Zeta functions of the join algebras over finite fields

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The Kuramoto model on networks of phase oscillators.

- Let G be a graph/network with adjacency matrix $A = (a_{ij})$. The Kuramoto model on G is described by the following differential equations

$$\dot{\theta}_i = \epsilon \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i)\,, \tag{0.1}$$

  where $\theta_i(t) \in [-\pi, \pi]$ is the state of oscillator $i \in [1, N]$ at time t and ϵ is the coupling strength.
- It is known that the structure of G strongly influences the dynamics governed by the Kuramoto model.

A multilevel network

- Many real-world networks are multilevel; namely, they are composed of several smaller “communities” joined together.

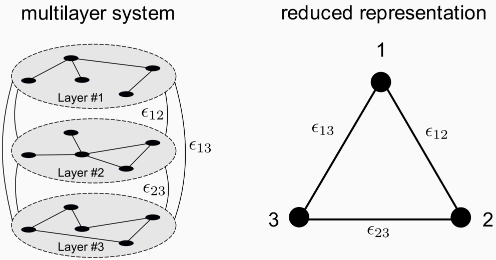

Figure 1: A multilevel network with three layers

- From both a theoretical and an applied perspective, it is interesting and important to study the spectra of these multilevel networks.

Let G_1, G_2 be two graphs. The join of G_1 and G_2 is defined pictorially as follows.

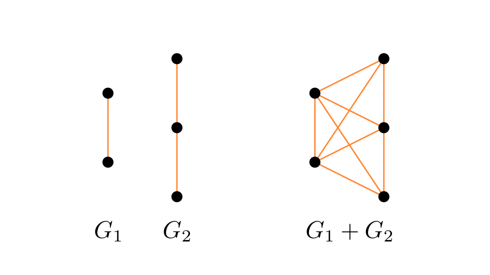

Figure 2: The join of two graphs

- More generally, suppose G is a (weighted) graph with d vertices $\{v_1, v_2, \ldots, v_d\}$. Let $G_1, G_2, \ldots, G_d$ be (weighted) graphs on $k_1, k_2, \ldots, k_d$ vertices. The **joined union** $G[G_1, G_2, \ldots, G_d]$ is obtained from the union of $G_1, \ldots, G_d$ by joining with an edge each pair of a vertex from G_i and a vertex from G_j whenever v_i and v_j are adjacent in G.
- The adjacency matrix of $G[G_1, G_2, \ldots, G_d]$ has the following form

$$A = \left[\begin{array}{c|c|c|c} A_{G_1} & a_{12}J_{k_1,k_2} & \cdots & a_{1d}J_{k_1,k_d} \\ \hline a_{21}J_{k_2,k_1} & A_{G_2} & \cdots & a_{2d}J_{k_2,k_d} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d1}J_{k_d,k_1} & a_{d2}J_{k_d,k_2} & \cdots & A_{G_d} \end{array}\right].$$

Here $J_{m,n}$ is the matrix of size $m \times n$ with all entries equal to 1, A_{G_i} is the adjacency matrix of G_i, and $A = (a_{ij})$ is the adjacency matrix of G.

Circulant matrices and group rings

Let R be a ring with unity and G a finite group of size n.

Definition

An $n \times n$ G-circulant matrix over R is an $n \times n$ matrix of the form

$$A = (a_{\tau^{-1}\sigma})_{\tau,\sigma \in G},$$

where $a_g \in R$ for all $g \in G$.

We see that A is uniquely determined by the vector $[a_g]_{g \in G}$. For convenience, we can write

$$A = \operatorname{circ}([a_g]_{g \in G}).$$

We will denote by $J_G(R)$ the set of all G-circulant matrices over R.

We can check that $J_G(R)$ is a subring of $M_n(R)$. Let us also recall

$$R[G] = \{\sum_{g \in G} a_g g\},$$

the group ring of G over R. We have the following

Proposition (Hurley)

The map

$$R[G] \to J_G(R),$$

$$\sum_{g \in G} a_g g \mapsto circ([a_g]_{g \in G}),$$

is a ring isomorphism.

The join group ring $J_{G_1,G_2,\ldots,G_d}(R)$

Definition

Let $G_1, G_2, \ldots, G_d$ be groups of size $k_1, k_2, \ldots, k_d$ respectively. A join of circulant matrices R is a matrix of the form

$$A = \left(\begin{array}{c|c|c|c} A_1 & a_{1,2}J & \cdots & a_{1,d}J \\ \hline a_{2,1}J & A_2 & \cdots & a_{2,d}J \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}J & a_{d,2}J & \cdots & A_d \end{array}\right),$$

where A_i is a G_i-circulant matrix and J denotes the matrix with all entries equal to 1.

We will denote by $J_{G_1,G_2,\ldots,G_d}(R)$ the set of all such A.

The join group ring $J_{G_1,G_2,\ldots,G_d}(R)$

We have the following observation.

Proposition

$J_{G_1,G_2,\ldots,G_d}(R)$ is a subring of $M_n(R)$ where $n = \sum_{i=1}^{d} |G_i|$. Furthermore, there is an augmentation map $J_{G_1,G_2,\ldots,G_d}(R) \to M_d(R)$ defined by

$$\varepsilon(A) = \begin{bmatrix} \epsilon(A_1) & k_2 a_{12} & \cdots & k_d a_{1d} \\ k_1 a_{21} & \epsilon(A_2) & \cdots & k_d a_{2d} \\ \vdots & \vdots & & \vdots \\ k_1 a_{n1} & k_2 a_{n2} & \cdots & \epsilon(A_d) \end{bmatrix}.$$

Here ϵ is the classical augmentation map on $R[G_i]$.

Zeta functions of $\mathbb{F}_q$-algebras

Let $\mathbb{F}_q$ be the finite field with $q = p^r$ elements and R a finite-dimensional $\mathbb{F}_q$-algebra.

Definition (Following Fukaya, Kato, Kurokawa)

The Hasse-Weil zeta function of R is given by the following Euler product

$$\zeta_R(s) = \prod_M (1 - |End_R(M)|^{-s})^{-1}, \tag{0.2}$$

where M runs over the isomorphism classes of (finite) simple left R-modules.

Note that for an $\mathbb{F}_q$-algebra, we always have

$$\zeta_R(s) = \zeta_{R^{ss}}(s),$$

where $R^{ss} = R/\mathrm{Rad(R)}$ is the semisimplification of R.

Some examples

1. Let $R = M_n(\mathbb{F}_q)$. By the Morita equivalence, we have
$$\zeta_R(s) = \zeta_{\mathbb{F}_q}(s) = (1 - q^{-s})^{-1}.$$
2. Suppose G is a p-group $R = \mathbb{F}_q[G]$. Then R is a local ring with
$$\mathrm{Rad}(R) = \ker(\epsilon : R \to \mathbb{F}_q).$$
In particular, $R^{\mathrm{ss}} = \mathbb{F}_q$ and $\zeta_R(s) = (1 - q^{-s})^{-1}$.
3. If $p \nmid |G|$ and G is split over $\mathbb{F}_q$, then by the Artin-Wedderburn theorem
$$R = \mathbb{F}_q[G] \cong \prod_{i=1}^{d} M_{n_i}(\mathbb{F}_q).$$
Therefore
$$\zeta_R(s) = (1 - q^{-s})^{-d}.$$

Zeta function of the join algebra $J_{G_1,G_2,\ldots,G_d}(\mathbb{F}_q)$

Up to an ordering, there exists a (unique) positive integer r such that

- $p \nmid |G_i|, 1 \leq i \leq r.$
- $p \mid |G_i|, r < i \leq d.$

Theorem

The zeta function of of the join algebra $J_{G_1,G_2,\ldots,G_d}(\mathbb{F}_q)$ is given by

$$\zeta_{J_{G_1,G_2,\ldots,G_d}(\mathbb{F}_q)}(s) = (1-q^{-s})^{r-1} \prod_{i=1}^{d} \zeta_{\mathbb{F}_q[G_i]}(s).$$

Sketch of the proof in the semisimple case

Assume that $|G_i|$ are all invertible in $\mathbb{F}_q$. Let

$$e_{G_i} = \frac{1}{|G_i|} \sum_{g \in G_i} g.$$

Then e_{G_i} is a central idempotent element in $\mathbb{F}_q[G]$. Therefore, we have the following decomposition

$$\mathbb{F}_q[G_i] \cong \mathbb{F}_q[G_i]e_{G_i} \times \mathbb{F}_q(1 - e_{G_i}) \cong \mathbb{F}_q \times \Delta_{G_i}(\mathbb{F}_q),$$

where $\Delta_{G_i}(\mathbb{F}_q) = \ker(\mathbb{F}_q[G_i] \to \mathbb{F}_q)$.

Using these idempotents and the generalized augmentation map, we can show that

$$J_{G_1, G_2, \ldots, G_d}(\mathbb{F}_q) \cong M_d(\mathbb{F}_q) \times \prod_{i=1}^{d} \Delta_{G_i}(\mathbb{F}_q).$$

The formula for the zeta function of $J_{G_1, G_2, \ldots, G_d}(\mathbb{F}_q)$ follows easily from this isomorphism.

Sketch of the proof in the general case

In general, we can show that

$$J_{G_1,G_2,\dots,G_d}(\mathbb{F}_q)^{\mathrm{ss}} \cong J_{G_1,\dots,G_r}(\mathbb{F}_q) \times \prod_{i=r+1}^{d} \mathbb{F}_q[G_i]^{\mathrm{ss}}.$$

The zeta function of $J_{G_1,G_2,\dots,G_d}(\mathbb{F}_q)$ can be computed via this isomorphism and the calculations done in the semisimple case.

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