Special values the Riemann zeta function at negative integers

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Plans

1. Motivations for zeta functions and their special values.
2. Special values of (generalized) Riemann zeta functions at negative integers.

A motivation: Fermat last theorem

- Fermat last theorem says that for odd prime p, the equation
$$x^p + y^p = z^p,$$
has no non-trivial solutions.
- In the 19th century, Kummer "almost" solved this problem by using the following factorization in $\mathbb{Z}[\zeta_p]$
$$x^p = (z^p - y^p) = \prod_{i=0}^{p-1}(z - \zeta^i y).$$
- The issue is that $\mathbb{Z}[\zeta_p]$ might not have the unique factorization property.
- The class number h of $\mathbb{Z}[\zeta_p]$ measures the failure of the unique factorization property of $\mathbb{Z}[\zeta_p]$. When $p \nmid h$, the above elementary approach works.

The class number formula

Let K be a number field and $\mathcal{O}_K$ be its ring of algebraic integers. Let

- r_1 (respectively r_2) be the number of real (respectively complex) places of K.
- h_K: class number of K.
- w_K: number of roots of unity in K.
- R_K: the regulator of K.
- D_K: the discriminant of K.

These numbers contain arithmetic information about K.

The class number formula

The Dedekind zeta function of K

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}.$$

Theorem (Class number formula)

$$\lim_{s \to 1} \frac{\zeta_K(s)}{s-1} = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{w_K \sqrt{|D_K|}}.$$

This formula provides a connection between the algebraic world and the analytic world.

The generalized class number formula (The Bloch-Kato's conjecture)

- More generally, Bloch-Kato made a precise conjecture about special values of the L-funtion attached to a motive.
- Roughly speaking, it says that the many arithmetic information of an algebraic variety X can be read off from its L-function.
- For example, if X is given by an cubic equation of the form

$$y^2 = x^3 + ax + b,$$

  with $a, b \in \mathbb{Q}$ then the L-function $L(E, s)$ can conjecturally tell how large $X(\mathbb{Q})$ is. This is known as the Birch and Swinnerton-Dyer conjecture.

Three phases of understanding of zeta values (according to K. Kato

1. Rationality of zeta values.
2. p-adic properties of zeta values.
3. Arithmetic significances of zeta values.

Today talk is mostly about phase 1 and a little bit about phase 2 and phase 3.

An explicit example: The Riemann zeta function

The Riemann zeta function is defined as (this is the same as the Dedekind zeta function when $K = \mathbb{Q}$)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is easy to show that $\zeta(s)$ is convergent for $Re(s) > 1$.
Furthermore, we have the following theorem.

Theorem (Riemann)

There is a meromorphic continuation of $\zeta(s)$ to the whole complex plan. The only (simple) pole of $\zeta(s)$ is $s = 1$. Furthermore, $\zeta(s)$ satisfies the following functional equation.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

An explicit example: The Riemann zeta function

In 1734, Euler discovered the following remarkable formula. For $n \in \mathbb{N}$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}(1)}{2(2n)!},$$

where $B_{2n}(x)$ are the Bernoulli polynomials defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k.$$

For example

$$B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

In particular, we have the following nice formula

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

An explicit example: The Riemann zeta function

From the functional equation, we can show that for all $n \geq 0$

$$\zeta(-n) = -\frac{B_{n+1}(1)}{n+1}.$$

In particular, this shows that $\zeta(-n)$ is a rational number for all $n \geq 0$. Furthermore, $\zeta(-2n) = 0$ for all $n > 0$. Here are some values of $B_n(1)$ (photo credits [2])

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

In particular,

$$\zeta(-1) = -\frac{1}{12}, \zeta(-2) = 0, \zeta(-3) = \frac{1}{120}.$$

Zeta values and power sums

For each $k \geq 0$, consider

$$S_k(n) = 1^k + 2^k + \ldots + (n-1)^k.$$

Some examples of $S_k(x)$ for small k are

$$S_1(x) = \frac{x(x-1)}{2}, S_2(x) = \frac{x(x-1)(2x-1)}{6}.$$

We can show that $S_k(x)$ are polynomials of degree $k+1$. It is closely related to $B_{k+1}(x)$. In fact, we have

$$S_k(x) = \frac{B_{k+1}(x) - B_{k+1}(1)}{k+1}.$$

We have a beautiful theorem due to Minac.

Theorem (Minac)

$$\zeta(-n) = \int_0^1 S_n(x)dx.$$

The first proof

There are two proofs for Minac's theorem. The first one uses Bernoulli polynomials.

$$\int_0^1 S_n(x)dx = \int_0^1 \frac{B_{n+1}(x) - B_{n+1}(1)}{n+1}dx$$
$$= \int_0^1 \frac{B_{n+1}(x)}{n+1}dx - \frac{B_{n+1}(1)}{n+1}.$$

It is known that $\int_a^{a+1} B_{n+1}(x)dx = a^{n+1}$ for all a. In particular, $\int_0^1 B_{n+1}(x)dx = 0$. Additionally, by Euler's theorem

$$\zeta(-n) = -\frac{B_{n+1}(1)}{n+1}.$$

Combining these two facts, we have

$$\zeta(-n) = \int_0^1 S_n(x)dx.$$

The second proof

We can also prove Minac's theorem without using Bernoulli polynomials. Instead, we use the following identity

$$1 = \sum_{q=0}^{\infty} \frac{(s-1)\dots(s+q-1)}{(q+1)!}(\zeta(s+q)-1).$$

By plugging $s = -1, -2, \dots$, we get some combinatorial relations between $\zeta(-n)$ for $n \geq 0$. Using the collapsing sum method, we have

$$\begin{aligned}(M-1)^{n+1} &= 1 + \sum_{m=2}^{M-1}(m^{n+1} - (m-1)^{n+1}) \\ &= \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k S_{n-k}(M).\end{aligned}$$

Integrating both sides from 0 to 1, we easily see that $\int_0^1 S_n(x)dx$ satisfies a similar combinatorial relation.

A generalization of Minac's theorem

For each $0 < a \leq 1$, we consider the Hurwitz zeta function

$$\zeta_H(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Consider the following power sum

$$S_{n,a}(M) = a^n + (a+1)^n + \ldots + (a+M-2)^n.$$

Then, we have the following theorem.

Theorem (Minac, Nguyen Duy Tan, N.)

For each $n \geq 0$

$$\zeta_H(-n, a) = \int_{1-a}^{2-a} S_{n,a}(x)dx.$$

There is also a "twisted" version of this theorem.

p-adic properties of zeta values

By Fermat's little theorem, we know that if $s \equiv t \pmod{p-1}$ then

$$n^s \equiv n^t \pmod{p}.$$

Ignoring convergence issue we have

$$\zeta(-s) = \sum_{n=1}^{\infty} n^s \equiv \sum_{n=1}^{\infty} n^t = \zeta(-t) \pmod{p}.$$

Amazingly, we can make this intuitive argument become rigorous.

In fact, we have the following theorem.

Theorem (Kummer)

Let r_1, r_2 be two positive integers. Suppose r_1 is not a multiple of $p-1$. If $r_1 \equiv r_2 \pmod{(p-1)p^{n-1}}$ then

$$(1-p^{r_1-1})\zeta(1-r_1) \equiv (1-p^{r_2-1})\zeta(1-r_2) \pmod{p^n}.$$

Arithmetic significance of zeta values

Recall that Fermat last theorem holds if $p \nmid h$ where h is the class number of $\mathbb{Z}[\zeta_p]$. The following criterion is due to Kummer.

Theorem (Kummer)

If $p \nmid \prod_{i=1}^{\frac{p-3}{2}} \zeta(1-2i)$ *then* $p \nmid h$*.*

There is a more refined version due to Herbrand-Ribet.

Ribet's proof is quite interesting: it uses Galois representations attached modular forms and congruence between modular forms. Further generalizations of his method have led to a proof of Fermat last theorem for all odd prime p (due to the work of Wiles, Taylor, and many others).

Some useful references

- S. Bloch, K. Kato, L-functions and Tamagawa numbers motives
- Roman J. Dwilewicz, Jan Minac, Values of the Riemann zeta function at integers
- Kazuya Kato, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}.

Thank you!