# raSAT: an SMT Solver for Polynomial Constraints over Reals

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**Abstract.** This paper presents an SMT solver **raSAT** for polynomial inequality. It consists of a simple iterative approximation refinement, **raSAT** loop, which is an extension of the standard ICP (Interval Constraint Propagation) with testing, aiming to accelerate SAT detection. If it fails to decide, input intervals are refined by decomposition.
ICP is robust for large degrees, but the number of boxes (products of intervals) may exponentially explode with respect to the number of variables. For boosting SAT detection, we design strategies, namely *SAT likelihood* and *sensitivity*, on the choice of a variable to decompose and a box to explore.
At last, **raSAT** loop is extended for polynomial equality based on the Intermediate Value Theorem. This extension is again evaluated on Zankl, Meti-tarski, and Keymaera families. A simple modification of **raSAT** loop to handle mixed integers will be also presented.

## 1 Introduction

*Polynomial constraint solving* over reals (resp. integers) is to find an instance from reals (resp. integers) that satisfies given polynomial inequality/equality. Solving polynomial constraints on reals is decidable [21], though that on integers is undecidable (*Hilbert's 10th problem*). Quantifier Elimination by Cylindrical Algebraic Decomposition (QE-CAD) [4] is a well known technique, and implemented in Mathematica, Maple/SynRac, Reduce/Redlog, QEPCAD-B, and recently in some SMT solvers [13, 6]. QE-CAD solves more than the satisfiability, and is DEXPTIME. By restricting on the satisfiability, *Variant quantifier elimination* [12] reduces to polynomial optimization problems, which are solved by Groebner basis in EXPTIME.

A practical alternative is Interval Constraint Propagation(*ICP*)[2], which is implemented in **iSAT3** [7], **dReal** [10], and **RSolver** [20]. ICP is an over-approximation by an interval arithmetic, and iteratively refines by interval decomposition. Although ICP is not complete for UNSAT detection with unbounded intervals, it is practically often more efficient than algebraic methods.

This paper presents an SMT solver **raSAT** for polynomial constraints over reals. It consists of a simple iterative approximation refinement, **raSAT** *loop*,

which adds testing to boost SAT detection to the standard ICP, aiming to accelerate SAT detection. If both the estimation by an interval arithmetic and testing fail, input intervals are refined by decomposition. The features of **raSAT** are,

- **raSAT** loop, which adds testing to boost SAT detection to a standard ICP,
- various interval arithmetic support, e.g., Affine intervals [15, 17, 14],
- sound use of floating point arithmetic, i.e., outward rounding in interval arithmetic [11], and confirmation of an SAT instance by an error-bound guaranteed floating point package **iRRAM**[3].

ICP and **raSAT** loop are robust for large degrees, but the number of boxes (products of intervals) grows exponentially. First, we target on polynomial inequations, and design SAT detection-directed strategies on the choice of a variable to decompose, a box to explore, and a variable to generate multiple test cases. They are based on heuristic measures, *SAT likelihood*, *sensitivity*, and the number of unsolved atomic polynomial constraints. Another strategy of **raSAT** is an incremental avoiding local optimal.

- **Incremental widening**. Starting **raSAT** loop with a smaller initial interval, and if it is UNSAT, enlarge the input intervals and restart.
- **Incremental deepening**. Starting with the bound that each interval will be decomposed no smaller than it. If neither SAT nor UNSAT is detected, set a smaller bound and restart.

The combinations are examined on Zankl, Meti-tarski and Keymaera benchmarks from QF_NRA of SMT-LIB, to find clear differences from random choices. We also show two extensions, (1) handling polynomial equations by using the Intermediate Value Theorem, and (2) polynomial constraints over integers (e.g., AProVE benchmark in QF_NIA). These results are also compared with **Z3 4.3**, **dReal-2.15.01** and **iSAT3**.

## 2 ICP overview and raSAT loop

Our target problem is solving nonlinear constraints. We mainly discuss on polynomial inequalities, and later in Section 5, we show an extension to cover polynomial equations based on the Intermediate Value Theorem. From now on, we use the following notations without redefine it. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^\infty = \mathbb{R} \cup \{-\infty, \infty\}$. The normal arithmetics on $\mathbb{R}$ is extended to those on $\mathbb{R}^\infty$ as in [16]. The set of all intervals is defined as $\mathbb{I} = \{[l, h] \mid l \leq h \in \mathbb{R}^\infty\}$.

**Definition 1.** *A polynomial inequality constraint is*

$$\varphi : \exists x_1 \in I_1 \cdots x_n \in I_n . \bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n)$$

*where $\psi_j(x_1, \cdots, x_n)$ is an atomic polynomial inequality (API) of the form $p_j(x_1, ..., x_n) > 0$ and $I_1, \cdots, I_n \in \mathbb{I}$. We denote the set of variables appearing in $p_j$ by $var(p_j)$.*

[3] `http://irram.uni-trier.de`

Note that $\varphi$ is equivalent to $\exists x_1 \dots x_n.(\bigwedge_{i=1}^{n} x_i \in I_i) \wedge (\bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n))$. We refer $\bigwedge_i x_i \in I_i$ and $\bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n)$ by $I_\varphi$ and $\psi(x_1, \cdots, x_n)$, respectively. We denote the set of solutions of the constraint $\psi(x_1, \cdots, x_n)$ as $\mathbb{S}(\psi(x_1, \cdots, x_n)) = \{(r_1, \cdots, r_n) \in \mathbb{R}^n \mid \psi(r_1. \cdots, r_n) \text{ holds}\}$.

We review Interval Constraint Propagation (ICP)[2], and then introduce **raSAT** (refinement of approximations for SAT) loop [14]. The main difference is that **raSAT** loop has testing after the estimation by Interval Arithmetic (IA) to boost SAT detection. Note that both ICP and **raSAT** are suffered from roundoff errors of the floating arithmetic. To guarantee the soundness, IA adopts outward rounding [11] on the lower/upper bounds of intervals. In **raSAT**, when testing says SAT, it is confirmed with the error bound guaranteed package **iRRAM**.

### 2.1 ICP overview

Algorithm 1 describe the basic ICP for solving polynomial inequations. Let $\psi = \bigwedge_{j=1}^{m} g_j(x_1, \cdots, x_n) > 0$ be a target constraint.

**Algorithm 1** ICP starting from the initial box $B_0 = I_1 \times \cdots \times I_n$

```
1:  S ← {B_0}                                              ▷ Set of boxes
2:  while S ≠ ∅ do
3:      B ← S.choose()                                     ▷ Get one box from the set
4:      B' ← prune(B, ψ)
5:      if B' = ∅ then                                     ▷ The box does not satisfy the constraint
6:          S ← S \ {B}
7:          continue
8:      else if B' satisfies ψ by using IA then
9:          return SAT
10:     else                          ▷ IA cannot conclude the constraint ⟹ Refinement Step
11:         {B_1, B_2} ← split(B')    ▷ split B' into two smaller boxes B_1 and B_2
12:         S ← (S \ {B}) ∪ {B_1, B_2}
13:     end if
14: end while
15: return UNSAT
```

where two functions $prune(B, \psi)$ and $split(B)$ satisfy

- If $B' = prune(B, \psi)$, then $B' \subseteq B$ and $B' \cap \mathbb{S}(\psi) = B \cap \mathbb{S}(\psi)$.
- If $\{B_1, B_2\} = split(B)$, then $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \emptyset$.

Since ICP concludes SAT (line 8) only when it finds a box in which the constraint becomes valid by IA. It is also suffered from roundoff errors, and the basic ICP cannot conclude the satisfiability of equations. In contrast, although

the number of boxes increase exponentially, ICP always detects SAT of an inequality constraint $\exists x_1 \in (a_1, b_1) \cdots x_n \in (a_n, b_n). \wedge_j g_j > 0$ if the *split* is fairly applied and each $I_j$ is bounded. However, ICP may miss to detect UNSAT. Limitations for detecting UNSAT come from the *kissing* and *convergent* cases in Fig. 1. The left shows a kissing case $x^2+y^2 < 2^2 \wedge (x-4)^2+(y-3)^2 < 3^2$ such that $\overline{\mathbb{S}(-x^2 - y^2 + 2^2 > 0)} \cap \overline{\mathbb{S}(-(x-4)^2 - (y-3)^2 + 3^2 > 0)} = \{(x, y) \mid (1.6, 1.2)\}$. Thus, it cannot be separated by the covering by (enough small) boxes. The right shows a convergent case $y > x + \frac{1}{x} \wedge y < x \wedge x > 0$, i.e., $xy > x^2 + x \wedge y < x \wedge x > 0$. The latter does not appear if all intervals $I_j$ are finitely bounded.

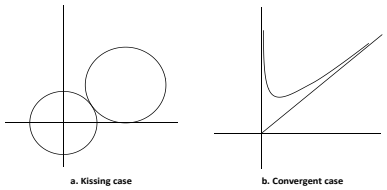



Fig. 1: Limitations for detection of UNSAT

### 2.2 raSAT loop

ICP is extended to **raSAT** loop, which adds testing to boost SAT detection [14].

Let $\psi = \bigwedge_{j=1}^{m} g_j(x_1, \cdots, x_n) > 0$ be a target constraint. Algorithm 2 displays **raSAT** loop. Its implementation **raSAT** adapts various IAs, and applies two main heuristics (the use of Affine intervals enable us the latter).

- Incremental widening and incremental deepening (Section 3.1).
- Heuristic measures *SAT-likelihood* and *sensitivity*, for selection of a variable at line 11 (of Algorithm 2) and a box at line 13 (Section 3.2).

### 2.3 Interval Arithmetic

Various Affine Intervals [5] are prepared in **raSAT**, adding to Classical Interval (CI) [16]. Although precision is incomparable, Affine interval partially preserves the dependency among values, which are lost in CI. For instance, $x - x$ is evaluated to $(-2, 2)$ for $x \in (2, 4)$ by CI, but 0 by an Affine interval.

Affine interval introduces *noise symbols* $\epsilon$, which are interpreted as values in $(-1, 1)$. For instance, $x = 3 + \epsilon$ describes $x \in (2, 4)$, and $x - x = (3 + \epsilon) - (3 + \epsilon)$ is evaluated to 0. The drawback is that the multiplication without dependency might be less precise than CI. Affine intervals also cannot represent infinite intervals, e.g., $(0, \infty)$, since it becomes $\infty + \infty\ \epsilon$. Forms of Affine intervals vary by choices how to approximate multiplications. They are,

(i) $\epsilon\epsilon'$ is replaced with a fresh noise symbol ($AF$) [5],
(ii) $\epsilon\epsilon'$ is reduced to the fixed error noise symbol $\epsilon_\pm$ ($AF_1$ and $AF_2$) [15],
(iii) $\epsilon\epsilon'$ is replaced with $(-1,1)\epsilon$ (or $(-1,1)\epsilon'$) ($EAI$) [17],
(iv) $\epsilon\epsilon$ is reduced to fixed noise symbols $\epsilon_+$ or $\epsilon_-$ ($AF_2$) [15],
(v) Chebyshev approximation of $x^2$ introduces a noise symbol $|\epsilon|$ as an absolute value of $\epsilon$ with $\epsilon\epsilon = |\epsilon||\epsilon| = |\epsilon| + (-\frac{1}{4}, 0)$ and $\epsilon|\epsilon| = \epsilon + (-\frac{1}{4}, \frac{1}{4})$ [14].

*Example 1.* Let $g = x^3 - 2xy$ with $x = (0,2)$ $(x = 1 + \epsilon_1)$ and $y = (1,3)$ $(y = 2 + \epsilon_2)$, we have,

- $AF_2$ estimates the range of $g$ as $-3 - \epsilon_1 - 2\epsilon_2 + 3\epsilon_+ + 3\epsilon_\pm$, thus $(-9, 6)$,
- $CAI$ estimates the range of $g$ as $(-4, -\frac{11}{4}) + (-\frac{1}{4}, 0)\epsilon_1 - 2\epsilon_2 + \mathbf{3}|\epsilon_1| + (-2, 2)\epsilon_\pm$, thus $(-8, 4.5)$.

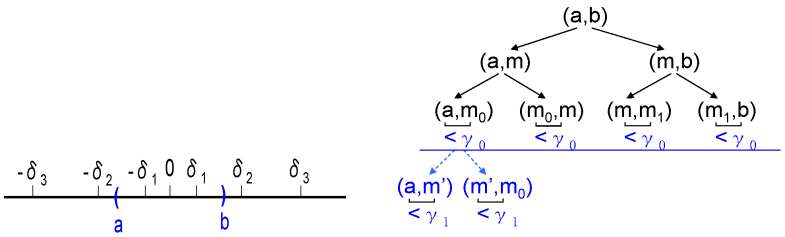
## 3 SAT directed Strategies of raSAT

ICP is affected less with the degree of polynomials, but affected most with the number of variables. Starting with $\varphi = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} g_j > 0$, $I_1 \times \cdots \times I_n$ is decomposed into exponentially many boxes, and $I_\varphi$ becomes the disjunction of existential formulae corresponding to these boxes. The detection of UNSAT requires exhaustive search on all boxes, and finding a small UNSAT core is the key. This is often observed by **Z3** where UNSAT either is quickly detected or leads to timeout. For SAT detection, the keys will be a strategic control not to fall into local optimal and a strategy to choose most likely decomposition/boxes.

### 3.1 Incremental search

Two incremental search strategies are prepared in **raSAT**, (1) *incremental widening*, and (2) *incremental deepening*. Let $\varphi = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} g_j > 0$ for $I_i = (a_i, b_i)$.

**Incremental widening** Given $0 < \delta_0 < \delta_1 < \cdots$, *incremental widening* starts with $I^0_\varphi = \exists x_1 \in I_1 \cap (-\delta_0, \delta_0) \cdots x_n \in I_n \cap (-\delta_0, \delta_0). \bigwedge_{j=1}^{m} g_j > 0$, and if it stays UNSAT, then enlarge the intervals as $I^1_\varphi = \exists x_1 \in I_1 \cap (-\delta_1, \delta_1) \cdots x_n \in I_n \cap (-\delta_1, \delta_1). \bigwedge_{j=1}^{m} g_j > 0$. This continues until either SAT, timeout, or a given bound of repetition (Fig. 2 (a)). Note that if $\delta_i = \infty$, we cannot use an Affine interval For instance, $(-\infty, \infty) = \infty\epsilon$ does not make sense. In **raSAT**, $AF_2$ is used if $\delta_i < \infty$, and $CI$ is used otherwise.




(a) Incremental widening (b) Incremental Deepening

Fig. 2: Two incremental search strategies

**Incremental deepening** To combine depth-first-search and breadth-first search among decomposed boxes, **raSAT** applied *incremental deepening*. Let $\gamma_0 > \gamma_1 > \cdots > 0$. It applies a threshold $\gamma$, such that no more decomposition occurs when a box becomes smaller than $\gamma$. $\gamma$ is initially $\gamma = \gamma_0$. If neither SAT nor UNSAT is detected, **raSAT** restarts with the threshold $\gamma_1$. This continues until either SAT, timeout, or a given bound of repetition (Fig. 2 (b)).

### 3.2 SAT directed heuristics measure

In **raSAT**, a strategy to select a variable to decompose is in the following two steps. (1) First select a most likely influential $API$, and (2) then choose a most likely influential variable in the selected API. For *most likely influential* measures, we apply the *SAT-likelihood* on APIs and the *sensitivity* on variables, respectively. Note that the latter measure is defined only by Affine intervals.

In line 3 of Algorithm 2, an IA will estimate the ranges of polynomials in a box $B$. We denote the estimated range of $g_j$ by $range(g_j, B)$. If an IA is an Affine interval, we assume that the estimated range of $g_j$ has the form $(c_1, d_1)\epsilon_1 + \cdots + (c_n, d_n)\epsilon_n$. By instantiating $(-1, 1)$ to $\epsilon_i$, we obtain $range(g_j, B)$. We define as follows.

- The *SAT-likelihood* of an API $g_j > 0$ is $|I \cap (0, \infty)|/|I|$ for $I = range(g_j, B)$.
- The *sensitivity* of a variable $x_i$ in an API $g_j > 0$ is $max(|c_i|, |d_i|)$.

*Example 2.* In Example 1,

- SAT-likelihood of $f$ is $0.4 = \frac{6}{9-(-6)}$ by $AF_2$ and $0.36 = \frac{4.5}{4.5-(-8)}$ by $CAI$.
- the sensitivity of $x$ is 1 by $AF_2$ and $3\frac{1}{4}$ by $CAI$, and that of $y$ is 2 by both $AF_2$ and $CAI$.

SAT-likelihood intends to estimate an API how likely to be SAT. There are two choices on the SAT-likelihood, either *the largest* or *the least*. The *sensitivity* of a variable intends to estimate how a variable is influential to the value of an API, and the largest sensitivity is considered to be the most influential. This selection of variables are used both for (1) *decomposition*, and (2) *test case generation*. For multiple test generation, we select multiple variables that have larger sensitivity.

At the decomposition, **raSAT** also examines the choice of the box. We define the *SAT-likelihood* of a box $B$ by the least SAT-likelihood of APIs. Since the SAT-likelihood of each box is computed when it is created by the decomposition, **raSAT** simply compares newly decomposed boxes with the previous ones. There are two choices of boxes, (1) a box with the largest SAT-likelihood, and (2) a box with the largest number of SAT (concluded by either IA or testing) APIs. These combinations of strategy choices are compared by experiments in Section 4.

**Test case generation strategy** The sensitivity of variables is also used for test case generation. That is, **raSAT** generates two test cases for the specified number of variables, and one for the rest. Such variables are selected from those with larger sensitivity. When two test cases are generated, **raSAT** also observes the sign of the coefficients of noise symbols. If positive, it takes the upper bound of possible values as the first test case; otherwise, the lower bound. The second test case is generated randomly.

*Example 3.* Let $g = -x_{15} * x_8 + x_{15} * x_2 - x_{10} * x_{16}0$ and consider a constraint $g > 0$. For $x_2 \in [9.9, 10], x_8 \in [0, 0.1], x_{10} \in [0, 0.1], x_{15} \in [0, 10]$, and $x_{16} \in [0, 10]$, $0.25\epsilon_2 - 0.25\epsilon_8 - 0.25\epsilon_{10} + 49.5\epsilon_{15} - 0.25\epsilon_{16} + 0.75\epsilon_{+-} + 49.25$ is the estimation of $g$ by $AF_2$. The coefficient of $\epsilon_2$ is 0.25, which is positive. Thus, the value of $g$ is likely to increase if $x_2$ increases. We take the upper bound of possible values of $x_2$ as a test case, i.e. 10. Similarly, we take the test cases for others, $x_8 = 0, x_{10} = 0, x_{15} = 10, x_{16} = 0$, which lead $g = 100 > 0$.

## 4 Experiments

We implement **raSAT** loop as an SMT solver **raSAT**, based on MiniSat 2.2 as a backbone SAT solver and the library in [1] for outward rounding in the IAs. They are also compared with **Z3 4.3**, **iSAT3** and **dReal-2.15.01** where the former is considered to be the state of the art ([13]), and the remaining ones are a popular ICP based tools. Note that our comparison in this section is only on polynomial inequality. Preliminary results on equality will be reported in Section 5. The experiments are with Intel Xeon E5-2680v2 2.80GHz and 4 GB of RAM.

### 4.1 Experiments on Strategy Combinations

Table 1 shows the comparison between random choices and choices with heuristics on Zankl and Meti-Tarski families of QF_NRA. The timeout is set to 500s, and time shows the total of successful cases (either SAT or UNSAT).

### 4.2 Results on SMT-COMP 2015

The **raSAT** loop is implemented in an SMT solver **raSAT** with the following extension to handle polynomial constraints over the integers:

| Benchmark | Random | | Strategy | |
|---|---|---|---|---|
| Matrix-1 (SAT ) | 19 | 230.39 (s) | **25** | 414.99(s) |
| Matrix-1 (UNSAT) | 2 | 0.01(s) | 2 | 0.01(s) |
| Matrix-2,3,4,5 (SAT) | 1 | 13.43 (s) | **11** | 1264.77(s) |
| Matrix-2,3,4,5 (UNSAT) | 8 | 0.37(s) | 8 | 0.38(s) |
| Meti-Tarski (SAT) | 3451 | 895.14 (s) | **3473** | 419.25 (s) |
| Meti-Tarski (UNSAT) | 1060 | 233.46 (s) | 1052 | 821.85 (s) |

Table 1: Effectiveness of Heuristics

– setting $\gamma_0 = 1$ in the incremental deepening, and
– restricting test data generation on integer numbers.

**raSAT** participated the SMT Competition 2015 on two logic divisions, namely QF_NRA and QF_NIA. The result is available at `http://smtcomp.sourceforge.net/2015/results-toc.shtml`. Briefly summarized,

– on QF_NRA, **raSAT** solved 7952 benchmarks over 10184, ending with the third place; and
– on QF_NIA, **raSAT** solved 7917 benchmarks over 8475, ending with the second place.

## 5 Extension for Equality Handling

For a polynomial constraint with equality:

$$\varphi = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n) \wedge \bigwedge_{j=1}^{m'} g_j(x_1, \cdots, x_n) = 0$$

one typical way to solve equations is an algebraic method, e.g., Groebner basis. In this section, we try a simple method based on *Intermediate Value Theorem*. It is illustrated by a single equation case. Note that before solving equations, we assume a box that makes $\bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n)$ IA-valid.

### 5.1 Single Equation

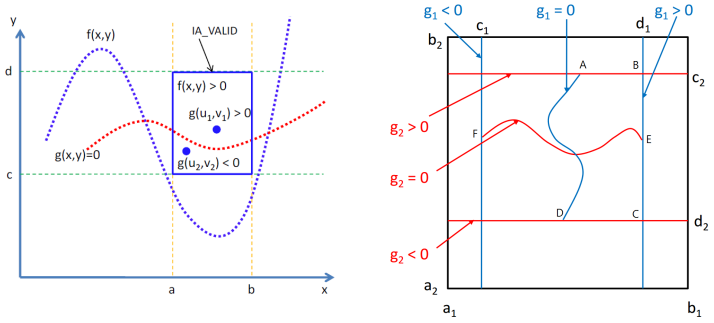
For solving polynomial constraints with a single equation ($g = 0$), we can apply in a simple way. That is, finding 2 test cases with $g > 0$ and $g < 0$ implies $g = 0$ somewhere in between.

**Lemma 1.** *For* $\varphi = \exists x_1 \in I_1 \cdots x_n \in I_n(\bigwedge_{j=1}^{m} g_j > 0 \wedge g = 0)$. *Suppose decomposition creates a box* $B = (l_1, h_1) \times \cdots \times (l_n, h_n)$ *such that* $(l_i, h_i) \subseteq I_i$ *and* $\bigwedge_{j=1}^{m} g_j > 0$ *is IA-VALID in the box. For* $(l_g, h_g) = range(g, B)$,

*(i) If $l_g > 0$ or $h_g < 0$, then $g = 0$ is UNSAT in the box.*
*(ii) If there are $\boldsymbol{t}, \boldsymbol{t'}$ in the box with $g(\boldsymbol{t}) > 0$ and $g(\boldsymbol{t'}) < 0$, then $g = 0$ is SAT.*

If neither (i) nor (ii) holds, **raSAT** continues the decomposition.

*Example 4.* Let $\varphi = f(x,y) > 0 \wedge g(x,y) = 0$. Suppose we find a box $B = (a,b) \times (c,d)$ such that $f(x,y) > 0$ is VALID in $B$. (Fig. 3a). In addition, inside the box, if we find two points $(u_1, v_1)$ and $(u_2, v_2)$ such that $g(u_1, v_1) > 0$ and $g(u_2, v_2) < 0$, then the constraint is satisfiable by Lemma 1.




(a) Single equation

(b) Multiple equations

Fig. 3: Example on solving equations using the Intermediate Value Theorem

**raSAT** first tries to find a box (by the decomposition) such that $\bigwedge_{j=1}^{m} g_j > 0$ is IA-VALID in the box. Then it tries to find 2 instances with $g > 0$ and $g < 0$ by testing. Intermediate Value Theorem guarantees the existence of an SAT instance in between. Note that this method does not find a SAT instance.

### 5.2 Multiple Equations

The idea of Intermediate Value Theorem is extended for solving multiple equations. Consider $m$ equations ($m \geq 1$): $\bigwedge_{j=1}^{m} g_j = 0$ and an box $B = (l_1, h_1) \times \cdots (l_n, h_n)$. For $V = \bigcup_{j=1}^{m} var(g_j)$ and $V' = \{x_{j_1}, \cdots x_{j_k}\} \subseteq V$, we denote $\{(r_1, \cdots, r_n) \in B \mid r_i = l_i \text{ for } i = j_1, ..., j_k\}$ and $\{(r_1, \cdots, r_n) \in B \mid r_i = h_i \text{ for } i = j_1, ..., j_k\}$ by $B \downarrow_{V'}$ and $B \uparrow_{V'}$, respectively.

**Definition 2.** *A sequence $(V_1, \cdots, V_m)$ of subsets of $V$ is a* check basis *of $(g_1, \cdots, g_m)$ on a box $B$, if, for each $j, j'$ with $1 \leq j, j' \leq m$,*

1. $V_j (\neq \emptyset) \subseteq var(g_j)$,

2. $V_j \cap V_{j'} = \emptyset$ *if* $j \neq j'$, *and*
3. *either* $g_j > 0$ *on* $B\uparrow_{V_j}$ *and* $g_j < 0$ *on* $B\downarrow_{V_j}$, *or* $g_j < 0$ *on* $B\uparrow_{V_j}$ *and* $g_j > 0$ *on* $B\downarrow_{V_j}$.

**Lemma 2.** *For a polynomial constraint containing multiple equations*

$$\varphi = \exists x_1 \in I_1 \cdots x_n \in I_n. \bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n) \wedge \bigwedge_{j=1}^{m'} g_j(x_1, \cdots, x_n) = 0$$

*and a box* $B \subseteq I_1 \times \cdots \times I_n$, *assume that*

1. $\bigwedge_{j=1}^{m} \psi_j(x_1, \cdots, x_n)$ *is IA-valid on* $B$, *and*
2. *there is a* check basis $(V_1, \cdots, V_m)$ *of* $(g_1, \cdots, g_m)$ *on* $B$.

*Then,* $\bigwedge_{j=1}^{m} g_j = 0$ *has a SAT instance in* $B$ *(and thus* $\varphi$ *is SAT).*

The idea is, from the Intermediate Value Theorem, each $g_j$ has a $n - |V_j|$ dimensional surface of null points of $g_j$ between $B\uparrow_{V_j}$ and $B\downarrow_{V_j}$. Since $V_j$'s are mutually disjoint (and $g_j$' are continuous), we have the intersection of all such surfaces of null points with the dimension $n - \sum_{j=1}^{m} |V_j|$. Thus, this method has a limitation that the number of variables must be greater than or equal to the number of equations.

*Example 5.* Consider two equations $g_1(x, y) = 0$ and $g_2(x, y) = 0$, and assume that $(\{x\}, \{y\})$ is a check basis of $(g_1, g_2)$ on $(c_1, d_1) \times (c_2, d_2)$ (Fig. 3b). Then, the blue line (null points of $g_1$) and the red line (null points of $g_2$) must have an intersection. We can explain this by Jordan curve theorem. Since $ABCD$ is a closed curve such that $E$ is inner and $F$ is outer, a continuous (red) line $EF$ must have an intersection by Jordan curve theorem.

| Benchmark | **raSAT** | | | | **Z3 4.3** | | | | **iSAT3** | | | | **dReal** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | SAT | | UNSAT | | SAT | | UNSAT | | SAT | | UNSAT | | $\delta$−SAT | | UNSAT | |
| Zankl (15) | **11** | 0.07 (s) | **4** | 0.17 (s) | **11** | 0.17 (s) | **4** | 0.02 (s) | 0 | 0.00 (s) | **4** | 0.05 (s) | 11 | 0.06 (s) | **4** | 0.02(s) |
| Meti-Tarski (3528/1573) | 875 | 174.90 (s) | 781 | 401.15 (s) | **1497** | 21.00 (s) | **1115** | 74.19 (s) | 1 | 0.28 (s) | 1075 | 22.6 (s) | 1497 | 72.85 (s) | 943 | 21.40 (s) |
| Keymaera (612) | 0 | 0.00 (s) | 312 | 66.63 (s) | 0 | 0.00 (s) | **610** | 2.92 (s) | 0 | 0.00 (s) | 226 | 1.63 (s) | 13 | 4.03 (s) | 318 | 1.96 (s) |

Table 2: Comparison among SMT solvers with equations

## 6 Related Work

There are many techniques appearing in various SMT solvers.

**QE-CAD** RAHD [19] and Z3 4.3 (nlsat in [13]) include QE-CAD.

**Virtual substitution (VS)** SMT-RAT toolbox [6] combines VS, incremental DPLL, and eager theory propagation. Z3 3.1 combines VS, ICP, and linearization.

**Bit-blasting** UCLID [3] and MiniSmt [22] give a bound on the number of bits to encode integers and rationals, respectively.

**Linearization** CORD [8] linearizes multiplications of reals by CORDIC encoding. Linearization suffers from the increase of the polynomial degrees.

ICP-based SMT solvers are iSAT3 and dReal, adding to **raSAT**.

**iSAT3** has tighter integration between DPLL procedure [18] and ICP. Fresh variables are introduced to decompose a polynomial to atomic representations, and each of them is assigned to an atomic proposition. A data structure is prepared to store intervals such that they correspond to the decision level one in DPLL. Its unit propagation is strengthened by combining with eager theory propagation. In a clause, if all except one literals are falsified, the remaining literal causes unit propagation and it becomes a candidate when next decomposition occurs. Note that **iSAT3** uses only CI as an IA.

**dReal** has different judgments on SAT, called $\delta$-SAT, which allows the deviation of the width $\delta$. Thus, $\delta$-SAT does not imply really SAT. This is the reason why dReal quite often concludes SAT (actually $\delta$-SAT) for UNSAT problems in SMT-LIB benchmarks. With the weakening of SAT to $\delta$-SAT, it obtains the completeness of $\delta$-SAT and $\delta$-UNSAT [9]. Note that **dReal** uses only CI as an IA, and lazy theory propagation as **raSAT**.

## 7 Conclusion

This paper presented **raSAT** loop, which extends ICP with testing to accelerate SAT detection and implemented as an SMT solver **raSAT**. With experiments on benchmarks from QF_NRA of SMT-LIB, we found two heuristic measures, *SAT-likelihood* and *sensitivity*, which lead an effective strategy for SAT detection. Currently, **raSAT** is in proto-type status, and lots of future work remain.

**UNSAT core**. Currently, **raSAT** focuses on SAT detection. For UNSAT detection, the target is to find a small UNSAT core in a large problem.

**Equality handling**. Section 5 shows equality handling, but current implementation has no effective strategies on finding a box on which all inequations are IA-valid and a decomposition $V_j$'s. We hope to find one based on the sensitivity. We also would like to additionally apply Groebner basis, to overcome the limitation that the number of variables is greater-than-equal to that of equations.

**Further strategy refinement**. Currently, raSAT uses information only from IA. We hope to refine strategies such that previous IA and testing results mutually guide. For instance, a box decomposition strategy can be a target.

## References


[1] Alliot, J.M., Gotteland, J.B., Vanaret, C., Durand, N., Gianazza, D.: Implementing an interval computation library for OCaml on x86/amd64 architectures. ICFP, ACM (2012)

[2] Benhamou, F., Granvilliers, L.: Continuous and Interval Constraints. Handbook of Constraint Programming (2006), pp.571–604

[3] Bryant, R.E., Kroening, D., Ouaknine, J., Seshia, S.A., Strichman, O., Brady, B.: Deciding bit-vector arithmetic with abstraction. TACAS 2007. LNCS 4424, pp. 358–372 (2007)

[4] Collins, G.: Quantifier elimination by cylindrical algebraic decomposition - twenty years of progress. Quantifier Elimination and Cylindrical Algebraic Decomposition, pp. 8–23 (1998)

[5] Comba, J.L.D., Stolfi, J.: Affine arithmetic and its applications to computer graphics (1993)

[6] Corzilius, F., Loup, U., Junges, S., brahm, E.: Smt-rat: An SMT-compliant nonlinear real arithmetic toolbox. SAT 2012, vol. 7317, pp. 442–448

[7] Frnzle, M., Herde, C., Teige, T., Ratschan, S., Schubert, T.: Efficient solving of large non-linear arithmetic constraint systems with complex boolean structure. JSAT 1, 209–236 (2007)

[8] Ganai, M., Ivancic, F.: Efficient decision procedure for non-linear arithmetic constraints using cordic. FMCAD 2009. pp. 61–68 (2009).

[9] Gao, S., Avigad, J., Clarke, E.M.: Delta-complete decision procedures for satisfiability over the reals. IJCAR'12, pp. 286–300 (2012)

[10] Gao, S., Kong, S., Clarke, E.: dReal: An SMT solver for nonlinear theories over the reals. CADE-24, LNCS 7898, pp. 208–214 (2013)

[11] Hickey, T., Ju, Q., Van Emden, M.H.: Interval Arithmetic: From principles to implementation. J. ACM 48(5), 1038–1068 (Sep 2001)

[12] Hong, H., Din, M.S.E.: Variant quantifier elimination. Journal of Symbolic Computation 47(7), 883 – 901 (2012)

[13] Jovanovi, D., de Moura, L.: Solving non-linear arithmetic. Automated Reasoning, LNCS 7364, pp. 339–354 (2012)

[14] Khanh, T.V., Ogawa, M.: SMT for polynomial constraints on real numbers. TAPAS 2012, ENTCS 289(0), pp.27 – 40 (2012)

[15] Messine, F.: Extensions of affine arithmetic: Application to unconstrained global optimization, Journal of Universal Computer Science 8(11), pp.992–1015 (2002)

[16] Moore, R.: Interval analysis. Prentice-Hall (1966)

[17] Ngoc, D.T.B., Ogawa, M.: Overflow and roundoff error analysis via model checking. SEFM 2009. pp. 105–114 (2009)

[18] Nieuwenhuis, R., Oliveras, A., Tinelli, C.: Abstract dpll and abstract dpll modulo theories. LPAR04, LNAI 3452. pp. 36–50 (2005)

[19] Passmore, G.O., Jackson, P.B.: Combined decision techniques for the existential theory of the reals. CALCULEMUS. LNCS 5625, pp. 122–137 (2009)

[20] Ratschan, S.: Efficient solving of quantified inequality constraints over the real numbers. TOCL 7(4), 723–748 (2006)

[21] Tarski, A.: A decision method for elementary algebra and geometry. Quantifier Elimination and Cylindrical Algebraic Decomposition, pp. 24–84 (1998)

[22] Zankl, H., Middeldorp, A.: Satisfiability of non-linear (ir)rational arithmetic. LPAR, LNCS 6355, pp. 481–500 (2010)

**Algorithm 2 raSAT** loop starting from the initial box $\Pi = \bigwedge_{i=1}^{n} x_i \in I_i^0$

1: **while** $\Pi$ is satisfiable **do** ▷ Some more boxes exist
2: $\pi = \{x_i \in I_{ik} \mid i \in \{1, \cdots, n\}, k \in \{1, \cdots, i_k\}\} \leftarrow$ a solution of $\Pi$
3: $B \leftarrow$ the box represented by $\bigwedge_{i=1}^{n} \bigwedge_{k=1}^{i_k} x_i \in I_{ik}$
4: **if** $B$ does not satisfy $\psi$ by using $IA$ **then**
5: $\Pi \leftarrow \Pi \wedge \neg(\bigwedge_{i=1}^{n} \bigwedge_{k=1}^{i_k} x_i \in I_{ik})$
6: **else if** $B$ satisfies $\psi$ by using $IA$ **then**
7: **return** SAT
8: **else if** $B$ satisfies $\psi$ by using *testing* **then** ▷ Different from ICP
9: **return** SAT
10: **else** ▷ Neither $IA$ nor *testing* conclude the constraint $\Longrightarrow$ *Refinement* Step
11: choose $(x_i \in I_{ik}) \in \pi$ such that $\forall k_1 \in \{1, \cdots, i_k\} I_{ik} \subseteq I_{ik_1}$
12: $\{I_1, I_2\} \leftarrow split(I_{ik})$ ▷ split $I_{ik}$ into two smaller intervals $I_1$ and $I_2$
13: $\Pi \leftarrow \Pi \wedge (x_i \in I_{ik} \leftrightarrow (x_i \in I_1 \vee x_i \in I_2)) \wedge \neg(x_i \in I_1 \wedge x_i \in I_2)$
14: **end if**
15: **end while**
16: **return** UNSAT