Research statement

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I am an interdisciplinary mathematician. My primary research interests are in computational number theory, non-commutative algebra, and nonlinear dynamics. I have also worked on several side projects on spectral graph theory, machine learning, and Galois modules. Since I started my postdoctoral position in June 2021, I have written 18 articles in various fields of pure and applied mathematics (12 papers were published, and 6 papers are under review). In my work, I enjoy collaborating with scientists from several different backgrounds: mathematicians, physicists, neuroscientists, and computer scientists.

In the domain of computational/algebraic number theory, I provided several fundamental insights into a new class of polynomials. More precisely, my collaborators and I also introduced a new mathematical object: generalized Fekete polynomials. In special cases, they are classical polynomials whose history can be traced to the 19th century in relation to the studies of Dirichlet L-functions. With the aid of classical methods and computer simulations, we have revealed much surprising hidden arithmetic and symmetric properties of these polynomials.

I am also deeply interested in several problems in non-commutative algebra and their applications to network theory and computational neuroscience. Specifically, my co-authors and I introduced a new class of algebra and we coined the term *join algebra*. This algebra represents a wide class of multilayer networks that appeared in the literature. We have applied the results of our study on this algebra to investigate the dynamics of networks of nonlinear oscillators described by the Kuramoto model. Originally introduced by Yoshiki Kuramoto in 1975, this model has been the subject of computational and mathematical study across many scientific fields such as physics, neuroscience, and biology. I believe that our new approach will open the possibilities for developing new theoretical advances in nonlinear dynamics, mathematical biology, spectral graph theory, Galois modules, and representation theory of finite-dimensional associative algebras.

1 Research accomplishments

1.1 Fekete polynomials and special values of L-functions

The class number formula is an inspiring pillar of number theory. The origin of this formula could be traced back to the 19th century through the work of Euler, Gauss, Dirichlet, and many other prominent mathematicians. From the beginning, it has had a strong influence on the development of mathematics. Through the work of many mathematicians, notably Deligne, Beilinson, Bloch, Kato, Fontaine, Perrin-Riou, Jannsen, and many others, we now have quite general (conjectural) class number formulas for motives, i.e., the Tamagawa number conjecture of Bloch-Kato. This conjecture provides a precise link between the L-function and the arithmetic of a motive (see [4, 5, 11, 29]).

The earliest examples of an L-function are those associated with Dirichlet characters. We recall that a Dirichlet character is a multiplicative complex-valued function $\chi : (\mathbb{Z}/D)^\times \to \mathbb{C}^\times$. The

L-function of χ is defined by the following infinite series

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

To further study this L-function, we (re)-introduced the following mathematical object.

Definition 1. Let $\chi : (\mathbb{Z}/D)^{\times} \to \mathbb{C}^{\times}$ be a Dirichlet character of modulus D. The generalized Fekete polynomial associated with χ is given by

$$F_{\chi}(x)=\sum_{a=1}^{D-1}\chi(a)x^a.$$

These polynomials were first considered in the 19th century concerning the studies of Dirichlet L-function $L(s,\chi_p)$ where $\chi_p(a)=\left(\frac{a}{p}\right)$ is the quadratic character associated with the Legendre symbol. There are many interesting investigations of behaviors of Fekete polynomials over the interval $[0,1]$, their roots in the complex plane, and their extremal property (see [3, 6, 13]). My main interest in these polynomials lies in exploring their arithmetic and Galois theoretic properties using the interesting interplay between the distribution of quadratic residues, arithmetic properties of Bernoulli numbers, class number formulas, and Galois theory. It is worth mentioning that our discoveries are guided and inspired by computational data and simulations (see [24] for some data that I generated for these projects). Below, I provide a brief summary of our discoveries which are discussed in [23, 25].

First, we showed that there is a bridge that connects $F_{\chi}(x)$ and $L(s,\chi)$ (see [25, Proposition 3.3]).

$$\Gamma(s)L(\chi,s)=\int_0^1\frac{(-\log(t))^{s-1}}{t}\frac{F_{\chi}(t)}{1-t^D}dt,$$

where $\Gamma(s)$ is the Gamma function. Additionally, by the theory of Gauss sums, we can show that if n is a proper divisor of D, then ζ_n is a root of $F_{\chi}(x)$ (see [25, Definition 3.8]). What is really surprising here is that the multiplicity of ζ_n is not always 1. Furthermore, sometimes there are other exceptional zeros that cannot be predicted by Gauss sums. Here is a concrete example of these phenomena.

Theorem 1. *[25, Section 9] Let p be a prime number, $\Delta=3p$, and $\chi=\chi_{\Delta}$ be the quadratic character associated with Δ. Suppose further that $p\equiv 2 \pmod 3$ and $p\equiv 3 \pmod 4$. Then*

(i) ζ_6 is a simple root of $F_{\chi}(x)$.

(ii) ζ_3 is a double root of $F_{\chi}(x)$.

We also observed that special values of certain polynomials associated with $F_{\chi}(x)$ contain much interesting arithmetic information such as the class numbers and the order of certain K-groups. These numbers are amongst the most important objects in number theory and arithmetic geometry. For instance, they play a fundamental role in the Bloch-Kato conjecture on special values of L-functions [5]. Our discoveries showed that these numbers are encoded in $F_{\chi_p}(x)$. More precisely

Theorem 2. *[23] Suppose p is a prime number such that $p\equiv 3 \pmod 4$. Let $g_p(x)$ be the reduced Fekete polynomial associated with $F_{\chi_p}(x)$. Let $h(-p)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$. Then*

1. $g_p(-2) = -\left(2\left(\frac{2}{p}\right) - 1\right) h(-p)$.

2. $g_p(-1) = -\frac{1}{2}\left(\left(\frac{p}{3}\right) + 3\right) h(-p)$.

3. $g_p(0) = g_p(-2) = -\left(2\left(\frac{2}{p}\right) - 1\right) h(-p)$.

4. $g_p(1) = -\frac{h(-p)}{2}\left(\frac{6}{p}\right)\left[6 - 3\left(\frac{2}{p}\right) - 2\left(\frac{3}{p}\right) + \left(\frac{6}{p}\right)\right]$.

A direct corollary of this theorem is that it provides an efficient way to compute $h(-p)$ in practice. Furthermore, our simulated data also suggests that the Galois groups of g_p are often as large as possible. More precisely, for $p \leq 1000$, the Galois group of $g_p(x)$ is the symmetric group S_n on $n = \deg(g_p)$ letters.

Our investigation into the Galois theoretic properties of these polynomials has also yielded some advancement in group theory and computational number theory. For instance, we discovered a significantly faster algorithm for computing the Galois group of a reciprocal polynomial (see [25, Proposition 11.11]). As a result, our work unlocks the potential for studying the arithmetic of a wide class of polynomials that naturally appear in the literature.

1.2 The join algebra and nonlinear dynamics on multilayer networks

Networks play a central role in modern neuroscience. From mapping the anatomy of connections between neurons to understanding how patterns of connections create computation in the brain, networks are a central mathematical tool (see [31, 32]). In this domain, the Kuramoto oscillator has emerged as the central mathematical model for studying these networks (see [14, 21, 30]). We recall that the Kuramoto model is described by the following differential equation

$$\dot{\theta}_i = \omega_i + \kappa \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \tag{1.1}$$

where $\theta_i(t) \in [-\pi, \pi]$ is the state of oscillator $i \in [1, N]$ at time t, ω_i is the intrinsic angular frequency, κ scales the coupling strength, and element $a_{ij} \in \{0, 1\}$ represents the connection between oscillators i and j.

In many applications, a natural model of a network of oscillators is a *ring graph*, in which nodes are regularly placed along a circle and, for a fixed number m, each oscillator is connected to its m closest neighbors on either side (see [18, 33]). These networks are represented by adjacency matrices which are circulant. The Circulant Diagonalization Theorem describes the eigenspectrum and eigenspaces of these circulant matrices explicitly via the discrete Fourier transform (see [16, 20]). Consequently, many problems involving circulant matrices have closed-form or analytical solutions. In our investigation of the non-linear dynamics on networks of oscillators, the multilayer network of (circulant) graphs appears quite often, and they provide new exciting phenomena (see Fig. 2 for an example of a multilayer network with three layers.)

It is known that the structure of these networks strongly influences their dynamics. This observation naturally led us to investigate their spectral properties. In [17], we generalized the Circulant Diagonalization Theorem to the join of several circulant matrices. We first recall the definition of the join of circulant matrices

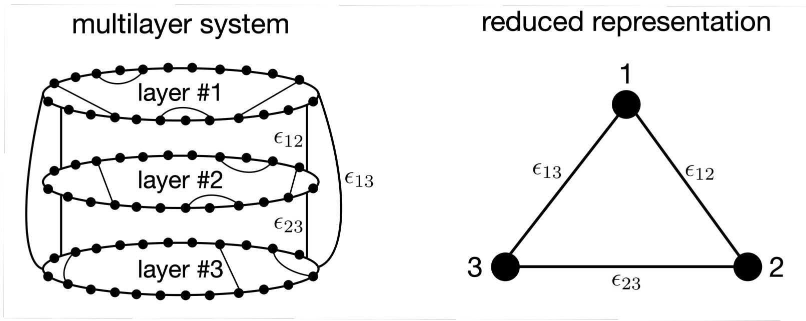

Figure 1: Graphic representation of a multilayer network with 3 layers.

$$C = \left(\begin{array}{c|c|c|c} C_1 & a_{1,2}\mathbf{1} & \cdots & a_{1,d}\mathbf{1} \\ \hline a_{2,1}\mathbf{1} & C_2 & \cdots & a_{2,d}\mathbf{1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{d,1}\mathbf{1} & a_{d,2}\mathbf{1} & \cdots & C_d \end{array} \right).$$

Here for each i, C_i is a circulant matrix of size $k_i \times k_i$ and it represents the inter-layer connections between oscillators within the i-th layer. Additionally, $a_{i,j}\mathbf{1}$ is the $k_i \times k_j$ matrix with all entries equal to a constant $a_{i,j}$ which describes the coupling strength of the connection between the i-th layer and the j-th layer. Our main result is the following.

Theorem 3 ([17], Generalized Circulant Diagonalization Theorem)**.** *The spectrum of C is the union of the following multisets, i.e.,*

$$Spec(C) = Spec(\overline{C}) \cup_{i=1}^{d} \{\lambda_j^{C_i} | 1 \leq j \leq n_i - 1\},$$

where $\overline{C}$ is an explicit $d \times d$ matrix, whose entries are the row sums of the blocks of C.

Furthermore, we also observed in [12] that the set of all such matrices has a structure of an algebra which we call the *join algebra* of $G_1, G_2, \ldots, G_d$ and denote by $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$. In [12], we laid out some foundational properties of this algebra. We mention here some of these properties.

Theorem 4. *Let k be a field ring and $\mathcal{J}_{G_1,\ldots,G_d}(k)$ the join algebra described above.*

1. *$\mathcal{J}_{G_1,\ldots,G_d}(k)$ has the structure of an unital ring and there is an augmentation map $\epsilon\colon \mathcal{J}_{G_1,\ldots,G_d}(k) \to M_d(k)$ that generalizes the augmentation map on group rings.*

2. *If $|G_i|$ is invertible in k for all $1 \leq i \leq d$, then the ring $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$ is semisimple.*

3. *If $|G_i| = 0$ in k for all $1 \leq i \leq d$, then*

$$(\mathcal{J}_{G_1,\ldots,G_d}(k))^{\times} \cong k^{d^2-d} \times \prod_{i=1}^{d} U_1(k[G_i])) \rtimes (k^{\times})^d,$$

where $U_1(k[G_i])$ is the group of principal units in $k[G_i]$.

4. *If k is algebraically closed and $char(k)$ is relatively prime to $\prod_{i=1}^{d} |G_i|$, then the number of irreducible modules over $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$ is $c(G_1) + c(G_2) + \ldots + c(G_d) - d + 1$, where $c(G_i)$ is the number of conjugacy classes of G_i.*

5. *If k is any field and $d \geq 2$, then $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$ is a Frobenius algebra if and only if $|G_i|$ is invertible in k for all $1 \leq i \leq d$.*

Using these properties of $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$, we discovered the following interesting phenomenon.

Theorem 5. *([10]) There exists a mechanism to broadcast a solution from the reduced system to the multilayer system. Furthermore, a solution is linearly stable iff the broadcast solution is.*

2 Future research plans

My first research experience in mathematics was in the first semester of my third year when I attended the Research Experience for Undergraduates program organized by Professor Moshe Rosenfeld at Vietnam National University in 2012. This program has changed my life. Consequently, I have a strong passion to bring similar opportunities to the next generation of students in mathematics. While writing this section, I have those students in mind. Projects marked with $(*)$ indicate that they are suitable for a student project.

2.1 Further properties of Fekete polynomials

In Section 1.1, I introduced the Fekete polynomials. While we have revealed many properties of these polynomials, there are several promising problems and projects that we plan to pursue. All of these problems are supported by extensive numerical data. Below, I provide a brief introduction to some of these problems.

Project 1. $(*)$ In [25], we determined the cyclotomic factors of $F_\chi(x)$ where χ is a quadratic character with conductor $D \in \{3p, 4p\}$. What about more general χ? For example, what if the conductor of χ is $8p$ (where p is a prime number)?

Students with a background in modular arithmetic and Galois theory can certainly enjoy Project 1. They will also have a chance to pick up some coding skills along the way (Sagemath, and more generally Python).

Project 2. $(*)$ As we demonstrated in [23], the Fekete polynomial $F_{\chi_p}(x)$ has interesting properties over a finite field. From our numerical data, we observe the following. Let Δ be an odd fundamental discriminant such that $gcd(\Delta, 6) = 1$. Let χ_Δ be the quadratic character defined by $\chi_\Delta(a) = \left(\frac{\Delta}{a}\right)$. Let p be a prime divisor of $D = |\Delta|$ and $d = \frac{D}{p}$. Then over $\mathbb{F}_p[x]$,

1. The multiplicity of Φ_d is $\frac{p-1}{2}$.
2. The multiplicity of Φ_n for $n|d$ and $1 < n < d$ is p.

Project 3. In [23] and [25], we focused on quadratic characters. From our data, it seems that this circle of ideas can be applied to a broader class of characters. To test this speculation, we plan to investigate the case of cubic characters and quartic characters. What can we say about the arithmetic of $F_\chi(x)$ in those cases?

2.2 The algebra of the join of circulant matrices and applications

In Section 1.2, I described the *join algebra* $\mathcal{J}_{G_1,G_2,\ldots,G_d}(k)$. While we have discovered many properties, there are lots of questions that we can ask about this algebra. The answers to these questions would open new exciting opportunities to study networks theory and Galois modules using the representation theory of finite groups.

Here are some questions that I am currently pursuing. They are quite accessible for both undergraduate and graduate students.

Question 1. Can we describe the categories of modules over $J_{G_1,G_2,\ldots,G_d}(k)$?

Satisfying answers to this question would lead to some new advances in the study of Galois modules (see [26, 27] for some old and recent advances on this topic).

When G_i are p-groups and k is a fields of characteristics p, we can show that the algebra $J_{G_1,G_2,\ldots,G_d}$ is basic. In this case, Auslander–Reiten theory is an important tool to study the join algebra (see [1, 2]). In particular, I plan to study the following question.

Question 2. Can we describe the ordinary quiver associated with $J_{G_1,G_2,\ldots,G_d}(k)$? What about the Auslander-Reiten quiver for this ring?

Question 3. $(*)$ Describe explicitly the spectrum of a join matrix $A \in J_{G_1,G_2,\ldots,G_d}(\mathbb{C})$ where $\mathbb{C}$ is the complex number field. Develop a non-abelian Circulant Diagonalization Theory for this class of matrices.

The answer to this question will provide an important tool to understand the dynamics of multilayer networks as discussed in Section 1.2.

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