Power sums and special values of L-functions

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Western University, Algebra Seminar

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Plans

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- Hurwitz zeta functions.
- Special values of Hurwitz zeta functions.
- Some further topics for future investigations.

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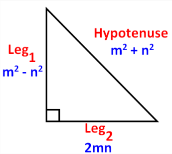

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These numbers contain arithmetic information about K.

Class number formula-The analytic side

The Dedekind zeta function for K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}.$$

It has a meromorphic continuation to $\mathbb{C}$ with a unique simple pole at $s = 1$.

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Theorem (Class number formula)

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{w_K \sqrt{|D_K|}}.$$

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This formula provides a connection between the algebraic world and the analytic world.

Euler's discoveries

- In 1734, Leonhard Euler found the following remarkable formula

$$\zeta(2) = \zeta_{\mathbb{Q}}(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.$$

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- Indeed, Euler did much more. In particular, he showed that
$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k},$$
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$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$
- Using the functional equation for $\zeta(s)$ we have
$$\zeta(1-n) = \zeta(-n) = (-1)^{n-1}\frac{B_n}{n}.$$

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Today talk is mostly about phase 1 and a little bit about phase 2 and phase 3.

The generalized class number formula (The Bloch-Kato's conjecture)

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- Roughly speaking, the generalized class number formula says that the many arithmetic information of an algebraic variety X can be read off from its L-function.
- For example, if X is given by an cubic equation of the form

  $$y^2 = x^3 + ax + b,$$

  with $a, b \in \mathbb{Q}$ then the L-function $L(E, s)$ can conjecturally tell how large $X(\mathbb{Q})$ is. This is known as the Birch and Swinnerton-Dyer conjecture.

Hurwitz zeta function

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- When $a = 1$, we have $\zeta(s,1) = \zeta(s)$.
- Like the Riemann zeta function, the Hurwitz zeta function has an analytic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.

Bernoulli polynomials

- Let $B_n(x)$ be the function given by the Taylor expansion

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n.$$

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  This can be seen by looking at the Taylor expansion

$$\frac{ze^{xz}}{e^z - 1} = e^{xz} \times \frac{z}{e^z - 1} = \left[\sum_{m=0}^{\infty} \frac{x^m}{m!} z^n\right] \times \left[\sum_{n=0}^{\infty} \frac{B_m(0)}{m!} z^m\right]$$

- $B_n(x)$ is called the n-th Bernoulli polynomial.

Special values of Hurwitz zeta functions, I

Using contour integrals, we can show that

Theorem (Special values of Hurwitz zeta functions)

For $n \geq 0$

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1}.$$

Some examples of Bernoulli numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

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is a polynomial in M of degree $k+1$. This prediction turns out be be correct! In fact, we have the following theorem.

Theorem

$$S_n(x) = \frac{B_{n+1}(x) - B_{n+1}(1)}{n+1}.$$

Minac's theorem

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$$\int_0^1 S_3(x)dx = \int_0^1 \frac{x^2(x-1)^2}{4}dx = \frac{1}{120}.$$

He observed that the numbers on the right hand sides are exactly zeta values $\zeta(-1), \zeta(-2), \zeta(-3)$.

From this observation, he proved that

Theorem

$$\zeta(-n) = \int_0^1 S_n(x)dx.$$

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It is known that $\int_a^{a+1} B_{n+1}(x)dx = a^{n+1}$ for all a. In particular, $\int_0^1 B_{n+1}(x)dx = 0$. Additionally, by Euler's theorem

$$\zeta(-n) = -\frac{B_{n+1}(1)}{n+1}.$$

Combining these two facts, we have

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By plugging $s = -1, -2, \dots$, we get some combinatorial relations between $\zeta(-n)$ for $n \geq 0$. Using the collapsing sum method, we have

$$\begin{aligned}(M-1)^{n+1} &= 1 + \sum_{m=2}^{M-1}(m^{n+1} - (m-1)^{n+1}) \\ &= \sum_{k=0}^{n} \binom{n+1}{k+1}(-1)^k S_{n-k}(M).\end{aligned}$$

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Integrating both sides from 0 to 1, we easily see that $\int_0^1 S_n(x)dx$ satisfies a similar combinatorial relation.

Generalied power sums and special values of the Hurwtiz zeta functions

Recently, we generalized Minac's theorem for the Hurwitz zeta functions. More precisely, let $S_{n,a}$ is the generalized power sum defined by

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Note that when $a = 1$, this gives a direct generalization of Minac's theorem.

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Additionally, $S_{n,a}(x) = a^n S_0(x) + \sum_{k=0}^{n-1} \binom{n}{k} a^k S_{n-k}(x-1)$.

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  Additionally, $S_{n,a}(x) = a^n S_0(x) + \sum_{k=0}^{n-1} \binom{n}{k} a^k S_{n-k}(x-1)$.
- We see that $\int_{1-a}^{2-a} S_{n,a}(x)dx$ satisfies a similar recursive relation.

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$$= \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s-1)(k+1)!} \frac{1}{(a+n)^{s+k}}.$$

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$$= \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s-1)(k+1)!} \frac{1}{(a+n)^{s+k}}.$$

- Using the collapsing sum, we have
$$\frac{1}{a^{s-1}} = \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s-1)(k+1)!}(\zeta(s+k,a) - \frac{1}{a^{s+k}}).$$

Sketchs of our proofs

For the third proof, we use the recursive relation between Hurwitz zeta values. The key idea is the following observation

- We have
$$\frac{1}{(n+a-1)^{s-1}} - \frac{1}{(n+a)^{s-1}} = \frac{1}{(n+a)^s}\left((1-\frac{1}{n+a})^{-(s-1)} - 1\right)$$
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- Using the same collapsing technique, we can show that $\int_{1-a}^{2-a} S_{n,a}(x)dx$ satisfies a similar recursive relation.

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In fact, we have the following theorem.

Theorem (Kummer)

Let r_1, r_2 be two positive integers. Suppose r_1 is not a multiple of $p-1$. If $r_1 \equiv r_2 \pmod{(p-1)p^{n-1}}$ then

$$(1-p^{r_1-1})\zeta(1-r_1) \equiv (1-p^{r_2-1})\zeta(1-r_2) \pmod{p^n}.$$

Arithmetic significance of zeta values

Recall that Fermat last theorem holds if $p \nmid h$ where h is the class number of $\mathbb{Z}[\zeta_p]$.

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Ribet's proof is quite interesting: it uses Galois representations attached modular forms and congruence between modular forms. Further generalizations of his method have led to a proof of Fermat last theorem for all odd prime p (due to the work of Wiles, Taylor, and many others).

Future investigations

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- In some ongoing joint projects with Lyle Muller, Jan Minac and many others, we are studying different aspects of zeta functions. In particular, we are quite interested in the Ihara zeta functions for graphs and their applications to neuroscience.
- The Galois groups play an important role in number theory. In joint work with Nguyen Duy Tan and Jan Minac, we are investing the structures of Galois groups over global fields.

Thank you!