Acyclic digraphs and an algebraic construction of cospectral digraphs

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Social network

Photo credit: Analytics Vidhya

What is a directed graph?

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Suppose the vertex set of G is $\{v_1, v_2, \ldots, v_n\}$. A convenient way to represent G is to use its adjacency matrix $A = (A_{ij})$ where

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \to v_j \\ 0 & \text{else.} \end{cases}$$

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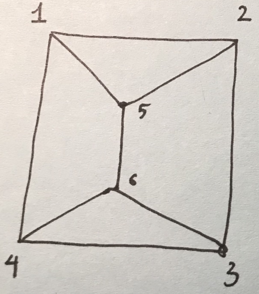
- Two graphs are called cospectral if they have the same spectrum.
- While two cospectral graphs may not be isomorphic, they share many common properties.

An example



$$
A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}
$$

$$
\chi_A(x) = x^6 - 9x^4 - 4x^3 + 12x^2
$$

$$
= x^2(x-3)(x-1)(x+2)^2
$$

So eigenvalues are $\boxed{0, 1, -2}$, 3.

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- Many spectral clustering techniques use the spectrum to perform dimensionality reduction on big data sets.
- And much more!

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- The main theme of these problems is motivated by the operator-based approach to random graph theory. This approach has been pioneered by Dr. Muller and Dr. Rudolph in their study of small-world networks.
- Through numerous experiments using computational software, we found several interesting patterns.
- During the summer of 2021, the FURSP students made more surprising discoveries.

Spectrum of complete digraphs

Let K_n be the complete digraph with n nodes. Without loss of generality, we assume that the nodes of K_n are

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The adjacency matrix of K_n is of the form

$$A_n = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & & & & \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}.$$

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Therefore $\lambda_n = n-1$. In summary, we have

Proposition

Let A_n be the adjacency matrix of K_n. Then the eigenvalues of A_n are -1 (with multiplicity $n-1$) and $n-1$ (with multiplicity 1). In particular, the characteristic polynomial of A_n is

$$p_{A_n}(t) = (t+1)^{n-1}(t-n+1).$$

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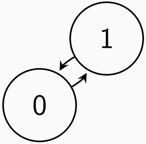
Proposition

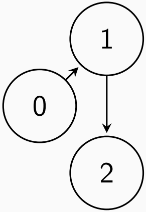
The eigenvalues of B_n are -1 (with multiplicity $n-2$), $\lambda_{1,n}$ (with multiplicity 1), and $\lambda_{2,n}$ (with multiplicity 1). Here $\lambda_{1,n}$ and $\lambda_{2,n}$ are given by

$$\lambda_{i,n} = \frac{(n-2) \pm \sqrt{n^2-4}}{2}, i = 1, 2.$$

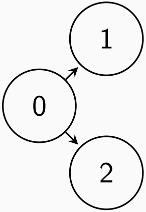
When we remove two edges, there are more configurations. Up to isomorphisms, we have the following types.



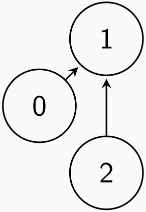
(a) Type 1



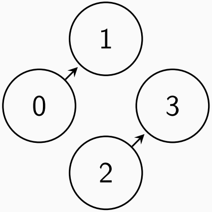
(b) Type 2



(c) Type 3



(d) Type 4



(e) Type 5

Complete digraphs with two edges removed: A quick summary

There are 3 clusters for the spectrum of K_n with two edges removed. They are summarized in the following table.

Cluster	Type of J_n	Spectrum of J_n
1	1	$-1, 0, \frac{(n-3)\pm\sqrt{n^2+2n-7}}{2}$
2	2	$-1, \lambda_{1,n}, \lambda_{2,n}, \lambda_{3,n}$
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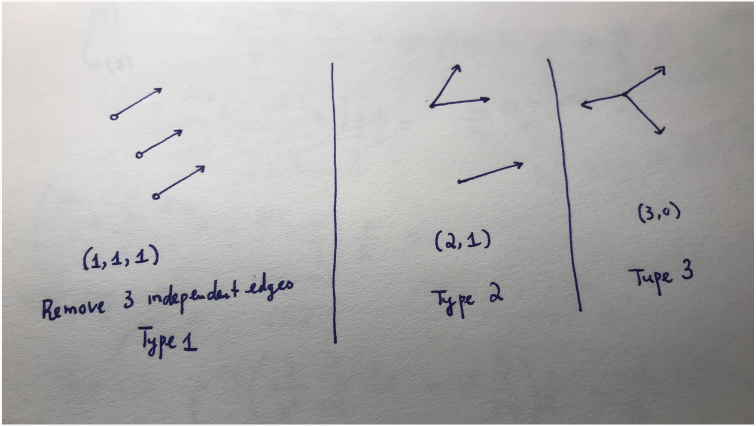
FURSP students: YES!

Suppose that we can now remove 3 edges. From the previous experience, it seems natural to consider the following types.



Let take $N = 6$. The adjacency matrix of the first type of graph is

$$A_1 = \begin{pmatrix} 0 & \boxed{0} & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & \boxed{0} & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & \boxed{0} \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Using Matlab, we can see that the characteristics polynomial of A_1 is

$$p_{A_1}(x) = (x+1)^4(x^2 - 4x - 2).$$

The adjacency matrix of the second type of graph is

$$A_2 = \begin{pmatrix} 0 & \boxed{0} & \boxed{0} & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \boxed{0} & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Using Matlab, we again see that

$$p_{A_2}(x) = (x+1)^4(x^2 - 4x - 2).$$

Finally, the adjacency matrix of the third type of graph is given by

$$A_3 = \begin{pmatrix} 0 & \boxed{0} & \boxed{0} & \boxed{0} & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

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We see that in all these cases, the resulting graphs are cospectral! Let us find an explanation for this phenomenon.

Some notations from graph theory

- A walk is a sequence $v_0, e_1, v_1, \cdots, e_k, v_k$ of graph vertices v_i and graph edges e_i such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i.

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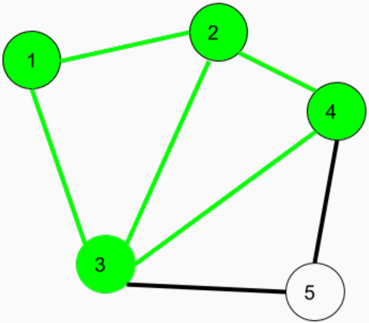
- A graph G is called acyclic if it has no closed walk.
- The complement of a graph G, often denoted by G^c, is a graph with the same vertices as G such that two distinct vertices of G^c are adjacent if and only if they are not adjacent in G.

An example



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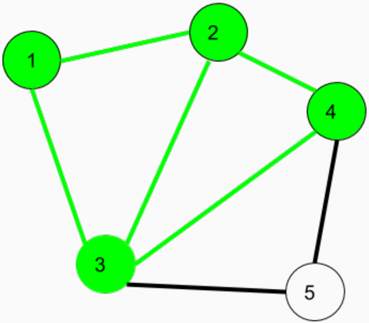

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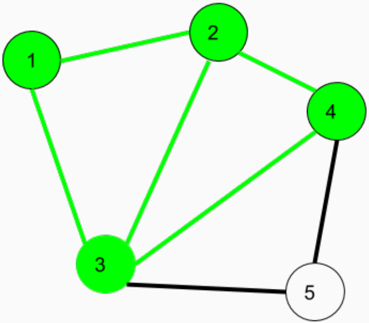

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Here are some corollaries of the previous proposition.

Corollary

The number of walks of length k is

$$S(A^k) = \sum_{i,j} (A^k)_{ij} = \mathbb{1}_n \, A^k \, \mathbb{1}_n^t \, .$$

where $\mathbb{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^{1 \times n}$.

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Corollary

The number of closed walks of length k is $Tr(A^k)$.

Spectrum and closed walk structure of a graph

Suppose that $\mathrm{Spec}(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then

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Therefore, the spectrum of G determines its closed walk structure.

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The converse is also true

Proposition

Let G_1, G_2 be two subgraphs of K_n with adjacency matrices A_1, A_2 respectively. Suppose that for all $k \geq 1$

$$Tr(A_1^k) = Tr(A_2^k).$$

Then G_1, G_2 are cospectral.

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Theorem

Let G_1, G_2 be two subgraphs of the complete digraph K_n. Let G_1^c and G_2^c be their complements. Suppose that for all $k \geq 0$, G_1^c and G_2^c have the same number of walks and closed walks length k, then G_1 and G_2 are cospectral.

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- The second and third proofs use tools from matrix algebra and generating functions.

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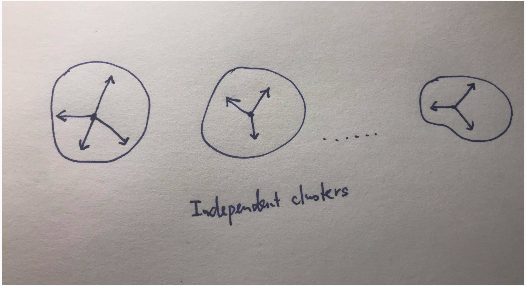
Let k, n be two positive integers. We start with the complete digraph K_n. Let $(k_1, k_2, \ldots, k_r)$ be an r-tuple of positive integers such that $k = \sum_{i=1}^{r} k_i$. We choose r vertices in K_n, say $v_1, v_2, \ldots, v_r$. For a vertex v_i, we remove k_i edges having v_i as the initial vertex. We require that all removed edges are independent, namely they pairwise do not share a terminal vertex.



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For all choices of $(k_1, k_2, \ldots, k_r)$, $G_{n,k_1,k_2,\ldots,k_r}$ are pairwisely cospectral. Their common characteristic polynomial is

$$p(t) = (t+1)^{n-2}[t^2 - (n-2)t - (n-k-1)].$$

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Proof of the first main theorem

Let G_1, G_2 be two subgraphs of K_n and G_1^c, G_2^c be their complements. Suppose the adjacency matrices of G_1^c and G_2^c are A_1, A_2 respectively.

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To show that G_1, G_2 are cospectral, it is enough to show that $J_n - A_1$ and $J_n - A_2$ are cospectral. It is sufficient to show that

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By a similar argument, we can see that for all $k \geq 1$

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Jordan forms of $A_{n,k_1,k_2,\ldots,k_r}$

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- *The spectrum of $A_{n,k_1,k_2,\ldots,k_r}$ is -1(with multiplicity $n-2$), λ_1 (with multiplicity 1) and λ_2 (with multiplicity 1).*

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- *The numer of Jordan blocks $J_{\lambda_i,1}$ is 1 for each $i \in \{1,2\}$.*

The second Theorem

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Theorem

The characteristic polynomial of A is given by

$$p_A(t) = (t+1)^n - a_1(t+1)^{n-1} + a_2(t+1)^{n-2} - \cdots + (-1)^n a_n$$

where a_{k+1} is the number of walks of length k in G^c for $0 \leq k \leq N-1$.

An example

Suppose G is a subgraph of K_n such that G^c is the subgraph of K_n with the following directed edges

$$0 \to 1 \to 2 \to 3.$$

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By the second theorem, the characteristic polynomial G is given by

$$\begin{aligned} p(t) &= (t+1)^n - n(t+1)^{n-1} + 3(t+1)^{n-2} - 2(t+1)^{n-3} + (t+1)^{n-4} \\ &= (t+1)^{n-4}[(t+1)^4 - n(t+1)^3 + 3(t+1)^2 - 2(t+1) + 1]. \end{aligned}$$

Some ongoing projects

In this work, we focus on the spectrum of graphs obtained by removing acyclic subgraphs of K_n. We can ask for other types of edge-removal. For example

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These types of questions have opened great opportunities to study spectral graph theory using tools and ideas from representation theory, matrix algebra, and discrete Fourier transforms.

Some references

- P. Van Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, Cambridge, 2010.
- Ján Mináč, Newton's Identities Once Again!, The American Mathematical Monthly Vol. 110, No. 3 (Mar., 2003), pp. 232-234.
- M. Rudolph-Lilith, L. Muller, Algebraic approach to small-world network models, Phys. Rev. E 89, 2014.
- A. Terras, Zeta functions of graphs: a stroll through the garden Cambridge Studies in Advanced Mathematics, Vol. 128, Cambridge University Press, 2011.

Thank you