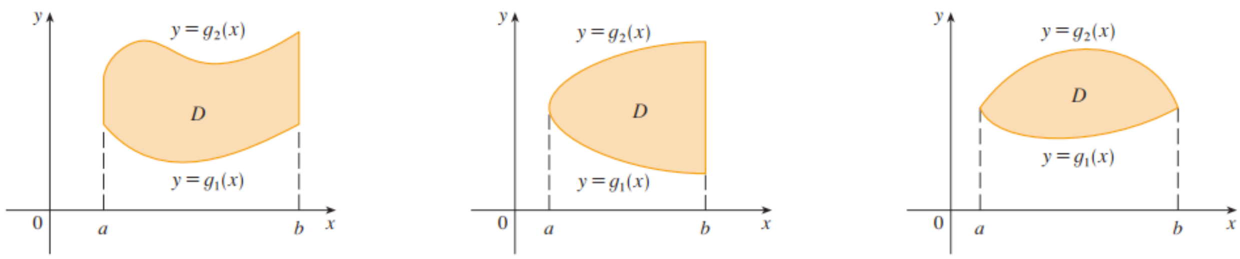
Theorem. If $f(x,y)$ is continuous throughout a region D then:

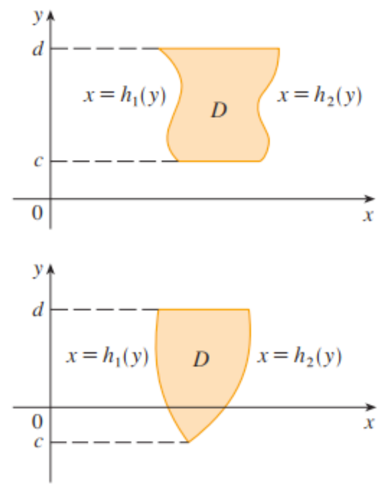
a) If $D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

then
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



b) If $D = \{(x,y) : c \le y \le d : h_1(y) \le x \le h_2(y)\}$

then
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



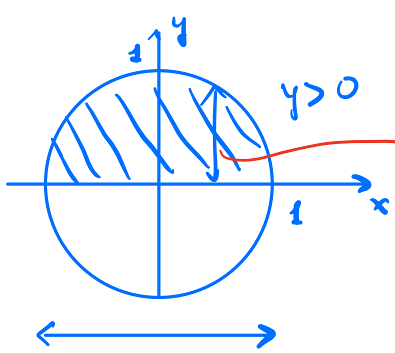
Ex1.

Sketch the region of integration and write an equivalent double integral with the order of integration reversed for
$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\,dx\,dy$$

(Proof : y x : $0 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \Rightarrow |x| \le \sqrt{1-y^2} \Rightarrow x^2 \le 1-y^2$

$\Rightarrow x^2+y^2 \le 1 \rightarrow$ circle



$0 \le y \le \sqrt{1-x^2}$

$-1 \le x \le 1$

To reverse : $-1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}$

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 3y\,dy\,dx = \int_{-1}^{1} \left. 3\frac{y^2}{2}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\cdot\left[2 - \left.\frac{x^3}{3}\right|_{-1}^{1}\right]$$

$$= \frac{3}{2}\left(2 - \frac{1-(-1)}{3}\right) = \frac{3}{2}\cdot 2 - \frac{2}{2} = 3-1 = \boxed{2}$$

If we use the original order:

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy$$

$$\int_0^1 3y \cdot 2\sqrt{1-y^2} \, dy$$

$$6 \int_0^1 y\sqrt{1-y^2} \, dy$$

Do $u = 1-y^2$

$du = -2y \, dy$

y	0	1
u	1	0

$$= 6 \cdot \int_1^0 \frac{1}{2} \sqrt{u} \, du$$

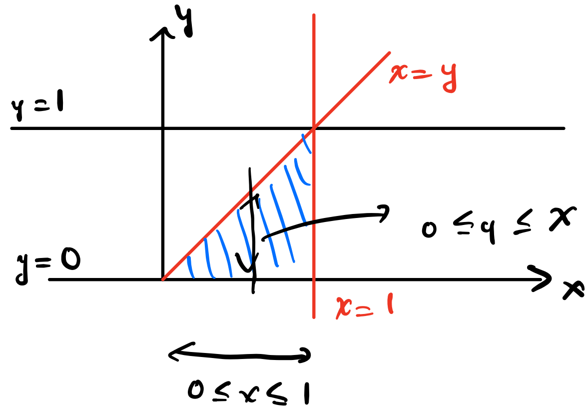
$$= 3 \int_0^1 \sqrt{u} \, du = 3 \cdot \frac{u^{3/2}}{3/2} \Big|_0^1 = 2u^{3/2}\Big|_0^1 = \boxed{2}$$

Ex 2 1. Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy$

$\uparrow y \quad \uparrow x$

$\int_y^1 \frac{\sin x}{x} dx$ is hard to compute

$D = \{ (x,y) : 0 \le y \le 1, \ y \le x \le 1 \}$



To reverse the order: go with x first

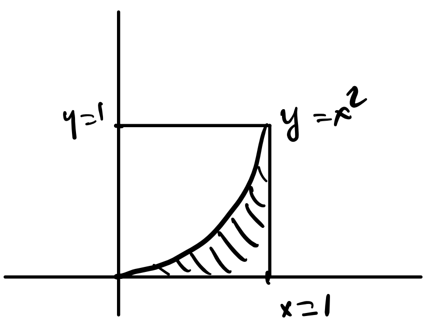
$0 \le x \le 1$, $0 \le y \le x$

$$\int_0^1 \int_0^x \frac{\sin x}{x} \, dy \, dx$$

$$\int_0^1 \left(\frac{\sin x}{x} \cdot x \right) dx = \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = \boxed{-\cos 1 + 1}$$

Ex 3 . Sketch the region of integration and evaluate the double integral

$$\iint_D \frac{y}{x^5+1} \, dA, \quad D = \{(x,y) \mid 0 \le x \le 1, \ 0 \le y \le x^2\}$$



$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} \, dy \, dx = \frac{1}{2} \int_0^1 \frac{x^4}{x^5+1} \, dx$$

$$\frac{1}{x^5+1} \cdot \frac{y^2}{2} \Big|_1^{x^2} = \frac{1}{x^5+1} \cdot \frac{x^4}{2}$$

$u = x^5 + 1$

$du = 5x^4 dx$

x	0	1
u	1	2

$$\frac{1}{2} \int_1^2 \frac{1/5 \, du}{u} = \frac{1}{10} \log u \Big|_1^2 = \boxed{\frac{\ln 2}{10}}$$

The other way $: 0 \le y \le 1 \,,\ \sqrt{y} \le x \le 1$

$$\underbrace{\int_0^1 \int_{\sqrt{y}}^1 \frac{y}{x^5+1}\,dx\,dy}$$

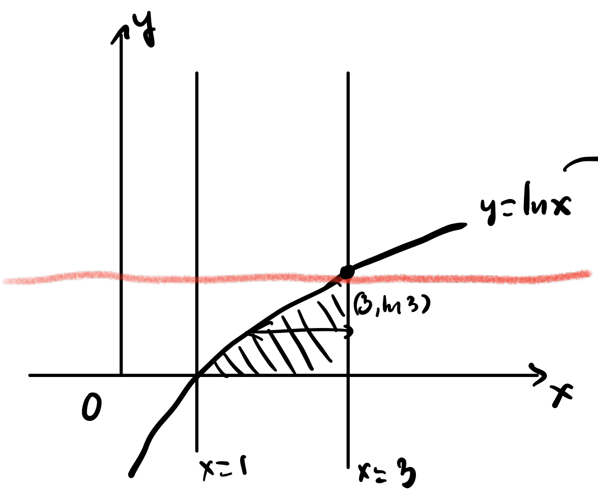
$\hookrightarrow$ hard to compute $\int \frac{dx}{x^5+1}$.

Ex 4.

Sketch the region of integration and change the order of integration: $\int_1^3 \int_0^{\ln x} f(x,y)\,dy\,dx$.

$D = \{\ 1 \le x \le 3 ,\ 0 \le y \le \ln x \}$

$\{ 0 \le y \le \ln 3 \,,\ e^y \le x \le 3 \}$



$e^y = x$

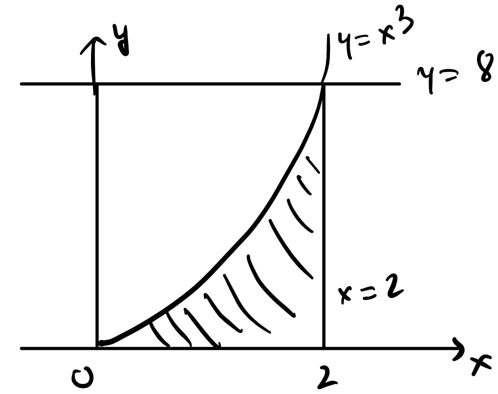
$$\boxed{\int_0^{\ln 3} \int_{e^y}^{3} f(x,y)\,dx\,dy}$$

Ex 5.

Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$

$D = \{\ 0 \le y \le 8 ,\ \sqrt[3]{y} \le x \le 2 \}$

$y = x^3$



$0 \le x \le 2 \,,\ 0 \le y \le x^3$

$$\int_0^2 \underbrace{\int_0^{x^3} e^{x^4}\,dy}\,dx$$

$$\int_0^2 e^{x^4} \cdot x^3\,dx$$

$u = x^4$

$du = 4x^3\,dx$

x	0	2
u	0	16

$$= \frac{1}{4}\int_0^{16} e^u\,du = \frac{1}{4} e^u \Big|_0^{16} = \boxed{\frac{e^{16}}{4}}$$