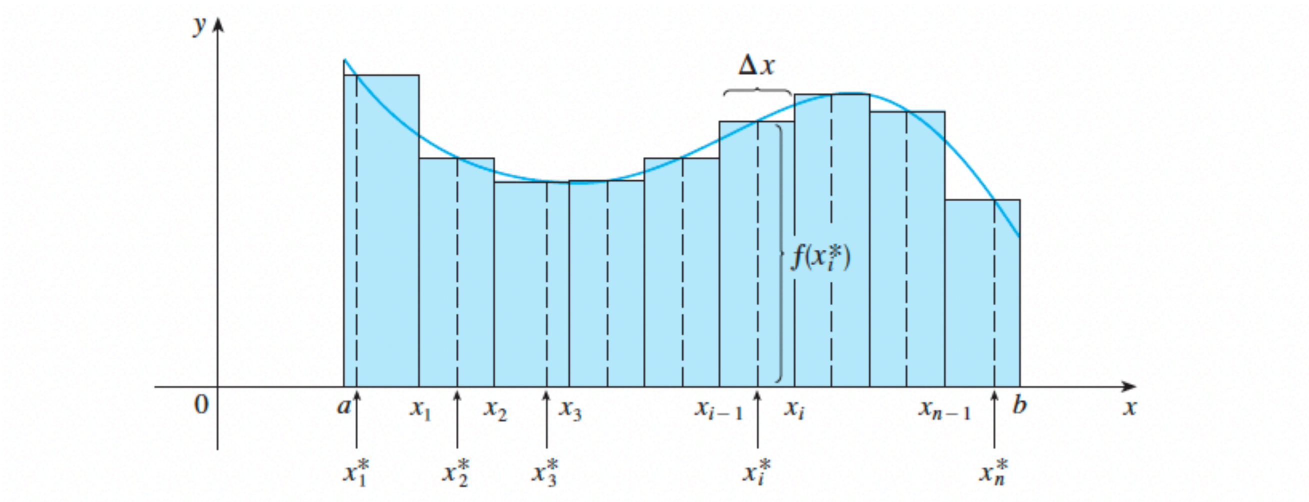
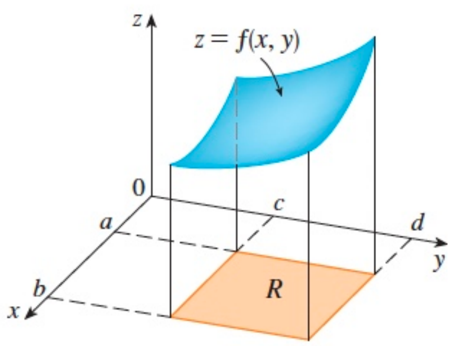
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

(Area under the curve)



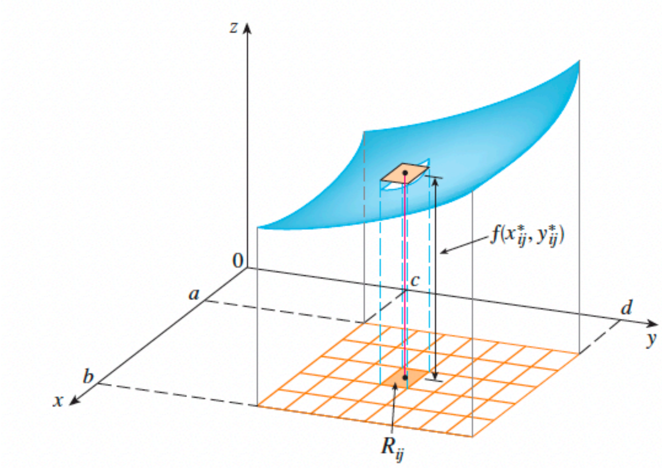


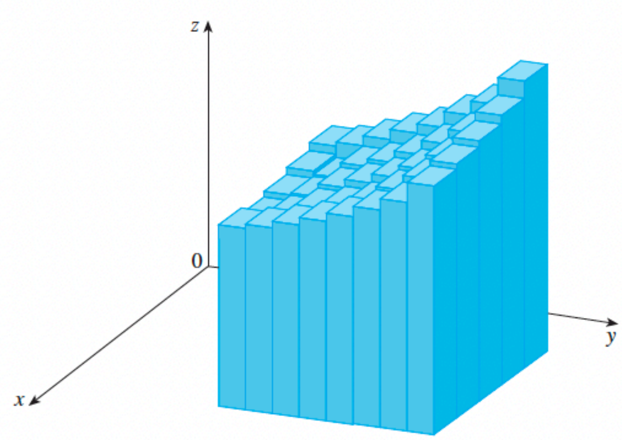
$$R = [a, b] \times [c, d]$$

The volume of the solid that lies under the surface $z = f(x, y)$ is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\triangle A$$

(volume under the surface)





1 Volumes and Double Integrals

1.1 Double Integrals over Rectangles - During Class

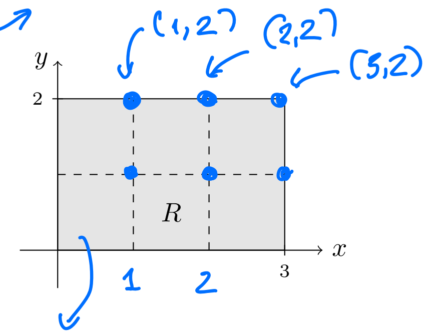
Objective(s):

- Define double integrals in all their technicalities.
- Define volume and average values of a function $f(x, y)$.
- Go over some properties of double integrals.

Definition(s) 1.1. If $f(x, y) \geq 0$ then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x,y)\,dA$$

Example 1.2. Use six rectangles of width and length 1 to approximate the volume under $f(x, y) = 6 - x - y$ over the region R shown below using a top-right-sum.



$$\approx f(1,2)\,dA + f(2,2)\,dA + f(3,2)\,dA$$

$$+ f(1,1)\,dA + f(1,2)\,dA + f(1,3)\,dA$$

$$\approx \boxed{f(1,2) + f(2,2) + f(3,2) + f(1,1) + f(1,2) + f(1,3)}$$

$$dA = 1 \times 1 = 1$$

Remark 1.3. Now in general for integrable function $f(x, y)$,

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

where

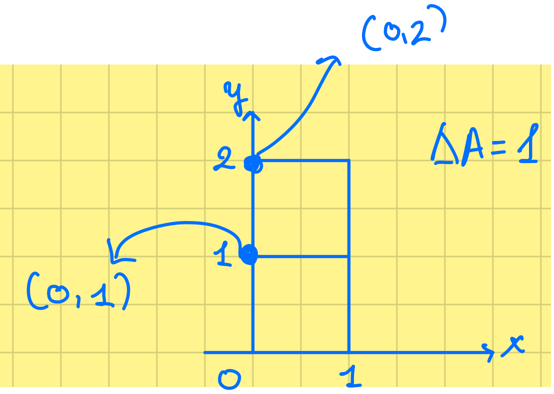
- n is the number of sub-rectangles.
- x_i^* and y_i^* are x and y values in the i^{th} sub-rectangle.
- $\triangle A$ is the area of the each sub-rectangle.

Example 1.4. Let $f(x,y) = x(1+y)$ and D be the rectangle determined by the inequalities $0 \le x \le 1$ and $0 \le y \le 2$. Choose a subdivision of D into squares with side length 1. Use sample points of the Riemann Sum at the top left corner of each square of the subdivision to approximate $\iint_D f(x,y)\, dA$

$$\approx f(0,2)\,\Delta A + f(0,1)\,\Delta A$$

$$\approx f(0,2) + f(0,1)$$

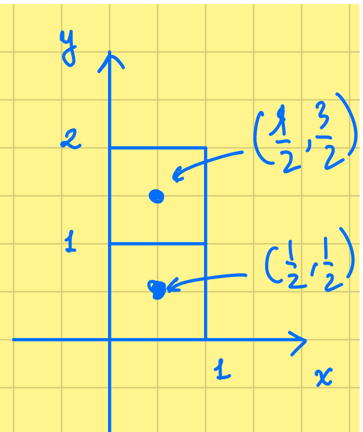
as $\Delta A = 1$

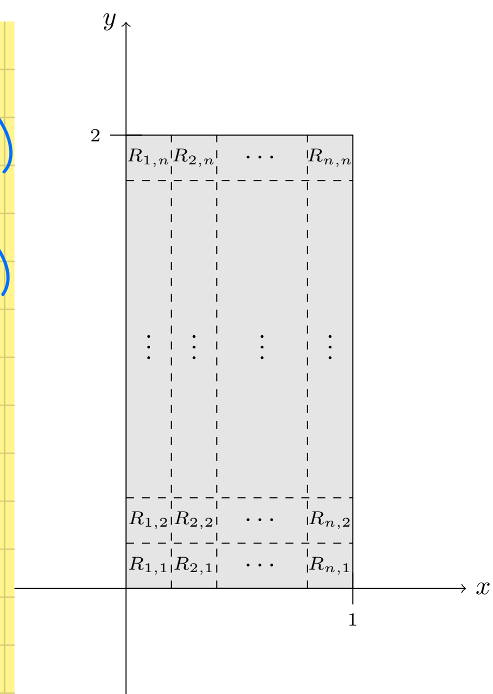


Example 1.5. Let $f(x,y) = x(1+y)$ and D be the rectangle determined by the inequalities $0 \le x \le 1$ and $0 \le y \le 2$. Formally compute $\iint_D f(x,y)\, dA$ using 2 Riemann sums (Hint: use **Remark 1.3** to inspire)

$$\approx f\left(\tfrac{1}{2},\tfrac{1}{2}\right) + f\left(\tfrac{1}{2},\tfrac{3}{2}\right)$$

as $\Delta A = 1$





Definition(s) 1.6. The average of a function f of two variables defined on R is given to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R)$ is the area of the region R

Theorem 1.7 (Properties of Double Integrals)**.** The properties are the same as for single integrals!

(a) $\iint_R [f(x,y) + g(x,y)]\ dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

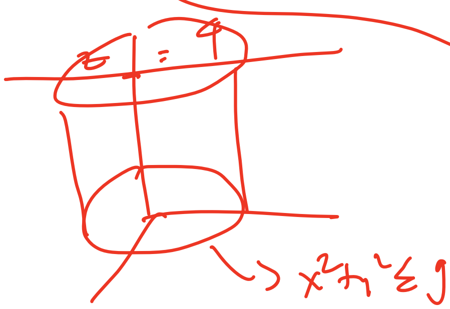
(b) $\iint_R cf(x,y)\ dA = c\iint_R f(x,y)\, dA$ where c is a constant

(c) If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\iint_R f(x,y)\, dA \geqslant \iint_R g(x,y)\, dA$$

Example 1.8. Find $\iint_R 4\ dA$ where $R = \{(x,y) \mid x^2 + y^2 \leq 9\}$.

↳ volume under $z = 4$, over $x^2+y^2 \leq 9$



volume of the cylinder

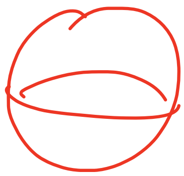
$$\boxed{(\pi \cdot 3^2) \times 4}$$

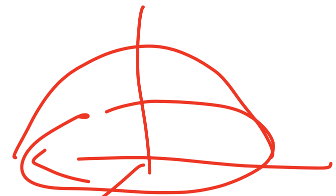
Example 1.9. Evaluate $\iint_D \sqrt{9-x^2-y^2}\, dA$ where D is disk $x^2+y^2 \leq 9$.

half the sphere

$z = \sqrt{9-x^2-y^2} \geq 0$

$\Rightarrow x^2+y^2+z^2 \leq 9$





$\frac{4}{3}\pi 3^3 / 2 = \frac{2}{3} \cdot 27 \cdot \pi$

Example 1.10. Find the average height of a point on the hemisphere $x^2+y^2+z^2 = 16$ with $z \geq 0$.

$\dfrac{1}{\text{area}(x^2+y^2 \leq 16)} \iint_{x^2+y^2 \leq 16} z = \ldots$

half volume of $x^2+y^2+z^2=16$

$\dfrac{1}{\pi \cdot 4^2} \times \frac{2}{3}\pi \cdot 4^3$

Example 1.11. Consider the disk D given by $x^2+y^2 \leq 25$. On average how far away is a point in D from the origin?

(***Hint:*** *remember the volume equations of cones and cylinders?*)

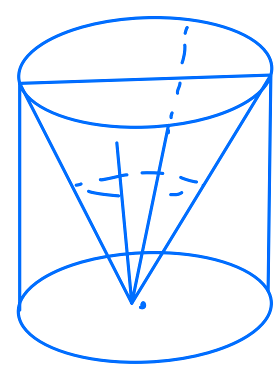
A point (x,y) has distance to $(0,0)$

$z = \sqrt{x^2+y^2}$

$f_{ave} = \dfrac{1}{\text{area}(D)} \iint_D \sqrt{x^2+y^2}\, dA$

$z = \sqrt{x^2+y^2} \Rightarrow z^2 = x^2+y^2$ (cone)

volume of the portion of cylinder after removing the cone



height = 5

cylinder − cone

$(\pi \cdot 5^2) \times 5 - \frac{1}{3} \cdot 5 \times (\pi 5^2)$

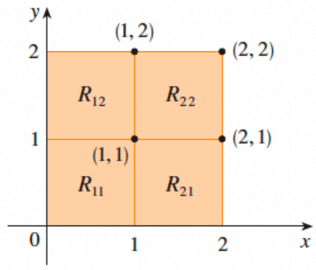


FIGURE 6

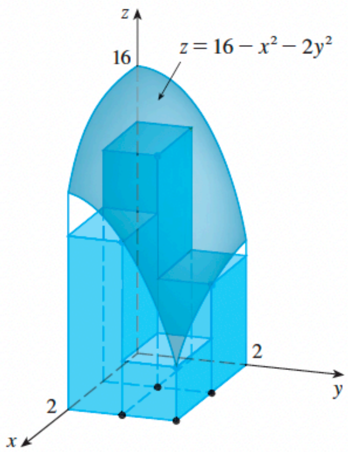


FIGURE 7

V EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

$$= f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

This is the volume of the approximating rectangular boxes shown in Figure 7. ■

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In the next section we will be able to show that the exact volume is 48.

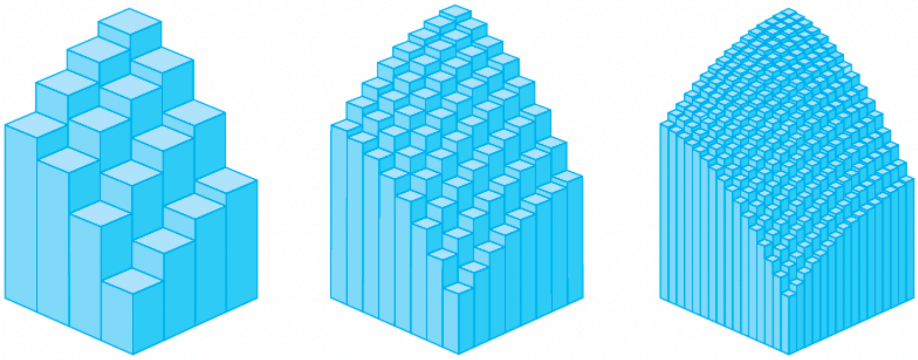


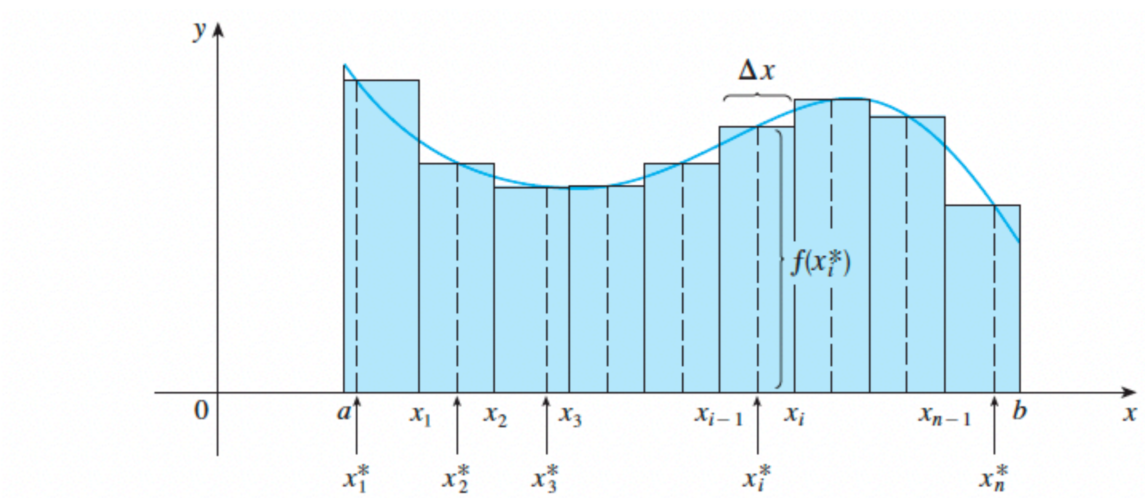
FIGURE 8

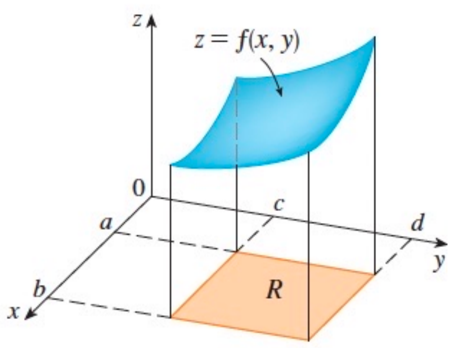
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

(Area under the curve)





$$R = [a,b] \times [c,d]$$

The volume of the solid that lies under the surface $z = f(x,y)$ is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\triangle A$$

(volume under the surface)