Rich Summer Intership 2022

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Abstract

We began our work this summer by inspecting the Veech groups of the covering spaces of various dilation surfaces. Using flatsurf, a SageMath based software package that works with flat surfaces, the Veech group of the covering spaces were generated. The process involves leveraging the UHP, which is naturally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, to construct Mobius transformations capable of taking distinct regions of the UHP to their neighbouring regions. If such a transformation exists, taking adjacent tiles to the opposite side of the region, then such a collection generates both the generating set of Veech elements and the entire fundamental domain. Our work further evolved into looking at the Veech group of a modified torus we called a *dimple surface* $D_{a,b}$. Using flatsurf to search for Veech groups of these dimple surfaces, we observed the program does not terminate, even after several thousand iterations. We suspect these Veech groups are infinitely generated and we are seeking to explain the reason why the generating sets are infinite.

1 Background

A flat surface is a collection of polygons who edges are glued via translation, dilation, rotation, or some combination thereof. Dilation surfaces are of particular interest to us, since Foster and a colleague Liu recently updated the SageMath based software package *flatsurf* to compute the Veech group of a dilation surface. [1]. This had only been done previously up to translation.

Observe that the group $\mathrm{GL}(2,\mathbb{R})$ acts naturally on the set of all surfaces. The Veech group of a surface S is $V(S) = \{A \in \mathrm{SL}(2,\mathbb{R}) : V(S) = S\}$. In other words, the Veech group $V(S)$, also written $\mathrm{PSL}(S)$, is the set of isometries in $\mathrm{SL}(2,\mathbb{R})$ that stabilize S.[1]

To each flat surface, there exists a unique (up to) tiling of the UHP which is described in Bowman's paper [2]. The matrix action from SL,GL? on each flat-surface is well-defined up to dilation (possibly 180 deg rotations – check) (say more(?)). Since SL,GL is naturally isomorphic to the UHP this makes an interesting connection between the two objects. Each cell of the UHP (in its tiling) is composed of a collection of matrices of SL,GL which share the same decomp/triangulation (gluing data/combinatorial info) when they act on our flat surface. This action is via matrix multiplication on the vertices of our flat surface. If there exists a Mobius transformation taking one tile to another which preserves this decomposition, then such a transformation represents an element of the Veech group. The program checks the gluing data at each vertex of our tile to check for equality. The maximally Veech distinct region is commonly referred to as the fundamental domain, in other words, a maximal region such that there exists no Veech symmetry taking one tile in this region to another preserving the gluing data/edge data. Since flatsurf scans outwards, it seeks a boundary of tiles of our fundamental domain, and if each tile on our boundary (how does it figure out it's on the boundary? Need to check) is able to be translated to the opposite side of our boundary (need to explain better), then such a collection taking the boundary to a region just adjacent the boundary makes up the generators of our Veech group.

This program was used upon ... blah blah blah

2 Our work

Our first task this summer was to design a dilation surface whose Veech group could be modeled using the flatsurf software, but one which was not already known. We experimented with several dilation

[1] Check with Hooper for an accurate definition – derivative matrix?

surfaces. The first set of candidates turned out to be too complicated (Figure 1), since the interior angles of the oblique rhombus faces are not rational multiples of π. Thus these quotient surfaces are not lattice surfaces and they are not suitable to be modeled by FlatSurf. The surfaces could be interesting for further study with respect to questions of genus and homotopy type.

Figure 1: Examples of early candidate surfaces - too complicated.

After a few more false starts, we decided on a surface we called a *dimple surface*, or $D_{a,b}$ with parameters a and b.

This is where we devoted most of our attention. A dimple surface is a modified torus with a square patch removed and a rectangle 'dimple' (of any aspect ratio) glued in its place. To construct a dimple surface, start with a square torus. Remove a smaller square patch and glue the four edges of the remaining hole to the four sides of a (not necessarily square) rectangle. This rectangle is the dimple and its corners are the four singularities of the surface, each with cone angle 2π.

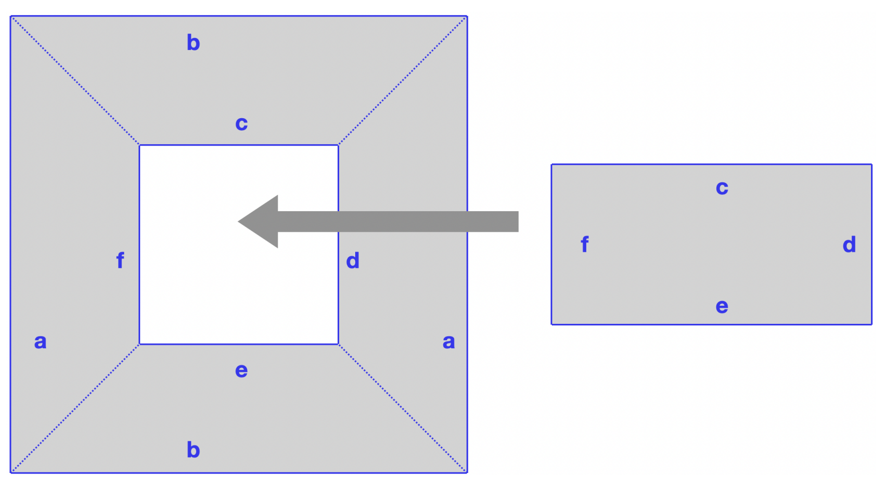

Figure 2: Construction of a dimple surface.

After some cut and paste transformations, we settled on the polygonal presentation that is similar to the standard presentation of a square torus (Example in figure 3). This similarity made it easier to analyze the surface using flatsurf. In this presentation, region 2 is always normalized to a unit square. The parameters a and b describe the the width and height of region 0 relative to region 2.

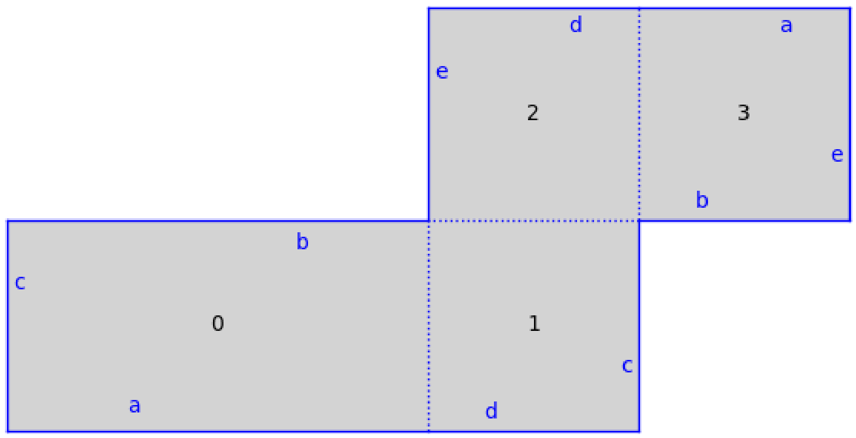

Figure 3: Polygonal presentation of the dimple surface $D_{2,1}$.

Using flatsurf and and the additional package by designed by Foster and Liu [1], we calculated Veech elements of $D_{a,b}$ for the parameters

$$(a, b) \in \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (5, 2), (1, 3), (2, 3)\}.$$

The flatsurf package is designed to take the surface S and return a generating set of the Veech group of S. The program terminates if the Veech group is a lattice. But the program did not terminate for any of the dimple surfaces, except for the trivial case $D_{1,1}$ which is simply thes unmodified torus. Thus these Veech groups are either (1) finitely generated but large enough that their generating sets are not quickly calculable (three hours running on a 2020 MacBook laptop); or, we suspect, (2) infinitely generated.

We were not able to fully characterize the generators of $\mathrm{PSL}(D_{a,b})$. However, we observed that a few patterns hold across each Veech group. First, for all $a, b \in \mathbb{Z}_{>0}$, each Veech group $\mathrm{PSL}(D_{a,b})$ contains the rotation by piπ. This symmetry follows from the fact that for all $a, b \in \mathbb{Z}_{>0}$, there exists a presentation of $D_{a,b}$ centered on region 0 with a rotational symmetry of order two.

Further, each surface $D_{a,b}$ contains a horizontal and vertical sheer whose matrices are (respectively)

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x = \mathrm{lcm}\Big(\frac{a+1}{b}, b+1\Big), \tag{1}$$

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \qquad y = \mathrm{lcm}\Big(\frac{1}{b+1}, \frac{ab}{b^2+a}\Big). \tag{2}$$

Bowman shows that whenever a flat surface decomposes into parallel, commensurable cylinders, its Veech group contains a parabolic element whose dimensions are determined by moduli of those cylinders[2]. In our example, $D_{a,b}$ comprises two horizontal cylinders and two vertical cylinders. Let C_1 be the horizontal cylinder of vectors crossing polygons 0 and 1 and let C_2 be the the horizontal cylinder of vectors crossing polygons 2 and 3. The modulus of a cylinder is its height divided by its circumference, thus the modulus of C_1 is $b/(a+1)$ and the modulus of C_2 is $1/(b+1)$. A Dehn twist on $D_{a,b}$ must respect the periodicity of both its cylinders, thus we have a horizontal parabolic element in $\mathrm{PSL}(D_{a,b})$ defined as in Equation (1) above. The derivation of the parabolic vertical sheer in Equation (2) follows similar reasoning.

Given the considerable amount of statistics that flatsurf can generate, we then decided to restrict our attention to the dimple surface $D_{2,1}$ (pictured in the figure 3 above). On a dilation surface, a dilation of a Veech matrix is also a Veech matrix, so we looked only at a normalized representative matrices with with determinant of one. We also considered the eigenvectors of each matrix, scaled to $(1, y)^T$.

Flatsurf generated approximately one hundred Veech matrices for $D_{a,b}$ in about a three hour run time, but did not terminate. This means it did not find a generating set during this time. Of the elements generated, we further narrowed our attention to those with rational eigenvectors $(1, y)$ where $y \in \mathbb{Q}$. Normalizing for rotations, these Veech elements and eigenvectors are

$$\begin{matrix} \begin{pmatrix} 17 & -32 \\ 8 & -15 \end{pmatrix}, & \begin{pmatrix} 23 & -11 \\ 44 & -21 \end{pmatrix}, & \begin{pmatrix} 23/2 & -63/2 \\ 7/2 & -19/2 \end{pmatrix}, & \begin{pmatrix} 121 & -180 \\ 80 & -119 \end{pmatrix}, & \begin{pmatrix} 31 & -20 \\ 45 & -29 \end{pmatrix}, \\ v = (1, 1/2)^T & v = (1, 2)^T & v = (1, 1/3)^T & v = (1, 2/3)^T & v = (1, 3/2)^T \end{matrix}$$

where each matrix has the single eigenvalue. We observed some patterns in these matrices and their eigenvectors, though we were not able to prove results about them. Under our observations of the statistics we saw, the matrix $M \in \mathrm{PSL}(D_{2,1})$ with corresponding eigenvector $(1, y)$ has $y \in \mathbb{Q}^\times$ if and only if it is of the form

$$M = I + z \begin{pmatrix} 1 & -\frac{1}{y} \\ y & -1 \end{pmatrix} \in \mathrm{PSL}(D_{2,1})$$

where $z \in \mathbb{Q}$ depends on y. Using a program written by Tupper [3] for the purpose of this project, we observed that, for each matrix above, the column vectors $(z, zy)^T$ and $(-\frac{z}{y}, -z)^T$ are related to the circumference of the cylinder in the direction of its corresponding eigenvector by a scaling factor of 1, 2, or 1/2. Since the value of z is completely determined by this eigenvector and the parameters of $D_{a,b}$, it should be possible to produce z, and hence M, simply from y, but we were not able to produce such a formula. Such an algorithm for generating Veech elements could be helpful either in proving $\mathrm{PSL}(D_{2,1})$ is infinitely generated or in enumerating all of its generators.

3 Future

We are left with a number of unanswered questions. Is $\mathrm{PSL}(D_{2,1})$ infinitely generated? Under what conditions is the Veech group of $D_{a,b}$ infinitely generated? How can the elements of these groups be characterized?

These questions have been answered for a number of translation surfaces, including the square torus, origamis, Teichmuller surfaces, [2], double regular polygons and certain triangles. Some progress has also been made classifying the Veech groups of dilation surfaces [5]. Completely characterizing the Veech group of dimple surface could help in the study of other dilation surfaces. Understanding the effect of a modification like *stretch*-and-paste procedure that we defined on the dimple surface could shed light on surfaces with similar dilation constructions.

Our inspection of the dimple surface did not dive deep into advanced or invented techniques. But authors such as Davis [4] and Duryev[5] have published results on translation and dilation surfaces using an array of different strategies. Adapting and implementing their methods could be useful in the study of the dimple surface.

References

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[5] E. Duryev, C. Fougeron, S. Ghazouani. Dilation surfaces and their Veech groups. Journal of Modern Dynamics, 2019, 14: 121-151. doi: 10.3934/jmd.2019005