

## Violating the exogeneity assumption of the linear regression model: When does it happen?

Econometrics for minor Finance, Lecture 7

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Linear regression model: Model assumption: Error is exogenous

When we introduced the linear regression model, we assumed **exogeneity**:

$$E[u_i | x_i] = 0$$

## Linear regression model: Model assumption: Error is exogenous: Implications

$$E[u_i | x_i] = 0$$

has a number of implications.

Linear regression model: Model assumption: Error is exogenous: Implications: Implication one

First, if

$$\mathbb{E}[u_i | x_i] = 0$$

then

$$\begin{aligned}\mathbb{E}[u_i x_i] &= \mathbb{E}_{x_i} [\mathbb{E}[u_i x_i | x_i]] \\ &= \mathbb{E}_{x_i} [x_i \mathbb{E}[u_i | x_i]] \\ &= 0\end{aligned}$$

using the LIE in the first equality.

Linear regression model: Model assumption: Error is exogenous: Implications: Implication two

Second, if

$$E[u_i | x_i] = 0$$

then

$$\begin{aligned} E[u_i] &= E_{x_i}[E[u_i | x_i]] \\ &= 0. \end{aligned}$$

by the LIE in the first equality. This says that if the average of  $u_i$  at all slices of the population determined by the values of  $x_i$  is zero, then the average of these zero conditional means must also be zero.

Linear regression model: Model assumption: Error is exogenous: Implications: Implication three

Third, if

$$E[u_i | x_i] = 0$$

then

$$\begin{aligned}\text{Cov}[u_i, x_i] &= E[u_i x_i] - E[u_i] E[x_i] \\ &= 0 - 0 E[x_i] \\ &= 0\end{aligned}$$

using the above results. That is, exogeneity implies that  $u_i$  are  $x_i$  are uncorrelated.

## Linear regression model: Model assumption: Error is endogenous

In this lecture we violate the exogeneity assumption. That is, we have

$$E[u_i | x_i] \neq 0$$

Linear regression model: Model assumption: Error is endogenous: When does this happen?

When does

$$E[u_i | x_i] \neq 0$$

happen?

## Linear regression model: Model assumption: Error is endogenous: The case of omitted variable

Consider the linear model

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$$

Assume that

$$\mathbb{E}[u_i | x_{1i}] = 0$$

and

$$\mathbb{E}[u_i | x_{2i}] = 0$$

Hence, the model is correctly specified.

## Linear regression model: Model assumption: Error is endogenous: The case of omitted variable

Suppose we do not observe  $x_{2i}$  so that it enters the error and we have

$$y_i = x_{1i}\beta_1 + u_i^*$$

and

$$u_i^* = x_{2i}\beta_2 + u_i$$

## Linear regression model: Model assumption: Error is endogenous: The case of omitted variable

Taking the expectation conditional on  $x_{1i}$ ,

$$u_i^* = x_{2i}\beta_2 + u_i$$

we have

$$\begin{aligned} E[u_i^* | x_{1i}] &= E[x_{2i}\beta_2 | x_{1i}] + E[u_i | x_{1i}] \\ &= \beta_2 E[x_{2i} | x_{1i}] \end{aligned}$$

## Linear regression model: Model assumption: Error is endogenous: The case of omitted variable

If

$$\beta_2 \neq 0$$

and

$$E[x_{2i} | x_{1i}] \neq 0$$

we have

$$E[u_i^* | x_{1i}] = \beta_2 E[x_{2i} | x_{1i}] \neq 0$$

- $\beta_2 \neq 0$  means that  $x_{2i}$  should enter the true model.
- $E[x_{2i} | x_{1i}] \neq 0$  means that  $x_{1i}$  and  $x_{2i}$  are correlated.

## Linear regression model: Model assumption: Error is endogenous: The case of omitted variable

$$E[u_i^* | x_{1i}] \neq 0$$

means that in the regression

$$y_i = x_{1i}\beta_1 + u_i^*$$

$x_{1i}$  is correlated with  $u_i^*$  due to leaving  $x_{2i}$  in  $u_i^*$ . The exogeneity assumption is violated for  $u_i^*$ .

## Linear regression model: Model assumption: Error is endogenous: The case of measurement error

Consider the linear model

$$y_i = \beta x_i^* + u_i$$

Suppose  $x_i^*$  is the true variable we do not observe. What we observe is  $x_i$ , a noisy version of  $x_i^*$  with unobserved **measurement error**  $\omega_i$  so that

$$x_i = x_i^* + \omega_i$$

Since we observe only  $x_i$ , replace  $x_i^*$  in the model to obtain

$$y_i = \beta x_i - \beta \omega_i + u_i$$

Define

$$-\beta \omega_i + u_i := u_i^*$$

so that we have

$$y_i = \beta x_i + u_i^*$$

$x_i$  is correlated with  $u_i^*$  due to  $\omega_i$ . The **exogeneity assumption is violated for  $u_i^*$** .

## Linear regression model: Model assumption: Error is endogenous: The case of simultaneity

Consider the **simultaneous equations model** given by

$$y_{1i} = \alpha_1 y_{2i} + \beta_1 z_{1i} + u_{1i}$$

$$y_{2i} = \alpha_2 y_{1i} + \beta_2 z_{2i} + u_{2i}$$

In each equation the constant is ignored for simplicity. Assume that

$$\mathbb{E}[u_{1i} | z_{1i}, z_{2i}] = 0$$

$$\mathbb{E}[u_{2i} | z_{1i}, z_{2i}] = 0$$

which imply

$$\mathbb{E}[u_{1i}] = 0$$

$$\mathbb{E}[u_{2i}] = 0$$

Hence,  $z_{1i}$  and  $z_{2i}$  are uncorrelated with  $u_{1i}$  and  $u_{2i}$ . Suppose that the **interest lies in estimating  $\alpha_1$**  in the first equation.

## Linear regression model: Model assumption: Error is endogenous: The case of simultaneity

Solve the two equations for  $y_{2i}$ , in terms of  $z_{1i}$ ,  $z_{2i}$ ,  $u_{1i}$ , and  $u_{2i}$ . First, replace  $y_{1i}$  in the equation for  $y_{2i}$ , and then solve for  $y_{2i}$  as

$$\begin{aligned}y_{2i} &= \alpha_2 y_{1i} + \beta_2 z_{2i} + u_{2i} \\&= \alpha_2 (\alpha_1 y_{2i} + \beta_1 z_{1i} + u_{1i}) + \beta_2 z_{2i} + u_{2i} \\&= \alpha_1 \alpha_2 y_{2i} + \beta_1 \alpha_2 z_{1i} + \alpha_2 u_{1i} + \beta_2 z_{2i} + u_{2i} \\(1 - \alpha_1 \alpha_2) y_{2i} &= \beta_1 \alpha_2 z_{1i} + \beta_2 z_{2i} + \alpha_2 u_{1i} + u_{2i} \\y_{2i} &= z_{1i} \frac{\beta_1 \alpha_2}{1 - \alpha_1 \alpha_2} + z_{2i} \frac{\beta_2}{1 - \alpha_1 \alpha_2} + u_{1i} \frac{\alpha_2}{1 - \alpha_1 \alpha_2} \\&\quad + u_{2i} \frac{1}{1 - \alpha_1 \alpha_2}\end{aligned}$$

assuming that  $\alpha_1 \alpha_2 \neq 1$ .

## Linear regression model: Model assumption: Error is endogenous: The case of simultaneity

The parameter of interest was  $\alpha_1$  in the equation

$$y_{1i} = \alpha_1 y_{2i} + \beta_1 z_{1i} + u_{1i}$$

and we have just shown that

$$y_{2i} = z_{1i} \frac{\beta_1 \alpha_2}{1 - \alpha_1 \alpha_2} + z_{2i} \frac{\beta_2}{1 - \alpha_1 \alpha_2} + u_{1i} \frac{\alpha_2}{1 - \alpha_1 \alpha_2} + u_{2i} \frac{1}{1 - \alpha_1 \alpha_2}$$

Note that we need

$$E[y_{2i} u_{1i}] = 0$$

to hold to estimate  $\alpha_1$  with no bias. Does it hold?

## Linear regression model: Model assumption: Error is endogenous: The case of simultaneity

$$y_{2i} = z_{1i} \frac{\beta_1 \alpha_2}{1 - \alpha_1 \alpha_2} + z_{2i} \frac{\beta_2}{1 - \alpha_1 \alpha_2} + u_{1i} \frac{\alpha_2}{1 - \alpha_1 \alpha_2} + u_{2i} \frac{1}{1 - \alpha_1 \alpha_2}$$

Multiply both sides with  $u_{1i}$ , take expectations, and use the earlier assumption that  $E[z_{1i}u_{1i}] = 0$  and  $E[z_{2i}u_{1i}] = 0$  to obtain

$$E[y_{2i}u_{1i}] = E[u_{1i}u_{1i}] \frac{\alpha_2}{1 - \alpha_1 \alpha_2} + E[u_{2i}u_{1i}] \frac{1}{1 - \alpha_1 \alpha_2}$$

If

$$\alpha_2 \neq 0, E[u_{2i}u_{1i}] = 0$$

or

$$\alpha_2 = 0, E[u_{2i}u_{1i}] \neq 0$$

we have

$$E[y_{2i}u_{1i}] \neq 0$$

## Linear regression model: Model assumption: Error is endogenous: The case of simultaneity

Go back to the simultaneous equations model:

$$y_{1i} = \alpha_1 y_{2i} + \beta_1 z_{1i} + u_{1i}$$

$$y_{2i} = \alpha_2 y_{1i} + \beta_2 z_{2i} + u_{2i}$$

We have just shown that

$$E[y_{2i}u_{1i}] \neq 0$$

meaning that, in the first equation,  $y_{2i}$  is correlated with  $u_{1i}$  due to simultaneity. The **exogeneity assumption is violated for  $u_{1i}$ .**