5 Equations of Lines and Planes (Part B)

5.4 Introduction to Planes in Space – Video Before Class

Objective(s):

- Define planes and determine their equations.
- Find where lines and planes intersect.
- Find an equation of a plane given three points.

Definition(s) 5.17.

(a) If $\mathbf{n} = \langle a, b, c\rangle$ is a specified vector, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ is the position vector for the point $P_0(x_0, y_0, z_0)$, and $\mathbf{r} = \langle x, y, z\rangle$ a vector of variables then

$$\{(x,y,z) : a(x-x_0)+b(y-y_0)+c(z-z_0)\}$$

Is the set of all vectors with the initial point $\mathbf{r}_0$ perpendicular to $\mathbf{n}$. More commonly this is the vector equation for the plane perpendicular to $\mathbf{n}$ through the point $\vec{r_0} = (x_0, y_0, z_0)$.

(b) Alternatively this plane can be expressed as:

$$(\vec{r}-\vec{r_0})\cdot\vec{n} = 0$$

(c) Finally if we collect all the non-variable terms on one side we can write:

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

$$\text{or} \quad ax+by+cz = d = ax_0+by_0+cz_0$$

Example 5.18. Find an equation of a plane orthogonal to $\langle 1, 4, -2\rangle$ and contains the point $(0, -3, 1)$.

$\vec{n} = (1,4,-2)$

Plane is

$$1\cdot(x-0)+4\cdot(y-(-3))+(-2)(z-1) = 0$$

$$\boxed{x+4y-2z+14=0}$$

Example 5.19. Find where the line $\mathbf{r}(t) = \langle 3-t, 2+t, 5t\rangle$ intersects the plane $x - y + 2z = 6$

$$\begin{cases} x = 3-t \\ y = 2+t \\ z = 5t \end{cases} \rightarrow \quad x - y + 2z$$

$$= (3-t) - (2+t) + 2(5t) = 6$$

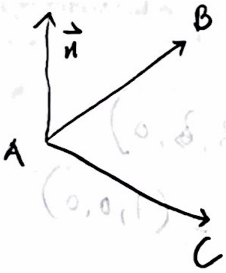
$$3 - t - 2 - t + 10t = 6$$

$$1 + 8t = 6$$

$$8t = 5 \Rightarrow t = \frac{5}{8}$$

the intersection is $(x,y,z) = \left(3 - \frac{5}{8},\ 2 + \frac{5}{8},\ \frac{25}{8}\right)$

Example 5.20. Find an equation of the plane that contains all three points $\underset{A}{(1,0,0)}$, $\underset{B}{(0,1,0)}$ and $\underset{C}{(0,1,3)}$.



form $\vec{AB} = (-1, 1, 0)$

$\vec{AC} = (-1, 1, 3)$

a normal vector can be chosen by $\vec{n} = \vec{AB} \times \vec{AC}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= i \underbrace{\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}}_{3} - j \underbrace{\begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix}}_{-3} + k \underbrace{\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}}_{0}$$

$$= (3, 3, 0) \parallel (1, 1, 0)$$

Choose $\vec{n} = (1, 1, 0)$

then the plane is (using A as the point) $A = (1, 0, 0)$

$$1 \cdot (x - 1) + 1 \cdot (y - 0) + 0 \cdot (z - 0) = 0$$

$$\boxed{x + y = 1}$$

5.5 Double the Planes, Double the Fun – During Class

Objective(s):

- Determine when two planes are parallel.
- If two planes intersect find the line of intersection.
- Calculate the angle of intersection between two planes.

Let's take a look at how planes interact: `https://tinyurl.com/mth234-003`

Definition(s) 5.21.

(a) The angle between two planes is defined to be the acute angle between their normal vector

parallel vectors parallel

(b) Two planes are perpendicular if their normal vectors are perpendicular

(c) If two planes are not parallel then they intersect at a line.

Example 5.22. Show that planes $x + y = 2z + 4$ and $4z - 2x = 2y + 5$ are parallel.

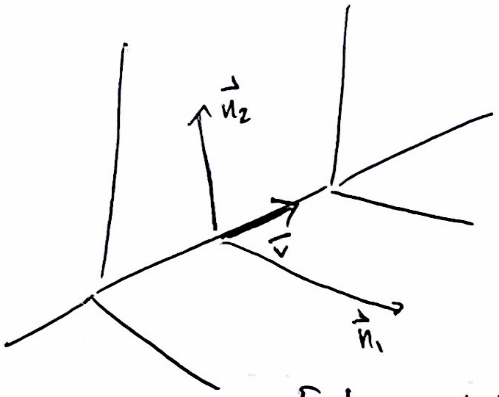
$\vec{n}_1 = (1, 1, -2)$ $\vec{n}_2 = (-2, -2, 4)$

$\vec{n}_2 = -2\,\vec{n}_1$ hence they are parallel

or using angle!

Example 5.23. Find the line of intersection for the planes $x + y + z = 1$ and $x + 2y + 2z = 1$.

https://tinyurl.com/mth234-004



$\vec{n}_1 = (1,1,1)$ $\vec{n}_2 = (1,2,2)$

$\begin{cases} \vec{v} \perp \vec{n}_1 \\ v \perp \vec{n}_2 \end{cases} \Rightarrow$ can choose $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= (0,-1,1)$$

Find a point P_0 s.t. $\begin{cases} x+y+z=1 \\ x+2y+2z=1 \end{cases}$

choose $P_0\,(1,0,0)$ $\Rightarrow$ $\boxed{(1,0,0) + t(0,-1,1),\ t \in \mathbb{R}}$

is the line of intersection

Theorem 5.24. If P_1 and P_2 are non-parallel planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ then their line of intersection has direction vector:

$$\mathbf{v} = \vec{n}_1 \times \vec{n}_2$$

Example 5.25. Find the angle between planes $3x - 6y - 2z = 15$ and $2x - 2z = 5 - y$

(acute angle, $\cos\theta > 0$)

$\vec{n}_1 = (3,-6,-2)$ $\vec{n}_2 = (2,1,-2)$

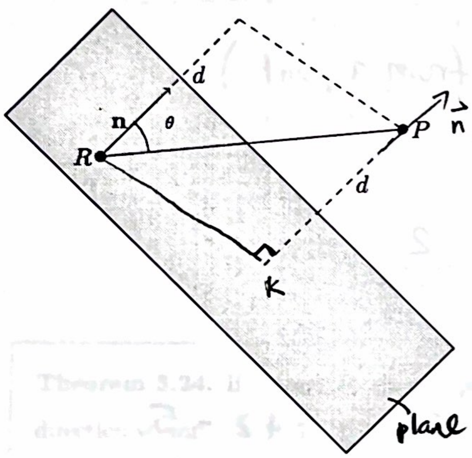
$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| . |\vec{n}_2|} = \frac{3.2 - 6.1 + 2.2}{\sqrt{3^2+6^2+2^2} . \sqrt{2^2+1^2+2^2}} = \frac{4}{7.3} = \frac{4}{21}$$

5.6 Distance from a Point to a Plane – During Class

Objective(s):

- Develop a formula to determine the distance from a point to a plane.
- Utilize the newly developed formula to calculate the distance from a point to a plane in space!
- Use the same formula to determine distance between a line and plane and between two planes.

Now we should develop a theorem for the distance between a point and a plane



$$\text{dist}(P, \text{plane}) = |PK| = |\text{proj}_{\vec{n}}(\vec{RP})|$$

$$= \left| \left(\frac{\vec{n} \cdot \vec{RP}}{\vec{n} \cdot \vec{n}} \right) \cdot \vec{n} \right|$$

$$= \frac{|\vec{n} \cdot \vec{RP}|}{|\vec{n}|}$$

Theorem 5.26. The distance from a Point P to a plane containing R with $\mathbf{n}$ normal to the plane is given by

$$d = \frac{|\vec{n} \cdot \vec{RP}|}{|\vec{n}|}$$

Example 5.27. Find the distance between the point $P(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$

$\vec{n} = (3, 2, 6)$

pick a pont $R(1, 1, 0)$ on $(3x + 2y + 6z = 5)$

$\vec{RP} = (0, -3, 4)$

$$d = \frac{|\vec{n} \cdot \vec{RP}|}{|\vec{n}|} = \frac{|(0, -3, 4) \cdot (3, 2, 6)|}{|(3, 2, 6)|}$$

$$= \frac{|-6 + 24|}{\sqrt{9 + 4 + 36}} = \frac{18}{7}$$

Another formula: $(P): ax + by + cz + d = 0$

$P_0: (x_0, y_0, z_0)$

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 5.28. Consider the line $\mathbf{r}(t) = \langle 7+4t, 2t, 1-t\rangle$ and the plane $x - y + 2z = 5$. If they intersect, find the point of intersection. If they don't intersect find the distance between them.

Check if they intersect: solve for t

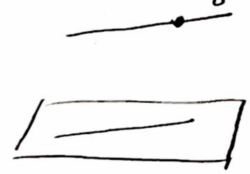
$(7+4t) - 2t + 2(1-t) = 5$

$7 + 4t - 2t + 2 - 2t = 5$

$9 = 5$ no solution

they do not intersect

P_0



distance: (only need to find distance from a point)

$t = 0 \Rightarrow P_0\,(7, 0, 1)$

$$\text{dist} = \frac{|7 - 0 + 2\cdot 1 - 5|}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{2}{\sqrt{6}}$$

Example 5.29. Consider the planes $x + y = 2z + 4$ and $4z - 2x = 2y + 5$. If they intersect, find the line of intersection. If they don't intersect find the distance between them.

$-2x - 2y + 4z = 5$

$\vec{n}_1 = (1, 1, -2)$ $\quad \vec{n}_2 = (-2, -2, 4)$

$\vec{n}_2 = -2\vec{n}_1$ they are parallel

distance: find a point $P_1 \in$ (plane 1)

dist = dist (P_1, plane 2)

$P_1\,(4, 0, 0)$

$$\text{dist}(P_1, \text{plane } 2) = \frac{|-2\cdot 4 - 2\cdot 0 + 4\cdot 0 - 5|}{\sqrt{2^2 + 2^2 + 4^2}}$$

$$= \frac{13}{\sqrt{24}}$$