Heights and Tamagawa numbers of motives

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Plans

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- Introduction to the Tamagawa number conjecture for motives.

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- Some ongoing joint work with Ján Mináč, Nguyễn Duy Tân, Lyle Muller and others.

Euler's discoveries

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- Indeed, Euler did much more. In particular, he showed that
$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \ldots = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k},$$
where $\{B_n\}$ are the Bernoulli numbers defined by following Taylor's expansion
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- Euler's discoveries is the beginning of many wonderful and mysterious studies.

Euler's discoveries from a modern perspective

In 1859, Bernhard Riemann studied the following (complex) function in his article "On the Number of Primes Less Than a Given Magnitude"

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad .$$

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Riemann shown that $\zeta(s)$ can be extended to a analytic function on $\mathbb{C}$ except for a simple pole at $s = 1$. Furthermore, it has a symmetric functional equation

$$\zeta(s) = 2^s \pi^{s-1} \ \sin\left(\frac{\pi s}{2}\right) \ \Gamma(1-s)\, \zeta(1-s).$$

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Question

What is the arithmetic significance of the zeta values $\zeta(1-n)$?

Class number formula

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These numbers contain arithmetic information about K.

Class number formula

A quite natural generalization of the Riemann zeta function is the Dedekind zeta function for K defined by

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}.$$

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Theorem (Class number formula)

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{w_K \sqrt{|D_K|}}.$$

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This formula provides a connection between the algebraic world and the analytic world.

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The L-function of E is defined by the Euler product

$$L(s, E) = \prod_p L_p(s, E)^{-1}.$$

The Birch and Swinnerton-Dyer conjecture

Let $E(\mathbb{Q})$ be the set of all rational solutions of E, i.e.

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Conjecture (BSD conjecture, weak form)

$$r_{al} = r_{an},$$

where r_{an} is the order of vanishing of the L-function $L(E, s)$ at $s = 1$.

Generalizations

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- Beilinson and Deligne made a breakthrough by putting this question in the framework of mixed motives and motivic cohomology. However, their work only predicts L-values up to an undetermined rational factor. Beilinson's approach is inspired by the work of Bloch on K_2 of CM elliptic curves.
- Bloch and Kato formulated a more precise conjecture using tools from p-adic Hodge theory discovered by Fontaine.

What is a (mixed) motive?

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- There is still no universally agreed definition of mixed motives even up to this day. Progress has been made by many mathematicians, notably Grothendieck, Jannsen, Bloch, Beilinson, Deligne and many others.
- A folklore definition: $H^n(X)(r)$ where X is a smooth variety over $\mathbb{Q}$. It has different realizations depending on different cohomology theories (de Rham, Betti, etale, crystalline.) These realizations are connected by comparison isomorphisms. For example, if X is smooth and projective, there is an isomorphism between Betti (singular) cohomology and etale cohomology

$$H^n_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell) \cong H^n_B(X(\mathbb{C}), \mathbb{Z}_\ell).$$

A working definition

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- A mixed motive with $\mathbb{Z}$-coefficients is a pair (M, Θ) where
  1. M is a mixed motive with $\mathbb{Q}$ coefficients.
  2. Θ is a free $\mathbb{Z}$-module of finite rank equipped with a linear action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{\Theta} = \widehat{\mathbb{Z}} \otimes \Theta$.
  3. There is an isomorphism $\mathbb{Q} \otimes \Theta \cong M_B$ where M_B is the Betti realization of M.

Period rings and the local Bloch-Kato Selmer groups

Let V be a $\mathbb{Q}_\ell$-vector space equipped with an action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and $T \subset V$ a Galois stable lattice. When $\ell = p$, we define the following subgroups of $H^1(\mathbb{Q}_p, V)$:

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- $H^1_e(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}}^{\varphi=1}))$.
- $H^1_f(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}}))$.
- $H^1_g(\mathbb{Q}_p, V) = \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}))$.

These are called the exponential, finite, and geometric parts of $H^1(\mathbb{Q}_p, V)$.

Local Bloch-Kato Selmer groups

Let $* \in \{e, f, g\}$. We define $H^1_*(\mathbb{Q}_p, T)$ to be the pre-image of of $H^1_*(\mathbb{Q}_p, V)$ under the canonical map

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If M is a $\widehat{\mathbb{Z}}$-module with a continuous $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\,\mathbb{Q}_p)$ action, we define

$$H^1_*(\mathbb{Q}_p, M) = \prod_{\ell} H^1_*(\mathbb{Q}_p, M_\ell).$$

Tamagawa measure

We define the following Selmer conditions on local Galois cohomology.

$$A(\mathbb{Q}_p) = \begin{cases} H^1_f(\mathbb{Q}_p, \widehat{\Theta}) & \text{if } p < \infty \\ (D_\infty \otimes_{\mathbb{R}} \mathbb{C})/(\mathrm{Fil}^0 D_\infty \otimes_{\mathbb{R}} \mathbb{C}) + \Theta)^+ & \text{if } p = \infty. \end{cases}$$

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The exponential map

$$\exp : D_p/D_p^0 \to A(\mathbb{Q}_p),$$

is an isomorphism if $\omega(M) \leq -1$ where D_p is the p-adic de Rham realization of M.

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The induced measure μ on the adelic space $\prod_p' A(\mathbb{Q}_p)$ is called the Tamagawa measure of M.

The Tamagawa number of M is defined as

$$\mathrm{Tam}(M) = \mu\left(\prod\nolimits'_{p\leq\infty} A(\mathbb{Q}_p)/A(\mathbb{Q})\right),$$

where $A(\mathbb{Q})$ can be thought as $\mathrm{Ext}^1(\mathbb{Z},(M,\Theta))$ in the category of motives with $\mathbb{Z}$-coefficients satisfying certain Selmer conditions.

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If $M = H^n(X)(r)$, then we should have $A(\mathbb{Q}) \subset A(\mathbb{Q}) \otimes \mathbb{Q} = \mathrm{gr}^r K_{2n-r-1}(\mathfrak{X}) \otimes \mathbb{Q}$ where $\mathfrak{X}/\,\mathbb{Z}$ is a regular flat model of X.

Tamagawa number conjecture, I

Conjecture (Tamagawa number conjecture)

$$Tam(M) = \frac{\#H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))}{\#\text{Ш}(M)},$$

where $\text{Ш}(M)$ *is the Tate-Shafarevich group associated with* M*.*
It is a conjectured to be a finite group.

Some known results

- $M = \mathbb{Z}(r)$. The class number formula is a special case of this when $r = 1$. (Bloch-Kato)
- $M = H^1(E)(2)$ where E is an elliptic curve with CM (Bloch-Kato)
- $M = H^1(E)(1)$ in some special cases by work of Coates-Wiles, Kolyvagin, Rubin, Kato, and many others. This case is equivalent to the BSD conjecture.
- M is the adjoint motive of a modular form (Diamond, Flach, Guo, Loeffler-Lei-Zerbes and others).
- Some progress for motives associated with automorphic forms on $GSp_4, GL_2 \times GL_2$ (Loeffler-Zerbes and others.)
- Most of these works construct special elements in $A(\mathbb{Q})$.
- Iwasawa theory plays an important role.

Tamagawa number conjecture, II

- In their work, Bloch-Kato mentioned that the Tamagawa number conjecture could be formulated using $B(\mathbb{Q})$ and $B(\mathbb{Q}_p)$ where $B(\mathbb{Q}_p)$ is the inverse image in $H^1_g(\mathbb{Q}_p, \widehat{\Theta})$ of

$$\operatorname{Im}\left(\Psi \to H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})\right),$$

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- We define $B(\mathbb{Q})$ similarly. As before, $B(\mathbb{Q})$ should be considered as the extension group $\mathrm{Ext}^1(\mathbb{Z}, (M, \Theta))$ in the category of motives with $\mathbb{Z}$-coefficients (with no Selmer conditions).

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- When $M = H^n(X)(r)$, we should take

$$\Psi = \mathrm{gr}^r K_{2r-m-1}(X) \otimes \mathbb{Q}.$$

Height functions

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- Height functions play important roles in many parts of mathematics. For example, G. Faltings defined height functions for abelian varieties and used them to give a proof of the Mordell conjecture.
- K. Kato recently defined height functions for motives. He dreams that these definitions will solve important problems in number theory.

Local height functions

Recall that we have the following inclusions

$$A(\mathbb{Q}_p) := H^1_f(\mathbb{Q}_p, \widehat{\Theta}) \subset B(\mathbb{Q}_p) \subset H^1_g(\mathbb{Q}_p, \widehat{\Theta}).$$

Furthermore, under some technical conditions then

$$H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell) = \begin{cases} \left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1} & \text{if } \ell \neq p \\ \left((\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1)\right)^{\varphi=1} & \text{if } \ell = p. \end{cases}$$

We use this identification and the polarization of M to define the height function

$$H_{\diamond,v,d} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}.$$

By definition, $H_{\diamond,v,d}$ is trivial on $A(\mathbb{Q}_p)$. Furthermore, it is expected that $H_{\diamond,v,d}$ is induced by a positive-definite quadratic form on $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$.

Local height functions

Under some conjectures as in Bloch-Kato's article, we have $B(\mathbb{Q}_p)/A(\mathbb{Q}_p) \cong \mathbb{Z}^{r_p}$ as topological groups. By choosing a basis, we have

$$B(\mathbb{Q}_p) = A(\mathbb{Q}_p) \times \mathbb{Z}^{s_p} .$$

The local Tamagawa measure μ on $B(\mathbb{Q}_p)$ is defined to be the product measure of the Tamagawa measure on $A(\mathbb{Q}_p)$ and the counting measure on $\mathbb{Z}^{s_p}$.

The following statement is straightforward.

Proposition

For each positive real number B

$$\mu(a \in B(\mathbb{Q}_p) | H_{\diamond,p,d}(a) \leq B) < \infty.$$

Adelic and global height functions

We define

$$H_{\diamond,d} : \prod\nolimits'_{p\leq\infty} B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}$$

as the product of local height functions.

Similarly, we define the height function on $B(\mathbb{Q})$ using the canonical embedding

$$B(\mathbb{Q}) \to \prod_{p\leq\infty} B(\mathbb{Q}_p).$$

By definition $H_{\diamond,d}$ is trivial on $A(\mathbb{Q})$.

We also define the Tamagawa measure on $\prod'_{p\leq\infty} B(\mathbb{Q}_p)$ as the product of the local Tamagawa measure on $B(\mathbb{Q}_p)$.

Adelic and global height functions

A simple observation.

Proposition

For each positive real number B, the set

$$\{a \in \prod\nolimits'_{p \leq \infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B\}.$$

is compact. In particular,

$$\mu(a \in \prod\nolimits'_{p \leq \infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B) < \infty,$$

and

$$\#\{a \in B(\mathbb{Q}) | H_{\diamond,d}(a) \leq B\} < \infty.$$

We have the the following theorem.

Theorem (Nguyen)

Let M be a pure motive with $\mathbb{Z}$-coefficients of weight $-d$ such that $d \geq 3$. We assume further that M has semistable reduction at all places. Then

$$\lim_{B\to\infty} \frac{\#\{x \in B(\mathbb{Q}) | H_{\diamond,d}(x) \leq B\}}{\mu\left(x \in \prod'_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(x) \leq B\right)} = \frac{1}{Tam(M)}.$$

Main idea of the proof: For a symmetric, convex region A in $\mathbb{R}^n$, the number of integer points in A is asymptotic to the area of A.

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- In an ongoing project with Lyle Muller et al, we are investigating the Ihara zeta functions for graphs, random graphs and their applications to neuroscience.

Thank you!