Math refresher A: Basic mathematical tools

Econometrics for minor Finance, Lecture 1

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Basic math tools: The summation operator and descriptive statistics: Properties

The summation operator is a shorthand to handle expressions involving sums of numbers, and is fundamental in statistics and econometrics.

If

$$\{x_i : i = 1, \ldots, n\}$$

denotes a sequence of n numbers, then we write the sum of these numbers as

$$\sum_{i}^{n} x_i := x_1 + x_2 + \cdots + x_n$$

Basic math tools: The summation operator and descriptive statistics: Properties

sum. ope. property 1. For any constant c,

$$\sum_{i=1}^{n} c = nc$$

Since c remains constant and does not depend on the index i, it is not written as c_i.

Basic math tools: The summation operator and descriptive statistics: Properties

sum. ope. property 2. For any constant c,

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

This shows that, since c is constant, it can be factored out of the summation. This reflects the distributive property of multiplication over addition.

Basic math tools: The summation operator and descriptive statistics: Properties

sum. ope. property 3. If

$$\{(x_i, y_i) : i = 1, 2, \ldots, n\}$$

is a set of n pairs of numbers, and a and b are constants, then

$$\sum_{i=1}^{n}(ax_i + by_i) = \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} by_i$$

This shows that summation distributes over addition and scalar multiplication, allowing each term to be summed independently.

Basic math tools: The summation operator and descriptive statistics: Example

In statistics, X typically denotes a random variable representing a quantity of interest. The lowercase x is used to denote realizations or observed values of X obtained from data.

Basic math tools: The summation operator and descriptive statistics: Example

Let $\{X_i : i = 1, \ldots, n\}$ be a sequence of independent and identically distributed random variables representing years of education. Suppose we observe their realizations $\{x_i : i = 1, \ldots, n\}$. We compute their average by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

When the x_i are a sample of data on a particular variable, we often call this the sample average to emphasize that it is computed from a particular set of data. The sample average is an example of a descriptive statistic; in this case, the statistic describes the central tendency of the observed values x_i.

Basic math tools: Properties of linear functions

Linear functions play an important role in econometrics because they are simple to interpret and manipulate. If x and y are two variables related by

$$y = \beta_0 + \beta_1 x$$

then we say that y is a linear function of x, and β_0 and β_1 are two parameters describing this relationship. The intercept is β_0, and the slope is β_1.

Basic math tools: Properties of linear functions

The defining feature of a linear function is that the change in y is always equal to a fixed multiple of the change in x:

$$\Delta y = \beta_1 \Delta x$$

Here, Δ denotes a discrete change, and β_1 represents the rate of change or slope, the amount by which y changes for each unit increase in x. In other words, the effect of x on y is constant and equal to β_1.

Basic math tools: Properties of linear functions: Example

Suppose that the relationship between monthly housing expenditure and monthly income is

$$\text{expenditure} = 164 + 0.27 \cdot \text{income}$$

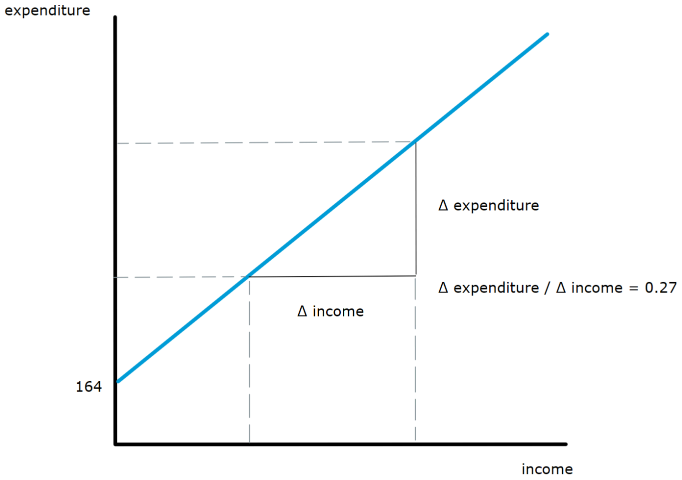
This shows that for each additional dollar of income,

$$\Delta\text{expenditure} = 0.27 \cdot \Delta\text{income} = 0.27 \cdot \$1$$

cents is spent on housing. If family income increases by \$200, then housing expenditure increases by

$$0.27 \cdot \$200 = \$54$$

Basic math tools: Properties of linear functions: Example



Basic math tools: Properties of linear functions: Example

A side note. According to the equation, a family with no income spends $164 on housing, which cannot be true. For low levels of income, this linear function would not describe the relationship well. We eventually need to use other types of functions to describe the relationship.

Basic math tools: Properties of linear functions

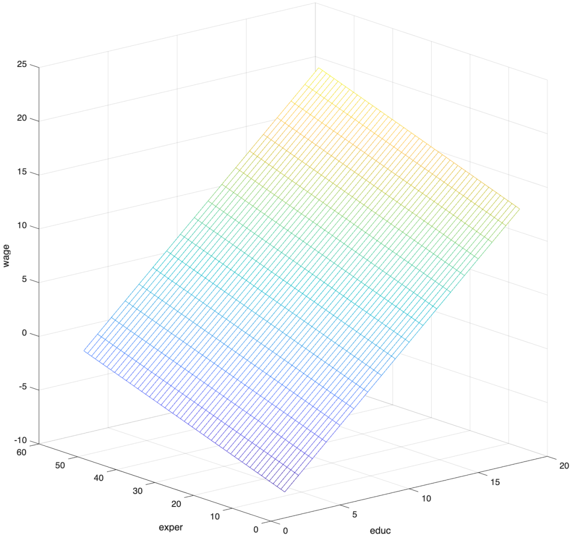
Linear functions are easily defined for more than two variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

β_0 is still the intercept, the value of y when $x_1 = 0$ and $x_2 = 0$. β_1 and β_2 measure particular slopes.

It is slightly more difficult to imagine this function because its graph is three-dimensional. But let's give it a try.

Basic math tools: Properties of linear functions: Example



Basic math tools: Properties of linear functions

From this, the change in y, for given changes in x_1 and x_2, is

$$\Delta y = \beta_1 \Delta x_1 + \beta_2 \Delta x_2$$

If x_2 does not change, that is, $\Delta x_2 = 0$, then we have

$$\Delta y = \beta_1 \Delta x_1$$

β_1 captures how y responds to changes in x_1 while holding x_2 constant. That is,

$$\beta_1 = \frac{\Delta y}{\Delta x_1}$$

This is why β_1 is called the partial effect of x_1 on y: it measures the effect of changing x_1 while keeping all other variables fixed. This idea is closely linked to the principle of ceteris paribus, which means "all else equal". Similarly, if $\Delta x_1 = 0$, then β_2 is the partial effect of x_2 on y.

Basic math tools: Proportions and percentages

In econometrics, we are often interested in measuring the changes in various quantities. Let x denote some variable, such as an individual's income, or profits of a firm. Let x_0 and x_1 denote two values for x:

- x_0 : initial value, and
- x_1 : subsequent value

For example, x_0 could be the annual income of an individual in 1994, and x_1 the income of the same individual in 1995. The proportionate change in x in moving from x_0 to x_1, is

$$\frac{x_1 - x_0}{x_0}$$

This is sometimes called the relative change.

Basic math tools: Proportions and percentages

It is common to state changes in terms of percentages. To do this, multiply and divide the proportionate change by 100:

$$\frac{x_1 - x_0}{x_0} \cdot 100 \cdot \frac{1}{100}$$

This is always denoted as:

$$\frac{x_1 - x_0}{x_0} \cdot 100\% \ \ \overset{!}{:=} \ \ \%\Delta x$$

Here, % means "per hundred", or in Latin, per centum. This is the percentage change in x when moving from x_0 to x_1.

Basic math tools: Proportions and percentages: Example

When income goes from \$30,000 to \$33,750, the proportionate change is

$$\frac{33{,}750 - 30{,}000}{30{,}000} = 0.125$$

The percentage change is

$$0.125 \cdot 100 \cdot \frac{1}{100} = 12.5 \cdot \frac{1}{100} = 12.5\%$$

which says that income has increased by 12.5 percent.

Basic math tools: Differential calculus

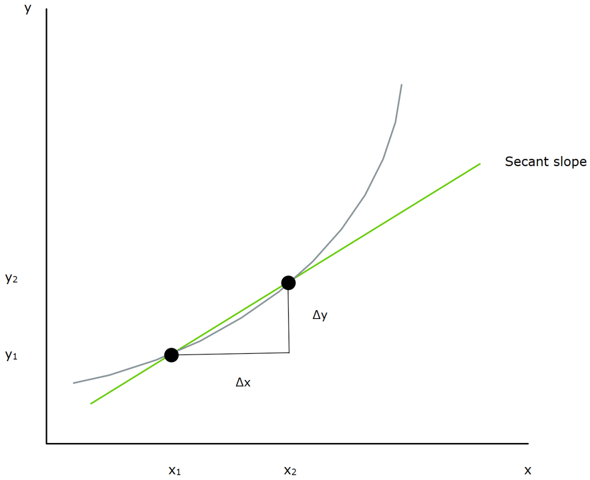
In differential calculus, the secant slope refers to the

- change in the value of a nonlinear function for a
- discrete change in its input variable:

$$\frac{y_2 - y_1}{x_2 - x_1} := \frac{\Delta y}{\Delta x}$$

where $y_2 := f(x_2)$ and same for y_1. This expression represents the slope of the secant line connecting two points on the graph of the function. It captures the average rate of change, as the ratio quantifies how much the output changes per unit change in the input.

Basic math tools: Differential calculus



Basic math tools: Differential calculus

Tangent slope refers to the

- change in the value of a nonlinear function for a
- marginal change in its variable:

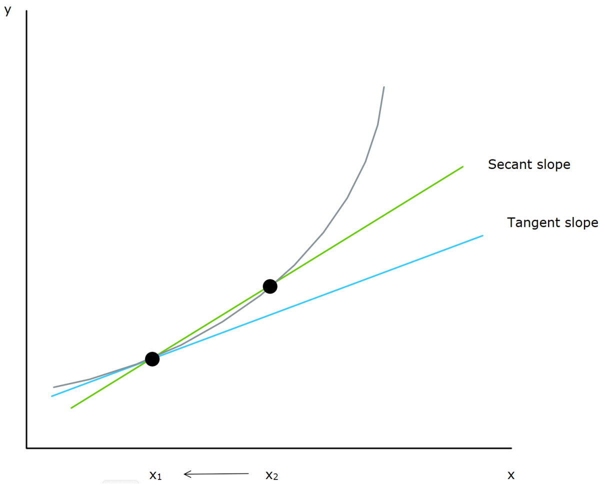
$$\frac{y_2 - y_1}{x_2 - x_1} := \frac{\delta y}{\delta x}$$

as

$$x_2 \text{ approaches } x_1$$

Geometrically, this gives the slope of the tangent line that touches the curve exactly at x_1, where the two values of the variable merge. It leads to the idea of a derivative, which measures how quickly a function changes at a single point.

Basic math tools: Differential calculus



Basic math tools: Differential calculus

When the change in x is small, the tangent slope provides a good estimate of the secant slope:

$$\frac{\delta y}{\delta x} \approx \frac{\Delta y}{\Delta x}$$

as

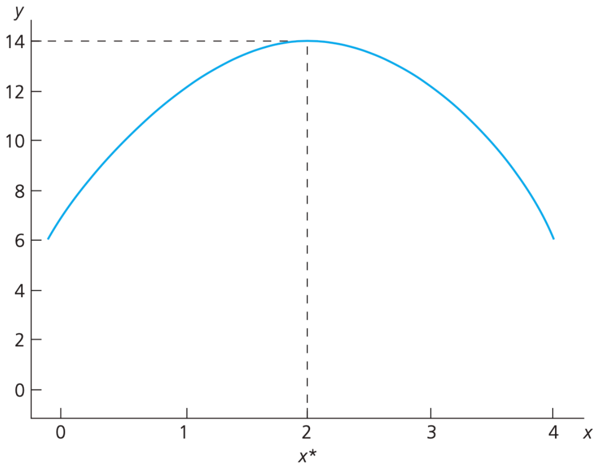
$$x_2 \text{ approaches } x_1$$

Basic math tools: Differential calculus

While the secant slope measures the average rate of change between two points, we are often more interested in the tangent slope, which reflects the instantaneous rate of change at a specific point. In many fields like economics or physics, we care about marginal changes, how a function behaves locally. The tangent slope shows how output reacts to small input shifts and reveals trends and curvature.

Basic math tools: Some special functions and their properties: Quadratic function

Graph of $y = 6 + 8x - 2x^2$.



Basic math tools: Some special functions and their properties: Quadratic function

Quadratic functions are useful in economic modeling because they naturally represent diminishing returns, such as decreasing marginal utility.

Basic math tools: Some special functions and their properties: Quadratic function

Consider the functional form

$$wage = \beta_0 + \beta_1 age + \beta_2 age^2$$

which is quadratic in age.

Basic math tools: Some special functions and their properties: Quadratic function

If age changes marginally, the change in wage is given approximately by:

$$\frac{\partial wage}{\partial age} = \beta_1 + 2 \cdot \beta_2 \cdot age$$

This is the derivative of the wage function with respect to age. If age changes by a discrete amount, the change in wage is exactly:

$$\Delta \mathrm{wage}_i = \left[\beta_0 + \beta_1(age + \Delta age) + \beta_2(age + \Delta age)^2\right] - \left[\beta_0 + \beta_1 age + \beta_2 age^2\right]$$

The derivative provides a good approximation for small changes in age, but becomes less accurate for larger changes, where the exact expression should be used instead.

Basic math tools: Some special functions and their properties: Quadratic function

In either case, the expression shows that the effect of age, that is, the slope of the relationship between *wage* and *age*, depends on the value of *age*.

Basic math tools: Some special functions and their properties: Quadratic function

By taking the derivative of the *wage* function with respect to *age* and setting it equal to zero, we can identify the critical point where the function may attain a maximum or minimum. That is,

$$\frac{\partial wage}{\partial age} = \beta_1 + 2 \cdot \beta_2 \cdot age = 0$$

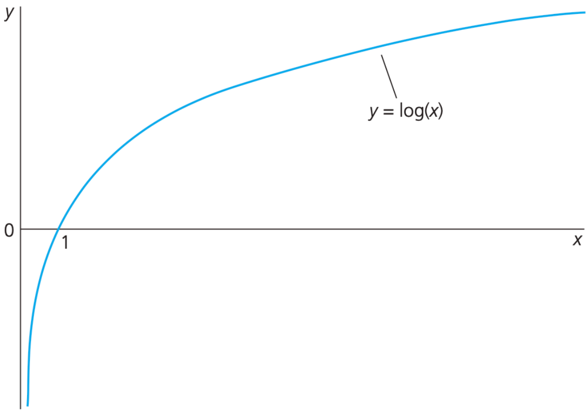
Solving for *age* gives:

$$age_{i,max} = -\frac{\beta_1}{2\beta_2}$$

Thus, the age at which wage reaches its maximum (or minimum) is determined by this critical point. If β_2 is negative, the point corresponds to a maximum; if positive, it corresponds to a minimum.

Basic math tools: Some special functions and their properties: Logarithmic function

Graph of $y = \log(x)$.



Basic math tools: Some special functions and their properties: Logarithmic function: Proportionate change

Consider the following logarithmic change:

$$\ln(41) - \ln(40) \approx 0.024$$

The proportionate change that we frequently calculate is

$$\frac{41-40}{40} = 0.025$$

The two quantities are very close. This shows that, for small changes, the logarithmic change is a good approximation of the proportionate change. We will make use of this result.

Basic math tools: Some special functions and their properties: Logarithmic function: Proportionate change

Consider the logarithmic change

$$\ln(60) - \ln(40) \approx 0.405.$$

The proportionate change is

$$\frac{60 - 40}{40} = 0.500$$

The two quantities are not close. This shows that, for large changes, the approximation is not accurate.

Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

Consider the log-linear function

$$\ln(y) = \beta x$$

We are interested in the change in $\ln(y)$ when we change x by some unit.

Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

$$\ln(y) = \beta x$$

Consider first a small change in x, which is considering the derivate:

$$\frac{\partial \ln(y)}{\partial x} = \beta$$

This change refers to the tangent slope.

We know that for a small change in x, the change in $\ln(y)$ closely approximates the proportionate change in y. For a large change in x, the linear approximation becomes less accurate. Then, what to do?

Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

In this case, we consider the exact proportionate change in y. This is derived from the change in $\ln(y)$:

$$\begin{aligned}
\Delta \ln(y) = \ln(y_1) - \ln(y_0) &= \beta \Delta x \\
e^{\ln(y_1) - \ln(y_0)} &= e^{\beta \Delta x} \\
e^{\ln\left(\frac{y_1}{y_0}\right)} &= e^{\beta \Delta x} \\
\frac{y_1}{y_0} &= e^{\beta \Delta x} \\
\frac{y_1 - y_0}{y_0} &= e^{\beta \Delta x} - 1
\end{aligned}$$

This exact proportionate change in y resulting from a change in x corresponds to the secant slope.

Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Implications of the logarithmic transformation for applied work are the following. Consider the log-linear model

$$\ln(y) = \beta x$$

Consider the change

$$\Delta \ln(y) = \beta \Delta x$$

For a small change in y, we can use the proportionate change approximation, which we know is good:

$$\Delta \ln(y) \approx \frac{y_1 - y_0}{y_0}$$

Replace to obtain

$$\frac{y_1 - y_0}{y_0} \approx \beta \Delta x$$

The LHS gives the proportionate change interpretation.

Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Multiply and divide by 100

$$\frac{y_1 - y_0}{y_0} \cdot 100 \cdot \frac{1}{100} \approx \beta \Delta x$$

to obtain

$$\frac{\% \Delta y}{\Delta x} \approx \beta$$

This gives the percentage change interpretation:

- If $\Delta x = 1$, then $\% \Delta y = \beta$ approx

Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Consider the linear-log model:

$$y = \beta \ln(x)$$

Consider the change

$$\Delta y = \beta \cdot \Delta \ln(x)$$

For a small change in x, we can use the proportionate change approximation, which we know is good:

$$\Delta \ln(x) \approx \frac{x_1 - x_0}{x_0}$$

Replace to obtain

$$\Delta y \approx \beta \cdot \frac{x_1 - x_0}{x_0}$$

Basic math tools: Some special functions and their properties: Logarithmic function: Implications

To express this proportionate change as a percentage change, multiply and divide by 100

$$\Delta y \approx \beta \cdot \frac{x_1 - x_0}{x_0} \cdot 100 \cdot \frac{1}{100}$$

to obtain

$$\Delta y \approx \beta \cdot \%\Delta x$$

or

$$\frac{\Delta y}{\%\Delta x} \approx \beta$$

Interpretation:

- If $\%\Delta x = 0.01$, then $\Delta y = \beta \cdot 0.01$ approx

Basic math tools: Some special functions and their properties: Logarithmic function: Implications

In the log-log model:

$$\ln(y) = \beta \ln(x)$$

Consider the change

$$\frac{\Delta \ln(y)}{\Delta \ln(x)} = \beta$$

By the same token as above, we can express this as

$$\frac{\%\Delta y}{\%\Delta x} \approx \beta$$

Interpretation:

- If $\%\Delta x = 0.01$, then $\%\Delta y = \beta \cdot 0.01$ approx

Basic math tools: Some special functions and their properties: Function with interaction terms

Suppose that the model of interest is

$$wage = \beta_0 + \beta_1 educ + \beta_2 female + \beta_3 educ \cdot female$$

where *educ* is a continuous variable, and *female* is a dummy variable.

Basic math tools: Some special functions and their properties: Function with interaction terms

Consider a change in either of the two interacting variables. For a unit change in educ:

$$\frac{\Delta wage}{\Delta educ} \approx \beta_1 + \beta_3 \cdot female$$

This equation indicates that the effect of *educ* on *wage* depends on whether someone is *female*. If so, the effect is $\beta_1 + \beta_3$. Here again, we approximate the exact change using the tangent slope, that is, the derivative from the linear model. This approximation is accurate for small changes in *educ*, but less reliable for large changes.

Basic math tools: Some special functions and their properties: Function with interaction terms

Consider a change in *female*. If $female = 1$:

$$wage = \beta_0 + \beta_2 + \beta_1 educ + \beta_3 educ$$

If $female = 0$:

$$wage = \beta_0 + \beta_1 educ$$

The two expressions differ by:

$$\beta_2 + \beta_3 educ$$

This shows that the effect of *female* on *wage* depends on both the constant β_2 and the level of *educ*.