Hurwitz zeta functions

A motivation

Let $\chi : (\mathbb{Z}/k)^\times \to \mathbb{C}^\times$

$$\underline{\underline{L(s,\chi)}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

<u>Goal</u> Understand $L(s,\chi)$ analytically / arithmetically

$$L(s,\chi) = \sum_{r=1}^{\ell} \left(\sum_{n \equiv r (\ell)} \frac{\chi(n)}{n^s} \right)$$

$$= \sum_{r=1}^{\ell} \left(\sum_{m=0}^{\infty} \frac{\chi(mk+r)}{(mk+r)^s} \right)$$

$$= \sum_{r=1}^{\ell} \chi(r)\, \ell^{s} \left(\sum_{m=0}^{\infty} \frac{1}{(m+\frac{r}{\ell})^s} \right)$$

We should study $\displaystyle \sum_{m=0}^{\infty} \frac{1}{(m+\frac{r}{\ell})^s}$

Defn Let $0 < a \leq 1$. The Hurwitz zeta function

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

Some examples

a=1

$$\zeta(s,1) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \zeta(s)$$

classical Riemann zeta function.

$a = \frac{1}{2}$

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^s} = 2^s \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}}$$

$$= 2^s \left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots \right)$$

$$\zeta(s) = \zeta(s,1) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

$$\prod_p (1-p^{-s})^{-1} = \left(\underbrace{1 + \frac{1}{3^s} + \frac{1}{5^s}} + \dots \right) + \left(\frac{1}{2^s} + \frac{1}{4^s} + \dots \right)$$

$$= 2^{+s} \zeta(s, \tfrac{1}{2}) + \frac{1}{2^s} \left(\underbrace{1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots} \right)$$

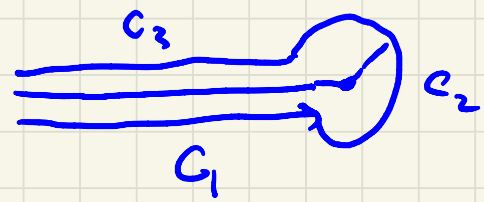
$$= 2^s \zeta(s, \tfrac{1}{2}) + 2^{-s} \zeta(s, 1)$$

$$\rightarrow \quad \underline{\zeta(s, \tfrac{1}{2})} = 2^{-s} (1 - 2^{-s}) \zeta(s)$$

An integral representation of $\zeta(a, s)$

$$\zeta(s,a) = \Gamma(1-s)\, I(s-a)$$

$$I(s) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}\, e^{az}}{1-e^{z}}\,dz$$



* $I(s,a)$ is holomorphic
* $\Gamma(s)$ is meromorphic, with simple at $0, -1, -2, -3, ..$

<u>A corollary</u> $\zeta(s,a)$ is meromorphic ~~at~~, except for a simple pole at $s=1$ with residue 1.

<u>Proof</u>

* $I(s,a)$ is an entire function ✓
* $\Gamma(1-s)$ has simple poles at $s = 1, 2, 3, \ldots$ ✓
* $\zeta(a,s)$ is homorphic if $re(s) > 1$
* $I(1,a) = \frac{1}{2\pi i}\int_{C_2} \frac{e^{az}}{1-e^{z}} = \operatorname{Res}_{z=0} \frac{e^{az}}{1-e^{z}} = -1$

So $\lim_{s\to 1} (s-1)\,\zeta(a,s) = \lim_{s\to 1} (s-1)\,\Gamma(1-s)\, I(s,a) = (-1)(-1)$

$= 1$

A corollary

(1) $\zeta(s)$ is meromorphic, except for a simple pole at $s=1$, with residue 1. $(a=1)$

(2) χ is a primitive character then $L(\chi, s)$ is holomorphic everywhere.

Proof

$$L(s,\chi) = k^s \sum_{r=1}^{K} \chi(r) \underbrace{\zeta\left(s, \frac{r}{\ell}\right)}$$

$s=1$ Sum of the residue at $s=1$

$$\underbrace{\sum_{r=1}^{\ell}} \chi(r) = 0$$

$\boxed{\text{Special values of Hurwitz zeta functions}}$

- Some motivating examples.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

Euler: $\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q}$. , $\zeta(3), \zeta(5), \zeta(7) \ldots$

Apéry: $\zeta(3)$ is irrational

- Three phases of understanding zeta values.

  * Rationality of zeta values. $\leftarrow$

  * p-adic properties of zeta value.

$$\zeta(-n) = 1^n + 2^n + 3^n + \cdots$$

$$m \equiv n \ (p-1) \qquad a^m \equiv a^n \pmod p$$

$$\zeta(-n) \equiv \zeta(-m) \pmod p \leftarrow \text{can be made rigorous.}$$

  * Arithmetic applications of zeta values (class NF, Bloch-Kato)

Evaluations of $\zeta(-n,a)$ (I)

$$\zeta(-n,a) = I(-n,a)\,\Gamma(n+1) = n!\,I(-n,a)$$

$$I(-n,a) = \frac{1}{2\pi i}\int_{C_2} \frac{z^{-n-1}e^{az}}{1-e^{z}}\,dz = \operatorname*{Res}_{z=0}\left(\frac{z^{-n-1}e^{az}}{1-e^{z}}\right)$$

Defn (1) $B_n(x)$ is the function given by the expansion

$$\frac{z e^{xz}}{e^{z}-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n$$

(2) $B_n(0)$ are called Bernoulli numbers.

Prop $B_n(x)$ are polynomials and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

and

$$B_m'(x) = m B_{m-1}(x) \quad (m \geq 1)$$

Proof

$$\frac{z e^{xz}}{1-e^{z}} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n$$

$$e^{xz}\frac{z}{1-e^{z}} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} z^n\right)\left(\sum_{n\geq 0} \frac{B_n}{n!} z^n\right)$$

Coeff of z^n: $\quad \frac{B_n(x)}{n!} = \sum_{k} \frac{1}{(n-k)!\,k!} B_K\, x^{n-K}$

Theorem For $n \geq 0$, we have

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}$$

Proof

$$I(-n, a) = n! \operatorname*{Res}_{z=0} \left(\frac{z^{-n-1} e^{az}}{1-e^{z}} \right)$$

$$= n! \operatorname*{Res}_{z=0} z^{-n-2} \left(\frac{z e^{az}}{1-e^{z}} \right)$$

$$= -n! \operatorname*{Res}_{z=0} \left(z^{-n-2} \sum_{m=0}^{\infty} \frac{B_m(a)}{m!} z^m \right)$$

$$= -\frac{B_{n+1}(a)}{n+1}$$

Reference Apostol: Introduction to analytic number theory

Some examples

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30}$$

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12},\quad \zeta(-2) = 0$$

$$\zeta(-3) = -\frac{B_4}{4} = +\frac{1}{120}$$

$\boxed{\text{Minac's observation}}$

$n \in \mathbb{N}$

$$S_n(M) = 1^n + \dots + (M-1)^n$$

$$S_1(M) = 1 + 2 + \dots + (M-1) = \frac{M(M-1)}{2}$$

$$S_2(M) = 1 + 2^2 + 3^2 + \dots + (M-1)^2 = \frac{M(M-1)(2M-1)}{6}$$

$$S_3(M) = \left(\frac{M(M-1)}{2}\right)^2$$

$S_n(M)$ is a polynomial of degree $(n+1)$, with leading coef $\frac{1}{n+1}$

$$\boxed{\zeta(-n) = \zeta(-n, 1) = \int_0^1 S_n(x)\,dx.}$$ (Minac)

$$S_1(x) = \frac{x(x-1)}{2}$$

$$\zeta(-1) = \int_0^1 \frac{x(x-1)}{2}\,dx$$

$$= \int_0^1 \frac{x^2}{2} - \frac{x}{2}\,dx$$

$$= \left[\frac{1}{6}x^3 - \frac{x^2}{4}\right]_1^0 = \frac{1}{6} - \frac{1}{4} = \boxed{-\frac{1}{12}}$$

Tautological proof

$$B_n(x+1) - B_n(x) = n x^{n-1}$$

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$$

$$\sum_{k=1}^{M-1} (n+1)\,k^n = \frac{B_{n+1}(M) - B_{n+1}(1)}{n+1}$$

$$\int_0^1 \underbrace{S_n(x)}\, dx = \int_0^1 \frac{B_{n+1}(x) - B_{n+1}(1)}{n+1}\, dx$$

$$= \int_0^1 \frac{B_{n+1}(x)}{n+1}\, dx - \frac{B_{n+1}(1)}{n+1}$$

$B_{n+2}'(x) = (n+2)\, B_{n+1}$

$B_{n+1}(x+1) - B_{n+1}(x) = n x^n$

$\downarrow$

$B_{n+1}(1) = B_{n+1}(0)$

$$= \underbrace{\left.\frac{B_{n+2}(x)}{?}\right|_0^1}_{0} - \underbrace{\frac{B_{n+1}(1)}{n+1}}$$

$$= \zeta(-n, 1)$$

- There is another proof w/t Bernoulli polynomials.

$$(s-1)(\zeta(s) - 1) = -\sum_{q=1}^{\infty} \frac{(s-1)s\ldots(s+q-1)}{(q+1)!}(\zeta(s+q) - 1)$$

Plug $s = -n$

$$\left|\zeta_{\mathbb{Q}}(1-m)\right|_p = \pm 2^r \frac{\left|K_{2n-2}(\mathbb{Z})\right|_p}{\left|K_{2n-1}(\mathbb{Z})\right|_p}$$

Known due to Mazur - Wiles proof of Iwasawa main conjecture.

$$\frac{1}{3}$$

$$\sum \frac{1}{\left(n+\frac{1}{3}\right)^s} = 3^s \sum \frac{1}{(3n+1)^s}$$

[1] mod 3

$$\sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{(N(\mathfrak{a}) + \mathfrak{a})^s}$$

$$\sum_{\alpha \equiv I()} \frac{}{N(\alpha)^s}$$