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raSAT: SMT Solver for Polynomial Inequality

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Abstract. This paper presents an SMT (Satisfiability Modulo Theory) solver **raSAT** for polynomial inequality. It consists of a simple iterative approximation refinement, called **raSAT** loop, which is an extension of the standard ICP (Interval Constraint Propagation) with testing. Two approximation schemes consist of interval arithmetic (over-approximation) and testing (under-approximation), to accelerate SAT detection. If both fails, input intervals are refined by decomposition.
ICP is robust for large degrees, but the number of boxes (products of intervals) to explore exponentially explodes with respect to the number of variables. We design strategies for boosting SAT detection on the choice of a variable to decompose and a box to explore.
Several heuristic measures, called *SAT likelihood*, *sensitivity*, and the number of unsolved atomic polynomial constraints, are compared on Zankl and Meti-tarski benchmarks from QF_NRA category of SMT-LIB. They are also evaluated by comparing **Z3 4.3**, **dReal-2.15.01** and **iSAT3**. raSAT loop is extended with the use of the Intermediate Value Theorem to solve equality. This extension is evaluated on equalities of Zankl, Meti-tarski and Keymaera families. We also show a simple modification to handle mixed integers, and experiments on AProVE benchmark from QF_NIA category of SMT-LIB.

1 Introduction

Polynomial Constraint solving over real (integer) numbers is to find an assignment from real (integer) numbers to variables that satisfies given polynomial inequality/equality. Many applications are reduced to solving polynomial constraints, such as

- **Locating roundoff and overflow errors**, which is our motivation [1, 2].
- **Automatic termination proving**, which reduces termination detection to finding a suitable ordering [3], e.g., T$_2$ [3], AProVE [4].
- **Loop invariant generation**. Farkas's lemma is a popular approach in linear loop invariant generation [4], and is reduced to degree 2 polynomials. Non-linear loop invariant [5] requires more complex polynomials.

[3] http://cl-informatik.uibk.ac.at/software/ttt2/
[4] http://aprove.informatik.rwth-aachen.de

- **Hybrid system**. SMT solvers for polynomial constraints over real numbers (QF_NRA) are often used as backend engines [6].
- **Mechanical contrnol design**. Proportional-integral-derivative controllers are simple but widely used, and designing parameters is reduced to polynomial constraints [7].

Solving polynomial constraints on real numbers is decidable [8], though that on integers is undecidable (*Hilbert's 10th problem*). Quantifier elimination by cylindrical algebraic decomposition (QE-CAD) [9] is a well known technique, and implemented in Mathematica, Maple/SynRac, Reduce/Redlog, QEPCAD-B, and recently in some SMT solvers [10]. It can solve general formula at the cost of DEXPTIME, which hardly work up to 8 variables and degree 10. Satisfiability targets on an existential problem, and *Variant quantifier elimination* [11] reduces polynomial constraint solving to polynomial optimization problems, which are solved by Groebner basis in EXPTIME.

A practical alternative is Interval Constraint Propagation (*ICP*), which are used in SMT solver community, e.g., **iSAT3** [12], **dReal** [13], and **RSolver** [14]. ICP is based on over-approximation by interval arithmetics, and iteratively refines by interval decompositions. It is practically often more efficient than algebraic computation with weaker theoretical completeness.

This paper presents an SMT solver **raSAT** for polynomial inequality. It consists of a simple iterative approximation refinement, called **raSAT** *loop*, which is an extension of the standard ICP with testing to accelerate SAT detection. Two approximation schemes consist of interval arithmetic (over-approximation) and testing (under-approximation), to accelerate SAT detection. If both fails, input intervals are refined by decomposition. Compared to typical ICP solvers, **raSAT**

- introduces testing (as an under-approximation) to accelerate SAT detection,
- applies various interval arithmetic, e.g., Affine intervals [15, 1, 16], which enables to analyze the effects of input values, and
- SAT confirmation step by an error-bound guaranteed floating point package **iRRAM**[5], to avoid soundess bugs caused by roundoff errors.

This design is more on SAT detection oriented, since from our preliminary experiences, if the target problems have several hundred variables, solvable cases in practice are either SAT or UNSAT with small UNSAT core. Thus, acceleration of SAT detection and finding UNSAT core will be keys for scalability.

As **iSAT3**, **raSAT** applies outward rounding [17] in Interval arithmetics to avoid soundless bugs due to round-off error of floating arithmetic operations. As a consequence, answers of raSAT (SAT or UNSAT) (SAT instances found in testing is verified by **iRRAM**) are guaranteed to be sound.

ICP is robust for larger degrees, but the number of boxes (products of intervals) to explore exponentially explodes when variables increase. Thus, design of strategies for selecting variables to decompose and boxes to explore is crucial for efficiency. Our strategy design is,

[5] `http://irram.uni-trier.de`

- a box with more possiblity to be SAT is selected to explore, which is estimated by several heuristic measures, called *SAT likelihood*, and the number of unsolved atomic polynomial constraints, and
- a more influential variable is selected for multiple test cases and decomposition, which is estimated by *sensitivity*.

Note that *SAT likelihood* and *sensitivity* are estimated using interval arithmetic. Especially, the latter can be applied only with Affine intervals. **raSAT** also applies incremental search, which is often faster in practice.

- **Incremental widening**. Starting **raSAT** loop with a smaller interval, and if it is UNSAT, enlarge the input intervals and restart.
- **Incremental deepening**. Starting with the bound that each interval will be decomposed no smaller than it. If neither SAT nor UNSAT is detected, take a smaller bound and restart.

Efficient UNSAT core is left for future work.

They are compared on Zankl and Meti-Tarski benchmarks from QF_NRA category of SMT-LIB[6]. They are also evaluated by comparing **Z3 4.3**[7] and **iSAT3**. Another advantage of **raSAT** is the ease to handle mixed intergers, and experiments on AProVE benchmark from QF_NIA category of SMT-LIB compares **raSAT** with **Z3 4.3**. Although **Z3 4.3** performs the best, **raSAT** shows comparable SAT detection on very large problems (e.g., with several hundred variables) with the combination of *SAT likelihood* and *sensitivity*.

Related Work

Non-linear constraints are still under development, and SMT solvers adapt several approachs other than ICP.

QE-CAD. RAHD [18] and Z3 4.3 (which is referred as nlsat in [10]) include QE-CAD. QE-CAD is precise and detects beyond SAT instances (e.g., SAT regions), scalability is still challenging, since it is DEXPTIME.

Virtual substitution (VS). SMT-RAT toolbox [19][20] combines VS, incremental DPLL, and eager theory propagation. Z3 (version 3.1), the winner of QF_NRA in SMT competition 2011, combines VS, ICP, and linearization.

Bit-blasting. Bid-blasting in bounded bit width is often used in SMTs for QF_NIA. UCLID [21] reduces the number of bits (i.e., narrowing bounds for SAT instances) as an under-approximation, and removes clauses as an over-approximation. They refine each other, which shares a similar sprit with **raSAT** loop. MiniSmt [22], the winner of QF_NRA in SMT competition 2010, applies it for rational numbers with symbolic representations for prefixed algebraic numbers. MiniSmt can show SAT quickly with small degree polynomials, but due to

[6] `http://www.smtlib.org/`
[7] `http://z3.codeplex.com`

the bounded bit encoding, it cannot conclude UNSAT. Bit-blasting also suffers a lot when the degree of polynomials increases.

Linearization. Linearization of polynomials is often used over integers, such as Barcelogic [23], which substitutes all possible integers in a given-bound to an argument of a multiplication. Then, multiplications are reduced to an exhaustive search on linear constraints. CORD [24] uses another linearization, called CORDIC (COrdinate Rotation DIgital Computer) for real numbers. Both Barcelogic and CORD apply Yices for solving linear constraints. Linearization also suffers a lot when the degree of polynomials increases. Because **raSAT** in the same category of using ICP with iSAT3 and dReal, next part is going to take a look at details of methodologies used in these solvers.

iSAT3

Although **iSAT3** also uses Interval Arithmetic (IA), its algorithm integrates IA with DPLL procedure [25] tighter than that of **raSAT**. During DPLL procedure, in addition to an assignment of literals, **iSAT3** also prepares a data structure to store interval boxes where each box corresponds to one decision level of DPLL procedure's assignment. In **UnitPropagation** rule, intead of using standard rule, **iSAT3** searches for clauses that have all but one atoms being inconsistence with the current interval box. When some atom are selected for the literals assignment, this tool tries to use the selected atoms to contract the corresponding box to make it smaller. In order to do this, **iSAT3** convert each inequality/equation in the given constraints into the conjunction of the atoms of the following form by introducing additional variables:

$$\begin{array}{ll} \text{atom} & ::= \text{bound} \mid \text{equation} \\ \text{bound} & ::= \text{variable relation rational\_constant} \\ \text{relation} & ::= <|\leq|=|\geq|> \\ \text{equation} & ::= \text{variable} = \text{variable bop variable} \\ \text{bop} & ::= + \mid - \mid \times \end{array}$$

In other words, the resulting atoms are of the form, e.g., either $x > 10$ or $x = y + z$. For example, the constraint

$$x^2 + y^2 < 1$$

is converted into:

$$\begin{cases} x_1 = x^2 \\ x_2 = y^2 \\ x_3 = x_1 + x_2 \\ x_3 < 1 \end{cases}$$

From the atoms of these form, the contraction can be easily done for interval boxes:

- For the bound atom of the form, e.g., $x > 10$, if the bound is $x \in \langle 0, 100 \rangle$, then the contracted box contain $x \in \langle 10, 100 \rangle$.

- For the equations of three variables $x = y$ bop z, from bounds of any two variables, we can infer the bound for the remaining one. For example, from

$$\begin{cases} x = y.z \\ x \in \langle 1, 10 \rangle \\ y \in \langle 3, 7 \rangle \end{cases}$$

we can infer that

$$z \in \langle \frac{1}{7}, \frac{10}{3} \rangle$$

When the **UnitPropagation** and contraction can not be done, **iSAT3** split one interval (decomposition) in the current box and select one decomposed interval to explore which corresponds to **decide** step. If the contraction yields an empty box, a conflict is detected and the complement of the bound selection in the last split needs to be asserted. This is done via **learn** the causes of the conflict and **backjump** to the previous bound selection of the last bound selection. In order to reason about causes of a conflict, **iSAT3** maintains an implication graph to represents which atoms lead to the asserting of one atom.

dReal

In stead of showing satisfiability/unsatisfiability of the polynomial constraints φ over the real numbers, **dReal** proves that either

- φ is unsatisfiable, or
- φ^δ is satisfiable.

Here, φ^δ is the δ-weakening of φ. For instance, the δ-weakening of $x = 0$ is $|x| \leq \delta$. Any constraint with operators in $\{<, \leq, >, \geq, =, \neq\}$ can be converted into constraints that contains only $=$ by the following transformations.

- **Removing** $\neq$: Each formula of the form $f \neq 0$ is transformed into $f > 0 \vee f < 0$.
- **Removing** $<$ **and** $\leq$: Each formula of the form $f < 0$ or $f \leq 0$ is transformed into $-f \geq 0$ or $-f > 0$ respectively.
- **Removing** $>$ **and** $\geq$: Each formula of the form $f > 0$ or $f \geq 0$ is transformed into $f - x = 0$ by introducing an auxiliary variable x that has bound $[0, m]$ or $(0, m]$ respectively. Here, m is any rational number which is greater than the maximum of f over intervals of variables. As the result, **dReal** requires the input that ranges of variables must be compact.

Note that the satisfiability of φ^δ does not imply that of φ. **dReal**'s methodology [26] also cooperates DPLL with ICP in the lazy manner as in **raSAT**.

2 Over and Under Approximation Theories and Their Refinement

2.1 Approximation Theory

Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n . \bigwedge_j \psi_j(x_1, \cdots, x_n)$, where $\psi_j(x_1, \cdots, x_n)$ is an atomic formula of the form $p_j(x_1, ..., x_n) \circ 0$ with $p_j(x_1, ..., x_n)$ is a polynomial over variables $x_1, ..., x_n$ and $\circ \in \{>, <\}$. F is equivalnet to $\exists x_1 \ldots x_n . (\bigwedge_i x_i \in I_i) \wedge (\bigwedge_j \psi_j(x_1, \cdots, x_n))$, and we call $\bigwedge_i x_i \in I_i$ *interval constraints*, and we refer $\bigwedge_j \psi_j(x_1, \cdots, x_n)$ by $\psi(x_1, \cdots, x_n)$. Initially, interval constraints have a form of the conjunction $\bigwedge_i x_i \in I_i$, and later by refinement, $x_i \in I_i$ is decomposed into a clause $\bigvee_k x_i \in I_{i_k}$, which makes a CNF.

As an SMT (SAT modulo theory) problem, boolean variables are assigned to each $x_i \in I_{i_k}$, and truth assignments is produced by a SAT solver, which are proved or disproved by a background theory T whether it satisfies $\psi(x_1, \cdots, x_n)$. As notational convention, m (the lower case) denotes an assignment $\{x_i \mapsto r_i \mid i \in \{1, \cdots, n\}\}$ from real numbers r_i's to x_i's, and M (the upper case) denotes a truth assignment on $x_i \in I_{i_k}$'s. We write $m \in M$ when for all $i \in \{1, \cdots, n\}$, we have $c_i \in I_{i_k}$ for all $x_i \in I_{i_k}$'s that are assigned true by M.

We assume *very lazy theory learning* [25], and a backend theory T is applied only for a full truth assignment M.

- If an instance m satisfies $\psi(x_1, \cdots, x_n)$, we denote $m \models_T \psi(x_1, \cdots, x_n)$.
- If each instance m with $m \in M$ satisfies $\psi(x_1, \cdots, x_n)$, we denote $M \models_T \psi(x_1, \cdots, x_n)$.

Definition 1. *Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n . \psi(x_1, \cdots, x_n)$. For a truth assignment on M, F is*

- *T-valid if $M \models_T \psi(x_1, \cdots, x_n)$,*
- *T-satisfiable (T-SAT) if $m \models_T \psi(x_1, \cdots, x_n)$ for some $m \in M$, and*
- *T-unsatisfiable (T-UNSAT) if $M \models_T \neg\psi(x_1, \cdots, x_n)$.*

If T is clear from the context, we simply say valid, satisfiable, and unsatisfiable.

Definition 2. *Let $T, O.T, U.T$ be theories.*

- *$O.T$ is an* over-approximation theory *(of T) if $O.T$-UNSAT implies T-UNSAT, and*
- *$U.T$ is an* under-approximation theory *(of T) if $U.T$-SAT implies T-SAT.*

We further assume that $O.T$-valid implies T-valid.

A typical ICP applies $O.T$ only as an interval arithmetic. Later in Section 3, we will instantiate interval arithmetic as $O.T$. Adding to $O.T$-valid, **raSAT** introduce testing as $U.T$ to accelerate SAT detection.

2.2 Over-Approximation Theory Refinement

From now on, We focus on a *polynomial inequality* such that I_i and $\psi_j(x_1, \cdots, x_n)$ are an open interval (a_i, b_i) and an atomic polynomial inequaltiy (API) $f_j > 0$, respectively. We denote $\mathbb{S}(f_j) = \{x \in \mathbb{R}^n \mid f_j > 0 \text{ holds}\}$.

For ICP, it is folklore that, for polynomial inequality $\exists x_1 \in (a_1, b_1) \cdots x_n \in (a_n, b_n). \wedge_j f_j > 0$,

- if $\exists x_1 \in (a_1, b_1) \cdots x_n \in (a_n, b_n). \wedge_j f_j > 0$ is SAT, ICP eventually detects it, and
- if $\exists x_1 \in [a_1, b_1] \cdots x_n \in [a_n, b_n]. \wedge_j f_j \geq 0$ is UNSAT, ICP eventually detects it,

under the assumptions of fair decomposition and bounded intervals (a_i, b_i) for all $i \in \{1, \cdots, n\}$. We will prepare terminology and briefly review this fact.

Definition 3. *An* open box *of dimension n is a set $(a_1, b_1) \times \cdots \times (a_n, b_n)$ where $a_i, b_i \in \mathbb{R}, a_i \leq b_i$. For $\mathfrak{a} = (a_1, \cdots, a_n)$ and $\mathfrak{b} = (b_1, \cdots, b_n)$, we denote $(a_1, b_1) \times \cdots \times (a_n, b_n)$ by $(\mathfrak{a}, \mathfrak{b})$.*

The set of all open boxes is a basis of Euclidean topology on $\mathbb{R}^n$. In $\mathbb{R}^n$, a set U is compact if, and only if, U is a bounded closed set. We denote a closure of a set U by $\overline{U}$. Since a polynomial is continuous, $\mathbb{S}(\bigwedge_{j=1}^{m} f_j > 0)$ is an open set. Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, and an SAT instance in reals can be replaced with one in rationals.

Initially, interval constraints consists of conjunction only. Later, by refinements, it becomes a CNF.

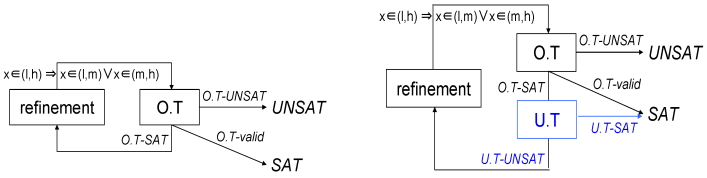
Example 1. $\exists x \in (-1, 3)\ y \in (2, 4).(x^3y - y^4 > 0) \wedge (y^3 - xy > 0)$ is an example of a polynomial inequality with 2 variables and 2 APIs.

For instance, $x \in (-1, 3)$ and $y \in (2, 4)$ are refined to smaller intervals such that $\exists x \in (-1, 1) y \in (2, 4).(x^3y - y^4 > 0) \wedge (y^3 - xy > 0) \vee \exists x \in (1, 3) y \in (2, 4).(x^3y - y^4 > 0) \wedge (y^3 - xy > 0)$, which results a CNF $(x \in (-1, 1) \vee x \in (1, 3)) \wedge (y \in (2, 4)) \wedge (x^3y - y^4 > 0) \wedge (y^3 - xy > 0)$.

Note that an interval arithmetic used in ICP is a converging theory.

Definition 4. *Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$ be a polynomial inequality such that each I_i is bounded. An over-approximation theory $O.T$ is* converging *if, for each $\delta > 0$ and $c = (c_1, \cdots, c_n) \in I_1 \times \cdots \times I_n$, there exists $\gamma > 0$ such that $\bigwedge_{i=1}^{n} x_i \in (c_i - \gamma, c_i + \gamma) \models_{O.T} \bigwedge_{j=1}^{m} (f_j(c) - \delta < f_j(x) < f_j(c) + \delta)$.*

$O.T$ refinemnet loop is shown in Fig. 1 (a). A standard ICP based algorithm of an SMT solver applies it with $O.T$ as a classical interval arithemtic. The variation of interval arithemtic will be presented in Section 3.



(a) $O.T$ refinement loop **raSAT** loop

Fig. 1. Rfinement loops

Definition 5. *Let* $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$ *for* $I_i = (a_i, b_i)$. *A refinement strategy is* fair, *if, for each* $c_i \in (a_i, b_i)$ *and* $\gamma > 0$, *a decomposition of* I_i *for each* i *eventually occurs in* $(c_i - \gamma, c_i + \gamma)$ *(as long as neither SAT nor UNSAT is detected).*

Theorem 1. *Let* $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$ *for* $I_i = (a_i, b_i)$. *Assume that an over-approximation theory* $O.T$ *is converging, each* (a_i, b_i) *is bounded, and a refinement strategy is fair. Then,*

- *if* $\exists x_1 \in (a_1, b_1) \cdots x_n \in (a_n, b_n). \wedge_j f_j > 0$ *is SAT,* $O.T$ *refinement loop eventually detects it, and*
- *if* $\exists x_1 \in [a_1, b_1] \cdots x_n \in [a_n, b_n]. \wedge_j f_j \geq 0$ *is UNSAT,* $O.T$ *refinement loop eventually detects it.*

Proof. The former is proved by the fact that, if F is SAT, there exists a non-empty neiborhood (open box) in $\cap\, \mathbb{S}(f_j)$. If the box decomposition strategy is fair, the refinemnet will eventually find such an open box.

For the latter, assume that $\overline{F} = \exists x_1 \in [a_1, b_1] \cdots x_n \in [a_n, b_n]. \wedge_j f_j \geq 0$ is UNSAT. Thus, $\cap\, \overline{\mathbb{S}(f_j)} = \emptyset$. Let $\delta_{j,k} = min\{|f_j(\bar{x}) - f_k(\bar{x})| \mid \bar{x} \in I_1 \times \cdots \times I_n\}$. Since f_i's are continuous and $\overline{I_i}$'s are compact, $\delta_{j,k}$ is well defined, and $\delta_{j.k} > 0$ for some j, k. Let $\delta = \frac{min\{\delta_{j,k}\}}{2}$. Since $O.T$ is converging, there exists $\gamma > 0$ for $\delta > 0$ satisfying Definition 4, and fair decomposition eventually finds open boxes such that $\mathbb{S}(f_j)$ and $\mathbb{S}(f_k)$ are separated. □

Limitations for detecting UNSAT occur on *kissing* and *convergent* cases. Fig. 2 left shows a kissing case $x^2 + y^2 < 2^2 \wedge (x-4)^2 + (y-3)^2 < 3^2$ such that $\overline{\mathbb{S}(-x^2 - y^2 + 2^2)} \cap \overline{\mathbb{S}(-(x-4)^2 - (y-3)^2 + 3^2)} = \{(x, y) \mid (1.6, 1.2)\}$. Thus, there are no coverings to separate them. Fig. 2 right shows a convergent case $y > x + \frac{1}{x} \wedge y < x \wedge x > 0$, which is equivalent to $xy > x^2 + x \wedge y < x \wedge x > 0$. There are no finite coverings to separate them.

2.3 raSAT loop

Although an $O.T$ refinement loop is enough to implement an ICP based SMT solver, we extend it as **raSAT** (SAT by refinement of approximations) loop to accelerate SAT detection by adding $U.T$, which works as in Fig. 1 (b).

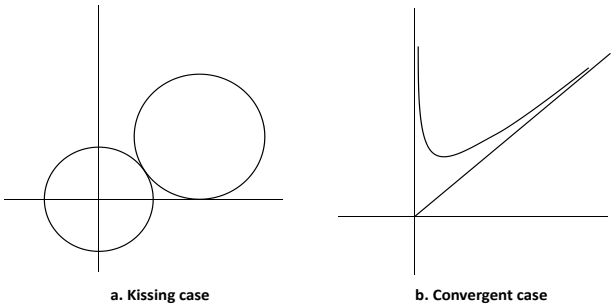


Fig. 2. Limitations for proving UNSAT

1. When an over-approximation theory $O.T$ detects $O.T$-UNSAT (resp. $O.T$-valid), answer UNSAT (resp. SAT).
2. When an under-approximation theory $U.T$ detects $U.T$-SAT, answer SAT.
3. If neither holds, a refinement is applied.

Our design of an SMT solver **raSAT** applies two heuristic features.

- Incremental widening intervals, and incremental deeping search (Section 4.1).
- Heurstic measures *SAT-likelihood* and *sensitivity*, for selection of a variable to decompose and a box to explore. (Section 4.2).

raSAT also prepares various interval arithmetics as $O.T$ as in Section 3, whereas currently only random tesing (*k-random ticks*, which consists of periodical k-test instances with a random offset) is prepared as $U.T$.

3 Interval arithmetic

A typical theory for $O.T$ and $U.T$ are an interval arithmetic and testing, respectively. We say *IA-valid*, *IA-SAT*, and *IA-UNSAT*, when it is $O.T$-valid, $O.T$-SAT, and $O.T$-UNSAT, respectively. Similarly, we say *test-SAT* when it is $U.T$-SAT and *test-UNSAT* when $U.T$-UNSAT. Note that either IA-valid or test-SAT implies SAT, and IA-UNSAT implies UNSAT, whereas IA-SAT and test-UNSAT can conclude neither.

raSAT prepares various Affine intervals, adding to classical interval (CI) [27], which keep lower and upper bounds. The weakness of CI is loss of dependency among values. For instance, $x - x$ is evaluated to $(-2, 2)$ for $x \in (2, 4)$.

Affine Interval [28, 29] introduces *noise symbols* ϵ, which are interpreted as values in $(-1, 1)$. For instance, $x = 3 + \epsilon$ describes $x \in (2, 4)$, and $x - x = (3 + \epsilon) - (3 + \epsilon)$ is evaluated to 0. The drawback is that the multiplication without dependency might be less precise than CI. Affine intervals also cannot represent infinite intervals, e.g., $(0, \infty)$, since it becomes $\infty + \infty\ \epsilon$. Forms of affine intervals vary by choices how to approximate multiplications. They are,

(i) $\epsilon\epsilon'$ is replaced with a fresh noise symbol (AF) [28, 29],
(ii) $\epsilon\epsilon'$ is reduced to the fixed error noise symbol $\epsilon_\pm$ (AF_1 and AF_2) [15],
(iii) $\epsilon\epsilon'$ is replaced with $(-1,1)\epsilon$ (or $(-1,1)\epsilon'$) (EAI) [1],
(iv) $\epsilon\epsilon$ is reduced to fixed noise symbols ϵ_+ or ϵ_- (AF_2) [15],
(v) Chebyshev approximation of x^2 introduces a noise symbol $|\epsilon|$ as an absolute value of ϵ with $\epsilon\epsilon = |\epsilon||\epsilon| = |\epsilon| + (-\frac{1}{4}, 0)$ and $\epsilon|\epsilon| = \epsilon + (-\frac{1}{4}, \frac{1}{4})$ [16].

Example 2. Let $f = x^3 - 2xy$ with $x = (0,2)$ $(x = 1 + \epsilon_1)$ and $y = (1,3)$ $(y = 2 + \epsilon_2)$, we have,

- AF_2 estimates the range of f as $-3 - \epsilon_1 - 2\epsilon_2 + 3\epsilon_+ + 3\epsilon_\pm$, thus $(-9, 6)$,
- CAI estimates the range of f as $(-4, -\frac{11}{4}) + (-\frac{1}{4}, 0)\epsilon_1 - 2\epsilon_2 + \mathbf{3}|\epsilon_1| + (-2, 2)\epsilon_\pm$, thus $(-8, 4.5)$.

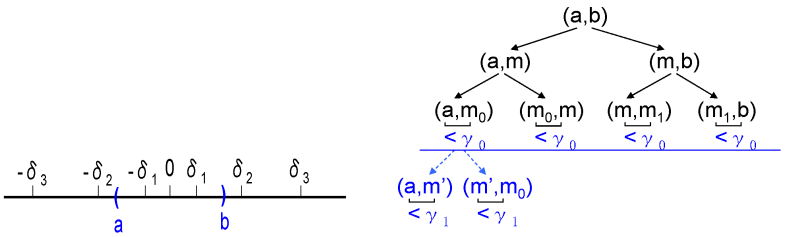
4 Strategies in raSAT

4.1 Incremental search

raSAT applies two incremental strategies, (1) *incremental windening*, and (2) *incremental deepening*. Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$ for $I_i = (a_i, b_i)$.

Incremental windening Given $0 < \delta_0 < \delta_1 < \cdots$, *incremental windening* starts with $F_0 = \exists x_1 \in I_1 \cap (-\delta_0, \delta_0) \cdots x_n \in I_n \cap (-\delta_0, \delta_0). \bigwedge_{j=1}^{m} f_j > 0$, and if it finishes with UNSAT, it runs with $F_1 = \exists x_1 \in I_1 \cap (-\delta_1, \delta_1) \cdots x_n \in I_n \cap (-\delta_1, \delta_1). \bigwedge_{j=1}^{m} f_j > 0$, and so on (Fig. 3 (a)).

If $\delta_i < \infty$, **raSAT** combines the result of an Affine interval (currently AF2) with that of CI by taking the conjunction of two results; otherwise, it uses CI only. Experiments in Section 5 are performed with $\delta_0 = 10$ and $\delta_1 = \infty$.



(a) Incremenal widening (b) Incremental Deepening

Fig. 3. Chebyshev approximation of x^2 and $x\ |x|$

Incremental deepening Starting with $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$, $I_1 \times \cdots \times I_n$ is decomposed into many boxes, and F becomes the disjunction of existential formulae corrsponding to these boxes. **raSAT** searches these boxes in depth-first manner, which may leads to exhaustive local search. To avoid it, **raSAT** applies a threshold γ, such that no more decomposition will be applied when a box becomes smaller than γ. If neither SAT nor UNSAT is detected, **raSAT** restarts with a smaller threshhold.

Let $\gamma_0 > \gamma_1 > \cdots > 0$, and **raSAT** incrementally deepens its search with these threshholds, i.e., starting with δ_0, and if it fails, restart with δ_1, and so on (Fig 3 (b)).

4.2 SAT directed heuristics measure

With several hundred variables, we observe that an SMT solver works when either SAT, or UNSAT with small UNSAT core. For the latter, we need an efficient heuristics to find an UNSAT core, which is left as future work. For the former, the keys are how to choose variables to decompose, and how to choose a box to explore. **raSAT** chooses such a variable in two steps; first it selects a *test-UNSAT API*, and then chooses a variable that appears in the API. We design SAT-directed heuristic measures based on the interval arithemtic ($O.T$).

Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} f_j > 0$ becomes $\vee\ (\exists x_1 \in I'_1 \cdots x_n \in I'_n. \bigwedge_{j=1}^{m} f_j > 0)$ after box decomposition. For $\exists x_1 \in I'_1 \cdots x_n \in I'_n. \bigwedge_{j=1}^{m} f_j > 0$, if some $f_j > 0$ is UNSAT, the box $I'_1 \times \cdots \times I'_n$ is UNSAT. If every $f_j > 0$ is SAT, F is SAT. Thus, if the box $I'_1 \times \cdots \times I'_n$ needs to be explore, it must contain a test-UNSAT API (thus IA-SAT).

We denote the estimated range of f_j for $x_1 \in I'_1 \cdots x_n \in I'_n$ with IA ($O.T$) by $range(f_j, I'_1 \times \cdots \times I'_n)$. If an IA is an affine interval, it is in the form $[c_1, d_1]\epsilon_1 + \cdots + [c_n, d_n]\epsilon_n$, and we can obtain $range(f_j, I'_1 \times \cdots \times I'_n)$ by instantiating ϵ_i with $[-1, 1]$ for $i \in \{1, \cdots, n\}$. We define

- *Sensitivity* of a variable x_i in a test-UNSAT API $f_j > 0$ is $max(|c_i|, |d_i|)$.
- *SAT-likelihood* of an API $f_j > 0$ is $|I \cap (0, \infty)|/|I|$ where $I = range(f_j, I'_1 \times \cdots \times I'_n)$, and
- *SAT-likelihood* of a box $I'_1 \times \cdots \times I'_n$ is the least SAT-likelihood of test-UNSAT APIs.

Example 3. In Example 2,

- sensitivity is estimated 1 for x and 2 for y by AF_2, and $3\frac{1}{4}$ for x and 2 for y.
- SAT-likelihood of f is estimated $0.4 = \frac{6}{9-(-6)}$ by AF_2 and $0.36 = \frac{4.5}{4.5-(-8)}$ by CAI.

SAT-likelihood intends to estimate APIs how likely to be SAT. For choosing variables, **raSAT** first choose a test-UNSAT API by SAT-likelihood. There are two choices, either *the largest* or *the least*. *Sensitivity* of a variable intends to estimate how a variable is influential to the value of an API. From a selected API by SAT-likelihood, **raSAT** selects a variable with the largest sensitivity. This selection of variables are used for (1) *multiple test instances generation*, and (2) *decomposition*. For test generation, we will select multiple variables by repeating the selection.

For choosing a box to explore, **raSAT** chooses one which is more likely to be SAT. There are two choice, (1) a box with the largest SAT-likelihood, and (2) a box with the largest number of SAT (either IA-valid or test-SAT) APIs.

Test case generation using variables sensitivity. The value of variables sensitivity can also be used to approximate how likely the value of a polynomial increases when the value of that variable increases. Consider the constraint $f = -x_{15} * x_8 + x_{15} * x_2 - x_{10} * x_{16} > 0$. With $x_2 \in [9.9, 10], x_8 \in [0, 0.1], x_{10} \in [0, 0.1], x_{15} \in [0, 10]$, and $x_{16} \in [0, 10]$. The result of AF2 for f is: $0.25\epsilon_2 - 0.25\epsilon_8 - 0.25\epsilon_{10} + 49.5\epsilon_{15} - 0.25\epsilon_{16} + 0.75\epsilon_{+-} + 49.25$. The coefficient of ϵ_2 is 0.25 which is positive, then we expect that if x_2 increases, the value of f is likely to increase. As the result, the test case of x_2 is expected as high as possible in order to satisfy $f > 0$. We will thus take the upper bound value of x_2, i.e. 10, as a test case. Similarly, we take the test cases for other variables: $x_8 = 0, x_{10} = 0, x_{15} = 10, x_{16} = 0$. With these test cases, we will have $f = 100 > 0$.

5 Experiments

We implement **raSAT** loop as an SMT solver **raSAT**, based on MiniSat 2.2 as a backend SAT solver. Various combinations of strategies of **raSAT** (in Section 4) and random stategies are compared on *Zankl*, *Meti-Tarski* in NRA category and *AProVE* in NIA category from SMT-LIB. The best combination of choices are

1. a test-UNSAT API by the least SAT-likelihood,
2. a variable by the largest sensitivity, and
3. a box by the largest SAT-likelihood,

and sometimes a random choice of a test-UNSAT API (instead of the least SAT-likelihood) shows an equally good result. They are also compared with **Z3 4.3** and **iSAT3**, where the former is considered to be the state of the art ([10]), and the latter is a popular ICP based tool. Note that our comparison is only on polynomial inequality. The experiments are on a system with Intel Xeon E5-2680v2 2.80GHz and 4 GB of RAM.

5.1 Benchmarks from SMT-LIB

In SMT-LIB[8], benchmark programs on non-linear real number arithmetic (QF_NRA) are categorized into Meti-Tarski, Keymaera, Kissing, Hong, and Zankl families. Until SMT-COMP 2011, benchmarks are only Zankl family. In SMT-COMP 2012, other families have been added, and currently growing. General comparison among various existing tools on these benchmarks is summarized in Table.1 in [10], which shows Z3 4.3 is one of the strongest.

From them, we extract problems of polynomial inequality only. The brief statistics and explanation are as follows.

- **Zankl** has 151 inequalities among 166, taken from termination provers. A Problem may contain several hundred variables, an API may contain more than one hundred variable, and the number of APIs may be over thousands, though the maximum degree is up to 6.
- **Meti-Tarski** contains 5101 inequalities among 7713, taken from elementary physics. They are mostly small problems, up to 8 varaibles (mostly up to 5 variables), and up to 20 APIs.
- **Keymaera** contains 161 inequalities among 4442.
- **Kissing** has 45 problems, all of which contains equality (mostly single equality).
- **Hong** has 20 inequalities among 20, tuned for QE-CAD and quite artificial.

The setting of the experiments are

- For test data generation, raSAT chooses 10 variables (1 variable from each of 10 APIs with largest SAT-likelihood) and apply random 2-ticks, and single random test data is generated for each of the rest of variables.
- For interval decomposition, raSAT applies the balanced decomposition.
- For incremental widening, $\delta_0 = 10, \delta_1 = \infty$
- For incremental deepening, $\gamma_i = 10^{-(i+1)}$ for $i \geq 0$.

5.2 Experiments on strategy combinations

We perform experiments only on inequalities of Zankl, and Meti-Tarski families. Table 1 shows the experimental results of above mentioned combination. The timeout is set to 500s, and each time is the total of successful cases (either SAT or UNSAT). Our combinations of strategies are,

Selecting a test-UNSAT API	Selecting a box (to explore):	Selcting a variable:
(1) Least SAT-likelihood.	(3) Largest number of SAT APIs.	(8) Largest sensitibity.
(2) Largest SAT-likelihood.	(4) Least number of SAT APIs.	
	(5) Largest SAT-likelihood.	
	(6) Least SAT-likelihood.	
(10) Random.	(7) Random.	(9) Random.

[8] `http://www.smt-lib.org`

Benchmark	(1)-(5)-(8)		(1)-(5)-(9)		(1)-(6)-(8)		(1)-(6)-(9)		(10)-(5)-(8)		(10)-(6)-(8)	
Matrix-1 (SAT)	20	132.72 (s)	18	101.07 (s)	15	1064.76 (s)	14	562.19 (s)	**21**	462.57 (s)	18	788.46(s)
Matrix-1 (UNSAT)	2	0.01 (s)	2	0.01 (s)	2	0.01 (s)	2	0.01 (s)	2	0.01 (s)	2	0.01 (s)
Matrix-2,3,4,5 (SAT)	**10**	632.37 (s)	3	140.27 (s)	1	3.46 (s)	0	0.00 (s)	5	943.08 (s)	0	0.00 (s)
Matrix-2,3,4,5 (UNSAT)	8	0.37 (s)	8	0.39 (s)	8	0.37 (s)	8	0.38 (s)	8	0.38 (s)	8	0.38 (s)

Benchmark	(2)-(5)-(8)		(2)-(5)-(9)		(2)-(6)-(8)		(2)-(6)-(9)		(2)-(7)-(8)		(10)-(7)-(9)	
Matrix-1 (SAT)	20	163.47 (s)	21	736.17 (s)	19	953.97 (s)	18	1068.40 (s)	19	799.79 (s)	19	230.39 (s)
Matrix-1 (UNSAT)	2	0.00(s)	2	0.00 (s)	2	0.00 (s)	2	0.00 (s)	2	0.00 (s)	2	0.00 (s)
Matrix-2,3,4,5 (SAT)	5	514.37 (s)	1	350.84 (s)	0	0.00 (s)	0	0.00 (s)	0	0.00 (s)	1	13.43 (s)
Matrix-2,3,4,5 (UNSAT)	8	0.43 (s)	8	0.37 (s)	8	0.40 (s)	8	0.38 (s)	8	0.37 (s)	8	0.38 (s)

Benchmark	(1)-(3)-(8)		(1)-(4)-(8)		(2)-(3)-(8)		(2)-(4)-(8)		(10)-(3)-(8)		(10)-(4)-(8)	
Matrix-1 (SAT)	18	1438.47 (s)	20	1537.9 (s)	19	1100.60 (s)	17	916.32 (s)	17	87.78 (s)	20	710.21 (s)
Matrix-1 (UNSAT)	2	0.00 (s)	2	0.00(s)	2	0.00 (s)	2	0.00 (s)	2	0.00 (s)	2	0.00 (s)
Matrix-2,3,4,5 (SAT)	0	0.00 (s)	1	33.17 (s)	1	201.32 (s)	2	328.03 (s)	0	0.00 (s)	1	20.94 (s)
Matrix-2,3,4,5 (UNSAT)	8	0.36 (s)	8	0.36 (s)	8	0.34 (s)	8	0.37 (s)	8	0.37 (s)	8	0.39 (s)

Benchmark	(1)-(5)-(8)		(1)-(5)-(9)		(10)-(5)-(8)		(10)-(7)-(9)	
Meti-Tarski (SAT)	3322	369.60 (s)	3456	644.21 (s)	**3454**	747.25 (s)	3451	895.14 (s)
Meti-Tarski (UNSAT)	1052	383.40 (s)	1044	957.71 (s)	**1061**	321.00 (s)	1060	233.46 (s)

Table 1. Combnations of **raSAT** strategies on NRA/Zankl,Meti-Tarski benchmark

Note that (10)-(7)-(9) means all random selection. Generally speaking, the combination of (5) and (8) show the best results, though the choice of (1),(2), and (10) shows different behavior on benchmarks. We tentatively prefer (1) or (10), but it needs to be investigated further.

Experiments in Table 1 are performed with random generation (k-random tick) for the former and the blanced decomposition (dividing at the exact middle) for the latter.

Experiments with test case generation using variables sensitivity

From above section, we can see that the combination (1)-(5)-(8) shows the best performance on benchmarks. This section is going to examine the effectiveness of variables sensitivity in generation of test cases which is named as (11). Table 2 presents the result of the experiments on QF_NRA/Zankl and QF_NRA/Meti-tarski benchmarks, which show that this strategy made some improvements.

Benchmark	(1)-(5)-(8)		(1)-(5)-(8)-(11)	
Matrix-1 (SAT)	20	132.72 (s)	**25**	414.99(s)
Matrix-1 (UNSAT)	2	0.01(s)	2	0.01(s)
Matrix-2,3,4,5 (SAT)	10	632.37 (s)	**11**	1264.77(s)
Matrix-2,3,4,5 (UNSAT)	8	0.37(s)	8	0.38(s)
Meti-Tarski (SAT)	3322	369.60 (s)	3322	369.60 (s)
Meti-Tarski (UNSAT)	3322	369.60 (s)	1052	383.40 (s)

Table 2. Effectiveness of variables sensitivity on test cases generation

Benchmark	raSAT				Z3 4.3				iSAT3				dReal			
	SAT		UNSAT		SAT		UNSAT		SAT		UNSAT		δ−SAT		UNSAT	
Zankl/matrix-1 (53)	25	414.99 (s)	2	0.01 (s)	**41**	2.17 (s)	12	0.00 (s)	11	4.68 (s)	3	0.00 (s)	46	3573.43 (s)	0	0.00 (s)
Zankl/matrix-2,3,4,5 (98)	11	1264.77 (s)	8	0.38 (s)	**13**	1031.68 (s)	11	0.57 (s)	3	196.40 (s)	12	8.06 (s)	19	2708.89 (s)	0	0.00 (s)
Meti-Tarski (5101)	3322	369.60 (s)	1052	383.40 (s)	3528	51.22 (s)	1568	78.56 (s)	2916	811.53 (s)	1225	73.83 (s)	3523	441.35 (s)	1197	55.39 (s)
Keymaera (68)	0	0.00 (s)	16	0.06 (s)	0	0.00 (s)	68	0.36 (s)	0	0.00 (s)	16	0.07 (s)	8	0.18 (s)	0	0.00 (s)

Table 3. Comparison among SMT solvers over inequalities

5.3 Comparison with other SMT solvers

We compare **raSAT** with other SMT solvers on NRA benchmarks, Zankl and Meti-Tarski. The timeout is $500s$. For **iSAT3**, ranges of all variables are uniformly set to be in the range $[-1000, 1000]$ (otherwise, it often causes segmentation fault). Thus, UNSAT detection of **iSAT3** means UNSAT in the range $[-1000, 1000]$, while that of **raSAT**, **dReal** and **Z3 4.3** means UNSAT over $[-\infty, \infty]$. Another note is that if **dReal** concludes SAT, the constraint is δ-SAT, which cannot imply the satisfiability of the constraint. For instances, with a number of UNSAT problems in Zankl, **dReal** still concludes SAT.

Among these SMT solvers, **Z3 4.3** shows the best performance. However, if we closely observe, there are certain tendency. **Z3 4.3** is very quick for small constraints, i.e., with short APIs (up to 5) and a small number of variables (up to 10). **raSAT** shows comparable performance on SAT detection with longer APIs (larger than 5) and a larger number of variables (more than 10), and sometimes outforms for SAT detection on vary long constraints (APIs longer than 40 and/or more than 20 variables). Such examples appear in Zankl/matrix-3-all-*, matrix-4-all-*, and matrix-5-all-* (total 74 problems), and **raSAT** solely solves

- *matrix-3-all-2* (47 variables, 87 APIs, and max length of an API is 27),
- *matrix-3-all-5* (81 variables, 142 APIs, and max length of an API is 20),
- *matrix-4-all-3* (139 variables, 244 APIs, and max length of an API is 73), and
- *matrix-5-all-01* (132 variables, 276 APIs, and max length of an API is 47).

Note that, for Zankl, when UNSAT is detected, it is detected very quickly. This is because SMT solvers detects UNSAT when they find small UNSAT cores, without tracing all APIs. However, for SAT detection with large problems, SMT solvers need to trace all problems. Thus, it takes much longer time.

6 Extensions

6.1 Extensions for Equality Handling

Single Equation For solving polynomial constraints with single equality ($g = 0$), we apply *Intermediate Value Theorem*. That is, if existing 2 test cases such that $g > 0$ and $g < 0$, then $g = 0$ is SAT somewhere in between.

Lemma 1. *For* $F = \exists x_1 \in I_1 \cdots x_n \in I_n(\bigwedge_j^m f_j > 0 \ \wedge \ g = 0)$. *Suppose decomposition creates a box* $I = (l_1, h_1) \times \cdots \times (l_n, h_n)$ *where* $(l_i, h_i) \subseteq I_i$ *for all* $i \in \{1, \cdots, n\}$, *such that* $\bigwedge_j^m f_j > 0$ *is IA-VALID in the box. Let* $(l_g, h_g) = range(g, I)$.

(i) If $l_g > 0$ *or* $h_g < 0$, *then* F *is UNSAT in the box.*
(ii) If there are two instances $\boldsymbol{t}, \boldsymbol{t'}$ *in the box with* $g(\boldsymbol{t}) > 0$ *and* $g(\boldsymbol{t'}) < 0$, *then* F *is SAT.*

Proof. (i) If $l_g > 0$ or $h_g < 0$, then $g = 0$ cannot be satisfied in box I. As a result, F is UNSAT in I.

(ii) If there are two instances $\boldsymbol{t}, \boldsymbol{t'}$ in the box with $g(\boldsymbol{t}) > 0$ and $g(\boldsymbol{t'}) < 0$, it is clear from the Intermediate Value Theorem that there exist one point $\boldsymbol{t_0}$ between $\boldsymbol{t}$ and $\boldsymbol{t'}$ such that $g(\boldsymbol{t_0}) = 0$. In addition, because $\bigwedge_j^m f_j > 0$ is IA-VALID in I, $\boldsymbol{t_0}$ also satisfies $\bigwedge_j^m f_j > 0$. As a result, F is satisfiable with $\boldsymbol{t_0}$ as the SAT instance.

In the case that both (i) and (ii) do not occur, then **raSAT** continue by decomposition.

Example 4. Consider the constraint $\varphi = f(x, y) > 0 \wedge g(x, y) = 0$. Suppose we can find a box represented by $\Pi = x \in \langle a, b\rangle \wedge y \in \langle c, d\rangle$ such that $f(x, y) > 0$ is $\Pi^p_{\mathbb{R}}$-VALID (Figure 4). In addition, inside that box, we can find two points (u_1, v_1) and (u_2, v_2) such that $g(u_1, v_1) > 0$ and $g(u_2, v_2) < 0$. By Lemma 1, the constraint is satisfiable.

raSAT first tries to find a box of variables' intervals (by refinements) such that $\bigwedge_j^m f_j > 0$ is VALID inside that box. Then it tries to find 2 instances for $g > 0$ and $g < 0$ by testing. Intermediate Value Theorem guarantees the existence of an SAT instance in between. Note that this method does not find an exact SAT instance.

Multiple Equations The idea of using the Intermediate Value Theorem can also be used for solving multiple equations. Consider m equations ($m \geq 1$): $\bigwedge_{j=1}^m g_j = 0$ and an box $I = (l_1, h_1) \times \cdots (l_n, h_n)$. If we can find a set $\{V_1, \cdots, V_m\}$ that satisfies the following properties, then we can conclude that $\bigwedge_{j=1}^m g_j = 0$ is satisfiable in I.

- For all $j \in \{1, \cdots, m\}$, we have $V_j \subseteq var(g_j)$.
- For all $j_1 \neq j_2 \in \{1, \cdots, m\}$, we have $V_{j_1} \cap V_{j_2} = \emptyset$.

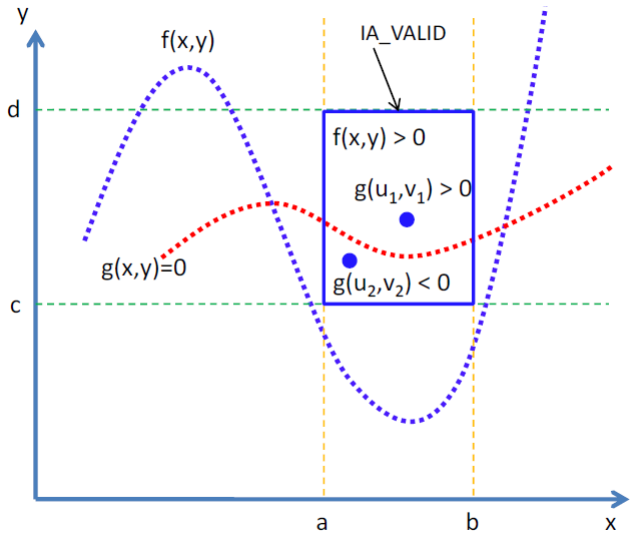


Fig. 4. Example on solving single equation using the Intermediate Value Theorem

- For all $j \in \{1, \cdots, m\}$, let $k_j = |V_j|$ and $V_j = \{v_{jk} \mid 1 \leq k \leq k_j\}$, then, there exist two combinations $(x_{j1}, \cdots, x_{jk_j}) = (t_{j1}, \cdots, t_{jk_j})$ and $(x_{j1}, \cdots, x_{jk_j}) = (t'_{j1}, \cdots, t'_{jk_j})$ where $t_{jk} \neq t'_{jk} \in (l_{jk}, h_{jk})$, $1 \leq k \leq k_j$ such that
$$g_j(t_{j1}, \cdots, t_{jk_j}, \cdots, x_{jk}, \cdots) > 0$$
and
$$g_j(t'_{j1}, \cdots, t'_{jk_j}, \cdots, x_{jk}, \cdots) < 0$$
for all values of x_{jk} in (l_{jk}, h_{jk}) where $x_{jk} \in var(g_j) \setminus V_j$. We denote $ivt(g_j, V_j, I)$ to represent that the polynomial g_j enjoy this property with respect to V_j and I.

By the first two properties, this method restricts that the number of variables must be greater than or equal to the number of equations.

Example 5. Consider two equations $g_1(x, y) = 0$ and $g_2(x, y) = 0$ (Figure 5) which satisfy the above restriction on the number of variables, and the variable intervals are $x \in (c_1, d_1)$ and $y \in (d_2, c_2)$. Let $V_1 = \{x\}$ and $V_2 = \{y\}$, we have:

$$g_1(c_1, y) < 0 \text{ and } g_1(d_1, y) < 0 \text{ for all } y \in \langle d_2, c_2 \rangle; \text{ and}$$

$$g_2(x, d_2) > 0 \text{ and } g_2(x, c_2) < 0 \text{ for all } x \in \langle c_1, d_1 \rangle$$

Thus we can conclude that $g_1(x, y) = 0 \wedge g_2(x, y) = 0$ has a solution inside the box $(c_1, d_1) \times (d_2, c_2)$. This is because $ABCD$ creates a Jordan Curve and the continuous graph of g_2 connects one point in the interior to one point in the the exterior of that curve, so the graph of g_2 must intersects the curve somewhere.

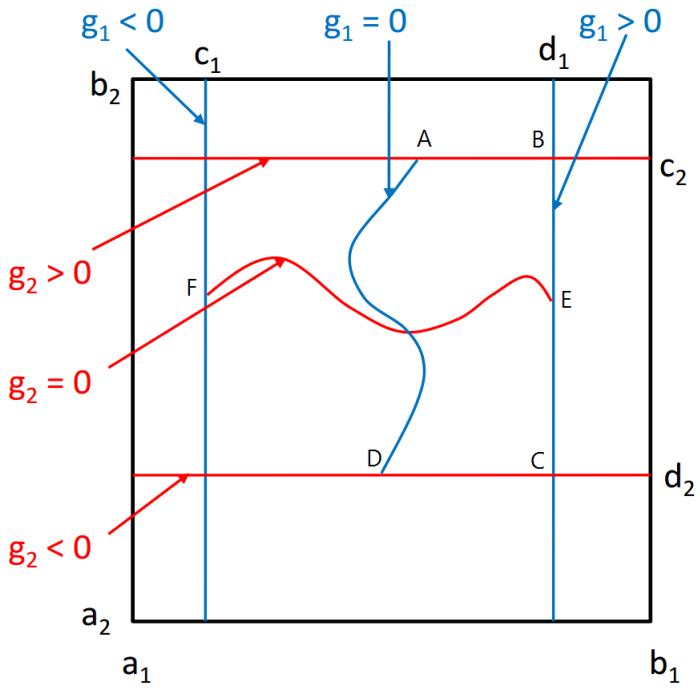


Fig. 5. Example on solving single equation using the Intermediate Value Theorem

Benchmark	raSAT				Z3 4.3				iSAT3				dReal			
	SAT		UNSAT		SAT		UNSAT		SAT		UNSAT		δ–SAT		UNSAT	
Zankl (15)	**11**	0.07 (s)	**4**	0.17 (s)	**11**	0.17 (s)	**4**	0.02 (s)	0	0.00 (s)	**4**	0.05 (s)	11	0.06 (s)	**4**	0.02(s)
Meti-Tarski (3528/1573)	3322	369.60 (s)	1052	383.40 (s)	1497	21.00 (s)	1115	74.19 (s)	1	0.28 (s)	1075	22.6 (s)	1497	72.85 (s)	943	21.40 (s)
Keymaera (612)	0	0.00 (s)	312	66.63 (s)	0	0.00 (s)	**610**	2.92 (s)	0	0.00 (s)	226	1.63 (s)	13	4.03 (s)	318	1.96 (s)

Table 4. Comparison among SMT solvers over equalities

Our current implementation of handling multiple equations is very naive which is described in Algorithm 1 because for each equality $g_j = 0$, **raSAT** checks every possible subsets of its variables as candidates for V_j. As a result, given the constraint $\bigwedge_{j=1}^{m} g_j = 0$, in the worst case **raSAT** will check $2^{|var(g_1)|} * \ldots * 2^{|var(g_m)|}$ cases. As a future work, we may use variables' sensitivity to give priority on subsets of variables.

We also do experiments on constraints with equalities from QF_NRA/Zankl and QF_NRA/Meti-tarski. Table 4

Algorithm 1 Solving multiple equations $\bigwedge_{i=1}^{n} g_i = 0$ with interval constraint $\Pi = \bigwedge_{v_i \in V} v_i \in \langle l_i, h_i \rangle$

```
1: function EQUATIONSPROVER(⋀_{i=j}^{n} g_i = 0, Π, V_0)
2:     if j > n then                                   ▷ All equations are checked
3:         return SAT
4:     end if
5:     for V_j ∈ P(var(g_j)) do                ▷ P(var(g_j)) is the powerset of var(g_j)
6:         if V_j ∩ V = ∅ and ivt(V', g_j, Π) then
7:             V_0 ← V_0 ∪ V'
8:             if EQUATIONSPROVER(⋀_{i=j+1}^{n} g_i = 0, Π, V_0) = SAT then
9:                 return SAT
10:            end if
11:        end if
12:    end for
13:    return UNSAT
14: end function
15: EQUATIONSPROVER(⋀_{i=1}^{n} g_i = 0, Π, ∅)
```

Benchmark	**raSAT**				**Z3 4.3**			
	SAT		UNSAT		SAT		UNSAT	
inequalities (6850)	**6784**	65.60 (s)	0	0.00 (s)	**6784**	97.77 (s)	**36**	32.46 (s)
equalities (1979)	891	33721.37 (s)	16	27.34 (s)	**900**	1951.01(s)	**250**	3104.74(s)

Table 5. Comparison on NIA/AProVE

6.2 Extension for Polynomial Constraints Over Integers

raSAT loop is easily modified to NIA (nonlinear arithmetic over integers) from NRA, by setting $\gamma_0 = 1$ in incremental deepening in Section 4.1 and restrcting testdata generation on intergers. We also compare **raSAT** (combination (1)-(5)-(8)-(11) with **Z3 4.3** on NIA/AProVE benchmark. **AProVE** consists of 6850 inequalities and 1979 equalities. Some has several hundred variables, but each API has few variables (mostly just 2 variables). Note that the use of the Intermediate Value Theorem cannot be applied for NIA constraints because the polynomials are not continuous. However, Interval Arithmetics can conclude UNSAT of QF_NIA benchmarks because UNSAT over real numbers simply implies UNSAT over integer numbers. For SAT benchmarks (both inequalities and equalities), **raSAT** concludes satisfiability only by testing.

The results are presented in Table 5 where the timeout is $500s$. **raSAT** does not detect unsatisfiability well since UNSAT problems have quite large coefficients which lead exhaustive search on quite large area.

7 Conclusion

This paper presented **raSAT** loop, which extends ICP with testing to accelerate SAT detection and implemented as an SMT solver **raSAT**. With experiments on benchmarks from QF NRA category of SMT-lib, we found two heuristic measures SAT-likelihood and sensitivity, which lead effective strategy combination for SAT detection. **raSAT** still remains in naive proto-type status, and there are lots of future work.

UNSAT core. Currently, **raSAT** focuses on SAT detection. For UNSAT detection, the target is to find a small UNSAT core in a large problem.

Equality handling. Section 6 shows equality handling where UNSAT constraints can be completely solved by ICP (with the assumption of bounded intervals). The Intermediate Value Theorem can be used to show satisfiability with restrictions on variables of polynomials. Moreover, the use of this theorem is not a complete in showing satisfiability. As a future work, we will apply Groebner basis.

Further strategy refiment. Currently, raSAT uses only information from O.T (interval arithmetic).We are planning to refine strategies such that previous O.T and U.T results mutually guide to each other. For instance, test generation and a box decomposition can be more focused.1

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