PERFECT STATE TRANSFERS ON GCD-GRAPHS OVER A FINITE FROBENIUS RING

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ABSTRACT. The existence of perfect state transfers (PST) on networks is an important problem in mathematics and physics. Various works in the literature have focused on PST in graphs with arithmetic origins, such as gcd-graphs over $\mathbb{Z}$ and cubelike graphs. In this article, building on our recent work on gcd-graphs over an arbitrary finite Frobenius ring, we investigate the existence of PST in these graphs. Our approach is algebraic in nature, which enables us to unify various existing results in the literature.

1. INTRODUCTION

Let G be an undirected graph with adjacency matrix A_G. Let $F(t)$ be the *continuous-time* quantum walk associated with G; namely $F(t) = \exp(iA_G t)$. There is a perfect state transfer (PST) in graph G if there are distinct vertices a and b and a positive real number t such that $|F(t)_{ab}| = 1$ (see [7] for some fundamental results about PST on graphs, and [1] for a survey on the state of the art of the problem). We note that some researchers define a PST on G using another matrix describing the *discrete-time* quantum walk on G (for example, see [3]). In this paper, we follow Godsil's definition as stated above. It is interesting to note that $F(t)$ also appears quite unexpectedly in the complex-valued Kuramoto model, a standard model in non-linear dynamics that describes the synchronization of phase oscillators ([4, 14]). By [7, Theorem 5.2], it is known that if a regular graph G has a PST, then G must be integral; i.e, all of its eigenvalues must be an integer. In particular, the class of Cayley graphs that have a PST is a subset of the class of integral Cayley graphs.

Let us briefly recall a special class of integral graphs that have a long history in algebraic graph theory; namely the gcd-graphs introduced by Klotz and Sander in [10]. Let n be a positive integer and D a subset of proper divisors of n. The gcd-graph $G_n(D)$ is the graph with the following data.

(1) The vertices of $G_n(D)$ are elements of the finite ring $\mathbb{Z}/n$.
(2) Two vertices a, b are adjacent if $\gcd(a - b, n) \in D$.

Using the theory of Ramanujan sums, Klotz and Sander explicitly describe the spectrum of $G_n(D)$ (see [10, Section 4]). Various works in the literature have utilized this

explicit spectral description to study perfect state transfer (PST) on gcd-graphs (see, for example, [2, 7, 18]).

In [13], inspired by the analogy between number fields and function fields, we introduce the notation of gcd-graphs over $\mathbb{F}_q[x]$ as an analog of the gcd-graphs over $\mathbb{Z}$. Using the theory of Ramanujan and Gauss sums developed by Lamprecht in [11], we are also able to describe the spectra of these gcd-graphs. Furthermore, we provide the necessary and sufficient conditions for a gcd-graph over $\mathbb{F}_q[x]$ to be integral (see [16]). We soon realize that the definition of gcd-graphs can in fact be defined over an arbitrary ring (we work out the details in [17]). When the underlying ring R is a finite Frobenius ring, their natural duality allows us to relate spectra of these generalized gcd-graphs to certain Ramanujan sums which can then be calculated explicitly (see [17, Theorem 4.16]). A direct consequence of this spectral analysis is that the gcd-graph $\Gamma(R, S)$ is integral, making it a promising candidate for graphs that have PST. We note that the integrality of $\Gamma(R, S)$ also follows from [16, Theorem 1.1], although that theorem does not provide a concrete spectral description. In this article, we develop an algebraic and number-theoretic approach to the problem of PST on gcd-graphs. As we will explain later in the article, this approach is more streamlined and, therefore, capable of unifying various existing results in the literature.

1.1. **Outline.** In Section 2, we describe the spectral theory for R-circulant matrices, where R is a finite Frobenius ring. We then apply this theory to establish a concrete criterion for the existence of a perfect state transfer (PST) on a gcd-graph defined over R (see Theorem 2.2). Utilizing this criterion, we develop in Section 3 some necessary conditions for the existence of a PST on a gcd-graph. In particular, we demonstrate that a PST can exist only between specific elements in R (see Theorem 3.5). In Section 4, we focus on the case where R is a local Frobenius ring. Here, we identify a sufficient condition for a gcd-graph to possess a PST (see Theorem 4.7). Furthermore, in certain instances, we show that this condition is also necessary. Finally, in Section 5, we classify all unitary Cayley graphs that exhibit a PST (see Theorem 5.3).

2. SPECTRAL ANALYSIS OF CAYLEY GRAPHS OVER A FINITE FROBENIUS RING

Let R be a finite Frobenius ring and n a positive integer such that $nR = 0$. We will fix n throughout this section, since we only need its exact value when describing the character group of R. Most of the time, n will only appear implicitly. In previous work [16, 17], we used the term *finite symmetric $\mathbb{Z}/n$-algebra* to refer to R. However, to maintain consistency with the literature, we will adopt the term Frobenius ring instead. The equivalence between these two seemingly different definitions follows from the work of Lamprecht in [11] (see [8] for historical insights and various ring and module theoretic characterizations of Frobenius rings).

By definition, being a Frobenius ring means that R is an $\mathbb{Z}/n$-algebra equipped with a nondegenerate linear functional $\psi\colon R \to \mathbb{Z}/n$. This non-degenerate functional is crucial in our spectral analysis. In fact, its existence implies that the character group of $(R,+)$ is parametrized by R itself, allowing us to treat R similarly to $\mathbb{Z}/n$ (see [16, Proposition 2.4]). Consequently, as explained below, several, although not all, techniques from [2, 7] can be naturally adapted to this more general setting.

Let G be a Cayley graph defined over R, namely $G = \Gamma(R, S)$ for some symmetric subset S of R. By definition, the adjacency matrix A_G is an R-circulant matrix (see [9, 5]). The entries of A_G are uniquely determined by the first row $\vec{c} = [c_s]_{s \in R}$ where

$$c_r = \begin{cases} 1 & \text{if } r \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Since R is a Frobenius ring and $nR = 0$, every character $\widehat{\psi}$ of R is of the form $\widehat{\psi}(s) = \zeta_n^{\psi(rs)}$ for some unique $r \in R$, where $\zeta_n = e^{2\pi i/n}$ (see [11, 16]). For such $r \in R$, let

$$\vec{v}_r = \frac{1}{\sqrt{|R|}}[\zeta_n^{\psi(rs)}]^T_{s \in R} \in \mathbb{C}^{|R|}.$$

Remark 2.1. By [15, Proposition 10], for each $r \in R$, $\vec{\theta}_r = \left(\frac{2\pi\psi(rs)}{n}\right)_{s \in R}$ is an equilibrium point of the Kuramoto model on $G = \Gamma(R, S)$. It would be interesting to study the stability of these points when S varies (see [19] for an example of this investigation).

By the circulant diagonalization theorem (see [9]), the set $\{\vec{v}_r : r \in R\}$ forms a normalized eigenbasis for all A_G. Furthermore, the eigenvalues of A_G are precisely

$$\lambda_r = \sum_{s \in R} c_s \zeta_n^{\psi(rs)} = \sum_{s \in S} \zeta_n^{\psi(rs)}.$$

We note that, by definition, λ_r depends on S. However, since S is often clear from the context, we will obmit in the definition of λ_r. Let $V = [v_r]_{r \in R} \in \mathbb{C}^{|R| \times |R|}$ be the matrix formed by this eigenbasis and V^* be the conjugate transpose of V. Then we can write

$$A_G = VDV^* = \sum_{r \in R} \lambda_r \vec{v}_r \vec{v}_r^*,$$

here $D = \mathrm{diag}([\lambda_r]_{r \in R})$ is the diagonal matrix formed by the eigenvalues λ_r. We then have

$$F(t) = \sum_{r \in R} e^{i\lambda_r t} \vec{v}_r \vec{v}_r^*.$$

Hence

$$F(t)_{s_1, s_2} = \frac{1}{|R|} \sum_{r \in R} e^{i\lambda_r t} \zeta_n^{\psi((s_1 - s_2)r)} = \frac{1}{|R|} \sum_{r \in R} e^{2\pi i\left(\lambda_r \frac{t}{2\pi} + \frac{\psi((s_1-s_2)r)}{n}\right)}.$$

By the triangle inequality, $|F(t)_{s_1, s_2}| = 1$ if and only if $\lambda_r \frac{t}{2\pi} + \frac{\psi(s_1-s_2)r}{n}$ are constant modulo 1. Furthermore, by symmetry, there exists a perfect state transfer between s_1 and s_2 if and only if there exists a perfect state transfer between 0 and $s_2 - s_1$. In summary,

our calculation shows the following criterion, which is a direct generalization of [2, Theorem 4].

Theorem 2.2. *There exists a perfect state transfer from 0 to s at time t if and only if for all $r_1, r_2 \in R$*

$$(\lambda_{r_1} - \lambda_{r_2})\frac{t}{2\pi} + \frac{\psi(s(r_1 - r_2))}{n} \equiv 0 \pmod 1.$$

3. Necessary conditions for the existence of PST

In this section, we apply Theorem 2.2 to impose some necessary conditions for gcd-graphs defined over R to have a PST. First, we recall their definition, which was introduced in [17].

Definition 3.1. Let R be a finite Frobenius ring. Let $D = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k\}$ be a set of distinct principal ideals in R. The gcd-graph $G_R(D)$ is equipped with the following data.

(1) The vertex set of $G_R(D)$ is R.
(2) Two vertices a, b are adjacent if there exists $I \in D$ such that $R(a - b) = I$ where $R(a - b)$ is the principal ideal generated by $a - b$.

We remark that that a Cayley graph $\Gamma(R, S)$ defined over R with a generating set S is a gcd-graph if and only if S is stable under the action of $R^\times$ (see [17, Proposition 2.5].)

Let $\Gamma(R, S) = G_R(D)$ be a gcd-graph. Let Δ be the abelian group generated by $r_1 - r_2$ where r_1 and r_2 are elements of R such that $\lambda_{r_1} = \lambda_{r_2}$. The following statement is a direct corollary of Theorem 2.2.

Corollary 3.2. *Let $\Gamma(R, S) = G_R(D)$ be a gcd-graph. Suppose that there exists a perfect state transfer from 0 to s at time t. Then $\psi(sd) = 0$ for all $d \in \Delta$.*

Proof. Theorem 2.2 implies that if there exists a perfect state transfer from 0 to s at time t then $\psi(s(r_1 - r_2)) = 0$. Since ψ is linear, we conclude that $\psi(sd) = 0$ for all $d \in \Delta$. $\square$

Let us now discuss a lower bound for the size of Δ. By [17, Corollary 4.17], $\lambda_{r_1} = \lambda_{r_2}$ if r_1 and r_2 are associates. In particular, Δ contains the subgroup Δ' generated by $u_1 - u_2$ where u_1, u_2 are units in R. Let us now describe the structure of Δ'. To do so, we recall that by the structure theorem for artinian rings, $R \cong \prod_{i=1}^d R_i$ where each R_i is a local ring. We can further write $R = R_1 \times R_2$ where R_1 (respectively, R_2) consists of local factors whose residue fields are $\mathbb{F}_2$ (respectively, $\neq \mathbb{F}_2$). Let $J(R_1)$ be the Jacobson radical of R_1. Because R_1 is a finite product of local rings, $J(R_1)$ is the kernel of the map $R_1 \to R_1/J(R_1) \cong \mathbb{F}_2^r$ where r is the number of local factors whose residue fields are $\mathbb{F}_2$. By [17, Corollary 3.5], every element in $J(R_1) \times R_2$ is a difference of two units. This shows that $J(R_1) \times R_2 \subset \Delta'$. In fact, this is an equality.

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Proposition 3.3. $\Delta' = J(R_1) \times R_2$ and hence $J(R_1) \times R_2 \subset \Delta$. Consequently, if there exists a PST between 0 and s then $s \in \mathrm{Ann}_R(J(R_1) \times R_2)$.

Proof. We already know that $J(R_1) \times R_2 \subset \Delta'$. Let us also prove that $\Delta' \subset J(R_1) \times R_2$. It is enough to show that if $u_1, u_2 \in R_1^\times$ then $u_1 - u_2 \in J(R_1)$. For this, it is sufficient to assume that R_1 is a local ring with maximal ideal $\mathfrak{m}$ and $|R/\mathfrak{m}| = 2$. In this case, $u_1 \equiv u_2 \equiv 1 \pmod{\mathfrak{m}}$ and hence $u_1 - u_2 \in \mathfrak{m} = J(R_1)$.

The last statement follows from Corollary 3.2 that $\psi(sd) = 0$ for all $d \in J(R_1) \times R_2$. This implies that $s(J(R_1) \times R_2)$ belongs to the kernel of ψ. Since ψ is non-degenerate, we must have $s(J(R_1) \times R_2) = 0$. In other words, $s \in \mathrm{Ann}_R(J(R_1) \times R_2)$. $\qquad \square$

Let us now focus on the structure of a local Frobenius ring. By [8, 11], a local ring $(S, \mathfrak{m})$ is Frobenius if and only if there exists an element $e \in \mathfrak{m}$ such that Se is a minimal ideal in S; namely Se is contained in every non-zero ideal in S. Said it differently, Se is precisely the socle module of S (see Definition 4.4 for the definition of the socle of a module). It turns out that if the residue field of S is $\mathbb{F}_2$, e must be unique.

Lemma 3.4. Let $(S, \mathfrak{m})$ be a local finite Frobenius ring such that the residue field of S is $\mathbb{F}_2$. Then, there exists a unique $e \in S$ such that $\mathrm{Ann}_R(e) = \mathfrak{m}$. Furthermore, if $\psi \colon S \to \mathbb{Z}/n$ be any non-degenerate linear functional then $\dfrac{\psi(e)}{n} \equiv \dfrac{1}{2} \pmod 1$.

Proof. Let e' be another element such that $Se = Se'$. Then $e' = ue$ for some $u \in S^\times$. Since the residue field of S is $\mathbb{F}_2$, $u - 1 \in \mathfrak{m}$. Furthermore, since $\mathrm{Ann}_S(e) = \mathfrak{m}$, we conclude that $(u-1)e = 0$. This implies that $e' = e$.

For the second statement, we know that $\psi(e) \neq 0$ since ψ is non-degenerate. Additionally, $0 = \psi(2e) = 2\psi(e)$. This shows that $\psi(e) \equiv \frac{n}{2} \pmod n$. Equivalently, $\frac{\psi(e)}{n} \equiv \frac{1}{2} \pmod 1$. $\qquad \square$

Combining Proposition 3.3 and Lemma 3.4 we have the following.

Theorem 3.5. Let R be a finite Frobenius ring. Suppose that R has the following Artin-Wedderburn decomposition: $R = (\prod_{i=1}^d S_i) \times R_2$. Here, $(S_i, \mathfrak{m_i})$ represents all local factors of R whose residue fields are $\mathbb{F}_2$. For each $1 \leq i \leq d$, let e_i be the unique minimal element of S_i as defined in Lemma 3.4. If there exists a PST between 0 and some $s \in R$, then s must be of the form $(a_1, a_2, \ldots, a_d, 0)$, where each a_i is 0 or e_i.

Proof. Suppose that there exists a PST between 0 and $s = (a_1, a_2, \ldots, a_d, a) \in R$. Then by Proposition 3.3, $a_i \in \mathrm{Ann}_{S_i}(m_i) = S_i e_i$ for each $1 \leq i \leq d$. By Lemma 3.4, this implies that $a_i \in \{0, e_i\}$. Additionally, $a \in \mathrm{Ann}_{R_2}(R_2) = 0$. $\qquad \square$

We discuss two direct corollaries of Theorem 3.5.

Corollary 3.6. There are at most $2^d - 1$ values of $s \in R$ such that there is a perfect state transfer from 0 to s. In particular, if R is a local ring, then $s = e$, where e is the minimal element of S.

Corollary 3.7. *If $r = 0$ then there is no state transfer on $G_R(D)$. In particular, if R is an $\mathbb{F}_q$-algebra with $q \neq 2$, then there are no perfect state transfers on any gcd-graphs defined over R.*

4. PST ON GCD-GRAPHS OVER A LOCAL FROBENIUS RING

Let $(R, \mathfrak{m})$ be a finite local Frobenius ring where $\mathfrak{m}$ is the maximal ideal of R. In this section, we study PST on gcd-graphs defined over R that have a PST. We remark that since R is Frobenius, it has the following elegant symmetry: for each ideal I in R, $\text{Ann}_R(\text{Ann}_R(I)) = I$ (see [8]). As a consequence, an ideal is uniquely determined by its annihilator ideal.

As explained in Lemma 3.4, since R is a Frobenius ring, there exists a unique $e \in R$ such that Re is the unique minimal ideal in R. By Lemma 3.4, a gcd-graph over R has PST only if the residue field $R/\mathfrak{m}$ is $\mathbb{F}_2$. For this reason, we will assume this property throughout this section.

As explained in Corollary 3.6, a PST can only exist between 0 and e. Furthermore, when R is a local ring, Theorem 2.2 can be simplified as follows.

Proposition 4.1. *There exists a PST between 0 and e if and only if for all $r \in \mathfrak{m}$ the following conditions hold*

(1) $(\lambda_r - \lambda_0)\frac{t}{2\pi} \in \mathbb{Z}$.
(2) $(\lambda_r - \lambda_1)\frac{t}{2\pi} + \frac{1}{2} \in \mathbb{Z}$.

Proof. This follows from the fact that $\lambda_u = \lambda_1$ if u is a unit and $\text{Ann}_R(e) = \mathfrak{m}$. $\square$

We will start our investigation with two concrete examples that will serve as a guide for what follows.

Example 4.2. Let $R = \mathbb{F}_2[x, y]/(x^2, y^2)$. Then R is a Frobenius ring with xy as the minimal element. Let $D = \{R, Rxy\}$. The characteristic polynomial of $G_R(D)$ is

$$(t - 9)(t + 7)(t - 1)^6(t + 1)^8.$$

Table 1 records the value of λ_r for $r \in R$. We can observe that $\lambda_u = -1$ for $u \in R^\times$. On

λ_r	r
9	0
-1	$1, xy + x + 1, xy + x + y + 1$
	$x + y + 1, x + 1, xy + 1, xy + y + 1, y + 1$
1	$xy + x + y, xy + x, x + y, x, xy + y, y$
-7	xy

TABLE 1. Spectrum of $G_R(D)$

the other hands, λ_u is of the form $4k + 1$ if $u \in \mathfrak{m}$. By Proposition 4.1, there exists a PST between 0 and xy at time $\frac{\pi}{2}$.

Example 4.3. Let us consider another example. Let $R = \mathbb{F}_2[x, y]/(x^2, y^2)$ and $D = \{R, Rx, Rxy\}$. In this case, the characteristic polynomial for $G_R(D)$ is

$$(t - 11)(t + 5)(t - 3)^2(t + 1)^{12}.$$

The eigenvalues of $G_R(D)$ are described by Table 2.

λ_r	r
11	0
-1	$1, xy + x + y, xy + x + 1$
	$xy + x + y + 1, x + y, x + y + 1, x + 1, xy + y$
	$xy + 1, xy + y + 1, y, y + 1$
3	$xy + x, x$
-5	xy

TABLE 2. Spectrum of $G_R(D)$

As we can see, in this case $\lambda_1 = \lambda_{xy+x+y}$ and hence the second conditions in Proposition 4.1 cannot happen. As a result, there is no PST on $G_R(D)$.

As we will see later, Example 4.3 represents a typical situation where the existence of $r \in \mathfrak{m}$ such that $\lambda_r = \lambda_1$ is one of the main obstructions to the existence of a PST on $G_R(D)$. It turns out that the socle series of R will play a fundamental role in our study of this phenomenon. For this reason, we first recall its definition.

Definition 4.4. Let M be an R-module. The socle modules $\mathrm{soc}^i(M)$ are defined inductively as follows.

(1) $\mathrm{soc}^1(M) := \mathrm{soc}(M)$ is the sum of all minimal non-zero submodules of M. By definition

$$\mathrm{soc}^1(M) = \{a \in M \mid xa = 0 \quad \forall x \in \mathfrak{m}\}.$$

(2) For each $i \geq 2$, $\mathrm{soc}^i(M)$ is the preimage of $\mathrm{soc}^1(M/\mathrm{soc}^{i-1}(M))$ in M.

When $M = R$, we know that $\mathrm{soc}^1(R) = Re$. For our investigation, $\mathrm{soc}^2(R)$ will be important. Since every ideal in R contains Re, a minimal ideal I containing Re is necessarily principal; namely $I = Ra$ for some $a \in \mathfrak{m}$ such that $a \neq e$. We provide below the necessary and sufficient condition for Ra to be minimal ideal containing Re.

Lemma 4.5. *Let $a \neq e$ be an element of R. The following statements are equivalent*

(1) Ra is a minimal ideal containing Re.

(2) $xa \in Re$ for all $x \in \mathfrak{m}$.

(3) $a \in \mathrm{Ann}_R(\mathfrak{m}^2)$.

(4) $a \in \text{soc}^2(R)$.

(5) $|R/\text{Ann}_R(a)| = 4$.

Proof. We first show that $(2) \iff (3) \iff (4)$. The first equivalence follows from the fact that $\text{Ann}_R(e) = \mathfrak{m}$. Since $a \neq e$, $(2) \implies (5)$ by the definition of $\text{soc}^2(R)$. Finally, let us prove that $(5) \implies (2)$. In fact, since $a \in \text{soc}^2(R)$, a can be expressed as an R-linear combination of a_i such that Ra_i is a minimal ideal containing Re. Since $xa_i \in Re$ for all a_i, the same holds true for a as well.

We will prove that $(1) \implies (2) \implies (5) \implies (1)$. First, we can see that $(1) \implies (2)$ by definition of the socle. Let us show that $(2) \implies (4)$. Since $\text{Ann}_R(a) \subsetneq \mathfrak{m}$ and $|R/\mathfrak{m}| = 2$, it is sufficient show that $[\mathfrak{m} : \text{Ann}_R(a)] = 2$. In fact, let $x_1, x_2 \in \mathfrak{m} \setminus \text{Ann}_R(a)$. Then $x_1 a, x_2 a \in Re \setminus \{0\} = \{e\}$ by Lemma 3.4. As a result, $(x_1 - x_2)a = 0$ and hence $x_1 - x_2 \in \text{Ann}_R(a)$. This shows that $[\mathfrak{m} : \text{Ann}_R(a)] = 2$. Finally, let us shows that $(5) \implies (1)$. Suppose to the contrary, that there exists an ideal J such that $Re \subsetneq J \subsetneq Ra$. Since $Re \subsetneq J$, we can find $b \in J$ such that $b \neq e$. We then have $Re \subsetneq Rb \subsetneq Ra$. By the duality between an ideal in R and its annihilator in R, we have the reversed inclusion $\text{Ann}_R(Ra) \subsetneq \text{Ann}_R(Rb) \subsetneq \text{Ann}_R(Re)$. This would imply that $|R/\text{Ann}_R(a)| \geq 8$, which is a contradiction. $\square$

Next, we show that Ramanujan sums satisfy some modular congruences (see also [12, Proposition 3.12] for some further congruences when R is a quotient of the polynomial ring $\mathbb{F}_q[x]$).

Lemma 4.6. *Let $r_1, r_2 \in \mathfrak{m}$. Then for all $x \in R$, we have the following congruence*

$$c(r_1, R/\text{Ann}_R(x)) \equiv c(r_2, R/\text{Ann}_R(x)) \pmod 4.$$

Proof. We recall the definition of the Ramanujan sum

$$c(r, \text{Ann}_R(x)) = \frac{\varphi(R/\text{Ann}_R(x))}{\varphi(R/\text{Ann}_R(rx))}\mu(R/\text{Ann}_R(rx)).$$

Furthrmore, we know that $\mu(R/\text{Ann}_R(rx)) \neq 0$ if and only $\text{Ann}_R(rx) \in \{\mathfrak{m}, R\}$. Consequently, $c(r, \text{Ann}_R(x)) \in \{0, \varphi(R/\text{Ann}_R(x)), -\varphi(R/\text{Ann}_R(x))\}$. We consider two cases.

Case 1. $Rx = Re$. In this case $r_1 e = r_2 e = 0$. Therefore

$$c(r_1, R/\text{Ann}_R(x)) \equiv c(r_2, R/\text{Ann}_R(x)) = 1.$$

Case 2. $x \neq e$. In this case $\text{Ann}_R(x)$ is a proper ideal of $\mathfrak{m}$ and hence $|R/\text{Ann}_R(x)| \geq 4$. Consequently, $2\varphi(R/\text{Ann}_R(x)) = |R/\text{Ann}_R(x)| \equiv 0 \pmod 4$. Therefore, if $r_1 x, r_2 x \in Re$, then both $c(r_1, \text{Ann}_R(x)), c(r_1, \text{Ann}_R(x)) \in \{\varphi(R/\text{Ann}_R(x)), -\varphi(R/\text{Ann}_R(x))\}$, and hence are the same modulo 4. Similarly, if both $r_1 x, r_2 x$ are not in Re then both these Ramanujan sums are zero. Finally, let us consider the case where exactly one of $r_1 x, r_2 x$ belongs to Re. In this case, $x \notin \text{soc}^2(R)$. By Lemma 4.5, we know that the order

of $R/\mathrm{Ann}_R(x)$ is at least 8 and therefore $\varphi(R/\mathrm{Ann}_R(x)) \equiv 0 \pmod 4$. As a result, $c(r_1, R/\mathrm{Ann}_R(x)) \equiv c(r_2, R/\mathrm{Ann}_R(x)) \equiv 0 \pmod 4$.

$\square$

Let $\mathrm{Isoc}^2(R)$ be the set of ideals of the form Ra where $a \in \mathrm{soc}^2(R)$. We remark that, by definition, Re is an element of $\mathrm{Isoc}^2(R)$. The following theorem explains the existence of a PST in Example 4.2.

Theorem 4.7. *Suppose that $|D \cap \mathrm{Isoc}^2(R)| \equiv 1 \pmod 2$. Then $G_R(D)$ has a PST between 0 and e at time $\frac{\pi}{2}$.*

Proof. Let $D = \{Rx_1, \ldots, Rx_d\}$. Let us first consider the case $Re \in D$. In this case, we have

$$\lambda_0 = \sum_{i=1}^{d} \varphi(R/\mathrm{Ann}_R(x_i)).$$

By Lemma 4.5, we know that $\varphi(R/\mathrm{Ann}_R(x_i))$ is a multiple of 4 unless $x_i \in \mathrm{soc}^2(R)$. We conclude that

$$\lambda_0 = 1 + \sum_{x_i \in \mathrm{soc}^2(R) \setminus \{e\}} \varphi(R/\mathrm{Ann}_R(x_i)) \equiv 1 \pmod 4.$$

By Lemma 4.6, we know that $\lambda_r \equiv 1 \pmod 4$ for all $r \in \mathfrak{m}$. On the other hands, we have

$$\lambda_1 = \sum_{i=1}^{d} \mu(R/\mathrm{Ann}_R(x_i)) = -1.$$

This is because $\mu(R/\mathrm{Ann}_R(x_i)) = 0$ unless $x_i = e$. By Proposition 4.1, there exists a PST between 0 and e at time $\frac{\pi}{2}$.

Similarly, in the case $Re \notin D$, we can show that $\lambda_r \equiv 2 \pmod 4$ for all $r \in \mathfrak{m}$ and $\lambda_1 = 0$. By Proposition 4.1, there exists a PST between 0 and e at time $\frac{\pi}{2}$.

$\square$

Corollary 4.8. *Suppose that $(R, \mathfrak{m})$ is a principal ideal local ring with a generator α and residue field $\mathbb{F}_2$. Let n be the smallest positive integer such that $\alpha^n = 0$. In this case, $\mathrm{Isoc}^2(R) = \{R\alpha^{n-2}, R\alpha^{n-1}\}$. Furthermore, the converse of Theorem 4.7 is also true. In other words, a gcd-graph $G_R(D)$ has a PST if and only if $|D \cap \{R\alpha^{n-1}, R\alpha^{n-2}\}| = 1$.*

Proof. By Theorem 4.7 if $|D \cap \{R\alpha^{n-1}, R\alpha^{n-2}\}| = 1$, then there exists a PST between 0 and $e = \alpha^{n-1}$ at time $\frac{\pi}{2}$. Suppose that $|D \cap \{R\alpha^{n-1}, R\alpha^{n-2}\}| \in \{0, 2\}$. We claim that there is no PST on $G_R(D)$. We consider two cases.

Case 1. $R\alpha^{n-1}, R\alpha^{n-2} \in D$. In this case, we have $\lambda_1 = -1$ and

$$\lambda_\alpha = \frac{\varphi(R/\mathrm{Ann}_R(\alpha^{n-2}))}{\varphi(R/\mathrm{Ann}_R(\alpha^{n-1}))} \mu(R/\mathrm{Ann}_R(\alpha^{n-1})) + \frac{\varphi(R/\mathrm{Ann}_R(\alpha^{n-1}))}{\varphi(R/\mathrm{Ann}_R(\alpha^n))} \mu(R/\mathrm{Ann}_R(\alpha^n))$$

$$= (-2) + 1 = -1.$$

By Proposition 4.1, there is no PST on $G_R(D)$.

Case 2. $R\alpha^{n-1}, R\alpha^{n-2} \notin D$. In this case, $\lambda_1 = \lambda_\alpha = 0$. As a result, Proposition 4.1 also implies that there is no PST on $G_R(D)$.

$\square$

We discuss another case where the converse of Theorem 4.7 also holds.

Corollary 4.9. *Suppose that $D = \{R, Ra\}$ for some $a \in \mathfrak{m}$. Then the converse of Theorem 4.7 holds. In other words $G_R(D)$ has a PST between 0 and e if and only if $|D \cap \mathrm{Isoc}^2(R)| = 1$.*

Proof. The forward direction is a consequence of Theorem 4.7. Let us prove the other direction, namely if $|D \cap \mathrm{Isoc}^2(R)| = 0$ then $G_R(D)$ has no PST. In fact, in this case we have $\lambda_1 = 0$. Furthermore, since $a \notin \mathrm{soc}^2(R)$, there exists $b \in \mathfrak{m}$ such that $ab \notin Re$. We then see that

$$\lambda_b = \frac{\varphi(R)}{\varphi(R/\mathrm{Ann}_R(b))}\mu(R/\mathrm{Ann}_R(b)) + \frac{\varphi(R/\mathrm{Ann}_R(a))}{\varphi(R/\mathrm{Ann}_R(ab))}\mu(R/\mathrm{Ann}_R(ab)) = 0.$$

The last equality follows from the fact that both $\mathrm{Ann}_R(b)$ and $\mathrm{Ann}_R(ab)$ is a proper ideal of $\mathfrak{m}$. By Proposition 4.1, we conclude that there is no PST on $G_R(D)$. $\square$

When R is an algebra $\mathbb{F}_2$, then, as an abelian group, R is isomorphic to $\mathbb{F}_2^r$ for some $r \geq 0$. In this case, $G_R(D)$ is an example of a cube-like graph. The study of PST on these graphs has an extensive literature (see [6] and the references therein for further discussions). We will show below that Theorem 4.7 is consistent with [6, Theorem 2.3]. To do so, we first need the following lemma.

Lemma 4.10. *Let $a \in R$. Then*

$$\sum_{b \sim a} b = \begin{cases} e & \text{if } |R/\mathrm{Ann}_R(a)| \in \{\mathbb{F}_2, \mathbb{F}_2[x]/x^2\}. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first prove this statement for when $a \in R^\times$. If $2 \neq 0$ in R then elements of $R^\times$ can be partitioned into pairs $\{u, -u\}$. As a result $\sum_{b \in R^\times} b = 0$. Let us consider the case the characteristic of R is 2. If $R = \mathbb{F}_2$ then $e = 1$ and hence the statement is true. Let us consider the case $R \neq \mathbb{F}_2$. In this case, the element of $R^\times$ can be partition into pairs $\{u, u+e\}$. For each pair, the sum is e. Therefore

$$\sum_{b \in R^\times} b = \frac{1}{2}\varphi(R)e.$$

This is 0 unless $\varphi(R) = 2$. Since the characteristics of R is 2, $\varphi(R) = 2$ can only happen when $R = \mathbb{F}_2$ or $\mathbb{F}_2[x]/x^2$.

Let us now consider the general case. First, since $a \neq 0$, we can find e_1 such that $e = ae_1$. While e_1 is not unique, it is unique in $R/\mathrm{Ann}_R(a)$. We claim that Re_1 is the minimal ideal in $R/\mathrm{Ann}_R(a)$ (this gives another proof for the fact that $R/\mathrm{Ann}_R(a)$ is a

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Frobenius ring. A different proof using a non-degenerate linear functional was previously discussed in [17, Lemma 4.3].) In fact, let $u \in R/\operatorname{Ann}_R(a)$ such that u is neither a unit or 0. Let $\hat{u}$ be a lift of u to R. Since $u \neq 0$, $\hat{u}a \neq 0$ and hence we can write $e = v(ua)$. We can then see that $e_1 = \hat{u}v$ in $R/\operatorname{Ann}_R(a)$. This shows that Re_1 is the unique minimal ideal in $R/\operatorname{Ann}_R(x)$.

By [17, Corollary 2.7] we have

$$\sum_{b \sim a} b = a \sum_{u \in (R/\operatorname{Ann}_R(a))^\times} u.$$

By the calculation done in the previous part, we know that modulo $\operatorname{Ann}_R(a)$

$$\sum_{u \in (R/\operatorname{Ann}_R(a))^\times} u = \begin{cases} e_1 = e & \text{if } R/\operatorname{Ann}_R(a) \in \{\mathbb{F}_2, \mathbb{F}_2[x]/x^2\}. \\ 0 & \text{otherwise.} \end{cases}$$

The statement follows easily from this.

$\square$

We can now relate Theorem 4.7 and [6, Theorem 2.3].

Proposition 4.11. *Suppose that R is an $\mathbb{F}_2$-algebra. Let S be the generating set of associated with D. Then the following statement are equivalent*

(1) $|D \cap \operatorname{Isoc}^2(R)| \equiv 1 \pmod 2$.

(2) $\sum_{s \in S} s \neq 0$.

Furthermore, if one of these conditions hold, there exists a PST between 0 and e at time $t = \frac{\pi}{2}$.

Proof. By the definition of S

$$\sum_{s \in S} s = \sum_{i=1}^{d} \left[\sum_{b \sim x_i} b. \right]$$

By Lemma 4.10, the sum

$$\sum_{b \sim x_i} b = \begin{cases} e & \text{if } |R/\operatorname{Ann}_R(x_i)| \in \{\mathbb{F}_2, \mathbb{F}_2[x]/x^2\}. \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by Lemma 4.5 implies that $|R/\operatorname{Ann}_R(x_i)| \in \{\mathbb{F}_2, \mathbb{F}_2[x]/x^2\}$ if and only if $x_i \in \operatorname{soc}^2(R)$. We conclude that

$$\sum_{s \in S} s = |D \cap \operatorname{Isoc}^2(R)|e.$$

This is nonzero if and only if $|D \cap \operatorname{Isoc}^2(R)|$ is odd.

$\square$

In all the examples that we consider, the converse of Theorem 4.7 always holds. More specifically, in the case where $|D \cap \operatorname{Isoc}^2(R)| \equiv 0 \pmod 2$, we can always find $a \in \mathfrak{m}$ such that $\lambda_a = \lambda_1$ (note that λ_1 is -1 or 0 depending on whether $Re \in D$ or not). It would be interesting to either prove or disprove this statement.

Remark 4.12. We note that, since the complete graph K_n has no PST unless $n = 2$, the converse of Theorem 4.7 would imply that if $R \neq \mathbb{F}_2$, $|\text{Isoc}^2(R)|$ must be an even number (otherwise, if we take D to be the union of all principal ideals in R, the converse of Theorem 4.7 would imply that the complete graph $K_{|R|}$ has PST). Fortunately, we can prove this statement unconditionally. Let $Re_1, Re_2, \ldots, Re_k$ be the elements of $\text{Isoc}^2(R) \setminus \{Re\}$. Then for Re_i we have exactly 4 elements. In addition, for $i \neq j$, $|Re_i \cap Re_j| = |Re| = 2$. We conclude that

$$|\text{soc}^2(R)| = |Re| + 2 + 2k = 4 + 2k.$$

When $R \neq \mathbb{F}_2$, $\text{soc}^1(R)$ is a proper subgroup of $\text{soc}^2(R)$. As a result, $4 \mid \text{soc}^2(R)$. This imply that k must be odd. In other words, $|\text{Isoc}^2(R)|$ must be even.

5. PST ON UNITARY CAYLEY GRAPHS

We recall that for a finite ring R, the unitary Cayley graph $G_R(1)$ associated with R is precisely the gcd-graph $G_R(\{R\})$. In other words, it is the graph whose vertex set is R and two vertices a, b are adjacent if $a - b \in R^\times$. In this section, we classify all finite Frobenius rings whose associated unitary Cayley graph has a PST. We remark that various authors study a similar problem in the context of discrete-time quantum walks (for example, see [3]). Thus, our results provide new insights that are independent of their findings.

We recall that the spectrum $G_R(1)$ is the multiset

$$\left\{ \lambda_g = c(g, R) \right\} = \left\{ \frac{\varphi(R)}{\varphi(R/\text{Ann}_R(g))} \mu(R/\text{Ann}_R(g)). \right\}$$

Let $R = \left(\prod_{i=1}^d S_i \right) \times R_2$ be the Artin-Wedderburn decomposition of R as described in Theorem 3.5. Since μ and φ are multiplicative, if we write $g = (g_1, g_2, \ldots, g_d, h)$ then

$$\lambda_g = c(h, R_2) \prod_{i=1}^d c(g_i, S_i).$$

Recall that Δ is the abelian group generated by $r_1 - r_2$ where $\lambda_{r_1} = \lambda_{r_2}$.

Proposition 5.1.

(1) *Suppose that there are at least two local factors T of R such that $\mu(T) = 0$. Then $\Delta = R$.*
(2) *Let $R = \left(\prod_{i=1}^d S_i \right) \times R_2$ as described above. If $\mu(R_2) = 0$ then $\Delta = R$.*

Proof. Let us prove the first statement. Let us write $R = T_1 \times T_2 \times T_3$ where T_1, T_2 are local rings such that $\mu(T_1) = \mu(T_2) = 0$. Let $t = (t_1, t_2, t_3)$ be an arbitrary element in R. We have

$$c(1, t_2 + 1, t_3) = c(1, T_1)c(t_1 + 1, T_2)c(t_3, T_3) = 0.$$

Similarly $c(1 - t_1, 1, 0) = 0$. This shows that $(t_1, t_2, t_3) = (1, t_1 + 1, t_3) - (1 - t_1, 1, 0) \in \Delta$. This shows that $\Delta = R$.

Let us now prove the second statement. For each $a \in \prod_{i=1}^{d} S_i$ we have $c((a,1), R) = c((0,1), R) = 0$. Consequently $(a,0) \in \Delta$ since $(a,0) = (a,1) - (0,1) \in \Delta$. Since $J(\prod_{i=1}^{d} S_i) \times R_2 \subset \Delta$, we know that $(a,b) \in \Delta$ for all $a \in \prod_{i=1}^{d} S_i$ and $b \in R_2$. In other words, $\Delta = R$.

$\square$

By Corollary 3.2, we have the following.

Corollary 5.2. *Suppose that there exists a PST on $G_R(1)$. Then the following conditions hold*

(1) R has at most one local factor T such that $\mu(T) = 0$.

(2) $\mu(R_2) \neq 0$.

In other words, R must have the form $S_1 \times \prod_{i=1}^{d-1} \mathbb{F}_2 \times \prod_{i=1}^{r} \mathbb{F}_{q_i}$ where S_1 is a Frobenius local ring with residue field $\mathbb{F}_2$ and $\mathbb{F}_{q_i}$ are finite fields with $q_i \geq 3$.

We can put further restrictions on $G_R(1)$ if there exists a PST on it. In fact, $G_R(1)$ has exactly 2^{d-1} connected components, each of which is isomorphic to $G_{S_1 \times \prod_{i=1}^{r} \mathbb{F}_{q_i}}(1)$. Since a PST cannot exist between vertices in different connected components, we can safely assume that $d = 0$ and $R = S_1 \times \prod_{i=1}^{r} \mathbb{F}_{q_i}$ where $(S_1, \mathfrak{m}_1)$ is a local ring with residue field $\mathbb{F}_2$.

Theorem 5.3. *Suppose that there exists a PST on $G_R(1)$ with $R = S_1 \times \prod_{i=1}^{r} \mathbb{F}_{q_i}$. Then the following conditions must hold*

(1) $S_1 \in \{\mathbb{F}_2, \mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$.

(2) $q_i \equiv 0 \pmod 4$ for all i.

Conversely, suppose that these conditions hold. Then there exists a PST between $(0,0)$ and $(e_1, 0)$ at time $\frac{\pi}{2}$. Here e_1 is the minimal element of S_1.

Proof. Let us prove the necessary conditions. Suppose that there exists a PST between 0 and some $s \in R$. By Theorem 3.5, s must equal to $(e_1, 0, \dots, 0)$. In this case, condition Theorem 2.2 can be described explicitly as

$$(5.1) \quad \begin{cases} (\lambda_{(x_1,a_1,\dots,a_r)} - \lambda_{(x_2,b_1,\dots,b_r)})\frac{t}{2\pi} + \frac{1}{2} \in \mathbb{Z} \text{ if } (x_1 - x_2) \notin \mathfrak{m}_1 \\ (\lambda_{(x_1,a_1,\dots,a_r)} - \lambda_{(x_2,b_1,\dots,b_r)})\frac{t}{2\pi} \in \mathbb{Z} \text{ if } (x_1 - x_2) \in \mathfrak{m}_1. \end{cases}$$

This follows from the fact that $\frac{\psi((e_1,0,\dots,0))}{n} = \frac{1}{2}$ as shown in Lemma 3.4. We also note that

$$\lambda_{(x_1,a_1,\dots,a_r)} = \frac{\varphi(S_1)}{\varphi(S_1/\operatorname{Ann}_{S_1}(x_1))} \mu(S_1/\operatorname{Ann}_{S_1}(x_1)) \prod_{i=1}^{r} \frac{\varphi(\mathbb{F}_{q_i})}{\varphi(\mathbb{F}_{q_i}/\operatorname{Ann}_{S_1}(a_i))} \mu(\mathbb{F}_{q_i}/\operatorname{Ann}_{\mathbb{F}_{q_i}}(a_i))$$

$$= \frac{\varphi(S_1)}{\varphi(S_1/\operatorname{Ann}_{S_1}(x_1))} \mu(S_1/\operatorname{Ann}_{S_1}(x_1)) \prod_{a_i=0}(q_i - 1) \prod_{a_i \neq 0}(-1)$$

$$= \frac{\varphi(S_1)}{\varphi(S_1/\operatorname{Ann}_{S_1}(x_1))} \mu(S_1/\operatorname{Ann}_{S_1}(x_1))(-1)^r \prod_{a_i=0}(1 - q_i).$$

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First, we claim that $S_1 \in \{\mathbb{F}_2, \mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$. If $S_1 = \mathbb{F}_2$ then we are done. Therefore, to avoid tautology, we will assume that $S_1 \neq \mathbb{F}_2$. We first claim that $\mathfrak{m}_1 = S_1 e_1$. In fact, suppose that it is not the case. Then we can find $x \in \mathfrak{m}_1 \setminus Re$ (since we are assuming that $S_1 \neq \mathbb{F}_2$, $S_1 e_1 \subset \mathfrak{m}_1$). We then have $\mathrm{Ann}_{S_1}(x) \neq \mathfrak{m}_1$ and hence $\lambda_{(x,0,\ldots,0)} = 0$. Similarly, we also have $\lambda_{(1,0,\ldots,0)} = 0$. These two equalities contradict the first condition. We conclude that $\mathfrak{m}_1 = S_1 e_1$. This would imply that $\mathfrak{m}_1^2 = 0$ and hence $1 \in \mathrm{Ann}_{S_1}(\mathfrak{m}_1^2)$. By Lemma 4.5, we conclude that $|S_1| = 4$ and hence $S_1 \in \{\mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$.

Next, we claim that $q_i \equiv 0 \pmod{4}$ for each $1 \leq i \leq r$.

Case 1. Let's first consider the case $S_1 = \mathbb{F}_2$. We have $\lambda_{(1,1,1,\ldots,1)} = (-1)^{r+1}$, $\lambda_{(0,1,\ldots,1)} = (-1)^r$, and $\lambda_{(1,0,1,\ldots,1)} = (-1)^{r+1}(1 - q_1)$. The first condition then implies that $2(\frac{t}{2\pi}) + \frac{1}{2} \in \mathbb{Z}$. The second condition then implies that $q_1 \frac{t}{2\pi} \in \mathbb{Z}$. This condition necessarily imply that q_1 must be even. By the same argument, q_i is even for all $1 \leq i \leq r$.

Case 2. Let us consider the case $S_1 \in \{\mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$. In this case $\lambda_{(e,1,1,\ldots,1)} = 2(-1)^{r+1}$, $\lambda_{(1,1,\ldots,1)} = 0$. and $\lambda_{(e,0,1\ldots,1)} = 2(-1)^r(1 - q_1)$. The first condition would imply that $2(\frac{t}{2\pi}) + \frac{1}{2} \in \mathbb{Z}$. The second condition implies that $2(-1)^{r+1}(1 - q_1)\frac{t}{2\pi} \in \mathbb{Z}$. We conclude that q_1 must be even. By the same argument, q_i are even for all $1 \leq i \leq r$.

Conversely, suppose that all the above conditions are satisfied. We claim that there exists a PST between 0 and $(e_1, 0, \ldots, 0)$ at time $\frac{\pi}{2}$. We consider two separate cases as above.

In the first case namely when $S_1 = \mathbb{F}_2$, we can see that $\lambda_{(1,a_1,a_2,\ldots,a_r)} \equiv (-1)^{r+1} \pmod{4}$ and $\lambda_{(0,a_1,\ldots,a_r)} \equiv (-1)^r \pmod{4}$ for every $(a_1, a_2, \ldots, a_r) \in \prod_{i=1}^r \mathbb{F}_{q_i}$. We can check that the conditions described in Eq. (5.1) are satisfied when $t = \frac{\pi}{2}$.

In the second case where $S_1 \in \{\mathbb{Z}/4, \mathbb{F}_2[x]/x^2\}$ we can see that

$$\lambda_{(0,a_1,\ldots,a_r)} \equiv \lambda_{(e_1,a_1,\ldots,a_r)} \equiv 2(-1)^r \pmod{4}.$$

On the other hand, if $u \in S_1^\times$ then $\lambda_{(u_1,a_1,\ldots,a_r)} \equiv 0 \pmod{4}$. These congruences show that all conditions described by Eq. (5.1) are met when $t = \frac{\pi}{2}$. $\qquad \square$

When R is a finite quotient of $\mathbb{F}_q[x]$, it is known that R is a Frobenius ring (see [13, Corollary 6.8]). In this case, Theorem 5.3 implies the following corollary.

Corollary 5.4. *Let $f \in \mathbb{F}_q[x]$ be a monic polynomial. Let $G_f(1) := G_{\mathbb{F}_q[x]/f}(1)$ be the unitary Cayley graph associated with $\mathbb{F}_q[x]/f$. Then there exists a PST on $G_f(1)$ if and only if the the following conditions holds*

(1) $q = 2$.

(2) Let $f = x^a(x+1)^b g(x)$. Then g is squarefree, $\gcd(g, x(x+1)) = 1$, $2 \geq \max\{a, b\} \geq 1$, and $(a, b) \neq (2, 2)$.

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