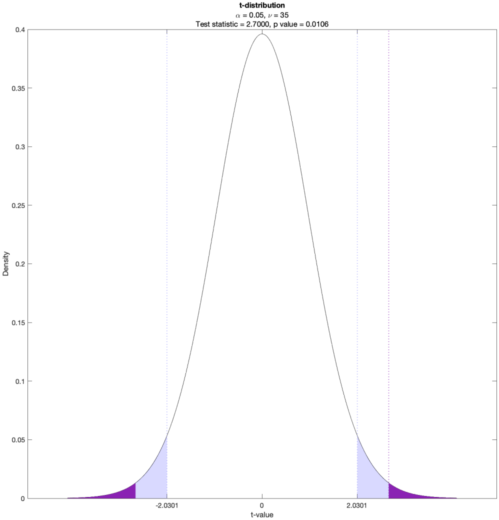
$$t < t^c_{\frac{\alpha}{2},n-K}$$

This is what is called a two-tailed t test.

Hypothesis testing in finite samples: Single restriction

Let's demonstrate for $\alpha = 0.05$, $n - K = 35$, and $t = 2.7$.

Hypothesis testing in finite samples: Single restriction



Hypothesis testing in finite samples: Single restriction: Example

Mincer (1974) considers the regression of the log of wage on experience, education, and IQ score. The data contains 935 observations. Estimation of this regression gives

$$\hat{\beta}_{IQ} = 0.0058$$

with

$$\text{SEE}\left[\hat{\beta}_{IQ} \,\middle|\, x\right] = 0.001$$

Hypothesis testing in finite samples: Single restriction: Example

Someone claims that each additional IQ point raises one's wage by 0.0075 on average. That is,

$$\beta_{IQ}^0 = 0.0075$$

We want to test this claim. The null and the alternative are

$$H_0 : \beta_{IQ} = \beta_{IQ}^0 = 0.0075$$
$$H_1 : \beta_{IQ} \neq \beta_{IQ}^0 = 0.0075$$

A two-tailed test.

Hypothesis testing in finite samples: Single restriction: Example

We need to calculate the

$$t$$

and

$$t^c$$

and compare the former to the latter to decide on the result of the test.

Hypothesis testing in finite samples: Single restriction: Example

t is calculated as

$$t = \frac{0.0058 - 0.0075}{0.001} = -1.75$$

Hypothesis testing in finite samples: Single restriction: Example

t^c is calculated as follows. Consider a significance level of 0.05. Then, for this two-tailed test

$$\frac{\alpha}{2} = 0.025$$

The degrees of freedom is

$$935 - 4 = 931$$

Then,

$$t^c_{0.025,931} = -1.9625$$

using the tabulated t distribution at the back of your textbook.

Hypothesis testing in finite samples: Single restriction: Example

Since

$$t > t^c$$

that is, since

$$-1.7500 > -1.9625$$

we fail to reject the null hypothesis.

Hypothesis testing in finite samples: Single restriction: Example

We can also compare

$$p,$$

which is the p value corresponding to the t value, to

$$p^c,$$

which is the critical p value corresponding to t^c, the critical t value.

p^c is what we call the significance level.

Hypothesis testing in finite samples: Single restriction: Example

p is calculated, for this two-tailed test, as

$$p = 2 * p_{-1.75,931} = 0.0805$$

using standard statistical software. The tabulated t distribution at the back of your textbook will not present this exact number because tabulations cannot be too detailed since there is no space to present them.

Hypothesis testing in finite samples: Single restriction: Example

p^c is calculated as

$$p^c = 2 * p_{-1.9625,931} = 0.05$$

using standard statistical software, or the tabulated t distribution at the back of your textbook will present this number because 0.05 is a conventional critical level.

Hypothesis testing in finite samples: Single restriction: Example

Since

$$p > p^c,$$

that is, since

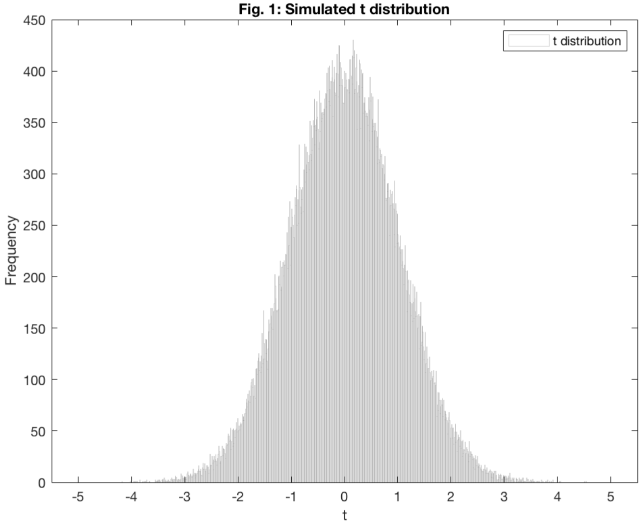
$$0.0805 > 0.0500,$$

we fail to reject the null hypothesis.

Hypothesis testing in finite samples: Single restriction: Example

O'Hara (2018) proposes that econometrics instructors move away from using the tabulated distribution of the test statistic at the back of the textbooks when teaching hypothesis testing. Instead, he proposes that instructors teach students to test hypotheses by using the simulated distribution of the test statistic which can be created using random number generators in statistical software. This provides students with a visual and intuitive understanding of the sampling distribution and the logic behind hypothesis testing. In the next few slides we will follow what O'Hara proposes.

Hypothesis testing in finite samples: Single restriction: Example

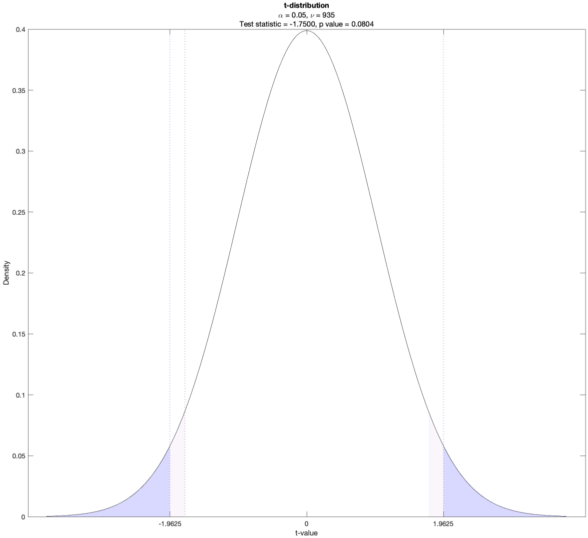


Hypothesis testing in finite samples: Single restriction: Example

In this plot, a probability area, p or p^c, represents the fraction that the t values occur up to some t value, t or t^c, in all t values in the distribution.

Now that we understand what we are doing, we can replace the simulated frequency distribution with the continuous PDF of the t statistic.

Hypothesis testing in finite samples: Single restriction: Example



Hypothesis testing in large samples

If

$$u \mid x \sim N\left(0, \sigma^2\right)$$

the exact sampling distribution of $\hat{\beta}$, conditional on x, is

$$\hat{\beta} \mid x \sim N\left[\beta,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

When $\hat{\beta}$ is normal, the t statistic has the exact t distribution, as shown above.

Hypothesis testing in large samples

If

$$u \mid x \sim N\left(0, \sigma^2\right)$$

does not hold, the t statistic does not have the exact t distribution in finite n.

What happens then?

Hypothesis testing in large samples: Single restriction

$$
\begin{aligned}
t &= \frac{\hat{\beta} - \beta^0}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \frac{\sqrt{n}}{\sqrt{n}} \\
&= \frac{\sqrt{n}\left(\hat{\beta} - \beta^0\right)}{\sqrt{\dfrac{\hat{\sigma}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}}}
\end{aligned}
$$

Hypothesis testing in large samples: Single restriction

Consider the numerator of

$$t = \frac{\sqrt{n}\left(\hat{\beta} - \beta^0\right)}{\sqrt{\dfrac{\hat{\sigma}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}}}$$

The derivation of the asymptotic normality of $\hat{\beta}$ showed that

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{d} N\left[0, \frac{\sigma^2}{\mathrm{E}\left[(x_i - \mu_x)^2\right]}\right]$$

In this derivation we did not assume that u is normal. The normal distribution is due to the CLT. Considering the scalar case, and that under the null $\beta = \beta^0$,

$$\sqrt{n}\left(\hat{\beta} - \beta^0\right) \xrightarrow{d} N\left[0,\ \sigma^2 \frac{1}{\mathrm{E}[(x_i - \mu_x)^2]}\right]$$

Hypothesis testing in large samples: Single restriction

Consider the denominator of

$$t = \frac{\sqrt{n}\left(\hat{\beta} - \beta^0\right)}{\sqrt{\frac{\hat{\sigma}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}}}$$

It can be shown that

$$\sqrt{\frac{\hat{\sigma}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}} \xrightarrow{d} \sqrt{\frac{\sigma^2}{\mathrm{E}\left[(x_i - \mu_x)^2\right]}}$$

Hypothesis testing in large samples: Single restriction

Using the ratio rule of limiting distributions,

$$t = \frac{\sqrt{n}\left(\hat{\beta} - \beta^0\right)}{\sqrt{\dfrac{\hat{\sigma}^2}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \xrightarrow{d} N(0,1)$$

Hypothesis testing in large samples: single restriction

Dropping the two instances of $\sqrt{n}$, we have

$$t = \frac{\hat{\beta} - \beta^0}{\sqrt{\dfrac{\hat{\sigma}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \xrightarrow{d} N(0,1)$$

So

$$t_k \overset{a}{\sim} N\left[0,1\right]$$

This shows that the t statistic approximately has a standard normal distribution in finite but large samples. Hence, if n is large, we can compare the t statistic with the critical values from a standard normal distribution. We do not need to assume that u is normal.