Towards universal gate synthesis and error correction in transmon qudits

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Gate-based quantum computers typically encode and process information in two-dimensional units called qubits. Using d-dimensional qudits instead may offer intrinsic advantages, including more efficient circuit synthesis, problem-tailored encodings and embedded error correction. In this work, we design a superconducting qudit-based quantum processor wherein the logical space of transmon qubits is extended to higher-excited levels. We propose a universal gate set featuring a two-qudit cross-resonance entangling gate, for which we predict fidelities beyond 99% in the d=4 case of ququarts with realistic experimental parameters. Furthermore, we present a decomposition routine that compiles general qudit unitaries into these elementary gates. As a proof-of-concept application, we numerically demonstrate that an embedded error correction sequence that encodes a qubit memory in a transmon ququart can successfully protect against pure dephasing noise. We conclude that universal qudit control – a valuable extension to the operational toolbox of superconducting quantum information processing – is within reach of current transmon-based architectures.

I. INTRODUCTION

In analogy to their classical counterparts, quantum computers encode information in binary systems – known as qubits – that consist of two physically distinct states. Many experimental implementations, including trapped ions, superconducting qubits, neutral atoms, and spin qubits, embed the logical qubit subspace in a much larger multi-level Hilbert space [1–4]. This full Hilbert space allows for a richer set of controls that is forgone by confining the computational space to qubits. By coherently controlling additional states we can enrich the set of available operations and make use of d-dimensional qudits as the local units of information.

Qudits have several conceptual advantages over their qubit counterparts. The number of qudits needed to reach the same Hilbert space dimension as a system of qubits is reduced by a factor of $log_2(d)$. For instance, ququarts, i.e. d = 4, cut the number of computational units in half. Moreover, qudits can synthesize arbitrary unitaries more efficiently than gubits with regards to the number of required entangling gates [5], an advantage that already emerges in the qutrit case of d=3 [6]. This has led to proposals for efficient implementations of quantum algorithms in the qudit space [7, 8]. Applications that are formulated in a product space of multivalued units particularly benefit from a qudit encoding. These include the quantum simulation of bosonic modes that arise, e.g., in light-matter interaction processes [9– 11], lattice gauge theories [12–14] and chemical vibrations and reactions [15, 16], but also classical problems like

multivalued integer optimization [17]. Moreover, qudit levels simplify the implementation of qubit gates [18, 19] and POVM measurements [20, 21]. Finally, qudits exhibit more complex entanglement than qubits [22], which can be leveraged to improve protocols such as superdense coding [23] and quantum error correction (QEC) codes [24–34]. As opposed to block-encoding QEC techniques, which use many physical qubits to encode a single logical one, an error-protected logical qubit can be encoded into the multi-level structure of a single qudit system, as proposed for molecular spins [29–31]. This simplifies the implementation of QEC by strongly reducing the number of controlled multi-qubit operations.

Qudit-based quantum information processing has recently been explored in trapped ions [35], photonic systems [36], Rydberg atoms [37], ultracold atomic mixtures [38], and molecular spins [29–32, 39–41]. Here, we conceptualize a superconducting qudit quantum processor, where d qudit levels are encoded into the d energetically lowest states of a transmon. We propose a concrete scheme to transpile arbitrary unitary circuits into a universal set of hardware-native single- and two-qudit gates, which in principle generalizes to any qudit dimension d.

Operating transmons as qutrits has already found many applications including multi-qutrit entanglement studies [42, 43], realization of multi-qubit gates [19, 44], excited state promotion readout [45, 46], quantum metrology [47], fast resets [48], and the realization of two-qutrit quantum algorithms [49]. The ququart case has been considered in the context of single-qudit applications [20, 50]. However, up to now, it was unclear how to drive general two-qudit unitaries. For qubits, a popular realization of the CNOT gate relies on driving cross-resonance pulses [51]. Here, we propose a generalization of the echoed cross-resonance (ECR) gate in the qudit space as the fundamental entangling gate between

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two qudits. We study this generalized ECR gate by numerically simulating the time-dynamics of the system, demonstrating that it can reach ququart gate fidelities of $\sim 99\%$ with simple pulse shapes.

This article is organized as follows. In Sec. II, we propose a universal set of qudit operations and numerically benchmark their fidelities with simulations that include leakage, crosstalk and charge-noise errors. Sec. III outlines how to decompose general two-qudit gates into the qudit ECR gate and single-qudit gates, enabling the synthesis of arbitrary unitary circuits. Finally, in Sec. IV, we show that the ECR gate forms the basis of a qudit-based QEC protocol and we demonstrate its basic implementation in the ququart case. We conclude with a discussion on current and future developments in Sec. V.

II. QUDIT CONTROL IN TRANSMONS

Superconducting circuits are a promising architecture to realize large-scale quantum information processors [52]. They have fast gates and measurement repetition rates [53] with comparatively long coherence times [54]. The transmon is a particularly popular type of superconducting circuit in which a Josephson junction of energy $E_{\rm J}$ shunted by a large capacitance with charging energy $E_{\rm C}$ creates an anharmonic oscillator. The eigenenergies E_n of this oscillator are characterized by the base excitation frequency $\omega = (E_1 - E_0)$ between the ground and first excited state, and the anharmonicity $\alpha = (E_2 - E_1) - \omega$ (setting $\hbar = 1$). Transmons are tuned towards large ratios of $E_{\rm J}/E_{\rm C}\gg 1$. This exponentially suppresses charge noise that causes fluctuations in the eigenenergies of the system, while maintaining sufficient anharmonicities for the selective driving of individual transitions [55]. Whereas the two lowest eigenstates of a transmon are typically employed as a qubit, in this work we encode a d-dimensional qudit in the d lowest energy states. We focus specifically on the ququart case of d=4 which represents the minimal unit for embedded QEC.

A. Single qudit control

We briefly review how single-qudit unitaries are realized in transmons, following Ref. [20]. Individual transmons are driven by microwave pulses whose carrier frequency can be adjusted to resonantly drive different transitions. With base frequencies of $\omega\sim 6\,\mathrm{GHz}$ and anharmonicities of $\alpha\sim -300\,\mathrm{MHz}$, the transition frequencies of neighboring levels among the first four excited states all lie within 1 GHz from the base frequency [see Fig. 1]. Such frequency shifts are routinely applied in existing microwave control stacks. We thus assume the ability to drive rotations between neighboring levels, e.g.,

$$R_x^{n(n+1)}(\varphi) = \mathbb{1}^{1,n-1} \oplus \exp(-i\frac{\varphi}{2}\sigma_x) \oplus \mathbb{1}^{n+2,d}$$
 (1)

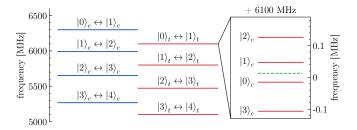


Figure 1. Transition frequencies of a two-transmon model. Blue and red lines denote the control and target, with bare frequencies of $\omega_c/(2\pi)=6.3\,\mathrm{GHz}$ and $\omega_t/(2\pi)=6.1\,\mathrm{GHz}$, respectively. The zoomed inset shows the dressing of the bare target frequency due to the capacitive coupling with the average transition frequency of $\overline{\omega}_t/(2\pi)$ in dashed green.

denotes an x-rotation between levels $|n\rangle$ and $|n+1\rangle$ and acts as the identity on all other levels. Here, σ_x is the Pauli x operator. Single-qudit phase gates can be applied "virtually" by adjusting the phases of subsequent drive pulses, which affects the polar angle of their rotation axes in the xy-plane. These virtual phase gates are near-perfect and come at no additional experimental cost [56]. By keeping track of the relative phase advances between all d levels, the correct frame changes can be implemented on the drives. In this setting, any single-qudit operation can be accomplished through at most d(d-1)/2 two-level rotations $R^{n,n+1}$ through a decomposition into Givens rotations.

B. Multi-qudit control

In addition to single qudit transformations, one entangling two-qudit gate is required to form a complete set of universal qudit operations [57]. We therefore investigate extensions of the popular cross-resonance gate to the qubit space.

1. The cross-resonance gate

The cross-resonance (CR) gate is applicable to transmons with a weak interaction mediated by a common resonator, since it is an all microwave-gate [51]. Here, one transmon, referred to as the control, is driven at the $|0\rangle_t \leftrightarrow |1\rangle_t$ transitions frequency ω_t of the second transmon, referred to as the target. This CR tone entangles the two systems through a complicated interaction dominated by a $Z_c \otimes X_t$ generator in the qubit space [58]. When tuning this rotation to $R_{ZX}(\pi/2) =$ $\exp(-i\frac{\pi}{4}\sigma_z\otimes\sigma_x)$, the CR gate is equivalent to a CNOT up to local Clifford gates. Analytical studies of CR tones based on perturbation theory show that the effective two-qubit interaction Hamiltonian contains various single-qubit terms $(I \otimes X, I \otimes Z, Z \otimes I)$, as well as a weak $Z \otimes Z$ term [59, 60]. A popular approach to largely cancel these unwanted terms employs the echoed pulse sequence shown in Fig. 2(a). In the echoed cross-resonance (ECR) gate the effects of the $Z\otimes I$, $Z\otimes Z$, and $I\otimes X$ destructively interfere, thus isolating the desired $Z\otimes X$ generator [51]. The echo sequence can further be improved with resonant rotary pulses on the target qubit [61]. Previous studies of the CR gate have focused on the qubit subspace [58–60]. In Ref. [19], an ECR sequence with the control prepared in $|2\rangle_c$ enables a pulse-efficient decomposition of the three-qubit Toffoli gate. Going beyond this, we now investigate the action of the CR gate in the full two-ququart subspace through a numerical simulation of the system's dynamics.

2. Numerical model

Typical transmon parameters of current IBM Quantum devices are $\omega/(2\pi) \sim 5$ GHz and $\alpha/(2\pi) \sim -300$ MHz. With a resulting $E_{\rm J}/E_{\rm C}$ -ratio of 40–50, the charge noise induced fluctuations of around 20 MHz in the $|2\rangle \leftrightarrow$ |3\ransition frequency are intolerable for full ququart operation [20]. We therefore choose a two-transmon model with frequencies of $\omega_c/(2\pi) = 6.3 \,\mathrm{GHz}, \,\omega_t/(2\pi) =$ 6.1 GHz, and anharmonicities of $\alpha_c/(2\pi) = -310 \,\mathrm{MHz}$, $\alpha_t/(2\pi) = -300 \,\mathrm{MHz}$. This increases the $E_\mathrm{J}/E_\mathrm{C}$ -ratio to ~ 70 , pushing the $|2\rangle \leftrightarrow |3\rangle$ frequency fluctuations down to $\sim 180\,\mathrm{kHz}$. Moreover, the chosen values of the detuning $\omega_c - \omega_t$ and α yield a large effective $Z \otimes X$ interaction strength [60]. Finally, the chosen parameters avoid crosstalk with a gap of at least 100 MHz between different transitions [see Fig. 1]. This is smaller than the anharmonicities of each gudit. Fortunately, any potential leakage is avoidable by shaping the control pulses [62, 63]. The ~ 3 MHz charge noise on the $|3\rangle \leftrightarrow |4\rangle$ transition renders high-fidelity control of this transition difficult. We thus focus on the ququart subspace.

The transmons are coupled by a weak exchange interaction of strength $J/(2\pi)=1.8\,\mathrm{MHz}$ that is routinely achieved in existing systems (see Appendix A 2 for a detailed definition of the model Hamiltonians). This always-on coupling J leads to a small shift of the eigenenergies of the joint two-transmon systems. From now on, when we denote a basis state as $|n\rangle_c\otimes|m\rangle_t$, we refer to these dressed basis states. In this basis, the $|0\rangle_t\leftrightarrow|1\rangle_t$ transition frequency varies by $\sim\pm100\,\mathrm{kHz}$ depending on the state of the control qudit [see inset of Fig. 1]. We therefore set the drive frequency of the CR tones at $\overline{\omega}_t/(2\pi)=\omega_t/(2\pi)+13\,\mathrm{kHz}$ obtained by averaging over the lowest four states of the control. This keeps the detuning to each transition as small as possible.

3. Simulation results

We now analyze the action of a single CR pulse by numerically integrating the full dynamics of the time-dependent Schrödinger equation of the two-transmon system in the d=4 subspace. We choose a Gaussian-square

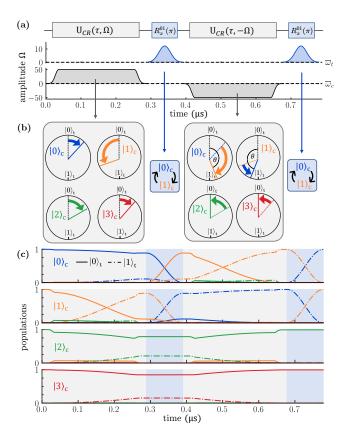


Figure 2. Qudit space action of the echoed cross-resonance gate. (a) The ECR pulse sequence applied to the control qudit consists of two cross-resonance tones (gray) played at the target qudit frequency $\overline{\omega}_t$ with opposite amplitudes, each followed by a π -pulse on the control. (b) Action of each pulse on the initial target state $|0\rangle_t$ depending on the control state $|\psi\rangle_c$ on the Bloch sphere spanned by $\{|0\rangle_t\,,|1\rangle_t\}.$ By construction of the echo sequence, a θ $(-\theta)$ rotation is applied in the $|0\rangle_c$ $(|1\rangle_c)$ case. (c) Evolution of the populations starting from $|0\rangle_t$ depending on the control state $|\psi\rangle_c$ for the pulse sequence shown in (a). Color denotes the state of the control while line style denotes the state of the target. Pulse durations are calibrated to $\theta=\pi.$

pulse envelope of amplitude $\Omega/(2\pi)=50\,\mathrm{MHz}$ and duration τ with a Gaussian rise and fall to suppress leakage out of the $|0\rangle_t-|1\rangle_t$ subspace, as shown in Fig 2(a) and detailed in Appendix A 3. Generalizing from the qubit case, we expect the resulting dynamics to create a $R_x^{01}(\varphi)$ rotation on the target, whose rotation angle and direction depend on the state of the control. We write this as

$$\begin{split} \operatorname{CR}(\vec{\varphi}) = & |0\rangle\!\langle 0|_c \otimes R_x^{01}(-\varphi_0) + |1\rangle\!\langle 1|_c \otimes R_x^{01}(\varphi_1) \\ + & |2\rangle\!\langle 2|_c \otimes R_x^{01}(-\varphi_2) + |3\rangle\!\langle 3|_c \otimes R_x^{01}(-\varphi_3) \end{split} \tag{2}$$

where each rotation angle φ_i is proportional to the total area under the pulse envelope. For the QEC application presented in Sec. IV, we require a rotation angle of $\pm \pi$ in the $|0\rangle_c$ case for the echoed sequence. We thus aim to calibrate the CR tones such that $\varphi_0 + \varphi_1 = \pi$. For the

chosen model parameters, this is achieved with a pulse duration of $\tau=289\,\mathrm{ns}$ shown in Fig. 2(a). Up to local phases on the control and target, the unitary resulting from our pulse simulation reaches an average gate fidelity of $\overline{\mathcal{F}}=99.93\%$ to Eq. (2) with angles $\vec{\varphi}=(\varphi_0,\ldots\varphi_3)\approx (0.22,0.78,0.30,0.26)\,\pi$. This result justifies the intuition behind the schematic illustration in Fig. 2(b). Note the difference in the rotation direction between the states $|0\rangle_c$, $|2\rangle_c$, $|3\rangle_c$ and $|1\rangle_c$.

 $|0\rangle_c\,, |2\rangle_c\,, |3\rangle_c$ and $|1\rangle_c.$ The echo π -pulse $R_x^{01}(\pi)$ in the ECR pulse sequence is simulated as a Gaussian at the (average) frequency of the control qudit $\overline{\omega}_c$. We fix the pulse duration at 100 ns and calibrate the amplitude to 12.3 MHz, obtaining a fidelity of $\overline{\mathcal{F}}=99.99\%$, for details see Appendix A 4. This is slower than current state-of-the-art X-gates in transmons [64, 65], to keep leakage minimal. For simplicity, we omit a careful calibration of DRAG pulses by which leakage errors and pulse duration could be further reduced [66]. Assuming that the reversed amplitude in the second CR tone of the ECR sequence reverses all rotation angles $\vec{\varphi}$ in Eq. (2), we expect that the unitary of the full ECR sequence is

$$\begin{aligned} \mathrm{ECR}(\theta) = & |0\rangle\langle 0| \otimes R_x^{01}(-\theta) + |1\rangle\langle 1| \otimes R_x^{01}(\theta) \\ & + |2\rangle\langle 2|_c \otimes \mathbb{1} + |3\rangle\langle 3|_c \otimes \mathbb{1} \end{aligned} \tag{3}$$

with $\theta = \varphi_0 + \varphi_1$. With the CR tones calibrated as described above, our simulation of the entire pulse sequence obtains an average gate fidelity of 99.6% for the targeted rotation angle $\text{ECR}(\theta = \pi)$ (up to local phase gates). This is the unitary error of the gate. To estimate the additional effect of incoherent error channels, we add amplitude damping with a T_1 -time of 500 μ s and pure dephasing with a T_2 time of 200 μ s to the simulation as detailed in Appendix A 6. This reduces the fidelity to 98.9%. In this work, we are primarily interested in understanding the limits to the unitary error of this gate. We thus leave exploring the trade-off between the unitary gate error – which is minimal for longer gate durations – and the incoherent gate error that increases with the gate duration for future work.

The evolution of the populations in the two-transmon system under the echoed CR sequence is shown in Fig. 2(c), with details given in Appendix A 5. The remaining unitary error of the gate originates mainly from the detuning of the CR tones to the target frequency in the cases where the control is in $|2\rangle_c$ and $|3\rangle_c$ [see Fig. 1]. This leads to a small Z-contribution in the effective rotation axis in each respective subspace, which is not fully reversed by the echoed CR sequence. These effects could potentially be resolved by adding rotary tones [61], including virtual phase gates into the sequence [56] or more advanced qudit-based optimal control techniques [67]. We find that leakage out of the $|0\rangle_t - |1\rangle_t$ subspace is not a relevant error source with populations of those levels remaining under 10^{-5} after the ECR sequence.

For qudit-based circuit decomposition, the ECR gate from Eq. (3) is particularly convenient, as it only depends on a single parameter θ which is tunable through

the duration of the CR pulses. In Sec. III, we present a decomposition routine that implements general qudit unitaries through the $ECR(\theta)$ gate and single-qudit gates.

C. Experimental requirements

Controlling additional states of a transmon beyond the qubit space comes with an increased complexity. We now comment on the requirements to operate the transmon as a ququart in the presence of charge noise, limited lifetimes, and imperfect ququart readout. Charge noise is exponentially larger in higher-excited states $|n\rangle$ [68]. However, charge noise is exponentially suppressed with increasing $E_{\rm J}/E_{\rm C}$ at the cost of a polynomial reduction of the anharmonicity α [55]. Our simulations suggest that reasonable anharmonicities remain with manageable charge noise for ququart operation when choosing $E_{\rm J}/E_{\rm C} \sim 70$. Scaling the system to more than two ququarts may require a careful design of the control pulses to mitigate leakage due to the weak anharmonicity and the frequency crowding introduced by the additional qudit levels.

The lifetime of higher-excited states $|d\rangle$ also generally decreases with d. Fortunately, their decay happens predominantly sequentially, e.g., following $|3\rangle \rightarrow |2\rangle \rightarrow |1\rangle \rightarrow |0\rangle$, which leads to workable coherence times for the lowest-lying qudit levels. For example, for devices with qubit lifetimes of $\sim 80\,\mu s$, which is below current state-of-the art of $\sim 500\,\mu s$, experiments have found lifetimes of $\sim 30\,\mu s$ for state $|3\rangle$ [20, 68]. In comparison our ECR pulse lasts about 1 μs .

Finally, transmons are measured with a dispersive readout by coupling to a resonator. This technique can distinguish between multiple qudit states, as demonstrated for ququarts [69]. In summary, our simulations along with previous experimental demonstrations of coherence times and readout confirm that high-fidelity ququart operation of transmons is possible.

III. UNIVERSAL QUDIT GATE SYNTHESIS

Quantum algorithms are described at the quantum circuit level with abstract gate instructions that typically do not match those of the hardware. A vast amount of work is dedicated to producing practical and efficient hardware-executable gate decompositions for qubits [19, 70–72]. However, little attention has been given to the qudit case aside for general algorithms. Here, we show how to transpile an arbitrary $d^n \times d^n$ unitary on a system of n d-dimensional transmon qudits into the hardware-native gate set of single-qudit rotations and the ECR gate. This constitutes the first practical blueprint for qudit unitary gate synthesis on superconducting qudits, as detailed in Appendix B.

Any set of arbitrary single-qudit gates combined with a single entangling two-qudit gate is in principle exactuniversal [57, 73]. Several constructive decomposition routines exists which rely on different choices of two-qudit gates [8, 74]. For qubits, the quantum Shannon decomposition (QSD) is a powerful tool to synthesize arbitrary unitaries [75]. We build on the multivalued QSD that generalizes QSD to the qudit setting [76]. Within this framework, the circuit complexity, quantified by the number of two-qudit gates required to achieve arbitrary N-qudit unitaries, is reduced by a factor of d-1 compared to the qubit setting of d=2, highlighting the comparative efficiency of qudit circuits [5]. The multivalued QSD iteratively reduces the desired unitary to block matrices that contain only one-qudit and two-qudit unitaries. The remaining two-qudit blocks consist of singly-controlled gates

$$C^{m}[U] = |m\rangle\langle m| \otimes U + \sum_{i \neq m} |i\rangle\langle i| \otimes \mathbb{1}$$
 (4)

that apply a unitary U on the target qudit if and only if the control qudit is in the basis state $|m\rangle$ [76].

We now present a decomposition routine to realize $C^m[U]$ through single-qudit gates and the $\mathrm{ECR}(\theta)$ gate. This decomposition holds for any d, as long as the ECR gate in Eq. (3) acts as $|n\rangle\langle n|\otimes 1$ for all states $n\geq 2$. We diagonalize $U=VDV^\dagger$ such that the diagonal matrix $D=e^{i\gamma}\mathrm{diag}(e^{-i(\sum_{j=0}^{d-1}\alpha_j)},e^{i\alpha_1},\dots,e^{i\alpha_{d-1}})$ can be decomposed into R_z^{0j} rotations as $D=e^{i\gamma}\prod_{j=1}^{d-1}R_z^{0j}(2\alpha_j)$. $C^m[U]$ can then be implemented as

$$C^{m}[U] = (S_{m} \otimes V) \prod_{j=1}^{d-1} C^{m}[R_{z}^{0j}(2\alpha_{j})] \left(\mathbb{1} \otimes V^{\dagger}\right)$$
 (5)

with a phase gate $S_m = \sum_{j=0}^{d-1} e^{i\gamma\delta_{jm}} |j\rangle \langle j|$ [see Fig. 3(a)]. Next, each $C^m[R_z^{0j}(2\alpha_j)]$ gate is expressed through two controlled x-rotations, yielding [see Fig. 3(b)]

$$C^{m}[R_{z}^{0j}(2\alpha_{j})] = C^{m}[R_{x}^{0j}(-\pi)] \left(\mathbb{1} \otimes R_{z}^{0j}(-\alpha_{j}) \right)$$
 (6)

$$\times C^{m}[R_{x}^{0j}(\pi)] \left(\mathbb{1} \otimes R_{z}^{0j}(\alpha_{j}) \right).$$

This shifts all angular dependence into local phase gates which can be implemented virtually. The general m-controlled R_x^{0j} gates from Eq. (6) can be realized through 0-controlled R_x^{01} rotations by applying single-qudit permutation gates on the control and the target, as shown in Fig. 3(c). Here, $X_{n(n+1)}$ denotes a swap of the levels i and i+1 which is equivalent to $R_x^{n(n+1)}(\pi)$ up to virtual local phases. Note that the permutation gates on the control qudit to shift the control from m to 0, highlighted in gray in Fig. 3(c), need to be applied only once at the beginning and the end of the entire sequence.

The final step is now to realize the remaining $C^0[R_x^{01}(\theta)]$ gates with the ECR gate defined in Eq. (3). Since the ECR gate acts non-trivially on the target for both control states $|0\rangle_c$ and $|1\rangle_c$, $C^0[R_x^{01}(\theta)]$ can not be implemented with a single ECR(θ) gate. Instead, we split the rotation into d-1 steps of ECR($-\theta/d$) and

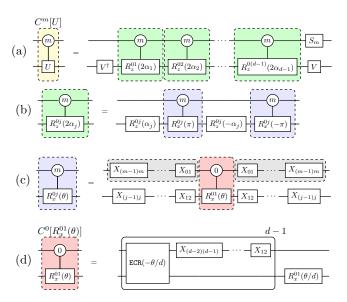


Figure 3. Gate decompositions to implement a general singly-controlled two-qudit gate $C^m[U]$. (a) Diagonalization of $C^m[U]$ leads to a sequence of controlled z-rotations R_z^{0j} between the 0^{th} and j^{th} level. (b) Each $C^m[R_z^{0j}]$ is implemented through two $C^m[R_x^{0j}]$ rotations and local phase gates. (c) Decomposition of $C^m[R_x^{0j}]$ into local permutations and a single $C^0[R_x^{01}]$ gate. (d) Realization of the $C^0[R_x^{01}(\theta)]$ gate through echoed cross-resonance gates.

permute the levels of the control between each step, as shown in Fig. 3(d). This way, the action on the target is $R_x^{01}(\theta(d-1)/d)$ when the control is in $|0\rangle_c$ and $R_x^{01}(-\theta/d)$ when the control is in any other state. Applying an $R_x^{01}(\theta/d)$ gate on the target finally recovers the desired $C^0[R_x^{01}(\theta)]$ rotation, since

$$\begin{split} C^0[R_x^{01}(\theta)] &= \left(\mathbbm{1} \otimes R_x^{01}\left(\frac{\theta}{d}\right)\right) & (7) \\ &\times \left(\left(\prod_{j=2}^{d-1} X_{(j-1)j} \otimes \mathbbm{1}\right) \operatorname{ECR}(-\frac{\theta}{d})\right)^{d-1}. \end{split}$$

Note that, while this construction adds an overhead in single-qudit gates, it makes efficient use of the entanglement generation rate of the CR effect as the total duration of the ECR pulses is proportional to $\theta(d-1)/d$.

In general, the presented decomposition routine requires $O(d^2)$ two-qudit entangling gates and $O(d^3)$ single-qudit gates. In the d=4 case, any $C^m[U]$ unitary can be synthesized with 18 ECR($\pm\pi/4$) gates and 56+2m single-qudit gates. The number of gates in our decomposition could potentially be improved by working with a direct (non-echoed) CR gate, which, however, increases the complexity as this depends on multiple parameters $\vec{\varphi}$. Note also that the step in Eq. (6) introduces controlled R_x gates with a maximal rotation angle of $\pm\pi$, irrespective of the original rotation angle α . Thus, the resulting pulse schedules could be shortened by pulse-efficient circuit transpilation techniques as developed in [71], resulting in better gate fidelities.

IV. APPLICATION: CORRECTION OF DEPHASING ERRORS

The multi-level transmon structure and the ECR gate can implement qudit-based QEC protocols [26, 28], which were recently discussed in the context of molecular spin qudits with embedded error correction [29–33]. We now first translate the same ideas to the here-proposed transmon setup and then present a detailed simulation of the full QEC routine. This provides a concrete use case for the universal qudit logic toolbox developed in the previous sections.

A. Embedding error correction into qudits

A 4-level qudit is the minimal unit needed to embed QEC against a single kind of noise source [30], such as amplitude damping or pure dephasing [27]. Indeed, for an error to be identified and corrected, its action on a superposition of logical states (code words) must bring them to distinguishable ones, thus requiring a Hilbert space of dimension ≥ 4 . Specifically, we consider a correction against pure dephasing errors, which for a bosonic system are well approximated by an error operator $\sqrt{\Gamma}n$, where $n=a^{\dagger}a$ and a^{\dagger} (a) is the creation (annihilation) operator of the bosonic mode [28]. We model dephasing as Markovian noise described by the Lindblad equation

$$\dot{\rho}(t) = -\frac{i}{\hbar} \left[H(t), \rho(t) \right] + \Gamma \left(2n\rho(t)n - \{n^2, \rho(t)\} \right) \tag{8}$$

where ρ is the system single-qudit density matrix subject to an external driving Hamiltonian H(t) and $\Gamma=1/T_2$ is the dephasing rate. Expanding the solution to this equation in series for small Γt yields a leading error term proportional to n in the Kraus representation [30]. An analogous result, with error operators represented by powers of n, can be derived for non-Markovian noise [28]. The same form of noise also describes pure dephasing in a slightly anharmonic transmon system as considered here, by simply making the replacement $n \to \sum_m m|m\rangle\langle m|$ in the truncated Hilbert space m=0,1,2,3 [77]. We require a pair of code words $|0_L\rangle$ and $|1_L\rangle$ protected against the set of errors $E_k \in \{I,n\}$, i.e., satisfying the Knill-Laflamme conditions [78]:

$$\langle 0_L | E_k E_j^{\dagger} | 0_L \rangle = \langle 1_L | E_k E_j^{\dagger} | 1_L \rangle$$

$$\langle 0_L | E_k E_j^{\dagger} | 1_L \rangle = 0.$$
(9)

A possible choice is

$$|0_L\rangle = \frac{|0\rangle + \sqrt{3}|2\rangle}{2}$$
 and $|1_L\rangle = \frac{\sqrt{3}|1\rangle + |3\rangle}{2}$. (10)

These code words fulfill Eq. (9) by construction and it can be easily checked that $\langle 0_L | n | 0_L \rangle = \langle 1_L | n | 1_L \rangle = 3/2$ and $\langle 0_L | n^2 | 0_L \rangle = \langle 1_L | n^2 | 1_L \rangle = 3$. Hence, the effect

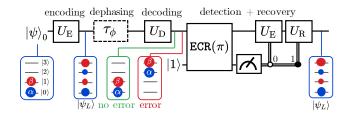


Figure 4. Sequence to correct pure dephasing errors in a transmon. The initial qubit state $|\psi\rangle_0$ is encoded into a ququart logical state $|\psi_L\rangle$. After dephasing, a decoding unitary U_D maps the ideal and erroneous cases to orthogonal states. The error can be detected by applying a $\text{ECR}(\pi)$ gate where the target is an ancilla qubit prepared in $|1\rangle$. In the ideal case where the ancilla is de-excited to 0, the state is re-encoded, while in the error case, where the ancilla is measured in 1, a recovery operation U_R restores $|\psi_L\rangle$.

of a n error is to bring a generic encoded logical state $|\psi_L\rangle = \alpha |0_L\rangle + \beta |1_L\rangle$ into the error state

$$\frac{n |\psi_L\rangle}{\sqrt{\langle \psi_L | n^2 |\psi_L\rangle}} = \frac{\sqrt{3}}{2} |\psi_L\rangle - \frac{1}{2} \underbrace{(\alpha |e_0\rangle + \beta |e_1\rangle)}_{=|\psi_e\rangle}. \quad (11)$$

This is a superposition of the code words and of the error words $|e_0\rangle = (\sqrt{3}\,|0\rangle - |2\rangle)/2$ and $|e_1\rangle = (|1\rangle - \sqrt{3}\,|3\rangle)/2$. Crucially, $|\psi_e\rangle$ preserves α and β . Note that the Lindblad dynamics of Eq. (8) yield the same time evolution as for a spin S subject to pure dephasing, see, e.g., Ref. [30]. This leads to an independent decay of each coherence $\rho_{mm'}$ with an exponential rate $(m-m')^2/T_2$. Indeed, the number operator appearing in Eq. (8) is equivalent to the spin operator S_z apart from an irrelevant shift, i.e. $S_z = n - S$.

To test the performance of this qudit code in protecting a transmon memory from dephasing, we consider a pair of transmons. The first one, used as a ququart, encodes the logical state defined by Eq. (10). The second one acts as an ancillary qubit to detect errors. The protocol is summarized in Fig. 4 and consists of the following steps: An initial qubit state $|\psi\rangle_0 = \alpha |0\rangle + \beta |1\rangle$ is encoded into the logical state $|\psi_L\rangle = \alpha |0_L\rangle + \beta |1_L\rangle$ by an encoding unitary realized with pulses on neighboring levels $U_{\rm E} = R_y^{12}(-\pi)R_y^{01}(-2\pi/3)R_y^{23}(\pi/3)R_y^{12}(\pi)$. The ququart then evolves freely for a memory time τ_ϕ subject to pure dephasing according to Eq. (8). Next, the decoding sequence $U_{\rm D} = U_{\rm E}^{\dagger}$ is applied to map the basis of the code and error words back into the basis of qudit eigenstates such that $|\psi_L\rangle$ is mapped back to $|\psi\rangle_0$ and $|\psi_e\rangle$ is mapped to $\alpha |2\rangle + \beta |3\rangle$.

To detect n errors, i.e., the qudit in $|2\rangle$ or $|3\rangle$, the qudit is coupled to a flag ancilla qubit prepared in $|1\rangle$. The ECR(π) gate with the data qudit as the control and the ancilla as the target de-excites the ancilla when the control is in $|0\rangle$ or $|1\rangle$. The ancilla is subsequently measured. This projects the qudit either to the 01-subspace or the 23-subspace while preserving the encoded superposition. If the ancilla is found in $|0\rangle$, i.e.,

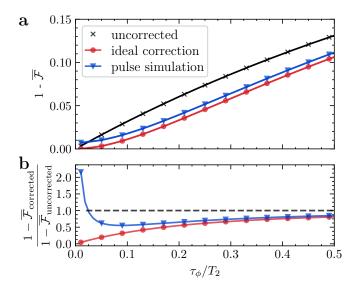


Figure 5. Numerical simulations of a single correction cycle applied after dephasing for a duration of τ_{ϕ} . (a) Qubit error $1-\overline{\mathcal{F}}$. (b) Error reduction compared to the uncorrected case.

no error has occurred, the encoding sequence $U_{\rm E}$ can be reapplied, whereas, if the ancilla is found in $|1\rangle$, a recovery sequence $U_{\rm R}$ is applied to restore $|\psi_L\rangle$ from $U_{\rm D}\,|\psi_e\rangle$. This is accomplished with two-level rotations as $U_{\rm R}=R_y^{12}(-\pi)R_y^{01}(\pi/3)R_y^{23}(-2\pi/3)R_y^{12}(\pi)$.

B. Implementation on transmons

We now benchmark the ququart error correction sequence in a transmon qudit through numerical simulations. We consider three cases: (1) The uncorrected case where the qubit state $|\psi\rangle_0$ dephases for a duration of τ_{ϕ} according to Eq. (8) with no external drives and no encoding, (2) the ideal correction sequence, where all gates in Fig. 4 are applied as instantaneous and perfect unitaries, and (3) the correction sequence with simulations of the pulses that implement each gate by integrating Eq. (8) with the appropriate drive Hamiltonians. We choose a dephasing time of 200 µs, which is comparable to state-of-the-art T_2 values for IBM Quantum devices [64]. In cases (2) and (3), the encoded qudit dephases for a time τ_{ϕ} before the correction cycle, after which a final decoding operation $U_{\rm D}$ and tracing out of the ancilla obtains the resulting qubit state. For case (3), we use the two-transmon model and calibration of the $ECR(\pi)$ as presented in Sec. IIB where the control gudit becomes the data qudit and the target qubit becomes the ancilla. For details on pulse shapes and numerics see Appendix A.

For all cases, we compute the channel \mathcal{E} that maps $|\psi\rangle_0$ to the final qubit state of the sequence by constructing its 4x4 Liouvillian superoperator matrix. The performance of each channel in preserving an input state is benchmarked by the average gate fidelity to the identity $\overline{\mathcal{F}}(\mathcal{E}, 1)$. In the uncorrected case, $\overline{\mathcal{F}}$ decreases expo-

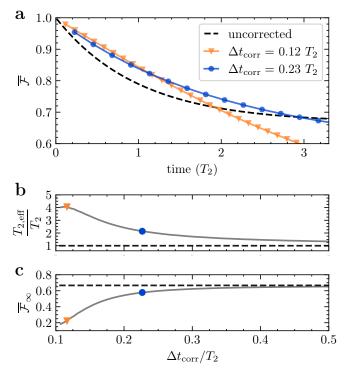


Figure 6. Numerical simulations of the error correction pulse sequence with repeated cycles where $\Delta t_{\rm corr}$ is the time between each cycle. (a) Exponential decay of the fidelity $\overline{\mathcal{F}}$ for the uncorrected case and two different correction cycle times. (b) Time constant of the exponential decay $T_{2,\rm eff}$ as a function of the cycle time $\Delta t_{\rm corr}$. (c) Asymptotic fidelity for $t \to \infty$.

nentially with a time constant given by T_2 . The ideal correction sequence always improves on this for all dephasing times τ_{ϕ} , see Fig. 5(a). The reduction in the error $1-\overline{\mathcal{F}}$ is most pronounced after only short dephasing times $\tau_{\phi} < 0.1T_2$ and becomes less and less significant for larger τ_{ϕ} , see Fig. 5(b). This is because higher-order powers of n become important for longer times, while the code only protects for first-order n-errors.

The pulse-level simulation introduces imperfections into the correction cycle that arise from finite unitary gate errors, leakage, non-zero durations of the gate and measurement pulses, during which the data qubit is unprotected and subject to dephasing. Therefore, after very short dephasing times, the correction sequence increases the qubit error compared to the uncorrected case [see Fig. 5(b)]. However, for $\tau_{\phi} > 0.025\,T_2$, the code breaks even. With our choice of parameters, the best achievable error reduction of 45% is found after a dephasing time $\tau_{\phi} = 0.09\,T_2$.

The error reduction can be extended to longer dephasing times by repeatedly applying the correction cycle, see Fig. 6(a). We denote the time between individual correction cycles as $\Delta t_{\rm corr}$. Ideally, the fidelity is kept arbitrarily high by making $\Delta t_{\rm corr}$ correspondingly small. In reality, each correction cycle introduces a small error. For our pulse simulations, we empirically find that the

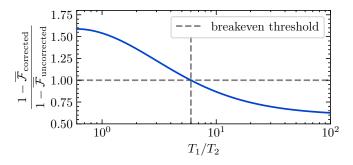


Figure 7. Effect of amplitude damping on the error reduction of one correction cycle. $\tau_{\phi}=0.23\,T_2$ and $T_2=200\,\mu s$ are fixed while the strength of T_1 is varied. The code breaks even for $T_1>6\,T_2$.

fidelity after each correction cycle is well-described by an exponential decay of the form

$$\overline{\mathcal{F}}(t) = (1 - \overline{\mathcal{F}}_{\infty}) \exp(-t/T_{2,\text{eff}}) + \overline{\mathcal{F}}_{\infty}.$$
 (12)

Here, $T_{2,\text{eff}}$ is an effective T_2 time and $\overline{\mathcal{F}}_{\infty}$ is the fidelity that is reached asymptotically for long times. We extract $T_{2,\text{eff}}$ and $\overline{\mathcal{F}}_{\infty}$ from fits to the data. Both quantities are plotted as a function of Δt_{corr} in Fig. 6(b) and Fig. 6(c), respectively. We find that for a short cycle time, the effective T_2 time reaches up to four times the uncorrected T_2 time. However, this comes at the expense of a much worse asymptotic fidelity at longer times. The optimal correction frequency thus comes with a trade-off of short-term gain vs. long term infidelity. Nonetheless, a significant reduction in the error can be upheld for a duration of $\approx T_2$ without drastically worsening the asymptotic fidelity as shown by the blue curve in Fig. 6.

In a more realistic setting, additional error terms such as amplitude damping are present, which is not handled by this code. We investigate how amplitude damping affects the protocol by adding jump operators of the form $\sqrt{n/T_1}|n-1\rangle\langle n|$ for $n\in\{1,2,3\}$ to the Lindbladian evolution of Eq. (8), see Appendix. A 6 for details. Here, T_1 is the time scale of the jumps which is chosen to decrease inverse proportionally with n. This model describes, in accordance with experiments [20, 68], a cascading decay from higher to lower-excited states. With additional errors present the correction sequence fails to reduce the error compared to the uncorrected case [see Fig. 7]. In our model, the code reaches a break even point for $T_1 > 6 T_2$, when pure dephasing is strongly dominant over amplitude damping. While these conditions are not necessarily realistic in typical transmon architectures, using more than four levels could enable a full-fledged error correction sequence. In that case, one might for example design code words correcting different errors simultaneously [27, 28] as pursued for molecular nanomagnets, where the system size can be increased without a significant impact on decoherence [29–31, 34]. Indeed, the degree of chemical control of these systems allows one to design multi-spin molecules which display decoherence not increasing with the number of encoding levels [34].

V. DISCUSSION

We have proposed a method to perform universal quantum computation in superconducting transmon qudits. Our universal basis gate set consists of general single-qudit unitaries and an entangling cross-resonance gate between two qudits. With this gate set, we have developed a decomposition routine to realize arbitrary m-controlled two-qudit gates $C^m[U]$ as building blocks to synthesize general qudit unitaries. This decomposition is the center piece of a general purpose qudit transpiler to map application-level qudit-based quantum circuits to hardware-native instructions. Compared to qubits, this qudit decomposition reduces the number of entangling gates required to synthesize general unitaries by a factor of d-1 [5].

The proposed entangling gate is suitable for dispersively coupled transmons. It represents a higher-dimensional extension of the echoed cross-resonance gate previously employed for qubits. Our numerical model that includes charge noise and leakage errors predicts average gate fidelities of up to 99.6% with simple Gaussian pulse profiles The main contribution to the unitary error is the frequency dependence of the dressed target states on the control states.

Our simulations are a first indication that high-fidelity gudit operations are within reach of transmon gudits. These findings would benefit from systematic complementary analytical studies of the qudit-space interactions. For example, we chose gudit frequencies and anharmonicities such that transition frequencies between neighboring states are well separated. Further research could therefore seek more optimal parameter regions to maximize the desired $Z \otimes X$ cross-resonance rate, akin to analytical studies of the qubit case [60]. Other open questions are whether a direct cross-resonance driving without echoes benefits qudit operation or whether rotary tones reduce the unitary error, as demonstrated for qubits [61]. Finally, we note that cross-Kerr interactions, tunable couplers, or tunable frequency transmons may offer different qudit gates [49, 69, 79].

Qudit operation of transmons is attractive from a theoretical standpoint, as it makes full use of the available quantum resources of the system. Compared to current standard transmon setups, the proposed gates require no additional microwave drive lines. Moreover, the fact that higher-excited states suffer increasingly from charge noise can be mitigated by moving the transmon parameters towards higher $E_{\rm J}/E_{\rm C}$ -ratios than those typically employed for qubit operation. Along with recent experimental demonstrations of extended qudit lifetimes [20, 68] and high-fidelity qudit readout [69], our numerical simulations suggest that high-fidelity ququart operations are possible under realistic experimental conditions. The maximum number of usable levels d_{max} in a transmon qudit is limited by frequency crowding, charge noise and decoherence. Identifying d_{max} is left to future work.

As an example application, the additional levels of a

qudit space can encode logical qubit states for quantum error correction. By embedding logical states within a single object, this scheme significantly reduces the complexity of QEC and may be a viable path towards an error-protected quantum computer. To date, its implementation has been proposed for molecular spin systems [30–33, 40], leveraging the large number of addressable levels and their intrinsic coherence. Our work shows that transmons can also embed self-corrected logical units. As a proof-of-principle, we used four gudit levels and an ancilla flag qubit to reduce pure dephasing errors for an encoded qubit memory. A fully-fledged quantum error correction code requires more complicated code words that cover more than four qudit states. Although going beyond d = 4 represents a non-trivial challenge for transmons, one may nevertheless envision optimal hybrid encoding schemes, where - making use of a two-qudit universal gate set as proposed here – the code space is spread over multiple qudits.

In conclusion, our results provide a blueprint for the implementation of multivalued quantum logic in transmons. By enabling a richer set of operation modes and embedded functionalities, this approach could open up

new design possibilities for the next generation of quantum processors, fueling the ongoing transition towards universal and fault tolerant quantum computation.

VI. ACKNOWLEDGMENTS

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Appendix A: Details on the numerical model

Here, we summarize the technical details of the numerical simulations presented in Sec. II and Sec. IV. Our analysis of the two-transmon system is largely based on Ref. [60]. We start form the standard transmon Hamiltonian

$$\hat{H} = 4E_{\mathcal{C}} \left(\hat{n} - n_g \right)^2 - E_{\mathcal{J}} \cos(\hat{\phi}), \tag{A1}$$

where \hat{n} and $\hat{\phi}$ are dimensionless conjugate variables for the charge and phase and n_q represents the offset charge.

1. Treatment of charge noise

The eigenenergies E_n of \hat{H} are subject to fluctuations with n_g , with a maximal value given by the charge dispersion $\Delta E_n = E_n(n_g=0) - E_n(n_g=1/2)$. We compute ΔE_n by diagonalizing \hat{H} in the charge representation truncating to 20 Fourier modes in $\hat{\phi}$ [80]. We can then estimate the deviation $\Delta \omega^{n,n+1}$ of the transition frequencies $\omega^{n,n+1} = E_{n+1}(n_g) - E_n(n_g)$ from their mean as a uniform average over n_g . For the chosen transmon model parameters with $E_J/E_C \sim 70$ as presented in Sec. II B 2, we obtain $\Delta \omega^{01}/(2\pi) = 0.2\,\mathrm{kHz}, \,\Delta \omega^{12}/(2\pi) = 7.7\,\mathrm{kHz}, \,$ and $\Delta \omega^{23}/(2\pi) = 185\,\mathrm{kHz}$. To account for this uncertainty caused by charge dispersion, we apply $\Delta \omega^{n,n+1}$ as a detuning to each pulse played on the $n\leftrightarrow n+1$ transition.

2. Model Hamiltonians

The transmon Hamiltonian from Eq. (A1) is commonly simplified in the Kerr approximation by expanding the cosine term to fourth order. Taking also the next-highest order into account, the Hamiltonian in its eigenbasis truncated to the d=4 subspace becomes [60]

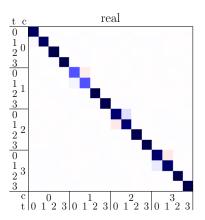
$$H' = \omega |1\rangle\langle 1| + (2\omega + \alpha) |2\rangle\langle 2|$$

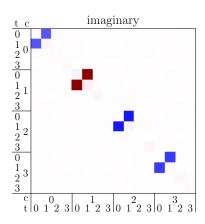
$$+3\left(\omega + \alpha - \frac{E_{\rm C}}{8E_{\rm J}}\sqrt{2E_{\rm C}E_{\rm J}}\right) |3\rangle\langle 3|$$
(A2)

with the qubit frequency ω and the anharmonicity α .

The capacitive coupling Hamiltonian between control (c) and target (t) is given by $H_J = Jy_c \otimes y_t$. It is expressed up to $O(\epsilon^3)$ in the unitless anharmonicity measure $\epsilon = \sqrt{2E_{\rm C}/E_{\rm J}} \sim 0.168$ through the unitless charge operator $y = -i(b-b^{\dagger})$ with

$$b = \begin{pmatrix} 0 & b_{01} & 0 & b_{03} \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & 0 & b_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A3}$$





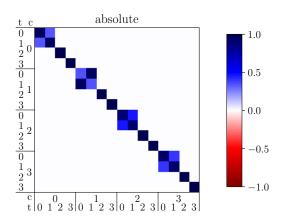


Figure 8. Numerical simulation of the unitary matrix $\tilde{U}_{\rm CR}$ of a single CR pulse in the two-transmon dressed basis. The resulting operation is well described by an R_x^{01} rotation on the target with a direction and angle that depend on the state of the control.

and

$$b_{01} = 1 - \frac{\epsilon}{8} - \frac{11\epsilon^2}{256} \tag{A4a}$$

$$b_{12} = \left(1 - \frac{\epsilon}{4} - \frac{73\epsilon^2}{512}\right)\sqrt{2}$$
 (A4b)

$$b_{23} = \left(1 - \frac{3\epsilon}{8} - \frac{79\epsilon^2}{256}\right)\sqrt{3}$$
 (A4c)

$$b_{03} = -\frac{\sqrt{6}\epsilon}{16} - \frac{5\sqrt{6}\epsilon^2}{128}.$$
 (A4d)

This always-on coupling leads to a dressing of the basis states of the two-transmon system. The states $|n\rangle_c \otimes |m\rangle_t$ referred to in the context of the coupled system, denote the dressed eigenstates of the time-independent Hamiltonian

$$H_0 = H_c' \otimes \mathbb{1} + \mathbb{1} \otimes H_t' + H_J. \tag{A5}$$

An external microwave drive with a carrier frequency of ω_d and pulse envelope $\Omega(t)$ leads to an interaction Hamiltonian (in the rotating wave approximation) of

$$H_{\rm int}(t) = \frac{\Omega(t)}{2} \left(b e^{i\omega_{\rm d}t} + b^{\dagger} e^{-i\omega_{\rm d}t} \right). \tag{A6}$$

We simulate the action of a pulse applied to the control by numerically solving the time-dependent Schrödinger equation for the full Hamiltonian $H_{\text{tot}} = H_0 + H_{\text{int}}(t) \otimes \mathbb{1}$ with the master equation solver provided by QuTip [81].

3. Single CR pulse

All CR pulses we apply to the control are played at a frequency of $\omega_{\rm d}=\overline{\omega}_{\rm t}$, i.e., the target qudit frequency in the dressed basis averaged over the four lowest levels of the control qudit. We choose a Gaussian-square pulse shape which consists of a plateau of duration τ_s between a Gaussian rise and fall with duration τ_g and standard

deviation σ . The pulse envelope is thus

$$\Omega(t) = \tilde{\Omega} \cdot \begin{cases} \frac{e^{-\frac{1}{2} \frac{(t - \tau_g)^2}{\sigma^2}} - \gamma}{1 - \gamma} &, 0 < t \le \tau_g \\ 1 &, \tau_g < t < \tau_g + \tau_s \\ \frac{e^{-\frac{1}{2} \frac{(t - \tau_g - \tau_s)^2}{\sigma^2}} - \gamma}{1 - \gamma} &, \tau_g + \tau_s \le t < \tau \end{cases}$$
(A7)

where $\tilde{\Omega}$ is the maximal amplitude, $\gamma = e^{-\frac{1}{2}\frac{(1+\tau_g)^2}{\sigma^2}}$ is a rescaling constant, and $\tau = \tau_s + 2\tau_g$ is the total duration. We choose parameter values of $\tilde{\Omega}/(2\pi) = 50\,\mathrm{MHz}$, $\tau_g = 36\,\mathrm{ns}$, and $\sigma = \tau_g/4$ for the simulated CR pulses, as we empirically find that this leads to a tolerable amount of leakage [see Figs. 8 to 10]. To calibrate the rotation angles, we fix $\tilde{\Omega}$, τ_g , and σ , adjusting only the width τ_s , and denote the obtained unitary as $U_{\mathrm{CR}}(\tau)$.

We benchmark how well the action of a CR pulse is described by the conditional R_x^{01} rotations from Eq. (2) (denoted $\operatorname{CR}(\vec{\varphi})$) through the average gate fidelity \mathcal{F} as defined in Ref. [82]. Under the time evolution of the CR pulse, each basis state $|n\rangle_c \otimes |m\rangle_t$ acquires a phase $e^{-i\alpha_{nm}}$, which we assume to be uncorrelated, i.e., $\alpha_{nm} = \alpha_n \alpha_m \, \forall n, m \in \{0,1,2,3\}$. We apply local phases after the action of the pulse to obtain the phase-corrected unitaries

$$\tilde{U}_{\text{CR}}(\tau) = [\operatorname{diag}(e^{i\alpha_{c_0}}, e^{i\alpha_{c_1}}, e^{i\alpha_{c_2}}, e^{i\alpha_{c_3}}) \otimes (A8)$$

$$\operatorname{diag}(e^{i\alpha_{t_0}}, e^{i\alpha_{t_1}}, e^{i\alpha_{t_2}}, e^{i\alpha_{t_3}})]U_{\text{CR}}(\tau).$$

In practice, these can be applied virtually on each qudit, as discussed in Sec. II A. For a duration of $\tau_{\pi}=289\,\mathrm{ns}$, we numerically optimize the angles $\vec{\varphi}$ and phases $\vec{\alpha_c}$, $\vec{\alpha_t}$, obtaining an optimal fidelity $\overline{\mathcal{F}}(\tilde{U}_{\mathrm{CR}}(\tau),\mathrm{CR}(\vec{\varphi}))=99.93\%$. Such a fidelity is possible since the optimized unitary $\tilde{U}_{\mathrm{CR}}(\tau)$ has a uniform phase structure with vanishing imaginary parts on the diagonal and little leakage, see Fig. 8, . The duration τ_{π} is calibrated such that $\varphi_0+\varphi_1=\pi$, which leads to an echoed CR sequence with a rotation angle of $\theta=\pi$.

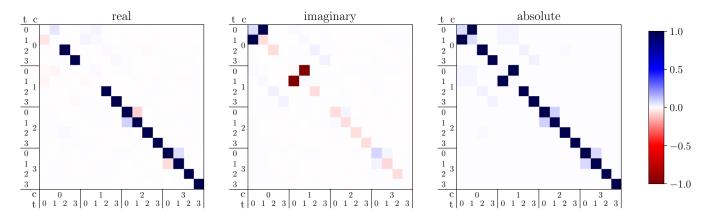


Figure 9. Numerical simulation of the unitary matrix of the full echoed CR sequence in the two-transmon dressed basis. The resulting operation is well described by an $ECR(\pi)$ gate.

4. Single qudit pulses

Here, we summarize details on the simulation of single-qudit pulses. The echoed CR sequence reported in Sec. II B 3 requires $R_x^{01}(\pm\pi)$ gates, whereas the single-qudit encoding, decoding, and recovery operations as defined in Sec. IV A are built from $R_y^{01}(\pm 2\pi/3)$, $R_y^{01}(\pi/3)$, $R_y^{12}(\pm\pi)$, $R_y^{23}(\pm\pi/3)$, and $R_y^{23}(-2\pi/3)$ rotations. For simplicity, we simulate the single-qudit system of the control with the Hamiltonian $H_c' + H_{\rm int}(t)$ and set the drive frequency to $\omega_d = \omega_c^{i,i+1}$ to drive $R^{i,i+1}$ rotations. We set the phase of the pulse envelope $\Omega(t)$ to define the axis of rotation. While a real and positive (negative) $\Omega(t)$ drives x(-x) rotations, an imaginary positive (negative) $\Omega(t)$ drives y(-y) rotations.

Each single qudit pulse is played with a Gaussian envelope given by Eq. (A7) when setting $\tau_s=0$ and $\sigma=\tau_g/4$ (up to a global phase setting the rotation direction). We fix the duration of the pulses τ_g to 100 ns, 66.6 ns, and 33.3 ns for rotation angles $\pm \pi$, $\pm 2\pi/3$, and $\pm \pi/3$, respectively. The amplitude $\tilde{\Omega}$ of the pulses is then tuned such that the area under the envelope matches the desired rotation angle. Within this model, we achieve average gate fidelities of > 99.99% (unitary error) in the ququart subspace for all single-qudit pulses.

5. Echoed CR sequence

Here, we discuss observations from the simulation of the echoed CR sequence in further detail. We simulate the CR pulses individually with the parameters given in Sec. A 3, applying the second CR pulse of the sequence with a negative amplitude. The durations of the CR tones τ_s are optimized such that the total rotation angle in Eq. (3) is $\theta = \pi$. The unitaries of the $R_x^{01}(\pi)$ rotations on the control are simulated as discussed in Sec. A 4. We apply local phase corrections for the final unitary to maximize the fidelity with ECR(π), similar to Eq. (A8). The final unitary, which achieves an average gate fidelity of

99.6%, is shown in Fig. 9. To further investigate the dominant contributions to the unitary error, we plot the evolution of all populations throughout the pulse sequence in Fig. 10. This is the same data as shown in Fig. 2(c), plotted on a logarithmic ordinate to highlight the small contributions in the undesired levels.

We can identify small non-zero off-diagonal entries in the $|2\rangle_c$ and $|3\rangle_c$ subspaces, which manifest in remaining populations of 0.005 - 0.01 for $|1\rangle_t$ and $|0\rangle_t$ in the bottom two panels of Fig. 10. This means that the second CR pulse does not fully reverse the rotations that were applied to these states by the first CR pulse. We attribute this to the fact that the driving frequency $\overline{\omega}_t$ is detuned by 120 kHz to the $|2\rangle_c\,|0\rangle_t \leftrightarrow |2\rangle_c\,|1\rangle_t$ transition and by $-110 \,\mathrm{kHz}$ to the $|3\rangle_c \,|0\rangle_t \leftrightarrow |3\rangle_c \,|1\rangle_t$ transition [see Fig. 1]. Therefore, both the x-rotation of the first CR pulse and the -x-rotation of the second CR pulse carry a small Z-component, which results in a misalignment between the rotation axis of the two CR pulses. We also attribute the non-trivial phase structure on the diagonal that is discernible in Fig. 9 to this effect. The populations of the higher-excited states of the target, shown in purple in Fig. 10, remain below 10^{-4} from which we conclude that the chosen pulse shape successfully limits leakage.

Finally, we summarize qualitatively the effect that the different parameters of our model have on the fidelity of the ECR gate. The transmon frequencies ω_c , ω_t and α_c , α_t define the level structure of the system. Through the E_J/E_C -ratio, this defines the amount of charge dispersion in each level. The closeness of frequencies in the system determine the susceptibility to leakage and crosstalk. The maximum drive amplitude $\tilde{\Omega}$ determines the entangling speed of the gate and has a strong influence on leakage and crosstalk. With increasing coupling strength J the entangling speed of the gate also increases. However, it also increases the dressing of the bare qudit eigenstates, which leads to a dependency of the $|0\rangle_t \leftrightarrow |1\rangle_t$ frequency on the state of the control and limits the unitary of the ECR gate in the idle levels $|2\rangle_c$ and $|3\rangle_c$ as detailed above.

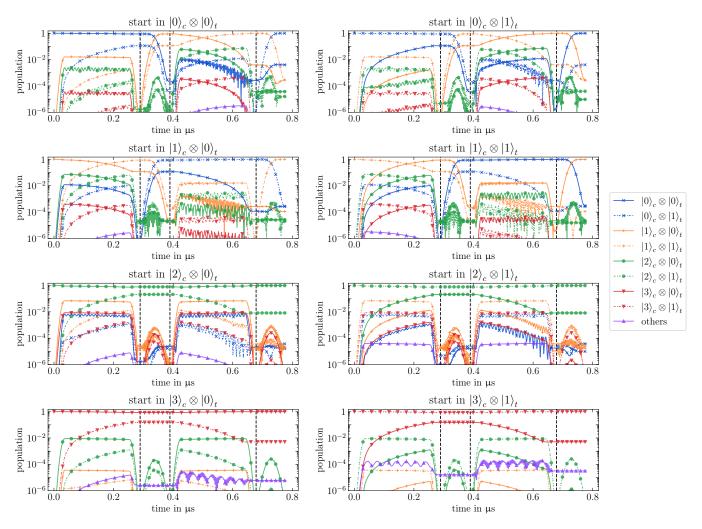


Figure 10. Evolution of the populations in the two-transmon dressed basis during the echoed CR sequence for different initial states. Color indicates the state of the control, while the linestyle indicates the state of the target. Vertical dashed black lines form four regions that correspond to the four total pulses of the ECR sequence.

6. Lindbladian dynamics

We model non-unitary dynamics under a Markovian noise approximation with a Lindblad master equation

$$\dot{\rho}(t) = -\frac{i}{\hbar} \left[H(t), \rho(t) \right] +$$

$$\frac{1}{2} \sum_{i} 2L_{i} \rho(t) L_{i}^{\dagger} - \left\{ L_{i}^{\dagger} L_{i}, \rho(t) \right\}$$
(A9)

where $\{L_i\}$ are a set of Lindblad jump operators. In Sec. IV, we consider two types of errors for ququarts, pure dephasing and amplitude damping. Firstly, we model pure dephasing with a jump operator $L_0 = \sqrt{2/T_2} \sum_{m=0}^{3} |m\rangle\langle m|$. The choice of T_2 is motivated by the fact that Eq. (A9) with only pure dephasing leads to a quantum channel $\mathcal{E}(t)$ whose fidelity $\overline{\mathcal{F}}(\mathcal{E}(t), 1)$ in the qubit subspace decays exponentially in time with a time constant of T_2 . Secondly, our model of amplitude damping contains three individual jump operators of the

form $L_n = \sqrt{n/T_1}|n-1\rangle\langle n|$ for $n\in\{1,2,3\}$. These lead to an exponential decay of the population in the n-th level, with a time constant given by T_1/n . Note that these definitions of T_1 and T_2 might differ from experimental usage in the context of measuring T_1 , T_2 , or T_2^* times for a qubit. Rather, in this work, T_1 and T_2 simply characterize the time-scale of dephasing and amplitude damping errors in our model. To simulate the action of the two-transmon ECR-gate under noise, we include the jump operators on each qudit individually, i.e., adding operators $L_i \otimes \mathbb{1}$ and $\mathbb{1} \otimes L_i$ to Eq. (A9).

The shortest current measurement pulses on IBM Quantum devices have a duration of $t_{\rm meas}=675\,\rm ns$ [64]. This affects the error correction sequence of Fig. 4 as the data qudit is in a decoded and thus unprotected state during the measurement of the ancilla. We account for the measurement duration in our simulations by adding an idle time of $t_{\rm meas}$ before processing the measurement.

We compute the quantum channel \mathcal{E} of a pulse sequence by simulating its action on a complete set of (not

necessarily physical) input states to obtain the full Liouvillian superoperator, leveraging the quantum information package of Qiskit [83]. The error correction sequence from Fig. 4 features a measurement of the ancilla and subsequent classically controlled operations on the data qudit. Simulating this measurement of the ancilla is problematic for non-physical input states as it can result in invalid measurement probabilities. We thus make use of the "delayed measurement principle" [84] and simulate the classically-controlled operations as quantum-controlled operations which are well-defined also for non-physical input states.

Appendix B: Details on unitary gate synthesis

Here, we point out technical details relating to the decomposition of a general unitary into the ECR and single-qudit gates and distinguish our work from previous state-of-the-art. Our transpilation routine presented in Sec. III builds on the multivalued quantum Shannon decomposition as developed in Ref. [76]. The authors of Ref. [76] show how to synthesize arbitrary unitaries with an m-controlled X gate

$$GCX = C^{m}[|0\rangle\langle 1| + |1\rangle\langle 0| + \sum_{k=2}^{d} |k\rangle\langle k|]$$
 (B1)

as the fundamental entangling gate. One might assume that existing implementations of the CNOT gate naturally generalize to the GCX gate. However, this is not the case for both direct and echoed cross-resonance CNOT gates, as discussed below. This necessitates the modification

we propose in Sec. III to the transpilation provided in Ref. [76].

As discussed in the main text, the cross-resonance effect applies an $R_x^{01}(\varphi_j)$ rotation to the target qudit, where φ_j depends on the control state $|j\rangle_c$ and the gate duration, see Eq. (2). To implement a direct Cnot, the angles are tuned such that $\varphi_0 - \varphi_1 = \pi$. In the qubit subspace, the resulting unitary is equivalent to the Cnot up to local operations. However, in contrast to Eq. (B1), the gate acts non-trivially on higher-excited states of the control; an effect that can not be reversed by single-qudit gates. For qutrits, this can be compensated for by tuning the drive strength such that $\varphi_0 = \varphi_2$, as demonstrated in Ref. [42]. This approach does not scale to higher qudit dimension.

The action of higher-excited control states can be cancelled by an echoed CR sequence, leading to the $\mathrm{ECR}(\theta)$ gate defined in Eq. (3). While $\mathrm{ECR}(\pi/2)$ is local-Clifford equivalent to the CNOT for qubits, it is again not possible to transform $\mathrm{ECR}(\pi/2)$ to the GCX gate with local operations. As shown in Sec. III, d-1 $\mathrm{ECR}(-\pi/d)$ gates can be combined to synthesize a $C^m[R_x^{01}(\pi)]$ gate. However, there is still a subtle difference to the GCX gate. A $C^m[R_x^{01}(\pi)]$ gate introduces a relative phase factor of i between the qubit subspace and higher-excited states, as

$$R_x^{01}(\pm \pi) = \mp i|0\rangle\langle 1| \mp i|1\rangle\langle 0| + \sum_{k=2}^d |k\rangle\langle k|.$$
 (B2)

Cancelling this relative phase on the target requires a controlled phase gate. The unitary synthesis proposed in Ref. [76] is thus not suitable for architectures that implement m-controlled x rotations, such as the ECR gate. We circumvent this issue by using pairs of $C^m[R_x^{01}(\pi)]$ and $C^m[R_x^{01}(-\pi)]$ gates instead of the GCX gate.