Join of circulant matrices

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Joint work with Jackie Doan, Jan Minac, Lyle Muller, Federico Pasini

Social network

Photo credit: Analytics Vidhya

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The spectrum of G, denoted by $\mathrm{Spec}(G)$, is the set of all eigenvalues of its adjacency matrix A. Equivalently, it is the set of all roots of the characteristic polynomial $p_A(t)$ of A where

$$p_A(t) = \det(tI_n - A).$$

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- In our study of networks of oscillators, the topology of the networks (i.e. the structure of connections) plays a fundamental role in understanding the existence and stability of the oscillators.
- And much more!

What is a circulant matrix?

A special type of network that appears often in our work in computational neuroscience is the ring graph.

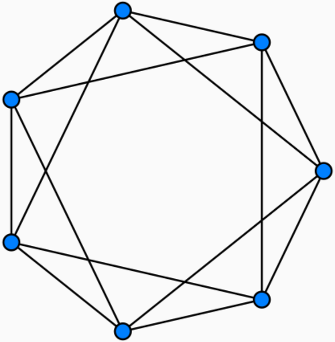

Photo credit: Wikimedia

What is a circulant matrix?

The adjacency matrix of this graph is given by

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

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We see that all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector. This is an example of a circulant matrix.

What is a circulant matrix?

More generally, a circulant matrix is a matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

In particular, a circulant matrix is completely determined by the first row vector $\vec{c} = (c_0, c_1, \ldots, c_{n-1})$.

The Circulant Diagonalization Theorem

Let us take a concrete example of a circulant matrix of size 3×3.

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Let ω_3 be 3-root of unity. Then we have

$$C \begin{pmatrix} 1 \\ \omega_3 \\ \omega_3^2 \end{pmatrix} = \begin{pmatrix} c_0 + c_1\omega_3 + c_2\omega_3^2 \\ c_2 + c_0\omega_3 + c_1\omega_3^2 \\ c_1 + c_2\omega_3 + c_0\omega_3^2 \end{pmatrix} = \begin{pmatrix} (c_0 + c_1\omega_3 + c_2\omega_3^2)1 \\ (c_0 + c_1\omega_3 + c_2\omega_3^2)\omega_3 \\ (c_0 + c_1\omega_3 + c_2\omega_3^2)\omega_3^2 \end{pmatrix}.$$

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We see that $(1, \omega_3, \omega_3^2)^t$ is an eigenvector of C associated with the eigenvalue $c_0 + c_1\omega_3 + c_2\omega_3^2$.

More generally we have the following theorem.

Theorem (The Circulant Diagonalization Theorem)

Let

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

be the circulant matrix formed by the vector $(c_0, c_1, \ldots, c_{n-1})^T$*.*

Let

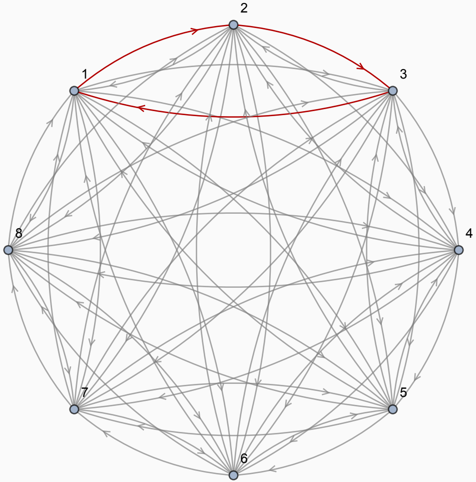
$$v_{n,j} = \left(1, \omega_n^j, \omega_n^{2j}, \ldots, \omega_n^{(n-1)j}\right)^T, \quad j = 0, 1, \ldots, n-1.$$

Then $v_{n,j}$ *is an eigenvector of* C *associated with the eigenvalue*

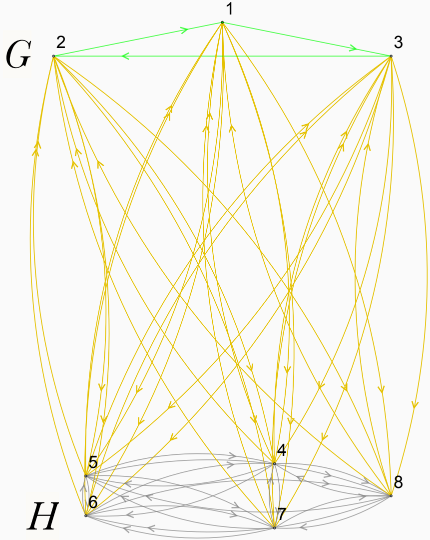
$$\lambda_j^C = c_0 + c_1\omega_n^j + c_2\omega_n^{2j} + \cdots + c_{n-1}\omega_n^{(n-1)j}$$

Join of circulant matrices

Originally, we are interested in graphs obtained by removing a directed cycle in a complete graph. Here is one example.



Here is another way to represent this graph



With this new representation, the adjacency matrix of this graph has the form

$$\left(\begin{array}{ccc|ccccc}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
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\end{array}\right)$$

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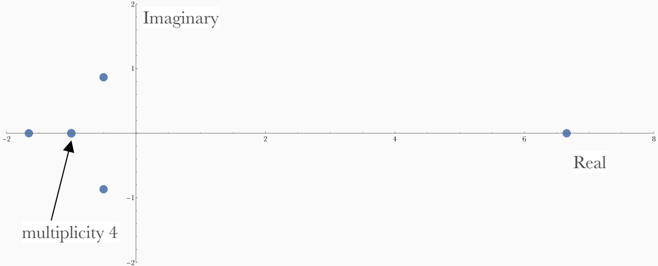

Join of circulant matrices

From this example, we are convinced that it would be quite interesting to study matrices of the following form

$$A = \begin{pmatrix} C & 1_{k_1,k_2} \\ 1_{k_2,k_1} & D \end{pmatrix},$$

with $C = Circ(c_0, \ldots, c_{k_1-1})$ and $D = Circ(d_0, \ldots, d_{k_2-1})$. Here $1_{m,n}$ is the matrix of size $m \times n$ with all entries equal to 1.

Join of circulant matrices

Here is a crucial observation.

Proposition

For $1 \leq j \leq k_1 - 1$ let

$$w_j = (1, \omega_{k_1}^{j}, \omega_{k_1}^{2j}, \ldots, \omega_{k_1}^{(k_1-1)j}, \underbrace{0 \ldots, 0}_{k_2 \text{ zeros}})^T = v_{k_1,j} * \vec{0}_{k_2}.$$

Then w_j is an eigenvector of A associated with the eigenvalue

$$\lambda_j^C = c_0 + c_{k_1-1}\omega_{k_1}^{j} + c_{k_1-2}\omega_{k_1}^{2j} + \cdots + c_1\omega_{k_1}^{(k_1-1)j}.$$

Join of circulant matrices

We have

$$Aw_j = \begin{pmatrix} C & 1_{k_1,k_2} \\ 1_{k_2,k_1} & D \end{pmatrix} \begin{pmatrix} v_{k_1,j} \\ 0 \end{pmatrix} = \begin{pmatrix} Cv_{k_1,j} \\ 1_{k_2,k_1} v_{k_1,j} \end{pmatrix}.$$

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In other words,

$$Aw_j = Cv_{k_1,j} * \underbrace{(t_j, t_j, \ldots, t_j)^T}_{k_2 \text{ terms}}. = \lambda_j^C v_{k_1,j} * \underbrace{(t_j, t_j, \ldots, t_j)^T}_{n-k \text{ terms}}$$

where

$$t_j = \sum_{i=0}^{k_1-1} \omega_k^{ij}.$$

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By the assumption $1 \leq j \leq k_1 - 1$, we have $t_j = 0$. Therefore, $Aw_j = \lambda_j^C w_j$. In other words, w_j is an eigenvector of A associated with the eigenvalue λ_j^C for $1 \leq j \leq k_1 - 1$.

By symmetry, we can also see that

Proposition

For $1 \leq j \leq k_2 - 1$, let

$$z_j = (\underbrace{0, \ldots, 0}_{k_1 \ zeros}, 1, \omega_{k_2}^{j}, \omega_{k_2}^{2j}, \ldots, \omega_{k_2}^{(k_2-1)j})^T = \vec{0}_{k_1} * v_{k_2,j}.$$

Then z_j is an eigenvector of A associated with the eigenvalue

$$\lambda_j^D = d_0 + d_1\omega_{k_2}^{j} + d_2\omega_{k_2}^{2j} + \cdots + d_{k_2-1}\omega_{k_2}^{(k_2-1)j}$$

Join of circulant matrices

In summary, we have been able to find $k_1 + k_2 - 2$ eigenvectors of A. We need to find two more. We will look for eigenvectors of the form

$$v = (\underbrace{x, x, \ldots, x}_{k_1 \text{ terms}}, \underbrace{y, y, \ldots, y}_{k_2 \text{ terms}})^T$$

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By explicit calculations, we have

$$Av = (\underbrace{C_s x + k_2 y, \ldots, C_s x + k_2 y}_{k_1 \text{ terms}}, \underbrace{k_1 x + D_s y, \ldots, k_1 x + D_s y}_{k_2 \text{ terms}})^T.$$

Here $C_s = \sum_{i=0}^{k_1-1} c_i, D_s = \sum_{i=0}^{k_2-1} d_i$.

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Here $C_s = \sum_{i=0}^{k_1-1} c_i, D_s = \sum_{i=0}^{k_2-1} d_i$. The condition $Av = \lambda v$ is equivalent to

$$\begin{pmatrix} C_s & k_2 \\ k_1 & D_s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

Join of circulant matrices

In other words, v is an eigenvector of A if and only if $(x, y)^T$ is an eigenvector of

$$\bar{A} = \begin{pmatrix} C_s & k_2 \\ k_1 & D_s \end{pmatrix}.$$

In summary, we have the following theorem.

Theorem

Let $\{(x_1, y_1)^T, (x_2, y_2)^T\}$ be an (generalized) eigenbasis for $\bar{A}$. Let

$$v_1 = (\underbrace{x_1, x_1, \dots, x_1}_{k_1 \text{ terms}}, \underbrace{y_1, y_1, \dots, y_1}_{k_2 \text{ terms}})^T,$$

$$v_2 = (\underbrace{x_2, x_2, \dots, x_2}_{k_1 \text{ terms}}, \underbrace{y_2, y_2, \dots, y_2}_{k_2 \text{ terms}})^T.$$

Then the system $\{\omega_j\}_{j=1}^{k_1-1} \cup \{z_j\}_{j=1}^{k_2-1} \cup \{v_1, v_2\}$ of eigenvectors of A is linearly independent. In other words, A is diagonalizable by these eigenvectors.

Further results

The above approach can be generalized to study the join of several circulant matrices

$$A = \left(\begin{array}{c|c|c|c} C_1 & a_{1,2}\mathbb{1} & \cdots & a_{1,d}\mathbb{1} \\ \hline a_{2,1}\mathbb{1} & C_2 & \cdots & a_{2,d}\mathbb{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbb{1} & a_{d,2}\mathbb{1} & \cdots & C_d \end{array}\right).$$

Here C_i is a circulant matrix of size $k_i \times k_i$ for each $1 \leq i \leq d$, and $a_{i,j}\mathbb{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j}$.

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Here C_i is a circulant matrix of size $k_i \times k_i$ for each $1 \leq i \leq d$, and $a_{i,j}\mathbb{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j}$. If we just care about the spectrum of A, we only need to assume that C_i is normal with constant row sums.

Thank you