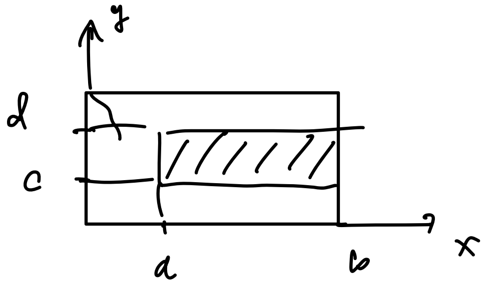



$R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA$$

volume under the surface.

**Theorem** (Fubini's theorem) $\quad R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \int_{\underset{x}{a}}^{b} \left( \int_{\underset{y}{c}}^{d} f(x,y)\,dy \right) dx = \int_c^d \left( \int_a^b f(x,y)\,dx \right) dy$$

(f has to be bounded, continuous on R
discontinuous on a very small set)

Ex 1. $\displaystyle \int_0^3 \underbrace{\int_1^2 \boxed{x^2 y}\,dy}\,dx$ = integral of $f(x,y) = x^2 y$ over $R = [0,3] \times [1,2]$

$$\boxed{x^2 y = f(x)\,g(y)}$$

$$\int_0^3 x^2 \left( \underbrace{\int_1^2 y\,dy} \right) dx$$

$$\int_0^3 x^2 \left( \frac{y^2}{2} \Big|_1^2 \right) dx$$

$$\int_0^3 x^2 \left( \frac{4-1}{2} \right) dx = \frac{3}{2} \cdot \frac{x^3}{3} \Big|_0^3 = \frac{3}{2} \cdot \frac{27}{3} = \boxed{\frac{27}{2}}$$

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 y \left( \int_0^3 x^2\,dx \right) dy$$

$$\int_1^2 y \left( \underbrace{\frac{x^3}{3} \Big|_0^3}_{\frac{27}{3}} \right) dy$$

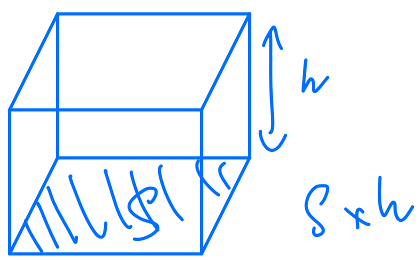
$$= \frac{27}{3} \cdot \frac{y^2}{2} \Big|_1^2 = \frac{27}{3} \cdot \left( \frac{4-1}{2} \right) = \frac{27}{2}$$

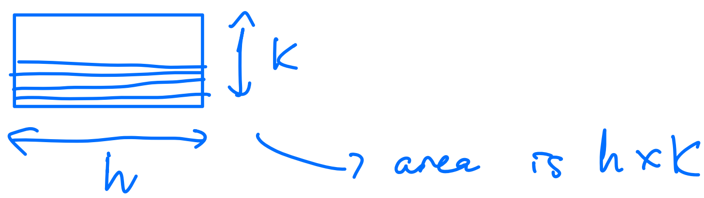
**Theorem:**

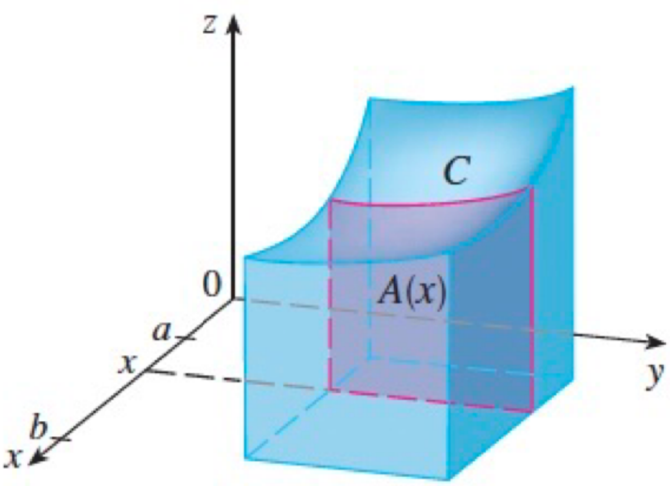
$$\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx = \int_a^b f(x) \left( \underbrace{\int_c^d g(y)\, dy}_{\text{const}} \right) dx$$

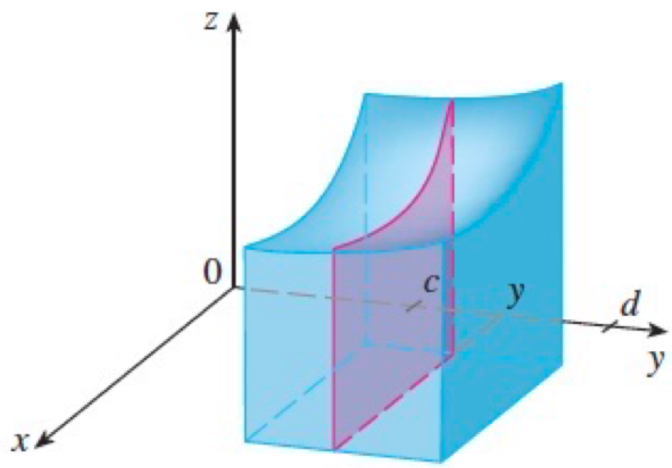
$$= \left( \int_c^d g(y)\, dy \right) \left( \int_a^b f(x)\, dx \right)$$

Note: geometric interpretation of volumes and areas




$$\int_a^b \underbrace{\left( \int_c^d f(x,y)\, dy \right)}_{A(x)} dx \qquad \int_c^d \underbrace{\left( \int_a^b f(x,y)\, dx \right)}_{A(y)} dy$$

Ex. Calculate the integral $f(x,y) = \frac{\sqrt{y}}{x^2}$ over

R = rectangle bounded by $x=1,\ x=3$
$y=0,\ y=1$

$$\left(\int_1^3 \frac{1}{x^2}\,dx\right)\left(\int_0^1 \sqrt{y}\,dy\right)$$

---

Ex. $\iint_R x\sin(x+y)\,dA$ where $R = \{(x,y) : |x-\frac{\pi}{2}| \leq \frac{\pi}{2}$ and $|y-\pi| \leq \frac{\pi}{2}\}$

Proof :

$$-\frac{\pi}{2} \leq x - \frac{\pi}{2} \leq \frac{\pi}{2} \Rightarrow 0 \leq x \leq \pi$$

$$-\frac{\pi}{2} \leq y - \pi \leq \frac{\pi}{2} \Rightarrow \pi - \frac{\pi}{2} \leq y \leq \pi + \frac{\pi}{2}$$

$$(x,y) \in [0,\pi] \times [\tfrac{\pi}{2}, \tfrac{3\pi}{2}]$$

$$\int_0^{\pi}\int_{\pi/2}^{3\pi/2} x\sin(x+y)\,dy\,dx \quad \text{v.s.} \quad \int_{\pi/2}^{3\pi/2}\int_0^{\pi} \underline{x\sin(x+y)}\,dx\,dy$$

$$\int_0^\pi x \boxed{\int_{\pi/2}^{3\pi/2} \sin(x+y)\,dy}\,dx$$

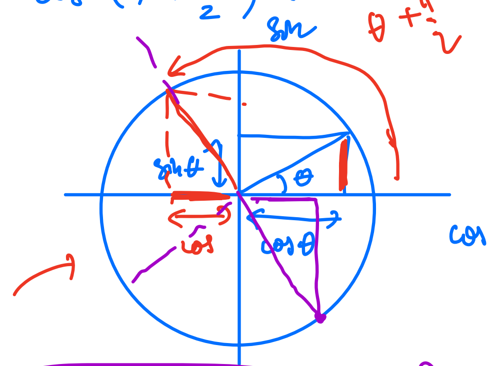
$$\int_0^\pi x\sin x\,dx$$

( hard )

$$\int_0^\pi x\left(-\cos(x+y)\Big|_{\pi/2}^{3\pi/2}\right)dx$$

$$\int_0^\pi x\left[\underbrace{-\cos\left(x+\tfrac{3\pi}{2}\right)}_{-\sin x}+\underbrace{\cos\left(x+\tfrac{\pi}{2}\right)}_{-\sin x}\right]dx$$

$\cos\left(x+\frac{\pi}{2}\right)$ ?




$$\boxed{\cos\left(x+\tfrac{\pi}{2}\right) = -\sin x}$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\cos\left(x+\tfrac{\pi}{2}\right) = \cos x\underbrace{\cos\tfrac{\pi}{2}}_{0} - \sin x\underbrace{\sin\tfrac{\pi}{2}}_{1}$$

$$= -\sin x$$

$$-2\int_0^\pi x\sin x\,dx$$

$u = x \qquad du = dx$

$dv = \sin x\,dx \qquad v = -\cos x$

$$\int u\,dv = uv - \int v\,du$$

$$= -x\cos x \ominus \int(\ominus\cos x)\,dx$$

$$-2\left[-x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx\right]$$

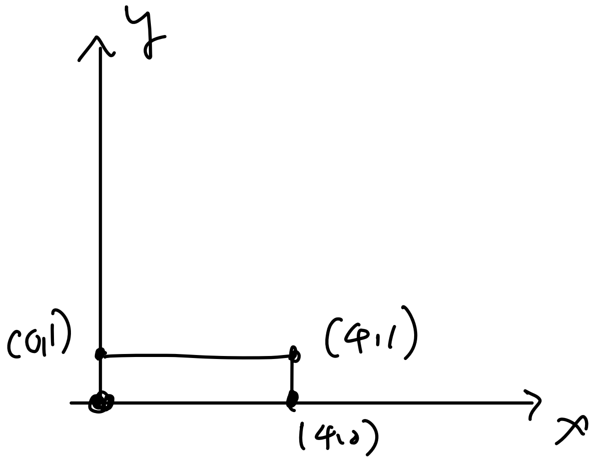
$$2x\cos x\Big|_0^\pi \qquad \sin(x)\Big|_0^\pi$$

$$2\pi(-1) - \cancel{2\cdot 0\cdot\cos 0}$$

$$\boxed{-2\pi}$$

Ex find the average value of $f(x,y) = e^y\sqrt{x+e^y}$

over the rectangle with




vertices $(0,0)$ $(4,0)$

$(0,1)$ $(4,1)$

$(x,y) \in [0,4] \times [0,1] = R$

$\text{area}(R) = 4 \times 1 = 4$

$$f_{ave} = \frac{1}{4} \int_0^4 \int_0^1 e^y \sqrt{x+e^y}\, dy\, dx$$

focus $\int_0^1 e^y \sqrt{x+e^y}\, dy$ (treat $x$ as a constant)

$u = e^y + x$

$du = e^y\, dy$

| $y$ | $0$ | $1$ |
|---|---|---|
| $u$ | $1+x$ | $e+x$ |

$$\int_{1+x}^{e+x} u^{1/2}\, du = \frac{2}{3} u^{3/2} \Big|_{1+x}^{e+x}$$

$$= \frac{2}{3}\left[(e+x)^{3/2} - (1+x)^{3/2}\right]$$

$$\left(\frac{1}{4}\right)\int_0^4 \left(\frac{2}{3}\right)\left[(e+x)^{3/2} - (1+x)^{3/2}\right] dx$$

$$= \frac{1}{6} \cdot \int_0^4 \left[(e+x)^{3/2} - (1+x)^{3/2}\right] dx$$

$$\frac{1}{6}\left(\frac{(e+x)^{5/2}}{5/2} - \frac{(1+x)^{5/2}}{5/2}\right)\Bigg|_0^4$$

$$\frac{1}{6}\cdot\frac{2}{5}\left[(e+4)^{5/2} - e^{5/2} - 5^{5/2} + 1^{5/2}\right]$$

<u>Read</u> .

$$\iint_R \frac{x}{x^2y^2+1}\, dA$$

(read solution in Ryan's note)

$R = [0,1] \times [0,1]$