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On the arithmetic of Generalized Fekete polynomials

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Motivations.

$\chi : (\mathbb{Z}/D)^\times \longrightarrow \mathbb{C}^*$ be a Dirichlet character.

$\Big\updownarrow$ Galois representation

$$\mathrm{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q}) \xrightarrow{\simeq} (\mathbb{Z}/D)^\times \xrightarrow{\chi} \mathbb{C}^*$$

Big goal : Use χ to understand $\mathbb{Q}(\zeta_D)$.

L- function associated with χ

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

Example 1 $D=1$, χ : trivial character

$$L(s,\chi)=\sum_{s=0}^{\infty}\frac{1}{n^s}$$ ← classical Riemann zeta function.

Example 2 d square free integer

$$\begin{tikzcd}
\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[rr, bend right=50, "\chi"'] & \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \arrow[r, "\chi"] \arrow[d, phantom, "\wr"] & \{\pm 1\} \subseteq \mathbb{C}^* \\
& \mathbb{Z}/2 &
\end{tikzcd}$$

χ : Dirichlet character , $D = |\Delta|$

$$\Delta = \begin{cases} d & \text{if } d \equiv 1 \ (4) \\ 4d & \text{if } d \equiv 2, 3 \ (4) \end{cases}.$$

- When $d = p$, $p \equiv 1 \ (4)$

$D = \Delta = p$

$$\chi(a) = \left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is square mod } p \\ -1 & \text{if } a \text{ is not } \text{———} \\ 0 & \text{if } p \mid a \end{cases}$$

$L(s, \chi)$ tells us about the class number of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$.

Fekete polynomials

χ: (quadratic) Dirichlet character.

$$\chi : (\mathbb{Z}/D)^{\times} \longrightarrow \mathbb{C}^*$$

Defn The Fekete polynomial associated with χ

$$F_\chi(x) = \sum_{a=0}^{D-1} \chi(a)\, x^a.$$

Why do we care about $F_\chi(x)$?

1. Direct relationship with $L(s,\chi)$

$$\Gamma(s) L(\chi, s) = \int_0^1 \frac{(-\log(t))^{s-1}}{t} \frac{F_\chi(t)}{1-t^D}\, dt$$

$\rightsquigarrow$ Siegel zero problem.

(2) Related to the theory of Gauss sum.

$$G(b,\chi) = \sum_{a=0}^{D-1} \chi(a)\,\zeta_D^{ab} = F_\chi(\zeta_D^{b})$$

$b \in \mathbb{Z}$, ζ_D: primitive D-root of 1.

<u>Theorem</u> $G(b,\chi) = \chi(b)\,G(1,\chi)$.

In particular, if $\gcd(b, D) > 1$ then

$$G(b,\chi) = 0.$$

<u>Cor</u> If $n \mid D$ and $n < D$ then ζ_n is a root of $F_\chi = F_\Delta$.

Remarks 1. The multiplicity of ϑ_n is **not** always 1.

2. There are "exceptional" zeros. (ϑ_n but $n \nmid D$)

Let $\Phi_n(x)$ be the n-th cyclotomic polynomial.

Defn

$r_\Delta(n) := $ multiplicity of ϑ_n is $F_x = F_\Delta$.
(or Φ_n)

Theorem A

$d = 3p$, $\Delta = D = 3p$, $p \equiv 3\ (4)$

Case 1 $p \equiv 2\ (3)$

	Multiplicity
$r_\Delta(1)$	2
$r_\Delta(2)$	1
$r_\Delta(3)$	2
$r_\Delta(6)$	1
$r_\Delta(p)$	1

Case 2 $p \equiv 1\ (3)$

	Multiplicity
$r_\Delta(1)$	2
$r_\Delta(2)$	1
$r_\Delta(3)$	1
$r_\Delta(6)$	0
$r_\Delta(p)$	1

Defn

$$f_\Delta(x) = \frac{F_\Delta(x)}{\text{cyclotomic factors}}$$

Prop $f_\Delta(x)$ is a reciprocal polynomial of even degree.

Theorem B $d = -3p$, $\Delta = -3p$, $D = 3p$

	Multiplicity
Φ_1	1
Φ_3	1
Φ_p	1

In general, if $\Delta < 0$ then $r_\Delta(n) = 1$ if $n \mid \Delta$.

$$f_\Delta(x) = \frac{F_\Delta(x)}{\text{cyclotomic factors}}.$$

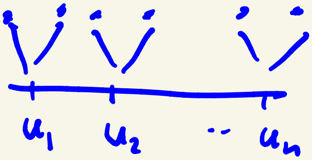
Prop $f_\Delta(x)$ is a reciprocal polynomial.

Galois Theory of f $\quad 2n = \deg f$

$$f = x^{\frac{\deg f}{2}}\, g\left(x + \frac{1}{x}\right)$$

$u_1, u_2, \ldots, u_n$ roots of g. Then roots of f:

$$x + \frac{1}{x} = u_i$$



$$\mathrm{Gal}(\mathbb{Q}(g)/\mathbb{Q}) \subseteq S_n$$

$$\mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q}) \subseteq (\mathbb{Z}/2)^n \rtimes S_n.$$

<u>Slogan</u> The Galois groups of f and g are usually as maximal as possible.

<u>Theorem</u> $\Delta = -3p$, Then for $p \leq 1000$

① If $p \equiv 1 \ (8)$, $2n = \deg f$

$$\mathrm{Gal}(\mathbb{Q}(g_\Delta)/\mathbb{Q}) \simeq S_n$$

$$\mathrm{Gal}(\mathbb{Q}(f_\Delta)/\mathbb{Q}) \simeq (\mathbb{Z}/2)^n \rtimes S_n.$$

② If $p \equiv 5 \ (8)$, $\Delta(f)$ is a square

$$\mathrm{Gal}(\mathbb{Q}(g_\Delta)/\mathbb{Q}) \simeq S_n$$

$$\mathrm{Gal}(\mathbb{Q}(f_\Delta)/\mathbb{Q}) \simeq \ker(\Sigma') \rtimes S_n$$

$$\Sigma' : (\mathbb{Z}/2)^n \longrightarrow \mathbb{Z}/2$$

THANK YOU!