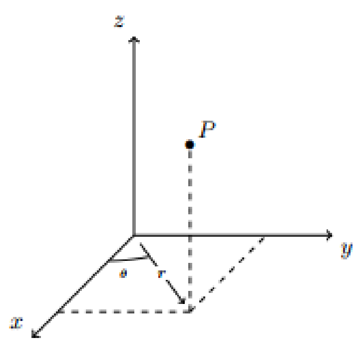
MICHIGAN STATE UNIVERSITY
MATH 234 – SPRING 2024

LECTURE NOTES

1 Cylindrical coordinates

- Cylindrical coordinates represent a point $P(x, y, z)$ in space by ordered triples (r, θ, z) in which (r, θ) is the polar coordinate of (x, y).
- z remains unchanged.

$$\boxed{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} r^2 = x^2 + y^2 \\ \dfrac{y}{x} = \tan\theta. \end{cases}}$$



- **Example.** Change $(x, y, z) = (-1, 1, 1)$ into cylindrical coordinates.

  Proof. $r^2 = x^2 + y^2 = 2$, thus $r = \sqrt{2}$. Then $\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$, thus $\theta = \frac{3\pi}{4}$. Hence

$$(-1, 1, 1) \mapsto \left(\sqrt{2}, \frac{3\pi}{4}, 1\right).$$

$\square$

- **Example.** Change $(\sqrt{2}, 3\pi/4, 2)$ to Cartesian coordinates.

  Proof. We have $x = r\cos\theta = \sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) = -1$ and $y = r\sin\theta = \sqrt{2} \times \left(\frac{1}{\sqrt{2}}\right) = 1$. Thus

$$(\sqrt{2}, 3\pi/4, 2) \mapsto (1, -1, 2).$$

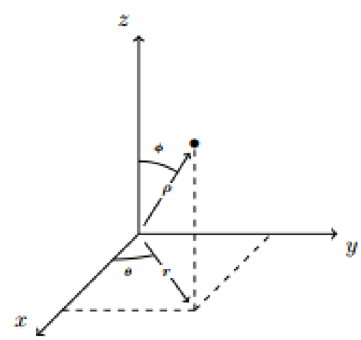
$\square$

2 Spherical coordinates

- $(x, y, z) \mapsto (\rho, \theta, \phi)$, where basically we repeat the polar coordinate first, and the *height* z is tracked via the variable ϕ, the angle with Oz. Note that the order is sometime written as (r, ϕ, θ). **Pay attention to the order!**

- The relations, still introducing an extra variable r as in polar coordinates (it will be very useful)

$$\boxed{\begin{cases} x = (\rho \sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ r = \rho\sin\phi \\ \dfrac{r}{z} = \tan\phi \end{cases}, \quad \theta \in [0, 2\pi], \phi \in [0, \pi]}$$



- If $\phi > \frac{\pi}{2}$ then $z < 0$, the angle make P lies below the Oxy-plane.

- **Example.** Convert $(1, 1, 0)$ into spherical coordinate.

  Proof. $\rho^2 = x^2 + y^2 + z^2 = 2$, thus $\rho = \sqrt{2}$. Now $z = \rho\cos\phi$ implies $0 = \sqrt{2}\cos\phi$, thus $\phi = \frac{\pi}{2}$. Finally $\tan\theta = \frac{y}{x} = 1$, thus $\theta = \frac{\pi}{4}$ (since $x > 0, y > 0$). We conclude

$$(1, 1, 0) \mapsto \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{2}\right) = (r, \theta, \phi).$$

  □

- **Example.** True/False: Consider the point with spherical coordinates $(\rho, \theta, \phi) = (4, \frac{3\pi}{4}, \frac{5\pi}{7})$. The product of the Cartesian coordinates, xyz, is positive.

  Proof. **True**. We see that $\phi = \frac{5\pi}{7} > \frac{\pi}{2}$, thus $z < 0$. Now $\theta = \frac{3\pi}{4}$, thus $x > 0, y < 0$ (draw a picture). Therefore $xyz > 0$. □

3 Practice

- **Example.** Convert the equation $z = \sqrt{x^2 + y^2}$ into cylindrical coordinates and spherical coordinates.

  Proof.

  - Cylindrical: $z = r$.
  - Spherical: $\rho\cos\phi = r = \rho\sin\phi$, thus $\tan\phi = 1$, thus $\phi = \frac{\pi}{4}$ is the equation of the cone!

  □

- **Example.** Identify the surface whose equation is $z = 4 - r^2$ in cylindrical coordinate.

Proof. We have $z = 4 - x^2 - y^2$, thus this is a elliptical paraboloid (one term of 1st order, two terms of second order having the same sign). $\square$

- **Example.** Convert to x, y, z the surface: $\rho = \sin\phi\cos\phi$.

  Proof. We can do
  $$(x^2+y^2+z^2)^{\frac{3}{2}} = \rho^3 = (\rho\sin\phi)(\rho\cos\phi) = rz = z\sqrt{x^2+y^2}.$$
  The answer is $(x^2+y^2+z^2)^{\frac{3}{2}} = z\sqrt{x^2+y^2}$. $\square$

- **Example.** Identify the surface whose equation is: $\rho = \sin\phi\cos\theta$.

  Proof. We can do
  $$x^2+y^2+z^2 = \rho^2 = (\rho\sin\phi)\cos\theta = r\cos\theta = x$$
  Therefore
  $$\left(x-\frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4}$$
  This is a sphere centered at $(\frac{1}{2}, 0, 0)$ with radius $\frac{1}{2}$, this is a *ellipsoid*. $\square$

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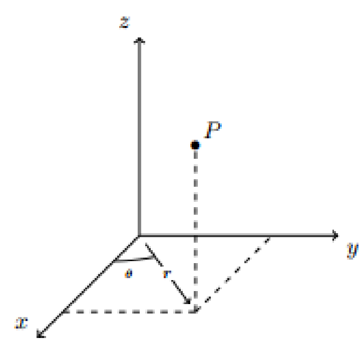

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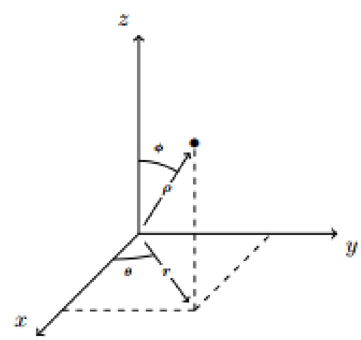

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7 Function of several variables

Definition 1.

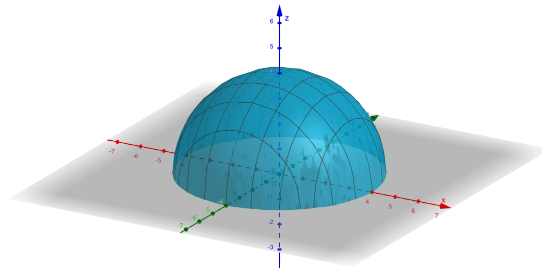
(a) *A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$.*

(b) *We often write $z = f(x, y)$.*

(c) *The graph of $z = f(x, y)$ is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.*

Example 1. *Consider $f(x, y) = \sqrt{16 - x^2 - y^2}$. Sketch the domain of f. Graph $z = f(x, y)$ using traces of $z = 0, x = 0, y = 0$.*

Proof. The domain us $D = \{(x, y) \in \mathbb{R}^2 : 16 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4^2\}$. This is the (closed) circle centered at $(0, 0)$ with radius 4. The trace of $z = 0, y = 0, x = 0$ give us $x^2 + y^2 = 16$, $z^2 + y^2 = 16$ and $z^2 + x^2 = 16$, i.e., in any cross-section it is a circle, therefore the graph of this function is a sphere of radius 4 in $\mathbb{R}^3$ (but only half of the sphere, the upper half as $z \geq 0$).

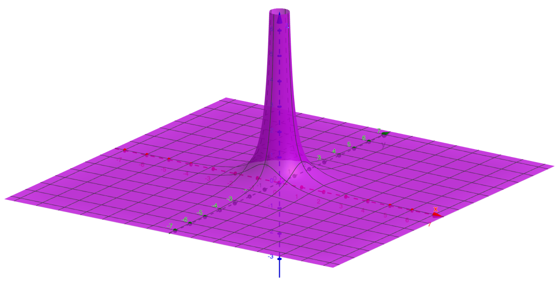


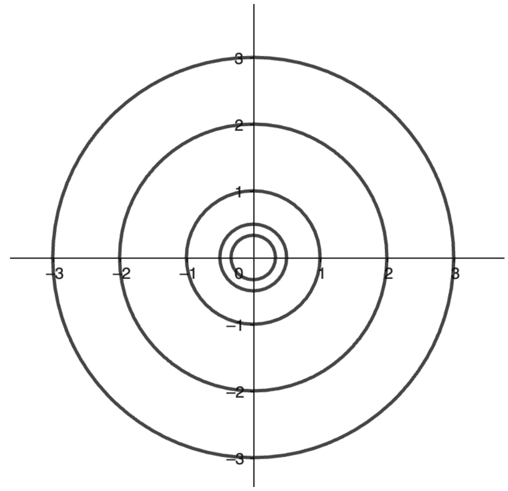
∎

Definition 2. *The contours of a function f of two variables are the curves with equations $f(x,y) = k$, where k is constant (in the range of f).*

Example 2. *Sketch the level curves of $f(x,y) = \frac{1}{x^2+y^2}$, with $k = \frac{1}{9}, \frac{1}{4}, 1, 4, 9$. Use these to attempt to sketch a 3D version of the graph.*

Proof. With $k = \frac{1}{3}$ we have $f(x,y) = \frac{1}{9}$ is equivalent to $x^2 + y^2 = 3^2$, it is a cirlce. Similarly with $k = \frac{1}{4}$ it is a circle $x^2 + y^2 = 4$. We have a set of cirles centered at $(0,0)$ with radius $3, 2, 1, \frac{1}{2}, \frac{1}{3}$, correspondingly to $k = \frac{1}{9}, \frac{1}{4}, 1, 4, 9$.



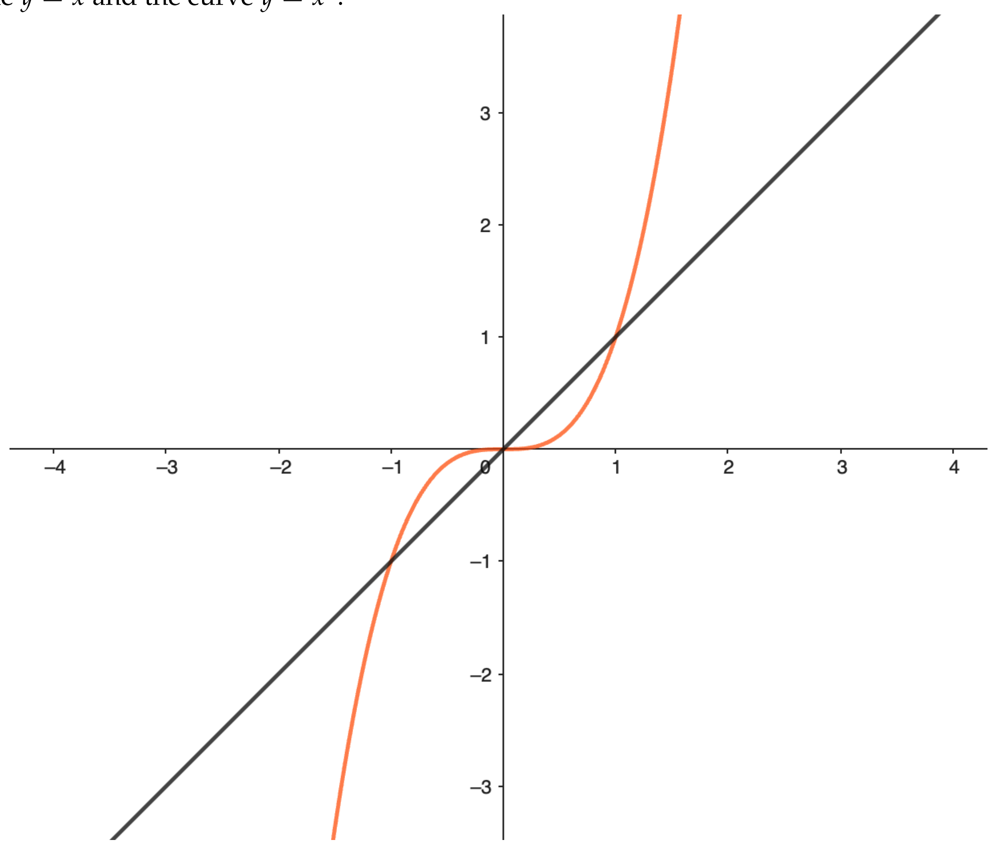


Note: if $f(x,y)$ is a 2-variables function then graph(f) is 3D (on the left), but its contours are 2D as in the picture (on the right). $\square$

Definition 3. *A function of 3 variables is $f(x,y,z)$ from a domain $D \subset \mathbb{R}^3$ to $\mathbb{R}$. The **level surfaces** of $f(x,y,z)$ are the surfaces with the equation $f(x,y,z) = k$ where k is a constant (by looking at level surfaces, we can view it in 3D, instead of the graph of f is in 4D).*

Example 3. *Find the domain of $f(x,y) = \frac{(x-1)(y+2)}{(y-x)(y-x^3)}$. Sketch and write the domain in set notation.*

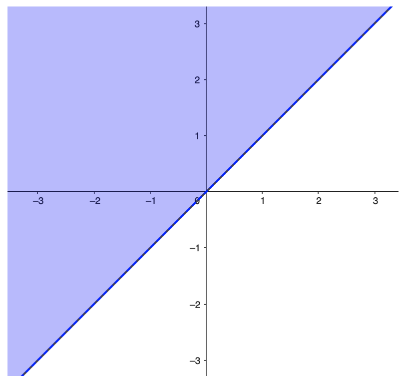
Proof. $D = \{(x,y) \in \mathbb{R}^2 : y \neq x, y \neq x^3\}$. The domain is the whole plane $\mathbb{R}$ (the xy-plane) removing the line $y = x$ and the curve $y = x^3$.



$\square$

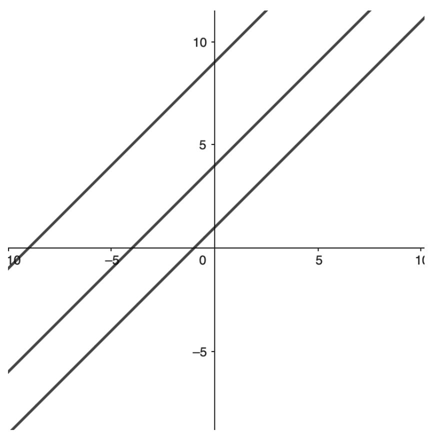
Example 4. *Consider the function $z = f(x,y) = \sqrt{y-x}$.*

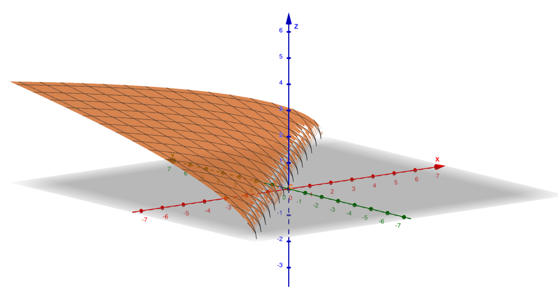
(a) Dmain $D = \{(x, y) : y - x \geq 0\} = \{(x, y) : y \geq x\}$. *(The line* $y = x$ *is included.)*



(b) The range is $z \in [0, +\infty)$.

(c) Sketch some level curves and the graph





8 Partial derivatives

Definition 4 (Partial Derivatives)**.**

(a) The partial derivatives of $f(x, y)$ *with respect to* x *at* (a, b) *is denoted by* $f_x(a, b)$ *and is given by*

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{|h|}.$$

This is equivalent to considering y *as a constant and taking derivative in* x.

(b) The partial derivatives of $f(x, y)$ *with respect to* y *at* (a, b) *is denoted by* $f_y(a, b)$ *and is given by*

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{|k|}.$$

This is equivalent to considering x *as a constant and taking derivative in* y.

(c) Other notations

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x, y) = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x, y) = \frac{\partial z}{\partial y} = D_y f.$$

(d) *Second-order derivatives:*

$$\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}.
\end{aligned}$$

Note the sequence: the first derivative is taken closest to the function.

(e) *(Clairaut's Theorem) The order of taking derivatives (around a point (a, b)) does not matter if the second order derivaties are continuous and defined around a point (a, b).*

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a,b) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a,b).$$

(f) *The gradient*

$$\nabla f(a,b) = \big(f_x(a,b), f_y(a,b)\big)$$

gives the direction in which the value of the function increases the fastest.

Example 5. *We have*

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(xy + x\sin y + \frac{y}{x}\right)\right) &= \frac{\partial}{\partial x}\left(x + x\cos y + \frac{1}{x}\right) = 1 + \cos y - \frac{1}{x^2} \\
\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(xy + x\sin y + \frac{y}{x}\right)\right) &= \frac{\partial}{\partial y}\left(y + \sin y - \frac{y}{x^2}\right) = 1 + \cos y - \frac{1}{x^2}
\end{aligned}$$

Example 6. *Let $v(x,y) = \frac{xy}{x-y}$. Compute v_x, v_{xx}, v_{xy}.*

Proof. We use product rule or quotient rule, or any rule from Calculus 1 and 2:

$$v_x = \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}, \qquad v_{xx} = \frac{-(-y^2)2(x-y)}{(x-y)^4} = \frac{2y^2}{(x-y)^3}$$

$$v_{xy} = \frac{-2y(x-y)^2 - (-y^2)2(x-y)}{(x-y)^4} = \frac{-2y+2y^2}{(x-y)^3}.$$

□

Example 7. *Find f_{xyz} for $f(x,y,z) = xyz + (x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}$.*

Proof. Note that $\frac{\partial}{\partial x}(\sin^{-1}(x)) = \frac{\partial}{\partial x}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$. We compute (product rule, then quotient rule)

$$f_x = yz + 2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right).$$

Let us not computing the derivative of that term for now, for a reason we will see soon. Now we have

$$\begin{aligned}
f_{xy} = \frac{\partial}{\partial y} f_x &= \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right)\right] \\
&= z + \frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right)\right].
\end{aligned}$$

Now we have

$$f_{xyz} = 1 + \frac{\partial}{\partial z}\underbrace{\frac{\partial}{\partial y}\left[2x\frac{\partial}{\partial x}\left((x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}\right)\right]}_{\text{no } z \text{ involved, thus this term is 0}}.$$

The intergral is zero, since there is no z involved, and we treat x, y as constants when taking $\frac{\partial}{\partial z}$. $\square$

Example 8. *Suppose you are surrounded by bees given by the bee density function*

$$B(x,y) = 100 - x^2 + y^2 + 3y \qquad \textit{bees/unit}^2.$$

You are currently standing at $(1,1)$. *Which of the four directions would be best to run in* $\{\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{j}, -\boldsymbol{j}\}$?

Proof. We have

$$\nabla B(x,y) = (-2x, 2y+3) \qquad \Longrightarrow \qquad \nabla B(1,1) = (-2,5).$$

The best direction to run would be the opposite of $(-2,5)$, i.e, $\mathbf{v} = (2,-5)$. Now among the 4 directions, we choose the one that is closest, i.e., taking the dot product to find the one with smallest angle, i.e., $\cos\theta$ the biggest:

- For $\mathbf{i} = (1,0)$ then $(1,0)\cdot(2,-5) = 2$.
- For $-\mathbf{i} = (-1,0)$ then $(-1,0)\cdot(2,-5) = -2$.
- For $\mathbf{j} = (0,1)$ then $(0,1)\cdot(2,-5) = -5$.
- For $-\mathbf{j} = (0,-1)$ then $(0,-1)\cdot(2,-5) = 5$.

Therefore we choose $-\mathbf{j}$. $\square$

9 Tangent planes

Example 9. *Find equation of the tangent plane of the surface* $z = f(x,y) = 3y^2 - 2x^2 + x$ *at the point* $(2,-1,3)$.

Proof.

- Step 1. We write the general form
$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$
- Step 2. Plug in $(x_0,y_0,z_0) = (2,-1,-3)$
$$\begin{aligned} z-(-3) &= f_x(2,-1)(x-2) + f_y(2,-1)(y-(-1)). \\ z+3 &= f_x(2,-1)(x-2) + f_y(2,-1)(y+1). \end{aligned}$$
- Step 3. Compute the partial derivatives
$$\begin{aligned} f_x(x,y) &= -4x+1, \qquad & f_x(2,-1) &= -7 \\ f_y(x,y) &= 6y, \qquad & f_y(2,-1) &= -6. \end{aligned}$$
- Step 4. Final answer
$$z+3 = -7(x-2) - 6(y+1).$$

$\square$

Example 10. *Find the linear approximation of* $f(x,y) = \frac{2x+3}{4y+1}$ *at* $(0,0)$*. Use it to approximate* $f(0.1,-0.2)$ *and* $f(0.01,-0.02)$*.*

Proof. Find the tangent plane at $(x_0,y_0) = (0,0)$ with $z_0 = f(x_0,y_0) = 3$.

- Step 1. We write the general form
$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$
- Step 2. Plug in $(x_0,y_0,z_0) = (0,0,3)$
$$\begin{aligned} z-3 &= f_x(0,0)(x-0) + f_y(0,0)(y-0. \\ z-3 &= f_x(2,-1)x + f_y(2,-1)y \end{aligned}$$
- Step 3. Compute the partial derivatives
$$\begin{aligned} f_x(x,y) &= \frac{2}{4y+1}, \qquad f_x(0,0) = 2 \\ f_y(x,y) = (2x+3)&\frac{-4}{(4y+1)^2}, \qquad f_y(0,0) = -12 \end{aligned}$$
- Step 4. The tangent plane
$$z+3 = 2x - 12y$$
- Step 5. The linear approximation
$$L(x,y) = 2x - 12y - 3$$
- Now plug in the value
$$L(0.1,-0.2) = 5.6 \qquad \text{while the true value} \qquad f(0.1,-0.2) = 16$$

  Here the change in x,y are 0.1 and -0.2. Now
$$L(0.01,-0.02) = 3.26 \qquad \text{while the true value} \qquad f(0.01,-0.02) = 3.28$$
  Here the change in x,y are 0.01 and -0.02.

We see that if the changes in x and y are small then the approximation is good! □

Example 11. *Consider* $z = f(x,y) = x^2 + 3xy - y^2$*. Find* dz*. If* x *changes from* $2 \to 2.05$ *and* y *changes from* $3 \to 2.96$*, compare the value of* Δz *(true differences) and* dz *(the total differential).*

Proof.

- Step 1. Here $(x_0,y_0) = (2,3)$ and $dx = 0.05$, $dy = -0.04$.
- Step 2. Write the total differential formula $dz = f_x(2,3)dx + f_y(2,3)dy$.
- Step 3. Compute the partial derivatives
$$\begin{aligned} f_x(x,y) &= 2x+3y, \qquad f_x(2,3) = 13 \\ f_y(x,y) &= 3x-2y, \qquad f_y(2,3) = 0 \end{aligned}$$
- Step 4. Plug in to the formula dz
$$dz = 13 \times (0.05) + 0 \times (-0.04) = 0.65$$
- Step 5. The true value
$$\begin{cases} f(2.05,-2.96) = (2.05)^2 - 3 \times (2.05) \times (-2.96) - (-2.96)^2 = 13.6449 \\ f(2,3) = 2^2 - 3 \times \times 3 - 3^2 = 13 \end{cases} \implies \Delta z = 0.06449$$
  We see that $\Delta z \approx dz$, but dz is much easier to compute.

□

Example 12. *The dimensions of a box are measure to be* $10cm, 5cm, 8cm$*. If each measurement is correct within* $0.2cm$*, approximate the largest possible error when the volume of the box is calculated from these measurements.*

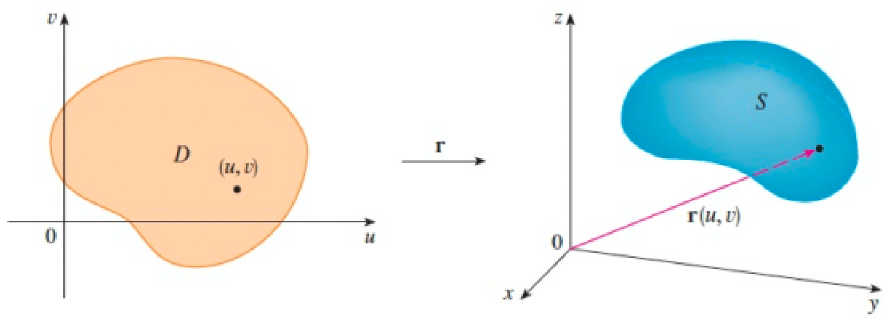
Proof. We have $V(x,y,z) = xyz$, thus

$$\begin{aligned} dV &= V_x dx + V_y dy + V_z dz \\ &= yz dx + xz dy + xy dz \\ &= 0.2(yz + xz + xy) = 0.2(10 \times 5 + t \times 8 + 8 \times 10) = 170 \times 0.2 = 34cm^3 \end{aligned}$$

The error is at most $34cm^3$. □

10 Parametric surfaces

- A curve is a function with one parameter $\mathbf{r}() = (x(t), y(t), z(t))$.
  1. Example 1. $\mathbf{r}(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$, this is a circle in xy-plane ($z = 0$).
  2. Example 2. $\mathbf{r}(t) = (t, 2t, 3t)$, $t \in \mathbb{R}$, this is a line going through $(0,0,0)$ with direction $\mathbf{v} = (1,2,3)$.
- A parametric surface is a function with two parameter $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in D$.



Example 13. $r(u,v) = (u, v, 1-u-v)$, $(u,v) \in \mathbb{R}^2$.

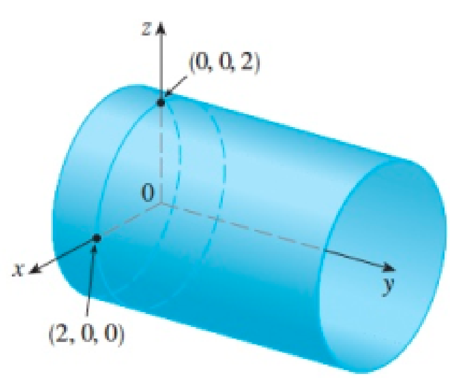
- *This is the plane* $x + y + z = u + v + (1 - u - v) = 1$.
- *We can also view it as*

  $$(u, v, 1-u-v) = (0,0,1) + (u, 0, -u) + (0, v, -v) = (0,0,1) + u(1,0,-1) + v(0,1,-1), \quad (u,v) \in \mathbb{R}^2.$$

  In this way, the plane is the one containing $(0,0,1)$ *and all vectors in the planes generated by* $(1,0,-1)$ *and* $(0,1,-1)$.

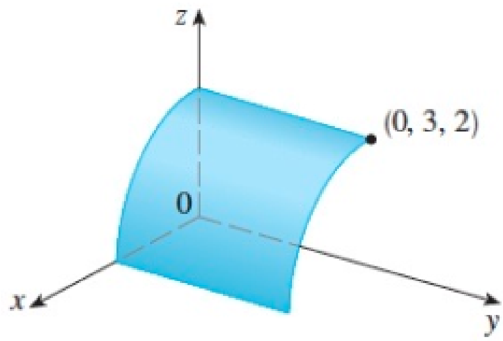
Example 14. $r(u,v) = (2\cos u, v, 2\sin u)$, $u \in [0, 2\pi]$, $v \in \mathbb{R}$.

- *Look at* $x = 2\cos u, y = v, z = 2\sin u$*, thus* $x^2 + z^2 = 4$*, while* $y \in \mathbb{R}$*. This is a cylinder.*



Example 15. $\boldsymbol{r}(u,v) = (2\cos u, v, 2\sin u)$, $u \in \left[0, \frac{\pi}{2}\right]$, $v \in [0,3]$.

- *Look at* $x = 2\cos u, y = v, z = 2\sin u$, *thus* $x^2 + z^2 = 4$, *while* $y \in \mathbb{R}$. *This is a cylinder.*
- *Note the angle* θ *in the Oxz-plane is* $\pi/4$, *thus only a quarter of the Oxz-plane is covered.*



11 Parametrize a surface in x, y, z

Example 16. *Find a parametric equation for* $x^2 + y^2 = 4, 0 \leq z \leq 1$.

Proof. We can use polar coordinates $x = 2\cos\theta, y = 2\sin\theta$ and $0 \leq z \leq 1$, thus

$$r(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

The domain is $D = \{(\theta, z) : 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1\}$. $\square$

Example 17. *Find a parametric equation for* $z = 2\sqrt{x^2 + y^2}, 0 \leq z \leq 1$.

Proof 1. We can just use the graph

$$r(x, y) = (x, y, z) = (x, y, 2\sqrt{x^2 + y^2}).$$

Note the condition $0 \leq z \leq 1$ means $0 \leq 2\sqrt{x^2 + y^2} \leq 1$, thus $x^2 + y^2 \leq \frac{1}{4}$. Therefore

$$D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \frac{1}{4}\right\}.$$

$\square$

Proof 2. We can use polar coordinates $x = r\cos\theta, y = r\sin\theta$ and $0 \leq z = 2r \leq 1$ which means $0 \leq r \leq \frac{1}{2}$, thus

$$\mathbf{r}(\theta, z) = (r\cos\theta, r\sin\theta, 2r)$$

The domain now is

$$D = \left\{(\theta, r) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq \frac{1}{2}\right\}.$$

$\square$

12 Grid

For a parametric surface $\mathbf{r}(u, v)$, if we:

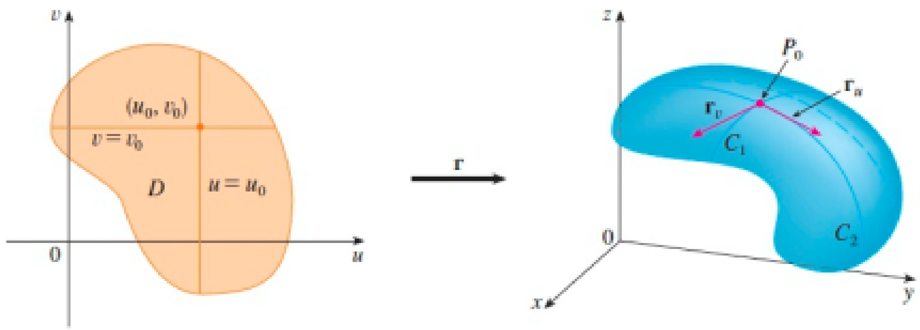
- Fix $u = u_0$, run v we get the images as a curvy grid on the surface
- Fix $v = v_0$, run u we get the images as a curvy grid on the surface

The two direction at each point $(x_0, y_0, z_0) = r(u_0, v_0)$ form a tangent plane at that point. The two directions here are the partial derivatives

$$\mathbf{r}_u \quad \text{and} \quad \mathbf{r}_v.$$

The normal vector of the tangent plane is

$$\mathbf{n} = r_{\mathbf{u}} \times r_{\mathbf{v}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$



Example 18. *Find the tangent plane of $x = u^2, y = v^2, z = u + 2v$ at $(1, 1, 3)$.*

Proof.

- Step 1. Solve for (u, v):

$$\begin{cases} x = u^2 = 1 \\ y = v^2 = 1 \\ z = u + 2v = 3 \end{cases} \implies \begin{cases} u = \pm 1 \\ v = \pm 1 \\ u + 2v = 3 \end{cases} \implies \begin{cases} u = 1 \\ v = 1 \end{cases}$$

- Step 2. Compute the partial derivatives of $\mathbf{r}(u, v) = (u^2, v^2, u + 2v)$

$$\mathbf{r}_u = (2u, 0, 1)$$
$$\mathbf{r}_v = (0, 2v, 2).$$

- Step 3. Plug in the value $u = v = 1$ to get

$$\begin{cases} \mathbf{r}_u = (2, 0, 1) \\ \mathbf{r}_u = (0, 2, 2) \end{cases}$$

- Step 4. Compute the normal by cross product

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2, -4, 4)$$

- The tangent plane with normal $(-2,-4,-4)$ going through $(1,1,3)$ is

$$\boxed{-2(x-1)-4(y-1)-4(z-3)=0.}$$

$\square$

Example 19. *Find the tangent plane of $x=u^2+1, y=v^3+1, z=u+v$ at $(5,2,3)$.*

Proof.

- Step 1. Solve for (u,v):

$$\begin{cases} x=u^2+1=5 \\ y=v^3+1=2 \\ z=u+v=3 \end{cases} \implies \begin{cases} u=2 \\ v=1. \end{cases}$$

- Step 2. Compute the partial derivatives of $\mathbf{r}(u,v)=(u^2+1,v^3+1,u+v)$

$$\begin{aligned} \mathbf{r}_u &= (2u,0,1) \\ \mathbf{r}_v &= (0,3v^2,1). \end{aligned}$$

- Step 3. Plug in the value $u=2, v=1$ to get

$$\begin{cases} \mathbf{r}_u = (4,0,1) \\ \mathbf{r}_u = (0,3,1) \end{cases}$$

- Step 4. Compute the normal by cross product

$$\mathbf{n}=\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}=(-3,-4,12)$$

- The tangent plane with normal $(-3,-4,12)$ going through $(5,2,3)$ is

$$\boxed{-3(x-5)-4(y-2)+12(z-3)=0.}$$

$\square$