MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE 03
2024-01-12

Key takeaways:

- Cross-product $\mathbf{a} \times \mathbf{b}$ computations, it is $\perp$ both $\mathbf{a}$ and $\mathbf{b}$.
- Examples

The Cross Product of

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3) \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = (b_1, b_2, b_3)$$

is a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \underbrace{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}_{a_2b_3 - b_2a_3} - \mathbf{j} \underbrace{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}_{a_1b_3 - b_1a_3} + \mathbf{k} \underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{a_1b_2 - b_1a_2}.$$

Example Evaluate $(1, 2, 3) \times (-2, 1, 0)$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -2 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= \vec{i}(-3) - \vec{j}(6) + \vec{k}(5)$$

$$= (-3, -6, 5)$$

Notes and Theorem $\mathbf{a}\cdot\mathbf{b}$ is a number, while $\mathbf{a}\times\mathbf{b}$ is a vector.

(a) $\mathbf{a}\times\mathbf{b}$ is both perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

(b) If θ is the angle between $\mathbf{a}\cdot\mathbf{b}$ $(0\leq\theta\leq\pi)$ then $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|.|\mathbf{b}|\sin\theta$.

(c) Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $|\mathbf{a}\times\mathbf{b}| = 0$.

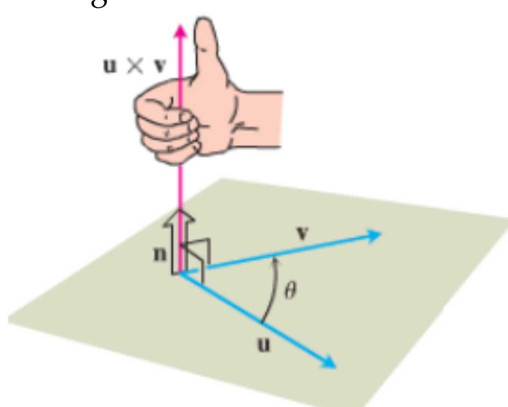
Example Find a vector $\mathbf{u}$ that satisfies $\mathbf{u}\cdot(9,3,1)=0$ and $\mathbf{u}\cdot(-2,4,0)=0$.

We can choose $\vec{u} = (9,3,1)\times(-2,4,0)$

$$= \begin{vmatrix} i & j & k \\ 9 & 3 & 1 \\ -2 & 4 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 9 & 1 \\ -2 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 9 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= (-4, -2, 42)$$

Theorem - Direction of the Cross Product Take two non-zero non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$. Then the direction of $\mathbf{a}\times\mathbf{b}$ is determined by the right-hand rule. That is: the way your right thumb handrub right points when your right-hand fingers curl through the angle θ from $\mathbf{a}$ to $\mathbf{b}$.



Theorem

(a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

(b) $(\alpha\vec{a}) \times (\beta\vec{b}) = (\alpha\beta)(\vec{a} \times \vec{b})$

(c) $\vec{0} \times \vec{a} = \vec{0}$

(d) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(e) $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

Example 4.8. Given that $\underbrace{\langle 1, 1, 0\rangle}_{\vec{a}} \times \underbrace{\langle 3, 4, -2\rangle}_{\vec{b}} = \langle -2, 2, 1\rangle$ quickly calculate the following:

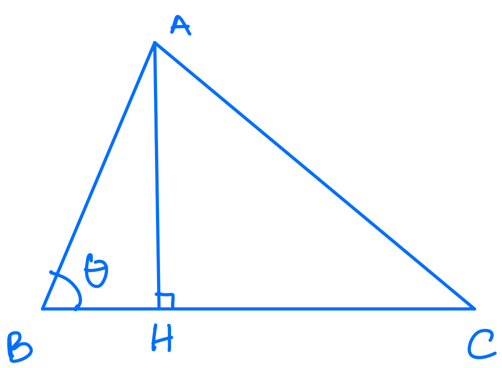
(a) $\langle 3, 4, -2\rangle \times \langle 1, 1, 0\rangle$

(b) $\langle 4, 4, 0\rangle \times \langle 3, 4, -2\rangle$

a) $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} = -(-2, 2, 1) = (2, -2, 1)$

b) $4\vec{a} \times \vec{b} = (-8, 8, 4)$

Area of triangle



$S_{ABC} = \frac{1}{2} AB \cdot BC \cdot \sin\theta$

why?

$AH = AB \sin\theta$

$S_{ABC} = \frac{1}{2} AH \cdot BC$

$= \frac{1}{2} AB \cdot BC \sin\theta.$

thus

$S_{ABC} = |\overrightarrow{BA} \times \overrightarrow{BC}|$

Example. Find the area of the triangle with vertices $P(1,0,1), Q(-2,1,3)$, and $R(4,2,5)$.

$$\vec{PQ} = (-2-1, 1-0, 3-1) = (-3,1,2) \qquad (\text{end} - \text{start})$$

$$\vec{PR} = (4-1, 2-0, 5-1) = (3,2,4)$$

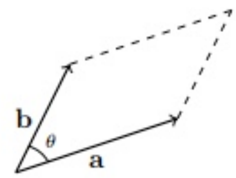
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (2, 15, -9)$$

$$\text{Area} = 1/2\sqrt{2^2 + 15^2 + 9^2}$$

Theorem 4.9. The parallelogram formed by vectors **a** and **b** with angle θ between them is given by:

Area of $\|$-ogram $= |\vec{a} \times \vec{b}| = |\vec{a}|.|\vec{b}|.\sin\theta$



Find the area of the parallelogram generated by $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 3\mathbf{k}$.

$$u = (1,-1,0) \qquad v = (0,1,3)$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= (-3, -3, 1)$$

thus area $= |(-3,-3,1)| = \sqrt{3^2 + 3^2 + 1^2} = \sqrt{19}$

Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

$$u = (0,1,-1) \qquad v = (1,1,0)$$

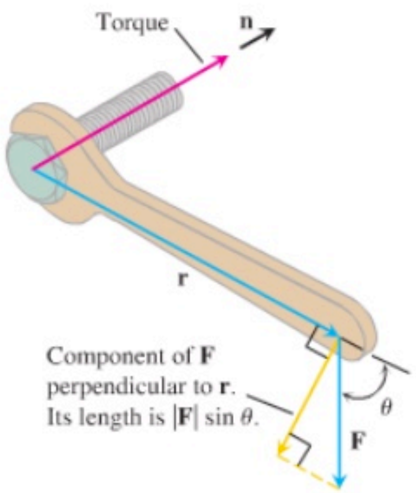
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (1,-1,-1)$$

magnitude $\sqrt{3}$

thus 2 solutions are $\frac{1}{\sqrt{3}}(1,-1,-1)$ and $\frac{-1}{\sqrt{3}}(1,-1,-1)$

So our application to the real world of the day is Torque! Here is the picture



Recall from you favorite physics class that

$Torque = (Force)(Distance$ from pivot$)$.

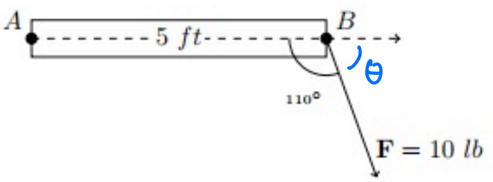
So long as the force is being applied perpendicular to the distance vector. But what if its not?

The magnitude of the torque vector is = $|\vec{r} \times \vec{F}|$

And what about direction?

The torque vector is of course given by:

Example 4.13. Find the magnitude of the torque generated by force **F** at the pivot point A in the figure below



$|F| \cdot |r| \cdot \sin\theta = (10 \cdot 9.8) \cdot 5 \times \sin(70^\circ)$

mg – gravity