Enumeration of rooted 3-connected bipartite planar maps

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Abstract: We provide the first solution to the problem of counting rooted 3-connected bipartite planar maps. Our starting point is the enumeration of bicoloured planar maps according to the number of edges and monochromatic edges, following Bernardi and Bousquet-Mélou [J. Comb. Theory Ser. B, 101 (2011), 315–377]. The decomposition of a map into 2- and 3-connected components allows us to obtain the generating functions of 2- and 3-connected bicoloured maps. Setting to zero the variable marking monochromatic edges we obtain the generating function of 3-connected bipartite maps, which is algebraic of degree 26. We deduce from it an asymptotic estimate for the number of 3-connected bipartite planar maps of the form $t\,n^{-5/2}\gamma^n$, where $\gamma = \rho^{-1} \approx 2.40958$ and $\rho \approx 0.41501$ is an algebraic number of degree 10.

Résumé: Nous apportons la première solution au problème du dénombrement des cartes enracinées planaires qui sont biparties et 3-connexes. Notre point de départ est l'énumération des cartes planaires bi-coloriées, d'après Bernardi et Bousquet-Mélou [J. Comb. Theory Ser. B, 101 (2011), 315–377]. La décomposition d'une carte en composantes 2- et 3-connexes nous permet ensuite d'obtenir les fonctions génératrices des cartes bi-coloriées 2- et 3-connexes. En évaluant à zéro la variable marquant le nombre d'arêtes monochromes, nous obtenons alors la fonction génératrice des cartes biparties 3-connexes. Cette dernière est algébrique de degré 26. Nous en déduisons une estimation asymptotique de la forme $t\,n^{-5/2}\gamma^n$ du nombre de cartes planaires biparties 3-connexes, avec $\gamma = \rho^{-1} \approx 2.40958$ et où $\rho \approx 0.41501$ est un nombre algébrique de degré 10.

1 Introduction

The theory of map enumeration was initiated by William Tutte in the 1960's, motivated by the Four Colour Problem, the most notorious open problem in graph theory at that time. In his own words [28, Chapter 10]:

> From time to time in a graph-theoretical career one's thoughts turn to the Four Colour Problem. It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the Problem. For example it might be possible to find the average number of 4-colourings, on vertices, for planar triangulations of a given size. One would determine the number of triangulations of $2n$ faces, and then the number of 4-coloured triangulations of $2n$ faces. Then one would divide the second number by the first to get the required average. I gathered that this sort of retreat from a difficult problem to a related average was not unknown in other branches of Mathematics, and that it was particularly common in Number Theory.

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The first task for Tutte was to decide what sort of triangulations to study. From the point of view of colouring the natural choice were 3-connected triangulations, which correspond to maximal planar graphs. He started drawing them and, again quoting [28]:

> Having made no progress with the enumeration of these diagrams I bethought myself of Cayley's work on the enumeration of trees. His first successes had been with the rooted trees, in which one vertex is distinguished as the "root". Perhaps I should root triangulations in some way and try to enumerate the rooted ones.

Tutte defined rooted maps and succeeded in counting rooted triangulations, thus starting his famous series of 'census' papers on map enumeration [23]. Later he counted all rooted maps as well as the 2-connected and 3-connected ones [24], and also Eulerian maps which by duality are in bijection with bipartite maps. From here it is not difficult to count those which are 2-connected and bipartite [13]. But counting 3-connected bipartite maps has remained an open problem.

After these preliminaries let us properly define our objects. A planar map is a connected multigraph with a given embedding in the sphere. All maps in this paper are rooted at a directed edge. Maps are counted according to the number of edges and up to orientation-preserving homeomorphisms of the sphere; see for instance [21] for basic concepts on planar maps. A map is 2-connected if it has at least two edges and no loop and no cut vertex (the smallest one is the double edge), and it is 3-connected if it has at least six edges and no double edge and no vertex separator of size two (the smallest one is the complete graph K_4). Tutte showed that the number of (rooted) maps with n edges is given by

$$A_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}. \tag{1}$$

Notice that $A_0 = 1$, corresponding to the map with one vertex and no edge. This surprisingly simple formula was explained much later by Schaeffer [20] in his Ph.D. thesis through a remarkable bijection with 'decorated' trees, drawing on previous work by Cori and Vauquelin [8]. This opened the way to the study of the asymptotic metric properties of maps and their connection with Brownian motion [7], culminating in the construction of the so-called Brownian map (see [12] for an overview).

Tutte originally proved Formula (1) using a correspondance with bicubic maps (defined later). But it can be proved more directly as follows. For a map m and an edge of m we denote by $\mathsf{m} - e$ the supression of e in m. Then, m is either the empty map (namely, with one vertex and no edges), or $\mathsf{m} - e$ is the union of two disjoint maps (which can be rooted canonically and determine m uniquely), or $\mathsf{m} - e$ is connected. In the latter case $\mathsf{m} - e$ can also be canonically rooted, but to recover m we need to know which vertex of the root face is the second vertex of the new root edge. Hence, one needs to refine the counting by considering the number $A_{n,k}$ of rooted maps with n edges and root face of degree k. This leads to a quadratic equation satisfied by the counting series (or generating function) $A(z,y) = \sum_{n,k} A_{n,k} z^n y^k$

$$A(z,y) = 1 + y^2 z A(z,y)^2 + yz \frac{yA(z,y) - A(z,1)}{y-1}. \tag{2}$$

In this context y is called a *catalytic* variable. Notice that we cannot solve (2) directly, computing $A(z) = \sum_n A_n z^n = A(z,1)$ as a specialisation of $A(z,y)$. Tutte already encounter this problem in his pioneering work on the enumeration of triangulations [23], and was solved completely and in a systematic way by Brown in [6]. Brown's paper introduced the so-called *quadratic method* (see [5] for a far-reaching generalisation to arbitrary polynomial equations). Applying this method one obtains the explicit formula

$$A(z) = \frac{18z - 1 + (1-12z)^{3/2}}{54z^2}.$$

From this expression it is straightforward to obtain (1) and the estimate

$$[z^n]A(z) \sim \frac{1}{\sqrt{\pi}} n^{-5/2}\, 12^n \qquad \text{as } n \to \infty,$$

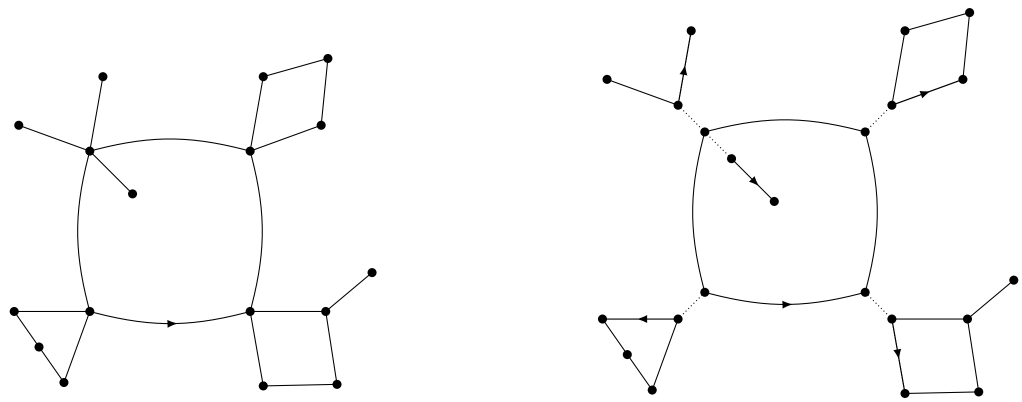

Figure 1: The decomposition of a rooted planar map m into a 2-connected map $\mathsf{C}(\mathsf{m})$ containing the root, the 2-core, and maps attached by their root vertex to the corners of $\mathsf{C}(\mathsf{m})$.

where we use the notation $[z^n]A(z)$ to denote the n-th coefficient of the power series $A(z)$. The exponent $-5/2$ is universal for 'naturally defined' classes of planar maps and has been explained in a number of different ways (see for instance [9] in relation to the quadratic method).

Once $A(z)$ is determined, one can obtain the generating functions $B(z)$ and $T(z)$ counting 2-connected and 3-connected planar maps, respectively, where z marks edges. We assume that the maps consisting of a single edge and a loop are 2-connected, and hence counted in $B(z)$. Given a rooted map m, let $\mathsf{C}(\mathsf{m})$ be the unique maximal (in the sense of submap) 2-connected component containing the root edge, called the *2-core* of m. Then m is recovered by placing a rooted map at each corner of $\mathsf{C}(\mathsf{m})$ (a corner consists of two consecutive edges with a common vertex). See Figure 1 for an example of such decomposition. Taking into account that the number of corners in a map is twice the number of edges, one obtains the relation

$$A(z) = 1 + B(zA(z)^2), \tag{3}$$

where the term 1 corresponds to the empty map and $B(zA(z)^2)$ corresponds to the substitution into the 2-core (the 2-connected component containing the root). Inverting this equation one obtains

$$B(z) = 2z + z^2 + 2z^3 + 6z^4 + 22z^5 + \dots,$$

where the term $2z$ corresponds to the edge and loop maps, while z^2 corresponds to the digon.

In order to find $T(z)$, one uses the decomposition of 2-connected graphs into 3-connected components (see [22] and [24] for details). Let $D(z) = B(z)/z$, meaning that the root edge is not counted. Then

$$D(z) = z + S(z) + P(z) + \frac{T(D(z))}{D(z)}, \tag{4}$$

where $S(z)$ and $P(z)$ encode *series* and *parallel* maps, respectively. A series map is a 2-connected map obtained by the series composition of an arbitrary 2-connected map and a non-series 2-connected map. We then have

$$S(z) = D(z)(D(z) - S(z)), \qquad P(z) = S(z),$$

where the second equation follows by duality. On the left of Figure 2 is an example of a series map decomposed into a smaller 2-connected map, here another series map, and a smaller non-series map, here the tetrahedron; on the right is its parallel dual map with the corresponding decomposition. The 2-connected maps m counted by the term $T(D(z))/D(z)$ in (4) are called *polyhedral* and are obtained after substituting 2-connected maps

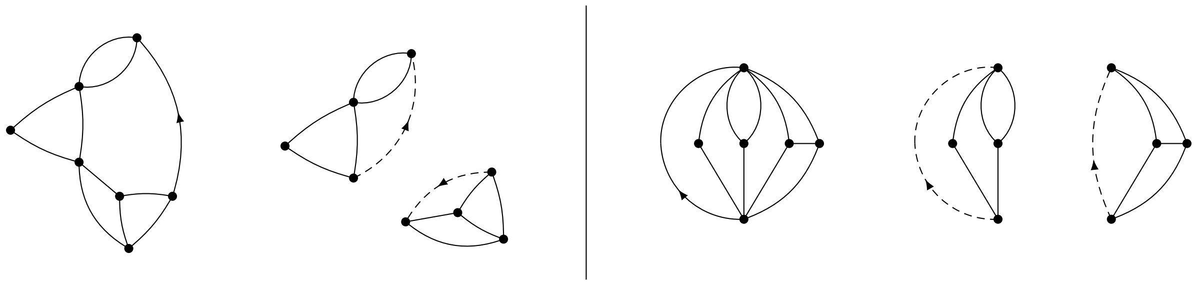

Figure 2: Decomposition of series and parallel maps into smaller 2-connected maps.

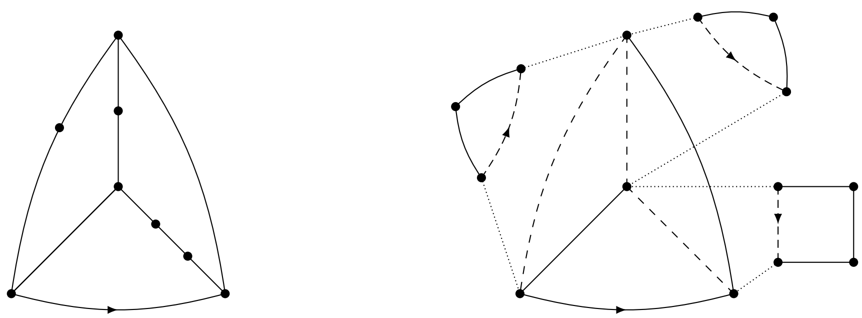

Figure 3: Decomposition of a polyhedral map m into its 3-core $\mathsf{T}(\mathsf{m})$ and smaller 2-connected maps.

for the non-root edges of a 3-connected map $\mathsf{T}(\mathsf{m})$. An edge e of $\mathsf{T}(\mathsf{m})$ is replaced by a 2-connected map m' with root edge r by identifying the two vertices of r with the two vertices of e, then deleting both e and r from the resulting map. In that case, $\mathsf{T}(\mathsf{m})$ is the maximal 3-connected component of m containing the root and is called the *3-core* of m. An illustration can be found in Figure 3. It follows that

$$\frac{T(z)}{z} = z - \frac{2z^2}{1+z} - f(z),$$

where $f(z)$ is the functional inverse of $D(z)$. From this expression, one obtains a quadratic equation satisfied by $T(z)$, which can be solved directly.

Using the previous algebraic expressions and by means of algebraic inversion, Tutte obtained implicit expressions for $B(z)$ and $T(z)$ (which are algrebraic series of degree 3 and 2, respectively) and deduced the estimates

$$[z^n]B(z) \sim \frac{1}{4\sqrt{\pi}} n^{-5/2} \left(\frac{27}{4}\right)^n \quad \text{and} \quad [z^n]T(z) \sim \frac{1}{720\sqrt{\pi}} n^{-5/2}\, 4^n, \qquad \text{as } n \to \infty.$$

From uncolored to bipartite maps. Our aim in this paper is to carry out the same program for *bipartite* maps. By convention we will always assume that vertices are coloured black and white, and that the root vertex is always black. A straightforward modification of the quadratic method (see for instance Equation (3.3) in [2]) provides the series of non-empty bipartite maps as

$$A_b(z) = \frac{-1 + 12z - 24z^2 + (1-8z)^{3/2}}{32z^2}. \tag{5}$$

Then the estimate for the number of bipartite maps with n edges is

$$[z^n]A_b(z) \sim \frac{15}{16\sqrt{\pi}}\, n^{-5/2}\, 8^n \qquad \text{as } n \to \infty.$$

The next step is also rather direct since a map (or graph) is bipartite if and only if its maximal 2-connected components are bipartite. The decomposition of a map into 2-connected components implies the following equation for the series B_b of 2-connected bipartite maps:

$$A_b(z) = 1 + B_b(zA_b(z)^2).$$

By elimination one obtains an algebraic equation for $B_b(z)$ of degree 5 from which (see [13]) one deduces the estimate

$$[z^n]B_b(z) \sim \frac{3}{16\sqrt{\pi}}\, n^{-5/2} \left(\frac{128}{25}\right)^n \qquad \text{as } n \to \infty.$$

However, the next step, i.e. to encode 3-connected maps starting from 2-connected maps, does not generalise at all since it is far from true that a 2-connected graph is bipartite if and only if its 3-connected components are bipartite (consider for instance the graph obtained from K_4 by subdividing once every edge).

Our approach is to enrich the framework by counting *bicoloured* maps, which are maps together with a 2-colouring (black and white) of the vertices such that the root vertex is always black. The 2-colouring is arbitrary and not necessarily proper. Let $\mathcal{A}$ be a class of bicoloured maps and let $A(z,\nu)$ be the associated series, where z marks all edges as before and ν marks monochromatic edges, that is, edges whose endpoints have the same colour. In fact we need to consider two series: $A_1(z,\nu)$ and $A_2(z,\nu)$ count those rooted at a monochromatic edge and at a bichromatic edge, respectively. It is then clear that $A_2(z,0)$ is the series of bipartite maps in $\mathcal{A}$. Our strategy is to compute $A_1(z,\nu)$ and $A_2(z,\nu)$ for general, 2-connected and 3-connected maps.

After his work on map enumeration in the 1960's, Tutte went back to the original motivation of counting 4-colourings of triangulations. We recall that the *chromatic polynomial* $\chi_G(q)$ of a graph G is a function equal to the number of proper q-colourings of G. For a class of maps $\mathcal{C}$ Tutte defined its 'chromatic sum' as

$$C(z,q) = \sum_{\mathsf{m}\in\mathcal{C}} z^{|\mathsf{m}|}\chi_A(q), \tag{6}$$

where $|\mathsf{m}|$ denotes the number of edges of m. If one could find a way of computing $C(z,q)$, then one could compute the average number of q-colourings of maps in $\mathcal{C}$. Tutte devoted a long series of papers to chromatic sums of triangulations for different values of q. Starting with [26], Tutte obtained that for any q, the generating function for triangulations weighted by their chromatic polynomial satisfies a differential equation from which one can obtain explicit recurrence relations (see [27, 18]). For $q = 4$, the differential equation obtained is somehow simple (see Equation (52) in [27]), so recursive enumerative formulas exist in this case. However, as he remarked years later in [28]

> I said near the beginning of this chapter that information about averages might be easier to obtain than a proof of the Four Colour Theorem. Yet now we have the Haken-Appel proof, and we still lack an explicit formula or even an asymptotic approximation for our four-colour average.

In fact, nowadays to find the asymptotic number of 4-coloured triangulations is still an open problem. The topic was revived much later by Bernardi and Bousquet-Mélou in a remarkable paper [2] (and its sequel [3]), where they write:

> This tour de force has remained isolated since then, and it is our objective to reach a better understanding of Tutte's rather formidable approach, and to apply it to other problems in the enumeration of colored planar maps.

The approach in [2] is to consider all q-colourings, proper or not, and take as a new parameter the number of monochromatic edges. Our starting point is Theorem 21 from [2], where the authors determine the series $M(z,\nu)$ of bicoloured maps, where z marks edges and ν monochromatic edges. It is an algebraic function of degree 6 whose first terms are

$$M(z,\nu) = 1 + (1+2\,\nu)\,z + \left(9\,\nu^2 + 8\,\nu + 3\right)z^2 + \left(42\,\nu^3 + 72\,\nu^2 + 51\,\nu + 12\right)z^3 + \cdots$$

For instance, the term z corresponds to the isthmus map with the unique proper 2-colouring (recall that the root vertex is by convention coloured black) and $2\nu z$ corresponds to the loop and isthmus maps coloured with a single color.

In Section 2 we obtain the series $M_1(z,\nu)$ of bicoloured maps where the root edge is monochromatic, and as a consequence the series $M_2(z,\nu) = M(z,\nu) - 1 - M_2(z,\nu)$ of bicoloured maps rooted at a bichromatic edge. Once M_1 and M_2 are determined, using the decomposition of maps into 2-connected components we obtain the generating functions $B_1(z,\nu)$ and $B_2(z,\nu)$ of 2-connected bicoloured maps. Then using the decomposition of 2-connected maps into 3-connected components we obtain the generating functions $T_1(z,\nu)$ and $T_2(z,\nu)$ of 3-connected bicoloured maps.

Finally, $T_b(z) = T_2(z,0)$ is the series of 3-connected bipartite maps, which is what we needed. $T_b(z)$ is algebraic of degree 26 and its minimal polynomial is too long to be reproduced here, but can be found, together with the relevant computations regarding this work, in the accompanying `Maple` session [1]. It is somehow surprising that the series of such a natural class of planar maps has this algebraic complexity. The first terms are

$$T_b(z) = z^{12} + 4z^{16} + 9z^{18} + 19z^{19} + 29z^{20} + 63z^{21} + 198z^{22} + 345z^{23} + 685z^{24} + 1775z^{25} + \cdots,$$

Recall that the coefficient of z^n counts the different rootings of 3-connected bipartite unrooted maps with n edges, where the number of rootings of a map is twice the number of edges divided by the number of symmetries. In particular, let us verify the first five coefficients following the order left to right then top to bottom of their illustration in Figure 4:

- The cube is the unique (unrooted) 3-connected bipartite map with 12 edges; it has 24 symmetries and thus admits a unique rooting.
- There are no 3-connected bipartite maps with 13, 14 and 15 edges.
- There is a unique 3-connected bipartite map with 16 edges; it has 8 symmetries hence the coefficient of z^{16} is equal to $2 \cdot 16/8 = 4$.
- There are two 3-connected bipartite maps with 18 edges, the hexagonal prism has 12 symmetries and the other one has 6.
- There is a unique 3-connected bipartite map with 19 edges and it has only 2 symmetries.
- The three 3-connected bipartite maps with 20 edges admit 8, 10 and 2 symmetries, respectively.

Remark that the hexagonal prism is the smallest 3-connected bipartite planar map that is not a quadrangulation (compare with the last table in [16]).

The computations are within the capabilities of a modern computer algebra system, and we can perform the corresponding singularity analysis to obtain our main result.

Theorem 1. *Let t_n be the number of 3-connected bipartite maps with n edges. Then*

$$t_n \sim t\,n^{-5/2}\,\gamma^n \qquad \text{as } n \to \infty,$$

with $t \approx 0.00412$ and $\gamma = \rho^{-1} \approx 2.40958$, and where $\rho \approx 0.41501$ is a root of the irreducible polynomial

$$\begin{aligned}&82796609536z^{10} - 125942890496z^9 - 278408523776z^8 + 430685329920z^7 + 381960569664z^6\\ &-519574865472z^5 - 253658186064z^4 + 253489557672z^3 + 78314553945z^2\\ &-51532664454z + 1113912729.\end{aligned}$$

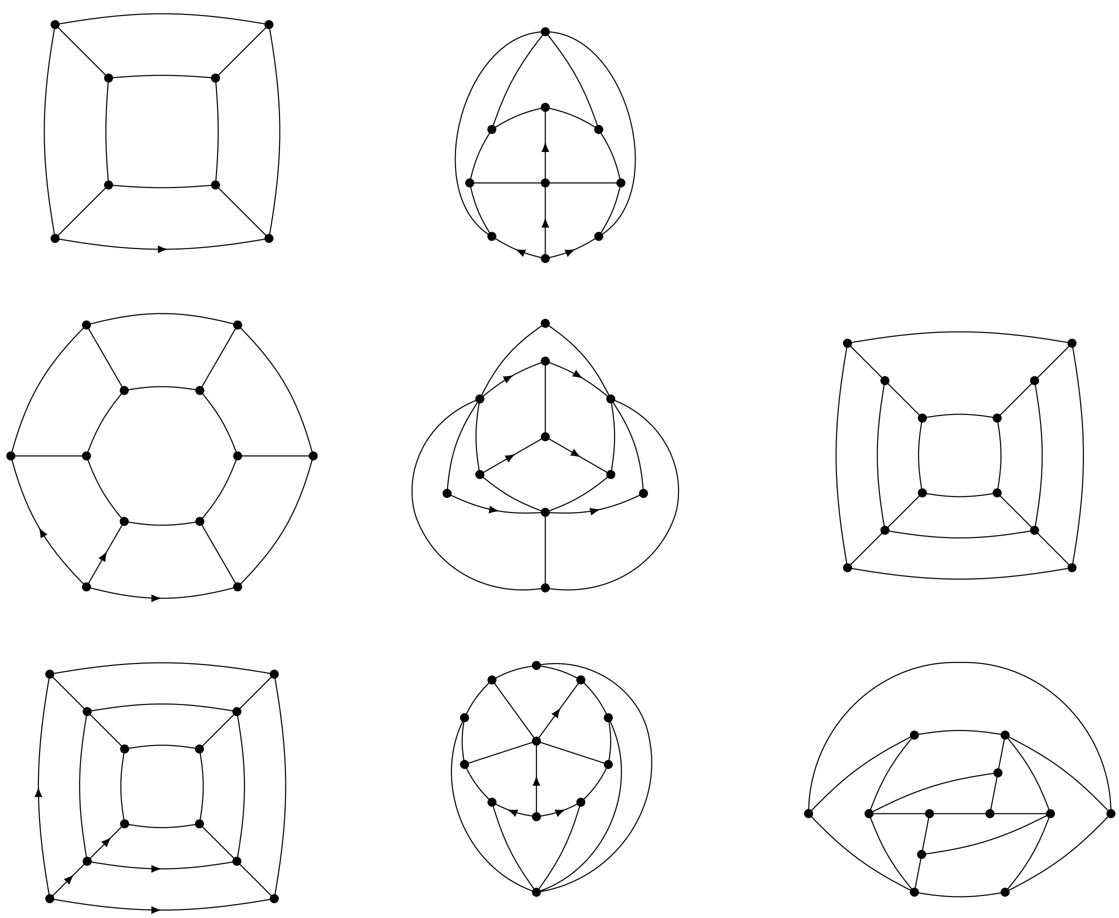

Figure 4: The smallest 3-connected bipartite planar maps pictured with their rootings. The rootings of the maps in the middle right and bottom right corners are not drawn for readability.

The class of 3-connected bipartite planar maps is a naturally defined class of planar maps that has not been counted before. Other natural classes of maps for which the enumeration problem remains open include those that are 4-connected, triangle-free, or 3-colourable.

We conclude this introduction by showing the growth constants of some of the classes of maps discussed above, in all cases counted by number of edges. The fact that the first two values in the third row are the same is because a connected bipartite cubic graph is necessarily 2-connected.

Class of maps	Arbitrary	2-connected	3-connected
Arbitrary	12	27/4	4
Bipartite	8	125/8	$\gamma \approx 2.40958$
Bipartite cubic	2	2	8/5

2 Counting bicoloured 3-connected planar maps

In this section we first determine the generating function of bicoloured maps rooted at a monochromatic edge or at a bichromatic edge. Then, using the decomposition of maps into maximal 2-and 3-connected components, we determine the series of 2-connected and 3-connected bicoloured maps.

2.1 Counting bicoloured maps

The Ising polynomial of a map. The partition function of the Potts model (a model in statistical physics for spin interactions) on a graph G is defined as

$$P_G(q,\nu) = \sum_{c\colon V(G)\to[q]} \nu^{m(c)},$$

where $[q]=\{1,2,\dots,q\}$, the sum is defined over all vertex colourings (with q colours) of G, and $m(c)$ is the number of monochromatic edges defined by c. When restricted to two colours, $P_G(2,\nu)$ is also known as the partition function of the Ising model (a model for ferromagnetism, where the two colours correspond to the two possible values of the "spin" of a particle).

The function $P_G(q,\nu)$ has the following properties. Let G_1 and G_2 be two vertex-disjoint graphs, and let G be the graph obtained by identifying a vertex of G_1 and a vertex of G_2. Then

$$P_{G_1\cup G_2}(q,\nu) = P_{G_1}(q,\nu)P_{G_2}(q,\nu) \quad \text{and} \quad P_G(q,\nu) = \frac{1}{q}P_{G_1}(q,\nu)P_{G_2}(q,\nu). \tag{7}$$

Furthermore, $P_G(q,\nu)$ can be computed recursively as follows. If G is the empty graph then $P_G(q,\nu) = q^{|V(G)|}$. Otherwise we have

$$P_G(q,\nu) = P_{G\backslash e}(q,\nu) + (\nu-1)P_{G/e}(q,\nu), \tag{8}$$

where $G\backslash e$ is the graph obtained by deleting the edge e from G and G/e is obtained by contracting e (when e is a loop, G/e is obtained by deleting e). A combinatorial explanation of this equation, as pointed out in [4, Section 2.3], is that $\nu P_{G/e}(q,\nu)$ counts colourings of G for which the edge e is monochromatic while $P_{G\backslash e}(q,\nu) - P_{G/e}(q,\nu)$ counts those where e is bichromatic. Observe that the relation (8) implies by induction that $P_G(q,\nu)$ is a polynomial in q and ν with no constant term, in particular it is divisible by q. Observe then that $P_G(q,\nu)$ encodes all q-colourings of G, while $P_G(q,\nu)/q$ will encode those whose root vertex is of a fixed colour. $P_G(q,\nu)$ is in fact equivalent to the ubiquitous Tutte polynomial $T_G(x,y)$ through the simple change of variables $y=\nu$ and $(x-1)(y-1)=q$, while $P_G(q,0)$ is the chromatic polynomial of G (see [4] for an extended account). We call $P_G(2,\nu)$ the Ising polynomial G.

Additionally, (8) allows us to define by induction the *Ising polynomial* $P_{\mathsf{m}}(2,\nu)$ *of a rooted map* m, as follows. The Ising polynomial of the empty map is equal to 2. The deletion and contraction operations are perdormed on the root edge e, and the resulting maps are denoted by $\mathsf{m}_\backslash$ and $\mathsf{m}_/$, respectively. The new root edge is defined canonically as the next edge encountered while walking along the boundary of the root face of m in the direction induced by e. In the case where e is a bridge of m, then $\mathsf{m}_\backslash$ is composed of two disjoint maps m_1 and m_2 and $P_{\mathsf{m}_\backslash}(2,\nu) = P_{\mathsf{m}_1}(2,\nu)P_{\mathsf{m}_2}(2,\nu)$.

As an illustration, we display in Figure 5 the 2-colourings of the triangle map m encoded by the Ising polynomial $P_{\mathsf{m}}(2,\nu) = 2\nu^3 + 6\nu$. On the left are the 2-colourings where the root edge is monochromatic, whose Ising polynomial is $P_{\mathsf{m}_1}(2,\nu) = 2\nu^3 + 2\nu$. On the right are the 2-colourings where the root edge is bichromatic, whose Ising polynomial is $P_{\mathsf{m}_2}(2,\nu) = 4\nu$. Summing the contributions from both parts gives $P_{\mathsf{m}_\backslash}(2,\nu) = 2\nu^2 + 4\nu + 2$ and $P_{\mathsf{m}_/}(2,\nu) = 2\nu^2 + 2$. Observe that indeed $P_{\mathsf{m}_1}(2,\nu) = \nu P_{\mathsf{m}_/}(2,\nu)$.

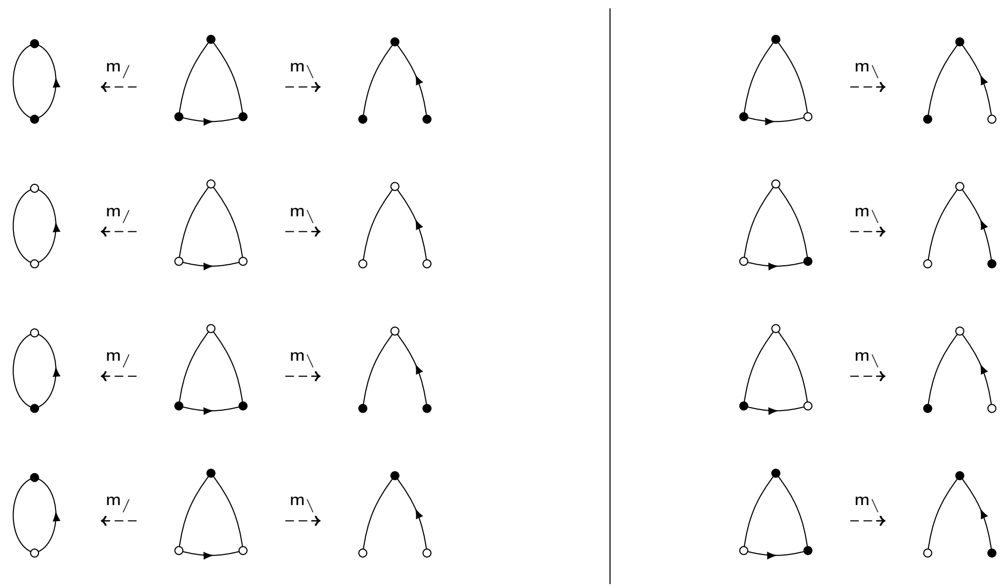

Figure 5: The eight different 2-colourings of the rooted triangle map m and their decompositions.

The Ising generating function of maps. In this section we use the common terminology in [2] and [4]. Let $\mathcal{M}$ be the class of all planar maps, and let $\mathcal{M}_0$ be equal to $\mathcal{M}$ minus the empty map. The generating function $M(z,\nu)$ of bicoloured maps can be obtained directly by weighting each map in $\mathcal{M}$ by its Ising polynomial (compare with (6)). Then we have

$$M(z,\nu) = \frac{1}{2}\sum_{\mathsf{m}\in\mathcal{M}} P_{\mathsf{m}}(2,\nu)\, z^{|\mathsf{m}|},$$

where $|\mathsf{m}|$ is the number of edges in m, and the factor $1/2$ is because the root vertex is colored black. The relation (8) applied to $P_{\mathsf{m}}(2,\nu)$ induces a functional equation for $M(z,\nu)$ analogous to the one for (uncoloured) maps in (2). In the same way that for edge deletion in (2) we encoded the degree of the root face in the variable y, the operation of edge contraction in (8) requires a second variable x encoding the degree of the root vertex. We remark that x and y play a dual role: the degree of the root face of a map m is the degree of the root vertex of the geometric dual m^*. Also, $(\mathsf{m}\backslash e)^* \simeq \mathsf{m}^*/e^*$, where e^* is the dual edge of e, and $(\mathsf{m}/e)^* \simeq \mathsf{m}^*\backslash e^*$. In terms of the Tutte polynomial, this implies that $T_{\mathsf{m}^*}(x,y) = T_{\mathsf{m}}(y,x)$.

We write now $N(x,y) = M(x,y;z,\nu)$ in order to emphasize the role of the catalytic variables. Let $\mathrm{dv}(\mathsf{m})$ and $\mathrm{df}(\mathsf{m})$ be the degree of the root vertex and the degree of the root face of m, respectively. Following [4], we define

$$N_\backslash(x,y) = \frac{1}{2}\sum_{\mathsf{m}\in\mathcal{M}_0} z^{|\mathsf{m}|}x^{\mathrm{dv}(\mathsf{m})}y^{\mathrm{df}(\mathsf{m})}P_{\mathsf{m}_\backslash}(2,\nu) \quad \text{and} \quad N_/(x,y) = \frac{1}{2}\sum_{\mathsf{m}\in\mathcal{M}_0} z^{|\mathsf{m}|}x^{\mathrm{dv}(\mathsf{m})}y^{\mathrm{df}(\mathsf{m})}P_{\mathsf{m}_/}(2,\nu).$$

The relation (8) directly implies that

$$N(x,y) = 1 + N_\backslash(x,y) + (\nu-1)N_/(x,y).$$

Furthermore, generalising (2) with the help of (7), one can write $N_\backslash(x,y)$ and $N_/(x,y)$ as functions of $N(x,y)$ and its evaluations (see [4, Proposition 5.1] for a detailed proof):

$$N_\backslash(x,y) = 2xy^2zN(1,y)N(x,y) + (x-1)xyzN(x,1)N(x,y) + xyz\frac{yN(x,y)-N(x,1)}{y-1} \tag{9}$$

and

$$N_/(x,y) = x^2yzN(x,1)N(x,y) + (y-1)xyzN(1,y)N(x,y) + xyz\frac{xN(x,y)-N(1,y)}{x-1}. \tag{10}$$

Combining (9) and (10) gives

$$\begin{aligned} N(x,y) = 1 &+ xyz(2y+(\nu-1)(y-1))N(x,y)N(1,y) + xyz(x\nu-1)N(x,y)N(x,1) \\ &+ xyz(\nu-1)\frac{xN(x,y)-N(1,y)}{x-1} + xyz\frac{yN(x,y)-N(x,1)}{y-1}. \end{aligned} \tag{11}$$

Equation (11) was first established in an equivalent form for maps weighted by their Tutte polynomial (called the *dichromate* by Tutte) in [25]. Using a remarkable method to solve this particular family (for general values of q) of functional equations with two catalytic variables, it is proven in [2] that $M(z,\nu)$ is an algebraic function of degree 6 and that $M(z,\nu)$ admits the rational parametrisation

$$\begin{aligned} M(z,\nu) = \frac{1+3\nu S-3\nu S^2-\nu^2S^3}{(1-2S+2\nu^2S^3-\nu^2S^4)^2}\Big(&1-(3+\nu)S+(1+2\nu)S^2-\nu(1-5\nu)S^3 \\ &+\nu(1-6\nu)S^4+2\nu^2(1-\nu)S^5+\nu^3S^6\Big), \end{aligned} \tag{12}$$

where $S = S(z,\nu)$ is defined as the unique solution of

$$S = z\frac{(1+3\nu S-3\nu S^2-\nu^2S^3)^2}{1-2S+2\nu^2S^3-\nu^2S^4}. \tag{13}$$

whose coefficients at $z=0$ are polynomials in ν with non-negative coefficients.

Rooting at a monochromatic edge. Let us denote by $M_1(z,\nu)$ and $M_2(z,\nu)$ the generating functions for bicoloured maps where the root edge is monochromatic and bichromatic, respectively, and where z marks edges and ν monochromatic edges, so that $M(z,\nu) = 1+M_1(z,\nu)+M_2(z,\nu)$. As observed above, $\nu P_{\mathsf{m}_/}(z,\nu)$ endcodes colourings of m for which the root edge is monochromatic while $P_{\mathsf{m}_\backslash}(z,\nu)-P_{\mathsf{m}_/}(z,\nu)$ encodes those where the root edge is bichromatic. Hence

$$M_1(z,\nu) = \nu N_/(1,1) \quad \text{and} \quad M_2(z,\nu) = N_\backslash(1,1) - N_/(1,1).$$

Setting $y=1$ in (10) gives

$$N_/(x,1) = x^2zN(x,1)^2 + xz\frac{xN(x,1)-N(1,1)}{x-1}.$$

The last summand of the above equation is a finite difference so that

$$\lim_{x\to1} N_/(x,1) = zN(1,1)^2 + zN(1,1) + z\frac{\partial}{\partial x}N(x,1)\Big|_{x=1}$$

which can be rephrased, using the identities $M_1(z,\nu) = \nu N_/(1,1)$ and $M(z,\nu) = N(1,1)$, as

$$M_1(z,\nu) = \nu zM(z,\nu)^2 + \nu zM(z,\nu) + \nu z\frac{\partial}{\partial x}N(x,1)\Big|_{x=1}. \tag{14}$$

As a byproduct of the method developed in [2], one can obtain a polynomial $R(X, Y, z, \nu)$ such that $R = 0$ when $X = M(z, \nu)$ and $Y = \partial N(x, 1)/\partial x|_{x=1}$. In fact, in the proof of [2, Theorem 21] the authors obtain an analogous polynomial for $\partial N(1, y)/\partial y|_{y=1}$, but a direct step-by-step adaptation of their method allows one to recover $R(X, Y, z, \nu)$ (see the `Maple` session [1]).

Eliminating from the system composed of Equations (12), (13), (14) and $R = 0$ yields an irreducible polynomial equation $Q^{(1)}(M_1, z, \nu) = 0$, also of degree 6 in M_1, defining $M_1 = M_1(z, \nu)$ implicitly as a function of z and ν. An equation $Q^{(2)}(M_2, z, \nu) = 0$ can be derived by elimination, using $M(z, \nu) = 1 + M_1(z, \nu) + M_2(z, \nu)$. The polynomial $Q^{(2)}$ has also degree 6 in $M_2 = M_2(z, \nu)$. Incidentally, both $M_1(z, \nu)$ and $M_2(z, \nu)$ also admit, as $M(z, \nu)$, rational parametrisations in S, given by

$$\begin{aligned}
M_1(z,\nu) = \frac{\nu S}{(1-2S+2\nu^2S^3-\nu^2S^4)^2}\Big(&\nu^4S^8+\nu^3(5-2\nu)S^7+\nu^2(7-16\nu)S^6+\nu(14\nu^2-24\nu+3)S^5\\
&+\nu(33\nu-8)S^4-(16\nu^2-7\nu-2)S^3-(8\nu+7)S^2+(3\nu+8)S-2\Big),\\
M_2(z,\nu) = \frac{S}{(1-2S+2\nu^2S^3-\nu^2S^4)^2}\Big(&\nu^4S^7-3\nu^4S^6+\nu^3(\nu-2)S^5+\nu^2(4\nu+5)S^4-5\nu^2S^3\\
&-\nu(\nu+4)S^2+(2\nu+3)S-1\Big).
\end{aligned}$$

2.2 Counting bicoloured 2- and 3-connected maps

Let $B_1(z, \nu)$ and $B_2(z, \nu)$ be the series counting 2-connected bicoloured maps rooted at a monochromatic and bichromatic edge, respectively, not counting the single edge nor the loop map. A straightforward generalisation of (3) gives

$$M_1(z,\nu) = B_1(zM(z,\nu)^2,\nu) + 2\nu z M(z,\nu)^2, \tag{15}$$

$$M_2(z,\nu) = B_2(zM(z,\nu)^2,\nu) + zM(z,\nu)^2. \tag{16}$$

Those equations reflect the fact that the root edge in the 2-core $\mathsf{C}(\mathsf{m})$ of a map m is actually the root edge of m, hence m and its 2-core are either both monochromatic or both bichromatic. The substitutions are in terms of $M(z, \nu)$ only, since the maps placed at the corners of $\mathsf{C}(\mathsf{m})$ share no edge with $\mathsf{C}(\mathsf{m})$. By elimination from Equations (12), (13), $Q^{(1)} = 0$ and (15), we obtain the minimal polynomial of $B_1(z, \nu)$ of degree 13. The same is done from Equations (12), (13), $Q^{(2)} = 0$ and (16) to obtain the minimal polynomial of $B_2(z, \nu)$, also of degree 13. Both polynomials can be found in [1].

Finally, we determine the series $T_2(z, \nu)$ counting 3-connected bicoloured maps rooted at a bichromatic edge, and incidentally the series $T_1(z, \nu)$ counting those rooted at a monochromatic edge. Let $D_1 = B_1(z, \nu)/(z\nu)$ and $D_2 = B_2(z, \nu)/z$. Let $S_i = S_i(z, \nu)$ and $P_i = P_i(z, \nu)$ encode series and parallel compositions as for bicoloured maps, where the index i has the same meaning as before. Then a straightforward generalisation of (4) gives

$$D_1 = z\nu + S_1 + P_1 + T_1\left(D_2, \frac{D_1}{D_2}\right), \qquad D_2 = z + S_2 + P_2 + T_2\left(D_2, \frac{D_1}{D_2}\right). \tag{17}$$

The terms $T_i\left(D_2, \frac{D_1}{D_2}\right)$ encode the replacement by 2-connected maps of the non-root edges of the 3-cores $\mathsf{T}(\mathsf{m})$, where monochromatic (resp. bichromatic) edges of $\mathsf{T}(\mathsf{m})$ have to be replaced by maps rooted at a monochromatic (resp. bichromatic) edge. We also have the following relations:

$$\begin{aligned}
S_1 &= (D_1 - S_1)D_1 + (D_2 - S_2)D_2, & S_2 &= (D_2 - S_2)D_1 + (D_1 - S_1)D_2,\\
P_1 &= (D_1 - P_1)D_1 & P_2 &= (D_2 - P_2)D_2.
\end{aligned} \tag{18}$$

For the equation for S_1, remark that in order to obtain a series map rooted at a monochromatic edge one must compose in series two maps that are either both rooted at a monochromatic edge or both rooted at a bichromatic edge. While for S_2 one root must be monochromatic and the other bichromatic. The equations for P_1 and P_2 are simpler since both root edges in the parallel compositions must be of the same kind.

Eliminating from the system composed of (17), (18) and the minimal polynomials of $B_1(z,\nu)$ and $B_2(z,\nu)$, one obtains the minimal polynomial of $T_2 = T_2(z,\nu)$, which turns out to be of degree 26 in T_2 (see [1]).

3 Proof of Theorem 1

We have obtained $T_2(z,\nu)$ in the previous section and, as discussed in the introduction, the series of 3-connected bipartite maps is equal to $T_b(z) = T_2(z,0)$. The minimal polynomial of $T_b(z)$ is of the form

$$P(z,T_b) = \sum_{i=0}^{26} p_i(z) T_b^i,$$

where the $p_i(z)$'s are polynomials in z which happen to be of degree $85-i$. In particular, the leading coefficient of $P(z,T_b)$ is given by

$$p_{26}(z) = 4096(z+3)^{11}(z-1)^{22}(z+1)^{26}. \tag{19}$$

Because it is algebraic, $T_b(z)$ can be represented at $z=0$ as a generating function with non-negative coefficients and radius of convergence ρ, for some $\rho > 0$, corresponding to a branch of the curve $P(z,T_b)=0$ passing through the origin. We refer the reader to the detailed discussion in [11, Chapter VII.7] for this and related facts on algebraic generating functions used in the rest of the proof.

Next we find the value of ρ. Note first that $\rho \in (0,1)$ since the growth of the class is exponential. Second, by Pringsheim's theorem (see [11, Theorem IV.6]), ρ must be a singularity of $T_b(z)$. Since $T_b(z)$ is algebraic its singularities can be of two types: either they are *algebraic poles*, i.e. points for which the degree of $P(z,T_b)$ decreases (by cancelling its leading coefficient), or they are *branch points*, that are roots of the discriminant $d(z)$ of $P(z,T_b)$ with respect to T_b. It is clear that ρ has to be a branch point singularity as none of the roots of (19) are in the interval $(0,1)$. Let us now consider the polynomial $d(z)$. It has 242 different roots and 10 of them are in $(0,1)$. They are approximately equal to

$$0.007,\ 0.018,\ 0.022,\ 0.079,\ 0.151,\ 0.192,\ 0.415,\ 0.649,\ 0.676,\ 0.850.$$

To determine the right one we discard other roots using combinatorial arguments. Since the radius of convergence of the series of all 3-connected maps is $1/4$ we must have $\rho \geq 1/4$. In order to get an upper bound for ρ we need a subclass of 3-connected bipartite maps which is large enough. This is provided by the class of 3-connected bipartite cubic maps (called bicubic maps by Tutte in [24]). As shown in [14] the radius of convergence of this class when maps are counted by the number of edges is $5/8$. It follows that $\rho \in (1/4, 5/8)$ and this leaves 0.415 as the only feasible root, which must then be the radius of convergence of $T_b(z)$ at $\rho \approx 0.415$. The irreducible factor of $d(z)$ having ρ as a root is precisely the one claimed in Theorem 1.

In order to obtain an estimate for $[z^n]T_b(z)$ we apply the following *transfer theorem* [11, Chapter VI.3]. Assume that $f(z)$ has radius of convergence $r > 0$ and is analytic in an open domain at $z = r$ of the form

$$\Delta(\theta,R) = \{z\colon |z| < R, z \neq \psi, |\arg(z-r)| > \theta\} \qquad \text{for some } R > r \text{ and } 0 < \theta < \pi/2.$$

Further assume that when $z \sim r$ for $z \in \Delta(\theta,R)$, $f(z)$ has a singular expansion of the form

$$f(z) \sim c \cdot \left(1 - \frac{z}{r}\right)^{-\alpha} \qquad \text{for some } c > 0 \text{ and } \alpha \notin \{0,-1,-2,\dots\}.$$

Then the coefficients of $f(z)$ satisfy

$$[z^n]f(z) \sim \frac{c}{\Gamma(\alpha)} n^{\alpha-1} r^{-n} \qquad \text{as } n \to \infty.$$

We compute first the 242 different complex roots of $d(z)$ and check that ρ is the only one having modulus ρ. As $T_b(z)$ is algebraic, there exists $R > \rho$ and $0 < \theta < \pi/2$ for which its representation at $z=0$ admits an analytic continuation to a domain at $z=\rho$ of the form $\Delta(\theta,R)$, which can be computed from $P(z,T_b)$ using Newton's polygon algorithm. This gives a singular expansion of the Puiseux type:

$$T_b(z) = t_0 - t_2\left(1-\frac{z}{\rho}\right) + t_3\left(1-\frac{z}{\rho}\right)^{3/2} + O\left(1-\frac{z}{\rho}\right)^2 \qquad \text{for } z\sim\rho \text{ and } z\in\Delta(\phi,R), \tag{20}$$

where $t_0 = T_b(\rho) \approx 0.000104$ and $t_2 \approx 0.002637$. It has the same form as the series of planar maps we considered before. Hence we can apply the transfer theorem and obtain the claimed estimate, with $t_3 \approx 0.009747$ and $t = t_3/\Gamma(-3/2) = 3t_3/(4\sqrt{\pi}) \approx 0.00412$. This concludes the proof. □

Note. An alternative for obtaining an upper bound on ρ is to use the class of 3-connected quadrangulations. This class was first counted by the present authors in [16, 15] as an intermediate step in the enumeration of labelled 4-regular planar graphs. The radius of convergence of 3-connected quadrangulations counted by number of faces was determined in [15] as being $\tau = \frac{88-12\sqrt{21}}{135} \approx 0.24451$. Since in a quadrangulation the number of edges is twice the number of faces, the radius of convergence in terms of edges is $\sqrt{\tau} \approx 0.49445$. Hence $\rho < 1/2$. This is a better upper bound than 5/8 but the conclusion regarding the value of ρ remains the same.

4 Concluding remarks

Our result could open the way to the enumeration of (labelled) bipartite planar *graphs*, an interesting open problem. In fact, this was the original motivation for embarking on this project. For this, one needs the generating functions $T_1(z,\nu,x)$ and $T_2(z,\nu,x)$ of bicoloured 3-connected bipartite maps counted additionally according to the number of vertices (marked by the variable x). We have been able to determine the minimal polynomials of both $T_1(z,\nu,x)$ and $T_2(z,\nu,x)$ but they are truly enormous, each containing about 10^7 monomials in z, ν and x. In [19], one can find equations, similar to the ones in Section 2, relating $T_1(z,\nu,x)$ and $T_2(z,\nu,x)$ to the series of 2-connected and connected bicoloured planar graphs. Solving these equations (as is done in [19] for series-parallel graphs, a simpler class having no 3-connected graphs, see also [10] for a discussion comparing with triangle-free graphs), and setting $\nu=0$, one could in principle obtain the series of bipartite planar graphs and deduce a precise asymptotic estimate. But the task appears daunting given the huge size of the equations and the intricated analysis needed to complete the project in this context.

However, we can solve the simpler problem of counting bipartite *cubic* planar graphs. Following Tutte, such graphs (or maps) are called *bicubic*. We use the fact [24, Section 11] that the series of rooted bicubic maps counted by half the number of vertices is precisely equal to $A_b(z)$ given in Equation (5) (a well-known bijection explains this fact). If $G(z)$ is the series of 3-connected bicubic maps, then we have (see [24]):

$$A_b(z) = G\left(z(1+A_b(z))^3\right).$$

As shown in [14], the radius of convergence of $G(z)$ is $\tau = 125/512$. From here one can proceed as in [17, Section 3.3]. The series $D(x)$ of bicubic planar *networks* (essentially edge-rooted 2-connected cubic planar graphs) satisfies the equation

$$F(x,D(x)) = D(x) + \frac{x^2}{2}(1+D(x)) - \frac{1}{2}G\big(x^2(1+D(x))^3\big) = 0.$$

The radius of convergence σ of $D(x)$ is then obtained by solving the system

$$\sigma^2(1+D(\sigma))^3 = \tau, \qquad F(\sigma,D(\sigma)) = 0.$$

After checking some rather simple analytic conditions (as in [17, Theorem 3]), and observing that a connected bicubic graph is necessarily 2-connected, we obtain the following result:

Theorem 2. *The number B_n of labelled bicubic planar graphs with n vertices satisfies*

$$B_n \sim g\, n^{-7/2} \delta^n n! \qquad \text{as } n \to \infty,$$

where $g > 0$, $\delta = 1/\sigma \approx 2.035614$, and $\sigma \approx 0.49125$ is the smallest positive root of the irreducible polynomial

$$125z^6 + 750z^4 - 4332z^2 + 1000.$$

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