$$\log q(l_j \mid \alpha'_{l_j} + d\alpha'_{l_j}) \simeq O(d\alpha'^2_{l_j}) + \log q(l_j \mid \alpha'_{l_j}) + d\alpha'^T_{l_j} \nabla_{\alpha'_{l_j}} \log q(l_j \mid \alpha'_{l_j})$$

$$\mathrm{Beta}(x; u, v) = \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} x^{u-1}(1-x)^{v-1}$$

$$f(X, Z \mid U, V, \Pi) = f(X \mid U, V, \Pi, Z) \cdot f(Z \mid \Pi) = \prod^N \prod^I [\pi_i \mathrm{Beta}(x; u, v)]^{z_{ni}}$$

$$f(x; \Pi, U, V) = \sum_{i=1}^{I} \pi_i \mathrm{Beta}(x; u_i, v_i) = \sum_{i=1}^{I} \pi_i \sum_{l=1}^{L} \mathrm{Beta}(x_l; u_{il}, v_{il})$$

$$f(X, Z, \Pi, U, V) = f(X \mid U, V, \Pi, Z) f(Z \mid \Pi) f(\Pi) f(U) f(V)$$

$$= \prod^N \prod^I [\pi_i \mathrm{Beta}(x; u, v)]^{z_{ni}} \cdot \boxed{C(c)} \prod^I \pi_i^{c_i - 1}$$

$$\prod^L \prod^I \left[\boxed{\frac{\alpha_{li}^{\mu_{li}}}{\Gamma(\mu_{li})}} u^{\mu_{li}-1} e^{-\alpha_{li} u_{li}} \right] \left[\boxed{\frac{\beta_{li}^{v_{li}}}{\Gamma(v_{li})}} v_{li}^{v-1} e^{-\beta_{li} v_{li}} \right]$$

$$f(u; \mu, \alpha) = \frac{\alpha^\mu}{\Gamma(\mu)} u^{\mu-1} e^{-\alpha u}$$

$$f(v; \nu, \beta) = \frac{\beta^\nu}{\Gamma(\nu)} v^{\nu-1} e^{-\beta v}$$

$$\mathcal{L}(X, Z) = \ln f(X, Z, \Pi, U, V)$$

$$= \sum^N \sum^I \left[\ln(\pi_i) + \sum^L \ln \frac{\Gamma(u_{li} + v_{li})}{\Gamma(u_{li})\Gamma(v_{li})} + \sum^L (u_{li} - 1) \ln(x_{ln}) + \sum^L (v_{li} - 1) \ln(1 - x_{ln}) \right] z_{ni}$$

$$+ \sum_{l=1}^{I} (c_i - 1) \ln \pi_i + \sum^L \sum^I \left[(\mu_{li} - 1)(\ln u_{li}) + u_{li} \alpha_{li} \right]$$

$$+ \sum^L \sum^I \left[(v_{li} - 1) \ln(v_{li}) + \beta_{li} v_{li} \right]$$

$$f(u, v \mid x) \simeq f(u \mid x) f(v \mid x)$$

$$f(Z \mid \Pi) = \prod^N \prod^I \pi_i^{z_{ni}}$$

$$f(\Pi) = \mathrm{Dir}(\Pi \mid C)$$

$$\ln f^*(u_{li}; \mu_{li}, \alpha_{li}) = E[\mathcal{L}(X, Z)] = \sum^N E\left[z_{ni} \left(\ln \frac{\Gamma(u_{li} + v_{li})}{\Gamma(u_{li})\Gamma(v_{li})} \right) \right] + \sum^N E\left[z_{ni} (u_{li} - 1) \ln x_{ln} \right]$$

$$+ \sum^N E\left[(\mu_{li} - 1) \ln u_{li} - u_{li} \alpha_{li} \right]$$

$$\ln f^*(v_{li}; v_{li}, \beta_{li}) = \sum^N E\left[z_{ni} \left(\ln \frac{\Gamma(u_{li} + v_{li})}{\Gamma(u_{li})\Gamma(v_{li})} \right) \right] + \sum^N E\left[z_{ni} (v_{li} - 1) \ln(1 - x_{ln}) \right]$$

$$+ \sum^N E\left[(v_{li} - 1) \ln v_{li} - \beta_{li} v_{li} \right]$$

$$F(x) \approx F(x_0) + \frac{\partial F(x)}{\partial \ln x}\Big|_{x=x_0} (\ln x - \ln x_0)$$

$$f(x) \approx F(a) + \frac{F'(a)}{1!}(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \dots$$

$$f(x,y) \approx f(a,b) + \frac{\partial f(x,y)}{\partial x}(x-a)\Big|_{\substack{x=a\\y=b}} + \frac{\partial f(x,y)}{\partial y}(y-b)\Big|_{\substack{x=a\\y=b}}$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 f(x,y)}{\partial^2 x}(x-a)^2 + 2\frac{\partial^2 f(x,y)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(x,y)}{\partial^2 y}(y-b)^2\right)$$

$$\ln\left(\frac{1}{Beta(x+y)}\right) \approx \ln\frac{1}{Beta(x_0+y)} + \frac{\partial F(x)}{\partial x}\frac{\partial x}{\partial \ln x}\Big|_{x=x_0}(\ln x - \ln x_0)$$

$$\approx \ln\frac{1}{Beta(x_0+y)} + \frac{\partial}{\partial x}\left(\ln\Gamma(x+y) - \ln\Gamma(x) - \ln\Gamma(y)\right)\frac{\partial x}{\partial \ln x}\Big|_{x=x_0}(\ln x - \ln x_0)$$

$$\approx \ln\frac{1}{Beta(x_0+y)} + \left[\psi(x_0+y) - \psi(x_0)\right]x_0(\ln x - \ln x_0)$$

$$E\left[\ln\frac{1}{Beta(x+y)}\right] \approx E\left[\ln\frac{1}{Beta(x_0+y)} + \left[\psi(x_0+y) - \psi(x_0)\right]x_0(\ln x - \ln x_0)\right]$$

$$\ln\left(\frac{1}{Beta(x+y)}\right) \geqslant \ln\left(\frac{1}{Beta(x_0+y_0)}\right) + x_0\left[\psi(x_0+y_0) - \psi(x_0)\right](\ln x - \ln x_0) + y_0\left[\psi(x_0+y_0) - \psi(y_0)\right](\ln y - \ln y_0)$$

$$+ \frac{1}{2!}x_0^2\left[\psi'(x_0+y_0) - \psi'(x_0)\right](\ln x - \ln x_0)^2 + \frac{1}{2!}y_0\left[\psi'(x_0+y_0) - \psi'(y_0)\right](\ln y - \ln y_0)^2$$

$$+ x_0 y_0 \cdot \psi'(x_0+y_0)(\ln x - \ln x_0)(\ln y - \ln y_0)$$

$$E\left[\ln\frac{1}{Beta(x+y)}\right] \geqslant \ln\frac{\Gamma(x_0+y_0)}{\Gamma(x_0)\Gamma(y_0)} + x_0\left[\psi(x_0+y_0) - \psi(x_0)\right](E[\ln x] - \ln x_0) + y_0\left[\psi(x_0+y_0) - \psi(y_0)\right] \cdot$$

$$(E[\ln y] - \ln y_0) + \frac{1}{2}x_0^2\left[\psi'(x_0+y_0) - \psi'(x_0)\right]E\left[(\ln x - \ln x_0)^2\right] + \frac{1}{2}y_0\left[\psi'(x_0+y_0) - \psi'(y_0)\right]$$

$$E\left[(\ln y - \ln y_0)^2\right] + x_0 y_0 \psi'(x_0+y_0)(E[\ln x] - \ln x_0)(E[\ln y] - \ln y_0)$$

$$\psi(x+y) \geqslant \psi(x_0+y) + \psi'(x_0+y).x_0(\ln(x) - \ln x_0)$$

$$f(x) = F(a) + \frac{F'(a)}{1!}(x-a) + \frac{F'(a)}{2!}(x-a)^2$$

$$E[\psi(x+y)] \geqslant \psi(x_0+y) + \psi'(x_0+y)\,x_0(E(\ln(x)) - \ln x_0)$$

$$E[(\ln x - \ln x_0)^2] = Var(\ln x - \ln x_0) + [E[\ln x - \ln x_0]]^2$$

$$= f(u;\mu,\alpha) = \frac{\alpha^\mu}{\Gamma(\mu)} u^{\mu-1} e^{-\alpha u}$$

$$E[(\ln u - \ln \bar{u})^2] = Var(\ln u - \ln \bar{u}) + [E[\ln u - \ln \bar{u}]]^2$$

$$\ln \bar{u} = \ln \frac{\mu}{\alpha}$$

$$E[(\ln u - \ln \bar{u})^2] = \psi'(\mu) + [\psi(\mu) - \ln(\alpha) - \ln(\mu) + \ln(\alpha)]^2$$

$$= \psi'(\mu) + [\psi(\mu) - \ln(\mu)]^2$$