3 The Dot Product

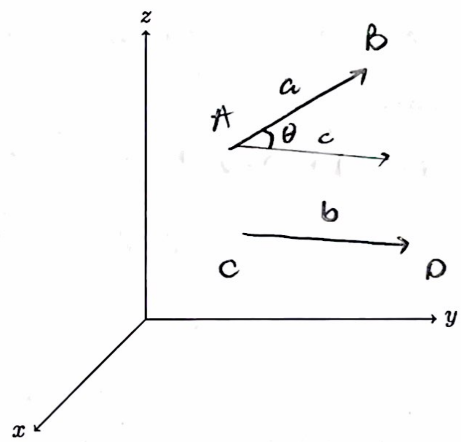
3.1 Definitions and Warmup – Video Before Class

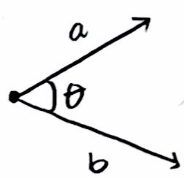
Objective(s):

- Define the dot product
- Understand what the dot product is used for.
- Get a little practice.

Interestingly enough there are 2 different (conventional) ways to multiply vectors together. Today we will talk about the Dot Product which is largely used to find the angle between two vectors.

So lets take our two vectors here ____________ and move them so that they have the same starting point. And draw in ____________. Our goal is to find ___. Let's add one more "natural" line to complete ________________. Call this vector **c**. Can anyone tell me what ______ in terms of ______? Hint: it's not **a** + **b**. Now lets consider the ________________________ to finish it out.





Theorem 3.1. The angle between two vectors **a** and **b** is given by:

$$\cos(\theta) = \frac{a \cdot b}{|a| \cdot |b|} \Rightarrow \theta = \arccos\left(\frac{a \cdot b}{|a| \cdot |b|}\right)$$

Definition(s) 3.2.

$$a_1b_1 + a_2b_2 + a_3b_3 = a \cdot b$$

This is called the dot product of vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.

Theorem 3.3. Two vectors **a** and **b** are **perpendicular** (or **orthogonal**) iff:

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \text{(scalar)}$$

Please notice that when you take the dot product between two vectors you get a number! Not a vector!

Example 3.4. Find $\mathbf{a} \cdot \mathbf{b}$ for the following vectors. Determine if the vectors are perpendicular or not.

(a) $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle -1, 3, 1\rangle$

$$a \cdot b = 1 \cdot (-1) + 0 \cdot 3 + 1 \cdot 1 = -1 + 0 + 1 = 0$$

perpendicular

(b) $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 4\mathbf{k}$ $\quad a = (3, -2, 1) \quad b = (0, 2, 4)$

$$a \cdot b = 3 \cdot 0 + (-2) \cdot 2 + 1 \cdot 4 = 0 - 4 + 4 = 0$$

perpendicular

(c) $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{CB}$ where $A = (3, -2)$, $B = (5, 6)$, and $C = (-1, -3)$

$$a = (5,6) - (3,2) = (2,4)$$

$$b = (5,6) - (-1,-3) = (6,9)$$

$$a \cdot b = 2 \cdot 6 + 4 \cdot 9 = 12 + 36 = 48$$

Not perpendicular

3.2 Dot Product Properties and Applications – During Class

Objective(s):

- Sketch simple surfaces in space.
- Determine when a point lies on a specified surface.

Theorem 3.5. This is the exact same as **Theorem 3.1** but with dot product notation.

The angle between two vectors **a** and **b** is given by:

$$\theta = \arccos\left(\frac{a \cdot b}{|a| \cdot |b|}\right)$$

or equivalently,

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Example 3.6. Find the angle between the following vectors.

(a) $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle -1, 3, 2\rangle$

$$a \cdot b = 1\cdot(-1) + 0\cdot 3 + 1\cdot 2 = -1 + 2 = 1$$

$$|a| = \sqrt{2}$$
$$|b| = \sqrt{1^2 + 9 + 4} = \sqrt{14}$$

$$\cos\theta = \frac{a \cdot b}{|a|\cdot|b|} = \frac{1}{\sqrt{2}\cdot\sqrt{14}} = \frac{1}{\sqrt{28}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{28}}\right)$$

(b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 4\mathbf{k}$

$$a = (1, -1, 3) \qquad |a| = \sqrt{1+1+9} = \sqrt{11}$$
$$b = (0, 2, 4) \qquad |b| = \sqrt{0+4+16} = \sqrt{20}$$

$$a \cdot b = 1\cdot 0 + (-1)\cdot 2 + 3\cdot 4 = 10$$

$$\cos\theta = \left(\frac{10}{\sqrt{11}\cdot\sqrt{20}}\right) \Rightarrow \theta = \arccos\left(\frac{10}{\sqrt{11}\cdot\sqrt{20}}\right)$$

So now that we have this great new form of multiplication between vectors we have to wonder what properties of multiplication hold!?

Theorem 3.7 (Properties of the Dot Product). Let **a, b, c** be vectors and c a scalar:

(a) $\mathbf{a}\cdot\mathbf{b} =$ $b\cdot a$

(b) $(c\mathbf{a})\cdot\mathbf{b} =$ $c(a\cdot b)$

(c) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) =$ $a\cdot b + a\cdot c$

(d) $\mathbf{0}\cdot\mathbf{a} =$ 0 (scalar)

(e) $\mathbf{a}\cdot\mathbf{a} =$ $|a|^2$

Proof of (a)

$\vec{a} = (x_1, y_1, z_1)$, $\vec{b} = (x_2, y_2, z_2)$

$\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$

$\vec{b}\cdot\vec{a} = x_2x_1 + y_2y_1 + z_2z_1$

thus $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

And now a nice physical application, **work**. Force does work. Recall that if we apply a constant force then we have the formula

And before we had it where force and distance were just numbers so you used regular old multiplication. However we can consider **F** =Force and **D** =*Displacement* to be vectors. The the constant (or scalar) W =Work is given by the formula:

$$W = F\cdot D$$

Example 3.8. A box is pushed with a constant force of $\mathbf{F} = \langle 1, 2, 3\rangle$ Newtons. How much work is done in moving the box from $(1, 0, 1)$ to $(2, 1, 1)$?

$$\vec{D} = (2,1,1) - (1,0,1) = (1,1,0)$$

$$W = (1,2,3)\cdot(1,1,0) = 1 + 2 + 0 = 3$$

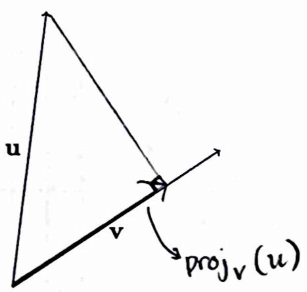
3.3 Projections – During Class

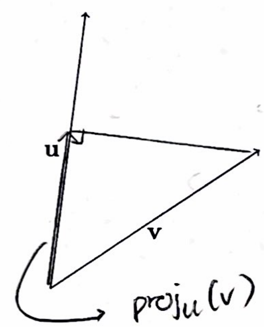
Objective(s):

- Be able to visualize and interpret the various projections.
- Define and calculate Scalar and Vector projections.
- Define and calculate orthogonal projections.

The first projection we will learn about is a standard vector projection.

Example 3.9. Given the vectors **u** and **v** below. In one picture draw $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and in the other draw $\text{proj}_{\mathbf{u}}(\mathbf{v})$.





Essentially this vector projection $\text{proj}_{\mathbf{v}}(\mathbf{u})$ is the amount of that vector **u** that is in the same direction as vector **v**. To calculate this vector you can use right triangles and dot products to get:

Definition(s) 3.10. The projection of **b** onto **a** is notated by $\text{proj}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$$

If you are only interested in a scalar answer you can compute the scalar projection rather than the full vector projection

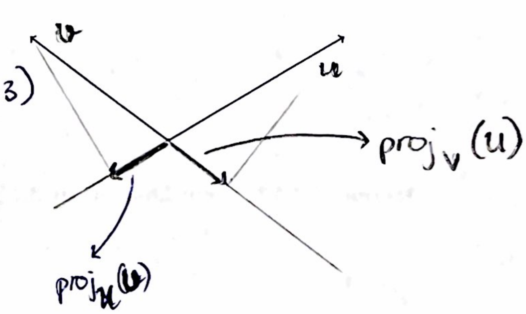
Definition(s) 3.11. The **scalar projection** of **b** onto **a** is notated by $\text{comp}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = |\text{proj}_a(b)| = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}|}$$

Example 3.12. Consider the vectors $\mathbf{u} = \langle 5, 3\rangle$ and $\mathbf{v} = \langle -4, 3\rangle$ sketched below

$\|v\| = \sqrt{25}$

(a) Draw $\text{proj}_{\mathbf{v}}(\mathbf{u})$



(b) Calculate $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = \frac{-20+9}{\sqrt{25}\cdot\sqrt{25}}(-4,3) = \frac{-11}{25}(-4,3)$

(c) Calculate $\text{comp}_{\mathbf{v}}(\mathbf{u})$

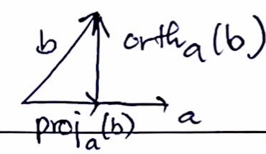
Proof 1 : $|\text{proj}_v(u)| = \frac{11}{25}\sqrt{4^2+3^2} = \frac{11}{\sqrt{25}}$

Proof 2 $\frac{|\vec{a}\cdot\vec{v}|}{|\vec{v}|} = \frac{|+11|}{\sqrt{25}}$

Finally to get the full picture we define the other side of this triangle.

Definition(s) 3.13. The orthogonal projection of **b** onto **a** is notated by $\text{orth}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{orth}_{\mathbf{a}}(\mathbf{b}) = \vec{b} - \text{proj}_a(b)$$



Note $\text{proj}_{\mathbf{a}}(\mathbf{b})$ is perpendicular to ~~a~~ $\text{orth}_a(b)$ and $\text{orth}_{\mathbf{a}}(\mathbf{b})$ is perpendicular to $\vec{a}$.

$\vec{b} = \text{proj}_a(\vec{b}) + \text{orth}_a(\vec{b})$

Example 3.14. Calculate the following then sketch the result on the grid below

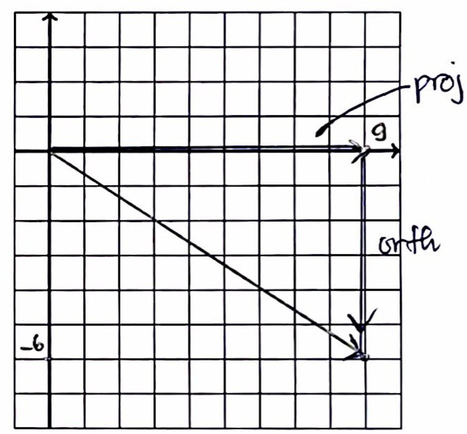
(a) $\text{proj}_{\mathbf{i}}(\langle 9, -6\rangle)$ $\vec{a}$

$i = (1,0)$

$$\frac{i \cdot a}{i \cdot i} \cdot i = \frac{9}{1} \cdot (1,0) = (9,0)$$

(b) $\text{orth}_{\mathbf{i}}(\langle 9, -6\rangle)$

$$(9,-6) - (9,0) = (0,-6)$$

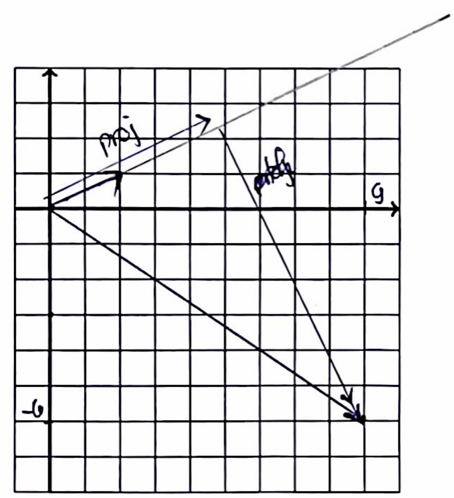


Example 3.15. Calculate the following then sketch the result on the grid below

(a) $\text{proj}_{\langle 2,1\rangle}(\langle 9, -6\rangle) = \frac{18-6}{\sqrt{5}\cdot\sqrt{5}} \cdot (2,1) = \frac{12}{5}(2,1)$

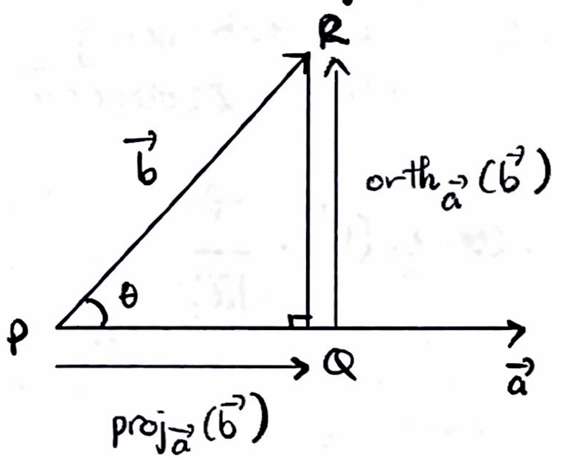
(b) $\text{orth}_{\langle 2,1\rangle}(\langle 9, -6\rangle)$

$$= (9,-6) - \frac{12}{5}(2,1)$$



Remark 3.16. From **Definition 3.13** we know: $\text{orth}_{\mathbf{a}}(\mathbf{b}) + \text{proj}_{\mathbf{a}}(\mathbf{b}) = \vec{b}$

* Projection



projection of $\vec{b}$ onto $\vec{a}$ is a vector that has the same direction as $\vec{a}$

$$\boxed{comp_{\vec{a}}(\vec{b}) = |proj_{\vec{a}}(\vec{b})|}$$

↳ the magnitude of the projection $= |PQ|$

Remember:

$$\vec{b} = proj_{\vec{a}}(\vec{b}) + orth_{\vec{a}}(\vec{b})$$

Note:

in the right-triangle PQR:

$|PQ| = |PR| \cdot \cos\theta$

$$comp_{\vec{a}}(\vec{b}) = \cancel{|\vec{b}|} \cdot \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot \cancel{|\vec{b}|}}\right) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

(you can cancel scalar $|\vec{b}|$)

$$\boxed{comp_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}} = \text{magnitude } |proj_a(b)|$$