

ON DIVISIBILITY RELATION GRAPHS

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ABSTRACT. For each positive integer n , we define the divisibility relation graph D_n whose vertex set is the set of divisors of n , and in which two vertices are adjacent if one is a divisor of the other. This type of graph is a special case of graphs associated with a partial order, which have been widely studied in the literature. In this work, we determine various graph-theoretic invariants of divisibility relation graphs, such as their clique and independence numbers, and their planarity. We also discuss various spectral properties that are discovered by our numerical experiments.

1. INTRODUCTION

The divisor graph G_n on n nodes is the graph whose vertices are $\{1, 2, \dots, n\}$ and in which edges are (a, b) where either $a \mid b$ or $b \mid a$. Due to their arithmetical nature, these graphs have been widely studied in the literature. For example, there exists an extensive line of work investigating the longest simple path on G_n (see [5, 11, 16, 17]). Additionally, the determination of the independence and clique numbers of G_n has also attracted significant interest (see, for example, [1, 2, 10]).

While studying gcd-graphs in [13, 15], we naturally rediscovered G_n . Specifically, in our attempts to understand certain induced structures on these gcd-graphs and to prove several statements about their primeness, we realized the importance of understanding the connection scheme between the given divisor subset. Therefore, quite naturally, for our purposes, we restrict ourselves to the induced subgraph D_n of G_n on the set of divisors of n . We remark that the vertices of D_n provide a complete set of representatives for the quotient space $(\mathbb{Z}/n\mathbb{Z})^\times \backslash (\mathbb{Z}/n\mathbb{Z})$, where the action of $(\mathbb{Z}/n\mathbb{Z})^\times$ on $(\mathbb{Z}/n\mathbb{Z})$ is via the natural multiplication. Consequently, these vertices also form a complete system of representatives for certain supercharacters on $\mathbb{Z}/n\mathbb{Z}$ (see [6]). It is our hope that the study of divisibility relation graphs will offer further insights into the study of these supercharacters and their associated Ramanujan sums.

We note that the definition of D_n is also introduced in [18, Definition 19]. To start our discussion, let us provide the following formal definition of D_n .

Definition 1.1. The divisibility relation graph D_n is the graph with the following data

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- (1) The vertex set of D_n is the set of all divisors of n .
- (2) Two vertices $a \neq b$ are adjacent if and only if $a \mid b$ or $b \mid a$.

The graph D_n is a special case of compatibility graphs; i.e. graphs which are associated with an order (we will explain this interpretation in more detail in Section 2.1). Therefore, D_n is somewhat more structured than G_n . For example, a classical theorem of Dilworth shows that compatibility graphs are perfect (see [4, 7]). In particular, D_n is a perfect graph. Furthermore, we can explicitly calculate some of its variants, such as its clique number (and hence the chromatic number since D_n is perfect), its independence number, and its planarity. More surprisingly, we found that the spectrum of D_n has several interesting properties. Although some of them can be explained by number-theoretic or set-theoretic arguments, others are somewhat more mysterious. We now summarize our main results. We refer the reader to the main text for more precise statements.

Theorem 1.2. *For each positive integer n , we denote by D_n the divisibility relation graph associated with n . Let f_n be the characteristic polynomial of D_n .*

- (1) D_n is a planar graph if and only if $n \in \{1, p, p^2, p^3, pq, p^2q\}$ for some distinct primes p, q .
- (2) Let p, q be two distinct primes such that $\gcd(n, pq) = 1$. Then $f_n \mid f_{npq}$. Furthermore, if there exists a prime r such that $r \parallel n$, then $f_n^2 \mid f_{npq}$.
- (3) -1 is always an eigenvalue of D_n .
- (4) Suppose that $\mu(n) = -1$ and $\Omega(n) \geq 2$ where μ and Ω are respectively the Mobius and degree functions. Then -2 and 1 are an eigenvalue of D_n .
- (5) Suppose that $\mu(n) = 1$. Then 0 is an eigenvalue of D_n .
- (6) Suppose that $n = pq^a$ where p, q are distinct primes and $a \geq 1$. Then 0 is an eigenvalue of D_n if and only if $a \equiv 1 \pmod{6}$.
- (7) Suppose that $n = p^a q^b$ where p, q are distinct primes and $a, b \geq 1$. If $a \equiv b \equiv 1 \pmod{6}$ then 0 is an eigenvalue of D_n .

1.1. Outline. In Section 2, we discuss various graph-theoretic properties of D_n . Most arguments in this section are quite elementary, which makes it accessible to a wide range of readers. Our main discoveries are in Section 3 where we discuss some variational properties of the spectrum of D_n when n varies. In particular, we find that the case where n has at most two prime factors has some quite surprising patterns.

1.2. Code. The code that we wrote to generate data and perform experiments with divisibility relation graphs can be found in [12]. We remark that we have also verified all statements in this work with various concrete and computable examples.

2. GRAPH THEORETIC PROPERTIES OF D_n

2.1. Isomorphism classes and equivalent interpretations. For each natural number n , we can factor n into a product of powers of distinct prime factors

$$n = p_1^{a_1} p_2^{a_2} \cdots p_d^{a_d}.$$

If we require further that $1 \leq a_1 \leq a_2 \leq \cdots \leq a_d$ and that $p_i < p_j$ when $i < j$ and $a_i = a_j$ then this factorization is unique.

Definition 2.1. We call (a_1, a_2, \dots, a_d) the factorization type of n . We will write

$$F(n) = (a_1, a_2, \dots, a_d).$$

Following the convention in [3], we also define the degree of n to be $\Omega(n) = \sum_{i=1}^d a_i$.

Remark 2.2. In this article, the term *degree* could refer to two distinct notions: the degree of a natural number n , denoted by $\Omega(n)$, and the degree of n considered as a vertex in D_n , denoted by $\deg(n)$. It should be clear from the context which one is being referenced.

We now describe an equivalent definition of D_n . Let S be the following set

$$S = \{(\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{Z}^d \mid 0 \leq \alpha_i \leq a_i \quad \forall 1 \leq i \leq d\}.$$

Definition 2.3. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_d)$ be two vectors in \mathbb{R}^d . We say that $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq d$.

Restricting this partial order on \mathbb{R}^d to S , we get a partially ordered set (S, \leq) . Let $\text{Div}(n)$ be the set of divisors of n . For each $d \in \text{Div}(n)$, d has a unique factorization of the form $\prod_{i=1}^d p_i^{\alpha_i}$ where $0 \leq \alpha_i \leq a_i$. As a result, we have a canonical map $\Phi: \text{Div}(n) \rightarrow S$ defined by $\Phi(d) = (\alpha_1, \dots, \alpha_d)$. By the fundamental theorem of arithmetic, Φ is a bijection. Furthermore, $d_1 \mid d_2$ if and only if $\Phi(d_1) \leq \Phi(d_2)$. We conclude that D_n is isomorphic to the compatibility graph associated with (S, \leq) . In particular, the isomorphism class of D_n only depends on the factorization type of n . While this interpretation of D_n may seem almost tautological, it proves useful—especially when constructing the graph using the Python library `networkx`, as it allows us to avoid explicit factorization of large numbers.

2.1.1. Equivalent description via lattice theory. We can also interpret D_n as a lattice of intermediate subfields of the finite field \mathbb{F}_{p^n} where p is a prime number. Indeed it is a fundamental basic result in theory of finite fields that there is a 1-1 correspondence (see [9, Theorem 2.6])

$$\{d \mid \text{such that } d \mid n\} \longleftrightarrow \{\text{all fields } \mathbb{F} \subset \mathbb{F}_q := \mathbb{F}_{p^n}\}$$

$$d \longrightarrow \mathbb{F}_{p^d}$$

We see therefore that each vertex $d \in D_n$ correspond exactly one subfield \mathbb{F}_{p^d} of \mathbb{F}_{p^n} . Furthermore, $a \mid b$ if and only if $\mathbb{F}_{p^a} \subset \mathbb{F}_{p^b}$. Therefore we attain an isomorphism

$$D_n \cong L(\mathbb{F}_q) := \text{Lattice of intermediate subfields of } \mathbb{F}_q \text{ viewed as a compatibility graph.}$$

Example 2.4. Let us discuss a concrete example. The lattice of all subfields of $\mathbb{F}_{2^{36}}$ is described in Fig. 1. We remark that we only draw edges between fields K and L where there is no proper subfield N , $K \subset N \subset L$ and $K \subset L$.

Considering the graph structure of $D_{36} \cong L(\mathbb{F}_{2^{36}})$, we have the following remark about the degree distribution: the vertices representing the trivial subfield \mathbb{F}_2 and the full field $\mathbb{F}_{2^{36}}$ both have degree 8. The vertices for \mathbb{F}_{2^4} and \mathbb{F}_{2^9} have degree 4. Finally, the vertices corresponding to $\mathbb{F}_{2^{12}}$, \mathbb{F}_{2^3} , \mathbb{F}_{2^6} , $\mathbb{F}_{2^{18}}$, and $\mathbb{F}_{2^{27}}$ each have degree 6.

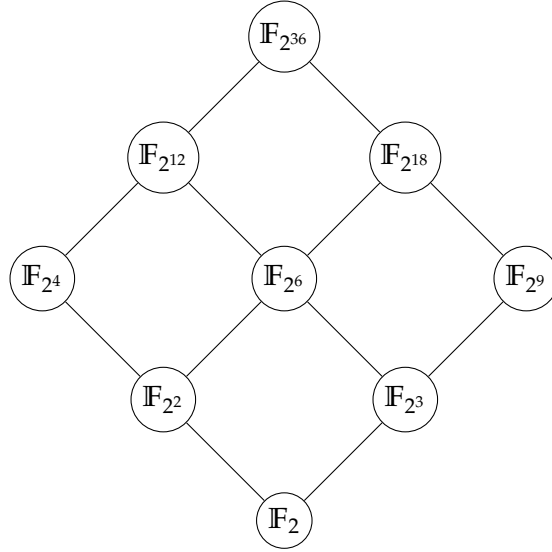


FIGURE 1. Lattice of subfields of $\mathbb{F}_{2^{36}}$.

2.1.2. *The case n is squarefree.* In the case where n is squarefree, we can give an explicit description of the adjacency matrix for D_n .

Proposition 2.5. *Let n be a squarefree positive integer, written as a product $p_0 p_1 \cdots p_{k-1}$ of distinct primes. Index the rows and columns of the $2^k \times 2^k$ adjacency matrix by $\{0, 1, \dots, 2^k - 1\}$. Then, using an appropriate ordering of vertices, for $0 \leq i, j \leq 2^k - 1$ the i, j entry of the adjacency matrix for D_n is $\binom{i}{j} + \binom{j}{i} \bmod 2$.*

Proof. Order the vertices to correspond to row and column indices as follows: for $0 \leq i \leq 2^k - 1$, find the base 2 representation $i = \sum_{r=0}^{k-1} \varepsilon_r^{(i)} 2^r$ ($\varepsilon_r^{(i)} \in \{0, 1\}$) and associate to this index the vertex $\prod_{r=0}^{k-1} p_r^{\varepsilon_r^{(i)}}$. Lucas' theorem with modulus 2 then states that $\binom{i}{j} \equiv \prod_{r=0}^{k-1} \binom{\varepsilon_r^{(i)}}{\varepsilon_r^{(j)}} \bmod 2$. Observing that $\binom{0}{0} = \binom{1}{0} = \binom{1}{1}$ and that $\binom{0}{1} = 0$, we see that $\binom{i}{j} = 1 \iff \varepsilon_r^{(i)} \geq \varepsilon_r^{(j)}$ for all r , and the result easily follows. \square

2.2. Degrees and edges in D_n .

Proposition 2.6. *If the factorization type of n is (a_1, \dots, a_d) then the graph D_n has $v := \prod_{i=1}^d (a_i + 1)$ vertices and $e := v \left(\frac{(a_1+2) \cdots (a_d+2)}{2^d} - 1 \right)$ edges.*

Proof. The vertex count is just the standard result on the number of divisors of an integer. To count edges, we orient the edges by considering an edge joining vertices b, c to originate at b and terminate at c if $b|c$. Then, if for $i = 1, \dots, d$ we have $0 \leq x_i \leq a_i$, the number of edges originating at $b = \prod_{i=1}^d p_i^{x_i}$ is $\prod_{i=1}^d (a_i + 1 - x_i) - 1$. (We subtract 1 as we do not include a loop at b .) So the total number of edges is given by

$$\sum_{x_1=0}^{a_1} \cdots \sum_{x_d=0}^{a_d} \left(\prod_{i=1}^d (a_i + 1 - x_i) - 1 \right)$$

Reindexing by setting $w_i = a_i + 1 - x_i$, we can rewrite this as

$$\begin{aligned}
\sum_{w_1=1}^{a_1+1} \cdots \sum_{w_d=1}^{a_d+1} \left(\prod_{i=1}^d w_i - 1 \right) &= \sum_{w_1=1}^{a_1+1} \cdots \sum_{w_d=1}^{a_d+1} \prod_{i=1}^d w_i - v \\
&= \left(\sum_{w_1=1}^{a_1+1} w_1 \right) \cdots \left(\sum_{w_d=1}^{a_d+1} w_d \right) - v \\
&= \frac{(a_1+1)(a_1+2)}{2} \cdots \frac{(a_d+1)(a_d+2)}{2} - v = \\
&= v \left(\frac{(a_1+2) \cdots (a_d+2)}{2^d} - 1 \right)
\end{aligned}$$

Alternatively, we could argue as follows. To count edges, add a loop to each vertex and orient the edges by viewing an edge connecting a to b , where $a|b$, to originate at a and terminate at b . Then the number of edges terminating at b is the number $d(b)$ of divisors of b . Thus, the total number of edges in D_n , plus a loop at each vertex, is

$$f(n) := \sum_{b|n} d(b)$$

which is a multiplicative function. Evaluating at a prime power,

$$f(p^a) = \sum_{i=0}^a d(p^i) = \sum_{i=0}^a (i+1) = \binom{a+2}{2},$$

so for $n = p_1^{a_1} \cdots p_d^{a_d}$ we have $f(n) = \prod_{i=1}^d \binom{a_i+2}{2}$, and, subtracting the loops, the number of edges in D_n is $e := \prod_{i=1}^d \binom{a_i+2}{2} - v = v \left(\frac{(a_1+2) \cdots (a_d+2)}{2^d} - 1 \right)$. \square

2.2.1. Vertices of Minimal Degree.

Definition 2.7. Let \mathbf{e}_i be the unit vector with 0 in position j , $1 \leq j (\neq i) \leq d$, and 1 in position i . Vectors $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{x} \pm \mathbf{e}_i$ for some i will be called close. A sequence $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a chain iff \mathbf{x}_i and \mathbf{x}_{i+1} are close, $1 \leq i \leq n-1$.

Definition 2.8. A vector $\mathbf{x} = (x_1, \dots, x_d)$ is called extremal in coordinate i if $x_i = 0$ or $x_i = a_i$, and is called extremal if it is extremal in all coordinate positions. The term nonextremal applies otherwise in both situations.

Remark 2.9. The degree of a vertex $\mathbf{x} = (x_1, \dots, x_d)$ is easily seen to be given by

$$f(\mathbf{x}) := \prod_{i=1}^d (a_i + 1 - x_i) + \prod_{i=1}^d (x_i + 1) - 2.$$

For $1 \leq i \leq d$ let $\Delta_i(\mathbf{x}) = f(\mathbf{x} + \mathbf{e}_i) - f(\mathbf{x}) = \prod_{j \neq i} (x_j + 1) - \prod_{j \neq i} (a_j + 1 - x_j)$, and note that $\Delta_i(\mathbf{x})$ does not depend on the value of x_i . It follows that if $\Delta_i(\mathbf{x}) < 0$,

$$\mathbf{x} - x_i \mathbf{e}_i, \mathbf{x} - (x_i - 1) \mathbf{e}_i, \dots, \mathbf{x} + (a_i - x_i) \mathbf{e}_i$$

is a chain through \mathbf{x} , joining the vectors $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$ and $(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_d)$ extremal in coordinate i , with strictly decreasing degrees. Similarly if $\Delta_i(\mathbf{x}) > 0$, we get such a chain with strictly increasing degrees, and if $\Delta_i(\mathbf{x}) = 0$, the chain maintains constant degree.

Proposition 2.10. *A vertex of minimal degree can always be found among the extremal vertices. In fact, any vertex \mathbf{x} of minimal degree satisfies the "stability condition" $\Delta_i(\mathbf{x}) = 0$ at every nonextremal coordinate, and admits a chain from \mathbf{x} to an extremal vertex of minimal degree where every member of this chain has minimal degree.*

Proof. The argument in the above remark implies that \mathbf{x} , if not already extremal, must satisfy $\Delta_i(\mathbf{x}) = 0$ at every nonextremal coordinate (else we can move to a close vertex of lower degree); choosing a nonextremal coordinate position i , \mathbf{x} occurs in a chain with constant degree leading to a vertex differing only in coordinate i where it is now extremal. Since all vertices in this chain are still of minimal degree, the stability condition is still intact. Repeat with any remaining nonextremal coordinate positions until arriving at an extremal vertex. The construction ensures that the degree is not changed in this process. \square

Proposition 2.11. *An extremal vertex \mathbf{x} has minimal degree is and only if $\prod_{i \in A} (a_i + 1) + \prod_{j \in B} (a_j + 1)$ is as small as possible among extremal vertices, where $A = \{i \mid x_i = a_i\}$ and $B = \{j \mid x_j = 0\}$.*

Proof. This follows simply from the formula for $f(\mathbf{x})$, which for an extremal vertex \mathbf{x} computes to $\prod_{i \in S} (a_i + 1) + \prod_{j \in T} (a_j + 1) - 2$. \square

2.3. Connectedness and bipartite property. The graph D_n is always connected since 1 and n are adjacent to all other vertices. Furthermore, we have the following property.

Proposition 2.12. *Let a, b be two vertices in D_n . Let $d(a, b)$ be the distance between a and b . Then*

$$d(a, b) = \begin{cases} 1 & \text{if } (a, b) \in E(D_n) \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If a and b are adjacent then $d(a, b) = 1$. Otherwise, we have the path $a \rightarrow 1 \rightarrow b$ and hence $d(a, b) = 2$. \square

Proposition 2.13. *Let G be the induced graph on $D_n \setminus \{1, n\}$. Then, G is connected unless $n = pq$ where p, q are distinct prime numbers; i.e, the factorization type of n is $(1, 1)$.*

Proof. If $n = pq$ where p, q are distinct primes then we can see that the induced subgraph on $\{p, q\}$ is the empty graph—that is, a graph with vertices but no edges—and hence it is not connected. Conversely, suppose that the factorization type of n is not $(1, 1)$. If n has exactly one prime divisor then the induced subgraph on $D_n \setminus \{1, n\}$ is a complete graph and therefore it is connected. Let us now assume that n has at least two distinct prime factors. Since a vertex is adjacent to all its divisors, it is sufficient to show that if p, q are distinct prime divisors of n then p, q are connected. This is clear since we have the path $p \rightarrow pq \rightarrow q$. \square

Proposition 2.14. *D_n is bipartite if and only if $F(n) = (1)$; i.e, n is a prime number.*

Proof. If n is not a prime number then there exists $1 < d < n$ such that $d \mid n$. In this case $\{1, d, n\}$ forms a triangle in D_n . As a result, D_n cannot be bipartite. \square

2.4. Clique and independent numbers of D_n .

Proposition 2.15. *Suppose that $F(n) = (a_1, a_2, \dots, a_d)$. Let X be a maximal clique in D_n . Then*

$$|X| = 1 + \sum_{i=1}^d a_i = 1 + \Omega(n).$$

Proof. We will prove by induction on d that there is a clique of size $1 + \Omega(n)$. For $d = 1$, $n = p_1^{a_1}$ is a prime power and hence D_n is the complete graph on $a_1 + 1$ nodes. In this case, $X = V(D_n)$ is a clique of size $1 + \Omega(n)$. Let us assume that the statement is true for d . Let us show that it is true for $d + 1$ as well. Let us write $n = \prod_{i=1}^{d+1} p_i^{a_i}$. Let $X = \{h_1, h_2, \dots, h_{1+\sum_{i=1}^d a_i}\}$ be a clique of size $1 + \sum_{i=1}^d a_i$ in $D_{\prod_{i=1}^d p_i^{a_i}}$. We can then check that $X' = \{1, p_{d+1}, \dots, p_{d+1}^{a_{d+1}-1}\} \cup p_{d+1}^{a_{d+1}} X$ is a clique of size $1 + \sum_{i=1}^{d+1} a_i$ in $D_{\prod_{i=1}^{d+1} p_i^{a_i}}$.

Conversely, we show that a clique in D_n has at most $1 + \Omega(n)$ elements. In fact, let $X = \{h_1, h_2, \dots, h_q\}$ be a clique, where $h_1 < h_2 < \dots < h_q$. We claim that $|X| \leq 1 + \Omega(n)$. Since X is a clique and $h_i < h_{i+1}$, $h_i \mid h_{i+1}$ for each $1 \leq i \leq q - 1$. This shows that $0 \leq \Omega(h_1) < \Omega(h_2) < \dots < \Omega(h_q) \leq \Omega(n)$. Consequently, we conclude that $|X| = q \leq 1 + \Omega(n)$. \square

For the independence number of D_n , we have the following result.

Proposition 2.16. (See [3, Theorem 1]) *Let S be a subset of D_n consisting of vertices of the same degree $\lfloor \frac{\Omega(n)}{2} \rfloor$. Then S is a maximal independence set in D_n . In particular, the independence number of D_n is precisely $|S|$.*

2.4.1. Perfectness and coloring of D_n . Since D_n is perfect, the chromatic number of D_n is equal to its clique number; i.e., $\chi(D_n) = 1 + \Omega(n)$. In fact, we can provide an explicit coloring on D_n with $1 + \Omega(n)$ colors. Specifically, we can define a coloring map $c : D_n \rightarrow \{0, 1, \dots, \Omega(n)\}$ by the rule $c(m) = \Omega(m)$. Clearly, two numbers with the same degree cannot be adjacent, so c is a coloring map.

2.5. Planarity of D_n . In this section, we classify all D_n which are planar. By exploring D_n with some small values of n , we can see that $F(n) = (a_1, a_2, \dots, a_d)$ with $d \leq 2$ and $\sum_{i=1}^d a_i \leq 3$ then D_n is planar. See Fig. 2 for a planar embedding when $F(n) = (1, 2)$ and Fig. 3 for a planar embedding when $F(n) = (3)$.

The following theorem completely classifies n such that D_n is planar.

Proposition 2.17. *The graph D_n is planar if and only if one of the following holds:*

- (1) $n = p^r$, p prime, $0 \leq r \leq 3$
- (2) $n = pq$, p, q prime
- (3) $n = p^2q$, p, q prime

Proof. We make use of the fact that a graph cannot be planar if it contains a subgraph isomorphic to $K_{3,3}$ or to a subdivision of K_5 . Since if D_n is planar and $m \mid n$, D_m (a subgraph) is also planar, and if D_n is not planar and $n \mid m$, D_m is also not planar, it suffices to check the following cases (where p, q, r are distinct primes)

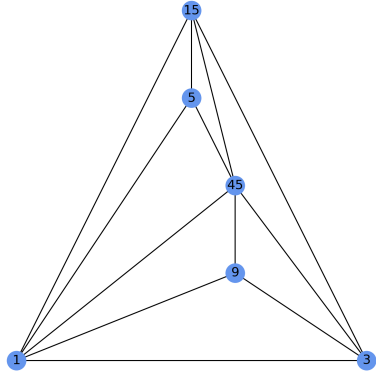


FIGURE 2. A planar embedding of D_{45}

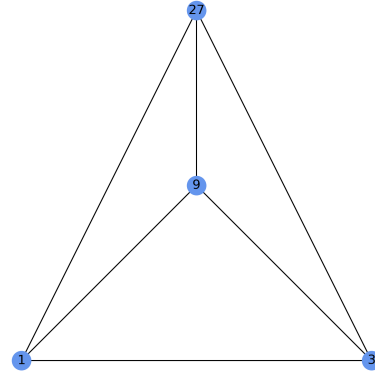
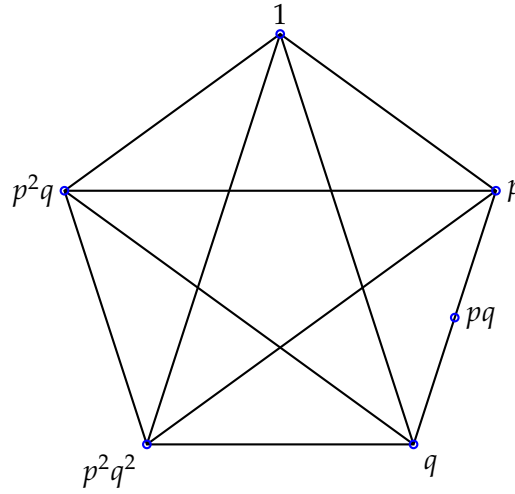
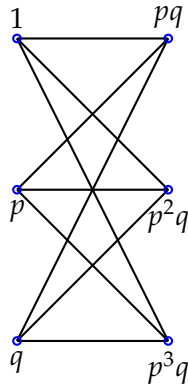


FIGURE 3. A planar embedding of D_{27} .

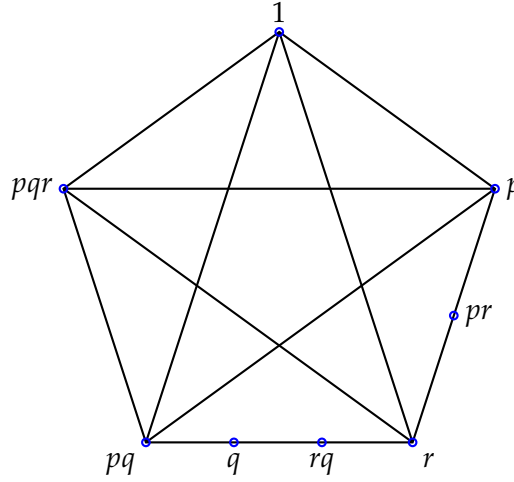
- (1) D_{p^3} is planar (see Fig. 3) and D_{p^4} is not. This is true since D_{p^r} is the complete graph K_{r+1} which is planar iff $r + 1 \leq 4$.
- (2) D_{pq} and D_{p^2q} are planar (see Fig. 2).
- (3) $D_{p^2q^2}$ is not planar. For it contains a subgraph which is a subdivision of K_5 .



- (4) D_{p^3q} is not planar. It contains the following subgraph which is isomorphic to $K_{3,3}$.



- (5) D_{pqr} is not planar.



It fails the known necessary planarity condition $e \leq 3v - 6$. Alternatively, we note that it contains a subgraph which is a subdivision of K_5 .

□

3. SPECTRUM OF D_n

In this section, we study the spectrum of D_n . We focus on two main themes:

- (1) How the spectra change when adding more prime factors to n ,
- (2) Some special eigenvalues of D_n .

3.1. Variation of the spectra of D_n . For each n , let f_n be the characteristic polynomial of D_n .

Theorem 3.1. *Let n be a natural number. Let p, q be distinct prime numbers such that $\gcd(n, pq) = 1$. Then $f_n \mid f_{npq}$.*

Proof. Suppose that p, q are distinct primes and that $\gcd(n, pq) = 1$. Let $1 = a_1 < \dots < a_k = n$ be the divisors of n , and let B be the adjacency matrix of D_n with respect to this order. Let the $k \times k$ matrix C be defined by $C_{ij} = 1$ if $a_i \mid a_j$ and 0 otherwise. (Then $B = C + C^T - 2I_k$.) Now list divisors of npq in the order $a_1, \dots, a_k, pa_1, \dots, pa_k, qa_1, \dots, qa_k, pqa_1, \dots, pqa_k$. With respect to this order, the adjacency matrix for D_{npq} is

$$M = \begin{bmatrix} B & C & C & C \\ C^T & B & 0 & C \\ C^T & 0 & B & C \\ C^T & C^T & C^T & B \end{bmatrix}$$

Write I_k for the $k \times k$ identity matrix and $\mathbf{0}_k$ for an $k \times k$ matrix of 0's. The characteristic polynomial of M can be analyzed as follows, where for the third equality, we have subtracted

the second row of blocks from the third:

$$\begin{aligned}
f_{npq} &= \det(M - \lambda I_{n+4}) = \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & \mathbf{0}_k & C \\ C^T & \mathbf{0}_k & B - \lambda I_k & C \\ C^T & C^T & C^T & B - \lambda I_k \end{bmatrix} \\
&= \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & \mathbf{0}_k & C \\ \mathbf{0}_k & \lambda I_k - B & B - \lambda I_k & \mathbf{0}_k \\ C^T & C^T & C^T & B - \lambda I_k \end{bmatrix} \\
&= \det \begin{bmatrix} I_k & \mathbf{0}_k & \mathbf{0}_k & \mathbf{0}_k \\ \mathbf{0}_k & I_k & \mathbf{0}_k & \mathbf{0}_k \\ \mathbf{0}_k & \mathbf{0}_k & B - \lambda I_k & \mathbf{0}_k \\ \mathbf{0}_k & \mathbf{0}_k & \mathbf{0}_k & I_k \end{bmatrix} \det \begin{bmatrix} B - \lambda I_k & C & C & C \\ C^T & B - \lambda I_k & 0 & C \\ \mathbf{0}_k & -I_k & I_k & \mathbf{0}_k \\ C^T & C^T & C^T & B - \lambda I_k \end{bmatrix} \\
&= \det(B - \lambda I_k) \cdot (\text{a polynomial in } \lambda) = f_n \cdot (\text{a polynomial in } \lambda).
\end{aligned}$$

This shows $f_n \mid f_{npq}$. □

By similar arguments, we can show:

Theorem 3.2. *Let n be a natural number, p, q distinct primes not dividing n . If there is a prime number r such that $r \parallel n$, then $f_n^2 \mid f_{npq}$. In particular, this is the case when $n > 1$ is squarefree.*

Proof. Let n be a natural number, p, q, r primes with $r \parallel n$ and $\gcd(n, pq) = 1$. Set $n = n_0 r$.

Let $a_1 = 1 < a_2 < \dots < a_k$ be the divisors of n_0 . Let B_0 be the adjacency matrix of D_{n_0} . Let C be the matrix defined by

$$C_{ij} = 1 \iff a_i \mid a_j.$$

List the divisors of $n_0 r p q$ in the order:

$$\begin{array}{cccc}
\underbrace{a_1, \dots, a_k}_{(1)} & \underbrace{ra_1, \dots, ra_k}_{(2)} & \underbrace{pa_1, \dots, pa_k}_{(3)} & \underbrace{pra_1, \dots, pra_k}_{(4)} \\
\underbrace{qa_1, \dots, qa_k}_{(5)} & \underbrace{qra_1, \dots, qra_k}_{(6)} & \underbrace{pqa_1, \dots, pqa_k}_{(7)} & \underbrace{pqra_1, \dots, pqra_k}_{(8)}.
\end{array}$$

We note that

$$C_{ij} = 1 \iff a_i \mid a_j \iff ra_i \mid ra_j.$$

Then the adjacency matrix for $D_{n_0 r p q}$ is of the form:

$$M = \begin{bmatrix} B_0 & C & C & C & C & C & C & C \\ C^T & B_0 & 0 & C & 0 & C & 0 & C \\ C^T & 0 & B_0 & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 & 0 & 0 & 0 & C \\ C^T & 0 & 0 & 0 & B_0 & C & C & C \\ C^T & C^T & 0 & 0 & C^T & B_0 & 0 & C \\ C^T & 0 & C^T & 0 & C^T & 0 & B_0 & C \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 \end{bmatrix} = \begin{bmatrix} B & A & A & A \\ A^T & B & 0 & A \\ A^T & 0 & B & A \\ A^T & A^T & A^T & B \end{bmatrix},$$

where

$$B = \begin{bmatrix} B_0 & C \\ C^T & B_0 \end{bmatrix}, \quad A = \begin{bmatrix} C & C \\ 0 & C \end{bmatrix}.$$

We note that B is precisely the adjacency matrix for D_n . We have

$$|M - \lambda I_N| = \begin{vmatrix} B_0 - \lambda I & C & C & C & C & C & C & C \\ C^T & B_0 - \lambda I & 0 & C & 0 & C & 0 & C \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ C^T & 0 & 0 & 0 & B_0 - \lambda I & C & C & C \\ C^T & C^T & 0 & 0 & C^T & B_0 - \lambda I & 0 & C \\ C^T & 0 & C^T & 0 & C^T & 0 & B_0 - \lambda I & C \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I \end{vmatrix} \\ = \pm \begin{vmatrix} B_0 - \lambda I & C & C & C & C & C & C & C \\ 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 & 0 & C & -C & 0 \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ 0 & 0 & -B_0 + \lambda I & -C & B_0 - \lambda I & C & 0 & 0 \\ 0 & 0 & -C^T & -B_0 + \lambda I & C^T & B_0 - \lambda I & 0 & 0 \\ 0 & C^T & -C^T & 0 & 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I \end{vmatrix}.$$

Here, $I = I_k$ and we the following (block) row operations:

- (2) : $R_2 \leftarrow R_2 - R_3$
- (5) : $R_5 \leftarrow R_5 - R_3$
- (6) : $R_6 \leftarrow R_6 - R_4$
- (7) : $R_7 \leftarrow R_6 - R_7$.

Thus

$$|M - \lambda I_N| = \pm \begin{vmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_0 - \lambda I & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C^T & B_0 - \lambda I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{vmatrix} \times$$

$$\begin{vmatrix} B_0 - \lambda I & C & C & C & C & C & C & C \\ 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 & 0 & C & -C & 0 \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ 0 & 0 & -I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & I & 0 & 0 \\ 0 & C^T & -C^T & 0 & 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I \end{vmatrix}.$$

In the above product of determinants, the first determinant is f_n . For the second determinant, which we denote temporarily as S , we move the block rows R_2 and R_7 to the top, we obtain

$$S = \pm \begin{vmatrix} 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 & 0 & C & -C & 0 \\ 0 & C^T & -C^T & 0 & 0 & B_0 - \lambda I & -B_0 + \lambda I & 0 \\ B_0 - \lambda I & C & C & C & C & C & C & C \\ C^T & 0 & B_0 - \lambda I & C & 0 & 0 & C & C \\ C^T & C^T & C^T & B_0 - \lambda I & 0 & 0 & 0 & C \\ 0 & 0 & -I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & I & 0 & 0 \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I \end{vmatrix}$$

Now in the above determinant, we move the block columns C_2 , C_6 , C_3 and C_7 (in this order) to the left, we imply that

$$\begin{aligned}
S &= \pm \begin{vmatrix} B_0 - \lambda I & C & -B_0 + \lambda I & -C & 0 & 0 & 0 & 0 \\ C^T & B_0 - \lambda I & -C^T & -B_0 + \lambda I & 0 & 0 & 0 & 0 \\ C & C & C & C & B_0 - \lambda I & C & C & C \\ 0 & 0 & B_0 - \lambda I & C & C^T & C & 0 & C \\ C^T & 0 & C^T & 0 & C^T & B_0 - \lambda I & 0 & C \\ 0 & 0 & -I & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & -I & 0 & 0 \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & B_0 - \lambda I \end{vmatrix} \\
&= \pm \begin{vmatrix} B_0 - \lambda I & C & 0 & 0 & 0 & 0 & 0 & 0 \\ C^T & B_0 - \lambda I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{vmatrix} \times \begin{vmatrix} I & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 & 0 & 0 \\ C & C & C & C & B_0 - \lambda I & C & C & C \\ 0 & 0 & B_0 \lambda I & C & C^T & C & 0 & C \\ C^T & 0 & C^T & 0 & C^T & B_0 - \lambda I & 0 & C \\ 0 & 0 & -I & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & -I & 0 & 0 \\ C^T & C^T & C^T & C^T & C^T & C^T & C^T & C^T \end{vmatrix} \\
&= \pm f_n \times (\text{a polynomial in } \lambda).
\end{aligned}$$

Hence f_{npq} is divisible by f_n^2 .

Of course if $n > 1$ is squarefree, we can choose for r any prime dividing n . \square

We discuss a different proof for Theorem 3.1. In essence, the two proofs are the same. However, this approach might be applied to other problems such as bounding the multiplicity of certain eigenvalues (see for example Corollary 3.15). We first discuss the general idea that leads to this approach. Let (S, \leq) be a partially ordered set and D_S the compatibility graph associated to S . Namely, the vertex set of D_S is S and two vertices a, b are adjacent if $a < b$ or $b < a$. As explained in Section 2.1, D_n is a special case of compatibility graphs. By definition, an eigenvector for D_S associated with an eigenvalue λ is the same as a function $f : S \rightarrow \mathbb{C}$ such that for each $\alpha \in S$

$$\sum_{\beta < \alpha} f(\beta) + \sum_{\beta > \alpha} f(\beta) = \lambda f(\alpha).$$

For two partially ordered sets S_1, S_2 , their cartesian product $S = S_1 \times S_2$ is defined in a canonical way. Specifically, as a set S is the cartesian product of S_1 and S_2 . In addition, we say that $(\beta_1, \beta_2) \leq (\alpha_1, \alpha_2)$ if and only if $\beta_1 \leq \alpha_1$ and $\beta_2 \leq \alpha_2$. With this perspective, we can see that the partially ordered set $\text{Div}(npq)$ is isomorphic to the product of $\text{Div}(n)$ and S_0 where

$$S_0 = \{(\alpha_1, \alpha_2) \in \mathbb{Z}^2 \mid 0 \leq \alpha_i \leq 1\}.$$

The following theorem is a natural generalization of Theorem 3.1.

Theorem 3.3. *Let S be a partially ordered set and $\hat{S} = S \times S_0$. Let f_S (respectively $f_{\hat{S}}$) be the characteristic polynomial of D_S (respectively $D_{\hat{S}}$). Then $f_S \mid f_{\hat{S}}$.*

To prove this statement, our idea is to show that each eigenvector of D_S lifts to an eigenvector for $D_{\hat{S}}$ with the same eigenvalue. Because we are dealing with direct product, a natural approach would be using the tensor product. Let us recall its formal definition.

Definition 3.4. Let $S = S_1 \times S_2$ and $g : S_1 \rightarrow \mathbb{C}$ and $h : S_2 \rightarrow \mathbb{C}$ be two functions. The tensor product $f := g \otimes h$ of g and h is the function $f : S_1 \times S_2 \rightarrow \mathbb{C}$ defined by the rule $f((s_1, s_2)) = g(s_1)h(s_2)$.

It is not true in general that if g, h are eigenvectors for S_1, S_2 then $g \otimes h$ is an eigenvector for S . However, there are cases where it is indeed the case. In fact, D_{S_0} has a special eigenvector that makes this work. More specifically, let $h : S_0 \rightarrow \mathbb{C}$ defined by the rule

$$h((0,0)) = h((1,1)) = 0, h((0,1)) = 1, h((1,0)) = -1.$$

Then h has the property that for each $\gamma \in S_0$

$$\sum_{\gamma < \epsilon} h(\epsilon) = \sum_{\gamma > \epsilon} h(\epsilon) = 0.$$

In particular, h is an eigenvector associated with the eigenvalue 0. We have the following observation.

Proposition 3.5. Let S be a partially ordered set and $\hat{S} = S \times S_0$. Let $g : S \rightarrow \mathbb{C}$ be an eigenvector of D_S associated with the eigenvalue λ . Then $\hat{g} := g \otimes h : \hat{S} \rightarrow \mathbb{C}$ is an eigenvector for $D_{\hat{S}}$ associated with λ .

Proof. Let $\hat{\alpha} = (\alpha, \gamma)$ where $\alpha \in S$ and $\gamma \in S_0$. We need to show that

$$\sum_{\hat{\beta} < \hat{\alpha}} \hat{g}(\hat{\beta}) + \sum_{\hat{\beta} > \hat{\alpha}} \hat{g}(\hat{\beta}) = \lambda \hat{g}(\hat{\alpha}).$$

For $\hat{\beta} = (\beta, \epsilon)$, we have $\hat{\beta} < \hat{\alpha}$ if and only if exactly one of the following conditions holds

- (1) $\alpha = \beta$ and $\epsilon < \gamma$.
- (2) $\epsilon = \gamma$ and $\beta < \alpha$.
- (3) $\beta < \alpha$ and $\epsilon < \gamma$.

Consequently, we can rewrite

$$\begin{aligned} \sum_{\hat{\beta} < \hat{\alpha}} \hat{g}(\hat{\beta}) &= \sum_{\epsilon < \gamma} \hat{g}(\alpha, \epsilon) + \sum_{\beta < \alpha} \hat{g}(\alpha, \gamma) + \sum_{\epsilon < \gamma, \beta < \alpha} \hat{g}(\beta, \epsilon) \\ &= g(\alpha) \sum_{\epsilon < \gamma} h(\epsilon) + h(\gamma) \sum_{\beta < \alpha} g(\beta) + \left(\sum_{\epsilon < \gamma} h(\epsilon) \right) \left(\sum_{\beta < \alpha} g(\beta) \right) \\ &= h(\gamma) \sum_{\beta < \alpha} g(\beta). \end{aligned}$$

The last equality follows from the fact that $\sum_{\epsilon < \gamma} h(\epsilon) = 0$. Similarly, we have

$$\sum_{\hat{\alpha} < \hat{\beta}} \hat{g}(\hat{\beta}) = h(\gamma) \sum_{\alpha < \beta} g(\beta).$$

Summing up and using the fact that g is an eigenvector associated with λ we conclude that

$$\sum_{\widehat{\beta} < \widehat{\alpha}} \widehat{g}(\widehat{\beta}) + \sum_{\widehat{\beta} > \widehat{\alpha}} \widehat{g}(\widehat{\beta}) = h(\gamma) \left(\sum_{\beta < \gamma} g(\beta) + \sum_{\alpha < \beta} g(\beta) \right) = \lambda h(\gamma) g(\alpha) = \lambda \widehat{g}(\widehat{\alpha}).$$

We conclude that \widehat{g} is an eigenvector of $D_{\widehat{S}}$ associated with λ . \square

We can now provide a proof of Theorem 3.3.

Proof. Since D_S is undirected, its adjacency matrix is diagonalizable. Let $\{g_1, g_2, \dots, g_m\}$ be a complete system of eigenvectors associated with the system of eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ (here $m = |S|$). For each $1 \leq i \leq m$, let $\widehat{g}_i = g_i \otimes h$ as discussed in Proposition 3.5. Then \widehat{g}_i is an eigenvector for $D_{\widehat{S}}$ associated with the eigenvalue λ_i . Furthermore, by a standard property of the tensor product, the system $\{\widehat{g}_i\}_{i=1}^m$ is linearly independent. This shows that $f_S \mid f_{\widehat{S}}$. \square

Remark 3.6. The analogues of Theorems 3.1 and 3.2 for partially ordered sets hold (one proof for 3.1 given above). Specifically, let (S, \leq) be a partially ordered set, and write $S' = S \times \{0, 1\}$, which we order by $(s, a) \leq (t, b) \leftrightarrow s \leq t \ \& \ a \leq b$, and use f_S for the characteristic polynomial of the compatibility graph corresponding to S . Then the analogue of 3.1 is $f_S \mid f_{S'}$ and that of 3.2 is $f_S^2 \mid f_{S''}$. Both of these can be proved by essentially the same arguments as those presented in Theorems 3.1 and 3.2.

3.2. Special eigenvalues. In this section, we discuss some special eigenvalues for D_n . While the existence of these eigenvalues typically follow from Theorem 3.1, and we may omit proofs below where this is clearly the case, we discuss some explicit eigenvectors which might be of independent interest. We also include some numerical data for the multiplicity of these eigenvalues which seem to follow some interesting patterns.

3.2.1. Eigenvalue -2 . Let λ be a real number. Then λ is an eigenvalue of D_n if there is a vector $(v_d)_{d|n}$ such that

$$\sum_{(d,m) \in E(D_n)} v_m = \lambda v_d.$$

The case $\lambda = -2$ is special because we can rewrite this as

$$(3.1) \quad \sum_{m|d} v_m + \sum_{d|m|n} v_m = 0.$$

We will use this fact to construct an eigenvector associated with $\lambda = -2$ under some conditions on n .

Let μ be the Moebius function. This function has the property that for each positive integer m

$$\sum_{d|m} \mu(d) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{else.} \end{cases}$$

Proposition 3.7. Suppose that $\mu(n) = -1$. Let $v = (v_d)_{d|n}$ be the vector defined by

$$v_d = \begin{cases} 0 & \text{if } d \in \{1, n\} \\ \mu(d) & \text{else.} \end{cases}$$

Then v is an eigenvector associated with the eigenvalue $\lambda = -2$.

Proof. We verify that Eq. (3.1) holds for all $d|n$. If $d \notin \{1, n\}$ then we have

$$\sum_{m|d} v_m + \sum_{d|m|n} v_m = \sum_{m|d} \mu(m) - \mu(1) + \mu(d) \sum_{m|\frac{n}{d}} \mu(m) - \mu(n) = 0.$$

If $d = 1$ then Eq. (3.1) becomes

$$v_1 + \sum_{m|n} v_m = \sum_{d|n} \mu(d) - \mu(1) - \mu(n) = 0.$$

Similarly, we can check that Eq. (3.1) holds for $d = n$. □

Let n be a squarefree number and $\omega(n)$ the distinct number of prime factors of n . The following table records the multiplicity of $\lambda = -2$ as an eigenvalue for D_n .

$\omega(n)$	2	3	4	5	6	7	8	9	10	11	12	13
m_{-2}	0	2	0	10	0	42	0	170	0	682	0	2730

TABLE 1. Multiplicity of -2 as an eigenvalue for D_n

Remark 3.8. It seems that -2 is an eigenvalue if and only $\mu(n) = (-1)^{\omega(n)} = -1$. Furthermore, a quick look on the The On-Line Encyclopedia of Integer Sequences reveals that the values

$$2, 10, 42, 170, 682, 2730$$

are the first few terms of the sequence A020988 (see <https://oeis.org/A020988>). We note that these terms satisfy the linear recurrence $x_{n+1} = 4x_n + 2$.

3.2.2. Eigenvalue -1 .

Lemma 3.9. *Let X be a clique set in a connected graph G . Suppose further that both X and $G \setminus X$ are homogeneous sets in G . Then -1 is an eigenvalue of G with multiplicity at least $|X| - 1$.*

Proof. By definition, G is a joined union of the induced graph on X and $G \setminus X$. The conclusion then follows from the proof of [14, Proposition 3.1]. More specifically, the adjacency matrix of G is of the form

$$A = \begin{pmatrix} A_X & J \\ J & A_{G \setminus X} \end{pmatrix}.$$

Let $v = (v_1, \dots, v_{|X|})$ be a vector such that $\sum_{i=1}^{|X|} v_i = 0$. Then, $w = (v_1, \dots, v_{|X|}, 0, \dots, 0)$ is an eigenvector for A_G associated with the eigenvalue $\lambda = -1$. □

Corollary 3.10. -1 is an eigenvalue of D_n for every $n \geq 2$.

Proof. $X = \{1, n\}$ satisfies the conditions of Lemma 3.9. □

The following table records the multiplicity of $\lambda = -1$ as an eigenvalue for D_n where n is a squarefree number.

$\omega(n)$	2	3	4	5	6	7	8	9	10	11	12	13
m_{-1}	1	3	4	10	15	35	56	126	212	462	814	1716

TABLE 2. Multiplicity of -1 as an eigenvalue for D_n

3.2.3. Eigenvalue 1.

Proposition 3.11. *Suppose that $\mu(n) = -1$. Then 1 is an eigenvalue of D_n .*

The following table records the multiplicity of $\lambda = 1$ as an eigenvalue for D_n where n is a squarefree number.

$\omega(n)$	2	3	4	5	6	7	8	9	10	11	12	13
m_1	0	2	0	5	0	14	0	42	0	132	0	429

TABLE 3. Multiplicity of 1 as an eigenvalue for D_n

3.2.4. Eigenvalue 0.

Proposition 3.12. *Suppose that $\mu(n) = 1$. Then 0 is an eigenvalue of D_n .*

The following table records the multiplicity of $\lambda = 0$ as an eigenvalue for D_n where n is a squarefree number.

$\omega(n)$	2	3	4	5	6	7	8	9	10	11	12	13
m_0	1	0	2	0	5	0	14	0	42	0	132	0

TABLE 4. Multiplicity of 0 as an eigenvalue for D_n

Remark 3.13. It seems that 0 (respectively 1) is an eigenvalue if and only $\mu(n) = 1$ (respectively $\mu(n) = -1$). Furthermore, a quick look on the The On-Line Encyclopedia of Integer Sequences reveals that the values 2, 5, 14, 42, 132, 429 are the first few terms of the Catalan sequence A000108 (see <https://oeis.org/A000108>). This observation has also been discovered independently in [8].

We provide below an partial explanation for this phenomenon. Let n be a squarefree number such that $\omega(n) = 2m$. We recall from the discussion before the proof of Proposition 3.5 that an eigenvector for D_n can be considered as a function on the powerset $\{0, 1\}^{2m} \rightarrow \mathbb{C}$ such that for all $\alpha \in \{0, 1\}^{2m}$

$$\sum_{\beta < \alpha} f(\beta) + \sum_{\beta > \alpha} f(\beta) = \lambda f(\alpha).$$

When $\alpha = 0$, this condition becomes

$$\sum_{\beta < \alpha} f(\beta) + \sum_{\beta > \alpha} f(\beta) = 0.$$

Let V_m be the space of all functions $f : \{0, 1\}^{2m} \rightarrow \mathbb{C}$ such that for all $\alpha \in \{0, 1\}^{2m}$

$$\sum_{\beta < \alpha} f(\beta) = \sum_{\beta > \alpha} f(\beta) = 0.$$

By the above discussion, V_m is a subspace of the nullspace of the adjacency matrix for D_n . We have the following observation whose proof is identical to the one given in Proposition 3.5.

Lemma 3.14. *Let $2m = 2m_1 + 2m_2$. For $f_1 \in V_{m_1}$ and $f_2 \in V_{m_2}$, let f be the tensor product of f_1 and f_2 as defined in Definition 3.4. Then $f \in V_m$.*

Corollary 3.15. *For each (m_1, m_2) such that $2m = 2m_1 + 2m_2$, we have $V_{m_1} \otimes V_{m_2} \subset V_m$.*

We remark while Corollary 3.15 sheds some insights, it is not sufficient to explain the dimension growth of the nullspace for D_n .

3.3. The case where n has at most two prime factors. If n is a prime power then D_n is a complete graph. In this case, the spectrum of D_n is well-known. In this section, we consider the next simplest case: namely n is a product of two prime numbers. First, we consider the case where $n = pq^a$ where p, q are distinct primes.

Proposition 3.16. *Let M_a be an adjacency matrix for D_{pq^a} where p, q are distinct primes, $a \geq 0$. Then $\det(M_a) = \det(M_{a+6})$.*

Proof. We note that $\det(M_5) = 1$, and explicitly that, with respect to the vertex ordering $1, q, \dots, q^5, p, pq, \dots, pq^5$

$$M_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$M_5^{-1} = \begin{bmatrix} -2 & 1 & 2 & 1 & -1 & -2 & -2 & -1 & 1 & 2 & 1 & -1 \\ 1 & -2 & -1 & 0 & 1 & 1 & 2 & 0 & -1 & -1 & 0 & 1 \\ 2 & -1 & -4 & -1 & 2 & 3 & 3 & 2 & -2 & -3 & -1 & 2 \\ 1 & 0 & -1 & -2 & 1 & 2 & 1 & 1 & 0 & -2 & -1 & 1 \\ -1 & 1 & 2 & 1 & -2 & -1 & -2 & -1 & 1 & 2 & 0 & -1 \\ -2 & 1 & 3 & 2 & -1 & -4 & -3 & -2 & 1 & 3 & 2 & -2 \\ -2 & 2 & 3 & 1 & -2 & -3 & -4 & -1 & 2 & 3 & 1 & -2 \\ -1 & 0 & 2 & 1 & -1 & -2 & -1 & -2 & 1 & 2 & 1 & -1 \\ 1 & -1 & -2 & 0 & 1 & 1 & 2 & 1 & -2 & -1 & 0 & 1 \\ 2 & -1 & -3 & -2 & 2 & 3 & 3 & 2 & -1 & -4 & -1 & 2 \\ 1 & 0 & -1 & -1 & 0 & 2 & 1 & 1 & 0 & -1 & -2 & 1 \\ -1 & 1 & 2 & 1 & -1 & -2 & -2 & -1 & 1 & 2 & 1 & -2 \end{bmatrix}$$

Order the vertices of M_a as $1, q, \dots, q^a, p, pq, \dots, pq^a$, and the vertices of M_{a+6} as $1, q, \dots, q^a, p, pq, \dots, pq^a, q^{a+1}, \dots, q^{a+6}, pq^{a+1}, \dots, pq^{a+6}$. Now if we write $\mathbf{1}_{r,s}$ for an $r \times s$ matrix of 1's and $\mathbf{0}_{r,s}$ for an $r \times s$ matrix of 0's, we have

$$M_{a+6} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A = M_a, B = \begin{bmatrix} \mathbf{1}_{a+1,6} & \mathbf{1}_{a+1,6} \\ \mathbf{0}_{a+1,6} & \mathbf{1}_{a+1,6} \end{bmatrix}, C = B^T$, and $D = M_5$. Since D is invertible,

$$\det(M_{a+6}) = \det(D) \det(A - BD^{-1}C).$$

But $\det(D) = 1$, and using the explicit computation of $D^{-1} = M_5^{-1}$ above, $BD^{-1}C = \mathbf{0}_{2a+2,2a+2}$. (We can check this by verifying $\mathbf{1}_{1,12}D^{-1}\mathbf{1}_{12,1} = [\mathbf{0}_{1,6} \ \mathbf{1}_{1,6}]D^{-1}\mathbf{1}_{12,1} = \mathbf{1}_{1,12}D^{-1} \begin{bmatrix} \mathbf{0}_{6,1} \\ \mathbf{1}_{6,1} \end{bmatrix} = [\mathbf{0}_{1,6} \ \mathbf{1}_{1,6}]D^{-1} \begin{bmatrix} \mathbf{0}_{6,1} \\ \mathbf{1}_{6,1} \end{bmatrix} = 0$.) Therefore $\det(M_{a+6}) = \det(A) = \det(M_a)$. \square

Proposition 3.17. For distinct primes p, q and $a \geq 0$, D_{pq^a} has eigenvalue 0 if and only if $a \equiv 1 \pmod{6}$.

Proof. After explicitly computing $\det(M_a) = -1, 0, 3, 5, 4, 1$ for $a = 0, 1, 2, 3, 4, 5$ respectively, this follows immediately from the previous proposition. \square

The more general case is somewhat more involved. We can only prove the if direction.

Proposition 3.18. If $n = p^a q^b$ and $a \equiv b \equiv 1 \pmod{6}$ then 0 is an eigenvalue of D_n .

Proof. Suppose that

$$n = p^u q^v \quad \text{with} \quad u \equiv v \equiv 1 \pmod{6}.$$

List the divisors of n in the order:

$$1, q, q^2, \dots, q^v, p, pq, pq^2, \dots, pq^v, \dots, p^u, p^u q, \dots, p^u q^v.$$

The adjacency matrix of D_n is of the form:

$$M = \begin{bmatrix} V & U & U & \cdots & U \\ U^T & V & U & \cdots & U \\ U^T & U^T & V & \cdots & U \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U^T & U^T & U^T & \cdots & V \end{bmatrix},$$

where

$$V = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(v+1) \times (v+1)} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(v+1) \times (v+1)}$$

Consider the following column matrices of size $(v+1) \times 1$:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 & \cdots & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}^T$$

$$A' = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & -1 & \cdots & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}^T.$$

(In A , the block $\begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$ is repeated $(v-1)/6$ times. The matrix A' is obtained from A by left-shifting.) Similarly, we consider following column matrices of size $(v+1) \times 1$:

$$B = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 & \cdots & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}^T,$$

$$B' = \begin{bmatrix} 0 & -1 & -1 & 0 & 1 & 1 & \cdots & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}^T$$

and

$$C = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 & \cdots & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}^T,$$

$$C' = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 & \cdots & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \end{bmatrix}^T.$$

Let X be the following column matrix of size $[(u+1)(v+1)] \times 1$:

$$X = \begin{bmatrix} A & A' & B & B' & C & C' & \cdots & A & A' & B & B' & C & C' & A & A' \end{bmatrix}^T.$$

(In X , the block $\begin{bmatrix} A & A' & B & B' & C & C' \end{bmatrix}$ is repeated $(u-1)/6$ times.)

We claim $MX = 0$. In particular, it shows that 0 is an eigenvalue of M .

We need to show that for every non-negative integers s, t with $s + t = u \equiv 1 \pmod{6}$, one has

$$\begin{bmatrix} U^T & \cdots & U^T & V & U & \cdots & U \end{bmatrix} X = 0.$$

Here in the first matrix, U^T appears s times and U appears t times.

Note that $A + A' + B + B' + C + C' = 0$. Hence $U^T(A + A' + B + B' + C + C') = 0 = U(B + B' + C + C' + A + A')$. We only need to consider the following six cases.

Case 1: $s \equiv 0 \pmod{6}$ and $t \equiv 1 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = 0.$$

$$\text{Let } \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix}. \text{ We have}$$

$$s_1 = \sum_{i \geq 2} a_i + \sum_{i \geq 1} a'_i = 1 + (-1) = 0.$$

On the other hand, the difference between the $(i+1)$ -st row and the i th row of $[V \mid U]$ is the row

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 & | & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, the first -1 is in the $i+1$ -th position and the second -1 is in the $(s+1)+i$ -th position. We deduce that

$$s_{i+1} - s_i = a_i - a_{i+1} - a'_i = 0.$$

Hence $s_i = 0$, for every i .

Case 2: $s \equiv 1 \pmod{6}$ and $t \equiv 0 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & V \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = 0.$$

$$\text{Let } \begin{bmatrix} U^T & V \end{bmatrix} \begin{bmatrix} A \\ A' \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \end{bmatrix}. \text{ We have}$$

$$s_1 = a_1 + \sum_{i \geq 2} a'_i = 0 + 0 = 0.$$

On the other hand, the difference between the $(i+1)$ -st row and the i th row of $[U^T \mid V]$ is the row

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & | & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, the first 1 is in the $i+1$ -th position and the second 1 is in the $(s+1)+i$ -th position. We deduce that

$$s_{i+1} - s_i = a_{i+1} + a'_i - a'_{i+1} = 0.$$

Hence $s_i = 0$, for every i .

Using similar arguments as in the previous two cases, we can show that

$$VB + UB' = \begin{bmatrix} s & s & \cdots & s \end{bmatrix}^T = U^T B + VB',$$

where $s = \sum_{i \geq 2} b_i + \sum_{i \geq 1} b'_i = -2 = b_1 + \sum_{i \geq 2} b'_i$, and that

$$VA' + UB = U^T A' + VB = \begin{bmatrix} -2 & -2 & \cdots & -2 \end{bmatrix}^T,$$

$$VB' + UC = U^T B' + VC = 0,$$

$$VC + UC' = U^T C + VC' = \begin{bmatrix} 2 & 2 & \cdots & 2 \end{bmatrix}^T.$$

Case 3: $s \equiv 2 \pmod{6}$ and $t \equiv 5 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & V & U & U & U & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$\begin{aligned} U^T(A + A') + VB + U(B' + C + C' + A + A') &= -VA' + U^T A' + VB - UB \\ &= (U^T A' + VB) - (VA' + UB) = 0. \end{aligned}$$

Case 4: $s \equiv 3 \pmod{6}$ and $t \equiv 4 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & U^T & V & U & U & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$\begin{aligned} U^T(A + A' + B) + VB' + U(C + C' + A + A') \\ &= U^T(A + A') + VB + U(B' + C + C' + A + A') + U^T B + VB' - VB - UB' \\ &= U^T B + VB' - (VB + UB') = 0. \end{aligned}$$

Case 5: $s \equiv 4 \pmod{6}$ and $t \equiv 3 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & U^T & U^T & V & U & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$\begin{aligned} U^T(A + A' + B + B') + VC + U(C' + A + A') \\ &= U^T(A + A' + B) + VB' + U(C + C' + A + A') + U^T B' + VC - VB' - UC \\ &= U^T B' + VC - (VB' + UC) = 0. \end{aligned}$$

Case 6: $s \equiv 5 \pmod{6}$ and $t \equiv 2 \pmod{6}$. In this case, it suffices to show that

$$\begin{bmatrix} U^T & U^T & U^T & U^T & U^T & V & U & U \end{bmatrix} \begin{bmatrix} A \\ A' \\ B \\ B' \\ C \\ C' \\ A \\ A' \end{bmatrix} = 0.$$

We have

$$\begin{aligned} & U^T(A + A' + B + B' + C) + VC' + U(A + A') \\ &= U^T(A + A' + B + B') + VC + U(C' + A + A') + U^TC + VC' - VC - UC' \\ &= U^TC + VC' - (VC + UC') = 0. \end{aligned}$$

□

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