

Rate of convergence for quasi-periodic homogenization of Hamilton–Jacobi equation and application

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- ① Introduction
- ② Homogenization
- ③ Rate of convergence
- ④ Application to Ergodic Estimate

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Ergodic estimate

- ① Given $\mathbb{F} \in C(\mathbb{T}^n)$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be a non-resonant vector, i.e., $\xi \cdot \kappa \neq 0$ for $\kappa \in \mathbb{Z}^n \setminus \{0\}$, then for $f(x) = \mathbb{F}(\xi x)$ in \mathbb{R}

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{F}(\xi x) dx = \mathcal{M}(f) := \int_{\mathbb{T}^n} \mathbb{F}(x) dx.$$

- ② If \mathbb{F} is **unbounded**, then what about

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} = \mathcal{M}(f^{-1}) := \int_{\mathbb{T}^n} \frac{dx}{\mathbb{F}(x)}$$

given that $x \mapsto \frac{1}{\mathbb{F}(\xi x)}$ is well-defined in \mathbb{R} ?

- ③ Rate of convergence? Example (result from our work):
 $\mathbb{F}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^3$ for $x = (x_1, x_2) \in \mathbb{T}^2$, then

$$\frac{1}{T} \int_0^T \frac{dx}{\mathbb{F}(\xi x)} \geq CT^{1/4} \quad \text{if } \frac{\xi_2}{\xi_1} \text{ badly approximable.}$$

- ④ Consequence from **homogenization of Hamilton–Jacobi equation**

Viscosity solutions - Definition

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega.$$

F is non-decreasing in u , non-increasing in D^2u (*degenerate elliptic*).

→ No integration by parts, only maximum principle.

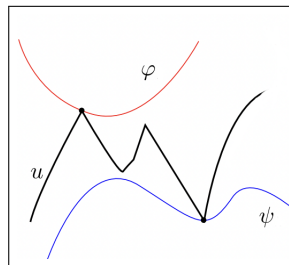
Subsolution: $\varphi \in C^2$, $u - \varphi$ *max* at x :

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

Supersolution: $\psi \in C^2$, $u - \psi$ *min* at x :

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.



→ *physically correct solution*

→ *value function in optimal control theory*

Vanishing viscosity - Eikonal equation

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \quad \text{in } (-1, 1) \quad \text{with } u(-1) = u(1) = 0.$$

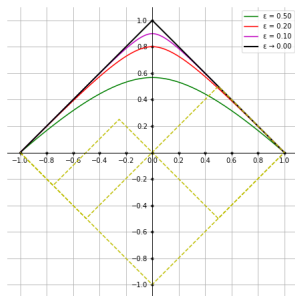
Infinitely many a.e. solutions, physically correct solution: $u(x) = 1 - |x|$.

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon(u^\varepsilon)'' & \text{in } (-1, 1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon \left(e^{-1/\varepsilon} - e^{-|x|/\varepsilon} \right) \rightarrow u(x)$$



Optimal control theory - An infinite horizontal example

Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0, \infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x, a) : \overline{\Omega} \times U \rightarrow \mathbb{R}^n & \text{velocity vector field} \\ f = f(x, a) : \overline{\Omega} \times U \rightarrow \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x, \alpha}(t)$ solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \quad t > 0, \quad \text{and} \quad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f(y^{x, \alpha}(s), \alpha(s)) \, ds.$$

Define $H(x, p) = \sup_{v \in U} (-b(x, v) \cdot p - f(x, v))$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in C^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!*

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Homogenization

In 1987, Lions, Papanicolaou and Varadhan [Lions-Papanicolaou-Varadhan'86] proved the homogenization result for a periodic, coercive Hamiltonian (possibly nonconvex)

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u^\varepsilon(x, 0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

As $\varepsilon \rightarrow 0^+$, $u^\varepsilon \rightarrow u$ and u solves

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{T}^n \times \mathbb{R}^n \\ u(x, 0) = u_0(x) & \text{in } \mathbb{T}^n. \end{cases}$$

$\overline{H}(p)$ is the unique *constant* such that the ergodic (cell) problem can be solve

$$H(x, p + Dv(x)) = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

$\overline{H}(p)$ is called:

- ① effective Hamiltonian
- ② ergodic constant
- ③ additive eigenvalue of H
- ④ α -function in dynamical system
- ⑤ Máně's critical value
- ⑥ ...

Homogenization - Example

In 1D, if

$$H(x, p) = \frac{|p|^2}{2} + V(x),$$

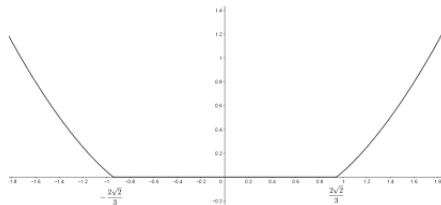
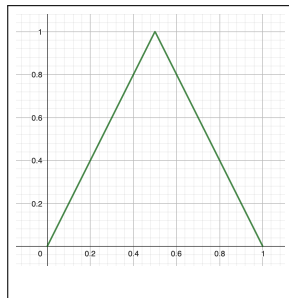
where

$$V(x) = \begin{cases} 2x & x \in \left[0, \frac{1}{2}\right], \\ -2x + 2 & x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then

$$|p| = \frac{2\sqrt{2}}{3} \left[\left(\overline{H}(p) + 1 \right)^{\frac{3}{2}} - \overline{H}(p)^{\frac{3}{2}} \right].$$

Then \overline{H} takes the form



Homogenization - Heuristic

- Introduce $y = \frac{x}{\varepsilon}$ as a fast variable, $x = \varepsilon y$ is a slow variable.
- *Ansatz*: $u^\varepsilon(x, t) = u^0(x, y, t) + \varepsilon u^1(x, y, t) + \varepsilon^2 u^2(x, y, t) + \dots$
- Plug in the equation $u_t + H(\frac{x}{\varepsilon}, Du) = 0$

$$u_t^0(x, y, t) + H\left(y, D_x u^0(x, y, t) + \varepsilon^{-1} D_y u^0(x, y, t) + D_y u^1(x, y, t)\right) = 0.$$

- $D_y u^0 = 0$, i.e., $u^0 = u^0(x, t)$ independent of y

$$H\left(y, \boxed{D_x u^0(x, t)} + D_y u^1(x, y, t)\right) = \boxed{-u_t^0(x, t)}$$

- Ergodic or cell problem (for a fixed (x, t))

$$H\left(y, \boxed{p} + D_y u^1(y)\right) = \boxed{\bar{H}(p)}$$

Homogenization

- The above ansatz gives

$$u^\varepsilon(x, t) \approx u^0(x, t) + \varepsilon u^1\left(\frac{x}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2).$$

- This means in homogenization as $\varepsilon \rightarrow 0$ then $u^\varepsilon \rightarrow u^0$.
- $v = u^1$ is a *corrector*

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon v\left(\frac{x}{\varepsilon}; Du(x, t)\right).$$

where

$$H(x, p + Dv(x; p)) = \overline{H}(p).$$

Solution v is not unique (up to adding a constant).

- If v is bounded then (the expected optimal rate)

$$|u^\varepsilon - u| = \mathcal{O}(\varepsilon).$$

- Via *doubling variable method*: can prove the convergence, but not the expansion.

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Literature

This received quite a lot of attention in the past twenty years.

Assume: $x \mapsto H(x, p)$ is Lipschitz locally in p

- [Capuzzo-Dolcetta-Ishii'01]: $\mathcal{O}(\varepsilon^{1/3})$, PDE method, nonconvex and multi-scale $H(x, \frac{x}{\varepsilon}, Du^\varepsilon) \rightarrow \bar{H}(x, Du)$. : many works use this method
- $\mathcal{O}(\varepsilon^{1/2})$ if there is a Lipschitz selection $p \mapsto v(\cdot, p)$ of the cell problem

$$H(x, p + Dv(x; p)) = \bar{H}(p).$$

Convex Hamiltonian

- $\mathcal{O}(\varepsilon)$ in 1D [Mitake-Tran-Yu'19] and [Tu'18] for 1D multi-scale.
- Conditional $\mathcal{O}(\varepsilon)$ under smoothness assumption of \bar{H} [Mitake-Tran-Yu'19].
first group utilized optimal control, optimal curve and metric distance
- Optimal rate $\mathcal{O}(\varepsilon)$ [Tran-Yu'21]. Burago Lemma and the metric distance.
- $\mathcal{O}(\varepsilon^{1/2})$ for multi-scale using Burago Lemma [Han-Jang'23].
- [Armstrong-Cardaliaguet-Souganidis'14]: followed [Capuzzo-Dolcetta-Ishii'01], $\mathcal{O}(\varepsilon^{1/8})$ for i.i.d, an abstract modulus $\omega(\varepsilon)$ for the almost periodic (PDE method).

Almost periodic homogenization

- For $f \in BUC(\mathbb{R}^n)$, we say it is almost periodic if $\{f(\cdot + z) : z \in \mathbb{R}^n\}$ is relatively compact in $BUC(\mathbb{R}^n)$.

periodic : $x \mapsto H(x, p)$ is \mathbb{Z}^n periodic

almost-periodic : $\{H(\cdot + z, \cdot) : z \in \mathbb{R}^n\}$ is relatively compact in $BUC(\mathbb{R}^n \times B_R(0))$.

- In one-dimensional case, for example

$$H(x, p) = \frac{|p|^2}{2} - V(x), \quad V(x) = 2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x).$$

- Quasi-periodic potential in 1D: $x \in \mathbb{R}$

$$V(x) = F(\xi x) \quad \text{where } F \in C^k(\mathbb{T}^k), \xi \in \mathbb{R}^k \text{ is nonresonant.}$$

- The corrector is replaced by *almost corrector* [Ishii'00]

$$\overline{H}(p) - \delta \leq H(y, p + Dv_\delta(y; p)) \leq \overline{H}(p) + \delta.$$

Almost periodic function in 1D

First studied by Bohr (1926):

- For $\varepsilon > 0$, τ is an ε -period, if

$$|f(x + \tau) - f(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

We say $E(\varepsilon, f) = \{\tau \in \mathbb{R} : |f(x + \tau) - f(x)| < \varepsilon\}$ the set of all ε -periods.

- $f \in \text{AP}(\mathbb{R})$ if for $\varepsilon > 0$, there exists l_ε such that, for every $a \in \mathbb{R}$

$$[a, a + l_\varepsilon] \cap E(\varepsilon, f) \neq \emptyset$$

any interval of length l_ε has an ε -period.

- We say l_ε is an *inclusion interval length* of $E(\varepsilon, f)$.
- Mean value property** If $f \in \text{AP}(\mathbb{R})$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) dx = \mathcal{M}(f).$$

- If $f(x) = F(\xi x)$ is quasi-periodic, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(x) dx = \mathcal{M}(f) = \int_{\mathbb{T}^n} F(\mathbf{x}) d\mathbf{x}.$$

Convergence to the mean value

If f is periodic of period 1, then $\mathcal{M}(f) = \int_0^1 f(x) dx$, and

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \left(\int_0^1 f(x) dx \right) \frac{1}{T}.$$

Key ingredient for periodic homogenization rate $\mathcal{O}(\varepsilon)$ in 1D [Mitake-Tran-Yu'19, Tu'18].

- (Almost-periodic) For every $\varepsilon > 0$

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})} \frac{l_\varepsilon(f)}{T}.$$

Need an estimate of $l_\varepsilon(f)$ with respect to ε , but good as only L^∞ is needed.

- (Quasi-periodic) If $f(x) = \mathbf{F}(\xi x)$ and $\mathbf{F} \in H^s(\mathbb{T}^n)$ for $s > \frac{n}{2} + \sigma_\xi$ then

$$\left| \frac{1}{T} \int_0^T \mathbb{F}(\xi x) dx - \int_{\mathbb{T}^n} \mathbf{F}(\mathbf{x}) d\mathbf{x} \right| \leq \frac{C(n, s) \|\mathbf{F}\|_{H^s(\mathbb{T}^n)}}{T}.$$

Here σ_ξ is a Diophantine condition of ξ :

$$\xi \cdot \kappa \geq \frac{C}{|\kappa|^\sigma} \quad \forall \kappa \in \mathbb{Z}^n.$$

Need higher regularity, not applicable for some potentials.

Diophantine Approximations

For almost periodic f

$$\left| \frac{1}{T} \int_0^T f(x) dx - \mathcal{M}(f) \right| \leq \varepsilon + 2\|f\|_{L^\infty(\mathbb{R})} \frac{l_\varepsilon(f)}{T}.$$

For quasi-periodic $f(x) = \mathbf{F}(\xi x)$ with $\mathbf{F} \in C^{0,\alpha}(\mathbb{T}^n)$

- ① [Nai96] $n = 2$, badly approximable (null set)

$$l_\varepsilon(f) \leq C\varepsilon^{\frac{-1}{\alpha}}$$

- ② [Ryn98] almost every n -frequencies

$$l_\varepsilon(f) \leq C\varepsilon^{-\frac{n-1}{\alpha}} |\log(\varepsilon)|^{3(n-1)}$$

Rate of convergence in 1D almost periodic

Theorem (Hu-Tu-Zhang '24): In 1D with H is convex, coercive ($\frac{1}{2}|p|^2$ for simplicity)

$$H(x, p) = \frac{|p|^2}{2} - V(x), \quad V(x) = \mathbb{V}(\xi x), \mathbb{V} \in C(\mathbb{T}^n), \mathbb{V} \geq 0.$$

There is $C(n, \alpha, \xi, V)$ such that

$$u^\varepsilon(x, t) - u(x, t) \geq \begin{cases} -C\varepsilon & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ -C\varepsilon^{\frac{\alpha}{\alpha+n-1}} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{V}^{1/2} \in C^\alpha(\mathbb{T}^n), \\ -C\varepsilon^{\frac{\alpha}{\alpha+1}} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

If $\bar{H} \in C^{1,\beta}(\mathbb{R})$ then

$$u^\varepsilon(x, t) - u(x, t) \leq \begin{cases} C\varepsilon^{\frac{\beta}{\beta+1}} & \mathbb{V}^{1/2} \in H^s(\mathbb{T}^n), s > n/2 + \sigma_\xi, \\ C\varepsilon^{\frac{\beta}{\beta+1}} \frac{\alpha}{\alpha+n-1} |\log(\varepsilon)|^{3(n-1)} & \text{for a.e. } \xi, \mathbb{V}^{1/2} \in C^\alpha(\mathbb{T}^n), \\ C\varepsilon^{\frac{\beta}{\beta+1}} \frac{\alpha}{\alpha+1} & n = 2, \xi \text{ badly approximable.} \end{cases}$$

Place in the literature

- 1 First **algebraic** rate for almost periodic setting (only abstract modulus rate, PDE method in the literature).
- 2 the relation between how irrational of ξ and the regularity of \mathbb{V} is intricate.

Case study

Examples $\mathbb{V}(x, y) = (2 - \sin(2\pi x) - \sin(2\pi y))^\gamma$ and $\xi = (1, \sqrt{2})$.

$$H(x, p) = \frac{|p|^2}{2} - \left(2 - \sin(2\pi x) - \sin(2\pi\sqrt{2}x)\right)^\gamma, \quad \gamma > 0.$$

Consider the homogenization problem in 1D

$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = 0 \\ u^\varepsilon(x, 0) = u_0(x) \end{cases} \longrightarrow \begin{cases} u_t + \bar{H}(Du) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Then

$$\boxed{\gamma > 2}$$

$$-C\varepsilon \leq u^\varepsilon - u \leq C\varepsilon^\tau, \quad \tau = \frac{\gamma - 2}{3\gamma - 2}$$

$$\boxed{\gamma = 2}$$

$$-C\varepsilon \leq u^\varepsilon - u \leq \frac{C}{|\log(\varepsilon)|}$$

$$\boxed{\gamma < 2}$$

$$u^\varepsilon - u \geq \begin{cases} -C\varepsilon^{\frac{\gamma}{\gamma+1}}, & \gamma \in (0, 1), \\ -C\varepsilon^{1/2}, & \gamma \in [1, 2]. \end{cases}$$

Idea of the proof

$$\boxed{\frac{v_p(t)}{t} = \mathcal{O}\left(\frac{1}{t^\alpha}\right) \text{ as } t \rightarrow \infty} \leq u^\varepsilon - u \leq \begin{cases} \text{shape and regularity of } \bar{H} \\ \text{averaging optimal path :} \\ \left| \frac{\eta(t)}{t} - \bar{H}'(p) \right| \leq \mathcal{O}\left(\frac{1}{t^\beta}\right). \end{cases}$$

- ① Lower bound is easy: decay rate of correctors and Hopf-Lax formula

$$\mathcal{M}(f)$$

- ② Upper bound is harder: long time average of characteristic (calibrated curve)

$$\mathcal{M}(f^{-1})$$

Shape of \overline{H}

To compute $\overline{H}(p)$, we look for a sublinear solution v_p to

$$H(x, p + Dv_p(x)) = \mu$$

Assume $\overline{H}(p) = \mu$, we look for p instead

$$\frac{|p + v'(x)|^2}{2} - \mathbb{V}(\xi x) = \mu \implies v(x) = \int_0^x \sqrt{2(\mu + \mathbb{V}(x))} \, dx - px$$

Then

$$\frac{v(x)}{x} = \frac{1}{x} \int_0^x \sqrt{2(\mu + \mathbb{V}(x))} \, dx - p \rightarrow 0$$

With

$$p_\mu = \mathcal{M}(\sqrt{2(\mu + \mathbb{V})}) = \int_{\mathbb{T}^n} \sqrt{2(\mu + \mathbb{V}(\mathbf{x}))} \, d\mathbf{x}.$$

Sketch of the proof - 1

- 1 If $H(x, p) = \frac{|p|^2}{2} + V(x)$ then the Lagrangian $L(x, v) = \frac{|v|^2}{2} - V(x)$.
- 2 Let $(x, t) = (0, 1)$, use optimal control formula (action minimizing)

$$A^\varepsilon[\eta] = \varepsilon \int_0^{\varepsilon^{-1}} L(\eta(s), -\dot{\eta}(s)) ds + u_0(\varepsilon\eta(\varepsilon^{-1}))$$

and

$$u^\varepsilon(0, 1) = \inf_{\eta(0)=0} A^\varepsilon[\eta]$$

- 3 A minimizer has conservation of energy

$$\frac{|\dot{\eta}(s)|^2}{2} + V(\eta(s)) = r$$

- 4 Rewrite

$$u^\varepsilon(0, 1) = \inf_r \left(\inf_{\eta_r} A^\varepsilon[\eta_r] \right)$$

- 5 For each energy r , averaging each terms of the action with rate

Sketch of the proof - 2

- ① Lower bound is easy

$$A^\varepsilon[\eta_r] \geq u(0, 1) + \inf_{|p| \geq p_0} \varepsilon v_p(\eta(\varepsilon^{-1}))$$

- ② Lower bound correspond to decay rate of corrector $\frac{v_p(x)}{|x|}$ as $|x| \rightarrow \infty$, i.e., convergence rate to the mean value

$$\left| \frac{1}{T} \int_0^T \mathbb{V}^{1/2}(\xi x) dx - \mathcal{M}(\mathbb{V}^{1/2}) \right| \leq \frac{C}{T^\theta}$$

- ③ For $|p| \geq p_0$

$$\left| \frac{v_p(t)}{t} \right| \leq \left| \frac{1}{t} \int_0^t \mathbb{F}_\mu(\xi x) dx - \mathcal{M}(\mathbb{F}_\mu) \right| \leq \begin{cases} C|t|^{-1} \\ C|t|^{-\frac{\alpha}{\alpha+n-1}} |\log(t)|^{3(n-1)} \end{cases}$$

- The first case happens for $\mathbb{F} \in H^s(\mathbb{T}^n)$ ($s > n/2 + \sigma_\xi$)
- The second case happens for a.e. $\xi \in \mathbb{R}^n$ with $\mathbb{F} \in C^{0,\alpha}(\mathbb{T}^n)$.

Sketch of the proof - 3

- 1 Upper bound is harder, obtainable when negative energy $r < 0$ does not play a role, i.e., $\bar{H} \in C^1$
- 2 Look at

$$A^\varepsilon[\eta_r] = (\varepsilon\eta_r(\varepsilon^{-1})) \underbrace{\left(\frac{1}{\eta_r(\varepsilon^{-1})} \int_0^{\eta_r(\varepsilon^{-1})} \sqrt{2(r - \mathbb{V}(\xi x))} \, dx \right)}_{p_r = \mathcal{M}(\sqrt{2(r - \mathbb{V})})} + u_0(\varepsilon\eta_r(\varepsilon^{-1})).$$

- 3 The difficult term is

$$\varepsilon\eta_r(\varepsilon^{-1}) \longleftrightarrow \frac{\eta(t)}{t} \rightarrow q \in \partial\bar{H}$$

This is the large time average of calibrated curve to a rotation vector.

- 4 Difficult to do directly in a uniform way as $r \rightarrow 0^+$, by Euler-Lagrange equation

$$\frac{1}{\varepsilon\eta(\varepsilon^{-1})} = \frac{1}{\eta(\varepsilon^{-1})} \int_0^{\eta(\varepsilon^{-1})} \frac{dx}{\sqrt{2(r - \mathbb{V}(\xi x))}} \rightarrow \mathcal{M}\left(\frac{1}{\sqrt{2(r - \mathbb{V})}}\right)$$

- 5 Using Hamilton–Jacobi equation: **uniform in $r \rightarrow 0^+$**

$$\bar{H} \in C^{1,\beta} \implies \left| \frac{\eta_r(t)}{t} - \bar{H}'_+(p_r) \right| \leq C\varepsilon^{\frac{\beta}{1+\beta}}.$$

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Application to ergodic estimate

For $\mathbb{V}(x_1, x_2) = (2 - \sin(2\pi x_1) - \sin(2\pi x_2))^\gamma$ and $\xi = (\xi_1, \xi_2)$ with $\frac{\xi_2}{\xi_1}$ is badly approximable, $H(x, p) = \frac{|p|^2}{2} - \mathbb{V}(\xi x)$, then

$$\left| \frac{\eta(t)}{t} \right| = \left| \frac{\eta(t)}{t} - \overline{H}'(p) \right| \leq \begin{cases} C|t|^{-\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ C|\log(t)|^{-1} & \gamma = 2. \end{cases}$$

Consequently

$$\frac{1}{T} \int_0^T \frac{dx}{\mathbb{V}^{1/2}(\xi x)} \geq \begin{cases} CT^{\frac{\gamma-2}{3\gamma-2}} & \gamma > 2 \\ C|\log(T)| & \gamma = 2. \end{cases}$$

Outlooks

- 1 For zero energy $r = 0$

$$\overline{H} \in C^{1,\alpha} \longrightarrow \varepsilon^{\frac{\alpha}{1+\alpha}} \longrightarrow \varepsilon^{\frac{\alpha(\alpha+1)}{\alpha(\alpha+1)+1}}$$

We have

$$\left| \frac{\eta_0(t)}{t} \right| \leq \left(\frac{1}{|t|} \right)^\tau \quad \text{where } \tau = \frac{(\gamma-2)(3\gamma-2)}{(\gamma-2)(3\gamma-2)+4\gamma^2}.$$

If this holds uniformly for η_r as $r \rightarrow 0^*$ then we can improve the rate of homogenization

- 2 Nonsmooth \overline{H} ?
- 3 Gaps in the quantitative estimate using two different methods?

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