Rich Summer Intership 2022

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1 Background

A *flat surface* is a collection of polygons whose edges are glued together via translation, dilation, rotation or some combination thereof. In particular, a *translation surface* is a type of flat surface whose side pairing rules identify parallel sides of opposite orientation and equal length. A *dilation surface* is a generalization of a translation surface where opposite identified edges are not required to be the same length (e.g. Figure 2). The group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on the set of all translation and dilation surfaces, sending polygons to polygons. The *Veech group* $V(S)$ of the surface S is the set of transformations in $\mathrm{SL}(2,\mathbb{R})$ that stabilize S. Thus $V(S) = \{A \in \mathrm{SL}(2,\mathbb{R}) : A(S) = S\}$ is the stabilizer of the surface S [4].

Dilation surfaces are of particular interest to us since flatsurf – a SageMath based software package that models flat surfaces – was recently updated to compute the Veech groups of dilation surfaces [6]. The program leverages the upper half-plane (UHP), defined $\mathcal{H} = \{x + iy \mid y > 0;\ x, y \in \mathbb{R}\}$, to compute the Veech group of both translation and dilation surfaces. This had previously only been done for translation surfaces. Our goal this summer was to learn about a novel dilation surface through experimentation with these new features of flatsurf.

2 Our work

Our first task was to design a dilation surface with a Veech group that could be modeled using the flatsurf software. Ideally, this surface would have a Veech group that was not already known. We experimented with several dilation surfaces. Our first set of candidates turned out to be too complicated (Figure 1). Each of these surfaces is built around an inner cube glued to an outer cube. A subset of faces from the inner cube and faces extending from the edges of the inner cube is chosen such that the quotient forms a surface. These were fun surfaces to construct: from a combinatorial standpoint, it was a challenge to enumerate all such surfaces [7]. Unfortunately, the interior angles of the oblique rhombus faces are not rational multiples of π, so these quotient surfaces are not lattice surfaces and they are not suitable to be modeled by flatsurf. The surfaces could be interesting for further study with respect to questions of genus and homotopy type.

Figure 1: Examples of early candidate surfaces - too complicated.

After a few more false starts, we decided on a surface we called a *dimple surface*, or $D_{a,b}$ with parameters a and b. This is where we devoted most of our attention. A dimple surface is a modified torus with a square patch removed and a rectangle 'dimple' (of any aspect ratio) glued in its place. To construct a dimple surface, start with a square torus. Remove a smaller square patch and glue the

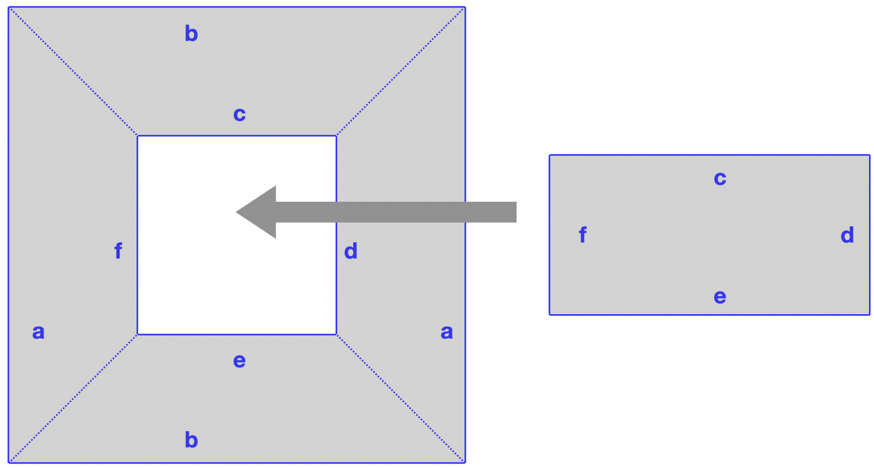

Figure 2: Construction of a dimple surface.

four edges of the resulting hole to the four sides of a (not necessarily square) rectangle. This rectangle is the dimple and its corners are the four singularities of the surface, each with cone angle 2π.

After some cut and paste transformations, we settled on the polygonal presentation that is similar to the standard presentation of a square torus (Figure 3). This presentation made it easier to analyze the surface using flatsurf. Region 2 is normalized to a unit square, while the parameters a and b describe the the width and height (respectively) of region 0 relative to the unit square.

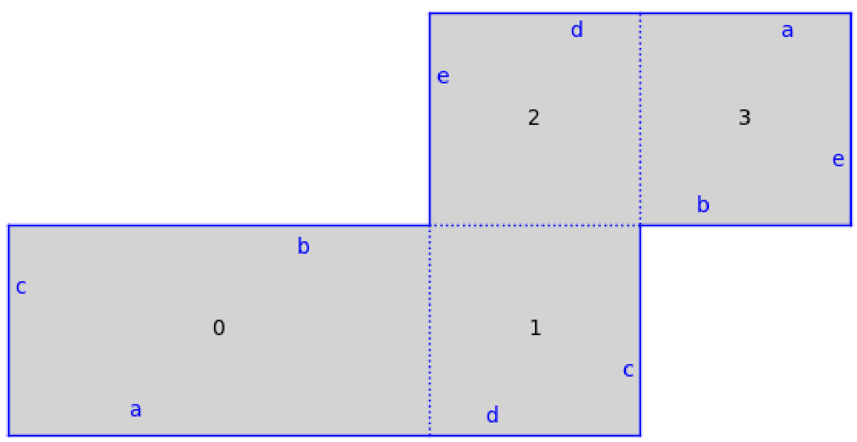

Figure 3: Polygonal presentation of the dimple surface $D_{2,1}$.

With the update designed by Foster and Liu [6], the flatsurf package can take a surface S and return a generating set of its Veech group $V(S)$. Using this software, we calculated Veech elements of $D_{a,b}$ for the parameters

$$(a, b) \in \Big\{(1,1),(2,1),(3,1),(4,1),(2,2),(3,2),(4,2),(5,2),(1,3),(2,3)\Big\}.$$

If a Veech group is a lattice, the program will eventually terminate. But the program did not terminate for any of these dimple surfaces, except for the trivial case $D_{1,1}$ which is simply the unmodified torus. Thus these Veech groups are either (1) finitely generated but large enough that their generating sets are not quickly calculable (three hours running on a 2020 MacBook Air laptop); or, we suspect, (2) infinitely generated.

We were not able to fully characterize the generators of $\mathrm{V}(D_{a,b})$, but we observed that a few patterns hold across each Veech group. First, for all $a, b \in \mathbb{Z}_{>0}$, each Veech group $\mathrm{V}(D_{a,b})$ contains the rotation by π. This symmetry follows from the fact that for all $a, b \in \mathbb{Z}_{>0}$, there exists a presentation of $D_{a,b}$ centered on region 0 with a rotational symmetry of order two.

Further, each surface $D_{a,b}$ above contains a horizontal and vertical sheer whose matrices are (respectively)

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad x = \operatorname{lcm}\Big(\frac{a+1}{b}, b+1\Big), \tag{1}$$

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \qquad y = \operatorname{lcm}\Big(\frac{1}{b+1}, \frac{ab}{b^2+a}\Big). \tag{2}$$

Bowman shows that whenever a flat surface decomposes into parallel, commensurable cylinders, its Veech group contains a parabolic element whose dimensions are determined by moduli of those cylinders [1]. In our example, $D_{a,b}$ comprises two horizontal cylinders and two vertical cylinders. Let C_1 be the horizontal cylinder crossing polygons 0 and 1 and let C_2 be the the horizontal cylinder crossing polygons 2 and 3. The modulus of a cylinder is its height divided by its circumference, thus the modulus of C_1 is $b/(a+1)$ and the modulus of C_2 is $1/(b+1)$. A Dehn twist on $D_{a,b}$ must respect the periodicity of both its cylinders, thus we have a horizontal parabolic element in $\mathrm{V}(D_{a,b})$ defined as in Equation (1) above. The derivation of the parabolic vertical sheer in Equation (2) follows similar reasoning.

Given the considerable amount of statistics that flatsurf can generate, we then decided to restrict our attention to the dimple surface $D_{2,1}$ (pictured in the figure 3 above). On a dilation surface, a dilation of a Veech matrix is also a Veech matrix, so we looked only at a normalized representative matrices with with determinant of one. We also considered the eigenvectors of each matrix, scaled to $(1, y)^T$.

Flatsurf generated approximately one hundred Veech matrices for $D_{2,1}$ in about a three hour run time, but did not terminate. This means it did not find a generating set during this time. We looked at these Veech elements from several different perspectives. The following is a graph of $V(D_{2,1})$ on the UHP produced by flatsurf [4]. To each flat surface, there exists a unique tiling of the UHP which is described in Bowman's paper [1]. Each cell of the UHP is composed of a collection of matrices in $\mathrm{SL}(2, \mathbb{R})$ which share the same decomposition and triangulation when they act on our flat surface.

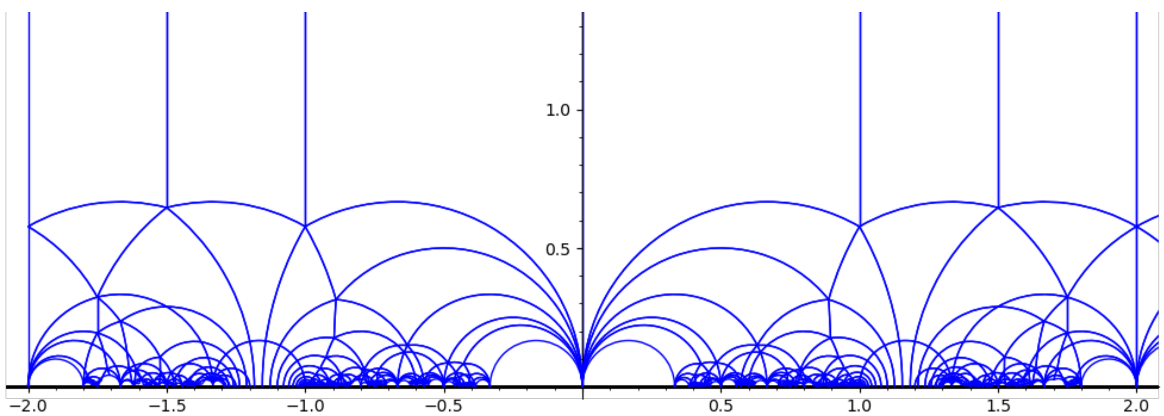

Figure 4: Graph of $V(D_{2,1})$ on the UHP.

Of the elements generated, we further narrowed our attention to those with rational eigenvectors $(1, y)$ where $y \in \mathbb{Q}$. Normalizing for rotations, these Veech elements and eigenvectors are

$$\left\{ \begin{pmatrix} 17 & -32 \\ 8 & -15 \end{pmatrix}, \begin{pmatrix} 23 & -11 \\ 44 & -21 \end{pmatrix}, \begin{pmatrix} 23/2 & -63/2 \\ 7/2 & -19/2 \end{pmatrix}, \begin{pmatrix} 121 & -180 \\ 80 & -119 \end{pmatrix}, \begin{pmatrix} 31 & -20 \\ 45 & -29 \end{pmatrix} \right\} \in V(D_{2,1})$$

$$v_\lambda = (1, 1/2)^T, \qquad (1, 2)^T, \qquad (1, 1/3)^T, \qquad (1, 2/3)^T, \qquad (1, 3/2)^T,$$

where each matrix has a one dimensional eigenspace with eigenvalue $\lambda = 1$. (In case one is tempted from these examples to conclude that every rational eigenvector will eventually be represented, it should be noted that $D_{2,1}$ does not decompose into cylinders in the direction of $(1, 3/4)^T \in \mathbb{Q}^2$. A fuller account of legal eigenvectors is included in the references below [9]). We observed some patterns in these matrices and their eigenvectors, though we were not able to prove results about them. Under our observations of the statistics we saw, the matrix $M \in V(D_{2,1})$ with corresponding eigenvector $(1, y)$ has $y \in \mathbb{Q}^\times$ if and only if it is of the form

$$M = I + z \begin{pmatrix} 1 & -\frac{1}{y} \\ y & -1 \end{pmatrix} \in V(D_{2,1})$$

where $z \in \mathbb{Q}$ depends on y. Using a program written by Tupper [8] for the purpose of this project, we observed that, for each matrix above, the column vectors $(z, zy)^T$ and $(-\frac{z}{y}, -z)^T$ are related to the circumference of the cylinder in the direction of its corresponding eigenvector by a scaling factor of 1, 2, or 1/2. Since the value of z is completely determined by this eigenvector and the parameters of $D_{2,1}$, it should be possible to produce z, and hence M, simply from y, but we were not able to produce such a formula. Such an algorithm for generating Veech elements could be helpful either in proving $V(D_{2,1})$ is infinitely generated or in enumerating all of its generators.

3 Future

We are left with a number of unanswered questions. Is $V(D_{2,1})$ infinitely generated? Under what conditions is the Veech group of $D_{a,b}$ infinitely generated? Does every Veech matrix $M \in V(D_{a,b})$ with rational eigenvector $(1, q)$ follow the pattern observed? How can the elements of these groups be characterized?

These questions have been answered for a number of translation surfaces, including the square torus, origamis, Teichmuller surfaces, double regular polygons and certain triangles. Some progress has also been made classifying the Veech groups of dilation surfaces [3]. Completely characterizing the Veech group of dimple surface could help in the study of other dilation surfaces. Understanding the effect of a modification like *stretch*-and-paste procedure that we defined on the dimple surface could shed light on surfaces with similar dilation constructions.

Our inspection of the dimple surface did not delve deep into advanced or invented techniques. But authors such as Davis [2] and Duryev [3] have published results on translation and dilation surfaces using an array of different strategies. Adapting and implementing their methods could be useful in further study of the dimple surface.

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