Michigan State University
Math 234 – Spring 2024

Lecture notes

1 Function of several variables

Definition 1.

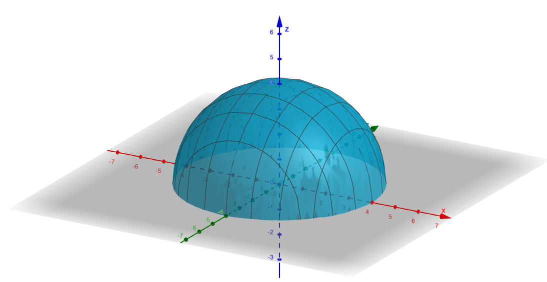
(a) *A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$.*

(b) *We often write $z = f(x, y)$.*

(c) *The graph of $z = f(x, y)$ is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.*

Example 1. *Consider $f(x, y) = \sqrt{16 - x^2 - y^2}$. Sketch the domain of f. Graph $z = f(x, y)$ using traces of $z = 0, x = 0, y = 0$.*

Proof. The domain us $D = \{(x, y) \in \mathbb{R}^2 : 16 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4^2\}$. This is the (closed) circle centered at $(0, 0)$ with radius 4. The trace of $z = 0, y = 0, x = 0$ give us $x^2 + y^2 = 16$, $z^2 + y^2 = 16$ and $z^2 + x^2 = 16$, i.e., in any cross-section it is a circle, therefore the graph of this function is a sphere of radius 4 in $\mathbb{R}^3$ (but only half of the sphere, the upper half as $z \geq 0$).

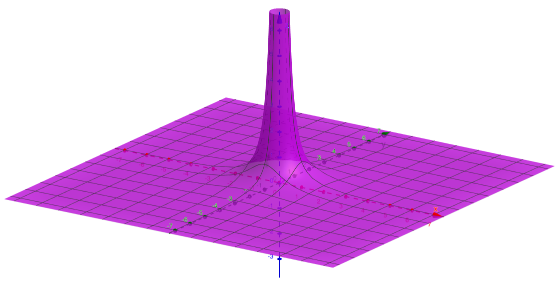
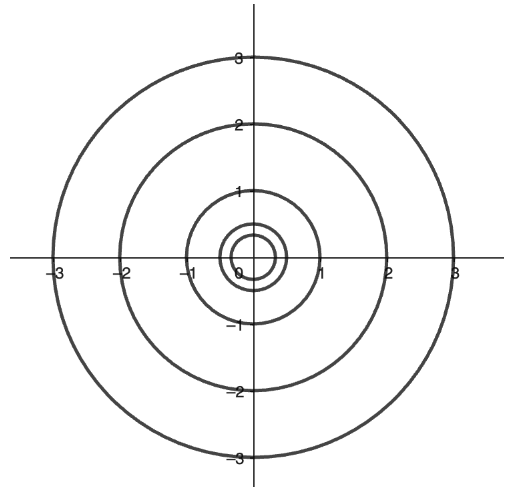


$\square$

Definition 2. *The contours of a function f of two variables are the curves with equations $f(x, y) = k$, where k is constant (in the range of f).*

Example 2. *Sketch the level curves of $f(x, y) = \frac{1}{x^2+y^2}$, with $k = \frac{1}{9}, \frac{1}{4}, 1, 4, 9$. Use these to attempt to sketch a 3D version of the graph.*

Proof. With $k = \frac{1}{3}$ we have $f(x, y) = \frac{1}{9}$ is equivalent to $x^2 + y^2 = 3^2$, it is a cirlce. Similarly with $k = \frac{1}{4}$ it is a circle $x^2 + y^2 = 4$. We have a set of cirles centered at $(0, 0)$ with radius $3, 2, 1, \frac{1}{2}, \frac{1}{3}$, correspondingly to $k = \frac{1}{9}, \frac{1}{4}, 1, 4, 9$.

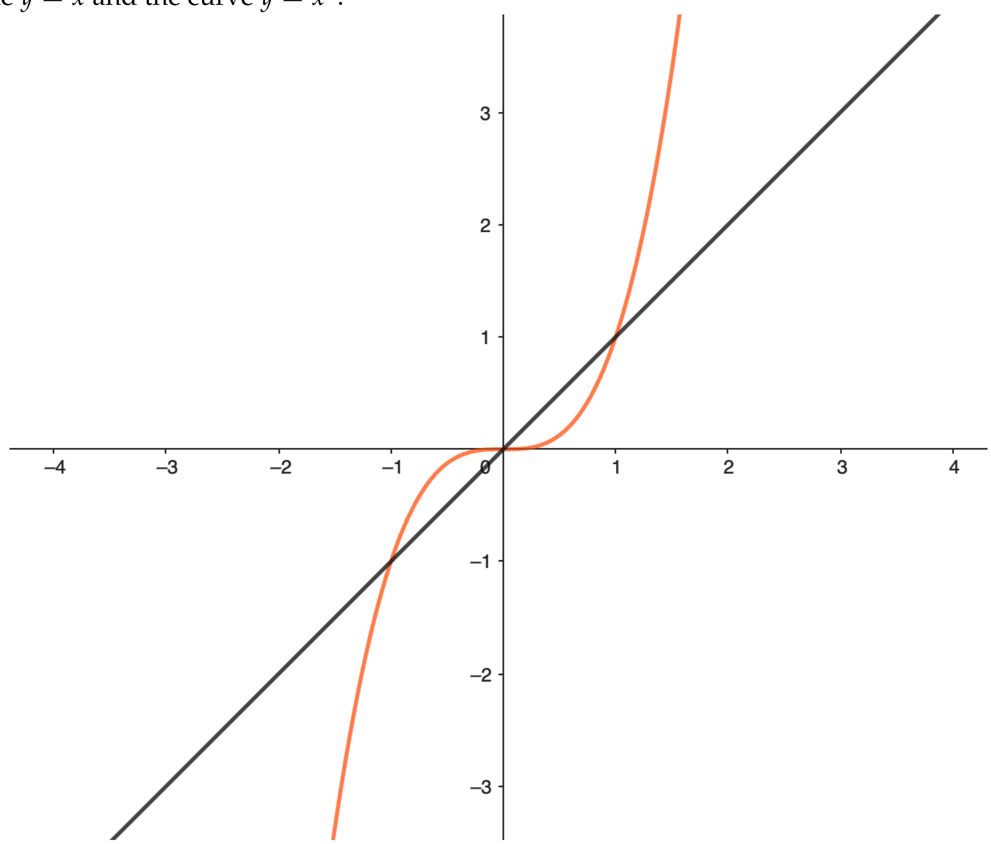



Note: if $f(x,y)$ is a 2-variables function then graph(f) is 3D (on the left), but its contours are 2D as in the picture (on the right). $\square$

Definition 3. *A function of 3 variables is $f(x,y,z)$ from a domain $D \subset \mathbb{R}^3$ to $\mathbb{R}$. The* ***level surfaces*** *of $f(x,y,z)$ are the surfaces with the equation $f(x,y,z) = k$ where k is a constant (by looking at level surfaces, we can view it in 3D, instead of the graph of f is in 4D).*

Example 3. *Find the domain of $f(x,y) = \frac{(x-1)(y+2)}{(y-x)(y-x^3)}$. Sketch and write the domain in set notation.*

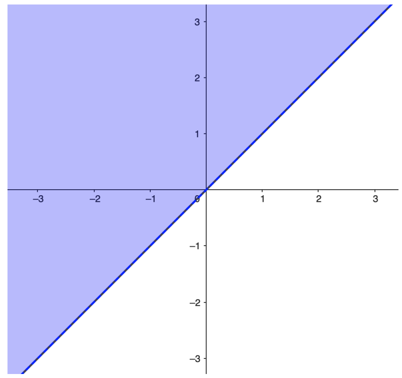
Proof. $D = \{(x,y) \in \mathbb{R}^2 : y \neq x, y \neq x^3\}$. The domain is the whole plane $\mathbb{R}$ (the xy-plane) removing the line $y = x$ and the curve $y = x^3$.



$\square$

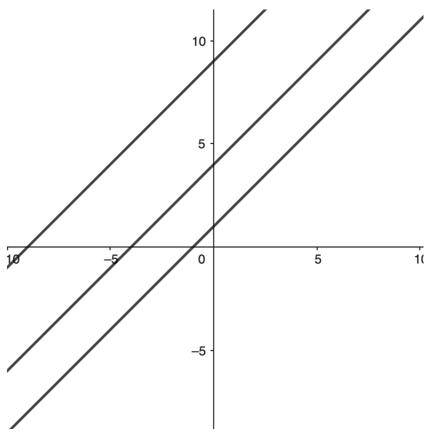
Example 4. *Consider the function $z = f(x,y) = \sqrt{y-x}$.*

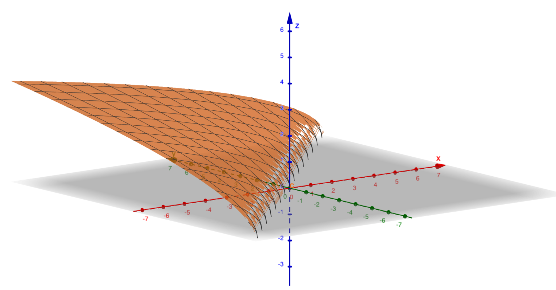
(a) *Dmain* $D = \{(x, y) : y - x \geq 0\} = \{(x, y) : y \geq x\}$. *(The line* $y = x$ *is included.)*



(b) *The range is* $z \in [0, +\infty)$.

(c) *Sketch some level curves and the graph*





2 Partial derivatives

Definition 4 (Partial Derivatives)**.**

(a) *The partial derivatives of* $f(x, y)$ *with respect to* x *at* (a, b) *is denoted by* $f_x(a, b)$ *and is given by*

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{|h|}.$$

This is equivalent to considering y *as a constant and taking derivative in* x.

(b) *The partial derivatives of* $f(x, y)$ *with respect to* y *at* (a, b) *is denoted by* $f_y(a, b)$ *and is given by*

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{|k|}.$$

This is equivalent to considering x *as a constant and taking derivative in* y.

(c) *Other notations*

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x, y) = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x, y) = \frac{\partial z}{\partial y} = D_y f.$$

(d) *Second-order derivatives:*

$$\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}\\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}\\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}\\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}.
\end{aligned}$$

Note the sequence: the first derivative is taken closest to the function.

(e) *(Clairaut's Theorem) The order of taking derivatives (around a point (a, b)) does not matter if the second order derivaties are continuous and defined around a point (a, b).*

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a,b) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a,b).$$

(f) *The gradient*

$$\nabla f(a,b) = \big(f_x(a,b), f_y(a,b)\big)$$

gives the direction in which the value of the function increases the fastest.

Example 5. *We have*

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(xy + x\sin y + \frac{y}{x}\right)\right) &= \frac{\partial}{\partial x}\left(x + x\cos y + \frac{1}{x}\right) = 1 + \cos y - \frac{1}{x^2}\\
\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(xy + x\sin y + \frac{y}{x}\right)\right) &= \frac{\partial}{\partial y}\left(y + \sin y - \frac{y}{x^2}\right) = 1 + \cos y - \frac{1}{x^2}
\end{aligned}$$

Example 6. *Let $v(x, y) = \frac{xy}{x-y}$. Compute v_x, v_{xx}, v_{xy}.*

Proof. We use product rule or quotient rule, or any rule from Calculus 1 and 2:

$$\begin{aligned}
v_x &= \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}, \qquad v_{xx} = \frac{-(-y^2)2(x-y)}{(x-y)^4} = \frac{2y^2}{(x-y)^3}\\
v_{xy} &= \frac{-2y(x-y)^2 - (-y^2)2(x-y)}{(x-y)^4} = \frac{-2y + 2y^2}{(x-y)^3}.
\end{aligned}$$

□

Example 7. *Find f_{xyz} for $f(x, y, z) = xyz + (x^2 + y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}$.*

Proof. Note that $\frac{\partial}{\partial x}(\sin^{-1}(x)) = \frac{\partial}{\partial x}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$. We compute (product rule, then quotient rule)

$$f_x = yz + 2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right).$$

Let us not computing the derivative of that term for now, for a reason we will see soon. Now we have

$$\begin{aligned}
f_{xy} = \frac{\partial}{\partial y} f_x &= \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right)\right]\\
&= z + \frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right)\right].
\end{aligned}$$

Now we have

$$f_{xyz} = 1 + \frac{\partial}{\partial z} \underbrace{\frac{\partial}{\partial y} \left[2x \frac{\partial}{\partial x} \left((x^2+y^2) \frac{\sin^{-1}(x\sqrt{y})}{\tan(x)} \right) \right]}_{\text{no } z \text{ involved, thus this term is } 0}.$$

The intergral is zero, since there is no z involved, and we treat x, y as constants when taking $\frac{\partial}{\partial z}$. $\square$

Example 8. *Suppose you are surrounded by bees given by the bee density function*

$$B(x,y) = 100 - x^2 + y^2 + 3y \qquad \textit{bees/unit}^2.$$

You are currently standing at $(1,1)$. Which of the four directions would be best to run in $\{\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{j}, -\boldsymbol{j}\}$?

Proof. We have

$$\nabla B(x,y) = (-2x, 2y+3) \qquad \Longrightarrow \qquad \nabla B(1,1) = (-2,5).$$

The best direction to run would be the opposite of $(-2,5)$, i.e, $\mathbf{v} = (2,-5)$. Now among the 4 directions, we choose the one that is closest, i.e., taking the dot product to find the one with smallest angle, i.e., $\cos\theta$ the biggest:

- For $\mathbf{i} = (1,0)$ then $(1,0) \cdot (2,-5) = 2$.
- For $-\mathbf{i} = (-1,0)$ then $(-1,0) \cdot (2,-5) = -2$.
- For $\mathbf{j} = (0,1)$ then $(0,1) \cdot (2,-5) = -5$.
- For $-\mathbf{j} = (0,-1)$ then $(0,-1) \cdot (2,-5) = 5$.

Therefore we choose $-\mathbf{j}$. $\square$