Research statement

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I am an interdisciplinary scientist. My research spans from algebraic number theory, representation theory of finite groups to spectral graph theory, computational neuroscience, and non-linear dynamics.

In the domain of algebraic number theory, I provided several new insights into the Bloch-Kato conjecture on special values of L-functions which is a central problem in the field. In particular, I showed that we could consider the Tamagawa number conjecture as to the problem of counting the numbers of mixed motives of bounded heights, which leads to new perspectives on the Tamagawa number conjecture. My collaborators and I also (re)-introduced a new mathematical object; namely the generalized Fekete polynomials. They are classical polynomials whose history can be traced to the 19th century in relation to the studies of Dirichlet L-functions. In particular, these polynomials already played a significant role in Gauss's original sixth proof of the quadratic reciprocity law. With the aid of classical methods and computer simulations, we have revealed several fundamental new insights about these polynomials.

I am also interested in several problems in network theory, spectral graph theory, non-linear dynamics, and computational neuroscience. Specifically, in recent work, my collaborators and I have developed fundamentally new insights into networks of nonlinear oscillators described by the Kuramoto model. Originally introduced by Yoshiki Kuramoto in 1975, this model has been the subject of computational and mathematical study across many scientific fields such as physics, neuroscience, and biology. The Kuramoto model has been a constant source for questions, ideas, and inspirations for my research. For example, our new results in spectral graph theory have provided exciting geometric insights into the existence and stability of these oscillators. This combined approach will open the possibilities for developing new theoretical advances in computational neuroscience, mathematical biology, spectral graph theory, algebraic number theory, and representation theory of finite-dimensional associative algebras. Furthermore, these research directions will provide some exciting projects suitable for both undergraduate and graduate students.

Below I provide a summary of my research accomplishments and future research plans.

1 Research accomplishments

1.1 Heights of motives and special values of L-functions

The class number formula is an inspiring pillar of number theory. By the work of many mathematicians, notably Deligne, Beilinson, Bloch, Kato, Fontaine, Perrin-Riou, Jannsen, and many others, we now have a quite general (conjectural) class number formulas for motives, i.e., the Tamagawa number conjecture of Bloch-Kato (see [4], [5], [12]).

In [21], K. Kato defined and studied heights of motives with an attempt to look at the Tamagawa number conjecture from another perspective; namely whether it is possible to consider the Tamagawa number conjecture as to the problem of counting the numbers of mixed motives of bounded heights. In [31], I answer some of Kato's questions in this direction. Here is one of them.

Theorem 1. *Let M be a pure motives with integer coefficients of weight $-d$ such that $d \geq 3$. We assume further that M has semistable reduction at all places. Then*

$$\lim_{B\to\infty} \frac{\#\{x \in B(\mathbb{Q}) | H_{\diamond,d}(x) \leq B\}}{\mu\left(x \in \prod'_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(x) \leq B\right)} = \frac{1}{Tam(M)}.$$

Here $B(\mathbb{Q})$ $B(\mathbb{Q}_p)$ are as defined in Bloch-Kato's paper [5].

1.2 Arithmetic of Fekete polynomials

For each prime p the Fekete polynomial $F_p(x)$ is defined by

$$F_p(x) = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) x^a.$$

These polynomials were first considered in the 19th century in relation to the studies of Dirichlet L-function

$$L(s, \chi_p) = \sum_{n=1}^{\infty} \frac{\chi_p(n)}{n^s},$$

where χ_p is the quadratic character associated with p. There are many interesting investigations of behaviors of Fekete polynomials over the interval $[0, 1]$, their roots in the complex plane, and their extremal property (see [3], [7],[13]). My main interest in these polynomials lies in exploring their arithmetic and Galois theoretic properties using the interesting interplay between the distribution of quadratic residues, arithmetic properties of Bernoulli numbers, class number formulas, and Galois theory. In [25], we compute some special values of the so called "reduced Fekete polynomial".

Theorem 2. *(See [25]) Suppose $p \equiv 3 \pmod 4$. Let $g_p(x)$ be the reduced Fekete polynomial associated with p. Let $h(-p)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. Then*

1. $g_p(-2) = -\left(2\left(\frac{2}{p}\right) - 1\right) h(-p)$.
2. $g_p(-1) = -\frac{1}{2}\left(\left(\frac{p}{3}\right) + 3\right) h(-p)$.
3. $g_p(0) = g_p(-2) = -\left(2\left(\frac{2}{p}\right) - 1\right) h(-p)$.
4. $g_p(1) = -\frac{h(-p)}{2}\left(\frac{6}{p}\right)\left[6 - 3\left(\frac{2}{p}\right) - 2\left(\frac{3}{p}\right) + \left(\frac{6}{p}\right)\right]$.

Remark 1. We also prove a similar theorem when $p \equiv 1 \pmod 4$.

Furthermore, we wrote some code to calculate some further arithmetic properties of $f_p(x)$ and $g_p(x)$. Our simulated data seems to suggest that the Galois groups of them are often as large as possible. More preicsely, for $p \leq 1000$, the Galois group of $g_p(x)$ is the symmetric group on $n = \deg(g_p)$ letters and the Galois group of $f_p(x)$ is the semi-direct product $(\mathbb{Z}/2)^n \rtimes S_n$.

Following this circle of ideas, we have also introduced and studied the arithmetic of generalized Fekete polynomials. Our findings are discussed in the preprint [26].

1.3 Analytical approach to the Kuramoto model

Networks play a central role in modern neuroscience. From mapping the anatomy of connections between neurons to understanding how patterns of connections create computation in the brain, networks are a central mathematical tool (see [34], [35]). In this domain, the Kuramoto oscillator has emerged as the central mathematical model for studying these networks (see [14], [23], [32]). We recall that the Kuramoto model is described by the following differential equation

$$\dot{\theta}_i = \omega_i + \kappa \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i)\,, \tag{1.1}$$

where $\theta_i(t) \in [-\pi, \pi]$ is the state of oscillator $i \in [1, N]$ at time t, ω_i is the intrinsic angular frequency, κ scales the coupling strength, and element $a_{ij} \in \{0, 1\}$ represents the connection between oscillators i and j.

This model, originally introduced by Yoshiki Kuramoto in 1975, has been the subject of numerical and analytical study across many scientific fields. However, due to the nonlinearity nature of these networks, their dynamics becomes challenging to study analytically. In our recent work [29], we provide a complex-valued matrix formulation of the Kuramoto model whose argument corresponds to the original Kuramoto model. This analytical model has the following form

$$\dot{\theta}_i = \gamma \sum_{j=1}^{N} a_{ij} \left[\sin(\theta_j - \theta_i) - \mathrm{i}\cos(\theta_j - \theta_i) \right]. \tag{1.2}$$

Using this analytical approach, we derive an explicit analytical solution for the dynamics using the eigenspectrum of the adjacency matrix. We also show that the original and the analytical Kuramoto model shows many common features. For example, in [11], we prove the following theorem.

Theorem 3. *Suppose $\boldsymbol{x}_0 = e^{\mathrm{i}\boldsymbol{\theta}_0}$ is an eigenvector of the adjacency matrix A in the Kuramoto model. Then*

1. *$\boldsymbol{\theta}_0$ is an equilibrium point of the original Kuramoto model 1.1.*
2. *$\boldsymbol{\theta}_0$ is also an equilibrium point of the analytical Kuramoto model 1.2.*

We note that our approach extends the analysis of equilibria in networks of Kuramoto oscillators to cases that could not be considered previously, demonstrating the utility of our complex-valued analytical approach.

1.4 Spectral graph theory and spectral matrix theory

In many applications, a natural model of a network of oscillators is a *ring graph*, in which nodes are regularly placed along a circle and, for a fixed number m, each oscillator is connected to its m closest neighbours on either side (see [18], [37]). These networks are representable by adjacency matrices which are circulant. The Circulant Diagonalization Theorem describes the eigenspectrum and eigenspaces of these circulant matrix explicitly via the discrete Fourier transform (see [16], [20]). Consequently, many problems involving circulant matrices have closed-form or analytical solutions. In our investigation of the non-linear dynamics on networks of oscillators, the joins of circulant graphs appear quite often, and they provide new exciting phenomena. This naturally led us to investigate the spectrum of these networks. In [17], we generalize the Circulant Diagonalization Theorem to the join of several circulant matrices. We first recall the definition of the join of circulant matrices.

$$C = \left(\begin{array}{c|c|c|c} C_1 & a_{1,2}\mathbf{1} & \cdots & a_{1,d}\mathbf{1} \\ \hline a_{2,1}\mathbf{1} & C_2 & \cdots & a_{2,d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbf{1} & a_{d,2}\mathbf{1} & \cdots & C_d \end{array}\right).$$

Here C_i are circulant matrix of sizes $k_i \times k_i$ and $a_{i,j}\mathbf{1}$ is a $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j}$. Our main result is the following.

Theorem 4 (See [17], Generalized Circulant Diagonalization Theorem)**.** *The spectrum of C is the union of the following multisets*

$$Spec(\overline{C}) \cup_{i=1}^{d} \{\lambda_j^{C_i} | 1 \leq j \leq n_i - 1\},$$

where $\overline{C}$ is an explicit $d \times d$ matrix.

When we join two ring graphs, we find new equilibria phenomena (see [11]). We are able to explain these phenomena via the following theorem.

Theorem 5. *The spectrum of $RG(k_1, m_1) + RG(k_2, m_2)$ is the union of three multisets*

$$(Spec(RG(k_1, m_1) \setminus \{2m_1\}) \cup (Spec(RG(k_2, m_2) \setminus \{2m_2\}) \cup \{\lambda_1, \lambda_2\},$$

with

$$\lambda_1, \lambda_2 = m_1 + m_2 \pm \sqrt{(m_1 - m_2)^2 + k_1 k_2}.$$

2 Future research plans

2.1 The algebra of the join of circulant matrices

In Section 1.4, I describe explicitly the spectrum of a matrix obtained by joining several circulant matrices. It turns out that the set of all such matrices has a structure of an algebra. More precisely, let $G_1, G_2, \ldots, G_d$ be finite groups, k be a field, and $J_{G_1,G_2,\ldots,G_d}$ be the join group algebra of $G_1, G_2, \ldots, G_d$. There are lots of questions that we can ask about this algebra. I would summarize this statement by the following slogan for my research program.

SLOGAN. For every problem for the group ring $k[G]$, there is a corresponding problem for $J_{G_1,G_2,\ldots,G_d}$.

Here are some questions that I am currently pursuing.

Question 1 (Generalization of Maschke's theorem)**.** When is $J_{G_1,G_2,\ldots,G_d}$ semisimple? When it is the case, describe the structure of $J_{G_1,G_2,\ldots,G_d}$.

The answer for this question would lead to a deeper understanding of the generalized Circulant Diagonalization Theorem 4. Additionally, it also opens new exciting opportunities to study networks theory using the representation theory of finite groups.

Question 2. Can we describe the categories of modules over $J_{G_1,G_2,\ldots,G_d}$?

Satisfying answers for this question would lead to some new advances in the study of Galois modules (see [27], [28] for some old and recent advances on this topic).

When G_i are p-groups and k is a fields of characteristics p, we can show that the algebra $J_{G_1,G_2,\ldots,G_d}$ is basic. In this case, Auslander–Reiten theory is an important tool to study the join algebra (see [1], [2]). In particular, I plan to study the following question.

Question 3. Can we describe the ordinary quiver associated with $J_{G_1,G_2,\ldots,G_d}$? What about the Auslander–Reiten quiver for this ring?

A classical problem in the representation theory of finite groups is the determination of the unit groups and their properties (see [6], [33]). In particular, there are recent several works on the unit groups for groups of small orders (see for example [15], [19]). It would be interesting to generalize these results to the unit groups of the join algebras $J_{G_1,G_2,\ldots,G_d}$.

Question 4. What is the structure of the unit group of $J_{G_1,G_2,\ldots,G_d}$?

2.2 Kuramoto models with time delays

It has been previously observed that time delays due to action potential conduction along the fiber tracts that connect different regions of the cortex may be an important mechanism underlying the generation of traveling waves (see [8], [30]). To understand this phenomenon, we consider a time-delay Kuramoto model (dKM) with delays τ_{ij} that depend on the distance between two oscillators i and j:

$$\dot{\theta}_i(t) = \omega + \epsilon \sum_{j=1}^{N} A_{ij} \sin\Big(\theta_j(t-\tau_{ij}) - \theta_i(t)\Big). \tag{2.1}$$

Below are some concrete questions and projects that I am currently pursuing.

Project 1. Develop an analytical approach to study the dynamics of nonlinear oscillator networks with heterogeneous time delays. How can we use this approach to capture the detailed wave patterns induced by time delays and the transient dynamics occurring before the system reaches the wave states?

Project 2. Describe an effective way to find twisted states in the time delay Kuramoto model 2.1.

These projects could potentially open a new road to the understanding and controlling of the dynamics in nonlinear oscillator networks, where one can express and change the behavior of the system by selecting specific eigenmodes.

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