

Broadcasting solutions on multilayer networks of phase oscillators

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Multilayer networks have been used to model and investigate the dynamical behavior of a variety of systems. On one hand, this approach offers a more realistic and rich modeling option than single-layer networks. On the other hand, it leads to an intrinsic difficulty in analyzing the system. Here, we introduce an approach to investigate the dynamics of Kuramoto oscillators on multilayer networks by considering the join of matrices, which represent the intra-layer and inter-layer connections. This approach provides a “reduced” representation of the original, multilayered system, where both systems have an equivalent dynamics. Then, we can find solutions for the reduced system and broadcast them to the multilayer network. Moreover, using the same idea we can investigate the stability of these states, where we can obtain information on the Jacobian of the multilayer system by analyzing the reduced one. This approach is general for arbitrary connection scheme between nodes within the same layer. Finally, our work opens the possibility of studying the dynamics of multilayer networks using a simpler representation, then leading to a better understanding of dynamical behavior of these systems.

I. INTRODUCTION

Networked systems have been used to model and understand collective behaviors of several systems [1–3]. Examples can be found spanning from neuroscience [4] to ecology [5] to physical systems, like power-grids [6–8]. This approach can be understood as single, undirected small networks, or network of networks, or even networks with higher-order interactions [9, 10]. In this context, one important approach to improve the modeling of real systems is given by multilayer networks, where two levels of connections are considering: connections within each layer (intra-layer), and connections between layers (inter-layer) [11, 12]. This approach allows the study of important systems such as broadcasting processes [13], animal behavior [14], neural dynamics [4, 15], and synchronization phenomena [16, 17]. On the other hand, the addition of an extra level of connections makes the analysis process more complicated, especially for analytical and mechanistic insights. In this paper, we introduce an approach that offers a simplified way to analyze the dynamics of multilayer networks, which allows us to obtain equilibrium points and analyze the stability for a large class of such systems.

Based on the framework developed in [18–20], we can represent the multilayer system as a join of matrices containing information about the intra-layer and inter-layer connections. We then introduce a reduced representation of the system, which has a clear connection with the join matrix that represents the multilayer network. Thus, this reduced system allows us to study the dynamics of the multilayer system in a simpler way. Then, we can find solutions for the reduced system and broadcast them to the multilayer networks. This construction offers a direct way to find equilibrium points in multilayer networks with an arbitrary number of layers and arbitrary connection scheme for nodes within the same layer (intra-layer connections).

Here, the individual dynamics of each node is modelled by the Kuramoto model [21–23], so each node is represented by a phase oscillator. The Kuramoto model is a central model for the studying of synchronization and the collective behavior of several systems. It has been used in problems spanning from social interactions to biology and physics [2, 24–27]. Particularly, a variety of synchronization phenomena and spatiotemporal patterns has been observed in Kuramoto systems: phase synchronization, partial and cluster synchronization, remote synchronization, twisted states, chimera states, and Bellerophon states [23, 28–34].

Furthermore, by considering the properties of joins of matrices and the idea of representing the multilayer network by the reduced system, we can investigate into the stability of equilibrium points for Kuramoto oscillators on multilayer networks. When the solutions for the multilayer systems are obtained through the broadcasting from the reduced system, we have found a relation between the Jacobian of both systems. In this sense, we can evaluate the spectrum

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of a simpler matrix - representing the reduced systems - and obtain information on the stability of the equilibrium points for the multilayer network.

In this paper, we first introduce Kuramoto oscillators on multilayer networks and expose our approach with the reduced system and also show how we broadcast solutions to the multilayer system - Sec. II; we extend the approach to the linear stability analysis of multilayer networks - in Sec. III; in addition to this, we show numerical simulations and examples of Kuramoto multilayer networks and their reduced version, which corroborate the main approach introduced in this paper - Sec. IV; finally, we discuss the results presented here and their implications, and draw our conclusions - Sec. V.

II. KURAMOTO OSCILLATORS ON MULTILAYER NETWORKS

The Kuramoto model [21–23] can be defined as the dynamical system governed by the equation

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i), \quad (\text{II.1})$$

where $\theta_i \in [-\pi, \pi]$ is the phase of the i^{th} oscillator, ω_i is its natural frequency, N is the number of oscillators in the system, and A_{ij} are the elements of the weighted adjacency matrix that represents the connections of this system. That is, $A_{ij} = 0$ if nodes i and j are not connected, and $A_{ij} > 0$ if nodes i and j are connected. Here, we consider the case where all oscillators have the same natural frequency $\omega_i = \omega$ for all $i \in [1, N]$.

A. Multilayer Kuramoto models

For multilayer systems, we can consider two levels of interactions: intra-layer (within each layer) and inter-layer (between layers) connections. In this case, the Kuramoto model for the i^{th} oscillator in the l^{th} layer can be written as:

$$\frac{d(\boldsymbol{\theta}_l)_i}{dt} = \omega + \underbrace{\sum_{j=1}^N (\mathbf{A}_{ll})_{ij} \sin((\boldsymbol{\theta}_l)_j - (\boldsymbol{\theta}_l)_i)}_{\text{intra-layer}} + \underbrace{\sum_{k=1, k \neq l}^M \sum_{j=1}^N (\mathbf{A}_{lk})_{ij} \sin((\boldsymbol{\theta}_k)_j - (\boldsymbol{\theta}_l)_i)}_{\text{inter-layer}}, \quad (\text{II.2})$$

where $i, j \in [1, N]$ and $l, k \in [1, M]$, M being the number of layers and N the number of nodes in each one. Here, the coupling strength between two oscillators can assume different values based on the layers these oscillators are in. Moreover, it is important to emphasize that the intra-layer coupling considers the coupling between oscillators within a given layer; the inter-layer coupling considers the coupling between oscillators in different layers.

In this paper, on the other hand, we consider a different representation of a multilayer system. Here, we use properties of joins of matrices to describe Kuramoto oscillators on multilayer networks as:

$$\frac{d\theta_i}{dt} = \omega + \sum_{j=1}^{NM} A_{ij} \sin(\theta_j - \theta_i), \quad (\text{II.3})$$

where now $i \in [1, NM]$, such that:

$$\boldsymbol{\theta} = (\underbrace{\theta_1, \theta_2, \dots, \theta_N}_{1^{\text{st}} \text{layer}}, \underbrace{\theta_{N+1}, \theta_{N+2}, \dots, \theta_{2N}}_{2^{\text{nd}} \text{layer}}, \dots, \underbrace{\theta_{N(M-1)+1}, \theta_{N(M-1)+2}, \dots, \theta_{NM}}_{M^{\text{th}} \text{layer}}). \quad (\text{II.4})$$

The adjacency matrix describing the multilayer system is the $NM \times NM$ matrix:

$$\mathbf{A} = \left(\begin{array}{c|c|c|c} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1M} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2M} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{A}_{M1} & \mathbf{A}_{M2} & \cdots & \mathbf{A}_{MM} \end{array} \right), \quad (\text{II.5})$$

where \mathbf{A}_{lk} is a $N \times N$ matrix, which describes the connections between the oscillators in the l^{th} and k^{th} layers. In case $l = k$, the block represents the intra-layer connections in the l^{th} layer. In case $l \neq k$, the block represents the

inter-layer connections between oscillators in layers l and k . Note that in our treatment these matrices absorb the information about the coupling strength. For simplicity, we consider here the case in which the coupling strength between oscillators in two fixed layers is constant. However, this coupling need not be the same for all layers.

For details on the joins of matrices and on the mathematical background of our approach, see [18–20].

B. Reduced Kuramoto models and broadcasted solutions

In this paper we focus on networks satisfying a regularity condition: namely that the off-diagonal blocks \mathbf{A}_{lk} for $l \neq k$ have the same row sum (that is, they are row-regular, or semimagic, matrices, [19]). In terms of the Kuramoto model, this assumption means that, for any pair of layers, each oscillator in the first layer is connected with the same number of oscillators in the other layer. To investigate the dynamics of such a system, we introduce an auxiliary reduced $M \times M$ network, in which each layer of the original system is condensed into one global oscillator. More concretely, the reduced system is described by the adjacency matrix

$$\bar{\mathbf{A}} = \begin{pmatrix} r_{\mathbf{A}_{11}} & r_{\mathbf{A}_{12}} & \cdots & r_{\mathbf{A}_{1M}} \\ r_{\mathbf{A}_{21}} & r_{\mathbf{A}_{22}} & \cdots & r_{\mathbf{A}_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\mathbf{A}_{M1}} & r_{\mathbf{A}_{M2}} & \cdots & r_{\mathbf{A}_{MM}} \end{pmatrix}, \quad (\text{II.6})$$

where $r_{\mathbf{A}_{lk}}$ is the row sum of \mathbf{A}_{lk} . At this point, it is important to emphasize that these matrices have information about the coupling strength for the intra-layer and inter-layer case, thus the line sum captures this information. We also remark that the row sums $r_{\mathbf{A}_{lk}}$ capture the information about the inter-layer and intra-layer coupling strengths; this information is therefore present in the reduced system.

We can now define the reduced Kuramoto model:

$$\frac{d\bar{\theta}_i}{dt} = \omega + \sum_{j=1}^M \bar{A}_{ij} \sin(\bar{\theta}_j - \bar{\theta}_i), \quad (\text{II.7})$$

where now $i \in [1, M]$. We emphasize that the main diagonal of $\bar{\mathbf{A}}$ does not affect the dynamics of the oscillators, since $\sin(\bar{\theta}_i - \bar{\theta}_i) = 0$. Therefore, in the following, we often make the harmless identification

$$\bar{\mathbf{A}} = \begin{pmatrix} r_{\mathbf{A}_{11}} & r_{\mathbf{A}_{12}} & \cdots & r_{\mathbf{A}_{1M}} \\ r_{\mathbf{A}_{21}} & r_{\mathbf{A}_{22}} & \cdots & r_{\mathbf{A}_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\mathbf{A}_{M1}} & r_{\mathbf{A}_{M2}} & \cdots & r_{\mathbf{A}_{MM}} \end{pmatrix} \simeq \begin{pmatrix} 0 & r_{\mathbf{A}_{12}} & \cdots & r_{\mathbf{A}_{1M}} \\ r_{\mathbf{A}_{21}} & 0 & \cdots & r_{\mathbf{A}_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\mathbf{A}_{M1}} & r_{\mathbf{A}_{M2}} & \cdots & 0 \end{pmatrix}. \quad (\text{II.8})$$

The Kuramoto network on the multilayer system - Eq. (II.3) - represented by \mathbf{A} - Eq. (II.5) - and the Kuramoto network on the reduced system - Eq. (II.7) - represented by $\bar{\mathbf{A}}$ - Eq. (II.6) - have the same dynamics. Thus, we can use the reduced system to better understand the dynamics on the more complicated, multilayer network.

More precisely, solutions for the reduced system can be *broadcast* to the original multilayer network: given a solution of the reduced Kuramoto model:

$$\bar{\boldsymbol{\theta}}^* = (\bar{\theta}_1^*, \bar{\theta}_2^*, \dots, \bar{\theta}_M^*), \quad (\text{II.9})$$

then

$$\boldsymbol{\theta}^* = (\underbrace{\bar{\theta}_1^*, \bar{\theta}_1^*, \dots, \bar{\theta}_1^*}_{1^{\text{st}} \text{layer}}, \underbrace{\bar{\theta}_2^*, \bar{\theta}_2^*, \dots, \bar{\theta}_2^*}_{2^{\text{nd}} \text{layer}}, \dots, \underbrace{\bar{\theta}_M^*, \bar{\theta}_M^*, \dots, \bar{\theta}_M^*}_{M^{\text{th}} \text{layer}}), \quad (\text{II.10})$$

is the corresponding *broadcasting* solution of the multilayer system described by \mathbf{A} .

Moreover, we note the dependence of the dynamics on the coupling strength. The inter-layer coupling controlled by ϵ is also presented in the reduced system, see Eq. (II.6). So, if we consider the same initial state, both systems evolve to the same final state. However, the stronger the coupling, the faster is the transition, which we summarize in the following proposition whose proof follows from direct calculations (valid for nonzero coupling only):

Proposition II.1. *Suppose $\bar{\boldsymbol{\theta}}^* = (\bar{\theta}_1^*, \bar{\theta}_2^*, \dots, \bar{\theta}_M^*)$ is a solution of the Kuramoto model*

$$\dot{\theta}_i = \epsilon \sum_j A_{ij} \sin(\theta_j - \theta_i),$$

with initial condition $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$. Let $\psi(t) = \bar{\boldsymbol{\theta}}^*(\frac{\epsilon'}{\epsilon}t)$. Then $\psi(t)$ is a solution of the Kuramoto model

$$\dot{\psi}_i = \epsilon' \sum_j A_{ij} \sin(\psi_j - \psi_i),$$

with the same initial condition $\psi(0) = \boldsymbol{\theta}_0$.

Remark II.2. We note that the idea presented in this section about the correspondence between the multilayer network and the reduced system is valid even when the layers have a different number of nodes. In this case, \mathbf{A} would be a $\mathcal{M} \times \mathcal{M}$ matrix, where $\mathcal{M} = N_1 + N_2 + \dots + N_M$. We presented here the version where the layers have the same number of nodes N for a simplicity purpose.

C. Kuramoto order parameter

In order to characterize the dynamical behavior of the system, we use the Kuramoto order parameter [22, 23]. The order parameter for the multilayer system is defined as

$$R(t) = \frac{1}{NM} \left| \sum_{j=1}^{NM} \exp(i\theta_j(t)) \right|, \quad (\text{II.11})$$

where $\theta(t)$ is given by Eq. (II.3). Here, $R(t) = 1$ means that all oscillators in all layers have the same phase at a given time t , which is defined as phase synchronization. If the oscillators are not phase synchronized, the order parameter is less than 1. For the asynchronous behavior, R assumes residual values. When the system is on a phase-locking solution, also called a twisted state, $R(t) = 0$.

We also measure the level of synchronization of the Kuramoto network on the reduced representation using the Kuramoto order parameter. In this case, the order parameter is written as

$$\bar{R}(t) = \frac{1}{M} \left| \sum_{j=1}^M \exp(i\bar{\theta}_j(t)) \right|, \quad (\text{II.12})$$

where $\bar{R}(t) = 1$ means the reduced system is phase synchronized at time t , an asynchronous behavior leads to a residual value of \bar{R} , and if the reduced system has a phase-locking solution (“twisted state”), then $\bar{R} = 0$.

By the definition of the broadcast solution and direct calculations, we have the following proposition.

Proposition II.3. Suppose that $\bar{\boldsymbol{\theta}}^*$ is a solution of the Kuramoto model on $\bar{\mathbf{A}}$ and $\boldsymbol{\theta}^*$ is the corresponding broadcasting solution of the Kuramoto model on \mathbf{A} . Then the two systems have the same Kuramoto order parameter for all times: $R(t) = \bar{R}(t)$, $t \geq 0$.

III. STABILITY OF EQUILIBRIUM POINTS

We now propose to analyze the existence and stability of some equilibria for multilayer networks through our knowledge of the existence and stability properties of equilibria in the reduced system. Of course, there is loss in complexity when considering the reduced system and we cannot guarantee that its analysis provides a full picture of the whole multilayer system. However, we will show that some crucial information can be obtained from the reduced system by simply broadcasting a known solution thereof to the multilayer network.

In this section, we require that connections between each pair of connected layers are dense, which we approximate by a mean-field (or all-to-all) coupling. In other words, we assume that if layers l and k are connected, then each node in layer l is connected to each node in layer k . Whereas these assumptions formally restrict the applicability of our method, they still cover a rather wide class of networked systems. It is furthermore quite standard to consider identical coupling strength within a networked system and to approximate unknown or uncertain couplings by mean-field interactions [31, 35–38]. Formally, the above assumptions translate as:

1. \mathbf{A}_{ll} is the adjacency matrix of an undirected, connected, weighted graph with positive weights.
2. For $l \neq k$, $\mathbf{A}_{lk} = \epsilon_{lk} \mathbf{1}_N$ where $\mathbf{1}_N$ is a matrix $N \times N$ with all entries equal to 1.

As observed in the previous section, the (simpler) reduced system can be used to study the dynamics of the multilayer network. In particular, equilibrium points of the reduced system can be broadcast to equilibrium points of the multilayer one.

However, for stability analysis, we cannot extend the idea in a straightforward way. The reason is that a neighborhood of an equilibrium point for the multilayer network $\boldsymbol{\theta}^*$ is NM -dimensional, while a neighborhood of an equilibrium point for the reduced system $\bar{\boldsymbol{\theta}}^*$ is M -dimensional. Broadcasting the neighborhood of $\bar{\boldsymbol{\theta}}^*$

$$B_\varepsilon(\bar{\boldsymbol{\theta}}^*) = \{ \bar{\boldsymbol{\theta}}^* + \varepsilon \mid \|\varepsilon\|_2 < \varepsilon \}, \quad (\text{III.1})$$

to the multilayer system yields an M -dimensional subset of the NM -dimensional neighborhood $B_\varepsilon(\boldsymbol{\theta})$.

In order to perform a stability analysis of the broadcasted fixed point, one needs a more in-depth analysis of the Jacobian matrix. Fortunately, we will show that such an analysis can still be performed, and leverages our knowledge of the fixed point for the reduced system.

In order to study on the stability of these solution we can use the Jacobian of each system. Here we denote the Jacobian for the multilayer system at the equilibrium point $\boldsymbol{\theta}^*$ as $J(\boldsymbol{\theta}^*) = J_{\mathbf{A}}(\boldsymbol{\theta}^*)$, and the Jacobian for the reduced system at the equilibrium point $\bar{\boldsymbol{\theta}}^*$ as $J(\bar{\boldsymbol{\theta}}^*) = J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*)$.

We first recall the matrix $\bar{\mathbf{A}}$, defined in Eq. (II.6). Then, we notice that main diagonal of $\bar{\mathbf{A}}$ does not affect the dynamics of the system (since $\sin(\bar{\theta}_i - \bar{\theta}_i) = 0$), so we can write the matrix as

$$\bar{\mathbf{A}} = \begin{pmatrix} 0 & r_{\mathbf{A}_{12}} & \cdots & r_{\mathbf{A}_{1M}} \\ r_{\mathbf{A}_{21}} & 0 & \cdots & r_{\mathbf{A}_{2M}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{\mathbf{A}_{M1}} & r_{\mathbf{A}_{M2}} & \cdots & 0 \end{pmatrix}, \quad (\text{III.2})$$

where, by definition, $r_{\mathbf{A}_{ij}} = N\epsilon_{ij}$. Based on this matrix, we can now write the Jacobian for the reduced system as

$$J(\bar{\boldsymbol{\theta}}^*) = \begin{pmatrix} -\lambda_1 & r_{\mathbf{A}_{12}} \cos(\bar{\theta}_2^* - \bar{\theta}_1^*) & \cdots & r_{\mathbf{A}_{1M}} \cos(\bar{\theta}_M^* - \bar{\theta}_1^*) \\ r_{\mathbf{A}_{21}} \cos(\bar{\theta}_1^* - \bar{\theta}_2^*) & -\lambda_2 & \cdots & r_{\mathbf{A}_{2M}} \cos(\bar{\theta}_M^* - \bar{\theta}_2^*) \\ \vdots & \vdots & \ddots & \vdots \\ r_{\mathbf{A}_{M1}} \cos(\bar{\theta}_1^* - \bar{\theta}_M^*) & r_{\mathbf{A}_{M2}} \cos(\bar{\theta}_2^* - \bar{\theta}_M^*) & \cdots & -\lambda_M \end{pmatrix}, \quad (\text{III.3})$$

where

$$\lambda_i = \sum_{j=1, j \neq i}^M r_{\mathbf{A}_{ij}} \cos(\bar{\theta}_j^* - \bar{\theta}_i^*). \quad (\text{III.4})$$

In particular, the above Jacobians are semimagic square matrices with line sum zero.

In order to compute the Jacobian for the multilayer system $J(\boldsymbol{\theta}^*)$, we first recall the definition of the Laplacian. For an undirected weighted graph H with adjacency matrix \mathbf{B}_H and degree matrix \mathbf{D}_H , the Laplacian is defined as:

$$\mathbf{L}_H = \mathbf{D}_H - \mathbf{B}_H. \quad (\text{III.5})$$

We note that, as long as H is undirected, \mathbf{L}_H is always a symmetric semi-magic square matrix (with row sum equals to zero). Regarding the spectrum of \mathbf{L}_H , we have the following proposition.

Proposition III.1. *Let \mathbf{L}_H be the weighted Laplacian matrix of an undirected, connected, weighted graph H . Then \mathbf{L}_H has a unique zero eigenvalue and all other eigenvalues are strictly positive.*

Proof. By Gershgorin's Discs Theorem [39, Theroem 6.1.1], all eigenvalues of \mathbf{L}_H are nonnegative. Note also that the eigenvectors of \mathbf{L}_H form an orthonormal basis of \mathbb{R}^n , because \mathbf{L}_H is symmetric. Now assume that \mathbf{v} is an eigenvector of \mathbf{L}_H associated to the eigenvalue 0. Then,

$$0 = \mathbf{v}^\top \mathbf{L}_H \mathbf{v} = \sum_{i < j} a_{ij} (v_i - v_j)^2, \quad (\text{III.6})$$

where a_{ij} is the weight of the edge linking nodes i and j . Each term in the sum above is necessarily zero. Therefore, for any par of nodes (i, j) that are connected, $v_i = v_j$. As the graph is connected, the eigenvector \mathbf{v} is necessarily a multiple of the vector $(1, \dots, 1)^\top$. Hence \mathbf{L}_H has a unique zero eigenvalue and all other eigenvalues are strictly positive. \square

We now compute the Jacobian $J(\boldsymbol{\theta}^*)$. By definition, for two indices (i_1, j_1) such that $i_1 \in V(G_i), j_1 \in V(G_j)$ where $i \neq j$ then

$$[J(\boldsymbol{\theta}^*)]_{i_1, j_1} = A_{i_1, j_1} \cos(\bar{\theta}_j^* - \bar{\theta}_i^*) = \epsilon_{ij} \cos(\bar{\theta}_j^* - \bar{\theta}_i^*). \quad (\text{III.7})$$

Here, G_i is the graph representing the i^{th} layer, and ϵ_{ij} is the coupling strength between nodes in the layer i and j . We emphasize that for the stability analysis, the coupling between layers is considered uniform, i.e., oscillators in layer i are connected to oscillators in layer j with the same coupling strength.

If $i_1, i_2 \in V(G_i)$ and $i_1 \neq i_2$, then

$$[J(\boldsymbol{\theta}^*)]_{i_1, i_2} = (\mathbf{A}_{ii})_{i_1, i_2} \cos(\bar{\theta}_i^* - \bar{\theta}_i^*) = (\mathbf{A}_{ii})_{i_1, i_2}. \quad (\text{III.8})$$

Finally, we need to consider the case $i_1 = i_2$ and $i_1 \in V(G_i)$. For this part, we observe that $J(\boldsymbol{\theta}^*)$ is a semimagic square matrix with line sum equal to zero. We can then see that

$$[J(\boldsymbol{\theta}^*)]_{i_1, i_1} = -\lambda_i - \deg(i_1). \quad (\text{III.9})$$

with $\deg(i_1)$ denoting the weighed degree of node i_1 .

By combining these facts, we can write the Jacobian for the multilayer system at the equilibrium point $\boldsymbol{\theta}^*$ as:

$$J(\boldsymbol{\theta}^*) = J_{\mathbf{A}}(\boldsymbol{\theta}^*) = \begin{pmatrix} -\lambda_1 \mathbf{I} - \mathbf{L}_{\mathbf{A}_{11}} & \epsilon_{12} \cos(\bar{\theta}_2^* - \bar{\theta}_1^*) \mathbf{1} & \cdots & \epsilon_{1M} \cos(\bar{\theta}_M^* - \bar{\theta}_1^*) \mathbf{1} \\ \epsilon_{21} \cos(\bar{\theta}_1^* - \bar{\theta}_2^*) \mathbf{1} & -\lambda_2 \mathbf{I} - \mathbf{L}_{\mathbf{A}_{22}} & \cdots & \epsilon_{2M} \cos(\bar{\theta}_M^* - \bar{\theta}_2^*) \mathbf{1} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{M1} \cos(\bar{\theta}_1^* - \bar{\theta}_M^*) \mathbf{1} & \epsilon_{M2} \cos(\bar{\theta}_2^* - \bar{\theta}_M^*) \mathbf{1} & \cdots & -\lambda_M \mathbf{I} - \mathbf{L}_{\mathbf{A}_{MM}} \end{pmatrix}. \quad (\text{III.10})$$

where, we recall that ϵ_{ij} is the coupling between oscillators in layer i and j .

One realizes, in particular, that $J(\boldsymbol{\theta}^*)$ is a joined union of semimagic square matrices (see [18, 19] for details). Furthermore, we have the following equality

$$\overline{J_{\mathbf{A}}(\boldsymbol{\theta}^*)} = J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*). \quad (\text{III.11})$$

It is important to emphasize that Eq. (III.11) is valid if the equilibrium point $\boldsymbol{\theta}^*$ of the multilayer network is obtained through the broadcasting of the solution $\bar{\boldsymbol{\theta}}^*$ of the reduced system - see Eqs. (II.9) and (II.10).

We summarize these ideas in the following proposition:

Proposition III.2. *Suppose $\bar{\boldsymbol{\theta}}^*$ is an equilibrium point of the Kuramoto model associated with the reduced system $(\bar{\mathbf{A}})$. Then $\boldsymbol{\theta}^*$ is an equilibrium point of the Kuramoto model on the multilayer system (\mathbf{A}) following Eq. (II.10). In addition to that, suppose that the matrices representing the inter-layer connections $(\mathbf{A}_{ij}, i \neq j)$ have all entries the same. The Jacobian $J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*)$ and $J_{\mathbf{A}}(\boldsymbol{\theta}^*)$ are given by Eq. (III.3) and Eq. (III.10), respectively. Furthermore, we have the following relation*

$$\overline{J_{\mathbf{A}}(\boldsymbol{\theta}^*)} = J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*).$$

For the next proposition, we need to introduce a couple of notations. For a set $S = \{s_1, s_2, \dots, s_m\} \subset \mathbb{C}^m$ and $a, b \in \mathbb{C}$, we denote

$$aS - b = \{as_1 - b, as_2 - b, \dots, as_m - b\}.$$

We also denote by $S \setminus s$ the set difference of S by s . By [19, Theorem 3.3], we have the following statement about the spectrum of $J_{\mathbf{A}}(\boldsymbol{\theta}^*)$:

Proposition III.3. *The spectrum of $J_{\mathbf{A}}(\boldsymbol{\theta}^*)$ can be defined in terms of:*

- The spectra of the Laplacian matrices of each layer, $\mathbf{L}_{\mathbf{A}_i}$;
- The Jacobian of the reduced system, $J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*)$.

Namely,

$$\text{Spec}(\mathbf{J}_{\mathbf{A}}(\boldsymbol{\theta}^*)) = \bigcup_{i=1}^d \{(-\text{Spec}(\mathbf{L}_{bmA_i}) \setminus \{0\}) - \lambda_i\} \cup \text{Spec}(J_{\bar{\mathbf{A}}}(\bar{\boldsymbol{\theta}}^*)). \quad (\text{III.12})$$

In Sec. II, we showed how we can use the reduced system to broadcast solutions to the multilayer Kuramoto network. We can now study the stability of these solutions.

Proposition III.4. *The fixed point $\bar{\theta}^*$ for the reduced system is linearly stable if and only if the corresponding broadcasted fixed point for the multilayer system is linearly stable.*

Proof. Assume that $\bar{\theta}^*$ is linearly stable, then $J_{\bar{A}}(\bar{\theta}^*)$ is symmetric negative-semidefinite. By Sylvester's criterion [40, Theorem 3.3.12], we know that $\lambda_i \geq 0$. Furthermore, by Proposition III.1, we know that all eigenvalues of L_{A_i} , except 0, are positive. These two facts show that all elements in

$$\bigcup_{i=1}^d \{(-\text{Spec}(L_{A_i}) \setminus \{0\}) - \lambda_i\},$$

are negative. By Proposition III.3 and the assumption that $\bar{\theta}^*$ is linearly stable, we conclude that all eigenvalues of $J_{\bar{A}}(\theta^*)$, other than 0, must be negative. Hence θ^* is linearly stable.

The other direction is straightforward. Indeed, by Proposition III.3, the spectrum of $J_{\bar{A}}(\bar{\theta}^*)$ is a subset of the spectrum of $J_A(\theta^*)$. Therefore, if θ^* is linearly stable, then $\bar{\theta}^*$ is linearly stable as well. \square

IV. NUMERICAL SIMULATIONS AND EXAMPLES

In this section, we show examples and numerical simulations of Kuramoto networks on multilayer systems and their respective reduced version that we introduce in this paper.

Figure 1 shows a graphic representation of this approach. We study Kuramoto oscillators on multilayer networks, where each layer is composed of N nodes and has an internal connection structure and coupling (intra-coupling). These layers (M in total) are then coupled with an external connection structure (inter-coupling), thus leading to a two-levels coupling scheme (left). In this paper, we introduced a reduced “inter-layers” representation that allows us to investigate on the dynamics of the multilayer system in a simplified way. In this simpler representation, each layer is given by a node, which is connected to other nodes following the inter-coupling scheme (right). As shown in Sec. II, the reduced system and the multilayer network have equivalent dynamics and we can broadcast solutions from the former to the latter.

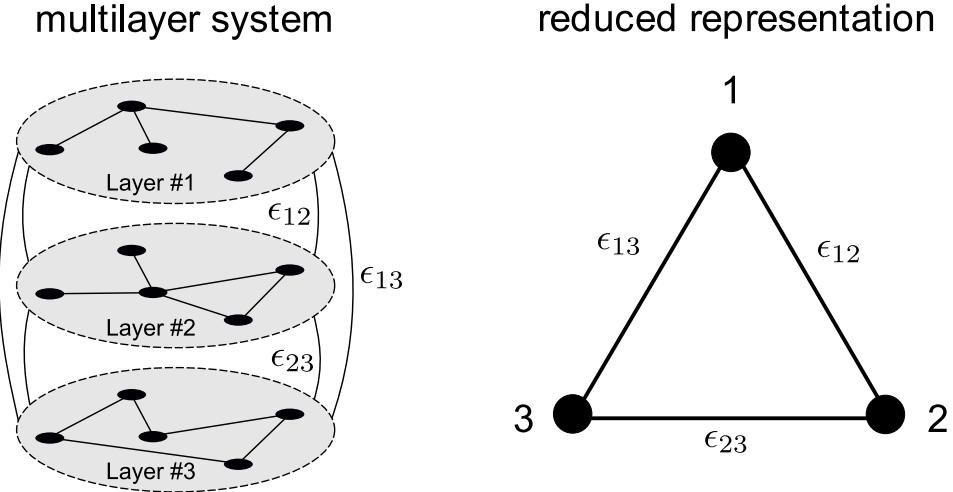


Figure 1. Graphic representation of the system. The multilayer network is given by M layers with N nodes each one. Here, we use a two-levels system, where each layer has internal connections in addition to the inter-layer connections. Our approach allows us to study the dynamics of these system using a reduced $M \times M$ system. This figure depicts an example with $M = 3$ layers.

A. Phase synchronization

We first consider a multilayer network with $M = 3$, where each layer is composed of $N = 100$ nodes. The internal connection structure of each layer is given by ring with periodic boundary conditions, described by a circulant matrix

where each node has 10 connections to each direction. The adjacency matrix for the internal coupling is given by $A_{ij, \text{intra}} = 1$ if nodes i and j are connected, and by $A_{ij, \text{intra}} = 0$ otherwise. The inter-layers coupling scheme is given by a complete graph, where all nodes in one layer are connected to all nodes in another layer with coupling strength given by ϵ .

The first example is depicted in Fig. 2. Here, the initial state is given by randomly selected phases for each layer, which are represented in the unitary circle in color-code (Fig. 2a). All oscillators in each layer have the same phase, therefore each layer is phase synchronized, but the entire, multilayer system is not – in the beginning of the simulation. As time evolves, the multilayer system transitions to phase synchrony due to the inter-layer coupling, which leads all oscillators in all layers to depict the same phase in the final state (Fig. 2b).

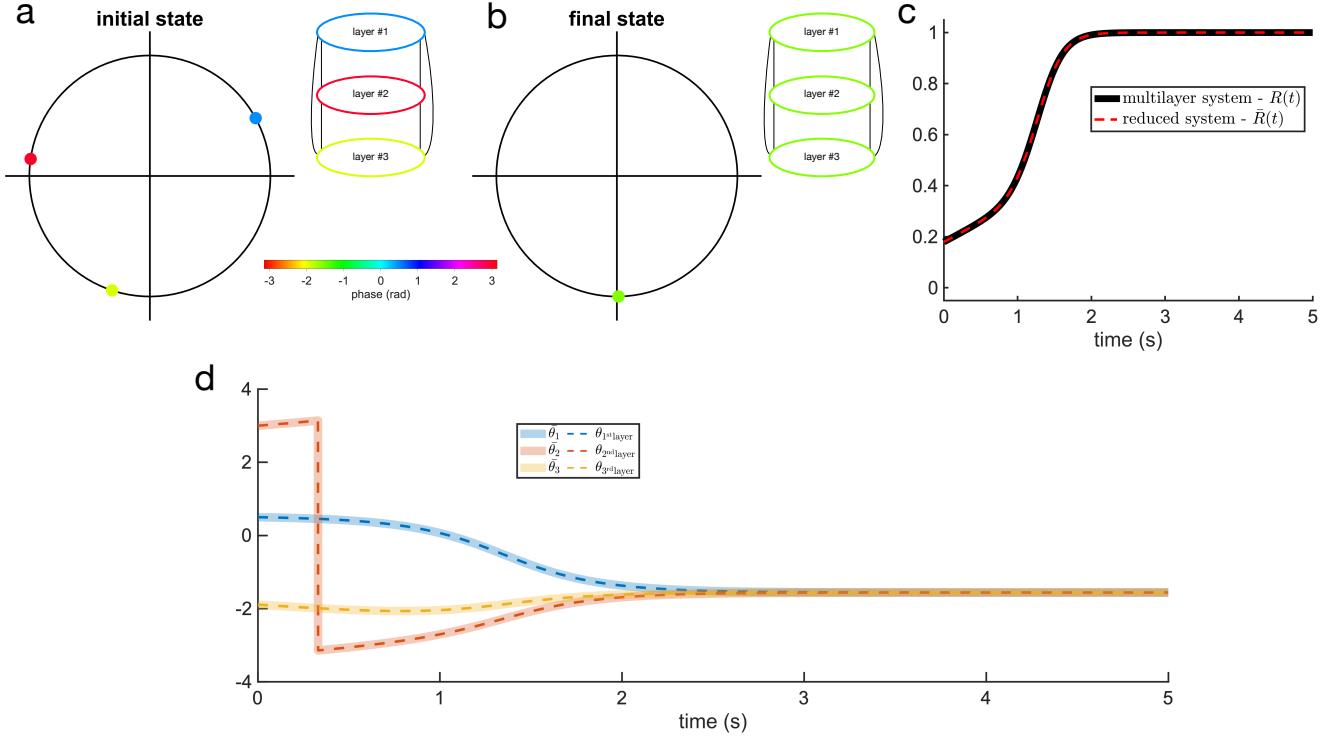


Figure 2. Dynamics on a 3 layers system and its reduced system with random initial conditions. In the beginning of the simulation, each layer is phase synchronized but in different phases (a), therefore the multilayer system is nonsynchronized. As time evolves, due to the inter-layer coupling, the internal dynamics changes and the multilayer system transition to phase synchronization (b). Here, the dynamics of the multilayer system and the reduced one are the same, which is expressed by the Kuramoto order parameter $R(t)$ (c). The trajectories for both the reduced system and the multilayer one are shown in (d), where we can observe that (i) these system have an equivalent dynamics; and (ii) the network transitions to phase synchrony as time evolves.

The change on the dynamics over time is represented by the Kuramoto order parameter (Fig. 2c), which is evaluated through Eq. (II.11) for the multilayer network and through Eq. (II.12) for the reduced system. Here, one can observe that both systems have an equivalent dynamics: at $t = 0$, R has a low value due to the random phases on the initial state; due to the coupling, $R(t)$ increases as time evolves and, at $t \approx 2$, the order parameter reaches the unity once both systems reach phase synchronization.

In order to show details on the dynamical equivalence between the reduced system and the multilayer network, we plot the trajectories given by the phases of the oscillators in Fig. 2d. Here, the phases in the reduced systems are represented by the solid-shaded lines, and the phases of the oscillators in the multilayer network are given by the dashed lines. This result emphasizes that: (i) these system have an equivalent dynamics, and also (ii) the network transitions to phase synchrony as time evolves, since the oscillators converge to the same phase.

B. Unstable twisted states

We use the same system to analyze a different kind of dynamical behavior: phase-locking states, where a constant phase difference is observed across oscillators [33]. This kind of state is also known as “twisted states”, and, for a

network with M units, the p^{th} twisted state is given by:

$$\boldsymbol{\theta}^{(p)} = \left(0, \frac{-2\pi p}{M}, \dots, \frac{-2\pi p(M-1)}{M} \right). \quad (\text{IV.1})$$

We use it equation to obtain the twisted states for the reduced system and then use the broadcasting approach to extend this to the multilayer system.

We consider here the same network: a 3 layered system, where each layer is composed of $N = 100$ nodes, and the internal connection structure of each layer is given by ring with periodic boundary conditions. The reduced system is then given by a 3 nodes networks, which are coupled in a all-to-all scheme. A possible phase-locking state for this network, following Eq. (IV.1) is represented in Fig. 3a, where constant difference of $2\pi/3$ is observed. As stated before, all oscillators within a layer have the same phase, so each layer is phase synchronized, but the entire system, in this particular case, has $R = 0$, given the phase-locking solution.

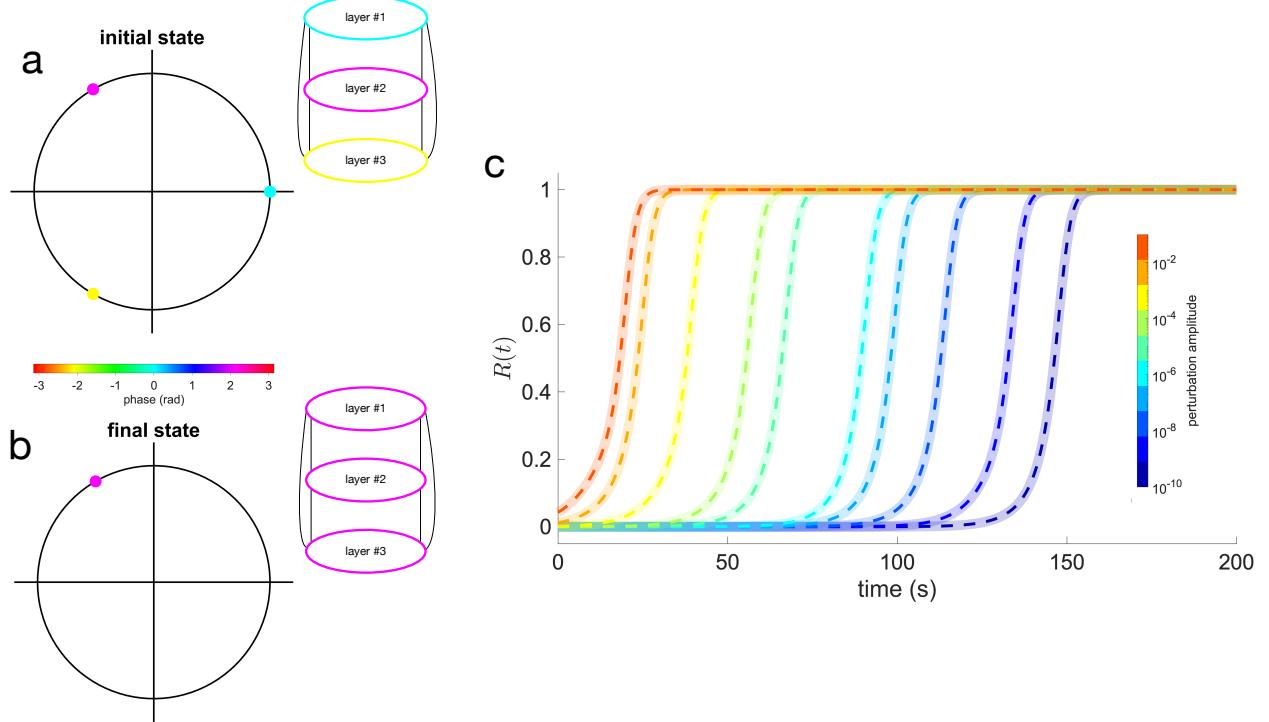


Figure 3. Dynamics on a 3 layers systems and its reduced system on a phase-locking state. We consider a twisted state for the reduced system following Eq. (IV.1) with $M = 3$ and $p = 1$. This leads to a initial state where a constant phase difference is observed across nodes (a). However, this twisted state is not stable, so as time evolves, a transition to phase synchrony is observed, and all oscillators have the same phase in the final state (b). The time to this transition occurs, however, depends on the perturbation applied to the system. We considered different perturbations amplitudes (color-code) and analyze the Kuramoto order parameter as a function of time for the reduced system (solid-shaded lines) and for the multilayer network (dashed lines) (c). The bigger the perturbation the faster the transition and our approach shows a perfect match between the dynamics of the reduced system and the multilayered system.

This state, however, is not stable, so small perturbation, e.g. due to the coupling or perturbation on the initial state, can lead the system to transition to phase synchronization (Fig. 3b). Our approach is able to capture the details on this transition. To show this point, we perform a detailed simulation protocol, where different perturbations are applied to the initial state. This perturbation is given by a uniform, random distribution of phases, multiplied by an amplitude factor. Mathematically, the perturbation can be described by: $\mathcal{A}\mathcal{U}(-\pi, \pi)$, where \mathcal{A} is the perturbation amplitude. Figure 3c depicts the Kuramoto order parameter as a function of time when different amplitude are considered, which are represented in color-code. Here, the order parameter for the reduced system is shown in the solid-shaded lines, and the order parameter for the multilayer systems is shown in the dashed lines. We observe that the smaller the perturbation, the longer it takes to the phase-locking state to transition to phase synchrony. We emphasize that our reduced approach has an equivalent dynamics to the multilayered system, so we observe a perfect match.

C. Higher number of layers

At this point, it is important to emphasize that the approach introduced here through the reduced system and the broadcasting process is general to study multilayer system with an arbitrary number of layers and internal connection scheme. To show this point, we then analyze a multilayer system composed of $M = 50$ layers, where each layer is given by a random network with $N = 100$ nodes. Here, all layers are connected, such that the reduced system is given by a complete graph with $M = 50$ nodes.

At first, we consider the case of random initial conditions for this system. Figure 4a show the trajectories for the reduced (solid-shaded lines) and the multilayer systems (dashed lines) based on the phases of the oscillators ($\bar{\theta}$ and θ). Due to the random initial conditions (see the inner panel for initial state), the systems are desynchronized in the beginning of the simulation. However, as time evolves, phase synchronization is reached and all oscillators have the same phase (see inner panel for final state). This result also shows the equivalence in the dynamical behavior of both systems.

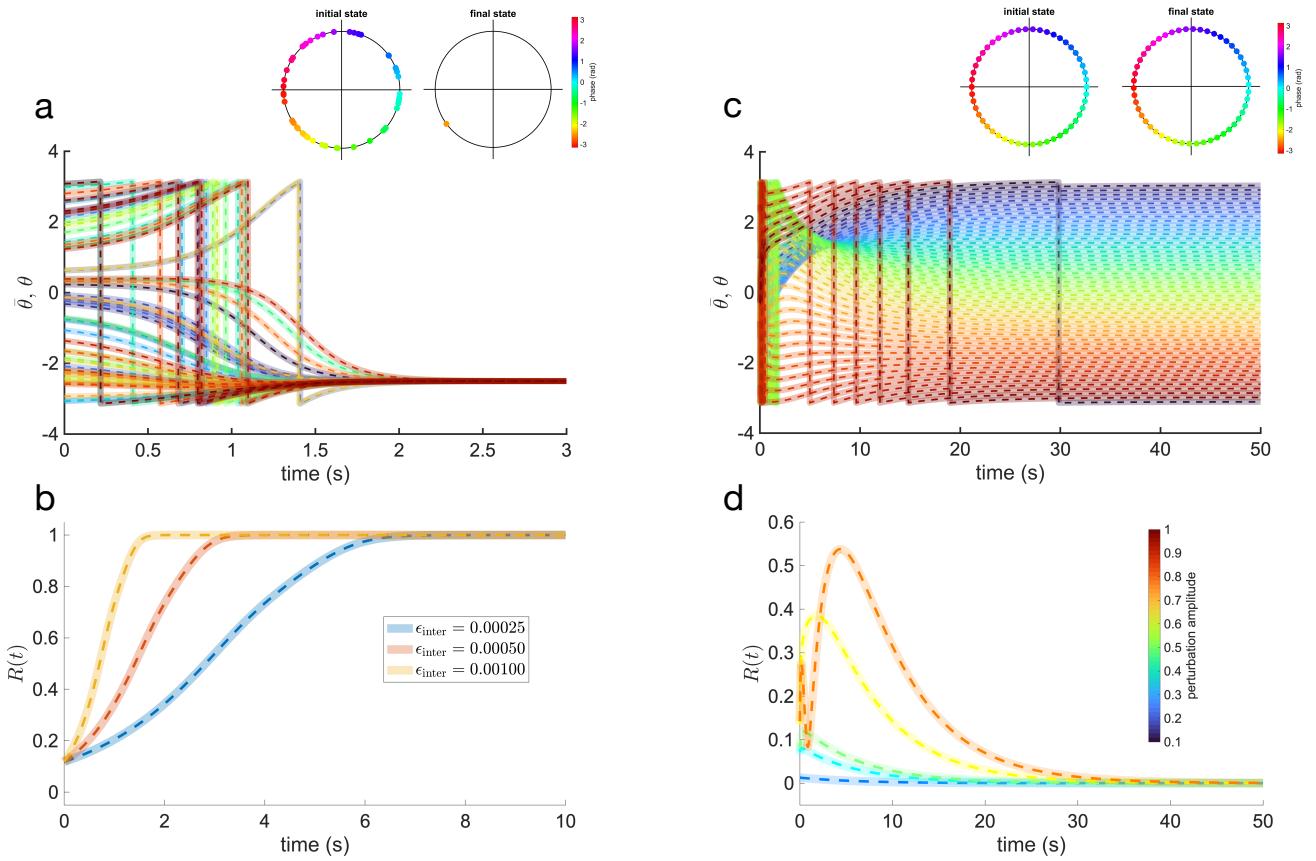


Figure 4. Dynamics on a 50 layers network and its reduced system. We first consider the case where the inter-layer coupling scheme is given by a complete graph (all-to-all). So, random initial conditions evolve to phase synchrony. Here, solid-shaded lines represent the reduced system and dashed lines represent the multilayer network. The trajectories of the systems emphasize this point (a) and also show the equivalent dynamics between the systems. When we consider higher coupling strengths, the systems transition to phase synchrony quicker (b). We then consider the case where the inter-layer connection scheme is given by a first-neighborhood architecture, where each layer is connected to two layers (one in each direction) with periodic boundary conditions. In this case, phase-locking solutions are stable. We then show that the systems assume the form of a twisted state (c); and that this is robust to different perturbations (d).

Moreover, if we consider different coupling strengths between the layers (inter coupling), the transition to phase synchrony occurs at different times, as described in Proposition (II.1). To show that our approach is able to capture these details, we consider the same random initial conditions and we change the inter coupling strength. Figure 4b shows the order parameter for the reduced system (solid-shaded lines) and for the equivalent multilayer network (dashed lines) as a function of time for different values of ϵ_{inter} . We observe that the higher the coupling the faster the transition to phase synchronization.

At last, we use the approach introduced in this paper to analyze the broadcasting a stable phase-locking states. To do so, we consider the multilayer system with $M = 50$ layers, each one being described by a random network with $N = 100$ nodes. Each layer, however, is connected to two layers only, in a first-neighborhood fashion. This leads to a reduced system being described as a ring graph with $N = 50$ nodes, where each node has degree 2 with periodic boundary conditions. This kind of network is known to support stable twisted states [33]. Particularly, we use Eq. (IV.1) with $M = 50$ and $p = 1$ to generate the stable phase-locking solution [33] for the reduced system, which is broadcast to the multilayer network following the approach described in this paper. Moreover, to show the stability of this solution and that our approach is able to capture these details, we apply perturbation to the this phase-locking state using the same approach as before introduced: $\mathcal{AU}(-\pi, \pi)$.

The trajectories for both, the reduced and the multilayer systems are shown in Fig. 4c. In this case, the initial state is given by a twisted state as shown in the inner panel. We then apply a perturbation to this state, which leads to a change in the phase configuration. However, given the stability of the 1st ($p = 1$) twisted state for this system, despite the perturbation, the phase-locking is recovered and we can observed the constant phase difference across the oscillators, which leads to the final state being exact the same as the initial one without perturbation (Fig. 4c inner panels). Again, the phases for the reduced system are represented by the solid-shaded lines and the phases for the multilayer networks are represented by the dashed lines.

We also consider different perturbation amplitudes applied on the initial state. Figure 4d shows the Kuramoto order parameter for both systems (solid-shaded lines represent the reduced system and dashed lines the multilayer network) as a function of time. The perturbation amplitude is shown in color-code. We observe that the stronger the amplitude the bigger the variation of the order parameter, but given the stability of this state, in the end $R = 0$ for all cases, which characterizes the solution for these systems.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, we have introduced an alternative approach to study and predict the dynamical behavior of Kuramoto oscillators on a wide class of multilayer networks. Based on the results presented in [18, 19], we can represent a multilayer system as a join of matrices, each one representing a different connection between nodes either in the same layer or in different layers. From this representation, we have developed a “reduced system” that holds the important information regarding the original, multilayer network. While the multilayer network is composed of M layers with N nodes, the reduced systems is given by M elements. We have shown that, when the initial state of the multilayer system has a relation with the initial state of the reduced one, both systems have an equivalent dynamics. Thus, it allows us to investigate into the dynamical behavior of the multilayer network in a simpler way.

Based on this result, we now can find solutions for the reduced system and broadcast them to the multilayer network. This method offers an alternative and simple way to find equilibrium point for Kuramoto oscillators on multilayer connection scheme. This approach is general to arbitrary topologies for intra-layer connections (within each layer). Furthermore, we can use a similar approach to obtain information on the linear stability of these equilibrium points. We can write the Jacobian matrix for both systems and, by using the results from [18], we can obtain the spectrum of the Jacobian of the multilayer network based on the spectrum of the Jacobian of the reduced system. Therefore, we are now able to investigate into the linear stability of equilibrium points of multilayer networks in a simple way.

Multilayer networks have been used to model a variety of systems. Extensive numerical studies have shown a rich repertory of dynamical behavior [41–43]. Moreover, experimental analyses confirm this feature [44, 45]. Furthermore, the use of multilayer networks have been helpful in the understanding of complex systems, e.g. neural systems [4, 46]. However, analytical treatment for this kind of system is still an open problem in these diverse fields. Our approach thus offers a novel path in the study of multilayer network, opening the possibility of analytical and mechanistic insights on the dynamics of these important systems.

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