# Heights and Tamagawa numbers of motives

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## Plans

- Introduction to the Tamagawa number conjecture for motives.
- Heights of motives.
- Connection between heights and Tamagawa numbers of motives.
- A concrete computation with mixed Tate motives and some speculations.

## Class number formula

Let $K$ be a number field and $\mathcal{O}_K$ be its ring of algebraic integers. Let

- $r_1$ (respectively $r_2$) be the number of real (respectively complex) places of $K$.
- $h_K$: class number of $K$.
- $w_K$: number of roots of unity in $K$.
- $R_K$: the regulator of $K$.
- $D_K$: the discriminant of $K$.

These numbers contain arithmetic information about $K$.

# Class number formula

The Riemann zeta function of $K$

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}.$$

It has a meromorphic continuation to $\mathbb{C}$ with a unique simple pole at $s = 1$.

**Theorem (Class number formula)**

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{w_K \sqrt{|D_K|}}.$$

This formula provides a connection between the algebraic world and the analytic world.

# Birch and Swinnerton-Dyer conjecture

Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$, i.e plane algebraic curve given by an equation of the form

$$y^2 = x^3 + ax + b.$$

The $L$-function of $E$ is defined by the Euler product

$$L(s,E) = \prod_p L_p(s,E)^{-1},$$

where, for a given prime p,

$$L_p(s,E) = \begin{cases} (1 - a_p p^{-s} + p^{1-2s}), & \text{if } p \nmid N \\ (1 - a_p p^{-s}), & \text{if } p \mid N \text{ and } p^2 \nmid N. \\ 1, & \text{if } p^2 \mid N \end{cases}$$

When $E$ has good reduction at $p$, $a_p = p + 1 - |E(\mathbb{F}_p)|$.

It is known that $L(E,s)$ has a analytic continuation to $\mathbb{C}$ (hard theorem!). Furthermore, it is expected that

**Conjecture (BSD conjecture)**

*$r_{al} = r_{an}$. Furthermore*

$$\frac{L^{(r)}(E,1)}{r!} = \frac{\#\text{Ш}(E) R_E \prod_{p|N\infty} c_p}{|G|^2},$$

*where*

- *$\text{Ш}(E)$ is the Tate-Shafarevich group of $E$.*
- *$R_E$ is the regulator.*
- *$c_p$ are Tamagawa factors.*
- *$G$ is the order of the torsion subgroup of $E$.*

# Generalization

- In the 70-80s, mathematicians realized that these two examples should be special cases of a more general phenomenon.
- Beilinson and Deligne made a breakthrough by putting this question in the framework of mixed motives and motivic cohomology. However, their work only predicts $L$-values up to an undetermined rational factor. Beilinson's approach is inspired by the work of Bloch on $K_2$ of CM elliptic curves.
- Bloch and Kato formulated a more precise conjecture using tools from $p$-adic Hodge theory discovered by Fontaine.

# What is a (mixed) motive?

- There is still no universally agreed definition of mixed motives even up to this day. Progress has been made by many mathematicians, notably Grothendieck, Jannsen, Bloch, Beilinson, Deligne and many others.
- A folklore definition: $H^n(X)(r)$ where $X$ is a smooth variety over $\mathbb{Q}$. It has different realizations depending on different cohomology theories (de Rham, Betti, etale, crystalline.) These realizations are connected by comparison isomorphisms. For example, if $X$ is smooth and projective, there is an isomorphism between Betti (singular) cohomology and etale cohomology

$$H^n_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) \cong H^n_B(X(\mathbb{C}), \mathbb{Z}_\ell).$$

# A working definition

- Jannsen's definition: a mixed motive with $\mathbb{Q}$-coefficients is a collection of realizations with weight and Hodge filtrations. These realizations are related by comparison isomorphisms that are compatible with both of these filtrations.
- A mixed motive with $\mathbb{Z}$-coefficients is a pair $(M, \Theta)$ where
    1. $M$ is a mixed motive with $\mathbb{Q}$ coefficients.
    2. $\Theta$ is a free $\mathbb{Z}$-module of finite rank equipped with a linear action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{\Theta} = \widehat{\mathbb{Z}} \otimes \Theta$.
    3. There is an isomorphism $\mathbb{Q} \otimes \Theta \cong M_B$ where $M_B$ is the Betti realization of $M$.

# Period rings and the local Bloch-Kato Selmer groups

Let $V$ be a $\mathbb{Q}_\ell$-vector space equipped with an action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and $T \subset V$ a Galois stable lattice. When $\ell = p$, we define the following subgroups of $H^1(\mathbb{Q}_p, V)$ :

- $H^1_e(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}}^{\varphi=1}))$.
- $H^1_f(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}}))$.
- $H^1_g(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}))$.

These are called the exponential, finite, and geometric parts of $H^1(\mathbb{Q}_p, V)$.

When $\ell \neq p$, we can define similar subgroups of $H^1(\mathbb{Q}_p, V)$

- $H^1_e(\mathbb{Q}_p, V) = 0$.
- $H^1_f(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(I, V))$.
- $H^1_g(\mathbb{Q}_p, V) = H^1(\mathbb{Q}_p, V))$.

# Local Bloch-Kato Selmer groups

Let $* \in \{e, f, g\}$. We define $H^1_*(\mathbb{Q}_p, T)$ to be the pre-image of of $H^1_*(\mathbb{Q}_p, V)$ under the canonical map

$$H^1(\mathbb{Q}_p, T) \to H^1(\mathbb{Q}_p, V).$$

If $M$ is a $\widehat{\mathbb{Z}}$-module with a continuous $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ action, we define

$$H^1_*(\mathbb{Q}_p, M) = \prod_\ell H^1_*(\mathbb{Q}_p, M_\ell).$$

# Tamagawa measure

We define the following Selmer conditions on local Galois cohomology.

$$A(\mathbb{Q}_p) = \begin{cases} H^1_f(\mathbb{Q}_p, \widehat{\Theta}) & \text{if } p < \infty \\ (D_\infty \otimes_{\mathbb{R}} \mathbb{C})/(\mathrm{Fil}^0 D_\infty \otimes_{\mathbb{R}} \mathbb{C}) + \Theta)^+ & \text{if } p = \infty. \end{cases}$$

The exponential map

$$\exp : D_p/D_p^0 \to A(\mathbb{Q}_p),$$

is an isomorphism if $\omega(M) \leq -1$ where $D_p$ is the $p$-adic de Rham realization of $M$.

# Tamagawa measure

By fixing a basis for $\det(D)$, we can define the Tamagawa measure on $\mu(A(\mathbb{Q}_p))$. Furthermore, under some technical conditions, we have

$$\mu(A(\mathbb{Q}_p)) = L_p(V, 0)^{-1},$$

where $L_p(V, p^{-s})$ is the local Euler factor at $p$.

The induced measure $\mu$ on the adelic space $\prod'_p A(\mathbb{Q}_p)$ is called the Tamagawa measure of $M$.

The Tamagawa number of $M$ is defined as

$$\mathrm{Tam}(M) = \mu\left(\prod_{p\leq\infty} A(\mathbb{Q}_p)/A(\mathbb{Q})\right),$$

where $A(\mathbb{Q})$ can be thought as $\mathrm{Ext}^1(\mathbb{Z},(M,\Theta))$ in the category of motives with $\mathbb{Z}$-coefficients satisfying certain Selmer conditions.

If $M = H^n(X)(r)$, then we should have $A(\mathbb{Q}) \subset A(\mathbb{Q}) \otimes \mathbb{Q} = \mathrm{gr}^r K_{2n-r-1}(\mathfrak{X}) \otimes \mathbb{Q}$ where $\mathfrak{X}/\mathbb{Z}$ is a regular flat model of $X$.

# Tamagawa number conjecture, I

**Conjecture (Tamagawa number conjecture)**

$$Tam(M) = \frac{\#H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))}{\#Ш(M)},$$

*where* $Ш(M)$ *is the Tate-Shafarevich group associated with* $M$*.*
*It is a conjectured to be a finite group.*

## Some known results

- $M = \mathbb{Z}(r)$. The class number formula is a special case of this when $r = 1$. (Bloch-Kato)
- $M = H^1(E)(2)$ where $E$ is an elliptic curve with CM (Bloch-Kato)
- $M = H^1(E)(1)$ in some special cases by work of Coates-Wiles, Kolyvagin, Rubin, Kato, and many others. This case is equivalent to the BSD conjecture.
- $M$ is the adjoint motive of a modular form (Diamond, Flach, Guo, Loeffler-Lei-Zerbes and others).
- Some progress for motives associated with automorphic forms on $GSp_4, GL_2 \times GL_2$ (Loeffler-Zerbes and others.)
- Most of these works construct special elements in $A(\mathbb{Q})$.
- Iwasawa theory plays an important role.

# Tamagawa number conjecture, II

- In their work, Bloch-Kato mentioned that the Tamagawa number conjecture could be formulated using $B(\mathbb{Q})$ and $B(\mathbb{Q}_p)$ where $B(\mathbb{Q}_p)$ is the inverse image in $H^1_g(\mathbb{Q}_p, \widehat{\Theta})$ of

$$\mathrm{Im}\left(\Psi \to H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})\right),$$

  and $B(\mathbb{R}) = A(\mathbb{R})$.
- We define $B(\mathbb{Q})$ similarly. As before, $B(\mathbb{Q})$ should be considered as the extension group $\mathrm{Ext}^1(\mathbb{Z}, (M, \Theta))$ in the category of motives with $\mathbb{Z}$-coefficients (with no Selmer conditions).
- When $M = H^n(X)(r)$, we should take

$$\Psi = \mathrm{gr}^r K_{2r-m-1}(X) \otimes \mathbb{Q}.$$

# Local height functions

Recall that we have the following inclusions

$$A(\mathbb{Q}_p) := H^1_f(\mathbb{Q}_p, \widehat{\Theta}) \subset B(\mathbb{Q}_p) \subset H^1_g(\mathbb{Q}_p, \widehat{\Theta}).$$

Furthermore, if $M_\ell$ satisfies the weight monodromy conjecture then

$$H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell) = \begin{cases} \left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1} & \text{if } \ell \neq p \\ \left((\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1)\right)^{\varphi=1} & \text{if } \ell = p. \end{cases}$$

We use this identification and the polarization of $M$ to define the height function

$$H_{\diamond,v,d} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}.$$

By definition, $H_{\diamond,v,d}$ is trivial on $A(\mathbb{Q}_p)$. Furthermore, it is expected that $H_{\diamond,v,d}$ is induced by a positive-definite quadratic form on $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$.

## Local height functions

Under some conjectures as in Bloch-Kato's article, we have $B(\mathbb{Q}_p)/A(\mathbb{Q}_p) \cong \mathbb{Z}^{r_p}$ as topological groups. By choosing a basis, we have

$$B(\mathbb{Q}_p) = A(\mathbb{Q}_p) \times \mathbb{Z}^{s_p}\,.$$

The local Tamagawa measure $\mu$ on $B(\mathbb{Q}_p)$ is defined to be the product measure of the Tamagawa measure on $A(\mathbb{Q}_p)$ and the counting measure on $\mathbb{Z}^{s_p}$.

The following statement is straightforward.

**Proposition**

*For each positive real number B*

$$\mu(a \in B(\mathbb{Q}_p) | H_{\diamond,p,d}(a) \leq B) < \infty.$$

## Adelic and global height functions

We define

$$H_{\diamond,d} : \prod\nolimits'_{p \leq \infty} B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}$$

as the product of local height functions.

Similarly, we define the height function on $B(\mathbb{Q})$ using the canonical embedding

$$B(\mathbb{Q}) \to \prod_{p \leq \infty} B(\mathbb{Q}_p).$$

By definition $H_{\diamond,d}$ is trivial on $A(\mathbb{Q})$.

We also define the Tamagawa measure on $\prod'_{p \leq \infty} B(\mathbb{Q}_p)$ as the product of the local Tamagawa measure on $B(\mathbb{Q}_p)$.

# Adelic and global height functions

A simple observation.

**Proposition**

*For each positive real number $B$, the set*

$$\{a \in \prod\nolimits'_{p \leq \infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B\}.$$

*is compact. In particular,*

$$\mu(a \in \prod\nolimits'_{p \leq \infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B) < \infty,$$

*and*

$$\#\{a \in B(\mathbb{Q}) | H_{\diamond,d}(a) \leq B\} < \infty.$$

Our first theorem is the following.

**Theorem (Nguyen)**

*Let $M$ be a pure motive with $\mathbb{Z}$-coefficients of weight $-d$ such that $d \geq 3$. We assume further that $M$ has semistable reduction at all places. Then*

$$\lim_{B\to\infty} \frac{\#\{x \in B(\mathbb{Q}) | H_{\diamond,d}(x) \leq B\}}{\mu\left(x \in \prod'_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(x) \leq B\right)} = \frac{1}{Tam(M)}.$$

## Sketch of the proof

We see that the numerator can be written explicitly as

$$\mathrm{N} = |A(\mathbb{Q})_{\mathrm{tor}}| \times N_B,$$

where

$$N_B = \#\{([a]_p) \in \prod_{p \leq \infty} \mathbb{Z}^{s_p}\},$$

such that

$$Q_{\diamond,\infty,d}(a_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}(a_p)^{1/d} \leq \ln B.$$

Also the numerator can be described as

$$\mathrm{D} = \mu(B(\mathbb{R})_{\mathrm{cpt}}) \times \prod_{p<\infty} \mu(A(\mathbb{Q}_p)) \times \mu(T_B),$$

# Sketch of the proof

where

$$T_B = \{([a]_p) \in \mathbb{R}^{s_\infty} \times \prod_{p<\infty} \mathbb{Z}^{s_p}\},$$

such that

$$Q_{\diamond,d}([a]_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}([a]_p)^{1/d} \leq \ln B.$$

From this, we see that

$$\frac{\mathrm{N}}{\mathrm{D}} = \frac{\mu((B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}})/A(\mathbb{Q})_{\mathrm{free}}) \times N_B}{\mathrm{Tam}(M) \times \mu(T_B)}.$$

Fixing a basis for $B(\mathbb{R})$, the above statement then becomes $\lim_{B\to\infty} \dfrac{N_B}{\mu(T_B)} = 1$ which is a consequence of the following general statement.

## Proposition

*Let* $f = (f_1, f_2) : \mathbb{R}^{r_1+r_2} = \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \to \mathbb{R}$*. Suppose for each* $i = 1, 2$ *the following conditions are satisfied*

1. $f_i(x) \geq 0$ *for all* $x$*. Moreover,* $f_i(x) = 0$ *if and only if* $x = 0$*.*
2. *There exists a positive real number* $c$ *such that* $f(\lambda x) = |\lambda|^c f(x)$ *for all* $x \in \mathbb{R}^{r_i}$ *and* $\lambda \in \mathbb{R}$*.*

*We equip* $\mathbb{Z}^{r_1}$ *with the counting measure* $d\mu_0$ *and* $\mathbb{R}^{r_1}, \mathbb{R}^{r_2}$ *the usual Lebesgue measures* $d\mu_1, d\mu_2$*. Let*

$$I(B) = ((n, y) \in \mathbb{Z}^{r_1} \times \mathbb{R}^{r_2} \,|\, f(n, y) = f_1(n) + f_2(y) \leq B),$$

*and*

$$V(B) = ((x, y) \in \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \,|\, f_1(x) + f_2(y) \leq B).$$

*Then*

$$\lim_{B \to \infty} \frac{\mu(I(B))}{\mu(V(B))} = 1.$$

## Mixed motives with more graded quotients

- In the above theorem, we consider the set of mixed motives $\widehat{M}$ that fits into the extension

$$0 \to M \to \widehat{M} \to \mathbb{Z} \to 0,$$

  where $M$ is a pure motive and $\mathbb{Z} = \mathbb{Z}(0)$ is the Tate motive.
- In this case, the set $B(\mathbb{Q})$ or $A(\mathbb{Q})$ have a structure of a group so the Tamagawa measure approach works.
- As we can see, the left hand side of theorem 1 makes sense even $B(\mathbb{Q})$ is just a set (as long as we can define Tamagawa measures!).

# Mixed motives with more graded quotients

- Let us fix motives $M_0, .., M_n$.
- We consider the set of all mixed motives $M$ with a decreasing filtration $M^i$ such that $M^0 = M, M^{n+1} = 0$, and $M^i/M^{i+1} = M_i$.
- $B(\mathbb{Q})$ is the set of all such $M$.
- $B(\mathbb{Q}_p)$ can be defined as a subset of the the $p$-adic period domain.
- We define the height of $M$ by

$$H_{\star,\diamond} = (\prod_{i=0}^{n} H_\star(M_i)) \times H_\diamond(M),$$

where

$$H_\diamond(M) = \prod_{\omega,d} H_{\diamond,\omega,d}(M).$$

Let us consider the case of mixed motives with graded quotients $\mathbb{Z}(12), \mathbb{Z}(3), \mathbb{Z}(0)$. A nice thing about this case is that.

**Lemma**

*Let $V$ be a mixed motive with the above graded quotients. As a Galois represenation of $Gal(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$, $V_p$ is unramifed if $v \nmid p$ and it is crystalline if $v \mid p$.*

Consequently, $A(\mathbb{Q}_p) = B(\mathbb{Q}_p)$ in this case and there is no height coming from non-Archimedean places.

## A concrete computation

Let $X$ be the set of all mixed motives with graded quotients $\mathbb{Z}(12), \mathbb{Z}(3), \mathbb{Z}$. Then, up to a factor of power of 2, we have

**Theorem**

$$\begin{aligned}\#\{x \in X | H_{\star,\diamond}(x) \leq B\} \sim & \frac{1}{8!2!\binom{12}{3}} \frac{Ш(3)}{\zeta(3)} \frac{Ш(9)}{\zeta(9)} \frac{Ш(12)}{\zeta(-11)} \\ & \times \left(\frac{2}{691} - \frac{1}{691^2}\right) \log(B)^{12}.\end{aligned}$$

# Key ideas of the proof

- The extension group $\mathrm{Ext}^1(\mathbb{Z}(12), \mathbb{Z}(3))$ is given by $K_{17}(\mathbb{Z})$.
- The extension group $\mathrm{Ext}^1(\mathbb{Z}(3), \mathbb{Z})$ is given by $K_5(\mathbb{Z})$.
- Modulo torsion, the group $K_{2m-1}$ is generated by the Soule element $b_m$. Furthermore,

  $$r_\infty(b_m) = 2^a \frac{(m-1)!\zeta(m)}{Ш(m)}.$$

  where $a$ is an unknown number.
- The cup product of $b_{17} \cup b_5 \in K_{22}(\mathbb{Z}) = \mathbb{Z}/691$ does not vanish.
- The result follows from the inclusion-exclusion principle.

## Some speculations and questions

- How to define the Tamagawa measure on $B(\mathbb{Q}_p)$?
- What is the relation between $\mu(B(\mathbb{Q}_p))$ and the local $L$-factor at $p$.
- How to define $Ш(M)$ for general $M$.

Thank you!