Some asymptotic problems on the theory of viscosity solutions of Hamilton–Jacobi equations

Ph.D. Thesis Presentation

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Viscosity solutions - Definition

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, we consider the fully nonlinear PDE

$$F(x, u, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

F is non-decreasing in u, non-increasing in D^2u (*degenerate elliptic*).

$\longrightarrow$ No integration by parts, only maximum principle.

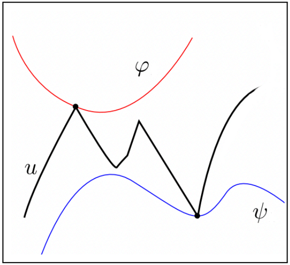
Subsolution: $\varphi \in C^2$, $u - \varphi$ *max* at x:

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

Supersolution: $\psi \in C^2$, $u - \psi$ *min* at x:

$$F(x, u(x), D\psi(x), D^2\psi(x)) \geq 0$$

Viscosity solution is *both* subsolution and supersolution.



$\longrightarrow$ *physically correct solution*

$\longrightarrow$ *value function in optimal control theory*

Vanishing viscosity - Eikonal equation

The minimal amount of time required to travel from a point to the boundary with constant cost 1 is model by

$$|u'(x)| = 1 \qquad \text{in } (-1,1) \qquad \text{with } u(-1) = u(1) = 0.$$

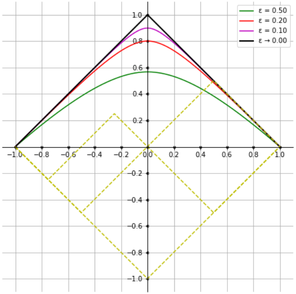
Infinitely many a.e. solutions, physically correct solution: $u(x) = 1 - |x|$.

Approximated equation with unique solution

$$\begin{cases} |(u^\varepsilon)'| = 1 + \varepsilon (u^\varepsilon)'' & \text{in } (-1,1), \\ u^\varepsilon(-1) = u^\varepsilon(1) = 0. \end{cases}$$

Vanishing viscosity

$$u^\varepsilon(x) = 1 - |x| + \varepsilon \left(e^{-1/\varepsilon} - e^{-|x|/\varepsilon} \right) \to u(x)$$



Optimal control theory - An infinite horizontal example

Let U be a compact metric space. A *control* is a Borel measurable map $\alpha : [0,\infty) \mapsto U$. We are given:

$$\begin{cases} b = b(x,a) : \overline{\Omega} \times U \to \mathbb{R}^n & \text{velocity vector field} \\ f = f(x,a) : \overline{\Omega} \times U \to \mathbb{R} & \text{running cost.} \end{cases}$$

For $x \in \mathbb{R}^n$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\dot{y}(t) = b(y(t), \alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$

Question. Minimize the cost functional ($\lambda \geq 0$)

$$u(x) = \inf_{\alpha(\cdot)} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s), \alpha(s)\right)\, ds.$$

Define $H(x,p) = \sup_{v\in U}\left(-b(x,v)\cdot p - f(x,v)\right)$ then

$$\lambda u(x) + H(x, Du(x)) = 0 \text{ in } \mathbb{R}^n$$

assuming that $u \in C^\infty$ (using optimality or dynamic programming principle). However the *value function is usually not smooth!*

Asymptotic problems

The development of viscosity solutions in the 1980s by P.-L Lions, L.C. Evans and M. G. Crandall connects the solution with optimal control.

Focus of the thesis: the study of asymptotic behavior of solutions with respect to the changing of parameters of the PDE

$$\lambda u + H(x, Du) - \varepsilon \Delta u = 0$$

where $\varepsilon \geq 0$ is the viscosity coefficient and $\lambda \geq 0$ is the discount factor.

$$\text{Asymptotic problems} \quad \longrightarrow \begin{cases} \text{Homogenization 1st-order} \\ \text{Changing domain 1st-order} \\ \text{Vanishing viscosity 2nd-order,} \\ \text{Vanishing discount 1st-order} \end{cases}$$

In this presentation, we focus on the asymptotic with respect to

1. vanishing viscosity 2nd-order
2. vanishing discount 1st-order

$\longleftarrow$ with *state-constraint*

State-constraint boundary condition

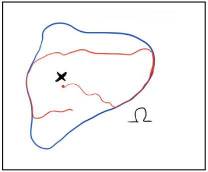
The name *state-constraint* comes from optimal control (Soner '86).

- Let $\Omega \subset \mathbb{R}^n$. Given a compact control set U, velocity field $b(x,v)$ and cost $f(x,v)$.
- For $x \in \Omega$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves
$$\dot{y}(t) = b(y(t),\alpha(t)), \qquad t > 0, \qquad \text{and} \qquad y(0) = x$$
- $\mathcal{A}_x$: control α such that $y^{x,\alpha}(t) \in \overline{\Omega}$ for all $t \geq 0$.

Question. Minimize the cost functional ($\lambda \geq 0$)

$$u(x) = \inf_{\alpha \in \mathcal{A}_x} \int_0^\infty e^{-\lambda s} f\left(y^{x,\alpha}(s),\alpha(s)\right)\, ds.$$

$$\begin{cases} \lambda u(x) + H(x,Du(x)) \leq 0 \text{ in } \Omega \\ \lambda u(x) + H(x,Du(x)) \geq 0 \text{ on } \overline{\Omega} \end{cases}$$



where

$$H(x,p) = \sup\left\{-b(x,v)\cdot p - f(x,v) : v \in U\right\}$$

State-constraint boundary condition

If H and u are smooth, the state-constraint boundary can be written as

$$\begin{cases} \lambda u(x) + H(x, Du(x)) = 0 \text{ in } \Omega \\ D_p H(x, Du(x)) \cdot \nu(x) \geq 0 \text{ on } \partial\Omega. \end{cases}$$

- Interpretation: $x \in \partial\Omega$, $\alpha^*(x)$ is optimal: $b(x, \alpha^*(x)) \cdot \nu(x) \leq 0$

$$H(x, Du(x)) \leq H(x, Du(x) + \beta\nu(x)) \qquad \text{for all } \beta \geq 0.$$

- Supersolution on $\partial\Omega$: $\varphi \in \mathrm{C}^1$ and $u - \varphi$ has min at $x \in \partial\Omega$.
  - $\partial\Omega$ around x is described $g(z) = 0$ and $\nu(x) = Dg(x)$
  - Lagrange multiplier: $D(u - \varphi)(x) = s\nu(x)$
  - $u - \varphi$ has min: $s \leq 0$,
  - $D\varphi(x) = Du(x) + \beta\nu(x)$ for $\beta = -s \geq 0$

  Assume $u \in \mathrm{C}^1$

$$\lambda u(x) + H(x, D\varphi(x)) \geq \lambda u(x) + H(x, Du(x)) = 0$$

Hamiltonian and Lagrangian

We have seen

$$\boxed{\text{optimal control}} \quad \Longrightarrow \quad \boxed{\begin{cases} \lambda u(x) + H(x, Du(x)) \leq 0 \text{ in } \Omega \\ \lambda u(x) + H(x, Du(x)) \geq 0 \text{ on } \overline{\Omega} \end{cases}}$$

The Lagrangian via Legendre's transform (convex Hamiltonian)

$$L(x,v) := \sup_{p \in \mathbb{R}^n} \left\{ p \cdot v - H(x,p) \right\}.$$

Using L as a cost, we can "inverse" an equation to the dynamic

$$\boxed{\begin{cases} \lambda u(x) + H(x, Du(x)) \leq 0 \text{ in } \Omega \\ \lambda u(x) + H(x, Du(x)) \geq 0 \text{ on } \overline{\Omega} \end{cases}} \quad \Longrightarrow \quad \boxed{\text{optimal control}}$$

by

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^{-\lambda s} L(\eta(s), -\dot{\eta}(s)) ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$

Summary of techniques

In the 4 type of asymptotic problems in this thesis, the techniques to study them can be summarized as follows.

$$\lambda u + H(x, Du) - \varepsilon\Delta u = 0$$

1. Homogenization 1st-order 2. Changing domain 1st-order	$\longleftarrow$	analysis of the optimal path $u(x) = \int_0^\infty e^{-\lambda s} L(\eta(s), -\dot{\eta}(s))\,ds$
3. Vanishing viscosity 2nd-order: $\varepsilon \to 0$	$\longleftarrow$	• Stochastic and deterministic optimal control • "Pure" PDE method
4. Vanishing discount 1st-order: $\lambda \to 0$	$\longleftarrow$	relax from optimal path to "optimal" measure $\lambda u_\lambda(x) = \int_{(x,v)} L(x,v)\, d\mu(x,v)$ and taking weak limit of measures.

The presentation focuses on Problem 3 & 4.

The 2nd-order state-constraint problem - stochastic optimal control

- Let $\Omega \subset \mathbb{R}^n$ be an open, bounded with $\partial\Omega \in \mathrm{C}^2$, $f \in \mathrm{C}(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.
- $\mathbb{B}_t \sim \mathcal{N}(0,t)$ is the Brownian motion. Given a control $\alpha(\cdot)$

$$\begin{cases} dX_t = \alpha\left(X_t\right) dt + \sqrt{2\varepsilon}\, d\mathbb{B}_t & \text{for } t > 0, \\ X_0 = x. \end{cases} \tag{1}$$

- To constraint $X_t \in \Omega$, we define

$$\widehat{\mathcal{A}}_x = \left\{ \alpha(\cdot) \in \mathrm{C}(\Omega) : \mathbb{P}(X_t \in \Omega) = 1 \text{ for all } t \geq 0 \right\}$$

- Minimize the cost function

$$u^\varepsilon(x) = \inf_{\alpha \in \widehat{\mathcal{A}}_x} \mathbb{E}\left[\int_0^\infty e^{-t} L\big(X_t, \alpha(X_t)\big)\, dt\right],$$

Here $L(x,v) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is the running cost, we consider the case Legendre's transform of L is $H(x,\xi) = |\xi|^p - f(x)$.

Theorem (Lasry-Lions '89): If $1 < p \leq 2$ then u^ε is the unique solution of

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0 & \text{in } \Omega, \\ u^\varepsilon(x) = +\infty \quad \text{as} \quad \mathrm{dist}(x,\partial\Omega) \to 0 \end{cases} \tag{PDE$_\varepsilon$}$$

The 2nd-order state-constraint problem

- **Viscosity solution framework**: Using the Dynamic Programming Principle

$$\begin{cases} u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \leq 0 & \text{in } \Omega, \\ u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) \geq 0 & \text{on } \overline{\Omega}, \end{cases} \qquad (\mathrm{PDE}_\varepsilon)$$

  in the viscosity sense.
- When $1 < p \leq 2$ u^ε is the unique solution with $u^\varepsilon(x) = +\infty$ on $\partial\Omega$.
- Let $\varepsilon \to 0$, we expect $u^\varepsilon \to u$ and u recovers thee deterministic optimal control, especially $u \in \mathrm{C}(\overline{\Omega})$

$$\begin{cases} u(x) + |Du|^p - f(x) \leq 0 & \text{in } \Omega, \\ u(x) + |Du|^p - f(x) \geq 0 & \text{on } \overline{\Omega} \end{cases} \qquad (\mathrm{PDE}_0)$$

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s))ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$

Literature

Qualitative

- As $\varepsilon \to 0$, it is natural to see that $u^\varepsilon \to u$ in some sense, it has been done
  - Lasry-Lions '89 (*PDEs approach*)
  - Capuzzo-Dolcetta and Lions '90 (*PDEs approach*)
  - Fabbri, Gozzi, and Swiech, '17 (*stochastic control approach*)
- In the limit, u is no longer blowing-up on the boundary.

Quantitative Can we quantify the rate of convergence?

- With Dirichlet boundary condition, the rate is $\mathcal{O}(\sqrt{\varepsilon})$

$$\boxed{\begin{cases} u^\varepsilon(x) + H(x, Du^\varepsilon) - \varepsilon \Delta u^\varepsilon(x) = 0 \\ u^\varepsilon(x) = 0 \qquad \text{on } \partial\Omega \end{cases}} \longrightarrow \boxed{\begin{cases} u(x) + H(x, Du) = 0 \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}}$$

- The one-sided rate can be $\mathcal{O}(\varepsilon)$ for convex Hamiltonian

$$-C\sqrt{\varepsilon} \le u^\varepsilon - u \le C\varepsilon$$

  - Fleming '64
  - Bardi and Capuzzo-Dolcetta '97
  - Crandall-Lions '84
  - Evans '10, Tran '11 (nonlinear adjoint method)

Properties of solutions

If $u^\varepsilon(x) + |Du^\varepsilon(x)|^p - f(x) - \varepsilon\Delta u^\varepsilon(x) = 0$ in Ω and $u = +\infty$ on $\partial\Omega$, assume $u^\varepsilon \approx Cd(x)^{-\alpha}$ near $\partial\Omega$ we find

$$\begin{cases} u^\varepsilon(x) \approx \dfrac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} & p < 2, \qquad \alpha = \frac{2-p}{p-1}, \quad C_\alpha = \frac{(\alpha+1)^{\alpha+1}}{\alpha} \\ u^\varepsilon(x) \approx -\varepsilon \log(d(x)) & p = 2 \end{cases}$$

Theorem (Han & Tu, 2021) Assume $f \geq 0$. Also assume f is Lipschitz.

1. If $f = 0$ on $\partial\Omega$ then $|u^\varepsilon - u| \leq C\sqrt{\varepsilon}$ in the interior of Ω. More precisely,

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\sqrt{\varepsilon} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad p < 2$$

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\sqrt{\varepsilon} + C\varepsilon|\log(d(x))|, \qquad p = 2.$$

2. If $\operatorname{supp} f \subset\subset \Omega$ then $-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon$ in the interior.
3. If $f \in \mathrm{C}^2(\Omega)$ such that $Df = 0$ and $f = 0$ on $\partial\Omega$ then

$$-C\sqrt{\varepsilon} \leq u^\varepsilon - u \leq C\varepsilon^{1/p} + \frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}, \qquad 1 < p < 2.$$

A heuristic argument

Viscosity solution is $\sim$ *weak solution in* $L^\infty \implies$ move the derivative to *test function without integration by parts*.

Heuristic: Doubling variable method

$$\Phi(x,y) = u^\varepsilon(x) - u(y) - \frac{|x-y|^2}{\sigma}, \qquad (x,y) \in \overline{\Omega} \times \overline{\Omega}$$

- Φ has max at x_σ, y_σ: $\Phi(x_\sigma, y_\sigma) \geq \Phi(x_\sigma, x_\sigma) \implies |x_\sigma - y_\sigma| \leq C\sigma$.
- Assume $x_\sigma \in \Omega$, subsolution test of u^ε with $|x - y_\sigma|^2/\sigma$ as a test function

$$u^\varepsilon(x_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(x_\sigma) - \varepsilon\frac{2n}{\sigma} \leq 0$$

- Supersolution test of u with $-|x - y_\sigma|^2/\sigma$ as a test function

$$u(y_\sigma) + \left|\frac{2(x_\sigma - y_\sigma)}{\sigma}\right|^p - f(y_\sigma) \geq 0$$

Subtract the two equations

$$u^\varepsilon(x) - u(x) \leq u^\varepsilon(x_\sigma) - u(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + f(x_\sigma) - f(y_\sigma) \leq \frac{2n\varepsilon}{\sigma} + C\sigma$$

and the best choice here is $\sigma = \sqrt{\varepsilon}$.

The $\mathcal{O}(\sqrt{\varepsilon})$ rate - idea

Write $\psi^\varepsilon(x) = u^\varepsilon(x) - \frac{C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha}$ we instead use

$$\Phi(x,y) = \psi^\varepsilon(x) - u(y) - \frac{C|x-y|^2}{\sigma}$$

then max happen at (x_σ, y_σ) where $x_\sigma \in \Omega$, also $|D\psi^\varepsilon(x)| \leq C$ if $d(x) \geq \varepsilon$ (Amstrong-Tran, '15) $\Longrightarrow$ layer is $\mathcal{O}(\varepsilon)$ from $\partial\Omega$

- Consider the case $f = 0$ first (then $u \equiv 0$), then $(\nu > 1)$

$$0 \leq u^\varepsilon \leq \frac{\nu C_\alpha \varepsilon^{\alpha+1}}{d(x)^\alpha} + C\varepsilon^{\alpha+2} \longrightarrow \text{supersolution}$$

- Compactly supported $\operatorname{supp}(f) \subset \Omega_\kappa = \{x \in \Omega : d(x) > \kappa\}$. If $\Phi(x,y)$ has max at (x_σ, y_σ)
  - (a) If $x_\sigma \in \Omega_\kappa$ then $d(x_\sigma) \geq C\kappa$, it is stronger than $\boxed{d(x_\sigma) \approx \varepsilon^\gamma}$.
  - (b) If $x_\sigma \in \Omega \backslash \Omega_\kappa$ we use a new *barrier*, bound solution by w that solves the PDE with $w = +\infty$ on $\partial\Omega_\kappa \cup \partial\Omega$.
- General case $f = 0$ on $\partial\Omega$: we do a cut-off $f_\kappa \to f$ as $\kappa \to 0$ and $\operatorname{supp}(f_\kappa) \subset \Omega_\kappa$. Since $f = 0$ on $\partial\Omega$, we can construct $\|f_\kappa - f\|_{L^\infty} \leq C\kappa$.

The $\mathcal{O}(\varepsilon)$ rate - idea

- To overcome the final bound in the above method $\kappa + \frac{C}{\kappa}$, which make the best rate is only $\mathcal{O}(\sqrt{\varepsilon})$ we use u^ε as a C^2 test function for u.
- Assume that $u^\varepsilon(x) - u(x)$ has a maximum over $\overline{\Omega}$ at some interior point $x_0 \in \Omega$, then

$$\max_{x\in\overline{\Omega}} \Big(u^\varepsilon(x) - u(x)\Big) \leq u^\varepsilon(x_0) - u(x_0) \leq \varepsilon \Delta u^\varepsilon(x_0).$$

- If u is uniformly semiconcave in $\overline{\Omega}$, then $\Delta u^\varepsilon(x_0) \leq \Delta u(x_0) \leq C$.

Difficulties

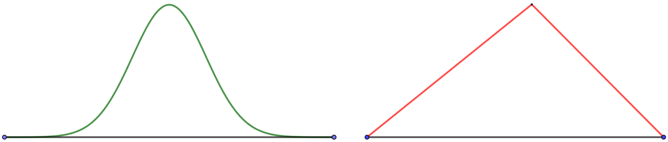
- $u^\varepsilon = +\infty$ on $\partial\Omega$, we can subtract by $\frac{C\varepsilon^{\alpha+1}}{d(x)^\alpha}$ to make maximum happen in the interior (then we need the barrier to handle the case $d(x)$ is small $\leftarrow$ the barrier still plays a crucial role).
- Unless $f \in \mathrm{C}^2_c(\Omega)$, in general, u is not uniformly semiconcave but only *locally semiconcave*. In fact

$$\Delta u(x) \leq \frac{C}{d(x)}$$

and this is enough to get $\mathcal{O}(\varepsilon)$ for compactly supported data.

Improved one-sided rate and the semiconcavity of u

- The improved one-sided rate is related to the semiconcavity modulus of u.
- If f is globally semiconcave or can be extended to a globally semiconcave function then u is globally semiconcave (see also new condition [Han, '22]).



- The one-sided rate $\mathcal{O}(\varepsilon)$ for compactly supported data (and thus $\mathcal{O}(\varepsilon^{1/p})$) is obtained as long as u is locally semiconcave with the bound

$$\Delta u(x) \leq \frac{C}{\mathrm{dist}(x, \partial\Omega)}.$$

If $f \in \mathrm{C}^2(\Omega)$ such that $Df = 0$ and $f = 0$ on $\partial\Omega$ then this condition holds, by using analysis of optimal path in optimal control of $u(x)$.

The problem

Let $\phi(\lambda) : (0, \infty) \nearrow (0, \infty)$ and $r(\lambda) : (0, \infty) \to \mathbb{R}$ be continuous

$$\lim_{\lambda \to 0^+} \phi(\lambda) = \lim_{\lambda \to 0^+} r(\lambda) = 0.$$

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1 + r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1 + r(\lambda))\overline{\Omega}. \end{cases} \qquad (S_\lambda)$$

Along $\lambda_j \to 0^+$, $\phi(\lambda)u_\lambda \to c_0$ and $u_\lambda - u_\lambda(x_0) \to u$

$$\text{(Ergodic problem)} \qquad \begin{cases} H(x, Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \qquad (S_0)$$

Difficulty: Solutions of (S_0) is not unique, even though c_0 is unique. Here c_0 is the so-called *additive eigenvalue*

$$c_0 = \inf \Big\{ c \in \mathbb{R} : H(x, Du(x)) \leq c \ \text{ in } \Omega \text{ has a solution} \Big\}.$$

Main questions of interest: Assume $\frac{\phi(\lambda)}{r(\lambda)} \to \gamma$ as $\lambda \to 0^+$.

1. Behavior of u_λ as $\lambda \to 0^+$?
2. Behavior of c_λ, the additive eigenvalue of H in $(1 + r(\lambda))\Omega$ as $\lambda \to 0^+$?

The problem - literature

The problem on the torus (periodic BC) is related to homogenization (Lions, Papanicolaou, Varadhan, 1986). Various results related vanishing discount

- 1st-order on the torus: [Davini-Fathi-Iturriaga-Zavidovique, 2016].
- 2nd-order on the torus: [Ishii-Mitake-Tran, 2017], [Mitake-Tran, 2017].
- Bounded domains with Neuumann boundary conditions [Al-Aidarous-Alzahrani-Ishii-Younas, 2016], [Ishii-Mitake-Tran, 2017].
- Problem in $\mathbb{R}^n$ [Ishii-Siconolfi, 2020].
- [Chen-Cheng-Ishii-Zhao, 2019] (vanishing discount with respect to changing Hamiltonians).

The problem - literature

Two normalizations for solutions

$$I_1 : \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}, \qquad \text{and} \qquad I_2 : \left\{ u_\lambda + \frac{c_\lambda}{\phi(\lambda)} \right\}_{\lambda>0}$$

where c_λ is the additive eigenvalues of H in $(1+r(\lambda))\Omega$.

Main results: Denote $\gamma = \lim \phi(\lambda)/r(\lambda)$.

1. If $\gamma = 0$ then both I_1, I_2 converge to the maximal solution u^0 in Ω.
2. If γ is finite then I_1 converges to u^γ with description in terms of Mather measures. If $\gamma = \infty$ then I_1 diverges (example).
3. The difference between I_1 and I_2 is $\dfrac{c_\lambda - c_0}{\phi(\lambda)}$. We show that
$$\lim_{r(\lambda)\to 0^+} \frac{c_\lambda - c_0}{r(\lambda)}, \qquad \text{and} \qquad \lim_{r(\lambda)\to 0^-} \frac{c_\lambda - c_0}{r(\lambda)}$$
exist (descriptions in terms of Mather measures). Thus B converges as well if γ is finite.
4. I_2 is bounded even if $\gamma = \infty$, but we have example showing divergence for I_2 when $\gamma = \infty$ and $r(\lambda) \leq 0$.

1. Introduction

2. Vanishing viscosity

3. Vanishing discount on changing domain
    - The problem
    - Summary
    - Mather measures
    - Main results
    - Divergence Example

Duality representation and Mather measures

A priori estimate: $|Du_\lambda| \leq h$, one can ignore $H(x,p)$ with $|p| > h$. For a measure μ defined on $\overline{\Omega} \times \overline{B}_h$, we define

$$\langle \mu, f \rangle := \int_{\overline{\Omega}\times\overline{B}_h} f(x,v)\, d\mu(x,v), \qquad \text{for } f \in \mathrm{C}(\overline{\Omega}\times\overline{B}_h).$$

For each $f \in \mathrm{C}(\overline{\Omega}\times\overline{B}_h)$, define

$$H_f(x,p) = \max_{|v|\leq h}\Big(p\cdot v - f(x,v)\Big), \qquad (x,p) \in \overline{\Omega}\times\overline{B}_h.$$

Let $\mathcal{R}(\overline{\Omega}\times\overline{B}_h)$: Radon measures on $\overline{\Omega}\times\overline{B}_h$. For $\lambda > 0$ and $z \in \overline{\Omega}$

$$\begin{cases} \mathcal{F}_{\lambda,\Omega} & = \Big\{(f,u) \in \mathrm{C}(\overline{\Omega}\times\overline{B}_h)\times\mathrm{C}(\overline{\Omega}) : \lambda u + H_f(x,Du) \leq 0 \text{ in } \Omega\Big\} \\ \mathcal{G}_{z,\lambda,\Omega} & = \Big\{f - \lambda u(z) : (f,u) \in \mathcal{F}_{\lambda,\Omega}\Big\} \\ \mathcal{G}'_{z,\lambda,\Omega} & = \Big\{\mu \in \mathcal{R}(\overline{\Omega}\times\overline{B}_h) : \langle \mu, f\rangle \geq 0 \text{ for all } f \in \mathcal{G}_{z,\lambda,\Omega}\Big\}. \end{cases}$$

Representation formula for u: (Mitake-Tran-Ishii, 2017)

$$\lambda u(z) = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{z,\lambda,\Omega}} \langle \mu, L\rangle = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{z,\lambda,\Omega}} \int_{\overline{\Omega}\times\overline{B}_h} L(x,v)\, d\mu(x,v).$$

Duality representation and Mather measures

$$\begin{cases} \mathcal{F}_{\lambda,\Omega} &= \Big\{(f,u) \in \mathrm{C}(\overline{\Omega}\times\overline{B}_h)\times\mathrm{C}(\overline{\Omega}) : \lambda u + H_f(x,Du) \le 0 \text{ in } \Omega\Big\} \\ \mathcal{G}_{z,\lambda,\Omega} &= \Big\{f - \lambda u(z) : (f,u) \in \mathcal{F}_{\lambda,\Omega}\Big\} \\ \mathcal{G}'_{z,\lambda,\Omega} &= \Big\{\mu \in \mathcal{R}(\overline{\Omega}\times\overline{B}_h) : \langle \mu, f\rangle \ge 0 \text{ for all } f \in \mathcal{G}_{z,\lambda,\Omega}\Big\}. \end{cases}$$

Let $\lambda \to 0^+$

$$\begin{cases} \mathcal{F}_{0,\Omega} &= \Big\{(f,u) \in \mathrm{C}(\overline{\Omega}\times\overline{B}_h)\times\mathrm{C}(\overline{\Omega}) : H_f(x,Du(x)) \le 0 \text{ in } \Omega\Big\} \\ \mathcal{G}_{0,\Omega} &= \Big\{f : (f,u) \in \mathcal{F}_{0,\Omega} \text{ for some } u \in \mathrm{C}(\overline{\Omega})\Big\} \\ \mathcal{G}'_{0,\Omega} &= \Big\{\mu \in \mathcal{R}(\overline{\Omega}\times\overline{B}_h) : \langle \mu, f\rangle \ge 0 \text{ for all } f \in \mathcal{G}_{0,\Omega}\Big\}. \end{cases}$$

$\mathcal{G}'_{0,\Omega}$: *holonomic measures*, s.t. $\langle \mu, v\cdot D\psi(x)\rangle = 0$ for all $\psi \in \mathrm{C}^1(\overline{\Omega})$.

Representation formula for c_0: (Mitake-Tran-Ishii, 2017)

$$-c_0 = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{0,\Omega}} \langle \mu, L\rangle = \min_{\mu\in\mathcal{P}\cap\mathcal{G}'_{0,\Omega}} \int_{\overline{\Omega}\times\overline{B}_h} L(x,v)\, d\mu(x,v). \qquad (2)$$

The set of all measures in $\mathcal{P}\cap\mathcal{G}'_{0,\Omega}$ that minimizing (2) is denoted $\mathcal{M}_0$.

Main results - Convergence of vanishing discount

Let c_λ bee the additive eigenvalues of H in $\Omega_\lambda = (1 - r(\lambda))\Omega$, i.e.,

$$c_\lambda = \inf \Big\{ c \in \mathbb{R} : H(x, Du(x)) \leq c \ \text{ in } \Omega_\lambda \text{ has a solution} \Big\}.$$

Denote $\gamma = \lim \phi(\lambda)/r(\lambda)$. The two normalizations

$$I_1 = \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}, \qquad \text{and} \qquad I_2 = \left\{ u_\lambda + \frac{c_\lambda}{\phi(\lambda)} \right\}_{\lambda>0}.$$

Convergence for $\gamma = 0$ (Tu, 2020):
If $\gamma = 0$ then I_1, I_2 converge to u^0 locally uniformly as $\lambda \to 0^+$.

$$\begin{cases} H(x, Du^0(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du^0(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \qquad \text{and} \qquad u^0 = \sup_{w \in \mathcal{E}} w$$

where

$$\mathcal{E} = \Big\{ w : H(x, Dw) \leq c_0 \text{ in } \Omega \ : \ \langle \mu, w \rangle \leq 0 \text{ for all } \mu \in \mathcal{M}_0 \Big\}.$$

Here $\mathcal{M}_0$ is the set of all $\mu \in \mathcal{P} \cap \mathcal{G}_0'$ such that $-c_0 = \langle \mu, L \rangle$.

Main results - Convergence with the first normalization

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, our normalization $I_1 = \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

Convergence of the I_1 (Tu, 2020): Assume $\lambda \mapsto L((1+\lambda)x, v)$ is C^1. If γ is finite then I_1 converges to u^γ locally uniformly in Ω

$$\begin{cases} H(x, Du^\gamma(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du^\gamma(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \qquad \text{and} \qquad u^\gamma = \sup_{w\in\mathcal{E}^\gamma} w,$$

where

$$\mathcal{E}^\gamma = \left\{ \begin{array}{l} w : H(x, Dw) \leq c_0 \text{ in } \Omega \ : \\ \langle \mu, (-x)\cdot D_x L(x, v) \rangle + \langle \mu, w \rangle \leq 0 \text{ for all } \mu \in \mathcal{M}_0 \end{array} \right\}.$$

- If $\gamma = \infty$ then I_1 is unbounded (counter example).
- The mapping $\gamma \mapsto u^\gamma(\cdot)$ from $\mathbb{R}$ to $\mathrm{C}(\overline{\Omega})$ is concave and decreasing.

Main results - The second normalization - eigenvalue behavior

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, $I_1 = \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}$ and $I_2 = \left\{ u_\lambda + \frac{c_\lambda}{\phi(\lambda)} \right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

The difference between I_2 and I_1 is $\lim\limits_{\lambda\to 0^+} \left(\dfrac{c_\lambda - c_0}{\phi(\lambda)} \right) = \gamma \lim\limits_{\lambda\to 0^+} \left(\dfrac{c_\lambda - c_0}{r(\lambda) - r(0)} \right)$.

$$c_\lambda = c_0 + c_{(1)}\lambda + \circ(\lambda), \quad \lambda \to 0^+.$$

Convergence of additive eigenvalue (Tu, 2020):

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)>0}} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) = \max_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_x L(x,v) \rangle,$$

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)<0}} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) = \min_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_x L(x,v) \rangle.$$

1. $\frac{c_\lambda - c_0}{r(\lambda)}$ converges $\iff \langle \mu, (-x)\cdot D_x L(x,v) \rangle = c_{(1)}$ for all $\mu \in \mathcal{M}_0$
2. To study $c_{(1)}$ we can assume $r(\lambda) = \lambda$.

Main results - Second normalization, divergence result

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, $I_1 = \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}$ and $I_2 = \left\{ u_\lambda + \frac{c_\lambda}{\phi(\lambda)} \right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

Convergence and divergence results (Tu, 2020):

- If $\gamma \neq 0$ then

$$\lim_{\lambda\to 0^+} \left(u_\lambda(x) + \frac{c_\lambda}{\phi(\lambda)} \right) = u^\gamma(x) + \gamma \underbrace{\lim_{\lambda\to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right)}_{c^\pm_{(1)}}$$

- I_2 is bounded even if $\gamma = \infty$, but we have a divergence example.
- There exists $H(x,p)$ where given any $r(\lambda) \leq 0$ we can construct $\phi(\lambda)$ such that

$$r(\lambda_j)/\phi(\lambda_j) \to -\infty, \quad \text{and} \quad I_2 \text{ diverges.}$$

Divergence example

Let $r(\lambda) \geq 0$, c_λ be the eigenvalue of H in $\Omega_\lambda = (1 - r(\lambda)\Omega$. Let

$$\boxed{H(x,p) = |p| - V(x)}, \qquad (x,p) \in \overline{\Omega} \times \mathbb{R}.$$

where $V : \overline{\Omega} \to \mathbb{R}$, $V \geq 0$ and $V \in \mathrm{BUC}(\overline{\Omega})$.

- **Goal:** Given $r(\lambda)$, construct $\phi(\lambda)$ s.t. $\{u_\lambda + \phi(\lambda)^{-1} c_\lambda\}_{\lambda > 0}$ is divergent.
- **Tool:** The instability of the Aubry set $\mathcal{A}_{\Omega_\lambda}$ of H on Ω_λ, when $\lambda \to 0^+$.
- **Semi-distance** Let

$$S_\Omega(x,y) = \sup\Big\{ u(x) - u(y) : u \text{ s.t. } H(x, Du(x)) \leq c_0 \text{ in } \Omega \Big\}$$

$$\begin{cases} H(x, Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du(x)) \geq c_0 & \text{on } \overline{\Omega} \backslash \{y\}. \end{cases}$$

- **Aubry set:**

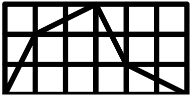
$$\mathcal{A}_\Omega = \Big\{ z \in \overline{\Omega} : x \mapsto S_\Omega(x,z) \text{ is a state-constraint solution} \Big\}$$

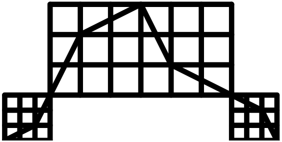
With $H(x,p) = |p| - V(x)$, then

$$-c_0 = \min_{\overline{\Omega}} V \qquad \text{and} \qquad \mathcal{A}_\Omega = \Big\{ x \in \overline{\Omega} : V(x) = \min_{\overline{\Omega}} V \Big\}.$$

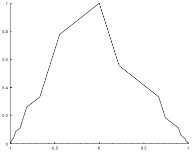
Key point: If $\mathcal{A}_\Omega = \{z_0\}$ then $u^0(x) \equiv S_\Omega(x, z_0)$ (maximal solution).

Divergence example





Switching the small box with this construction, we obtain



> **Lemma** (Tu, 2020) Let $\Omega_\lambda = (-1 + r(\lambda), 1 - r(\lambda))$. Then the maximal solution on Ω_λ, denoted by $u^0_\lambda(x)$, does not converge as $\lambda \to 0^+$.

This is intuitive as we can choose two subsequences where the mimimum points of V over Ω_λ converge to the two vertices.

Divergence example

Consider

$$\begin{cases} \delta u_\delta(x) + H(x, Du_\delta(x)) \le 0 & \text{in } \Omega_\lambda, \\ \delta u_\delta(x) + H(x, Du_\delta(x)) \ge 0 & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{3}$$

Let c_λ be the eigenvalue of H over Ω_λ

$$\lim_{\delta\to 0^+} \left(u_\delta(x) + \frac{c_\lambda}{\delta}\right) \to u^0_\lambda(x)$$

where $u^0_\lambda(x)$ is a maximal solution on Ω_λ. For each $\lambda > 0$, let $\tau(\lambda) > 0$ such that

$$\sup_{x\in\overline{\Omega}_\lambda} \left| \left(u_\delta(x) + \frac{c_\lambda}{\delta}\right) - u^0_\lambda(x)\right| \le r(\lambda) \qquad \text{for all } \delta \le \tau(\lambda). \tag{4}$$

Set $\phi(\lambda) = \tau(\lambda) r(\lambda)^2$, then $\phi(\lambda) \to 0$ as $\lambda \to 0^+$ and $\gamma = \infty$ ($u_\lambda \equiv u_{\phi(\lambda)}$). Along two subsequences λ_j and δ_j we have

$$\lim_{\lambda_j\to 0^+} \left(u_{\lambda_j}(x) + \frac{c_{\lambda_j}}{\phi(\lambda_j)}\right) = S_\Omega(x,-1) \neq S_\Omega(x,1) = \lim_{\delta_j\to 0^+} \left(u_{\delta_j}(x) + \frac{c_{\delta_j}}{\phi(\delta_j)}\right).$$

Thank You