2. Consider the parametric surface

$$\mathbf{r}(u,v) = (u^2+1, v^3+1, u+v).$$

Find the normal vector at $(2,2,0)$ and the tangent plane at that point to the surface.

$$r_u = (2u, 0, 1) \qquad r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix}$$

$$r_v = (0, 3v^2, 1)$$

$$= (-3v^2, -2u, 6uv^2)$$

At $(2,2,0) = (x,y,z)$, we solve for (u,v) s.t.

$$\begin{cases} x = u^2+1 = 2 \\ y = v^3+1 = 2 \\ z = u+v = 0 \end{cases} \Rightarrow \begin{cases} u^2 = 1 \\ v^3 = 1 \\ u+v = 0 \end{cases} \Rightarrow \begin{cases} u = -1 \\ v = 1 \end{cases}$$

Now plug in $\vec{n} = r_u \times r_v = (-3, 2, -6)$

The tangent plane is

$$-3(x-2) + 2(y-2) - 6(z-0) = 0.$$

7. Let C be the union of the line segments from $P(2,0,\pi)$ to $R(0,0,0)$ and from R to $Q(1,1,\pi/2)$. Evaluate

$$\int_C (2x+5y^2z)\,dx + (10xyz - 3e^{3y}\cos z)\,dy + (5xy^2 + e^{3y}\sin z)\,dz$$

Here this is a line integral of the second type

$$\vec{F} = \left(2x+5y^2z,\ 10xyz-3e^{3y}\cos z,\ 5xy^2+e^{3y}\sin z\right)$$

We can use the fundamental theorem of line integral if we can find a scalar function f s.t.

$$\nabla f = F$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2x+5y^2z \quad , \quad \frac{\partial f}{\partial y} = 10xyz - 3e^{3y}\cos z \quad , \quad \frac{\partial f}{\partial z} = 5xy^2 + e^{3y}\sin z$$

(1) $\dfrac{\partial f}{\partial x} = 2x+5y^2z \Rightarrow f(x,y,z) = x^2 + 5xy^2z + C(y,z)$

integrating w.r.t x

(2) $\dfrac{\partial f}{\partial y} = 10xyz - 3e^{3y}\cos z \Rightarrow \dfrac{\partial f}{\partial y} = 10xyz + \dfrac{\partial C}{\partial y}(y,z)$

match

$$\Rightarrow \quad \frac{\partial C}{\partial y}(y,z) = -3e^{3y}\cos z$$

(integrate w.r.t y) $\Rightarrow \quad C(y,z) = -e^{3y}\cos z + C(z)$

thus $f(x,y,z) = x^2 + 5xy^2z - e^{3y}\cos z + C(z)$

(3) $\dfrac{\partial f}{\partial z} = 5xy^2 + e^{3y}\sin z \quad , \quad \dfrac{\partial f}{\partial z} = 5xy^2 + e^{3y}\sin z + C'(z)$

match $\Rightarrow \quad C'(z) = 0 \Rightarrow C(z) = 0$

choose

Thus

$$f(x,y,z) = x^2 + 5xy^2z - e^{3y}\cos z$$

By the fundamental theorem of line integral $\quad F = \nabla f$

$$\int_C F \cdot dr = f(\text{end}) - f(\text{start})$$

$$= f\left(1, 1, \tfrac{\pi}{2}\right) - f(2, 0, \pi)$$

$$\left(1 + 5 \cdot \frac{\pi}{2} - e^{3} \underbrace{\cos\frac{\pi}{2}}_{0}\right) - \left(4 + 0 - \underbrace{\cos\pi}_{-1}\right)$$

$$= 1 + \frac{5\pi}{2} - 5$$

$$= \boxed{\frac{5\pi}{2} - 4}$$

8. Find the value of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at the point $(1,1,1)$ if z as a function of x and y is defined by the equation $z^3 - xy + yz + y^3 - 2 = 0$.

$\frac{\partial}{\partial x}$:

$$3z^2 \cdot \frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + y\right) = y \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}}$$

$\frac{\partial}{\partial y}$:

$$3z^2 \cdot \frac{\partial z}{\partial y} - x + \left(z + y \cdot \frac{\partial z}{\partial y}\right) + 3y^2 = 0$$

$$\frac{\partial z}{\partial y}\left(3z^2 + y\right) = x - z - 3y^2$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{x - z - 3y^2}{3z^2 + y}}$$

9. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint

$$f(x,y) = 3x + y, \qquad x^2 + y^2 = 10.$$

Write down the values of possible λ.

$\nabla f(x,y) = \lambda \nabla g(x,y)$ $\qquad$ $g(x,y) = x^2 + y^2 - 10$

$$\begin{cases} (3,1) = \lambda(2x, 2y) \\ x^2 + y^2 = 10 \end{cases} \rightarrow \begin{cases} 3 = 2\lambda x \\ 1 = 2\lambda y \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2\lambda} \\ y = \frac{1}{2\lambda} \end{cases}$$

$$x^2 + y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{1}{\lambda^2} \cdot \left(\frac{9}{4} + \frac{1}{4}\right) = 10$$

$$\Rightarrow \frac{1}{\lambda^2} \cdot \frac{10}{4} = 10$$

$$\Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \frac{1}{2} \text{ or } \lambda = \frac{-1}{2}$$

$\lambda = \frac{1}{2} \Rightarrow (x,y) = (3,1) \Rightarrow$ max

$\lambda = \frac{-1}{2} \Rightarrow (x,y) = (-3,-1) \Rightarrow$ min

6. Let C be the curve whose parametrization is given by the vector equation $r(t) = (2t, t)$, $0 \leqslant t \leqslant 1$. Find

$$\int_C (xy + y)\, ds.$$

$$r'(t) = \sqrt{x'(t)^2 + y'(t)^2}$$
$$= \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\int_0^1 (2t \cdot t + t) \cdot \sqrt{5}\, dt$$

$$\sqrt{5} \int_0^1 (2t^2 + t)\, dt = \sqrt{5} \left(\frac{2t^3}{3} \Big|_0^1 + \frac{t^2}{2} \Big|_0^1 \right)$$

$$= \sqrt{5} \left(\frac{2}{3} + \frac{1}{2} \right)$$

$$= \sqrt{5} \cdot \left(\frac{4+3}{6} \right) = \boxed{\sqrt{5} \cdot \frac{7}{6}}$$

4. Let $f(x,y) = \frac{x}{x+y}$, find the linearization of f at (2,1). Use your answer to approximate $f(2.2, 0.9)$. What is the minimum rate of change of f at (2,1)?

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x+y-x}{(x+y)^2}, \frac{-x}{(x+y)^2}\right)$$

$$= \left(\frac{y}{(x+y)^2}, \frac{-x}{(x+y)^2}\right)$$

at $(x,y) = (2,1)$ $\quad \left(\frac{1}{9}, \frac{-2}{9}\right)$

The linearization at (2,1)

$$\boxed{L(x,y) = f(2,1) + \frac{1}{9}(x-2) - \frac{2}{9}(y-1)}$$

Approximation

$$f(2.2, 0.9) \approx L(2.2, 0.9)$$

$$\approx f(2,1) + \frac{1}{9} * 0.2 - \frac{2}{9}(-0.1)$$

$$\approx \frac{2}{3} + \frac{0.2 + 0.2}{9}$$

$$\approx \frac{4 + 0.4}{9} \approx \frac{4.4}{9}$$