Remarks on the vanishing viscosity process of state-constraint Hamilton-Jacobi equations

Rate of convergence

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Presentation Overview

A model problem: escape of a light ray

- Let Ω be open with smooth boundary $\partial\Omega$ (the medium).
- A light ray starting from $x \in \Omega$ is a path $\gamma : [0, t] \to \Omega$ with $\gamma(0) = x$ for some $t > 0$.
- $c : \overline{\Omega} \to [0, +\infty)$ the medium constraint of the speed of light (inhomogeneity).
- $T_\gamma = \inf\{s \geq 0 : \gamma(s) \notin \Omega\}$: first time the light ray exists the medium and $T_\gamma = +\infty$ if $\gamma([0, \infty)) \subset \Omega$.

The light ray takes the path that exists the medium in the least amount of time with the speed constraint

$$|\dot{\gamma}(s)| \leq c(\gamma(s)), \qquad s \geq 0.$$

This leads to the introduction of the minimum time function

$$u(x) = \inf\left\{T_\gamma : \gamma(0) = x, |\dot{\gamma}(s)| \leq c(\gamma(s))\right\}$$

for $x \in \Omega$. *Assume* that $\nabla u(x)$ exists at all points, then using Bellman's optimality principle and a Taylor expansion:

$$\begin{cases} c(x)|Du(x)| = 1 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

This is *Eikonal equation*.

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \qquad \text{in } (-1,1) \qquad \text{with } u(-1) = u(1) = 0.$$

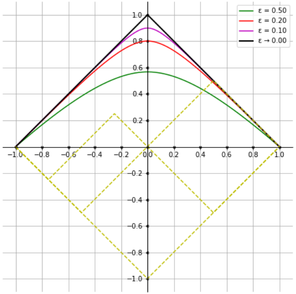
Infinitely many a.e. solutions, physically correct solution: $u(x) = 1 - |x|$.

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon (u^\varepsilon)'' & \text{in } (-1,1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon\left(e^{-1/\varepsilon} - e^{-|x|/\varepsilon}\right) \to u(x)$$



Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0,\infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x,a) : \overline{\Omega} \times U \to \mathbb{R}^n & \text{velocity vector field} \\ f = f(x,a) : \overline{\Omega} \times U \to \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\dot{y}(t) = b(y(t),\alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$ - the discount factor)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s), \alpha(s)\right)\, ds.$$

Define $H(x,p) = \sup_{v\in U}\left(-b(x,v)\cdot p - f(x,v)\right)$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in \mathrm{C}^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!* $\longrightarrow$ *viscosity solution*.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

F is non-decreasing in u, non-increasing in D^2u (*degenerate elliptic*).

$\longrightarrow$ No integration by parts, only maximum principle.

Subsolution: $\varphi \in C^2$, $u - \varphi$ *max* at x:

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

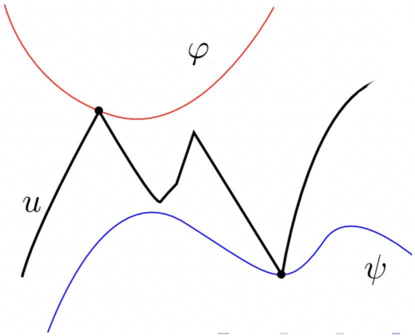
Supersolution: $\psi \in C^2$, $u - \psi$ *min* at x:

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.

$\longrightarrow$ *physically correct solution*

$\longrightarrow$ *value function in optimal control theory*



We consider

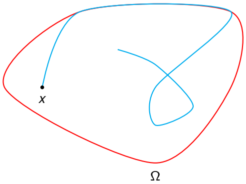
$$\begin{cases} u(x) + |Du|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du|^p - f(x) \geq 0 & \text{on } \overline{\Omega} \end{cases} \qquad (PDE_0)$$

This is the state-constrain Hamilton-Jacobi equation Soner (1986), which describe the value function of a deterministic optimal control problem

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^s L(\eta(s), -\dot{\eta}(s)) ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$

Here $L(x,v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is the running cost, Legendre's transform of $H(x,\xi) = |\xi|^p - f(x)$. Generally, if H is smooth and u is smooth

$$\begin{cases} u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ D_p H(x, Du(x)) \cdot \nu(x) \geq 0 & \text{on } \partial\Omega. \end{cases}$$



Given stochastic control $\alpha(\cdot)$, we solve

$$\begin{cases} dX_t = \alpha\left(X_t\right) dt + \sqrt{2\varepsilon}\, d\mathbb{B}_t & \text{for } t > 0, \\ X_0 = x. \end{cases} \tag{1}$$

$\mathbb{B}_t \sim \mathcal{N}(0, t)$ is the Brownian motion, to constraint $X_t \in \Omega$, we define

$$\widehat{\mathcal{A}}_x = \left\{\alpha(\cdot) \in \mathrm{C}(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0\right\}$$

Minimize the cost function

$$u^\varepsilon(x) = \inf_{\alpha \in \widehat{\mathcal{A}}_x} \mathbb{E}\left[\int_0^\infty e^{-t} L(X_t, \alpha(X_t))\; dt\right],$$

If $1 < p \leq 2$, $u^\varepsilon \in \mathrm{C}^2(\Omega)$ Lasry and Lions (1989) is the solution to

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x, \partial\Omega) \to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

If $p > 2$ then $u^\varepsilon \in \mathrm{C}(\overline{\Omega})$. We focus on the subquadratic case $1 <\leq 2$.

Using the stochastic, Lasry and Lions (1989) Dynamic Programming Principle, u^ε solves

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\ u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases} \tag{2}$$

- u^ε is a viscosity subsolution in Ω, that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\Omega)$ with $u^\varepsilon - \varphi$ has a maximum over Ω at x_0, then
$$u^\varepsilon(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon\Delta\varphi(x_0) \leq 0.$$
- u^ε is a viscosity supersolution on $\overline{\Omega}$, that is that is if $x_0 \in \Omega$ and $\varphi \in \mathrm{C}^2(\overline{\Omega})$ with $u^\varepsilon - \varphi$ has a maximum over $\overline{\Omega}$ at x_0, then
$$u^\varepsilon(x_0) + |D\varphi(x_0)|^p - f(x_0) - \varepsilon\Delta\varphi(x_0) \geq 0.$$

When $1 < p \leq 2$, u^ε is the unique solution with $u^\varepsilon(x) = +\infty$ on $\partial\Omega$.

Literature

Qualitative: As $\varepsilon \to 0$, $u^\varepsilon \to u$ in some sense, and in the limit, u is no longer blowing-up on the boundary:

1. Lasry and Lions (1989) (*PDEs approach*)
2. Capuzzo-Dolcetta and Lions (1990) (*PDEs approach*)
3. Fabbri et al. (2017) (*stochastic control approach*)

In the literature the solution is also called *large solutions*, and has been studied extensively. Blow-up rate of gradient is studied in Porretta (2004); Porretta and Véron (2006).
Quantitative: Rate of convergence: not yet done for state-constraint but for Dirichlet BC:

$$\begin{cases} u^\varepsilon(x) + H(x, Du^\varepsilon) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = 0 & \text{on } \partial\Omega \end{cases} \longrightarrow \begin{cases} u(x) + H(x, Du) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The rate is $\mathcal{O}(\sqrt{\varepsilon})$, $\|u^\varepsilon - u\|_{L^\infty(\overline{\Omega})} \leq C\sqrt{\varepsilon}$ and the one-sided rate can be $\mathcal{O}(\varepsilon)$ for convex Hamiltonians

1. Fleming (1961)
2. Bardi and Capuzzo-Dolcetta (1997)
3. Crandall and Lions (1984)
4. Evans (2010), Tran (2011) (nonlinear adjoint method)

Properties of solutions

A blow up rate of u^ε near $\partial\Omega$

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

When $H(x,\xi) = |\xi|^p - f(x)$, we can compute the asymptotic expansion of u^ε near $\partial\Omega$. Assume

$$u^\varepsilon \sim \frac{C}{d(x)^\alpha}$$

we find

$$\boxed{u^\varepsilon(x) \sim \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}}, \qquad p < 2, \qquad \alpha = \frac{2-p}{p-1}, \quad C_\alpha = \frac{(\alpha+1)^{\alpha+1}}{\alpha}$$

$$\boxed{u^\varepsilon(x) \sim -\varepsilon \log(d(x))}, \qquad p = 2$$

Summary of main results

Theorem (Han and Tu (2022))

Without loss of generality, we can assume $f \geq 0$. Also assume f is Lipschitz.

1. *Assume $f = 0$ on $\partial\Omega$ then $|u^\varepsilon - u| \leq C\sqrt{\varepsilon}$ in the interior of Ω. More precisely,*

$$\begin{aligned} -C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\sqrt{\varepsilon} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, &\qquad p < 2 \\ -C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\sqrt{\varepsilon} + C\varepsilon|\log(d(x))|, &\qquad p = 2 \end{aligned}$$

2. *If f is compactly supported in Ω then*

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}.$$

3. *If $f \in \mathrm{C}^2(\Omega)$ such that $Df = 0$ and $f = 0$ on $\partial\Omega$ then*

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon^{1/p} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad 1 < p < 2.$$

Difficulties and contributions

Difficulties

- The blow-up behaviors at the boundary makes it a nontrivial task to apply conventional method: doubling variables.
  $\longrightarrow$ We construct a new blow-up solution near the boundary and glue things together
- A uniform bound for the Laplacian of u^ε is complicated with blow-up behavior.
  $\longrightarrow$ We avoid this by using a bound for the Laplacian of u instead. This is related to the semi-concavity of the solution u^ε and u.

Contributions

- The rate $\mathcal{O}(\varepsilon^{1/p})$ is new!
- Construct a new blow-up solution to deal with the blow-up behavior of u^ε (major difficulty)
- Specific (blow-up rate) of semi-concavity behavior of u of improve the one-sided rate.

A heuristic argument

Heuristic: Doubling variable method

$$\Phi(x,y) = u^{\varepsilon}(x) - u(y) - \frac{|x-y|^2}{\sigma}, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}$$

- Viscosity solution $\sim$ *weak solution in* $L^{\infty} \Longrightarrow$ move the derivative to *test function without integration by parts* by maximum principle.
- Assume Φ has a maximum at x_σ, y_σ and $x_\sigma \in \Omega$ then $\Phi(x_\sigma, y_\sigma) \geq \Phi(x_\varepsilon, x_\varepsilon)$ implies that $|x_\sigma - y_\sigma| \leq C\sigma$
  - $\dfrac{|x-y_\sigma|^2}{\varepsilon}$ as a test function for $u^{\varepsilon}(x)$ in (PDE$_\varepsilon$) to get

$$u^{\varepsilon}(x_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(x_\sigma) - \varepsilon\frac{2n}{\sigma} \leq 0$$

  - $-\dfrac{|x_\sigma - y|^2}{\varepsilon}$ as a test function for $u(y)$ in (PDE_0) to obtain

$$u(y_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(y_\sigma) \geq 0$$

$$u^{\varepsilon}(x) - u(x) \leq u^{\varepsilon}(x_\sigma) - u(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + f(x_\sigma) - f(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + C\sigma$$

and the best choice here is $\sigma = \sqrt{\varepsilon}$.

- To overcome the difficulties in the argument, we instead use

$$\Phi(x,y) = \underbrace{u^\varepsilon(x) - \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}}_{\psi^\varepsilon(x)} - u(y) - \frac{C|x-y|^2}{\sigma} \tag{3}$$

- This forces the maximum happen at (x_σ, y_σ) where $x_\sigma \in \Omega$ (make C_α bigger). We also have

$$D\psi^\varepsilon(x) = Du^\varepsilon(x) + C_\alpha \alpha \left(\frac{\varepsilon}{d(x)}\right)^{\alpha+1} Dd(x). \tag{4}$$

- $|D\psi^\varepsilon(x)| \leq C$ if $d(x) \geq \varepsilon$ Armstrong and Tran (2015) $\Longrightarrow$ boundary layer is $\mathcal{O}(\varepsilon)$ from the boundary.
- However, we need $d(x_\sigma) \approx \varepsilon^\gamma$ for $\gamma \in (0,1)$. We need fine control of this after using some penalty to force the max happens.

Recall the equation

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ \lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u^\varepsilon(x) = +\infty. \end{cases} \tag{PDE$_\varepsilon$}$$

- Consider the case $f = 0$ first (then $u \equiv 0$), then ($\nu > 1$)

$$0 \le u^\varepsilon \le \underbrace{\frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\varepsilon^{\alpha+2}}_{\text{supersolution}}$$

- Compactly supported $\mathrm{supp}(f) \subset \Omega_\kappa = \{x \in \Omega : d(x) > \kappa\}$. If $\Phi(x,y)$ has max at (x_σ, y_σ)
  - (a) If $x_\sigma \in \Omega_\kappa$ then $d(x_\sigma) \ge C\kappa$, it is stronger than $d(x_\sigma) \approx \varepsilon^\gamma$.
  - (b) If $x_\sigma \in \Omega\backslash\Omega_\kappa$ we use a new *barrier*, bound solution by w that solves the PDE with $w = +\infty$ on $\partial\Omega_\kappa \cup \partial\Omega$.
- General case $f = 0$ on $\partial\Omega$: we do a cut-off $f_\kappa \to f$ as $\kappa \to 0$ and $\mathrm{supp}(f_\kappa) \subset \Omega_\kappa$. Since $f = 0$ on $\partial\Omega$, we can construct $\|f_\kappa - f\|_{L^\infty} \le C\kappa$.

Heuristic

- To overcome $\kappa + \frac{C}{\kappa}$, which make the best rate is only $\mathcal{O}(\sqrt{\varepsilon})$ we use u^ε as a C^2 test function for u.
- Assume that $u^\varepsilon(x) - u(x)$ has a maximum over $\overline{\Omega}$ at some interior point $x_0 \in \Omega$, then

$$\max_{x\in\overline{\Omega}} \Big(u^\varepsilon(x) - u(x) \Big) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0).$$

- If u is uniformly semiconcave in $\overline{\Omega}$, then $\Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C$.

Difficulties

1. $u^\varepsilon = +\infty$ on $\partial\Omega$, we can subtract by $\frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}$ to make maximum happen in the interior (then we need the barrier to handle the case $d(x)$ is small $\leftarrow$ the barrier still plays a crucial role).
2. Unless $f \in \mathrm{C}^2_c(\Omega)$, in general, u is not uniformly semiconcave but only locally semiconcave. In fact

$$\Delta u(x) \leq \frac{C}{d(x)}$$

and this is enough to get $\mathcal{O}(\varepsilon)$ for compactly supported data.

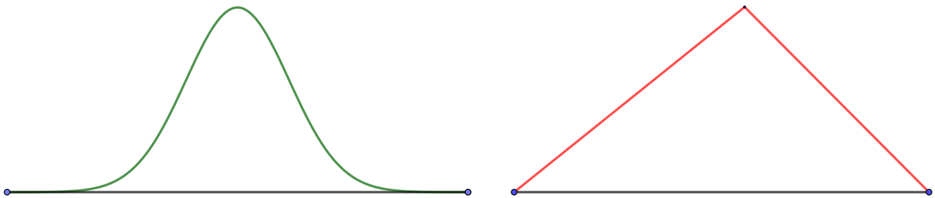


Figure: The different data that lead to different semiconcavities of u

- One the left: If f can be extended to a semiconcave function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by setting $f = 0$ on Ω^c, then u is uniformly semiconcave, i.e., $|Du| \leq C$, and hence an improvement on the rate happens.
- One the right: the best we can do is $|Du| \leq Cd(x)^{-1}$.

We want to show that if $x, x-h, x+h \in \overline{\Omega}$ then

$$u(x+h) - 2u(x) + u(x-h) \leq C|h|^2$$

- If f can be extended to $f \in \mathbb{R}^n$ by setting $f = 0$ outside Ω and $\tilde{f}$ is semiconcave then u is the restriction of $\tilde{u}$ where
$$\tilde{u}(x) + |D\tilde{u}(x)|^p - \tilde{f}(x) = 0 \qquad \text{in } \mathbb{R}^n.$$
Equation in the whole space is easier to deal with, see Calder (2018).
- If $f = 0$ on $\partial\Omega$ but cannot be extended to semiconcave function globally by setting $f = 0$ outside Ω, we relies on *optimal control formula and* $p \leq 2 \implies q = p^* > 2$. Take a minimizer η of x and let η hits $\partial\Omega$ at time T, then
$$u(x) = \int_0^T e^{-s}\left(c|\dot{\eta}(s)|^q + f(\eta(s))\right) ds.$$
- $\xi \mapsto |\xi|^q$ is C^2 if $q > 2$, thus locally semiconcave.
- Bounded velocity $|\dot{\eta}| \leq C$ implies $d(x) \leq CT$.

Discussion

Open questions

1. Can we remove the assumption $f = 0$ on $\partial\Omega$?
2. What is the optimal rate of convergence? (The semiconcavity of u in a more general setting was studied in a recent paper Han (2022)).
3. What is the rate of convergence for the super-quadratic case $p > 2$?
4. More general form of Hamiltonians?
5. A finer control of solution locally (which could leads to better rate) by using stochastic approach?

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Thank you!

The End

Questions & Comments

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