

# Rate of convergence for quasi-periodic homogenization of Hamilton–Jacobi equation and application

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- ① Introduction
- ② Homogenization
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## Ergodic estimate

- ① Given  $\mathbb{F} \in C(\mathbb{T}^n)$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$  be a non-resonant vector, i.e.,  $\xi \cdot \kappa \neq 0$  for  $\kappa \in \mathbb{Z}^n \setminus \{0\}$ , then for  $f(x) = \mathbb{F}(\xi x)$  in  $\mathbb{R}$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{F}(\xi x) dx = \mathcal{M}(f) := \int_{\mathbb{T}^n} \mathbb{F}(x) dx.$$

- ② If  $\mathbb{F}$  is **unbounded**, then what about

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} = \mathcal{M}(f^{-1}) := \int_{\mathbb{T}^n} \frac{dx}{\mathbb{F}(x)}$$

given that  $x \mapsto \frac{1}{\mathbb{F}(\xi x)}$  is well-defined in  $\mathbb{R}$ ?

- ③ Rate of convergence? Example (result from our work):  
 $\mathbb{F}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^3$  for  $x = (x_1, x_2) \in \mathbb{T}^2$ , then

$$\frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} \geq \frac{C}{T^{1/4}} \quad \text{if } \frac{\xi_2}{\xi_1} \text{ badly approximable.}$$

- ④ Consequence from **homogenization of Hamilton–Jacobi equation**

# Viscosity solutions - Definition

Let  $\Omega \subset \mathbb{R}^n$  be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega.$$

$F$  is non-decreasing in  $u$ , non-increasing in  $D^2u$  (*degenerate elliptic*).

→ No integration by parts, only maximum principle.

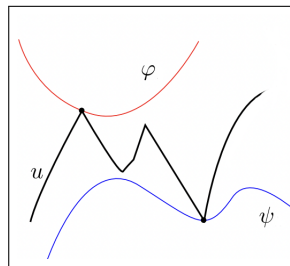
**Subsolution:**  $\varphi \in C^2$ ,  $u - \varphi$  *max* at  $x$ :

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

**Supersolution:**  $\psi \in C^2$ ,  $u - \psi$  *min* at  $x$ :

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

**Viscosity solution** is *both* subsolution and supersolution.



→ *physically correct solution*

→ *value function in optimal control theory*

# Vanishing viscosity - Eikonal equation

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \quad \text{in } (-1, 1) \quad \text{with } u(-1) = u(1) = 0.$$

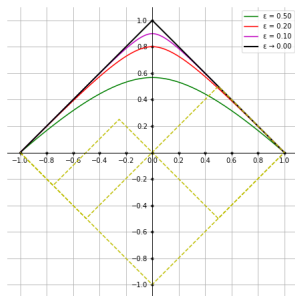
*Infinitely many a.e. solutions*, physically correct solution:  $u(x) = 1 - |x|$ .

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon(u^\varepsilon)'' & \text{in } (-1, 1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon \left( e^{-1/\varepsilon} - e^{-|x|/\varepsilon} \right) \rightarrow u(x)$$



# Optimal control theory - An infinite horizontal example

Let  $U$  be a compact metric space. A *control* is a Borel measurable map  $\alpha : [0, \infty) \mapsto U$ . We are given:

$$\begin{cases} b = b(x, a) : \overline{\Omega} \times U \rightarrow \mathbb{R}^n & \text{velocity vector field} \\ f = f(x, a) : \overline{\Omega} \times U \rightarrow \mathbb{R} & \text{running cost.} \end{cases}$$

For  $x \in \mathbb{R}^n$  and a control  $\alpha(\cdot)$ , let  $y^{x, \alpha}(t)$  solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \quad t > 0, \quad \text{and} \quad y(0) = x$$

**Question.** Minimize the cost functional ( $\lambda \geq 0$ )

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f(y^{x, \alpha}(s), \alpha(s)) \, ds.$$

Define  $H(x, p) = \sup_{v \in U} (-b(x, v) \cdot p - f(x, v))$  then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that  $u \in C^\infty$  (using optimality or dynamic programming principle). However the *value function is usually not smooth!*

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# Homogenization

In 1987, Lions, Papanicolaou and Varadhan [Lions-Papanicolaou-Varadhan'86] proved the homogenization result for a periodic, coercive Hamiltonian (possibly nonconvex)

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u^\varepsilon(x, 0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

As  $\varepsilon \rightarrow 0^+$ ,  $u^\varepsilon \rightarrow u$  and  $u$  solves

$$\begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u(x, 0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

$\bar{H}(p)$  is the unique *constant* such that the ergodic (cell) problem can be solve

$$H(x, p + Dv(x)) = \bar{H}(p) \quad \text{in } \mathbb{T}^n.$$

$\bar{H}(p)$  is called:

- ① effective Hamiltonian
- ② ergodic constant
- ③ additive eigenvalue of  $H$
- ④  $\alpha$ -function in dynamical system
- ⑤ Máně's critical value
- ⑥ ...

# Homogenization - Example

In 1D, if

$$H(x, p) = \frac{|p|^2}{2} + V(x),$$

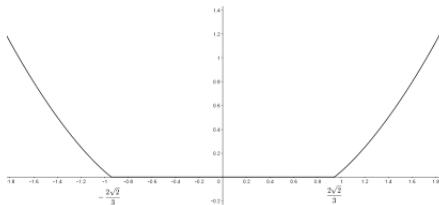
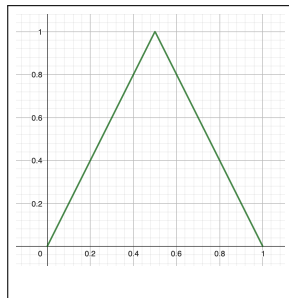
where

$$V(x) = \begin{cases} 2x & x \in \left[0, \frac{1}{2}\right], \\ -2x + 2 & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then

$$|p| = \frac{2\sqrt{2}}{3} \left[ \left( \overline{H}(p) + 1 \right)^{\frac{3}{2}} - \overline{H}(p)^{\frac{3}{2}} \right].$$

Then  $\overline{H}$  takes the form



# Homogenization - Heuristic

- Introduce  $y = \frac{x}{\varepsilon}$  as a fast variable,  $x = \varepsilon y$  is a slow variable.
- *Ansatz*:  $u^\varepsilon(x, t) = u^0(x, y, t) + \varepsilon u^1(x, y, t) + \varepsilon^2 u^2(x, y, t) + \dots$
- Plug in the equation  $u_t + H(\frac{x}{\varepsilon}, Du) = 0$

$$u_t^0(x, y, t) + H\left(y, D_x u^0(x, y, t) + \varepsilon^{-1} D_y u^0(x, y, t) + D_y u^1(x, y, t)\right) = 0.$$

- $D_y u^0 = 0$ , i.e.,  $u^0 = u^0(x, t)$  independent of  $y$

$$H\left(y, \boxed{D_x u^0(x, t)} + D_y u^1(x, y, t)\right) = \boxed{-u_t^0(x, t)}$$

- Ergodic or cell problem (for a fixed  $(x, t)$ )

$$H\left(y, \boxed{p} + D_y u^1(y)\right) = \boxed{\bar{H}(p)}$$

# Homogenization

- The above ansatz gives

$$u^\varepsilon(x, t) \approx u^0(x, t) + \varepsilon u^1\left(\frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2).$$

- This means in homogenization as  $\varepsilon \rightarrow 0$  then  $u^\varepsilon \rightarrow u^0$ .
- $v = u^1$  is a *corrector*

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon v\left(\frac{x}{\varepsilon}; Du(x, t)\right).$$

where

$$H(x, p + Dv(x; p)) = \overline{H}(p).$$

**Solution  $v$  is not unique (up to adding a constant).**

- If  $v$  is bounded then (the expected optimal rate)

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon).$$

- Via *doubling variable method*: can prove the convergence, but not the expansion.

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## Literature

This received quite a lot of attention in the past twenty years.

**Assume:**  $x \mapsto H(x, p)$  is Lipschitz locally in  $p$

- [Capuzzo-Dolcetta-Ishii'01]:  $\mathcal{O}(\varepsilon^{1/3})$ , PDE method, nonconvex and multi-scale  $H(x, \frac{x}{\varepsilon}, Du^\varepsilon) \rightarrow \bar{H}(x, Du)$ . : many works use this method
- $\mathcal{O}(\varepsilon^{1/2})$  if there is a Lipschitz selection  $p \mapsto v(\cdot, p)$  of the cell problem

$$H(x, p + Dv(x; p)) = \bar{H}(p).$$

## Convex Hamiltonian

- $\mathcal{O}(\varepsilon)$  in 1D [Mitake-Tran-Yu'19] and [Tu'18] for 1D multi-scale.
- Conditional  $\mathcal{O}(\varepsilon)$  under smoothness assumption of  $\bar{H}$  [Mitake-Tran-Yu'19].  
first group utilized optimal control, optimal curve and metric distance
- Optimal rate  $\mathcal{O}(\varepsilon)$  [Tran-Yu'21]. Burago Lemma and the metric distance.
- $\mathcal{O}(\varepsilon^{1/2})$  for multi-scale using Burago Lemma [Han-Jang'23].
- [Armstrong-Cardaliaguet-Souganidis'14]: followed [Capuzzo-Dolcetta-Ishii'01],  $\mathcal{O}(\varepsilon^{1/8})$  for i.i.d, an abstract modulus  $\omega(\varepsilon)$  for the almost periodic (PDE method).

# Almost periodic homogenization

- For  $f \in BUC(\mathbb{R}^n)$ , we say it is almost periodic if  $\{f(\cdot + z) : z \in \mathbb{R}^n\}$  is relatively compact in  $BUC(\mathbb{R}^n)$ .

periodic :  $x \mapsto H(x, p)$  is  $\mathbb{Z}^n$  periodic

almost-periodic :  $\{H(\cdot + z, \cdot) : z \in \mathbb{R}^n\}$  is relatively compact in  $BUC(\mathbb{R}^n \times B_R(0))$ .

- In one-dimensional case, for example

$$H(x, p) = \frac{|p|^2}{2} - V(x), \quad V(x) = 2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x).$$

- Quasi-periodic potential in 1D:  $x \in \mathbb{R}$

$$V(x) = F(\xi x) \quad \text{where } F \in C^k(\mathbb{T}^k), \xi \in \mathbb{R}^k \text{ is nonresonant.}$$

- The corrector is replaced by *almost corrector* [Ishii'00]

$$\overline{H}(p) - \delta \leq H(y, p + Dv_\delta(y; p)) \leq \overline{H}(p) + \delta.$$

# Almost periodic function in 1D

First studied by Bohr (1926):

- For  $\varepsilon > 0$ ,  $\tau$  is an  $\varepsilon$ -period, if

$$|f(x + \tau) - f(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

We say  $E(\varepsilon, f) = \{\tau \in \mathbb{R} : |f(x + \tau) - f(x)| < \varepsilon\}$  the set of all  $\varepsilon$ -periods.

- $f \in \text{AP}(\mathbb{R})$  if for  $\varepsilon > 0$ , there exists  $l_\varepsilon$  such that, for every  $a \in \mathbb{R}$

$$[a, a + l_\varepsilon] \cap E(\varepsilon, f) \neq \emptyset$$

any interval of length  $l_\varepsilon$  has an  $\varepsilon$ -period.

- We say  $l_\varepsilon$  is an *inclusion interval length* of  $E(\varepsilon, f)$ .
- Mean value property** If  $f \in \text{AP}(\mathbb{R})$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) dx = \mathcal{M}(f).$$

- If  $f(x) = F(\xi x)$  is quasi-periodic, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) dx = \mathcal{M}(f) = \int_{\mathbb{T}^n} F(\mathbf{x}) d\mathbf{x}.$$



# Convergence to the mean value

If  $f$  is periodic of period 1, then  $\mathcal{M}(f) = \int_0^1 f(x)dx$ , and

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \left( \int_0^1 f(x)dx \right) \frac{1}{T}.$$

**Key ingredient** for periodic homogenization rate  $\mathcal{O}(\varepsilon)$  in 1D [Mitake-Tran-Yu'19, Tu'18].

- (Almost-periodic) For every  $\varepsilon > 0$

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})} \frac{l_\varepsilon(f)}{T}.$$

Need an estimate of  $l_\varepsilon(f)$  with respect to  $\varepsilon$ , but good as only  $L^\infty$  is needed.

- (Quasi-periodic) If  $f(x) = \mathbf{F}(\xi x)$  and  $\mathbf{F} \in H^s(\mathbb{T}^n)$  for  $s > \frac{n}{2} + \sigma_\xi$  then

$$\left| \frac{1}{T} \int_0^T \mathbb{F}(\xi x) dx - \int_{\mathbb{T}^n} \mathbf{F}(\mathbf{x}) d\mathbf{x} \right| \leq \frac{C(n, s) \|\mathbf{F}\|_{H^s(\mathbb{T}^n)}}{T}.$$

Here  $\sigma_\xi$  is a Diophantine condition of  $\xi$ :

$$\xi \cdot \kappa \geq \frac{C}{|\kappa|^\sigma} \quad \forall \kappa \in \mathbb{Z}^n.$$

Need higher regularity, not applicable for some potentials.

# Diophantine Approximations

For almost periodic  $f$

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})} \frac{l_\varepsilon(f)}{T}.$$

For quasi-periodic  $f(x) = \mathbf{F}(\xi x)$  with  $\mathbf{F} \in C^{0,\alpha}(\mathbb{T}^n)$

- ① [Nai96]  $n = 2$ , badly approximable (null set)

$$l_\varepsilon(f) \leq C\varepsilon^{\frac{-1}{\alpha}}$$

- ② [Ryn98] almost every  $n$ -frequencies

$$l_\varepsilon(f) \leq C\varepsilon^{-\frac{n-1}{\alpha}} |\log(\varepsilon)|^{3(n-1)}$$

# Rate of convergence in 1D almost periodic

**Theorem** (Hu-Tu-Zhang '24): In 1D with  $H$  is convex, coercive ( $\frac{1}{2}|p|^2$  for simplicity)

$$H(x, p) = \frac{|p|^2}{2} - V(x), \quad V(x) = \mathbb{V}(\xi x), \mathbb{V} \in C(\mathbb{T}^n), \mathbb{V} \geq 0.$$

There is  $C(n, \alpha, \xi, V)$  such that

$$u^\varepsilon(x, t) - u(x, t) \geq \begin{cases} -C\varepsilon & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ -C\varepsilon^{\frac{\alpha}{\alpha+n-1}} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{V}^{1/2} \in C^\alpha(\mathbb{T}^n), \\ -C\varepsilon^{\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

If  $\bar{H} \in C^{1,\beta}(\mathbb{R})$  then

$$u^\varepsilon(x, t) - u(x, t) \leq \begin{cases} C\varepsilon^{\frac{\beta}{\beta+1}} & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ C\varepsilon^{\frac{\beta}{\beta+1}} \frac{\alpha}{\alpha+n-1} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{V}^{1/2} \in C^\alpha(\mathbb{T}^n), \\ C\varepsilon^{\frac{\beta}{\beta+1}} \frac{\alpha}{\alpha+1} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

## Place in the literature

- 1 First **algebraic** rate for almost periodic setting (only abstract modulus rate, PDE method in the literature).
- 2 the relation between how irrational of  $\xi$  and the regularity of  $\mathbb{V}$  is intricate.

## Case study

**Examples**  $\mathbb{V}(x, y) = (2 - \sin(2\pi x) - \sin(2\pi y))^\gamma$  and  $\xi = (1, \sqrt{2})$ .

$$H(x, p) = \frac{|p|^2}{2} - \left(2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x)\right)^\gamma, \quad \gamma > 0.$$

Consider the homogenization problem in 1D

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \\ u^\varepsilon(x, 0) = u_0(x) \end{cases} \longrightarrow \begin{cases} u_t + \bar{H}(Du) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Then

$$\boxed{\gamma > 2}$$

$$-C\varepsilon \leq u^\varepsilon - u \leq C\varepsilon^\tau, \quad \tau = \frac{\gamma - 2}{3\gamma - 2}$$

$$\boxed{\gamma = 2}$$

$$-C\varepsilon \leq u^\varepsilon - u \leq \frac{C}{|\log(\varepsilon)|}$$

$$\boxed{\gamma < 2}$$

$$u^\varepsilon - u \geq \begin{cases} -C\varepsilon^{\frac{\gamma}{\gamma+1}}, & \gamma \in (0, 1), \\ -C\varepsilon^{1/2}, & \gamma \in [1, 2]. \end{cases}$$

## Idea of the proof

$$\boxed{\frac{v_p(t)}{t} = \mathcal{O}\left(\frac{1}{t^\alpha}\right) \text{ as } t \rightarrow \infty} \leq u^\varepsilon - u \leq \begin{cases} \text{shape and regularity of } \bar{H} \\ \text{averaging optimal path :} \\ \left| \frac{\eta(t)}{t} - \bar{H}'(p) \right| \leq \mathcal{O}\left(\frac{1}{t^\beta}\right). \end{cases}$$

- ① Lower bound is easy: decay rate of correctors and Hopf-Lax formula

$$\mathcal{M}(f)$$

- ② Upper bound is harder: long time average of characteristic (calibrated curve)

$$\mathcal{M}(f^{-1})$$

# Shape of $\overline{H}$

To compute  $\overline{H}(p)$ , we look for a sublinear solution  $v_p$  to

$$H(x, p + Dv_p(x)) = \mu$$

Assume  $\overline{H}(p) = \mu$ , we look for  $p$  instead

$$\frac{|p + v'(x)|^2}{2} - \mathbb{V}(\xi x) = \mu \implies v(x) = \int_0^x \sqrt{2(\mu + \mathbb{V}(x))} \, dx - px$$

Then

$$\frac{v(x)}{x} = \frac{1}{x} \int_0^x \sqrt{2(\mu + \mathbb{V}(x))} \, dx - p \rightarrow 0$$

With

$$p_\mu = \mathcal{M}(\sqrt{2(\mu + \mathbb{V})}) = \int_{\mathbb{T}^n} \sqrt{2(\mu + \mathbb{V}(\mathbf{x}))} \, d\mathbf{x}.$$

# Sketch of the proof - 1

- 1 If  $H(x, p) = \frac{|p|^2}{2} + V(x)$  then the Lagrangian  $L(x, v) = \frac{|v|^2}{2} - V(x)$ .
- 2 Let  $(x, t) = (0, 1)$ , use optimal control formula (action minimizing)

$$A^\varepsilon[\eta] = \varepsilon \int_0^{\varepsilon^{-1}} L(\eta(s), -\dot{\eta}(s)) ds + u_0(\varepsilon\eta(\varepsilon^{-1}))$$

and

$$u^\varepsilon(0, 1) = \inf_{\eta(0)=0} A^\varepsilon[\eta]$$

- 3 A minimizer has conservation of energy

$$\frac{|\dot{\eta}(s)|^2}{2} + V(\eta(s)) = r$$

- 4 Rewrite

$$u^\varepsilon(0, 1) = \inf_r \left( \inf_{\eta_r} A^\varepsilon[\eta_r] \right)$$

- 5 For each energy  $r$ , averaging each terms of the action with rate

## Sketch of the proof - 2

- ① Lower bound is easy

$$A^\varepsilon[\eta_r] \geq u(0, 1) + \inf_{|p| \geq p_0} \varepsilon v_p(\eta(\varepsilon^{-1}))$$

- ② Lower bound correspond to decay rate of corrector  $\frac{v_p(x)}{|x|}$  as  $|x| \rightarrow \infty$ , i.e., convergence rate to the mean value

$$\left| \frac{1}{T} \int_0^T \mathbb{V}^{1/2}(\xi x) dx - \mathcal{M}(\mathbb{V}^{1/2}) \right| \leq \frac{C}{T^\theta}$$

- ③ For  $|p| \geq p_0$

$$\left| \frac{v_p(t)}{t} \right| \leq \left| \frac{1}{t} \int_0^t \mathbb{F}_\mu(\xi x) dx - \mathcal{M}(\mathbb{F}_\mu) \right| \leq \begin{cases} C|t|^{-1} \\ C|t|^{-\frac{\alpha}{\alpha+n-1}} |\log(t)|^{3(n-1)} \end{cases}$$

- The first case happens for  $\mathbb{F} \in H^s(\mathbb{T}^n)$  ( $s > n/2 + \sigma_\xi$ )
- The second case happens for a.e.  $\xi \in \mathbb{R}^n$  with  $\mathbb{F} \in C^{0,\alpha}(\mathbb{T}^n)$ .



## Sketch of the proof - 3

- 1 Upper bound is harder, obtainable when negative energy  $r < 0$  does not play a role, i.e.,  $\bar{H} \in C^1$
- 2 Look at

$$A^\varepsilon[\eta_r] = (\varepsilon\eta_r(\varepsilon^{-1})) \underbrace{\left( \frac{1}{\eta_r(\varepsilon^{-1})} \int_0^{\eta_r(\varepsilon^{-1})} \sqrt{2(r - \mathbb{V}(\xi x))} \, dx \right)}_{p_r = \mathcal{M}(\sqrt{2(r - \mathbb{V})})} + u_0(\varepsilon\eta_r(\varepsilon^{-1})).$$

- 3 The difficult term is

$$\varepsilon\eta_r(\varepsilon^{-1}) \longleftrightarrow \frac{\eta(t)}{t} \rightarrow q \in \partial\bar{H}$$

This is the large time average of calibrated curve to a rotation vector.

- 4 Difficult to do directly in a uniform way as  $r \rightarrow 0^+$ , by Euler-Lagrange equation

$$\frac{1}{\varepsilon\eta(\varepsilon^{-1})} = \frac{1}{\eta(\varepsilon^{-1})} \int_0^{\eta(\varepsilon^{-1})} \frac{dx}{\sqrt{2(r - \mathbb{V}(\xi x))}} \rightarrow \mathcal{M}\left(\frac{1}{\sqrt{2(r - \mathbb{V})}}\right)$$

- 5 Using Hamilton–Jacobi equation: **uniform in  $r \rightarrow 0^+$**

$$\bar{H} \in C^{1,\beta} \implies \left| \frac{\eta_r(t)}{t} - \bar{H}'_+(p_r) \right| \leq C\varepsilon^{\frac{\beta}{1+\beta}}.$$

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# Application to ergodic estimate

For  $\mathbb{V}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^\gamma$  and  $\xi = (\xi_1, \xi_2)$  with  $\frac{\xi_2}{\xi_1}$  is badly approximable,  $H(x, p) = \frac{|p|^2}{2} - \mathbb{V}(\xi x)$ , then

$$\left| \frac{\eta(t)}{t} \right| = \left| \frac{\eta(t)}{t} - \bar{H}'(p) \right| \leq \begin{cases} C|t|^{-\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ C|\log(t)|^{-1} & \gamma = 2. \end{cases}$$

Consequently

$$\frac{1}{T} \int_0^T \frac{dx}{\mathbb{V}^{1/2}(\xi x)} \geq \begin{cases} C \left( \frac{1}{T} \right)^{\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ \frac{C}{|\log(T)|} & \gamma = 2. \end{cases}$$

# Outlooks

- 1 For zero energy  $r = 0$

$$\overline{H} \in C^{1,\alpha} \longrightarrow \varepsilon^{\frac{\alpha}{1+\alpha}} \longrightarrow \varepsilon^{\frac{\alpha(\alpha+1)}{\alpha(\alpha+1)+1}}$$

We have

$$\left| \frac{\eta_0(t)}{t} \right| \leq \left( \frac{1}{|t|} \right)^\tau \quad \text{where } \tau = \frac{(\gamma-2)(3\gamma-2)}{(\gamma-2)(3\gamma-2)+4\gamma^2}.$$

If this holds uniformly for  $\eta_r$  as  $r \rightarrow 0^*$  then we can improve the rate of homogenization

- 2 Nonsmooth  $\overline{H}$ ?
- 3 Gaps in the quantitative estimate using two different methods?

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