

Equality handling and efficiency improvement of SMT for non-linear constraints over reals

By VU XUAN TUNG

A thesis submitted to
School of Information Science,
Japan Advanced Institute of Science and Technology,
in partial fulfillment of the requirements
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Graduate Program in Information Science

Written under the direction of
Professor Mizuhito Ogawa

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Abstract

This thesis presents strategies for efficiency improvement and extensions of an SMT solver **raSAT** for polynomial constraints.

raSAT which initially focuses on polynomial inequalities over real numbers follows ICP methodology and adds testing to boost satisfiability detection [11]. In this work, in order to deal with exponential exploration of boxes, several heuristic measures, called *SAT likelihood*, *sensitivity*, and the number of unsolved atomic polynomial constraints, are proposed. Extensions for handling equations and handling constraints over integer number are also presented.

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Chapter 1

Introduction

1.1 Polynomial constraint solving

Polynomial constraint solving over real numbers is a task of computing an assignment of real values to variables that satisfies given polynomial inequalities/equations. If such an assignment exists, the constraint is said to be satisfiable (SAT) and the assignment is called SAT instance; otherwise we mention it as unsatisfiable (UNSAT).

Example 1 $x^2 + y^2 < 1 \wedge xy > 1$ is an example of an UNSAT constraint. While the set of satisfiable points for the first inequality ($x^2 + y^2 < 1$) is the red circle in Figure 1.1, the set for the second one is the blue area. Because these two areas do not intersect, the conjunction of two equalities is UNSAT.

Example 2 Figure 1.2 illustrates the satisfiability of the constraint: $x^2 + y^2 < 4 \wedge xy > 1$. Any point in the purple area is a SAT instance of the constraint, e.g. (1.5, 1).

Solving polynomial constraints has many application in Software Verification, such as

- **Locating roundoff and overflow errors**, which is our motivation [14, 15]
- **Automatic termination proving**, which reduces termination detection to finding a suitable ordering [13], e.g., $\mathsf{T}\mathsf{T}\mathsf{T}_2$ ¹, AProVE².
- **Loop invariant generation**. Farkas’s lemma is a popular approach in linear loop invariant generation [4], and is reduced to degree 2 polynomials. Non-linear loop invariant [20] requires more complex polynomials.
- **Hybrid system**. SMT for QF_NRA are often used as backend engines [19].
- **Mechanical contrnol design**. PID control is simple but widely used, and designing parameters is reduced to polynomial constraints [1].

¹<http://cl-informatik.uibk.ac.at/software/ttt2/>

²<http://aprove.informatik.rwth-aachen.de>



Figure 1.1: Example of UNSAT constraint

1.2 Existing approaches

Although solving polynomial constraints on real numbers is decidable [21], current methodologies have their own pros and cons. They can be classified into the following categories:

1. **Quantifier elimination by cylindrical algebraic decomposition (QE-CAD)** [3] is a complete technique, and is implemented in Mathematica, Maple/SynRac, Reduce/Redlog, QEPCAD-B, and recently in Z3 4.3 (which is referred as nlsat in [10]). Although QE-CAD is precise and detects beyond SAT instances (e.g., SAT regions), scalability is still challenging, since its complexity is doubly-exponential with respect to the number of variables.
2. **Virtual substitution** eliminates an existential quantifier by substituting the corresponding quantified variable with a very small value ($-\infty$), and either each root (with respect to that variable) of polynomials appearing in the constraint or each



Figure 1.2: Example of SAT constraint

root plus an infinitesimal ϵ . Disjunction of constraints after substitutions is equivalent to the original constraint. Because VS needs the formula for roots of polynomials, its application is restricted to polynomials of degree up to 4. SMT-RAT and Z3 [17] applies VS.

3. **Bit-blasting.** In this category of methodology, numerical variables are represented by a sequence of binary variables. The given constraint is converted into another constraint over the boolean variables. SAT solver is then used to find a satisfiable instance of binary variables which can be used to calculate the values of numerical variables. MiniSmt [22], the winner of QF_NRA in SMT competition 2010, applies it for (ir)-rational numbers. It can show SAT quickly, but due to the bounded bit encoding, it cannot conclude UNSAT. In addition, high degree of polynomial results in large SAT formula which is an obstacle of bit-blasting.
4. **Linearization.** CORD [8] uses COordinate Rotation DIgital Computer (CORDIC)

for real numbers to linearizes multiplications into a sequence of linear constraints. Each time one multiplication is linearized, a number of new constraints and new variables are introduced. As a consequence, high degree polynomials in the original constraint lead to large number of linear constraints.

5. **Interval Constraint Propagation (ICP)** which are used in SMT solver community, e.g., iSAT3 [7], dReal [9], and RSOLVER [18]. ICP combines over-approximation by interval arithmetics and constraint propagation to prune out the set of unsatisfiable points. When pruning does not work, decomposition (branching) on intervals is applied. ICP which is capable of solving "multiple thousand arithmetic constraints over some thousands of variables" [7] is practically often more efficient than algebraic computation.

1.3 Proposed Approach and Contributions

Our aim is an SMT solver for solving polynomial constraint. We first focus on strict inequalities because of the following reasons.

1. Satisfiable inequalities allow over-approximation. An over-approximation estimates the range of a polynomial f as $O.range(f)$ that covers all the possible values of f , i.e. $range(f) \subseteq O.range(f)$. For an inequality $f > 0$, if $O.range(f)$ stays in the positive side, it can be concluded as SAT. On the other hand, over-approximation cannot prove the satisfiability of SAT equations.
2. Satisfiable inequalities allow under-approximation. An under-approximation computes the range of the polynomial f as $U.range(f)$ such that $range(f) \supseteq U.range(f)$. If $U.range(f)$ is on the positive side, $f > 0$ can be said to be SAT. Due to the continuity of f , finding such an under-approximation for solving $f > 0$ is more feasible than that for $f = 0$.
 - If $f(\bar{x}) > 0$ has a real solution \bar{x}_0 , there exist rational points near \bar{x}_0 which also satisfy the inequality. Solving inequalities over real numbers thus can be reduced to that over rational numbers.
 - The real solution of $f(\bar{x}) = 0$ cannot be approximated to any rational number.

For UNSAT constraint (both inequalities and equations) can be solved by over-approximation. Suppose $O.range(f)$ is the result of an over-approximation for a polynomial f .

1. If $O.range(f)$ resides on the negative side, $f > 0$ is UNSAT.
2. If $O.range(f)$ stays on either negative or positive side, $f = 0$ is UNSAT.

Our approach of "iterative approximation refinement" - **raSAT** loop for solving polynomial constraint was proposed and implemented as an SMT solver named raSAT in [11]. This work improves the efficiency of the tool and extend it to handle equations. The summary of the proposed method in [11] is:

1. Over-approximation is used for both disproving and proving polynomial inequalities. In addition, under-approximation is used for boosting SAT detection. When both of these methods cannot conclude the satisfiability, the input formula is refined so that the result of approximation become more precise.
2. Interval Arithmetic (IA) and Testing are instantiated as an over-approximation and an under-approximation respectively. While IA defines the computations over the intervals, e.g. $[1, 3] +_{IA} [3, 6] = [2, 9]$, Testing attempts to propose a number of assignments from variables to real numbers and check each assignment against the given constraint to find a SAT instance.
3. In refinement phase, intervals of variables are decomposed into smaller ones. For example, $x \in [0, 10]$ becomes $x \in [0, 4] \vee x \in [4, 10]$.
4. Khanh and Ogawa [11] also proposed a method for detecting satisfiability of equations using the Intermediate Value Theorem.

The contributions of this work are as follows:

1. Although the method of using IA is robust for large degrees of polynomial, the number of boxes (products of intervals) grows exponentially with respect to the number of variables during refinement (interval decomposition). As a result, strategies for selecting variables to decomposed and boxes to explore play a crucial role in efficiency. We introduce the following strategies:
 - A box with more possibility to be SAT is selected to explore, which is estimated by several heuristic measures, called *SAT likelihood*, and the number of unsolved atomic polynomial constraints.
 - A more influential variable is selected for multiple test cases and decomposition. This is estimated by *sensitivity* which is determined during the computation of IA.
2. Two schemes of incremental search are proposed for enhancing solving process:
 - **Incremental deepening.** raSAT follows the depth-first-search manner. In order to escape local optimums, it starts searching with a threshold that each interval will be decomposed no smaller than it. If neither SAT nor UNSAT is detected, a smaller threshold is taken and raSAT restarts.
 - **Incremental widening.** Starting with a small interval, if raSAT detects UNSAT, input intervals are enlarged and raSAT restarts. For SAT constraint, small (finite) interval allows sensitivity to be computed because Affine Interval [11] requires finite range of variables. As a consequence, our above strategies will take effects on finding SAT instance. For the UNSAT case, combination of small intervals and incremental deepening helps raSAT quickly determines the threshold in which unsatisfiability may be proved by IA.

3. SAT confirmation step by an error-bound guaranteed floating point package **iR-RAM**³, to avoid soundness bugs caused by roundoff errors.
4. This work also implement the idea of using Intermediate Value Theorem to show the satisfiability of equations which was suggested in [11].
5. We also extend raSAT to handle constraint over integer numbers. For this extension, we only generate the integer values for variables in testing phase. In addition, the threshold used for stopping decomposition is set to 1.

1.4 Thesis outline

Coming soon...

³<http://irram.uni-trier.de>

Chapter 2

Preliminaries

2.1 Abstract DPLL

2.2 SMT

2.2.1 Syntax

Definition 1 A signature Σ is a 4-tuple (S, P, F, α) containing a set S of sorts, a set P of predicate symbols, a set F of function symbols, and a map α which associates symbols to their sorts.

- $\forall p \in P; \alpha(p)$ is a n -tuple argument sorts of p .
- $\forall f \in F; \alpha(f)$ is a n -tuple of argument and returned sorts of f .

We denote V as the set of variables and extend α to include the sort of each variable, i.e. $\forall v \in V; \alpha(v)$ returns the sort of v . Followings are the definitions of terms and formulas.

Definition 2 The set $TERM$ of terms is the set that satisfies the properties

1. $v \in V \Rightarrow v \in TERM$
2. $t_1, \dots, t_n \in TERM; f \in F \Rightarrow f(t_1, \dots, t_n) \in TERM$

Definition 3 The set $FORM$ of formulas is the set that satisfies the properties:

1. $\perp \in FORM$
2. $t_1, \dots, t_n \in TERM; p \in P \Rightarrow p(t_1, \dots, t_n) \in FORM$
3. $\varphi, \psi \in FORM \Rightarrow (\varphi \circ \psi) \in FORM$ where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
4. $\varphi \in FORM \Rightarrow \neg\varphi \in FORM$
5. $\varphi \in FORM; v \in V \Rightarrow (\forall v)\varphi, (\exists v)\varphi \in FORM$

2.2.2 Semantics

Definition 4 Let Σ is a signature. A model M of Σ is a pair (U, I) in which U is the universe and I is the interpretation of symbols.

- $\forall s \in S; I(s) \subseteq U$ specifies the possible values of sort s .
- $\forall f \in F; I(f) = \{(t_1, \dots, t_n) | t_1 \in I(s_1), \dots, t_n \in I(s_{n-1})\} \mapsto I(s_n)$ with $\alpha(f) = (s_1, \dots, s_n)$
- $\forall p \in P; I(p) = \{(t_1, \dots, t_n) | t_1 \in I(s_1), \dots, t_n \in I(s_n)\} \mapsto \{true, false\}$ with $\alpha(p) = (s_1, \dots, s_n)$
- $\forall v \in V; I(v) \in I(\alpha(v))$

Definition 5 Let Σ is a signature. A Σ -theory T is a (infinite) set of Σ -models.

A theory T' is a subset of theory T iff $T' \subseteq T$.

Definition 6 Let $\Sigma = (S, P, F, \alpha)$, t , φ and $M = (U, I)$ are a signature, a Σ -formula and a Σ -model respectively. The evaluations of t against M which is denoted by t^M is defined recursively as:

1. If $t = v \in V$, then $t^M = I(v)$.
2. If $t = f(t_1, \dots, t_n)$, then $t^M = I(f)(t_1^M, \dots, t_n^M)$ for $f \in F$ and $t_1, \dots, t_n \in TERM$

Similarly, the evaluation φ^M of φ is defined as:

1. If $\varphi = \perp$, then $\varphi^M = false$.
2. If $\varphi = p(t_1, \dots, t_n)$, then $\varphi^M = I(p)(t_1^M, \dots, t_n^M)$ for $p \in P$ and $t_1, \dots, t_n \in TERM$
3. If $\varphi = \varphi_1 \circ \varphi_2$, then $\varphi^M = \varphi_1^M \circ \varphi_2^M$ for $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ and $\varphi_1, \varphi_2 \in FORM$
4. If $\varphi = \neg \varphi'$, then $\varphi^M = \neg(\varphi'^M)$ for $\varphi' \in FORM$
5. If $\varphi = \forall v \varphi'$, then $\varphi^M = \bigwedge_{v_i \in I(\alpha(v))} \varphi'^{(U, I < v \mapsto v_i >)}$ for $v \in V$, $\varphi' \in FORM$ and $I < v \mapsto v_i >$ denotes the map after updating the map from v to v_i into I .
6. If $\varphi = \exists v \varphi'$, then $\varphi^M = \bigvee_{v_i \in I(\alpha(v))} \varphi'^{(U, I < v \mapsto v_i >)}$ for $v \in V$ and $\varphi' \in FORM$.

We say that M satisfies φ which is denoted by $\models_M \varphi$ iff $\varphi^M = true$. If $\varphi^M = false$, $\not\models_M \varphi$ is used to denote that M does not satisfy φ .

Lemma 1 Given any Σ -model M and Σ -formula φ , we have $\models_M \varphi \iff \not\models_M \neg \varphi$

Proof 1 $\models_M \varphi \iff \varphi^M = true \iff \neg \varphi^M = false \iff (\neg \varphi)^M = false \iff \not\models_M \neg \varphi$

Definition 7 Let T be a Σ -theory. A Σ -formula φ is:

- *satisfiable in T or T -SAT* iff $\exists M \in T; \models_M \varphi$
- *valid in T or T -VALID* iff $\forall M \in T; \models_M \varphi$
- *unsatisfiable in T or T -UNSAT* iff $\forall M \in T; \not\models_M \varphi$

Lemma 2 If T be a Σ -theory, then φ is T -VALID $\iff \neg\varphi$ is T -UNSAT

Proof 2 φ is T -VALID $\iff \forall M \in T; \models_M \varphi \iff \forall M \in T; \not\models_M \neg\varphi$ (Lemma 1) $\iff \neg\varphi$ is T -UNSAT.

Lemma 3 If $T' \subseteq T$, then φ is T' -SAT $\implies \varphi$ is T -SAT.

Proof 3 φ is T' -SAT $\implies \exists M \in T'; \models_M \varphi \implies \exists M \in T; \models_M \varphi$ (because $T' \subseteq T$) $\implies \varphi$ is T -SAT.

2.3 Polynomial constraints over real numbers

2.3.1 Syntax

We instantiate the signature $\Sigma^p = (S^p, P^p, F^p, \alpha^p)$ in Section 2.2.1 for polynomial constraints as following:

1. $S^p = \{Real\}$
2. $P^p = \{\succ, \prec, \succeq, \preceq, \approx, \not\approx\}$
3. $F^p = \{\oplus, \ominus, \otimes, \mathbf{1}\}$
4. $\forall p \in P^p; \alpha^p(p) = (Real, Real)$ and $\forall f \in F^p \setminus \{\mathbf{1}\}; \alpha^p(f) = (Real, Real, Real); \alpha^p(\mathbf{1}) = Real$

A polynomial and a polynomial constraint are a Σ^p -term and a Σ^p -formula respectively. However, for simplicity, we currently focus on small portion of Σ^p -formulas

2.3.2 Semantics

A model $M_{\mathbb{R}}^p = (\mathbb{R}, I_{\mathbb{R}}^p)$ over real numbers contains the set of reals number \mathbb{R} and a map I that satisfies the following properties.

1. $I_{\mathbb{R}}^p(Real) = \mathbb{R}$.
2. $\forall p \in P^p; I_{\mathbb{R}}^p(p) = \mathbb{R} \times \mathbb{R} \mapsto \{true, false\}$ such that $I_{\mathbb{R}}^p(p)(r_1, r_2) = r_1 \text{ } p_{\mathbb{R}} \text{ } r_2$ where $(\succ_{\mathbb{R}}, \prec_{\mathbb{R}}, \succeq_{\mathbb{R}}, \preceq_{\mathbb{R}}, \approx_{\mathbb{R}}, \not\approx_{\mathbb{R}}) = (>, <, \geq, \leq, =, \neq)$.

3. $\forall f \in F^p \setminus \{\mathbf{1}\}; I_{\mathbb{R}}^p(f) = \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ such that $I_{\mathbb{R}}^p(f)(r_1, r_2) = r_1 \ f_{\mathbb{R}} \ r_2$ where $(\oplus_{\mathbb{R}}, \ominus_{\mathbb{R}}, \otimes_{\mathbb{R}}) = (+, -, *)$.
4. $I_{\mathbb{R}}^p(\mathbf{1}) = 1$
5. $\forall v \in V; I_{\mathbb{R}}^p(v) \in \mathbb{R}$.

The theory of real numbers is $T_{\mathbb{R}}^p = \{M_{\mathbb{R}}^p | M_{\mathbb{R}}^p \text{ is a model of real numbers} \}$. By this instantiation, each model differs to another by the mapping from variables to real numbers. As a result, an assignment of real numbers to variables can be used to represent a model $M_{\mathbb{R}}^p$; i.e. $\{v \mapsto r \in \mathbb{R} | v \in V\}$ represents a model. Because a theory is a set of models, it can also be represented by a set of variables assignments. For example, a map from variables to intervals $\Pi = \{v \in [l, h] | v \in V\}$ is enough to represent a theory which is a subset of $T_{\mathbb{R}}^p$. We denote this theory as $\Pi_{\mathbb{R}}^p$.

Chapter 3

Over-Approximation and Under-Approximation

3.1 Approximation Theory

Definition 8 Let T, T' be Σ -theories and φ be any Σ -formula.

- T' is an over-approximation theory (of T) iff T' -UNSAT of φ implies T -UNSAT of φ .
- T' is an under-approximation theory (of T) iff T' -SAT of φ implies T -SAT of φ .

Theorem 1 If T_O be an over-approximation theory of T , then for any Σ -formula φ : φ is T_O -VALID $\implies \varphi$ is T -VALID.

Proof 4 φ is T_O -VALID $\implies \neg\varphi$ is T_O -UNSAT (Lemma 2) $\implies \neg\varphi$ is T -UNSAT (Definition 8) $\implies \varphi$ is T -VALID (Lemma 2)

A typical ICP applies $O.T$ only as an interval arithmetic. Later in Section ??, we will instantiate interval arithmetic as $O.T$. Adding to $O.T$ -valid, **raSAT** introduce testing as $U.T$ to accelerate SAT detection.

3.2 Interval Arithmetic as an Over-Approximation Theory

A model $M_{IA}^p = (U_{IA}^p, I_{IA}^p)$ over intervals contains a set of all intervals $U_{IA}^p = \{[l, h] | l, h \in \mathbb{R} \text{ and } l \leq h\}$ and a map I_{IA}^p that satisfies the following conditions.

1. $I_{IA}^p(\text{Real}) = U_{IA}^p$
2. $\forall p \in P^p; I_{IA}^p(p) = U_{IA}^p \times U_{IA}^p \mapsto U_{IA}^p$ where $I_{IA}^p(p)(i_1, i_2) = i_1 \text{ p}_{IA} i_2$. The definition of p_{IA} is as follow:

- $[l_1, h_1] \succ_{IA} [l_2, h_2] = \text{true} \iff l_1 > h_2$
 - $[l_1, h_1] \succ_{IA} [l_2, h_2] = \text{false} \iff h_1 \leq l_2$
 - $[l_1, h_1] \prec_{IA} [l_2, h_2] = \text{true} \iff h_1 < l_2$
 - $[l_1, h_1] \prec_{IA} [l_2, h_2] = \text{false} \iff l_1 \geq h_2$
 - $i_1 \succeq_{IA} i_2 = \neg(i_1 \prec_{IA} i_2)$
 - $i_1 \preceq_{IA} i_2 = \neg(i_1 \succ_{IA} i_2)$
 - $i_1 \approx_{IA} i_2 = i_1 \succeq_{IA} i_2 \wedge i_1 \preceq_{IA} i_2$
 - $i_1 \not\approx_{IA} i_2 = \neg(i_1 \approx_{IA} i_2)$
3. $\forall f \in F^p \setminus \{\mathbf{1}\}; I_{IA}^p(f) = U_{IA}^p \times U_{IA}^p \mapsto U_{IA}^p$ such that $I_{IA}^p(f)(i_1, i_2) = i_1 \text{ } f_{IA} \text{ } i_2$ where f_{IA} satisfies the following properties:
- $i_1 \oplus_{IA} i_2 \supseteq \{r_1 + r_2 | r_1 \in i_1 \text{ and } r_2 \in i_2\}$.
 - $i_1 \ominus_{IA} i_2 \supseteq \{r_1 - r_2 | r_1 \in i_1 \text{ and } r_2 \in i_2\}$.
 - $i_1 \otimes_{IA} i_2 \supseteq \{r_1 * r_2 | r_1 \in i_1 \text{ and } r_2 \in i_2\}$.
4. $I_{IA}^p(\mathbf{1}) = [1, 1]$
5. $\forall v \in V; I_{IA}^p \in U_{IA}^p$

Theory $T_{IA}^p = \{M_{IA}^p | M_{IA}^p \text{ is a model over intervals}\}$. Each model differs to another by the mapping from variables to intervals. As a consequence, one assignment from variables to intervals can be used to describe an model. We denote Π_{IA}^p as the model represented by $\Pi = \{x \in [l, h] | v \in V\}$.

Lemma 4 $\forall i_1, i_2 \in U_{IA}^p; \forall r_1 \in i_1, r_2 \in i_2; \forall p \in P^p(i_1 \text{ } p_{IA} \text{ } i_2 = \text{false} \implies \neg(r_1 \text{ } p_{\mathbb{R}} \text{ } r_2))$

Proof 5 Let $i_1 = [l_1, h_1]$ and $i_2 = [l_2, h_2]$ where $l_1 \leq h_1$ and $l_2 \leq h_2$. We have:

- $r_1 \in i_1 \implies l_1 \leq r_1 \leq h_1$.
- $r_2 \in i_2 \implies l_2 \leq r_2 \leq h_2$.

Suppose that $i_1 \text{ } p_{IA} \text{ } i_2 = \text{false}$, we need to show $\neg(r_1 \text{ } p_{\mathbb{R}} \text{ } r_2)$ by considering all the possible cases of p :

1. If p is \succ , the we have $[l_1, h_1] \succ_{IA} [l_2, h_2] = \text{false} \implies h_1 \leq l_2 \implies r_1 \leq r_2$ (because $r_1 \leq h_1$ and $l_2 \leq r_2$) $\implies \neg(r_1 > r_2) \implies \neg(r_1 \succ_{\mathbb{R}} r_2)$.
2. If p is \prec , the we have $[l_1, h_1] \prec_{IA} [l_2, h_2] = \text{false} \implies l_1 \geq h_2 \implies r_1 \geq r_2$ (because $r_1 \geq l_1$ and $r_2 \leq h_2$) $\implies \neg(r_1 < r_2) \implies \neg(r_1 \prec_{\mathbb{R}} r_2)$.
3. If p is \succeq , the we have $[l_1, h_1] \succeq_{IA} [l_2, h_2] = \text{false} \implies \neg([l_1, h_1] \prec_{IA} [l_2, h_2]) = \text{false} \implies [l_1, h_1] \prec_{IA} [l_2, h_2] = \text{true} \implies h_1 < l_2 \implies r_1 < r_2$ (because $r_1 \leq h_1$ and $r_2 \geq l_2$) $\implies \neg(r_1 \geq r_2) \implies \neg(r_1 \succeq_{\mathbb{R}} r_2)$.

4. If p is \preceq , then we have $[l_1, h_1] \preceq_{IA} [l_2, h_2] = \text{false} \implies \neg([l_1, h_1] \succ_{IA} [l_2, h_2]) = \text{false} \implies [l_1, h_1] \succ_{IA} [l_2, h_2] = \text{true} \implies l_1 > h_2 \implies r_1 > r_2$ (because $r_1 \geq l_1$ and $r_2 \leq h_2$) $\implies \neg(r_1 \leq r_2) \implies \neg(r_1 \preceq_{\mathbb{R}} r_2)$.
5. If p is \approx , then we have $i_1 \approx_{IA} i_2 = \text{false} \implies i_1 \succeq_{IA} i_2 \wedge i_1 \preceq_{IA} i_2 = \text{false} \implies i_1 \succeq_{IA} i_2 = \text{false}$ or $i_1 \preceq_{IA} i_2 = \text{false} \implies r_1 < r_2$ or $r_1 > r_2$ (as the third and fourth case of this proof) $\implies \neg(r_1 = r_2) \implies \neg(r_1 \approx_{\mathbb{R}} r_2)$.
6. If p is $\not\approx$, then we have $i_1 \not\approx_{IA} i_2 = \text{false} \implies \neg(i_1 \approx_{IA} i_2) = \text{false} \implies \neg(i_1 \succeq_{IA} i_2 \wedge i_1 \preceq_{IA} i_2) = \text{false} \implies \neg(i_1 \succeq_{IA} i_2) \vee \neg(i_1 \preceq_{IA} i_2) = \text{false} \implies i_1 \prec_{IA} i_2 \vee i_1 \succ_{IA} i_2 = \text{false} \implies i_1 \prec_{IA} i_2 = \text{false}$ and $i_1 \succ_{IA} i_2 = \text{false} \implies r_1 \geq r_2$ and $r_1 \leq r_2$ (as the first and second case of this proof) $\implies r_1 = r_2 \implies \neg(r_1 \neq r_2) \implies \neg(r_1 \not\approx_{\mathbb{R}} r_2)$.

Theorem 2 If $\Pi = \{v \in [l, h] | v \in V\}$ is a map from variables to intervals, then $\{\Pi_{IA}^p\}$ is an over-approximation of $\Pi_{\mathbb{R}}^p$.

Lemma 5 $\forall t \in \text{TERM}^p \forall M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p t^{M_{\mathbb{R}}^p} \in t^{\Pi_{IA}^p}$.

Proof 6 Proof is done by induction on structure of term. Let $t \in \text{TERM}^p$ and $M_{\mathbb{R}}^p = (\mathbb{R}, I_{\mathbb{R}}^p) \in \Pi_{\mathbb{R}}^p$

1. If $t = v \in V$, then $t^{M_{\mathbb{R}}^p} = I_{\mathbb{R}}^p(v) \in \Pi(v)$ because $M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p$. In addition, $t^{\Pi_{IA}^p} = \Pi_{\mathbb{R}}^p$.
2. If $t = \mathbf{1}$, then $t^{M_{\mathbb{R}}^p} = 1 \in [1, 1] = t^{\Pi_{IA}^p}$
3. If $t = t_1 f t_2$ for some $f \in F^p \setminus \{0, 1\}$

$$t^{M_{\mathbb{R}}^p} = t_1^{M_{\mathbb{R}}^p} f_{\mathbb{R}} t_2^{M_{\mathbb{R}}^p}$$

$$t^{\Pi_{IA}^p} = t_1^{\Pi_{IA}^p} f_{IA} t_2^{\Pi_{IA}^p}$$

By induction hypothesis, we have $t_1^{M_{\mathbb{R}}^p} \in t_1^{\Pi_{IA}^p}$ and $t_2^{M_{\mathbb{R}}^p} \in t_2^{\Pi_{IA}^p}$. In addition, due to the properties of f_{IA} : $t_1^{M_{\mathbb{R}}^p} f_{\mathbb{R}} t_2^{M_{\mathbb{R}}^p} \in t_1^{\Pi_{IA}^p} f_{IA} t_2^{\Pi_{IA}^p}$, or $t^{M_{\mathbb{R}}^p} \in t^{\Pi_{IA}^p}$

Proof 7 In order to prove Theorem 2, we need to prove that $\forall \varphi^p (\varphi^p \text{ is } \{\Pi_{IA}^p\}\text{-UNSAT} \implies \varphi^p \text{ is } \Pi_{\mathbb{R}}^p\text{-UNSAT})$.

Given an Σ^p -formula φ^p and suppose that φ^p is $\{\Pi_{IA}^p\}$ -UNSAT. Suppose φ^p is not $\Pi_{\mathbb{R}}^p$ -UNSAT, that means it is either $\Pi_{\mathbb{R}}^p$ -SAT or $\Pi_{\mathbb{R}}^p$ -VALID. In either case, there exist at least a model $M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p$ such that $\models_{M_{\mathbb{R}}^p} \varphi^p \iff (\varphi^p)^{M_{\mathbb{R}}^p} = \text{true}$.

1. If $\varphi^p = p(t_1, t_2)$ for some $p \in P^p$, then $(\varphi^p)^{M_{\mathbb{R}}^p} = \text{true} \implies t_1^{M_{\mathbb{R}}^p} p_{\mathbb{R}} t_2^{M_{\mathbb{R}}^p}$.

On the other hand, φ^p is $\{\Pi_{IA}^p\}$ -UNSAT $\implies (\varphi^p)^{\Pi_{IA}^p} = \text{false} \implies t_1^{\Pi_{IA}^p} p_{IA} t_2^{\Pi_{IA}^p} = \text{false}$. In addition, because $t_1^{M_{\mathbb{R}}^p} \in t_1^{\Pi_{IA}^p}$ and $t_2^{M_{\mathbb{R}}^p} \in t_2^{\Pi_{IA}^p}$ (Lemma 5), $t_1^{\Pi_{IA}^p} p_{IA} t_2^{\Pi_{IA}^p} = \text{false} \implies \neg(t_1^{M_{\mathbb{R}}^p} p_{\mathbb{R}} t_2^{M_{\mathbb{R}}^p})$ (Lemma 4). This is a contradiction. As the result, φ^p must be $\Pi_{\mathbb{R}}^p$ -UNSAT.

2. If $\varphi^p = \varphi_1^p \wedge \varphi_2^p$, then we have φ^p is $\{\Pi_{IA}^p\}$ -UNSAT $\implies \not\models_{\Pi_{IA}^p} (\varphi_1^p \wedge \varphi_2^p) \implies (\varphi_1^p \wedge \varphi_2^p)^{\Pi_{IA}^p} = \text{false} \implies (\varphi_1^p)^{\Pi_{IA}^p} \wedge (\varphi_2^p)^{\Pi_{IA}^p} = \text{false} \implies (\varphi_1^p)^{\Pi_{IA}^p} = \text{false}$ and $(\varphi_2^p)^{\Pi_{IA}^p} = \text{false}$. By induction hypothesis, φ_1^p and φ_2^p are $\Pi_{\mathbb{R}}^p$ -UNSAT $\implies \forall M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p \not\models_{M_{IA}^p} \varphi_1^p$ and $\forall M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p \not\models_{M_{IA}^p} \varphi_2^p \implies \forall M_{\mathbb{R}}^p \in \Pi_{\mathbb{R}}^p \not\models_{M_{IA}^p} (\varphi_1^p \wedge \varphi_2^p) \implies \varphi_1^p \wedge \varphi_2^p$ is $\Pi_{\mathbb{R}}^p$ -UNSAT.

3.3 Testing as an Under-Approximation Theory

Definition 9 Let $T^p \subseteq T_{\mathbb{R}}^p$ be a sub-theory of real numbers. Any sub-theory T_T^p of T^p , i.e. $T_T^p \subseteq T^p$ is call a theory of testing with respect to T^p .

Theorem 3 If T_T^p is a theory of testing w.r.t T^p , T_R^p is an under-approximation of T^p .

Proof 8 Let φ^p is a Σ^p -formula and suppose it is T_T^p -SAT. We need to prove φ^p is T^p -SAT. We have φ^p is T_T^p -SAT $\implies \exists M^p \in T_T^p \models_{M^p} \varphi^p \implies \exists M^p \in T^p \models_{M^p} \varphi^p$ (because $T_T^p \subseteq T^p$) $\implies \varphi^p$ is T^p -SAT.

Given a theory T^p , we randomly select a number of its models to form a theory of testing T_T^p w.r.t T^p .

3.4 raSAT loop

$\text{bound_atom} ::= \text{variable} \in [\text{real}, \text{real}] \mid \text{variable} \in (-\text{inf}, \text{real}] \mid \text{variable} \in [\text{real}, \text{inf}) \mid \text{variable} \in (-\text{inf}, \text{inf})$

$\text{bound_constraint} ::= \bigwedge_i \bigvee_j \text{bound_atom}_{ij}$

In other word, each model can be represented by an assignment of one real number to each variable. State of search procedure contains $(\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U)$ where

- Π is the interval constraint
- φ represents the polynomial constraint.
- φ^V contains the constraints that are valid under over-approximation.
- φ^U is the set of constraints which are UNKNOWN under over-approximation.

The transition rules are described in Table 3.1. Figure 4.1 illustrates the transition system.

$$\begin{array}{c}
\frac{\Pi \models_{SAT} \perp}{(\Pi, \varphi, \emptyset, \emptyset, \emptyset) \rightarrow UNSAT} \quad \Pi_UNSAT \\
\\
\frac{\Pi \models_{SAT} \Pi^c \quad \Pi' = flattern(\Pi^c)}{(\Pi, \varphi, \emptyset, \emptyset, \emptyset) \rightarrow (\Pi, \varphi, \Pi^c, \emptyset, \emptyset)} \quad \Pi_SAT \\
\\
\frac{\Pi^c \neq \emptyset \quad \varphi^V \wedge \varphi^U = \varphi \quad \varphi^V \cup \varphi^U = \emptyset \quad \varphi^V \text{ is } \Pi_{IA}^c\text{-VALID}}{(\Pi, \varphi, \Pi^c, \emptyset, \emptyset) \rightarrow (\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U)} \quad IA_SAT \\
\\
\frac{\varphi^V = \varphi}{(\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U) \rightarrow SAT} \quad IA_VALID \\
\\
\frac{\Pi^c \neq \emptyset \quad \varphi^U \neq \emptyset \quad \varphi^U \text{ is } \Pi_T^c\text{-SAT}}{(\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U) \rightarrow SAT} \quad TEST_SAT \\
\\
\frac{\varphi^U \text{ is } \Pi_T^c\text{-UNSAT} \quad x_j \in var(\varphi^U) \quad (I_j = x_j \in [l_j, h_j]) \in \Pi^c \quad l_j < d \in \mathbb{R} < h_j \quad I_{j1} = x_j \in [l_j, d] \quad I_{j2} = x_j \in [d, h_j]}{(\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U) \rightarrow (\Pi \wedge (\neg I_j \vee I_{j1} \vee I_{j2}) \wedge (I_j \vee \neg I_{j1}) \wedge (I_j \vee \neg I_{j2}) \wedge (\neg I_{j1} \vee \neg I_{j2}), \varphi, \emptyset, \emptyset, \emptyset)} \quad REFIN \\
\\
\frac{\{\varphi_i \mid \varphi_i \wedge \varphi'_i = \varphi; \varphi_i \text{ is } \Pi_{IA}^c\text{-UNSAT}\} \quad var(\varphi_i) = \{x_{ij} \mid j=1, \dots, m_i\} \quad (I_{ij} = x_{ij} \in [l_{ij}, h_{ij}]) \in \Pi^c}{(\Pi, \varphi, \Pi^c, \emptyset, \emptyset) \rightarrow (\Pi \wedge \bigwedge_{i=1}^{m_i} \neg I_{ij}, \varphi, \emptyset, \emptyset, \emptyset)} \quad IA_UNSAT
\end{array}$$

Table 3.1: Transition rules

3.5 Soundness - Completeness

3.5.1 Soundness

Theorem 4 *Let $(\Pi, \varphi, \Pi^c, \varphi^V, \varphi^U)$ be any state of our system, then the following properties are invariants:*

1. $\Pi_{\mathbb{R}}^c \subseteq T_{\mathbb{R}}^p$
2. φ^V is Π_R^c -VALID.
3. $\varphi^U = \emptyset \vee (\varphi = \varphi^U \wedge \varphi^V)$
4. φ is $\Pi_{\mathbb{R}}$ -UNSAT $\implies \varphi$ is $T_{\mathbb{R}}^p$ -UNSAT

Proof 9 1. Easy from the definition.

2. Easy to see.

3. Easy from the transitions.

4. The proof is done inductively:

- Initial state: $(\bigwedge_{v \in V} v \in (-\infty, +\infty), \varphi, \emptyset, \emptyset, \emptyset)$ and by definition, $((\bigwedge_{v \in V} v \in (-\infty, +\infty))_{\mathbb{R}} = T_{\mathbb{R}}^p$. Then, the invariant trivially holds.
- Each transition is considered:
 - For rules Π_SAT , IA_SAT the interval constraint Π does not changed, so if the properties holds for the former state, it also does for the later one.
 - For $REFINE$ transition: Denote $\Pi' = \Pi \wedge (\neg I_j \vee I_{j1} \vee I_{j2}) \wedge (I_j \vee \neg I_{j1}) \wedge (I_j \vee \neg I_{j2}) \wedge (\neg I_{j1} \vee \neg I_{j2})$. We will prove that φ is $\Pi'_{\mathbb{R}}\text{-UNSAT} \implies \varphi$ is $\Pi_{\mathbb{R}}\text{-UNSAT}$. First suppose that φ is $\Pi'_{\mathbb{R}}\text{-UNSAT}$. Let
 - for IA_UNSAT transition,

Theorem 5 Let φ be the polynomial constraint to be solved. Starting with the state $(\Pi, \varphi, \emptyset, \emptyset, \emptyset)$, if our transitional system terminates and output:

- SAT then φ is $T_{\mathbb{R}}^p\text{-SAT}$.
- UNSAT then φ is $T_{\mathbb{R}}^p\text{-UNSAT}$.

Proof 10 1. If the system output SAT, there are two possibles transition to SAT:

- In the case of IA_VALID , we have φ^V is $\Pi_{\mathbb{R}}^c\text{-VALID}$ (invariant 2) $\implies \varphi^V$ is $\Pi_{\mathbb{R}}^c\text{-SAT} \implies \varphi$ is $\Pi_{\mathbb{R}}^c\text{-SAT}$ (because $\varphi^V = \varphi$ is the condition of this transition). In addition, following invariant 1, we have $\Pi_{\mathbb{R}}^c \subseteq T_{\mathbb{R}}^p$, then φ is $T_{\mathbb{R}}^p\text{-SAT}$ (Lemma 3).
 - In the case of $TEST_SAT$, φ^U is $\Pi_{\mathbb{R}}^c\text{-SAT} \implies \exists M \in \Pi_{\mathbb{R}}^c \models_M \varphi^V$. Let $M_{\mathbb{R}}^c \in \Pi_{\mathbb{R}}^c$ such that $\models_{M_{\mathbb{R}}^c} \varphi^V$ which implies $(\varphi^V)^{M_{\mathbb{R}}^c} = \text{true}$. In addition, because φ^V is $\Pi_{\mathbb{R}}^c\text{-VALID}$ (invariant 2) and $M_{\mathbb{R}}^c \in \Pi_{\mathbb{R}}^c$, we have $\models_{M_{\mathbb{R}}^c} \varphi^U$ or $(\varphi^U)^{M_{\mathbb{R}}^c} = \text{true}$. Consider the evaluation of φ under the model $M_{\mathbb{R}}^c$: $(\varphi)^{M_{\mathbb{R}}^c} = (\varphi^U \wedge \varphi^V)^{M_{\mathbb{R}}^c}$ (invariant 3) $= (\varphi^U)^{M_{\mathbb{R}}^c} \wedge (\varphi^V)^{M_{\mathbb{R}}^c} = \text{true} \wedge \text{true} = \text{true} \implies \models_{M_{\mathbb{R}}^c} \varphi \implies \varphi$ is $\Pi_{\mathbb{R}}^c\text{-SAT} \implies \varphi$ is $T_{\mathbb{R}}^p\text{-SAT}$ (because of invariant 1 and Lemma 4)
2. If the system output UNSAT, there is only one transition of rule Π_UNSAT . Because $\Pi \models_{SAT} \perp$, $\Pi_{\mathbb{R}}$ is empty (Lemma ?). As a result, φ is $\Pi_{\mathbb{R}}\text{-UNSAT}$ which implies that φ is $T_{\mathbb{R}}^p\text{-UNSAT}$ (invariant 4).

3.5.2 Completeness

Chapter 4

Interval Arithmetic

We design an SMT solver named **raSAT**¹ to solve polynomial constraint.

Although an $O.T$ refinement loop is enough to implement an ICP based SMT solver, we extend it as **raSAT** (SAT by refinement of approximations) loop to accelerate SAT detection by adding $U.T$, which works as in Fig. ?? (b).

1. When an over-approximation theory $O.T$ detects $O.T$ -UNSAT (resp. $O.T$ -valid), answer UNSAT (resp. SAT).
2. When an under-approximation theory $U.T$ detects $U.T$ -SAT, answer SAT.
3. If neither holds, a refinement is applied.

Our design of an SMT solver **raSAT** applies two heuristic features.

- Incremental widening intervals, and incremental deeping search (Section 5.1).
- Heuristic measures *SAT-likelihood* and *sensitivity*, for selection of a variable to decompose and a box to explore. (Section 5.2).

raSAT also prepares various interval arithmetic as $O.T$ as in Section ??, whereas currently only random testing (k -random ticks, which consists of periodical k -test instances with a random offset) is prepared as $U.T$.

A typical theory for $O.T$ and $U.T$ are an interval arithmetic and testing, respectively. We say *IA-valid*, *IA-SAT*, and *IA-UNSAT*, when it is $O.T$ -valid, $O.T$ -SAT, and $O.T$ -UNSAT, respectively. Similarly, we say *test-SAT* and *test-UNSAT*, when it is $U.T$ -SAT and $U.T$ -UNSAT, respectively. Note that either IA-valid or test-SAT implies SAT, and IA-UNSAT implies UNSAT, whereas IA-SAT and test-UNSAT can conclude neither.

raSAT prepares various Affine intervals, adding to classical interval (CI) [?], which keep lower and upper bounds. The weakness of CI is loss of dependency among values. For instance, $x - x$ is evaluated to $(-2, 2)$ for $x \in (2, 4)$.

Affine Interval [? ?] introduces *noise symbols* ϵ , which are interpreted as values in $(-1, 1)$. For instance, $x = 3 + \epsilon$ describes $x \in (2, 4)$, and $x - x = (3 + \epsilon) - (3 + \epsilon)$ is

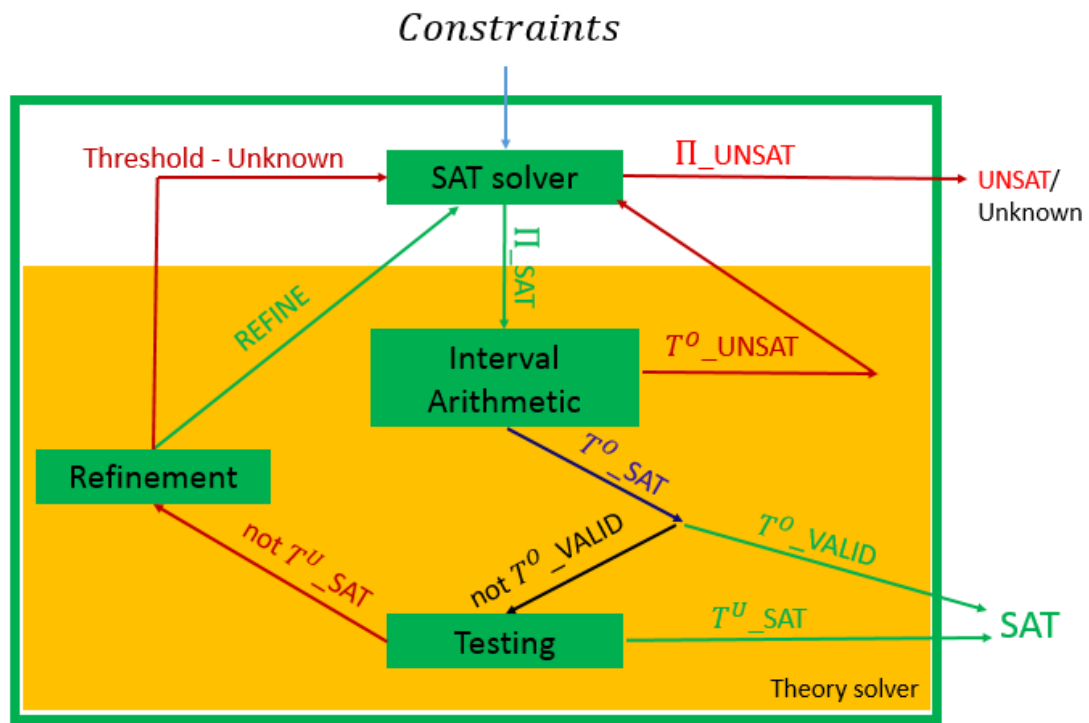
¹<http://www.jaist.ac.jp/~mizuhito/tools/rasat.html>

evaluated to 0. The drawback is that the multiplication without dependency might be less precise than CI. Affine intervals also cannot represent infinite intervals, e.g., $(0, \infty)$, since it becomes $\infty + \infty \epsilon$. Forms of affine intervals vary by choices how to approximate multiplications. They are,

- (i) $\epsilon\epsilon'$ is replaced with a fresh noise symbol (AF) [? ?],
- (ii) $\epsilon\epsilon'$ is reduced to the fixed error noise symbol ϵ_{\pm} (AF_1 and AF_2) [?],
- (iii) $\epsilon\epsilon'$ is replaced with $(-1, 1)\epsilon$ (or $(-1, 1)\epsilon'$) (EAI) [?],
- (iv) $\epsilon\epsilon$ is reduced to fixed noise symbols ϵ_+ or ϵ_- (AF_2) [?],
- (v) Chebyshev approximation of x^2 introduces a noise symbol $|\epsilon|$ as an absolute value of ϵ with $\epsilon\epsilon = |\epsilon||\epsilon| = |\epsilon| + (-\frac{1}{4}, 0)$ and $\epsilon|\epsilon| = \epsilon + (-\frac{1}{4}, \frac{1}{4})$ [?].

Example 3 Let $f = x^3 - 2xy$ with $x = (0, 2)$ ($x = 1 + \epsilon_1$) and $y = (1, 3)$ ($y = 2 + \epsilon_2$), we have,

- AF_2 estimates the range of f as $-3 - \epsilon_1 - 2\epsilon_2 + 3\epsilon_+ + 3\epsilon_{\pm}$, thus $(-9, 6)$,
- CAI estimates the range of f as $(-4, -\frac{11}{4}) + (-\frac{1}{4}, 0)\epsilon_1 - 2\epsilon_2 + 3|\epsilon_1| + (-2, 2)\epsilon_{\pm}$, thus $(-8, 4.5)$.

Figure 4.1: **raSAT** design

Chapter 5

Design strategies

We implemented a number of strategies for improving efficiency of raSAT: incremental search and refinement heuristics.

5.1 Incremental search

raSAT applies three incremental strategies, (1) *incremental widening*, (2) *incremental deepening* and (3) *incremental testing*. Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^m f_j > 0$ for $I_i = (a_i, b_i)$.

5.1.1 Incremental widening

Given $0 < \delta_0 < \delta_1 < \cdots$, *incremental widening* starts with $F_0 = \exists x_1 \in I_1 \cap (-\delta_0, \delta_0) \cdots x_n \in I_n \cap (-\delta_0, \delta_0). \bigwedge_{j=1}^m f_j > 0$, and if it finishes with UNSAT, it runs with $F_1 = \exists x_1 \in I_1 \cap (-\delta_1, \delta_1) \cdots x_n \in I_n \cap (-\delta_1, \delta_1). \bigwedge_{j=1}^m f_j > 0$, and so on (Fig. 5.1 (a)).

Note that if $\delta_i < \infty$, **raSAT** applies an Affine interval; otherwise, it uses CI. Experiments in Section ?? are performed with $\delta_0 = 10$ and $\delta_1 = \infty$.

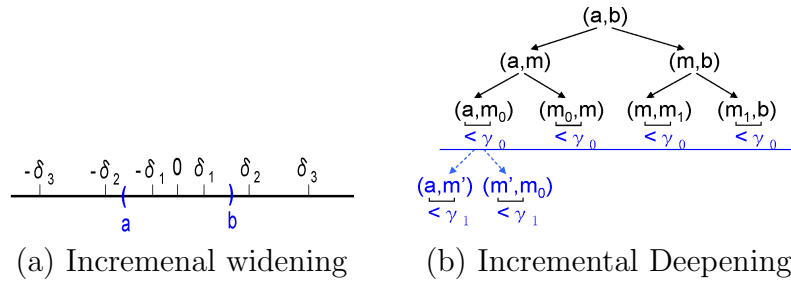


Figure 5.1: Incremental Widening and Deepening

5.1.2 Incremental deepening

Starting with $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^m f_j > 0$, $I_1 \times \cdots \times I_n$ is decomposed into many boxes, and F becomes the disjunction of existential formulae corresponding to these boxes. **raSAT** searches these boxes in depth-first manner. Figure 5.2 illustrates the search procedure of raSAT. First, the initially box I is examined (by IA and testing) and suppose raSAT detects neither SAT or UNSAT here. As a consequence, I is decomposed into 2 boxes I_{11} and I_{12} such that $\bar{x} \in I \leftrightarrow \bar{x} \in I_{11} \vee \bar{x} \in I_{12}$. raSAT will choose one of the two boxes to explore, for example I_{11} . If IA and testing cannot detect satisfiability of the given constraint in the box I_{11} , decomposition of this box takes place: $\bar{x} \in I_{11} \leftrightarrow \bar{x} \in I_{21} \vee \bar{x} \in I_{22}$. Either I_{21} or I_{22} is selected next. The process continues until SAT/UNSAT is detected which may leads to exhaustive local search. To avoid it, **raSAT** applies a threshold γ , such that no more decomposition will be applied when a box becomes smaller than γ . If neither SAT nor UNSAT is detected, **raSAT** restarts with a smaller threshold.

Let $\gamma_0 > \gamma_1 > \cdots > 0$, and **raSAT** incrementally deepens its search with these thresholds, i.e., starting with δ_0 , and if it fails, restart with δ_1 , and so on (Fig 5.1 (b)).

5.1.3 Incremental testing

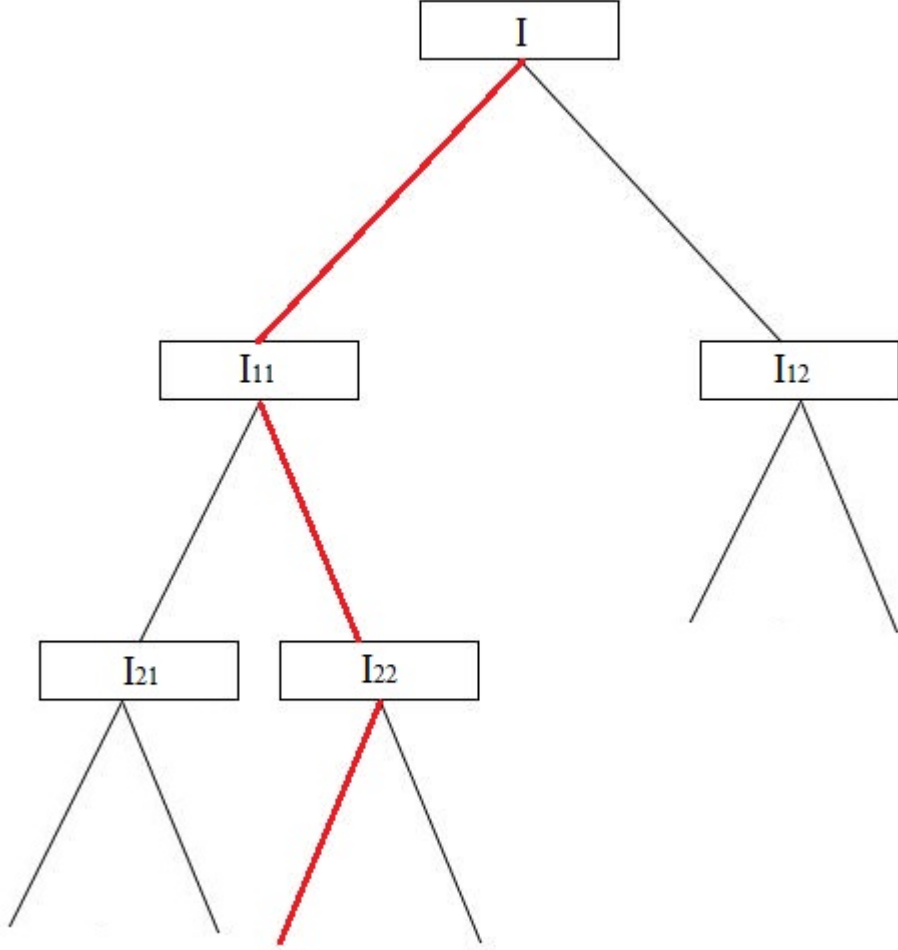
One obstacle in testing is the exponentially large number of test instances. If 2 values are generated for each of n variables, 2^n test cases (combinations of generated values) will present.

Example 4 Suppose $\{x, y\}$ is the set of variables which appears in the input constraint and let $\{2, 9\}$ and $\{5, 8\}$ are generated values for x and y respectively. In total 4 test cases arise: $(x, y) = (2, 5), (2, 8), (9, 5), (9, 8)$.

In order to tackle the problem, the following strategies are proposed:

1. Restrict the number of test cases to 2^{10} by choosing most 10 influential variables which are decided by the following procedures for generating multiple (2) test values.
 - Select API by SAT-likelihood
 - Select 1 variable of each selected API.
2. Incrementally generate test values for variables to prune test cases that do not satisfy an API. This was proposed by Khanh and Ogawa in [12]:
 - Dynamically sort the IA-SAT APIs.
 - Generate the test values for variables of selected APIs.

Example 5 Let $x^2 > 4$ and $x * y > 0$ are two IA-VALID APIs to be tested and somehow they are sorted in that order, i.e. $x^2 > 4$ is selected before $x * y > 0$. Suppose $\{1, 3\}$ are generated as test values for x which are is enough to test the

Figure 5.2: **raSAT** design

first selected API, i.e. $x^2 > 4$. As a result of testing, $x = 1$ is excluded from the satisfiable test cases whilst $x = 3$ is not. Next, when $x * y > 0$ is considered, y needs to be generated 2 values, e.g. $\{-3, 4\}$ and two test cases $(x, y) = (3, -3), (3, 4)$ come out to be checked. In this example, $(x, y) = (1, -3)$ and $(x, y) = (1, 4)$ are early pruned by only testing $x = 1$ against $x^2 > 4$.

5.2 SAT directed heuristics measure

With several hundred variables, we observe that an SMT solver works when either SAT, or UNSAT with small UNSAT core. For the latter, we need an efficient heuristics to find an UNSAT core, which is left as future work. For the former, the keys are how to choose variables to decompose, and how to choose a box to explore. **raSAT** chooses such

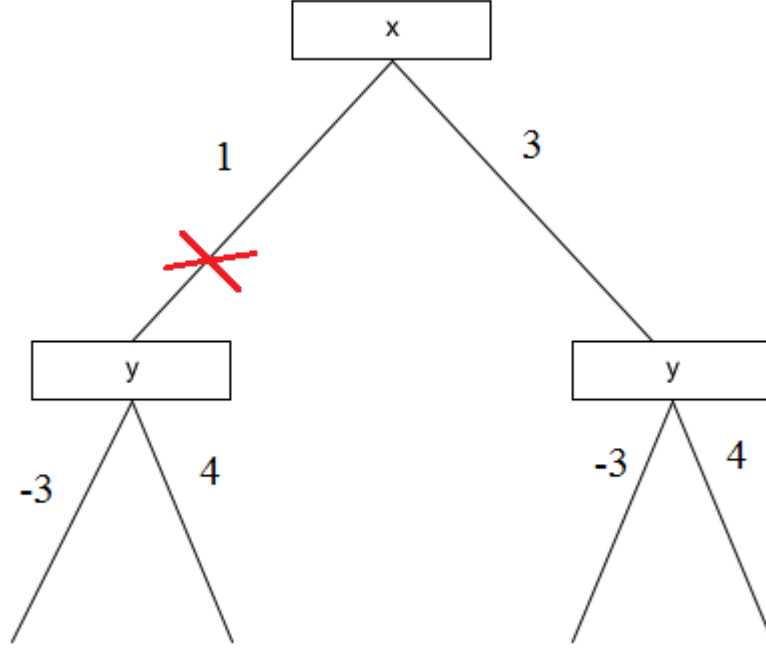


Figure 5.3: Incremental Testing Example

a variable in two steps; first it selects a *test-UNSAT API*, and then chooses a variable that appears in the API. We design SAT-directed heuristic measures based on the interval arithmetic (*O.T*).

Let $F = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^m f_j > 0$ becomes $\vee (\exists x_1 \in I'_1 \cdots x_n \in I'_n. \bigwedge_{j=1}^m f_j > 0)$

after box decomposition. For $\exists x_1 \in I'_1 \cdots x_n \in I'_n. \bigwedge_{j=1}^m f_j > 0$, if some $f_j > 0$ is UNSAT, the box $I'_1 \times \cdots \times I'_n$ is UNSAT. If every $f_j > 0$ is SAT, F is SAT. Thus, if the box $I'_1 \times \cdots \times I'_n$ needs to be explore, it must contain a test-UNSAT API (thus IA-SAT).

We denote the estimated range of f_j for $x_1 \in I'_1 \cdots x_n \in I'_n$ with IA (*O.T*) by $range(f_j, I'_1 \times \cdots \times I'_n)$. If an IA is an affine interval, it is in the form $[c_1, d_1]\epsilon_1 + \cdots + [c_n, d_n]\epsilon_n$, and by instantiating ϵ_i with $[-1, 1]$, the resulting classical interval coincides with $range(f_j, I'_1 \times \cdots \times I'_n)$. We define

- *Sensitivity* of a variable x_i in a test-UNSAT API $f_j > 0$ is $max(|c_i|, |d_i|)$.
- *SAT-likelihood* of an API $f_j > 0$ is $|I \cap (0, \infty)|/|I|$, and
- *SAT-likelihood* of a box $I'_1 \times \cdots \times I'_n$ is the least SAT-likelihood of test-UNSAT APIs.

Example 6 In Example 3,

- *sensitivity is estimated 1 for x and 2 for y by AF_2 , and $3\frac{1}{4}$ for x and 2 for y .*
- *SAT-likelihood of f is estimated $0.4 = \frac{6}{9-(-6)}$ by AF_2 and $0.36 = \frac{4.5}{4.5-(-8)}$ by CAI .*

SAT-likelihood intends to estimate APIs how likely to be SAT. For choosing variables, **raSAT** first choose a test-UNSAT API by SAT-likelihood. There are two choices, either *the largest* or *the least*. *Sensitivity* of a variable intends to estimate how a variable is influential to the value of an API. From a selected API by SAT-likelihood, **raSAT** selects a variable with the largest sensitivity. This selection of variables are used for (1) *multiple test instances generation*, and (2) *decomposition*. For test generation, we will select multiple variables by repeating the selection.

For choosing a box to explore, **raSAT** chooses more likely to be SAT. There are two choice, (1) a box with the largest SAT-likelihood, and (2) a box with the largest number of SAT (either IA-valid or test-SAT) APIs.

5.3 Other

UNSAT core, test case generation, interval decomposition

Chapter 6

Experiments

This chapter is going to present the experiments results which reflect how effective our designed strategies are. In addition, comparison between raSAT, Z3 and iSAT3 will be also shown. The experiments were done on a system with Intel Xeon E5-2680v2 2.80GHz and 4 GB of RAM. In the experiments, we exclude the problems which contain equalities because currently raSAT focuses on inequalities only.

6.1 Experiments on strategy combinations

We perform experiments only on Zankl, and Meti-Tarski families.

Our combinations of strategies mentioned in Section ?? are,

Selecting a test-UNSAT API	Selecting a box (to explore):	Selecting a variable:
(1) Least SAT-likelihood.	(3) Largest number of SAT APIs.	(8) Largest sensitivity.
(2) Largest SAT-likelihood.	(4) Least number of SAT APIs.	
	(5) Largest SAT-likelihood.	
	(6) Least SAT-likelihood.	
(10) Random.	(7) Random.	(9) Random.

Table 6.1 shows the experimental results of above mentioned combination. The timeout is set to 500s, and each time is the total of successful cases (either SAT or UNSAT).

Note that (10)-(7)-(9) means all random selection. Generally speaking, the combination of (5) and (8) show the best results, though the choice of (1),(2), and (10) shows different behavior on benchmarks. We tentatively prefer (1) or (10), but it needs to be investigated further.

Other than heuristics mentioned in Section ??, there are lots of heuristic choices. For instance,

- how to generate test instances (in $U.T$),
- how to decompose an interval,

and so on.

Benchmark	(1)-(5)-(8)			(1)-(5)-(9)			(1)-(6)-(8)			(1)-(6)-(9)			(10)-(5)-(8)			(10)-(6)-(8)		
Matrix-1 (SAT)	20	132.72	(s)	21	21.48		19	526.76		18	562.19		21	462.57		19	155.77	
Matrix-1 (UNSAT)	2	0.00		3	0.00		3	0.00		3	0		3	0.00		3	0.00	
Matrix-2,3,4,5 (SAT)	11	632.37		1	4.83		0	0.00		1	22.50		9	943.08		1	30.48	
Matrix-2,3,4,5 (UNSAT)	8	0.37		8	0.39		8	0.37		8	0.38		8	0.38		8	0.38	
Benchmark	(2)-(5)-(8)			(2)-(5)-(9)			(2)-(6)-(8)			(2)-(6)-(9)			(2)-(7)-(8)			(10)-(7)-(9)		
Matrix-1 (SAT)	22	163.47	(s)	19	736.17		20	324.97		18	1068.40		21	799.79		21	933.39	
Matrix-1 (UNSAT)	2	0		2	0.00		2	0.00		2	0.00		2	0.00		2	0.00	
Matrix-2,3,4,5 (SAT)	5	202.37		1	350.84		1	138.86		0	0.00		0	0.00		0	0.00	
Matrix-2,3,4,5 (UNSAT)	8	0.43		8	0.37		8	0.40		8	0.38		8	0.37		8	0.38	
Benchmark	(1)-(3)-(8)			(1)-(4)-(8)			(2)-(3)-(8)			(2)-(4)-(8)			(10)-(3)-(8)			(10)-(4)-(8)		
Matrix-1 (SAT)	20	738.26	(s)	21	1537.9		18	479.60		21	867.99		20	588.78		19	196.21	
Matrix-1 (UNSAT)	2	0.00		2	0.00		2	0.00		2	0.00		2	0.00		2	0.00	
Matrix-2,3,4,5 (SAT)	0	0.00		2	289.17		1	467.12		1	328.03		1	195.18		2	354.94	
Matrix-2,3,4,5 (UNSAT)	8	0.36		8	0.36		8	0.34		8	0.37		8	0.37		8	0.39	
Benchmark	(1)-(5)-(8)			(1)-(5)-(9)			(10)-(5)-(8)			(10)-(7)-(9)								
Meti-Tarski (SAT, 3528)	3322	369.60	(s)	3303	425.37		3325	653.87		3322	642.04							
Meti-Tarski (UNSAT, 1573)	1052	383.40		1064	1141.67		1100	842.73		1076	829.43							

Table 6.1: Combnations of **raSAT** strategies on NRA/Zankl, Meti-Tarski benchmark

Experiments in Table 6.1 are performed with randome generation (k -random tick) for the former and the blanced decomposition (dividing at the exact middle) for the latter. Further investigation is left for future.

6.2 Comparison with other SMT solvers

We compare **raSAT** with other SMT solvers on NRA benchmarks, Zankl and Meti-Tarski. The timeouts for Zankl and Meti-tarski are 500s and 60s, respectively. For **iSAT3**, ranges of all variables are uniformly set to be in the range $[-1000, 1000]$ (otherwise, it often causes segmentation fault). Thus, UNSAT detection of **iSAT3** means UNSAT in the range $[-1000, 1000]$, while that of **raSAT** and **Z3 4.3** means UNSAT over $[-\infty, \infty]$.

Among these SMT solvers, **Z3 4.3** shows the best performance. However, if we closely observe, there are certain tendency. **Z3 4.3** is very quick for small constraints, i.e., with short APIs (up to 5) and a small number of variables (up to 10). **raSAT** shows comparable performance on SAT detection with longer APIs (larger than 5) and a larger number of variables (more than 10), and sometimes outforms for SAT detection on vary long constraints (APIs longer than 40 and/or more than 20 variables). Such examples appear in Zankl/matrix-3-all-*, matrix-4-all-*, and matrix-5-all-* (total 74 problems), and **raSAT** solely solves

- *matrix-3-all-2* (47 variables, 87 APIs, and max length of an API is 27),
- *matrix-3-all-5* (81 variables, 142 APIs, and max length of an API is 20),
- *matrix-4-all-3* (139 variables, 244 APIs, and max length of an API is 73), and

Benchmark	raSAT				Z3 4.3				iSAT3			
	SAT		UNSAT		SAT		UNSAT		SAT		UNSAT	
Zankl/matrix-1 (53)	20	132.72 (s)	2	0.00	41	2.17	12	0.00	11	4.68	3	0.00
Zankl/matrix-2,3,4,5 (98)	11	632.37	8	0.37	13	1031.68	11	0.57	3	196.40	12	8.06
Meti-Tarski (3528/1573)	3322	369.60	1052	383.40	3528	51.22	1568	78.56	2916	811.53	1225	73.83

Table 6.2: Comparison among SMT solvers

- *matrix-5-all-01* (132 variables, 276 APIs, and max length of an API is 47).

Note that, for Zankl, when UNSAT is detected, it is detected very quickly. This is because SMT solvers detects UNSAT only when they find small UNSAT cores, without tracing all APIs. However, for SAT detection with ;arge problems, SMT solvers need to trace all problems. Thus, it takes much longer time.

6.3 Experiments on equality handling

6.4 Polynomial constraints over integers

raSAT loop is easily modified to NIA (nonlinear arithmetic over integers) from NRA, by setting $\gamma_0 = 1$ in incremental deepening in Section 5.1 and restricting testdata generation on intergers. We also compare **raSAT** (combination (1)-(5)-(8)) with **Z3 4.3** on NIA/AProVE benchmark. **AProVE** contains 6850 inequalities among 8829. Some has several hundred variables, but each API has few variables (mostly just 2 variables).

The results are,

- **raSAT** detects 6764 SAT in 1230.54s, and 0 UNSAT.
- **Z3 4.3** detects 6784 SAT in 103.70s, and 36 UNSAT in 36.08s.

where the timeout is 60s. **raSAT** does not successfully detect UNSAT, since UNSAT problems have quite large coefficients which lead exhaustive search on quite large area.

Chapter 7

Equality handling

7.1 SAT on Equality by Intermediate Value Theorem

For solving polynomial constraints with single equality ($g = 0$), we apply *Intermediate Value Theorem*. That is, if existing 2 test cases such that $g > 0$ and $g < 0$, then $g = 0$ is SAT somewhere in between, as in Fig. ??.

Lemma 6 For $F = \exists x_1 \in (a_1, b_1) \wedge \cdots \wedge x_n \in (a_n, b_n) \cdot \bigwedge_j^m f_j > 0 \wedge g = 0$, F is SAT, if there is a box $(l_1, h_1) \times \cdots \times (l_n, h_n)$ with $(l_i, h_i) \subseteq (a_i, b_i)$ such that

(i) $\bigwedge_j^m f_j > 0$ is IA-VALID in the box, and

(ii) there are two instances \vec{t}, \vec{t}' in the box with $g(\vec{t}) > 0$ and $g(\vec{t}') < 0$.

raSAT first tries to find an IA-VALID box for $\bigwedge_j^m f_j > 0$ by refinements. If such a box is found, it tries to find 2 instances for $g > 0$ and $g < 0$ by testing. Intermediate Value Theorem guarantees the existence of an SAT instance in between. Note that this method works for single equality and does not find an exact SAT instance. If multiple equalities do not share variables each other, we can apply Intermediate Value Theorem repeatedly to decide SAT. In Zankl benchmarks in SMT-lib, there are 15 gen-*.smt2 that contain equality (among 166 problems), and each of them satisfy this condition.

In Table 7.1 we show preliminary experiment for 15 problems that contain polynomial equalities in Zankl family. **raSAT** works well for these SAT problems and it can detect all SAT problems (11 among 15). At the current implementation, raSAT reports *unknown* for UNSAT problems. The first 4 columns indicate *name of problems*, *the number of variables*, *the number of polynomial equalities* and *the number of inequalities* in each problem, respectively. The last 2 columns show comparison results of **Z3 4.3** and **raSAT**.

Problem Name	No. Variables	No. Equalities	No. Inequalities	Z3 4.3 (15/15)		raSAT (11/15)	
				Result	Time(s)	Result	Time(s)
gen-03	1	1	0	SAT	0.01	SAT	0.015
gen-04	1	1	0	SAT	0.01	SAT	0.015
gen-05	2	2	0	SAT	0.01	SAT	0.046
gen-06	2	2	1	SAT	0.01	SAT	0.062
gen-07	2	2	0	SAT	0.01	SAT	0.062
gen-08	2	2	1	SAT	0.01	SAT	0.062
gen-09	2	2	1	SAT	0.03	SAT	0.062
gen-10	1	1	0	SAT	0.02	SAT	0.031
gen-13	1	1	0	UNSAT	0.05	unknown	0.015
gen-14	1	1	0	UNSAT	0.01	unknown	0.015
gen-15	2	3	0	UNSAT	0.01	unknown	0.015
gen-16	2	2	1	SAT	0.01	SAT	0.062
gen-17	2	3	0	UNSAT	0.01	unknown	0.031
gen-18	2	2	1	SAT	0.01	SAT	0.078
gen-19	2	2	1	SAT	0.05	SAT	0.046

Table 7.1: Experimental results for 15 equality problems of Zankl family

We also apply the same idea for multiple equalities $\bigwedge_i g_i = 0$ such that $Var(g_k) \cap Var(g_{k'}) = \emptyset$ where $Var(g_k)$ is denoted for the set of variables in the polynomial g_k . In the next section we will present idea for solving general cases of multiple equalities.

7.2 Equality handling using Grobner basis

Coming soon...

Chapter 8

UNSAT core

Chapter 9

Conclusion

This paper presented **raSAT** loop, which mutually refines over and under-approximation theories. For polynomial inequality, we adopted interval arithmetic and testing for over and under-approximation theories, respectively. **raSAT** loop is implemented as an SMT **raSAT**. The result of Experiments on QF_NRA in SMT-lib is encouraging, and **raSAT** shows comparable and sometimes outperforming to existing SMTs, e.g., Z3 4.3, HySAT, and dReal. For instance, `*****` which has `**` variables and degree `**` was solved by **raSAT**, whereas none of above mentioned SMTs can.

9.1 Observation and Discussion

From experimental results in Section ?? and ??, we observe the followings.

- The degree of polynomials will not affect much.
- The number of variables are matters, but also for Z3 4.3. The experimental results do not show exponential growth, and we expect the strategy of selection of an API in which related intervals are decomposed seems effective in practice. By observing Zankl examples, we think the maximum number of variables of each API seems a dominant factor.
- Effects of the number of APIs are not clear at the moment. In simple benchmarks, **raSAT** is faster than Z3 4.3, however we admit that we have set small degree $n = 6$ for each API.

For instance, *matrix-2-all-5,8,11,12* in Zankl contain a long monomial (e.g., 60) with the max degree 6, and relatively many variables (e.g., 14), which cannot be solved by Z3 4.3, but **raSAT** does. As a general feeling, if an API contains more than $30 \sim 40$ variables, **raSAT** seems saturating. We expect that, adding to a strategy to select an API (Section ??), we need a strategy to select variables in the focus. We expect this can be designed with sensitivity (Example 3) and would work in practice. Note that sensitivity can be used only with noise symbols in Affine intervals. Thus, iSAT and RSOLVER cannot use this strategy, though they are based on IA, too.

9.2 Future Work

UNSAT core

Exact confirmation. Currently, **raSAT** uses iRRAM to verify SAT instances only. We are planning to implement confirmation phase to UNSAT cases.

Further strategy refinement. Test data generation, blanced box decomposition

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