SECOND-ORDER STATE-CONSTRAINT HAMILTON-JACOBI EQUATIONS

YUXI HAN AND SON N. T. TU

ABSTRACT. We investigate qualitatively the convergence of large, or state-constraint solution to nonlinear elliptic equation as the viscosity vanish.

CONTENTS

sec:intro

1. INTRODUCTION

1.1. **Motivation.** Let Ω be an open, bounded and connected with C^2 boundary domain of $\mathbb{R}^n$. Let us consider the following Hamiltonian $H(x,\rho) = |\rho|^p - f(x)$ for $(x,p) \in \overline{\Omega}\times\mathbb{R}^n$, $f \in C(\overline{\Omega})\cap W^{1,\infty}(\Omega)$. Let $u^\varepsilon \in C^2(\Omega)$ (see [3]) be the solution to

$$\begin{cases} \lambda u^\varepsilon(x) + H(x, Du^\varepsilon(x)) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \displaystyle\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$ eq:PDEeps

When $1 < p \leqslant 2$, equation (PDE$_\varepsilon$) describes the (expectation) value function associated with a minimization of a stochastic optimal control problem with state-constraint. We are interested in studying the asymptotic behavior of $\{u^\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$. Heuristically, the state-constraint second-order problem converges to the state-constraint first-order, which is associated with deterministic optimal control, which is described in the framework of viscosity solution as follows:

$$\begin{cases} \lambda u(x) + H(x, Du(x)) \leqslant 0 & \text{in } \Omega, \\ \lambda u(x) + H(x, Du(x)) \geqslant 0 & \text{on } \overline{\Omega}. \end{cases} \tag{PDE$_0$}$$ eq:PDE0

Date: February 3, 2021.
2010 *Mathematics Subject Classification.* 35B40, 35D40, 49J20, 49L25, 70H20 .
Key words and phrases. first-order Hamilton–Jacobi equations; state-constraint problems; optimal control theory; rate of convergence; viscosity solutions.
The authors are supported in part by NSF grant DMS-1664424.

Equation (PDE$_0$) admits a unique viscosity solution in the space $C(\overline{\Omega})$, which is also the maximal viscosity subsolution among all viscosity subsolution $v \in C(\overline{\Omega})$. The problem is interesting since in the limit we no longer have blowing up behavior. In this paper, we are interested in the rate of convergence of $u^\varepsilon \to u$ as $\varepsilon \to 0$. A boundary layer is expected to describe the behavior of convergence near the boundary.

1.2. **Assumptions.** We will always assume Ω is an open, bounded and connected with C^2 boundary in $\mathbb{R}^n$ and $H : \mathbb{R}^n \times \mathbb{R}^n$ is a continuous Hamiltonian.

(A1) $H(x, \rho) = |\rho|^p - f(x)$ where $1 < p \leqslant 2$ and $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

We list the assumptions on a general Hamiltonian as follow.

(H1) There exists $C_1 > 0$ such that $H(x, p) \geqslant -C_1$ for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$.
(H2) There exists $C_2 > 0$ such that $|H(x, 0)| \leqslant C_2$ for all $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$.
(H3) For each $R > 0$ there exists a modulus $\omega_R[0, \infty) \to [0, \infty)$ such that $\omega_R(0^+) = 0$ and

$$\begin{cases} |H(x,p) - H(y,p)| \leqslant \omega_R(|x-y|), \\ |H(x,p) - H(x,q)| \leqslant \omega_R(|p-q|), \end{cases} \qquad \text{for all } x, y \in \overline{\Omega}, p, q \in \mathbb{R}^n \text{ with } |p|, |q| \leqslant R.$$

(H4) $H(x, p) \to \infty$ as $|p| \to \infty$ uniformly in $x \in \overline{\Omega}$.

2. Preliminaries

sec:prelim

2.1. **Setting and simplifications.** Let Ω be an open, bounded and connected subset of $\mathbb{R}^n$ with boundary ∂ is of class C^2. For $\delta > 0$, let us define $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \Omega) > \delta\}$ and $\Omega^\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \overline{\Omega}) < \delta\}$. We consider δ small enough so that the distance function $x \mapsto \mathrm{dist}(x, \partial\Omega)$ is

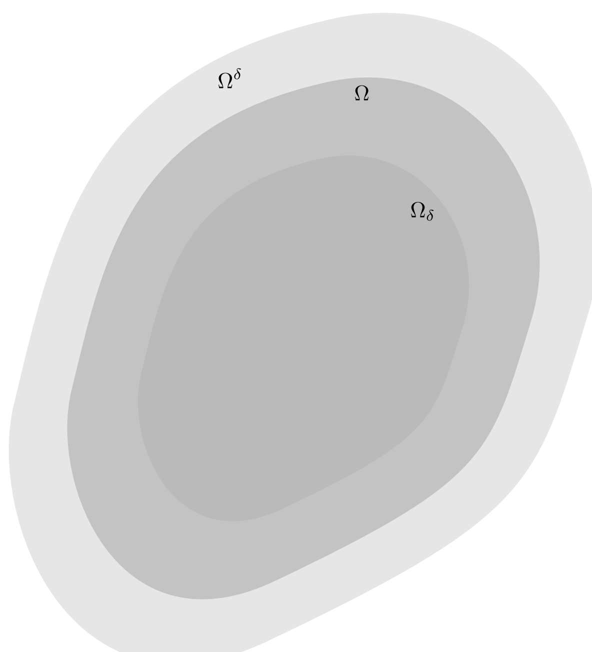

fig:Domains

Figure 2.1. The domain Ω variations $\Omega_\delta, \Omega^\delta$.

C^2 in the strip $\Omega^\delta \backslash \overline{\Omega}_\delta$, we recall that $|Dd(x)| = 1$ in that region. We extend the signed distance function into a $C^2(\mathbb{R}^n)$ function, denoted by $d(x)$ such that

$$\begin{cases} d(x) \geqslant 0 \text{ for } x \in \Omega \text{ with } d(x) = +\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega \backslash \Omega_\delta, \\ d(x) \leqslant 0 \text{ for } x \notin \Omega \text{ with } d(x) = -\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega^\delta \backslash \Omega. \end{cases}$$

We can choose $d(\cdot)$ so that $\Omega_\delta = \{x \in \mathbb{R}^n : d(x) - \delta > 0\}$ and $\Omega^\delta = \{x \in \mathbb{R}^n : d(x) + \delta > 0\}$.

Let $\delta_0 > 0$ be a fixed number such that $x \mapsto \mathrm{dist}(x, \partial\Omega)$ is of class C^2 on $\Omega \backslash \Omega_{\delta_0}$, i.e., in the region where $0 < \mathrm{dist}(x, \partial\Omega) \leqslant \delta_0$. We then extend $\mathrm{dist}(x, \partial\Omega)$ to a function $d(x) \in C^2(\mathbb{R}^n)$ such that

$$\begin{cases} d(x) \geqslant 0 \text{ for } x \in \Omega \text{ with } d(x) = +\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega \backslash \Omega_{\delta_0}, \\ d(x) \leqslant 0 \text{ for } x \notin \Omega \text{ with } d(x) = -\mathrm{dist}(x, \partial\Omega) \text{ for } x \in \Omega^{\delta_0} \backslash \Omega. \end{cases}$$

Let us denote

$$K := \max_{x \in \overline{\Omega}} |\Delta d(x)|.$$

Let us denote by $\mathcal{L}^\varepsilon : C^2(\Omega) \to C^2(\Omega)$ the operator

$$\mathcal{L}^\varepsilon[u](x) := \lambda u(x) + H(x, Du(x)) - \varepsilon \Delta u(x), \qquad x \in \Omega.$$

2.2. **Gradient estimate.**

2.3. **Well-posedness of large solution for subquaratic case.** In this section, with the specific form of the Hamiltonian $H(x,p) = |p|^p - f(x)$ where $1 < p \leqslant 2$ and $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, we show the existence and uniqueness of solutions to (PDE_ε). We note that the assumption of f can be relaxed to $f \in L^\infty(\Omega)$ only, but for the clarity of the proof we will assume $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

Theorem 2.1. *If $H(x,p) = |p|^p - f(x)$ where $1 < p < 2$ and $f \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ then there exists a unique solution $u^\varepsilon \in C^2(\Omega)$ of (PDE_ε) such that*

$$\lim_{d(x) \to 0} u^\varepsilon(x)\, d(x)^\alpha = C_\varepsilon$$

where

$$\alpha = \frac{2-p}{p-1} \qquad \textit{and} \qquad C_\varepsilon = \left(\frac{1}{\alpha} (\alpha+1)^{\frac{1}{p-1}} \right) \varepsilon^{\frac{1}{p-1}}.$$

Proof. To find a candidate for subsolution and supersolution to (PDE_ε), we use the *ansatz*

$$u(x) = C_\varepsilon d(x)^{-\alpha}, \qquad x \in \Omega \tag{2.1}$$ eq:ansatz

as we expect it blows up near the boundary like some thing proportionate to inverse of the distance function, due to the structure of $H(x,p) = |p|^p - f(x)$. Let us plug (2.1) into (PDE_ε) we obtain that

$$\begin{aligned} &\lambda u(x) \approx C_\varepsilon d(x)^{-\alpha}, \\ &\frac{\partial u}{\partial x_i}(x) \approx -\alpha C_\varepsilon d(x)^{-(\alpha+1)} \frac{\partial d}{\partial x_i}(x) \qquad \Longrightarrow \qquad |Du(x)|^p \approx \alpha^p C_\varepsilon^p d(x)^{-p(\alpha+1)} |\nabla d(x)|^p, \\ &\varepsilon \Delta u(x) \approx \frac{\varepsilon C_\varepsilon \alpha(\alpha+1)}{d(x)^{\alpha+2}} |\nabla d(x)|^2 - \frac{\varepsilon C_\varepsilon \alpha}{d(x)^{\alpha+1}} \Delta d(x). \end{aligned}$$

As $|\nabla d(x)| = 1$ for x near $\partial\Omega$, we see that as $x \to \partial\Omega$, the highest explosive order terms are

$$-\varepsilon C_\varepsilon \alpha(\alpha+1) d^{-(\alpha+2)} + C_\varepsilon^p \alpha^p d^{-(\alpha+1)p}.$$

Setting them to zero, we deduce that

$$\alpha = \frac{2-p}{p-1} \qquad \text{and} \qquad C_\varepsilon = \left(\frac{1}{\alpha} (\alpha+1)^{\frac{1}{p-1}} \right) \varepsilon^{\frac{1}{p-1}}.$$

We can obtain the following families of supersolution on Ω_δ and subsolution on Ω^δ as follow. Let us denote

$$G = \left[(K\delta_0 + (\alpha+1))^{\frac{1}{p-1}} - (\alpha+1)^{\frac{1}{p-1}} \right] \alpha.$$

lem:subsln **Lemma 2.2.** *For each $< \delta < \frac{1}{2}\delta_0$, if $\eta \geqslant G\varepsilon^{\frac{1}{p-1}}$, we have*

$$\overline{w}_{\eta,\delta}(x) = \frac{C_\varepsilon + \eta}{(d(x)-\delta)^\alpha} + \frac{M}{\lambda}, \qquad x \in \Omega_\delta$$

is a supersolution of (PDE$_\varepsilon$) *in Ω_δ where*

$$M = \max_{\overline{\Omega}} f(x) + 2^{\alpha+1} K\alpha\varepsilon\delta_0^{-(\alpha+1)}.$$

Proof. We compute

$$\begin{aligned}\mathcal{L}^\varepsilon\left[\overline{w}_{\eta,\delta}\right](x) = \frac{\lambda(C_\varepsilon+\eta)}{(d(x)-\delta)^\alpha} + M_\eta + \frac{(C_\varepsilon+\eta)^p\alpha^p}{(d(x)-\delta)^{p(\alpha+1)}}|\nabla d(x)|^p - f(x)\\ - \frac{\varepsilon(C_\varepsilon+\eta)\alpha(\alpha+1)}{(d(x)-\delta)^{\alpha+2}}|\nabla d(x)|^2 + \frac{\varepsilon(C_\varepsilon+\eta)\alpha}{(d(x)-\delta)^{\alpha+1}}\Delta d(x).\end{aligned}$$

to be filled in... □

em:supersln **Lemma 2.3.** *For each $\eta > 0$ there exists $M_\eta > 0$ such that*

$$\overline{w}_{\eta,\delta}(x)(\underline{w}_{\eta,\delta}) = \frac{C_\varepsilon - \eta}{(d(x)+\delta)^\alpha} - \frac{M_\eta}{\lambda}, \qquad x \in \Omega^\delta$$

is a supersolution (subsolution) of (PDE$_\varepsilon$) *in Ω^δ.*

We divide the rest of the proof into 3 steps. We first construct a minimal solution, then a maximal solution to (PDE$_\varepsilon$), and finally show that they are equal to conclude the existence and uniqueness of solution to (PDE$_\varepsilon$).

:minimalsol **Proposition 2.4.** *There exists a minimal solution $\underline{u} \in C^2(\Omega)$ of* (PDE$_\varepsilon$) *such that $v \geqslant \underline{u}$ for any other solution $v \in C^2(\Omega)$ solving* (PDE$_\varepsilon$).

Proof. Let $w_{\eta,\delta} \in C^2(\Omega)$ solves

$$\begin{cases} \mathcal{L}\left[w_{\eta,\delta}\right] = 0 & \text{in } \Omega, \\ \quad w_{\eta,\delta} = \underline{w}_{\eta,\delta} & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$ e:w_def

Fix $\eta > 0$, as $\delta \to 0$ the value of $\underline{w}_{\eta,\delta}$ blows up on the boundary, therefore by comparison principle we have $\delta_1 \leqslant \delta_2$ implies $w_{\eta,\delta_1} \geqslant w_{\eta,\delta_2}$ on $\overline{\Omega}$.
Since $\underline{w}_{\eta,\delta'}$ is a subsolution in $\overline{\Omega}$ with finite boundary, we obtain that

$$0 < \delta \leqslant \delta' \qquad \Longrightarrow \qquad \underline{w}_{\eta,\delta'} \leqslant w_{\eta,\delta'} \leqslant w_{\eta,\delta} \qquad \text{on } \overline{\Omega}. \tag{2.3}$$ e:cp_delta1

For $\delta' > 0$, since $\overline{w}_{\eta,\delta}$ is a supersolution on $\Omega_{\delta'}$ with infinity value on the boundary $\partial\Omega_{\delta'}$, by comparison principle

$$w_{\eta,\delta} \leqslant \overline{w}_{\eta,\delta'} \quad \text{in } \Omega_{\delta'} \qquad \Longrightarrow \qquad w_{\eta,\delta} \leqslant \overline{w}_{\eta,0} \quad \text{in } \Omega. \tag{2.4}$$ e:cp_delta2

From (2.3) and (2.4) we have

$$0 < \delta \leqslant \delta' \qquad \Longrightarrow \qquad \underline{w}_{\eta,\delta'} \leqslant w_{\eta,\delta'} \leqslant w_{\eta,\delta} \leqslant \overline{w}_{\eta,0} \qquad \text{in } \Omega. \tag{2.5}$$ e:cp_delta3

Thus $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $L^\infty_{loc}(\Omega)$. Using the gradient estimate for $w_{\eta,\delta}$ solving (2.2) we deduce that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{1,\infty}_{loc}(\Omega)$. Plug it back into the defining equation

(2.2) to bound the Laplacian and the second derivative terms, we deduce that $\{w_{\eta,\delta}\}_{\delta>0}$ is locally bounded in $W^{2,r}_{\mathrm{loc}}(\Omega)$ for all $r<\infty$.
Local boundedness of $\{u_{\eta,\delta}\}_{\delta>0}$ in $W^{2,r}_{\mathrm{loc}}(\Omega)$ implies weak* compactnes, that is there exists a function $u\in W^{2,r}(\Omega)$ such that (via subsequence and monotonicity)

$$\begin{cases} w_{\eta,\delta}\rightharpoonup u & \text{weakly in } W^{2,r}_{\mathrm{loc}}(\Omega),\\ w_{\eta,\delta}\to u & \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega).\end{cases}$$

In particular, $w_{\eta,\delta}\to u$ in $C^1_{\mathrm{loc}}(\Omega)$ thanks to compact embedding. Let us rewrite the equation $\mathcal{L}\left[u_{\eta,\delta}\right]=0$ as $\varepsilon\Delta w_{\eta,\delta}(x)=F[w_{\eta,\delta}](x)$ in U for $U\subset\subset\Omega$ where

$$F[w_{\eta,\delta}](x)=\lambda w_{\eta,\delta}(x)+H(x,Dw_{\eta,\delta}(x)).$$

As $u_{\eta,\delta}\to u$ in $C^1(U)$ as $\delta\to 0$, we have $F[w_{\eta,\delta}](x)\to F(x)$ uniformly in U where

$$F(x)=\lambda u(x)+H(x,Du(x)).$$

In the limit we see that $u\in L^2$ is a weak solution of $\varepsilon\Delta u=F$ in U where F is continuous, thus $u\in C^2(\Omega)$ as well and by stability we see that u solves $\mathcal{L}[u]=0$ in Ω. From (2.5) we also have

$$\underline{w}_{\eta,0}\leqslant u\leqslant \overline{w}_{\eta,0}\qquad \text{in } \Omega.$$

It is clear that $u(x)\to\infty$ as $\operatorname{dist}(x,\partial\Omega)\to 0$ with the precise rate like (2.1). We note that at this point, by construction, u may depend on η. We show that u is the maximal solution of $\mathcal{L}[u]=0$ in Ω such that $u=+\infty$ on $\partial\Omega$, and thus consequently showing that u is independent of η.
Let $v\in W^{2,r}(\Omega)$ for all $r<\infty$ solving (PDE_ε). Fix $\delta>0$, then for all $\vartheta>0$ small we see that on $\partial\overline{\Omega}_\vartheta$ the value of v blows up as $\vartheta\to 0$, while $w_{\eta,\delta}$ remains bounded (we fixed $\delta>0$). Comparison principle gives us that $v\geqslant w_{\eta,\delta}$ on Ω_ϑ, thus as $\vartheta\to 0$ we obtain $v\geqslant w_{\eta,\delta}$ on Ω. Let $\delta\to 0$ we deduce that $v\geqslant u$ on Ω. This concludes that u is the minimal solution in $W^{2,r}(\Omega)(\forall\, r<\infty)$ and thus independent of $\eta>0$. □

Proposition 2.5. *There exists a maximal solution $\overline{u}\in C^2(\Omega)$ of (PDE_ε) such that $v\leqslant\overline{u}$ for any other solution $v\in C^2(\Omega)$ solving (PDE_ε).*

Proof. For each $\delta>0$, let us denote $u_\delta\in C^2(\Omega_\delta)$ be the minimal solution to $\mathcal{L}[u_\delta]=0$ in Ω and $u_\delta=+\infty$ on $\partial\Omega_\delta$ according to Proposition 2.4. By comparison principle, for every $\eta>0$ there holds

$$\underline{w}_{\eta,\delta}\leqslant u_\delta\leqslant\overline{w}_{\eta,\delta}\qquad\text{in } \Omega_\delta,$$

and

$$0<\delta<\delta'\qquad\Longrightarrow\qquad u_\delta\leqslant u_\delta'\qquad\text{in } \Omega_{\delta'}.$$

We have the local boundedness of $\{u_\delta\}_{\delta>0}$ in $W^{2,r}(\Omega)$ as well, thus together with monotonicity there exists $u\in W^{2,r}(\Omega)$ for all $r<\infty$ such that $u_\delta\to u$ strongly in $C^1_{\mathrm{loc}}(\Omega)$, therefore using the equation $\mathcal{L}[u_\delta]=0$ in Ω_δ and the regularity of Laplace equation we deduce that $u\in C^2(\Omega)$ solving (PDE_ε) and

$$\underline{w}_{\eta,0}\leqslant u\leqslant\overline{w}_{\eta,0}\qquad\text{in } \Omega$$

for all $\eta>0$. From the construction, as u_δ is independent of η, it is clear that u is also independent of η. We now show that u is the maximal solution of (PDE_ε). Let $v\in C^2(\Omega)$ solves (PDE_ε), then clearly $v\leqslant u_\delta$ on Ω_δ and therefore as $\delta\to 0$ we have $v\leqslant u$. □

In conclusion we have found a minimal solution $\underline{u}$ and a maximal solution $\overline{u}$ in $C^2(\Omega)$ such that

$$\underline{w}_{\eta,0}\leqslant\underline{u}\leqslant\overline{u}\leqslant\overline{w}_{\eta,0}\qquad\text{in } \Omega \tag{2.6}$$ e:chain

for any $\eta>0$. This extra parameter η now enables us to show that $\overline{u}=\underline{u}$ in Ω.

Proposition 2.6 (Uniqueness)**.** *We have* $\overline{u} \equiv \underline{u}$ *in* Ω *and therefore solution to* (PDE$_\varepsilon$) *in* $C^2(\Omega)$ *is unique.*

Proof. Let $\theta \in (0,1)$, we cook up a subsolution by convex combination $w_\theta = \theta\overline{u} + (1-\theta)\lambda^{-1}(\inf_\Omega f)$, which if we can use comparison principle then

$$w_\theta = \theta\overline{u} + (1-\theta)\lambda^{-1}\left(\inf_\Omega f\right) \leqslant \underline{u} \qquad \text{in } \Omega.$$

If that is the case, let $\theta \to 1$ we conclude that $\overline{u} \leqslant \underline{u}$. There is a problem here, as they are both explosive solutions, to use comparison principle we need to show that $w_\theta \leqslant \underline{u}$ in a neighborhood of $\partial\Omega$. From (2.6) we see that

$$1 \leqslant \frac{\overline{u}(x)}{\underline{u}(x)} \leqslant \frac{\overline{w}_{\eta,0}(x)}{\underline{w}_{\eta,0}(x)} = \frac{(C_0+\eta) + \lambda^{-1}C_\eta d(x)^\alpha}{(C_0-\eta) - \lambda^{-1}C_\eta d(x)^\alpha} \qquad \text{for } x \in \Omega$$

and therefore

$$1 \leqslant \lim_{d(x)\to 0}\left(\frac{\overline{u}(x)}{\underline{u}(x)}\right) \leqslant \frac{C_0+\eta}{C_0-\eta} \qquad \Longrightarrow \qquad \lim_{d(x)\to 0}\left(\frac{\overline{u}(x)}{\underline{u}(x)}\right) = 1$$

since $\eta > 0$ is chosen arbitrary. This means for any $\sigma \in (0,1)$ we can find $\delta(\sigma) > 0$ small such that on $\Omega\backslash\Omega_\delta$ one has

$$\frac{\overline{u}(x)}{\underline{u}(x)} \leqslant (1+\sigma) \qquad \Longrightarrow \qquad \frac{1}{1+\sigma}\overline{u}(x) \leqslant \underline{u}(x) \qquad \text{in } \Omega\backslash\Omega_\delta$$

For given $\theta \in (0,1)$, we can always choose σ small enough such that $(1+\sigma)^{-1} > \theta$ which renders $\underline{u}(x) \geqslant \theta\overline{u}(x) + (1-\theta)\lambda^{-1}(\inf_\Omega f)$ for $0 < d(x) < \delta' < \delta$. ☐

A final remark is that, we can also repeat the proof for $p = 2$. ☐

2.4. **Convergence result.** We show the convergence (qualitatively) of (PDE$_\varepsilon$) to (PDE$_0$). We first state the following Lemma ([1]), which characterizes the state-constraint of first-order equation

$$\lambda u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega. \tag{S_0}$$ S_0

lem:max **Lemma 2.7.** *Let* $u \in C(\overline{\Omega})$ *be a viscosity subsolution of* (S_0) *such that, for all viscosity subsolution* $v \in C(\overline{\Omega})$ *of* (S_0) *one has* $v \leqslant u$ *in* $\overline{\Omega}$*, then* u *is a viscosity supersolution of* (S_0) *on* $\overline{\Omega}$.

Lemma 2.7 is a variation of Perron's method. We give a proof in Appendix for the sake of completeness. Let $u^\varepsilon \in C^2(\Omega)$ be a solution to (PDE$_\varepsilon$), we claim that $\{\lambda u^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded from below by a constant independent of ε and λ. We have

$$u^\varepsilon(x) = \lim_{m\to\infty} u^\varepsilon_m(x), \qquad x \in \Omega,$$

where $u^{\varepsilon,m} \in C^2(\Omega)\cap C(\overline{\Omega})$ solves the Drichlet problem

$$\begin{cases} \lambda u(x) + H(x, Du(x)) - \varepsilon\Delta u(x) = 0 & \text{in } \Omega, \\ u(x) = m & \text{on } \partial\Omega. \end{cases} \tag{PDE$_{\varepsilon,m}$}$$ e:uepsm

If we assume (H2) then we can take $C_2 = \|f\|_{L^\infty(\Omega)}$ and take $\varphi(x) \equiv -\lambda^{-1}C_2$ for $x \in \overline{\Omega}$ as a subsolution to (PDE$_{\varepsilon,m}$), hence by comparison principle for (PDE$_{\varepsilon,m}$) we have $u^\varepsilon_m(x) \geqslant -\lambda^{-1}C_2$ for all $x \in \overline{\Omega}$, thus as $m \to \infty$ we obtain $u^\varepsilon(x) \geqslant -\lambda^{-1}C_2$ for all $x \in \Omega$.

thm:qual **Theorem 2.8** (Qualitative result)**.** *Assume* (A1)*. Let* u^ε *be the solution to* (PDE$_\varepsilon$)*, then there exists* $u^0 \in C(\overline{\Omega})$ *such that* $u^\varepsilon \to u^0$ *locally uniformly in* Ω *as* $\varepsilon \to 0$ *and* u^0 *solves* (PDE$_0$)*. Furthermore*

$$u^0 \leqslant u^\varepsilon \qquad \text{in } \Omega.$$

Proof of Theorem 2.8. By the priori estimate

$$|u^\varepsilon(x)| + |Du^\varepsilon(x)| \leqslant C_\delta \qquad \text{for } x \in \overline{\Omega}_\delta$$

we deduce from the Arzelá–Ascoli theorem that there exists a subsequence $\varepsilon_j \to 0$ and a function $u^0 \in C(\Omega)$ such that $u^{\varepsilon_j} \to u^0$ locally uniformly in Ω. By the stability of viscosity solution we easily deduce that

$$\lambda u^0(x) + H(x, Du^0(x)) = 0 \qquad \text{in } \Omega. \tag{2.7}$$ eq:u0int

Since $u^\varepsilon(x) \geqslant -C_2$ for all $x \in \Omega$ we also have $u^0(x) \geqslant -C_2$ for all $x \in \Omega$, thus together with (2.7) we obtain $H(x, \xi) \leqslant C_2$ for all $\xi \in D^+u^0(x)$ and $x \in \Omega$. The coercivity (H4) implies that there exists $K > 0$ such that

$$|u^0(x) - u^0(y)| \leqslant K \qquad \text{for all } x, y \in \Omega.$$

Thus we can extend u^0 uniquely to $u^0 \in C(\overline{\Omega})$. We will use Lemma 2.7 to show that u^0 is a supersolution of (S_0) on $\overline{\Omega}$. It suffices to to show that $u^0 \geqslant w$ on $\overline{\Omega}$ where $w \in C(\overline{\Omega})$ is the unique solution to (PDE_0).

For $\delta > 0$, let $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \Omega) < \delta\}$ and $v_\delta \in C(\overline{\Omega}_\delta)$ be the state-constraint viscosity subsolution to the problem $\lambda u(x) + H(x, Du(x)) = 0$ in Ω_δ. As $v_\delta \to w$ locally uniformly as $\delta \to 0^+$ (see [2]) and w is bounded, therefore $\{v_\delta\}_{\delta>0}$ is uniformly bounded. Let $v_\delta^\varepsilon \in C^2(\Omega_\delta) \cap C(\overline{\Omega}_\delta)$ be the unique solution to the Dirichlet problem

$$\begin{cases} \lambda v_\delta^\varepsilon(x) + |Dv_\delta^\varepsilon(x)|^p - f(x) = \varepsilon \Delta v_\delta^\varepsilon(x) & \text{in } \Omega_\delta, \\ \qquad\qquad\qquad\qquad v_\delta^\varepsilon = v_\delta & \text{on } \partial\Omega_\delta. \end{cases} \tag{2.8}$$ eq:vv_eps

For all δ small enough $v_\delta \leqslant u^\varepsilon$ on $\partial\Omega_\delta$, thus by maximum principle $v_\delta^\varepsilon \leqslant u^\varepsilon$ on $\overline{\Omega}_\delta$. Let $\varepsilon \to 0$ we have $v_\delta \leqslant u^0$ on $\overline{\Omega}_\delta$. Let $\delta \to 0$ we obtain $w \leqslant u^0$ in Ω, which implies $w \leqslant u^0$ on $\overline{\Omega}$ since both w, u^0 belong to $C(\overline{\Omega})$. □

3. Rate of convergence via Doubling variables - a toy case

Let $\Omega = B(0,1)$. Let $\beta, \sigma, \delta \in (0,1)$ to be chosen later, let us define

$$\Phi^\sigma(x,y) = u^\varepsilon(x) - u_\delta(y) - \frac{L_0|x-y|^2}{\sigma} - \frac{C_\varepsilon}{d(x)^{\alpha+\beta}}, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}_\delta$$

where L_0 is the Lipschitz constant of u^0 and $u_\delta \in C(\overline{\Omega}_\delta)$ is the state-constraint solution to

$$\begin{cases} \lambda u_\delta(x) + H(x, Du_\delta(x)) \leqslant 0 & \text{in } \Omega_\delta \\ \lambda u_\delta(x) + H(x, Du_\delta(x)) \geqslant 0 & \text{on } \overline{\Omega}_\delta. \end{cases}$$

As we are working with the ball, note that $\Omega_\delta = (1-\delta)\Omega$. Let $\mathfrak{F}$ be the Lipschitz constant of f.

Step 1. Since $u^\varepsilon(x)d(x)^\alpha \to C_\varepsilon$ as $d(x) \to 0$, we see that $\Phi^\sigma(x,y) \to -\infty$ as $d(x) \to 0$, therefore for a fixed $\varepsilon > 0$, there is $(x_\sigma, y_\sigma) \in \Omega \times \overline{\Omega}_\delta$ such that

$$\max_{\overline{\Omega} \times \overline{\Omega}_\delta} \Phi^\sigma = \Phi^\sigma(x_\sigma, y_\sigma).$$

Step 2. $\Phi^\sigma(x_\sigma, y_\sigma) \geqslant \Phi^\sigma(x_\sigma, x_\sigma)$ gives us that

$$\mathfrak{C}\frac{|x_\sigma - y_\sigma|^2}{\sigma} \leqslant u_\delta(x_\sigma) - u_\delta(y_\sigma) \leqslant \mathfrak{C}|x_\sigma - y_\sigma| \qquad \Longrightarrow \qquad |x_\sigma - y_\sigma| \leqslant \sigma. \tag{3.1}$$ e:est_sigma

This gives us that

$$|x_\sigma| \leqslant |y_\sigma| + \sigma \qquad \Longrightarrow \qquad d(x_\sigma) \geqslant d(y_\sigma) - \sigma \geqslant \delta - \sigma. \tag{3.2}$$ e:est_sigma

Let

$$\boxed{\delta = 2\sigma.}$$

then

$$\boxed{d(x_\sigma) \geqslant \sigma.}$$

Step 3. $x \mapsto \Phi^\sigma(x, y_\sigma)$ has a maximum at $x_\sigma \in \Omega$, thus

$$\lambda u^\varepsilon(x_\sigma) + \left|\frac{2\mathfrak{C}(x_\sigma - y_\sigma)}{\sigma} - \frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\nabla d(x_\sigma)\right|^p - f(x_\sigma)$$
$$- \varepsilon\left(\frac{2n}{\sigma} + \frac{C_\varepsilon(\alpha+\beta)(\alpha+\beta+1)|\nabla d(x_\sigma)|^2}{d(x_\sigma)^{\alpha+\beta+2}} - \frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\Delta d(x_\sigma)\right) \leqslant 0.$$

Step 4. $y \mapsto \Phi^\sigma(x_\sigma, y)$ has a maximum at $y_\sigma \in \overline{\Omega}$, thus

$$\lambda u_\delta(y_\sigma) + \left|\frac{2\mathfrak{C}(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(y_\sigma) \geqslant 0$$

Step 5. Gradient estimate, let

$$\xi_\sigma = \frac{2\mathfrak{C}(x_\sigma - y_\sigma)}{\sigma} \qquad \text{and} \qquad \zeta_\sigma = -\frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\nabla d(x_\sigma)$$

then $|\xi_\sigma| \leqslant 2\mathfrak{C}$ already and furthermore

$$\xi_\sigma + \zeta_\sigma \in D^- u^\varepsilon(x_\sigma),$$

which, by the local gradient estimate, as $x_\sigma \in \Omega_\sigma$, this is bounded in terms of σ and will blow up as $\sigma \to 0$. Nevertheless, we can do better as following.

$$|\zeta_\sigma| = \left|\frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\nabla d(x_\sigma)\right| \leqslant \frac{C_\varepsilon(\alpha+\beta)}{\sigma^{\alpha+\beta+1}} \leqslant \frac{\alpha+\beta}{\alpha}(\alpha+1)^{\frac{1}{p-1}}\frac{\varepsilon^{\frac{1}{p-1}}}{\sigma^{\alpha+\beta+1}}.$$

Thus by choosing $\sigma = \varepsilon^\gamma$ appropriately, we can make this tends to zero as $\varepsilon \to 0$. Indeed, we have

$$|\zeta_\sigma| \leqslant \frac{(\alpha+1)^{\frac{1}{p-1}}}{\alpha}\varepsilon^{\frac{1}{p-1}-\gamma(\alpha+\beta+1)} = \frac{\alpha+\beta}{\alpha}(\alpha+1)^{\alpha+1}\varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)},$$

where we made use of the fact that $\alpha+1 = \frac{1}{p-1}$. We see that the first condition we want to impose is

$$\boxed{0 < \gamma < \frac{\alpha+1}{\alpha+1+\beta}.} \tag{3.3}$$ e:condition

As long as (3.3) holds then

$$|\zeta_\sigma| \leqslant \mathfrak{C}_\alpha \qquad \text{where} \qquad \mathfrak{C}_\alpha = \frac{\alpha+\beta}{\alpha}(\alpha+1)^{\alpha+1}. \tag{3.4}$$ e:bddp_sigm

Step 6. Recall $|\nabla d(x_\sigma)| = 1$, we have

$$\lambda u^\varepsilon(x_\sigma) - \lambda u_\delta(y_\sigma) \leqslant \underbrace{\left|\frac{2\mathfrak{C}(x_\sigma - y_\sigma)}{\sigma}\right|^p}_{|\xi_\sigma|^p} - \underbrace{\left|\frac{2\mathfrak{C}(x_\sigma - y_\sigma)}{\sigma} - \frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\nabla d(x_\sigma)\right|^p}_{|\xi_\sigma+\zeta_\sigma|^p}$$
$$+ \underbrace{f(y_\sigma) - f(x_\sigma)}_{\mathfrak{F}|x_\sigma - y_\sigma|} + \varepsilon\left(\frac{2n}{\sigma} + \underbrace{\frac{C_\varepsilon(\alpha+\beta)(\alpha+\beta+1)}{d(x_\sigma)^{\alpha+\beta+2}}}_{\frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\cdot\frac{\alpha+\beta+1}{d(x_\sigma)}} - \underbrace{\frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}\Delta d(x_\sigma)}_{\frac{C_\varepsilon(\alpha+\beta)}{d(x_\sigma)^{\alpha+\beta+1}}.\Delta d(x_\sigma)}\right)$$
$$\leqslant |\xi_\sigma|^p - |\xi_\sigma + \zeta_\sigma|^p + \mathfrak{F}\sigma + \frac{2n\varepsilon}{\sigma} + (\alpha+\beta+1)|\zeta_\sigma|\frac{\varepsilon}{\sigma} + K|\zeta_\sigma|\varepsilon \tag{3.5}$$ e:est3

where we recall that $K = \max_{x\in\overline{\Omega}} \Delta d(x)$. Let's recall a simple estimate, let $f(t) = (x+ty)^p$ then

$$f(1) - f(0) = \int_0^1 f'(s)\, ds = p \int_0^1 (x+sy)^{p-1} y\, ds,$$

therefore

$$\begin{aligned} ||x+y|^p - |x|^p| &\leqslant p \left(\int_0^1 |x+sy|^{p-1} ds \right) |y| \\ &\leqslant p \left(\int_0^1 \Big(|x| + s|y|\Big)^{p-1} ds \right) |y| \leqslant p \Big(|x|+|y|\Big)^{p-1} |y|. \end{aligned}$$

Using this for $x = \xi_\sigma$ and $y = \zeta_\sigma$ we deduce that

$$\begin{aligned} \Big||\xi_\sigma + \zeta_\sigma|^p - |\xi|^p\Big| &\leqslant p\Big(|\xi_\sigma| + |\zeta_\sigma|\Big)^{p-1} |\zeta_\sigma| \\ &\leqslant p \left(2\mathfrak{C} + \frac{(\alpha+1)^{\alpha+1}}{\alpha} \varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} \right)^{p-1} \frac{(\alpha+1)^{\alpha+1}}{\alpha} \varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} \\ &\leqslant \underbrace{\Big(p\big(2\mathfrak{C} + \mathfrak{C}_\alpha\big)^{p-1} \mathfrak{C}_\alpha \Big)}_{\mathfrak{B}} \varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} \end{aligned} \tag{3.6}$$

e:est2

where we make use of (3.4).

Step 7. Combining stuffs. From (3.6) and (3.5) and $\sigma = \varepsilon^\gamma$ we deduce that

$$\begin{aligned} \lambda u^\varepsilon(x_\sigma) - \lambda u_\delta(y_\sigma) &\leqslant \mathfrak{B}\varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} + \mathfrak{F}\varepsilon^\gamma + 2n\varepsilon^{1-\gamma} \\ &\quad + (\alpha+\beta+1)\mathfrak{C}_\alpha \varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} \varepsilon^{1-\gamma} + K\mathfrak{C}_\alpha \varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)}\varepsilon. \end{aligned}$$

Let us denote

$$\mathfrak{X} := \max\Big\{\mathfrak{B}, \mathfrak{F}, 2n, (\alpha+\beta+1)\mathfrak{C}_\alpha, K\mathfrak{C}_\alpha\Big\}$$

then

$$\lambda u^\varepsilon(x_\sigma) - \lambda u_\delta(y_\sigma) \leqslant \mathfrak{X}\Big(\varepsilon^{\alpha+1-\gamma(\alpha+1+\beta)} + \varepsilon^\gamma + \varepsilon^{1-\gamma} + \varepsilon^{\alpha+2-\gamma(\alpha+2+\beta)} + \varepsilon^{\alpha+2-\gamma(\alpha+1+\beta)}\Big).$$

Take $\gamma = \frac{1}{2}$ to balance $\varepsilon^\gamma \approx \varepsilon^{1-\gamma}$ (which normally gives the rate $\sqrt{\varepsilon}$ for zero Dirichlet boundary problem), we have

$$\begin{cases} \alpha + 1 - \frac{1}{2}(\alpha+1+\beta) > \frac{1}{2} \\ \alpha + 2 - \frac{1}{2}(\alpha+2+\beta) > \frac{1}{2} \\ \alpha + 2 - \frac{1}{2}(\alpha+1+\beta) > \frac{1}{2} \end{cases} \iff \begin{cases} \alpha+1-\beta > 1 \\ \alpha+2-\beta > 1 \\ \alpha+3-\beta > 1 \end{cases}$$

which is always correct as long as $0 < \beta < \alpha$. We note that as $1 < p < 2$, we have $\alpha \in (0, \infty)$ and therefore such a choice of β is always possible. We conclude that

$$\lambda u^\varepsilon(x_\sigma) - \lambda u_\delta(y_\sigma) \leqslant \mathfrak{X}\varepsilon^{\frac{1}{2}} \tag{3.7}$$

e:crucial

where $\sigma = \varepsilon^{\frac{1}{2}}$.

Step 8. Deriving the conclusion. Using $\Phi^\sigma(x_\sigma, y_\sigma) \geqslant \Phi^\sigma(x,x)$ for all $x \in \overline{\Omega}_\delta = \overline{\Omega}_{2\sigma}$, we have

$$\lambda u^\varepsilon(x) - \lambda u_\delta(x) - \frac{C_\varepsilon}{d(x)^{\alpha+\beta}} \leqslant \Phi^\sigma(x_\sigma, y_\sigma) \leqslant \lambda u^\varepsilon(x_\sigma) - \lambda u_\delta(y_\sigma) \leqslant \mathfrak{X}\varepsilon^{\frac{1}{2}}.$$

Using $d(x) \geqslant 2\sigma = 2\varepsilon^{\frac{1}{2}}$ we deduce that

$$\lambda u^\varepsilon(x) - \lambda u_\delta(x) \leqslant \mathfrak{X}\varepsilon^{\frac{1}{2}} + \frac{(\alpha+1)^{\alpha+1}}{2^{\alpha+\beta}\alpha}\varepsilon^{\alpha+1-\frac{\alpha+\beta}{2}} \leqslant \tilde{\mathfrak{X}}\varepsilon^{\frac{1}{2}} \qquad \text{for all } x \in \overline{\Omega}_{2\sigma} \tag{3.8}$$ e:almost

where (we see that $\alpha + 1 - \frac{\alpha+\beta}{2} = 1 + \frac{\alpha-\beta}{2} > 1$)

$$\tilde{\mathfrak{X}} = \max\left\{\mathfrak{X}, \frac{(\alpha+1)^{\alpha+1}}{2^{\alpha+\beta}\alpha}\right\}.$$

Finally, in the case of the ball, $\Omega_\delta = (1-\delta)\Omega$, by [2, Theorem 1.5] we know that, optimally,

$$0 \leqslant \lambda u_\delta(x) - \lambda u^0(x) \leqslant C_H\delta \qquad \text{for all } x \in \overline{\Omega}_\delta. \tag{3.9}$$ e:citepaper

From (3.8) and (3.9) and $\delta = 2\sigma = 2\varepsilon^{\frac{1}{2}}$, and $u^\varepsilon \geqslant u^0$ from Theorem 2.8 we deduce that

$$\boxed{0 \leqslant \lambda u^\varepsilon(x) - \lambda u^0(x) \leqslant \left(\tilde{\mathfrak{X}} + 2C_H\right)\sqrt{\varepsilon} \qquad \text{for all } x \in \overline{\Omega}_{2\sqrt{\varepsilon}}.}$$

It appears that the boundary layer is the strip

$$\Gamma_\varepsilon = \{x \in \Omega : 0 < \operatorname{dist}(x, \partial\Omega) < 2\sqrt{\varepsilon}\}.$$

The next questions and tasks:

(1) Check all the constants, make sure things are correct!
(2) It seems that $\sqrt{\varepsilon}$ is the best using this method. Now we can improve this by either:
   (a) Make the boundary layer to $\sqrt{\varepsilon}$ only, instead of $2\sqrt{\varepsilon}$.
   (b) **Important.** Generalize this to more general domain, possibly star-shaped domains where we can do scaling is fine (just my guess).
(3) Other formulations of the doubling variable method here, for example, in the first step, it seems that making $\Phi^\sigma(x,y) \to -\infty$ as $x \to \partial\Omega$ is "*too wasteful*". One may need only to do something like

$$\Phi^\sigma(x,y) < \Phi^\sigma(0,0) \qquad \text{if} \qquad d(x) < \text{ something close to the boundary.}$$

   Can we improve that to get a better rate? My guess is not even if we can improve this, since the dominating term coming from ε^γ and $\varepsilon^{1-\gamma}$ later. Nevertheless, it maybe still interesting to see why and how the *wasteful* formulation here can be improved.
(4) **Important.** Using nonlinear adjoint method.
(5) Now once we have some rate, can we do bootstrap to improve it as we discussed earlier (the simple ideas, ...)?
(6) **Important.** Can we find an example where $|u^\varepsilon - u^0| = \mathcal{O}(1)$ in the strip where $0 < \operatorname{dist}(x, \partial\Omega) < 2\sqrt{\varepsilon}$? Or can we even make more layers inside with more scales, as $|u^\varepsilon - u^0| = \mathcal{O}(1)$ is not possible near the boundary $\partial\Omega$.

4. Rate of convergence via a simple idea

Let $\Omega = B(0,1)$, the scaling and distance near boundary are the same, i.e., $(1-\delta)\Omega = \Omega_\delta$.
Let $u_\delta \in C(\overline{\Omega}_\delta)$ be the state-constraint solution of the first-order equation, still $H(x,\rho) = |\rho|^p - f(x)$,

$$\begin{cases} \lambda u_\delta(x) + H(x, Du_\delta(x)) \leqslant 0 & \text{in } (1-\delta)\Omega, \\ \lambda u_\delta(x) + H(x, Du_\delta(x)) \geqslant 0 & \text{on } (1-\delta)\overline{\Omega}. \end{cases}$$

We know that the optimal rate of convergence of u_δ to u^0 is given by ([2])

$$0 \leqslant u_\delta - u^0 \leqslant C\delta$$

where C depends on H only (check this).
Let $g_{\varepsilon,\delta}(x) = u^\varepsilon(x)$ for $x \in \partial\Omega_\delta$ which is finite. We have $u_\delta \leqslant g_{\varepsilon,\delta}$ for all ε and also for each fixed $\delta > 0$ then

$$\lim_{\varepsilon\to 0} g_{\varepsilon,\delta}(x) = u_\delta(x).$$

On the domain Ω_δ, we consider the problem

$$\begin{cases} \mathcal{L}[v^\varepsilon] = 0 & \text{in } \Omega_\delta, \\ v^\varepsilon = g_{\delta,\varepsilon} & \text{on } \partial\Omega_\delta. \end{cases}$$

Question: can we quantify $\|v^\varepsilon - u_\delta\|_{L^\infty}$ in term of ε?

5. Rate of convergence via nonlinear adjoint method

6. Questions and Ideas

Question 1 (Jan 12, 2021). *Why do we use the distance functions to get boundary estimates?*

Question 2 (Jan 13, 2021). *Maximum principle for sub-quadratic case.*

Question 3 (Jan 20, 2021). *Here is an idea to estimate $\|u^\varepsilon - u^0\|_{L^\infty_{loc}(\Omega)}$. We start first by assuming star-shaped and consider*

$$\begin{cases} \lambda u^\varepsilon_\eta(x) + H(x, Du^\varepsilon_\eta(x)) - \varepsilon\Delta u^\varepsilon_\eta(x) = 0 & \textit{in } (1-\eta)\Omega, \\ u^\varepsilon_\eta(x) = +\infty & \textit{on } (1-\eta)\partial\Omega. \end{cases} \tag{S_η}$$

e:S_eta

Can we estimate $0 \leqslant u^\varepsilon_\eta - u^\varepsilon \leqslant \omega(\varepsilon,\eta)$ and then chose $\eta = \omega'(\varepsilon)$ to conclude? One way is to approximate infinity boundary by finite boundary first, it may be too naive, but whatever!

$$\begin{cases} \lambda v^\varepsilon_\eta(x) + H(x, Dv^\varepsilon_\eta(x)) - \varepsilon\Delta v^\varepsilon_\eta(x) = 0 & \textit{in } (1-\eta)\Omega, \\ v^\varepsilon_\eta(x) = m & \textit{on } (1-\eta)\partial\Omega. \end{cases} \tag{S^m_η}$$

e:S_eta_m

In contrast, let us consider

$$\begin{cases} \lambda v^\varepsilon(x) + H(x, Dv^\varepsilon(x)) - \varepsilon\Delta v^\varepsilon(x) = 0 & \textit{in } \Omega, \\ v^\varepsilon(x) = m & \textit{on } \partial\Omega. \end{cases} \tag{S^m}$$

e:S_0

How can we compare v^ε and v^ε_η?

Appendices

A. Gradient bounds

A.1. **Local interior gradient bound for elliptic equation.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with C^2 boundary and $H(x,p) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable

Hamiltonian satisfying

$$\lim_{|p\to\infty}\left(\frac{1}{2}H(x,p)^2 + D_xH(x,p)\cdot p\right) = +\infty \qquad \text{uniformly in } x\in\overline{\Omega}. \tag{H1}$$ eq:grow

We consider the following equation

$$\lambda u^\varepsilon(x) + H(x, Du^\varepsilon(x)) - \varepsilon\Delta u^\varepsilon(x) = 0 \qquad \text{in } \Omega. \tag{A.1}$$ eq:C_eps

Let $u^\varepsilon \in C^2(\Omega)\cap C^1(\overline{\Omega})$ be a bounded solution to (A.1), say $|\lambda u^\varepsilon(x)|\leqslant C_1$ for all $x\in\overline{\Omega}$. In this section we show that an interior gradient bound also holds, i.e., if $x\mapsto |Du^\varepsilon(x)|$ has a maximum over $\overline{\Omega}$ at $x_0\in\Omega$ then $|Du^\varepsilon(x_0)|\leqslant C_2$ for all $x\in\Omega$. Here C_1, C_2 are independent of $\varepsilon>0$.

We will use the classical Bernstein's argument. Let $\varphi(x) = \frac{1}{2}|Du^\varepsilon(x)|^2$ for $x\in\Omega$. Differentiate (A.1) in x_i then multiply the resulting equation with $u^\varepsilon_{x_i}$ and then sum over $i = 1,2,\dots,n$ we obtain

$$2\lambda\varphi(x) + D_pH(x, Du^\varepsilon(x))\cdot D\varphi(x) - \varepsilon\Delta\varphi(x) + \Big(\varepsilon|D^2u^\varepsilon(x)| + D_xH(x,Du^\varepsilon(x))\cdot Du^\varepsilon(x)\Big) = 0.$$

If $\varphi(x)$ achieves its maximum over $\overline{\Omega}$ at $x_0\in\Omega$, then $D\varphi(x_0)=0$ and $\Delta\varphi(x_0)\leqslant 0$, together with $\varepsilon|D^2u^\varepsilon(x)|^2 \geqslant \frac{1}{n\varepsilon}(\varepsilon\Delta u^\varepsilon(x))^2 \geqslant (\varepsilon\Delta u^\varepsilon(x))^2$ if $n\varepsilon<1$ we deduce that

$$\lambda|Du^\varepsilon(x_0)|^2 + \Big(\lambda u^\varepsilon(x_0) + H(x_0, Du^\varepsilon(x_0))\Big)^2 + D_xH(x_0, Du^\varepsilon(x_0))\cdot Du^\varepsilon(x_0)\leqslant 0.$$

Assume $|\lambda u^\varepsilon(x_0)|\leqslant C_1$, using (H1) we deduce that

$$\frac{1}{2}H(x_0,Du^\varepsilon(x_0))^2 + D_xH(x_0,Du^\varepsilon(x_0))\cdot Du^\varepsilon(x_0) + \left(\frac{1}{\sqrt{2}}H(x_0,Du^\varepsilon(x_0) - \sqrt{2}C_1\right)^2 - (C_1)^2\leqslant 0$$

which gives us $|Du^\varepsilon(x_0)|\leqslant C_2$ for some C_2 independent of ε.

Remark 1. Assumption (H1) is weaker than the combination of $p\mapsto H(x,p)$ is superlinear and $|D_xH(x,p)|\leqslant C(1+|p|)$.

B. Differentiability with respect to the parameter

For the vanishing viscosity problem with the Dirichlet boundary condition,

$$\begin{cases} H(x, Du^\epsilon(x)) = \epsilon\Delta u^\epsilon & \text{in } U,\\ \qquad\qquad u^\epsilon = 0 & \text{on } \partial U,\end{cases} \tag{B.1}$$ dir

where $H(x,p)$ is $C^\infty(\overline{U}\times\mathbb{R}^n)$, $\dfrac{H(x,p)}{|p|}\to\infty$ uniformly in x as $|p|\to\infty$ and $\sup\limits_{x\in U}|D_xH(x,p)|\leqslant C(1+|p|)$, we want to show the smooth dependence of u^ϵ on ϵ. Formally, if we differentiate (B.1) with respect to ϵ, we get

$$\begin{cases} D_pH(x,Du^\epsilon(x))\cdot Du^\epsilon_\epsilon = \epsilon\Delta u^\epsilon_\epsilon + \Delta u^\epsilon & \text{in } U,\\ \qquad\qquad\qquad u^\epsilon_\epsilon = 0 & \text{on } \partial U.\end{cases} \tag{B.2}$$ dir_dif

By Schaefer's fixed point theorem and the maximal principle, u^ϵ_ϵ is the unique solution in $C^{2,\alpha}(\overline{U})$ of (B.2).

The main idea is, we look at the difference quotients $\dfrac{u^{\epsilon+h}-u^\epsilon}{h}$ and prove that as $h\to 0^+$, they converge to a limiting function w^* in the uniform norm such that w^* solves (B.2). Since (B.2) has a unique solution, we have

$$u^\epsilon_\epsilon = \lim_{h\to 0^+}\frac{u^{\epsilon+h}-u^\epsilon}{h}.$$

B.1. **Solution $u^\epsilon \in C^{2,\alpha}(\overline{U})$ exists.** We use Schaufer's fixed point theorem as follows.

Theorem B.1. *Suppose* X *is a Banach space. Let* $A : X \to X$ *be continuous and compact. Assume the set* $\{u \in X : u = \lambda A[u]$ *for some* $0 \leqslant \lambda \leqslant 1\}$ *is bounded. Then* A *has a fixed point* $u = A[u]$.

Fix $0 < \alpha < 1$. Let $X = C^{1,\alpha}(\overline{U})$. Given $u \in X = C^{1,\alpha}(\overline{U})$, we look at the linear PDE

$$\begin{cases} \epsilon\Delta v = H(x, Du) & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases} \tag{B.3}$$

fix

Estimate the Holder norm of RHS

$$\|H(x, Du)\|_{C^{0,\alpha}(\overline{U})} := \sup_{x\in\overline{U}} |H(x, Du(x))| + \sup_{x,y\in\overline{U}} \frac{|H(x, Du(x)) - H(y, Du(y))|}{|x-y|^\alpha}.$$

Since Du is bounded and H is smooth,

$$\sup_{x\in\overline{U}} |H(x, Du(x))| \leqslant C. \tag{B.4}$$

$$\begin{aligned} &\sup_{x,y\in\overline{U}} \frac{|H(x, Du(x)) - H(y, Du(y))|}{|x-y|^\alpha} \\ \leqslant &\sup_{x,y\in\overline{U}} \frac{|H(x, Du(x)) - H(y, Du(x))|}{|x-y|^\alpha} + \sup_{x,y\in\overline{U}} \frac{|H(y, Du(x)) - H(y, Du(y))|}{|x-y|^\alpha} \\ = &\sup_{x,y\in\overline{U}} \frac{|\int_0^1 D_x H(y + \theta(x-y), Du(x))d\theta \cdot (x-y)|}{|x-y|^\alpha} \\ &+ \sup_{x,y\in\overline{U}} \frac{|\int_0^1 D_p H(y, Du(y) + \theta(Du(x) - Du(y)))d\theta \cdot (Du(x) - Du(y))|}{|x-y|^\alpha} \\ \leqslant &C \sup_{x,y\in\overline{U}} (1 + |Du(x)|)|x-y|^{1-\alpha} + C \sup_{x,y\in\overline{U}} \frac{|(Du(x) - Du(y))|}{|x-y|^\alpha} \\ \leqslant &C(1 + \|u\|_{C^{1,\alpha}(\overline{U})}) \end{aligned} \tag{B.5}$$

since Du is bounded on $\overline{U}$, $D_p H \in C^\infty(\overline{U} \times \mathbb{R}^n)$ and $\sup_{x\in U} |D_x H(x,p)| \leqslant C(1 + |p|)$. Therefore,

$$\|H(x, Du)\|_{C^{0,\alpha}(\overline{U})} \leqslant C(1 + \|u\|_{C^{1,\alpha}(\overline{U})}). \tag{B.6}$$

By Schauder estimates, there exists a unique solution $v \in C^{2,\alpha}(\overline{U})$ such that

$$\|v\|_{C^{2,\alpha}(\overline{U})} \leqslant C\|H(x, Du)\|_{C^{0,\alpha}(\overline{U})}. \tag{B.7}$$

schauder

Define operator A on $X := C^{1,\alpha}(\overline{U})$ by $A[u] = v$. So

$$\|A[u]\|_{C^{2,\alpha}(\overline{U})} \leqslant C(1 + \|u\|_{C^{1,\alpha}(\overline{U})}), \tag{B.8}$$

and thus A is continuous and compact. (i.e., if $\{u_k\}_{k=1}^\infty$ is bounded in $X = C^{1,\alpha}(\overline{U})$, then $\{A[u_k]\}_{k=1}\infty$ is bounded in $C^{2,\alpha}(\overline{U})$, thus precompact in $C^{1,\alpha}(\overline{U})$. Lemma 6.36 in Gilbarg and Trudinger.)

Next we try to bound $\{u \in X : u = \lambda A[u]$ for some $0 \leqslant \lambda \leqslant 1\}$. If $u = \lambda A[u]$, the PDE becomes

$$\begin{cases} \epsilon\Delta u = \lambda H(x, Du) & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{B.9}$$

lamb

Calderon-Zygmund estimates tell us that if we have

$$\begin{cases} -\Delta v = \tilde{f} & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases} \tag{B.10}$$

cald

and $\tilde{f} \in L^p(U)$ for some $p \in (1, \infty)$, then $v \in W^{2,p}(U)$ and

$$\|v\|_{W^{2,p}(U)} \leqslant C\|\tilde{f}\|_{L^p(U)}.$$

Apply to (B.9) and we get

$$\|u\|_{W^{2,p}(U)} \leqslant C\|H(x, Du)\|_{L^p(U)} \tag{B.11}$$

(We want RHS to be bounded by some constant so that later we can choose p larger than n to conclude $\|u\|_{C^{1,\alpha}(U)} \leqslant C\|u\|_{W^{2,p}(U)} \leqslant C$ by Morrey's estimate.) By a priori estimate, if we assume the solution of (B.1) u exists, then $\|Du\|_{L^\infty} \leqslant C_0$ where C_0 is independent of ϵ. We can modify H to get a new $\tilde{H}$ so that it is smooth, $\tilde{H} = H$ for $|p| < C_1$ and $H(x, p) = C_1 + 1$ for $|p| > C_1 + 1$ for some constant C_1 (C_1 is definitely larger than C^0). Moreover, the same prior estimate is correct for $\tilde{H}$. Namely, if the solution $\tilde{u}$ to (B.1) with H replaced by $\tilde{H}$ exists, then $\|D\tilde{u}\|_{L^\infty} \leqslant C_0$. (The way we choose this C_1 is that we work back from the beginning of the proof of a priori estimate for u, on both the boundary and interior of Ω, and modify H so that all the proofs go through and the same a priori estimate holds for the equation with $\tilde{H}$.) Now with $\tilde{H}$, we go through the same argument of Schaufer's fixed point theorem from the very beginning, and (B.10) reads

$$\|\tilde{u}\|_{W^{2,p}(U)} \leqslant C\|\tilde{H}(x, D\tilde{u})\|_{L^p(U)} \leqslant C(1 + \|D\tilde{u}\|_{L^\infty}) \leqslant C. \tag{B.12}$$

Choose $p = 2n$ and $\alpha = \dfrac{1}{2}$. We have $\{\tilde{u} \in X : \tilde{u} = \lambda A[\tilde{u}]$ for some $0 \leqslant \lambda \leqslant 1\}$ is bounded in $X = C^{1,\frac{1}{2}}(\overline{U})$. Thus Schaefer's fixed point theorem implies the equation (B.1) with H replaced by $\tilde{H}$ has a solution $\tilde{u} \in C^{2,\alpha}(\overline{\Omega})$. Since $\|D\tilde{u}\|_{L^\infty} \leqslant C_0$, $\tilde{u}$ also solves the original equation (B.1).

B.2. **Uniqueness.** Let u and v be two solutions to (B.1) and $w := u - v$. Then we have

$$\begin{aligned} &-\epsilon\Delta(u - v) = H(x, Dv) - H(x, Du) \\ \Rightarrow &-\epsilon\Delta w = \int_0^1 D_p H(x, tDv + (1-t)Du) \cdot (Dv - Du)dt \\ \Rightarrow &-\epsilon\Delta w + \int_0^1 D_p H(x, tDv + (1-t)Du)dt \cdot Dw = 0. \end{aligned} \tag{B.13}$$

By the strong maximum principle, $w \equiv 0$.

B.3. **Smooth dependence on ϵ.** Fix $\epsilon > 0$. Let

$$w^h(x) := \frac{u^{\epsilon+h}(x) - u^\epsilon(x)}{h} \in C^{2,\alpha}(\overline{U}).$$

A little computation shows that w^h solves

$$\begin{cases} \epsilon\Delta w^h(x) + \dfrac{\epsilon}{\epsilon+h}\Delta u^\epsilon = \dfrac{\epsilon}{\epsilon+h}\displaystyle\int_0^1 D_p H(x, Du^\epsilon + \theta(Du^{\epsilon+h} - Du^\epsilon))d\theta \cdot Dw^h & \text{in } U, \\ w^h = 0 & \text{on } \partial U. \end{cases} \tag{B.14}$$

dir_quo

From the existence proof, we know $\|u^\epsilon\|_{C^{2,\alpha}(\overline{U})} \leqslant C$ uniformly in ϵ. So $\|Du^{\epsilon+h} - Du^\epsilon\|_{C^{0,\alpha}(\overline{U})}$ and $\|\Delta u\|_{C^{0,\alpha}(\overline{U})}$ is uniformly bounded in h.

By Schauder estimates, $\{w^h\}_{h>0} \subset C^{2,\alpha}(\overline{U})$ are bounded, hence is precompact in $C^{2,\beta}(\overline{U})$ for any $\beta < \alpha$. Therefore, there exits a subsequence $\{w^{h_j}\}_{j=1}^{\infty}$ such that $w^{h_j} \to w^*$ for some $w^* \in C^{2,\beta}(\overline{U})$ and w^* solves (B.2). This implies $w^h \to w^*$ in $C^{2,\beta}(\overline{U})$.

C. Proofs of some Lemmas and Propositions

Proof of Lemma 2.7. The proof is a variation of Perron's method (see [1]). Let $\varphi \in C(\overline{\Omega})$ and $x_0 \in \overline{\Omega}$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a global strict minimum over $\overline{\Omega}$ at x_0 and that

$$\lambda\varphi(x_0) + H(x_0, D\varphi(x_0)) < 0. \tag{C.1}$$ eq:max_a1

Let $\varphi^\varepsilon(x) = \varphi(x) - |x - x_0|^2 + \varepsilon$ for $x \in \overline{\Omega}$. Let $\delta > 0$, we see that for $x \in \partial B(x_0, \delta) \cap \overline{\Omega}$ then

$$\varphi^\varepsilon(x) = \varphi(x) - \delta^2 + \varepsilon \leqslant \varphi(x) - \varepsilon$$

if $2\varepsilon \leqslant \delta^2$. We observe that

$$\begin{aligned} \varphi^\varepsilon(x) - \varphi(x_0) &= \varphi(x) - \varphi(x_0) + \varepsilon - |x - x_0|^2 \\ D\phi^\varepsilon(x) - D\phi(x_0) &= D\varphi(x) - D\varphi(x_0) - 2(x - x_0) \end{aligned}$$

for $x \in B(x, \delta) \cap \overline{\Omega}$. We deduce from (C.1), the continuity of $H(x, p)$ near $(x_0, D\varphi(x_0))$ and the fact that $\varphi \in C^1(\overline{\Omega})$ that if δ is small enough and $0 < 2\varepsilon < \delta^2$ then

$$\lambda\varphi^\varepsilon(x) + H(x, D\varphi^\varepsilon(x)) < 0 \qquad \text{for } x \in B(x_0, \delta) \cap \overline{\Omega}. \tag{C.2}$$ eq:max_a2

We have found $\phi^\varepsilon \in C^1(\overline{\Omega})$ such that $\varphi^\varepsilon(x_0) > u(x_0)$, $\varphi^\varepsilon < u$ on $\partial B(x_0, \delta) \cap \overline{\Omega}$ and (C.2). Let

$$\tilde{u}(x) = \begin{cases} \max\left\{u(x), \phi^\varepsilon(x)\right\} & x \in B(x_0, \delta) \cap \overline{\Omega}, \\ u(x) & x \notin B(x_0, \delta) \cap \overline{\Omega}, \end{cases}$$

We see that $\tilde{u} \in C(\overline{\Omega})$ is a subsolution of (S_0) in Ω with $\tilde{u}(x_0) > u(x_0)$, which is a contradiction, thus u is a supersolution of (S_0) on $\overline{\Omega}$. □

References

[1] Capuzzo-Dolcetta, I., and Lions, P.-L. Hamilton–Jacobi equations with state constraints. *Transactions of the American Mathematical Society 318*, 2 (1990), 643–683.

[2] Kim, Y., Tran, H. V., and Tu, S. N. State-Constraint Static Hamilton–Jacobi Equations in Nested Domains. *SIAM Journal on Mathematical Analysis 52*, 5 (Jan. 2020), 4161–4184. Society for Industrial and Applied Mathematics.

[3] Lasry, J. M., and Lions, P.-L. Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. the model problem. *Mathematische Annalen 283*, 4 (1989), 583–630.

(Y. Han) Department of Mathematics, University of Wisconsin Madison, 480 Lincoln Drive, Madison, WI 53706, USA

Email address: yuxi.han@wisc.edu

(S. N.T. Tu) Department of Mathematics, University of Wisconsin Madison, 480 Lincoln Drive, Madison, WI 53706, USA

Email address: thaison@math.wisc.edu