

## Math refresher A: Basic mathematical tools

Econometrics for minor Finance, Lecture 1

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## Basic math tools: The summation operator and descriptive statistics: Properties

The **summation operator** is a shorthand to handle expressions involving sums of numbers, and is fundamental in statistics and econometrics.

If

$$\{x_i : i = 1, \dots, n\}$$

denotes a sequence of  $n$  numbers, then we write the sum of these numbers as

$$\sum_i^n x_i := x_1 + x_2 + \dots + x_n$$

## Basic math tools: The summation operator and descriptive statistics: Properties

**sum. ope. property 1.** For any constant  $c$ ,

$$\sum_{i=1}^n c = nc$$

Since  $c$  remains constant and does not depend on the index  $i$ , it is not written as  $c_i$ .

## Basic math tools: The summation operator and descriptive statistics: Properties

**sum. ope. property 2.** For any constant  $c$ ,

$$\sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$$

This shows that, since  $c$  is constant, it can be factored out of the summation. This reflects the distributive property of multiplication over addition.

## Basic math tools: The summation operator and descriptive statistics: Properties

**sum. ope. property 3.** If

$$\{(x_i, y_i) : i = 1, 2, \dots, n\}$$

is a set of  $n$  pairs of numbers, and  $a$  and  $b$  are constants, then

$$\sum_{i=1}^n (ax_i + by_i) = \sum_{i=1}^n ax_i + \sum_{i=1}^n by_i$$

This shows that summation distributes over addition and scalar multiplication, allowing each term to be summed independently.

## Basic math tools: The summation operator and descriptive statistics: Example

In statistics,  $X$  typically denotes a **random variable** representing a quantity of interest. The lowercase  $x$  is used to denote **realizations** or observed values of  $X$  obtained from data.

## Basic math tools: The summation operator and descriptive statistics: Example

Let  $\{X_i : i = 1, \dots, n\}$  be a sequence of independent and identically distributed **random variables** representing years of education. Suppose we observe their **realizations**  $\{x_i : i = 1, \dots, n\}$ . We compute their **average** by

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

When the  $x_i$  are a sample of data on a particular variable, we often call this the **sample average** to emphasize that it is computed from **a particular set of data**. The sample average is an example of a descriptive statistic; in this case, the statistic describes the central tendency of the observed values  $x_i$ .

## Basic math tools: Properties of linear functions

Linear functions play an important role in econometrics because they are simple to interpret and manipulate. If  $x$  and  $y$  are two **variables** related by

$$y = \beta_0 + \beta_1 x$$

then we say that  $y$  is a **linear function** of  $x$ , and  $\beta_0$  and  $\beta_1$  are two **parameters** describing this relationship. The **intercept** is  $\beta_0$ , and the **slope** is  $\beta_1$ .

## Basic math tools: Properties of linear functions

The defining feature of a linear function is that the change in  $y$  is always equal to a **fixed multiple** of the change in  $x$ :

$$\Delta y = \beta_1 \Delta x$$

Here,  $\Delta$  denotes a **discrete change**, and  $\beta_1$  represents the **rate of change** or **slope**, the amount by which  $y$  changes for each unit increase in  $x$ . In other words, the **effect** of  $x$  on  $y$  is constant and equal to  $\beta_1$ .

## Basic math tools: Properties of linear functions: Example

Suppose that the relationship between monthly housing expenditure and monthly income is

$$\text{expenditure} = 164 + 0.27 \cdot \text{income}$$

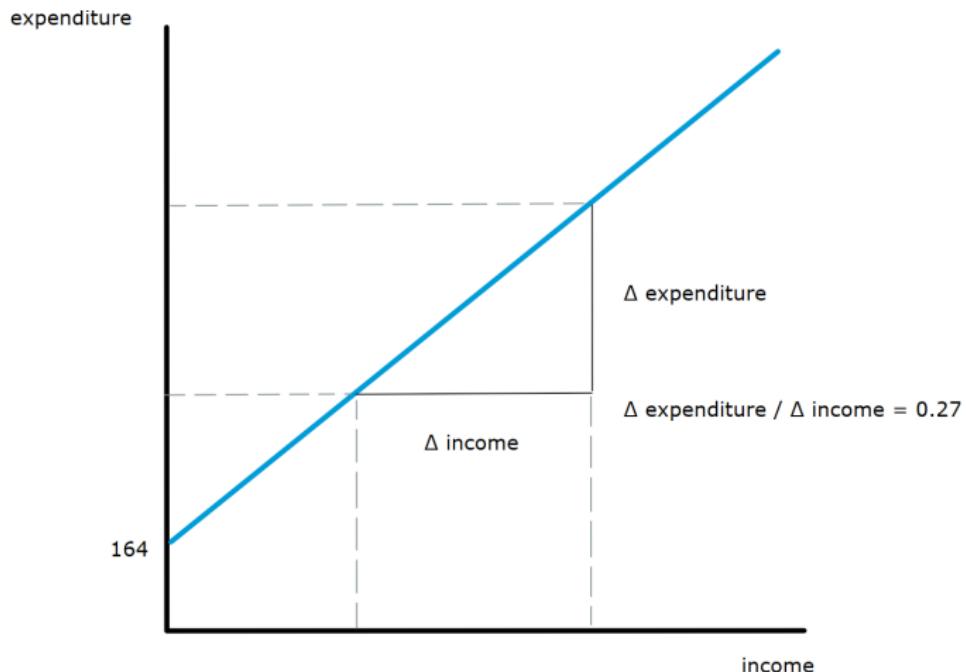
This shows that for each additional dollar of income,

$$\Delta \text{expenditure} = 0.27 \cdot \Delta \text{income} = 0.27 \cdot \$1$$

cents is spent on housing. If family income increases by \$200, then housing expenditure increases by

$$0.27 \cdot \$200 = \$54$$

# Basic math tools: Properties of linear functions: Example



## Basic math tools: Properties of linear functions: Example

A side note. According to the equation, a family with no income spends \$164 on housing, which cannot be true. For low levels of income, this linear function would not describe the relationship well. We eventually need to use other types of functions to describe the relationship.

## Basic math tools: Properties of linear functions

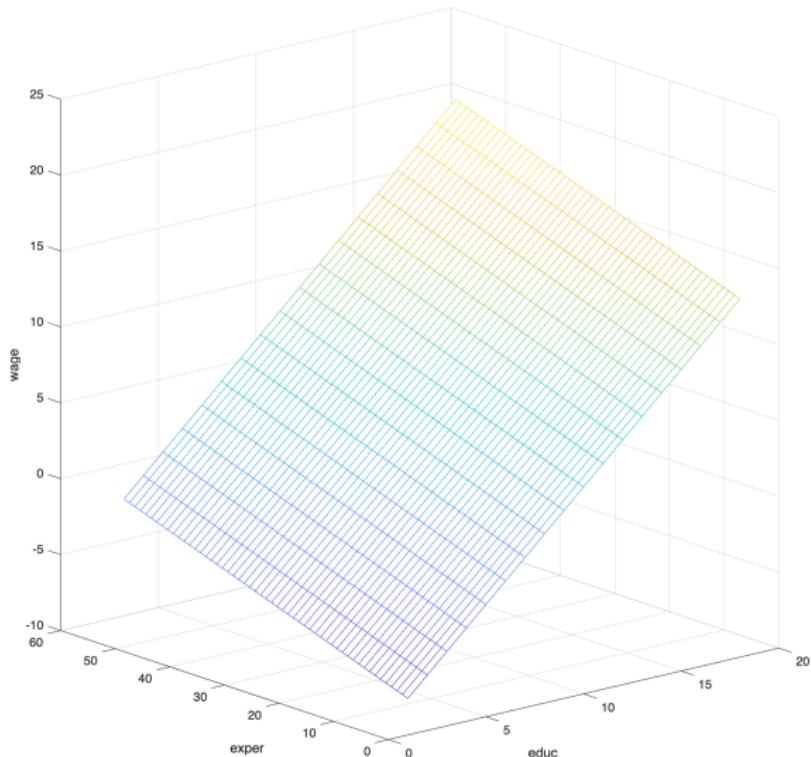
Linear functions are easily defined for more than two variables:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

$\beta_0$  is still the intercept, the value of  $y$  when  $x_1 = 0$  and  $x_2 = 0$ .  $\beta_1$  and  $\beta_2$  measure particular slopes.

It is slightly more difficult to imagine this function because its graph is three-dimensional. But let's give it a try.

# Basic math tools: Properties of linear functions: Example



## Basic math tools: Properties of linear functions

From this, the change in  $y$ , for given changes in  $x_1$  and  $x_2$ , is

$$\Delta y = \beta_1 \Delta x_1 + \beta_2 \Delta x_2$$

If  $x_2$  does not change, that is,  $\Delta x_2 = 0$ , then we have

$$\Delta y = \beta_1 \Delta x_1$$

$\beta_1$  captures how  $y$  responds to changes in  $x_1$  while holding  $x_2$  constant. That is,

$$\beta_1 = \frac{\Delta y}{\Delta x_1}$$

This is why  $\beta_1$  is called the **partial effect** of  $x_1$  on  $y$ : it measures the effect of changing  $x_1$  while keeping all other variables fixed. This idea is closely linked to the principle of **ceteris paribus**, which means “all else equal”. Similarly, if  $\Delta x_1 = 0$ , then  $\beta_2$  is the partial effect of  $x_2$  on  $y$ .

## Basic math tools: Proportions and percentages

In econometrics, we are often interested in measuring the changes in various quantities. Let  $x$  denote some variable, such as an individual's income, or profits of a firm. Let  $x_0$  and  $x_1$  denote two values for  $x$ :

- $x_0$  : initial value, and
- $x_1$  : subsequent value

For example,  $x_0$  could be the annual income of an individual in 1994, and  $x_1$  the income of the same individual in 1995. The **proportionate change** in  $x$  in moving from  $x_0$  to  $x_1$ , is

$$\frac{x_1 - x_0}{x_0}$$

This is sometimes called the relative change.

## Basic math tools: Proportions and percentages

It is common to **state** changes in terms of percentages. To do this, multiply and divide the proportionate change by 100:

$$\frac{x_1 - x_0}{x_0} \cdot 100 \cdot \frac{1}{100}$$

This is always denoted as:

$$\frac{x_1 - x_0}{x_0} \cdot 100\% \stackrel{!}{=} \% \Delta x$$

Here, **%** means “per hundred”, or in Latin, **per centum**. This is the **percentage change** in  $x$  when moving from  $x_0$  to  $x_1$ .

## Basic math tools: Proportions and percentages: Example

When income goes from \$30,000 to \$33,750, the **proportionate change** is

$$\frac{33,750 - 30,000}{30,000} = 0.125$$

The **percentage change** is

$$0.125 \cdot 100 \cdot \frac{1}{100} = 12.5 \cdot \frac{1}{100} = 12.5\%$$

which says that income has increased by 12.5 percent.

## Basic math tools: Differential calculus

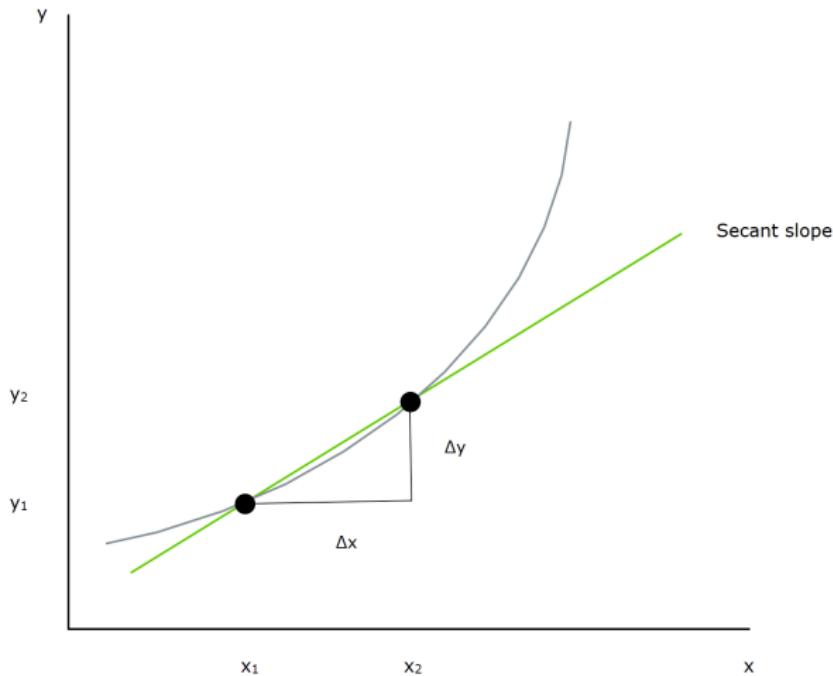
In differential calculus, the **secant slope** refers to the

- change in the value of a nonlinear function for a
- **discrete** change in its input variable:

$$\frac{y_2 - y_1}{x_2 - x_1} := \frac{\Delta y}{\Delta x}$$

where  $y_2 := f(x_2)$  and same for  $y_1$ . This expression represents the slope of the secant line connecting two points on the graph of the function. It captures the **average rate of change**, as the ratio quantifies how much the output changes per unit change in the input.

# Basic math tools: Differential calculus



## Basic math tools: Differential calculus

Tangent slope refers to the

- change in the value of a nonlinear function for a
- marginal change in its variable:

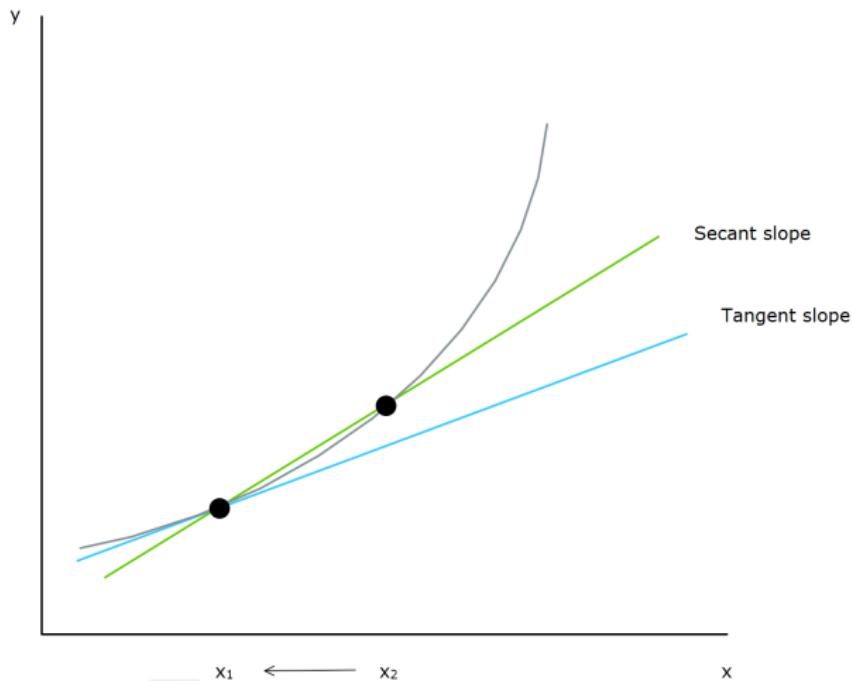
$$\frac{y_2 - y_1}{x_2 - x_1} := \frac{\delta y}{\delta x}$$

as

$x_2$  approaches  $x_1$

Geometrically, this gives the slope of the tangent line that touches the curve exactly at  $x_1$ , where the two values of the variable merge. It leads to the idea of a **derivative**, which measures how quickly a function changes at a single point.

# Basic math tools: Differential calculus



## Basic math tools: Differential calculus

When the change in  $x$  is small, the tangent slope provides a good estimate of the secant slope:

$$\frac{\delta y}{\delta x} \approx \frac{\Delta y}{\Delta x}$$

as

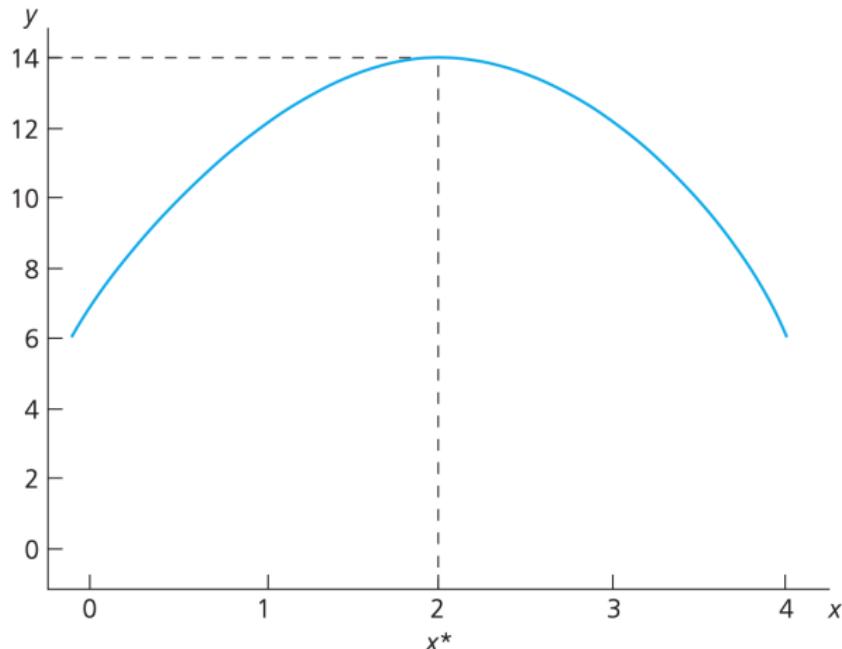
$x_2$  approaches  $x_1$

## Basic math tools: Differential calculus

While the **secant slope** measures the average rate of change between two points, we are often more interested in the **tangent slope**, which reflects the **instantaneous rate of change** at a specific point. In many fields like economics or physics, we care about **marginal changes**, how a function behaves **locally**. The tangent slope shows how output reacts to small input shifts and reveals trends and curvature.

## Basic math tools: Some special functions and their properties: Quadratic function

Graph of  $y = 6 + 8x - 2x^2$ .



## Basic math tools: Some special functions and their properties: Quadratic function

Quadratic functions are useful in economic modeling because they naturally represent diminishing returns, such as decreasing marginal utility.

## Basic math tools: Some special functions and their properties: Quadratic function

Consider the functional form

$$wage = \beta_0 + \beta_1 age + \beta_2 age^2$$

which is quadratic in age.

## Basic math tools: Some special functions and their properties: Quadratic function

If age changes **marginally**, the change in wage is given approximately by:

$$\frac{\partial \text{wage}}{\partial \text{age}} = \beta_1 + 2 \cdot \beta_2 \cdot \text{age}$$

This is the **derivative** of the wage function with respect to age. If age changes by **a discrete amount**, the change in wage is **exactly**:

$$\Delta \text{wage}_i = [\beta_0 + \beta_1(\text{age} + \Delta \text{age}) + \beta_2(\text{age} + \Delta \text{age})^2] - [\beta_0 + \beta_1 \text{age} + \beta_2 \text{age}^2]$$

The derivative provides a good approximation for small changes in age, but becomes less accurate for larger changes, where the exact expression should be used instead.

## Basic math tools: Some special functions and their properties: Quadratic function

In either case, the expression shows that the effect of age, that is, the slope of the relationship between *wage* and *age*, depends on the value of *age*.

## Basic math tools: Some special functions and their properties: Quadratic function

By taking the derivative of the *wage* function with respect to *age* and setting it equal to zero, we can identify the critical point where the function may attain a maximum or minimum. That is,

$$\frac{\partial \text{wage}}{\partial \text{age}} = \beta_1 + 2 \cdot \beta_2 \cdot \text{age} = 0$$

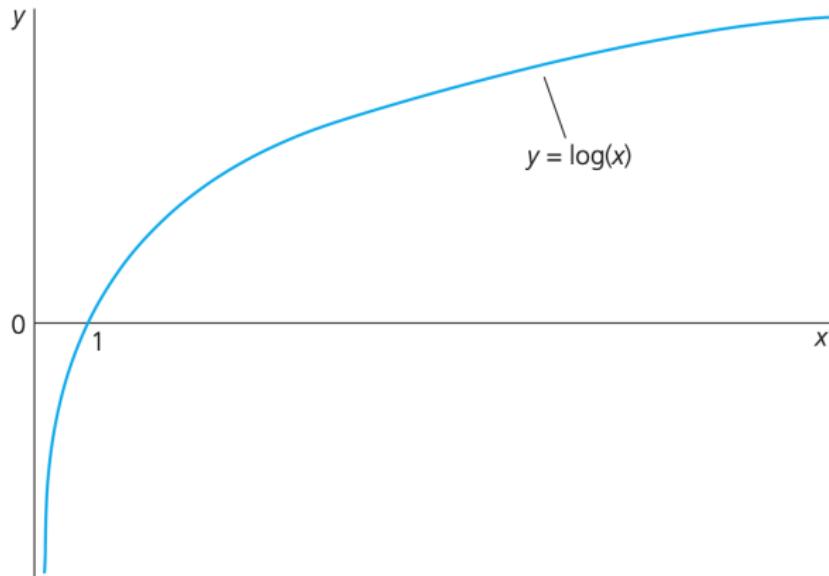
Solving for *age* gives:

$$\text{age}_{i,\max} = -\frac{\beta_1}{2\beta_2}$$

Thus, the age at which wage reaches its maximum (or minimum) is determined by this critical point. If  $\beta_2$  is negative, the point corresponds to a maximum; if positive, it corresponds to a minimum.

# Basic math tools: Some special functions and their properties: Logarithmic function

Graph of  $y = \log(x)$ .



## Basic math tools: Some special functions and their properties: Logarithmic function: Proportionate change

Consider the following **logarithmic change**:

$$\ln(41) - \ln(40) \approx 0.024$$

The **proportionate change** that we frequently calculate is

$$\frac{41 - 40}{40} = 0.025$$

The two quantities are very close. This shows that, **for small changes, the logarithmic change is a good approximation of the proportionate change**. We will make use of this result.

## Basic math tools: Some special functions and their properties: Logarithmic function: Proportionate change

Consider the logarithmic change

$$\ln(60) - \ln(40) \approx 0.405.$$

The proportionate change is

$$\frac{60 - 40}{40} = 0.500$$

The two quantities are not close. This shows that, for large changes, the approximation is not accurate.

## Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

Consider the log-linear function

$$\ln(y) = \beta x$$

We are interested in the change in  $\ln(y)$  when we change  $x$  by **some** unit.

## Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

$$\ln(y) = \beta x$$

Consider first a **small** change in  $x$ , which is considering the derivative:

$$\frac{\partial \ln(y)}{\partial x} = \beta$$

This change refers to the **tangent slope**.

We know that for a small change in  $x$ , the change in  $\ln(y)$  closely approximates the proportionate change in  $y$ . For a **large** change in  $x$ , the linear approximation becomes less accurate. Then, what to do?

## Basic math tools: Some special functions and their properties: Logarithmic function: Small and large changes

In this case, we consider the exact **proportionate change** in  $y$ . This is derived from the change in  $\ln(y)$ :

$$\Delta \ln(y) = \ln(y_1) - \ln(y_0) = \beta \Delta x$$

$$e^{\ln(y_1) - \ln(y_0)} = e^{\beta \Delta x}$$

$$e^{\ln\left(\frac{y_1}{y_0}\right)} = e^{\beta \Delta x}$$

$$\frac{y_1}{y_0} = e^{\beta \Delta x}$$

$$\frac{y_1 - y_0}{y_0} = e^{\beta \Delta x} - 1$$

This **exact proportionate change** in  $y$  resulting from a change in  $x$  corresponds to the **secant slope**.

## Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Implications of the logarithmic transformation for applied work are the following. Consider the log-linear model

$$\ln(y) = \beta x$$

Consider the change

$$\Delta \ln(y) = \beta \Delta x$$

For a small change in  $y$ , we can use the proportionate change approximation, which we know is good:

$$\Delta \ln(y) \approx \frac{y_1 - y_0}{y_0}$$

Replace to obtain

$$\frac{y_1 - y_0}{y_0} \approx \beta \Delta x$$

The LHS gives the **proportionate change** interpretation.

## Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Multiply and divide by 100

$$\frac{y_1 - y_0}{y_0} \cdot 100 \cdot \frac{1}{100} \approx \beta \Delta x$$

to obtain

$$\frac{\% \Delta y}{\Delta x} \approx \beta$$

This gives the **percentage change** interpretation:

- If  $\Delta x = 1$ , then  $\% \Delta y = \beta$  approx

## Basic math tools: Some special functions and their properties: Logarithmic function: Implications

Consider the linear-log model:

$$y = \beta \ln(x)$$

Consider the change

$$\Delta y = \beta \cdot \Delta \ln(x)$$

For a small change in  $x$ , we can use the proportionate change approximation, which we know is good:

$$\Delta \ln(x) \approx \frac{x_1 - x_0}{x_0}$$

Replace to obtain

$$\Delta y \approx \beta \cdot \frac{x_1 - x_0}{x_0}$$

## Basic math tools: Some special functions and their properties: Logarithmic function: Implications

To express this proportionate change as a percentage change, multiply and divide by 100

$$\Delta y \approx \beta \cdot \frac{x_1 - x_0}{x_0} \cdot 100 \cdot \frac{1}{100}$$

to obtain

$$\Delta y \approx \beta \cdot \% \Delta x$$

or

$$\frac{\Delta y}{\% \Delta x} \approx \beta$$

Interpretation:

- If  $\% \Delta x = 0.01$ , then  $\Delta y = \beta \cdot 0.01$  approx

## Basic math tools: Some special functions and their properties: Logarithmic function: Implications

In the log-log model:

$$\ln(y) = \beta \ln(x)$$

Consider the change

$$\frac{\Delta \ln(y)}{\Delta \ln(x)} = \beta$$

By the same token as above, we can express this as

$$\frac{\% \Delta y}{\% \Delta x} \approx \beta$$

Interpretation:

- If  $\% \Delta x = 0.01$ , then  $\% \Delta y = \beta \cdot 0.01$  approx

## Basic math tools: Some special functions and their properties: Function with interaction terms

Suppose that the model of interest is

$$wage = \beta_0 + \beta_1 \text{educ} + \beta_2 \text{female} + \beta_3 \text{educ} \cdot \text{female}$$

where *educ* is a continuous variable, and *female* is a dummy variable.

## Basic math tools: Some special functions and their properties: Function with interaction terms

Consider a change in either of the two interacting variables. For a unit change in *educ*:

$$\frac{\Delta \text{wage}}{\Delta \text{educ}} \approx \beta_1 + \beta_3 \cdot \text{female}$$

This equation indicates that the effect of *educ* on *wage* depends on whether someone is *female*. If so, the effect is  $\beta_1 + \beta_3$ . Here again, we approximate the exact change using the tangent slope, that is, the derivative from the linear model. This approximation is accurate for small changes in *educ*, but less reliable for large changes.

## Basic math tools: Some special functions and their properties: Function with interaction terms

Consider a change in *female*. If *female* = 1:

$$\text{wage} = \beta_0 + \beta_2 + \beta_1 \text{educ} + \beta_3 \text{educ}$$

If *female* = 0:

$$\text{wage} = \beta_0 + \beta_1 \text{educ}$$

The two expressions differ by:

$$\beta_2 + \beta_3 \text{educ}$$

This shows that the effect of *female* on *wage* depends on both the constant  $\beta_2$  and the level of *educ*.