SPECTRAL PERTURBATION BY RANK-m MATRICES

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Abstract. Let A and B designate $n \times n$ matrices with coefficients in a field F. In this paper, we completely answer the following question: For A fixed, what are the possible characteristic polynomials of $A + B$, where B ranges over matrices of rank $\leq m$?

1. Introduction

The perturbation of a given matrix by another low-rank matrix is an important topic in mathematics, physics, and engineering. For example, it has been used to study the stability and controllability of dynamical systems, the Baik-Ben Arous-Péché (BBP) phase transition, and quantum chaotic scattering (see [1], [3], [4], [8], [11]). Consequently, the spectral perturbation problem has been extensively studied in the literature. For interested readers, we refer to some prominent works on this topic (see for example [2], [5], [6], [7], [9], [10]).

A particularly interesting question in the low-rank perturbation study is the following: For a fixed $n \times n$ matrix A with coefficients in a field F, what are the possible characteristic polynomials of $A + B$, where B ranges over matrices with coefficients in F and of rank $\leq m$? Previously, our group has studied this problem in the case where $m = 1$ and F is algebraically closed (see [9]). In this article, we answer this question completely without any restriction on the field F.

To state our main theorem, we first introduce some notation. For a polynomial $p(t) \in F[t]$ and $\lambda \in \overline{F}$ (a fixed algebraic closure of F) we write $m_\lambda(p(t))$ for the multiplicity of λ as a zero of $p(t)$ (taking this to be 0 if λ is not a zero of $p(t)$). The characteristic polynomial of a matrix A will be designated p_A; for $\lambda \in \overline{F}$ we denote by $alg_\lambda(A)$ the algebraic multiplicity of λ as an eigenvalue of A, that is, $alg_\lambda(A) = m_\lambda(p_A)$. Our principal result is the following.

Theorem 1. *Let A be an $n \times n$ matrix over a field F and $q(t) \in F[t]$ monic of degree n. Then there exists an $n \times n$ matrix B over F of rank $\leq m$ such that $p_{A+B} = q$ if and only*

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if for each eigenvalue λ of A,

$$m_\lambda(q) \geq alg_\lambda(A) - \sum_{j=1}^{m} k_{\lambda,j}.$$

where $k_{\lambda,1} \geq k_{\lambda,2} \geq \cdots \geq k_{\lambda,m}$ are the sizes of the largest m Jordan blocks for λ in the Jordan form for A.

The structure of our article is as follows. In Section 2, we derive the necessary condition in the theorem for the existence of the matrix B, using a rank estimate. In Section 3, we show that the necessary condition is also sufficient. Additionally, we provide a concrete demonstration of our proof in the case $m = 2$.

2. Necessary conditions using Jordan forms

As indicated above, we will fix a matrix A, and let B be a matrix of rank less than or equal to the postive integer m. In this section we develop a necessary condition on a monic polynomial q of degree n for $q = p_{A+B}$ for some such B.

Lemma 1. *The following identity holds:*

$$(B + A)^k = \left[\sum_{m=0}^{k-1} A^m B(B + A)^{k-m-1}\right] + A^k.$$

Remark 1. The result in fact holds for arbitrary elements A, B in any ring, as the proof below shows.

Proof. Let us prove this by induction. For $k = 1$, the left hand side and the right hand side are both $B + A$. Let's consider $k = 2$. The left hand side is

$$(B + A)^2 = (B + A)(B + A) = B(B + A) + A(B + A) = B(A + B) + AB + A^2.$$

Suppose the formula is true for k. Let us show that it is true for $k + 1$. Indeed we have

$$\begin{aligned}(B + A)^{k+1} &= (B + A)(B + A)^k = B(B + A)^k + A(A + B)^k \\ &= B(B + A)^k + A\left[A^k + \sum_{m=0}^{k-1} A^m B(B + A)^{k-m-1}\right] \\ &= B(B + A)^k + A^{k+1} + \sum_{m=0}^{k-1} A^{m+1} B(B + A)^{k-m-1}.\end{aligned}$$

For the last term, let $n = m + 1$

$$\sum_{m=0}^{k-1} A^{m+1} B (B+A)^{k-m-1} = \sum_{n=1}^{k} A^{n} B (B+A)^{(k+1)-n-1}.$$

Therefore, we can see that

$$(B+A)^{k+1} = \left[\sum_{m=0}^{k} A^{m} B (B+A)^{k-m} \right] + A^{k+1}.$$

By induction, the above formula is true for all k.

$\square$

We provide another pictorial proof for Lemma 1.

Proof. Terms in the expression of $(A+B)^k$ corresponds to paths of length $k-1$ in the following labelled graphs (with $2k$ nodes).

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A \arrow[r] \arrow[dr] & A \dots \arrow[r] & A \\
B \arrow[r] \arrow[ur] & B \arrow[r] \arrow[ur] & B \dots \arrow[r] & B
\end{tikzcd}$$

Apart from the term A^k, each term has an initial block of the form $A^m B$, $0 \leq m \leq k-1$. Visualizing that block in the graph above (starting from the left), and considering all terms which begin with that block, we see that they correspond to continuing paths through the expansion $(B+A)^{k-m-1}$. $\square$

Corollary 1. *For each k*

$$rank((A+B)^k) \leq k \, rank(B) + rank(A^k).$$

Proof. This is a direct consequence of Lemma 1 and the facts that for two matrices M, N

$$\operatorname{rank}(MN) \leq \min\{\operatorname{rank}(M), \operatorname{rank}(N)\},$$

and

$$\operatorname{rank}(M+N) \leq \operatorname{rank}(M) + \operatorname{rank}(N).$$

$\square$

Let C be a matrix defined over F and $\lambda \in \overline{F}$. As in the statement of the main theorem, we denote by $\operatorname{alg}_\lambda(C)$ the algebraic multiplicity of λ with respect to C. More precisely,

$$\operatorname{alg}_\lambda(C) = m_\lambda(p_C(t)).$$

Let $k_{\lambda,1} \geq k_{\lambda,2} \geq \ldots$ be the sizes of all Jordan blocks of A with λ on the diagonal (here, to avoid burdening the notation, we do not explicitly fix the number of Jordan blocks, but we think of $k_{\lambda,j}$ as an eventually zero sequence of integers or equivalently we adopt the convention that a 0×0 Jordan block is an empty block.

By Corollary 1, we have

$$\operatorname{rank}((A+B-\lambda I_n)^{k_{\lambda,i}}) \leq k_{\lambda,i} \operatorname{rank}(B) + \operatorname{rank}((A-\lambda I_n)^{k_{\lambda,i}}) \leq m k_{\lambda,i} + \operatorname{rank}((A-\lambda I_n)^{k_{\lambda,i}}).$$

It is straightforward to see that

$$\operatorname{rank}((A-\lambda I_n)^{k_{\lambda,i}}) = (n - \operatorname{alg}_\lambda(A)) + \sum_{j=1}^{i} (k_{\lambda,j} - k_{\lambda,i}).$$

Therefore

$$\operatorname{rank}((A+B-\lambda I_n)^{k_{\lambda,i}}) \leq n - \operatorname{alg}_\lambda(A) + (m-i)k_{\lambda,i} + \sum_{j=1}^{i} k_{\lambda,j}. \tag{2.1}$$

Our goal here is to make sure that the right hand side is as small as possible as a function of i. Equivalently, we want to minimize the sum $s_i = (m-i)k_{\lambda,i} + \sum_{j=1}^{i} k_{\lambda,j}$.

Proposition 1. *The sum*

$$s_i = (m-i)k_{\lambda,i} + \sum_{j=1}^{i} k_{\lambda,j},$$

attains its minimum at $i = m$.

Proof. Since

$$\begin{aligned} s_i - s_{i+1} &= \left[(m-i)k_i + \sum_{j=1}^{i} k_i\right] - \left[(m-i-1)k_{i+1} - \sum_{j=1}^{i+1} k_i\right] \\ &= (m-i)(k_i - k_{i+1}). \end{aligned}$$

We see that the sequence s_i is decreasing for $i \leq m$ and increasing for $i > m$. Therefore, it attains its minimum at $i = m$.

□

Taking $i = m$ in estimate (2.1), which is the optimal choice by Proposition 1, we see that

$$\operatorname{rank}((A+B-\lambda I_n)^{k_{\lambda,m}}) \leq n - \operatorname{alg}_\lambda(A) + \sum_{j=1}^{m} k_{\lambda,j}.$$

Consequently, we have

Proposition 2. *Let A be a given matrix. Suppose B is a matrix with rank at most m such that $p_{A+B}(t) = q(t)$. Then*

$$\text{(2.2)} \qquad \textit{for each eigenvalue } \lambda \textit{ of } A,\ m_\lambda(q) \geq alg_\lambda(A) - \sum_{j=1}^{m} k_{\lambda,j}.$$

3. Sufficient conditions using rational canonical forms

We now show that the condition (2.2) is sufficient for the existence of B defined over F (of rank $\leq m$) with $p_{A+B} = q$. As above, A is a fixed $n \times n$ matrix over F and $q(x) \in F[x]$ is monic of degree n.

Assume now without loss of generality that A is in rational canonical form,

$$A = \begin{bmatrix} \boxed{p_s} & & & 0 \\ & \boxed{p_{s-1}} & & \\ & & \ddots & \\ 0 & & & \boxed{p_1} \end{bmatrix}$$

where $p_1 | p_2 | \cdots | p_s$ and $\boxed{p_i}$ is the companion matrix of p_i. (Note that $p_1, \cdots, p_s \in F[x]$.) We have $p_A = \prod_{i=1}^{s} p_i$.

We first reformulate the necessary condition (2.2) in terms of $p_1, \cdots, p_s$.

Proposition 3. *For $q(t) \in F[t]$, the condition (2.2) is equivalent to $p_1 p_2 \ldots p_{s-m} | q$.*

Proof. The Jordan form for A is the direct sum of Jordan blocks from the Jordan decompositions of the $\boxed{p_i}$. But $\boxed{p_i}$ has p_i as its minimal and characteristic polynomial, and so there can be at most one Jordan block with a given eigenvalue in the Jordan decomposition of $\boxed{p_i}$. Since $p_1 | p_2 | \cdots | p_s$, the largest m Jordan blocks for an eigenvalue λ come from $\boxed{p_s}, \cdots, \boxed{p_{s-m+1}}$; thus

$$\sum_{j=1}^{m} k_{\lambda,j} = \sum_{j=s-m+1}^{s} m_\lambda(p_j)$$

while

$$alg_\lambda(A) = \sum_{j=1}^{s} m_\lambda(p_j).$$

So (2.2) is equivalent to $m_\lambda(q) \geq \sum_{i=1}^{s-m} m_\lambda(p_i)$ for each eigenvalue λ of A . Since the p_i's have only eigenvalues of A as roots, this amounts to $p_1 \cdots p_{s-m} \mid q$ □

Proposition 4. *If $q(t) \in F[t]$ is monic of degree n and satisfies condition (2.2) in Proposition 2 then there exists a matrix B over F of rank at most m with $p_{A+B} = q$.*

Proof. If condition (2.2) holds then by Proposition 3 we have $p_1 \cdots p_{s-m} \mid q$; set $h = q/(p_1 \cdots p_{s-m})$. Let d_i be the degree of p_i for $i = 1, \cdots, s$.

Certainly our goal is accomplished if we are able to create a matrix with characteristic polynomial q by replacing m columns of A with new columns whose entries are in F. Let A_i be the ith column of A and let e_i be the column vector with 1 in position i and 0 elsewhere. Also for $1 \leq i \leq m$ let $\delta_i = \sum_{j=0}^{i-1} d_{s-j}$. (Note that $\deg h = \delta_m$.) We claim that we can alter columns $\delta_1, \delta_2, \cdots, \delta_m$ of A so that the first δ_m rows and columns constitute $\boxed{h}$, the companion matrix of h. For each $i \notin \{\delta_1, \delta_2, \cdots, \delta_m\}$ where $1 \leq i \leq \delta_m$ we already have $A_i = e_{i+1}$. To create $\boxed{h}$ we need only replace A_i with e_{i+1} for $i = \delta_1, \delta_2, \cdots, \delta_{m-1}$ and replace A_{δ_m} with $[-b_0, -b_1, \cdots, -b_{\delta_m - 1}, 0, \cdots, 0]^T$ where $h(x) = x^{\delta_m} + b_{\delta_m - 1} x^{\delta_m - 1} + \cdots + b_1 x + b_0$.

The resulting matrix, though no longer necessarily in rational canonical form, is the direct sum of blocks $\boxed{h}, \boxed{p_{s-m}}, \ldots, \boxed{p_1}$ and so has characteristic polynomial $h p_{s-m} \cdots p_1 = q$ as desired. □

It may be helpful to look at the above proof in the special cases $m = 1$ and $m = 2$.

In the case $m = 1$ we have $q = h p_1 p_2 \cdots p_{s-1}$. Setting $d = d_s$ which is the degree of both p_s and h, we may write

$$p_s = x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

and

$$h = x^d + b_{d-1} x^{d-1} + \cdots + b_1 x + b_0,$$

So

$$\boxed{p_s} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d-1} \end{bmatrix}$$

and

$$\boxed{h} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{d-1} \end{bmatrix}$$

Taking B to be a matrix with 0 everywhere but the first d entries of column d, and having for those entries $[a_0 - b_0 \quad a_1 - b_1 \quad \cdots \quad a_{d-1} - b_{d-1}]^T$, we find in forming $A + B$ that we have simply replaced $\boxed{p_s}$ with $\boxed{h}$, and it's clear that B has coefficients in F, that B has rank one and that the characteristic polynomial of $A + B$ is $hp_1p_2 \cdots p_{s-1} = q$.

In the case $m = 2$ we have $q = hp_1 \cdots p_{s-2}$, and we can modify just 2 columns of the (rational canonical form of the) matrix A to transform the leftmost two blocks

$$\begin{bmatrix} \boxed{p_s} & \\ & \boxed{p_{s-1}} \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 & 0 & \cdots & 0 & -a_0^{(s)} \\ 1 & 0 & \cdots & 0 & -a_1^{(s)} \\ 0 & 1 & \cdots & 0 & -a_2^{(s)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d_s-1}^{(s)} \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & \cdots & 0 & -a_0^{(s-1)} \\ 1 & 0 & \cdots & 0 & -a_1^{(s-1)} \\ 0 & 1 & \cdots & 0 & -a_2^{(s-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{d_{s-1}-1}^{(s-1)} \end{matrix} \end{bmatrix}$$

into

$$\boxed{h} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & & & & & -b_0 \\ 1 & 0 & \cdots & 0 & 0 & & & & & -b_1 \\ 0 & 1 & \cdots & 0 & 0 & & & 0 & & -b_2 \\ \vdots & \vdots & & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & -b_{d_s-1} \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & -b_{d_s} \\ & & & 0 & 0 & 1 & 0 & \cdots & 0 & -b_{d_s+1} \\ & & 0 & & & & \ddots & & & \vdots \\ & & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -b_{d_s+d_{s-1}-1} \end{bmatrix}$$

where $p_i = x^{d_i} + a^{(d_i)}_{d_i-1} x^{d_i-1} + \cdots + a^{(d_i)}_0$ and $h = x^{d_s+d_{s-1}} + b_{d_s+d_{s-1}-1} x^{d_s+d_{s-1}-1} + \cdots + b_1 x + b_0$. (The two altered columns are column d_s and column $d_s + d_{s-1}$.) This yields a matrix B of rank 2 such that the characteristic polynomial of $A + B$ is q.

We can now prove our principal result.

Proof of Main Theorem. Combine the sufficient condition of Proposition 4 with the necessary condition of Proposition 2. □

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References

[1] J. Baik, G. Ben Arous, and S. Péché, Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices, The Annals of Probability 33 (2005), no. 5, 1643–1697.

[2] L. Batzke, C. Mehl, A. Ran, L. Rodman, Generic rank-k perturbations of structured matrices. In: Eisner, T., Jacob, B., Ran, A., Zwart, H. (eds.) Operator Theory, Function Spaces, and Applications IWOTA. Springer, Berlin (2016).

[3] Y.V. Fyodorov and H.J. Sommers, Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance, Journal of Mathe- matical Physics 38 (1997), no. 4, 1918–1981.

[4] J. Kautsky and N.K. Nichols. Robust pole assignment in linear state feedback. Int. J. Control, 41:1129–1155, 1985.

[5] M. Krupnik, Changing the spectrum of an operator by perturbation. Sixth Haifa Conference on Matrix Theory (Haifa, 1990), Linear Algebra Appl. 167 (1992), 113–118.

[6] C. Mehl, V. Mehrmann, A. Ran, L. Rodman, Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations, Linear Algebra Appl., 435 (2011), pp. 687-716.

[7] C. Mehl and A. Ran, Low rank perturbations of quaternion matrices, Electron. J. Linear Algebra 32 (2017), 514–530.

[8] S. Péché, The largest eigenvalue of small rank perturbations of Hermitian random matrices, Probability Theory and Related Fields 134 (2006), no. 1, 127–173.

[9] Merzel, J., Minac, J., Muller, L., Pasini, F.W. and Nguyen, T.T., 2021. Spectral perturbation by rank one matrices. arXiv preprint arXiv:2111.10624.

[10] A. Ran and M. Wojtylak, Eigenvalues of rank one perturbations of unstructured matrices, Linear Algebra Appl. 437 (2012), no. 2, 589–600.

[11] Shinners SM. Modern control system theory and design. John Wiley and Sons; 1998 May 6.

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