University Of Primorska, Graph Theory, 2025

Project Title: Classification of all isomorphism classes of highly irregular graphs of degree $1 \leq n \leq 11 + 12(***)$.

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Introduction

In this paper, we focus on the classification of **highly irregular graphs** of order up to 12 using computational methods. We begin by formally defining what constitutes a highly irregular graph, then prove two key properties that enable efficient computation. Next, we describe the algorithm used for classification, present the resulting graphs, and discuss the possible degree sequences that can arise in such graphs.

A graph is defined as **highly irregular** if it is simple (i.e., has no loops or multiple edges), connected (i.e., there exists a path between any pair of vertices), and satisfies the following condition: for every vertex $v \in V(G)$, no two neighbors of v have the same degree. Equivalently, denoting the neighborhood of v as $N(v)$:

$$\forall v \in V(G),\ \forall u, w \in N(v),\ u \neq w \implies \deg_G(u) \neq \deg_G(w).$$

We now present and prove two bounds on highly irregular graphs, which will later be used. These facts were previously stated and proven in [1] and [2].

Fact (i) [1] **:** A highly irregular graph H with maximum degree d has at least $2d$ vertices.

Proof. Let v be a vertex of maximal degree d in a highly irregular graph H. By the definition of a highly irregular graph, all neighbors of v must have *distinct degrees.* Let $\{u_1, u_2, \ldots, u_d\}$ be the set of neighbors of v.

Since the degrees of the u_i are distinct, their degrees must be d different integers. Moreover, each degree must lie in the range 1 to d because d is the maximal degree. Thus, without loss of generality, we may assume that:

$$\{\deg(u_1), \deg(u_2), \ldots, \deg(u_d)\} = \{1, 2, \ldots, d\}.$$

Now, consider a neighbor u_d of v with degree d. Since H is highly irregular, the neighbors of u_d must also all have distinct degrees.

Then, since H is highly irregular, u_d and v cannot have a common neighbor w since u_d and v would then be two neighbors of w with the same degree . Therefore, if H has n vertices, then $n \geq d + d = 2d$. □

Fact (ii) [2] **:** The size of a highly irregular graph of order n is at most $\frac{n(n+2)}{8}$.

Proof. By **Fact (i)**, every highly irregular graph of order n has maximum degree d at most $\frac{n}{2}$. Moreover, from the proof of **Fact (i)** we know that there are at least two vertices of the maximal degree with each having distinct neighbors - with these neighbors having degrees $\{1, 2, \ldots, d\}$.

Now assume that $n = 2 \cdot k$ for some $k \in \mathbf{Z}$, then the maximum size is attained when the maximum degree is k and hence every degree being present in the degree sequence exactly twice. Applying **HandShaking Lemma** we get:

$$2 \cdot m(H) = \sum_{v \in V(H)} deg(v) = 2 \cdot (1 + 2 + ... + k) \iff$$

$$m(H) = (1 + 2 + ... + k) \overset{\text{sum of } n \text{ integers}}{=} \frac{k \cdot (k+1)}{2} =$$

$$\frac{\frac{n}{2} \cdot (\frac{n}{2} + 1)}{2} = \frac{n \cdot (n+2)}{8} = \frac{n^2 + 2n}{8} \quad (1)$$

If $n = 2 \cdot k + 1$ for some $k \in \mathbf{Z}$, then the maximum is attained when each degree is present exactly twice except the k, which is present one extra time, that is, three times in total. Then, again applying **HandShaking Lemma** we get:

$$2 \cdot m(H) = 2(1 + 2 + ... + k) + k \iff$$

$$m(H) = \frac{k}{2} + \frac{k \cdot (k+1)}{2} = \frac{k \cdot (k+2)}{2}_{k = \frac{n-1}{2}}$$

$$\frac{(n-1)(n+3)}{8} = \frac{n^2 + 2n - 3}{8} \quad (2)$$

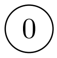
As a consequence, as (2) is less than (1), the upper bound for the size of highly irregular graph of order n is $\frac{n \cdot (n+2)}{8}$ □

Methodology

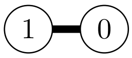
We begin this section by manually analyzing small cases and proceed with presenting an algorithm to computationally determine larger graphs.

Trivial Cases

$n = 1$:
The only graph is K_1 (isolated vertex), which is trivially highly irregular with degree sequence $\{0\}$.



$n = 2$:
The only connected graph is K_2 (Path of length 1). It is highly irregular because both vertices have single neighbor of degree 1. Degree sequence: $\{1, 1\}$.



$n = 3$:
There are two connected graphs:

- P_3 (Path of length 2): Not highly irregular because the central vertex has two neighbors of degree 1 both.
- K_3 (Complete graph on 3 vertices): Not highly irregular because every vertex has two neighbors of degree 2.

Thus, no highly irregular graphs exist for $n = 3$.

Cases for $n = 4, 5, ..., 11$

For larger graph orders we use a computational approach to identify all non-isomorphic highly irregular graphs up to order $n = 11$.

1. Checking Highly Irregularity

We define the following function to determine whether a graph G on n vertices is highly irregular by checking that each vertex has neighbors with distinct degrees:

Algorithm 1 IS_HIGHLY_IRREGULAR(G, n)

```
Let deg be the degree mapping of graph G
for each vertex v in G do
   Initialize empty set seen
   for each neighbor u of v do
      Let d ← deg[u]
      if d ∈ seen then
         return False
      end if
      Add d to seen
   end for
end for
return True
```

2. Enumerating Candidate Graphs

We generate all simple, connected graphs of order n that meet the necessary bounds derived in **Fact (i)** and **Fact (ii)**. Each candidate is tested using the previous function:

Algorithm 2 FIND_HIGHLY_IRREGULAR_GRAPHS(n)

```
Initialize empty list result
Set count ← 0
Compute max_edges ← ⌊n(n+2)/8⌋
Set max_degree ← ⌊n/2⌋
for each connected simple graph that attains the
max_edges and max_degree bounds G do
   if IS_HIGHLY_IRREGULAR(G, n) then
      Append G to result
      Increment count
   end if
end for
return result
```

Since we check all the graphs that admit to necessarily conditions derived previously, we exhaustively find every Highly Irregular graph of a given order.

Case for $n = 12$ (***)

In this section, we present the possible degree sequences.

Note: Due to computational constraints, the computations were not performed. Thus in this section we just outline a proposed procedure for computations.

Algorithm 3 FIND_DEGREE_SEQUENCES

```
Initialize empty list ds_list
for d ← 1 to 6 do
   Initialize empty list base
   for i ← 1 to d do
      Append i twice to base
   end for
   r ← 12 − 2 · d
   for each multiset comb of length r from {1, . . . , d}
   with replacement do
      ds ← base ∪ comb
      Append ds to ds_list
   end for
end for
for each degree sequence seq in ds_list do
   if IS_GRAPHICAL(seq) then
      FIND_HI_ORDER_GRAPHS_WITH_DS(seq)
   end if
end for
```

We generate all possible degree sequences and check which admit a Highly Irregular graph. IS_GRAPHICAL uses the Havel–Hakimi algorithm [3], and FIND_HI_ORDER_GRAPHS_WITH_DS exhaustively searches for valid realizations.

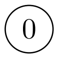
Results

Results, namely graphs and distinct degree sequences, are presented separately for cases when: $n = 1, 2, ..., 9$; when $n = 10$ and when $n = 11$ (as stated before, there are no reported results for $n = 12$).

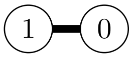
n = 1, 2, ..., 9

We already considered the cases when $n = 1, 2, 3$; but just for the sake of completeness they are included:

$n = 1$ **(1 In Total):**
Distinct Degree Sequence (DDS): $\{(0)\}$

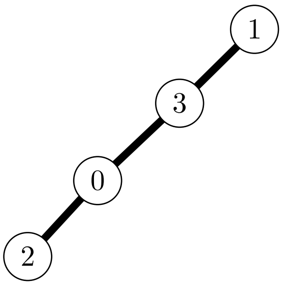


$n = 2$ **(1 In Total):**
DDS: $\{(1, 1)\}$



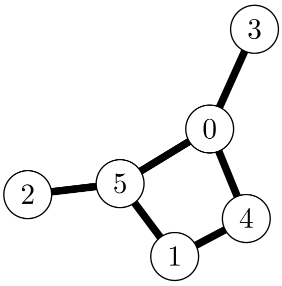
$n = 3$ **(0 In Total):**
None

$n = 4$ **(1 In Total):**
DDS: $\{(1, 1, 2, 2)\}$



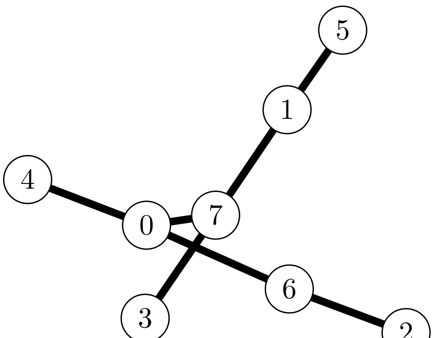
$n = 5$ **(0 In Total):**
None

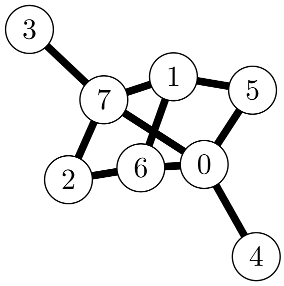
$n = 6$ **(1 In Total):**
DDS: $\{(1, 1, 2, 2, 3, 3)\}$

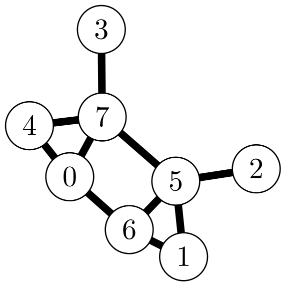


$n = 7$ **(0 In Total):**
None

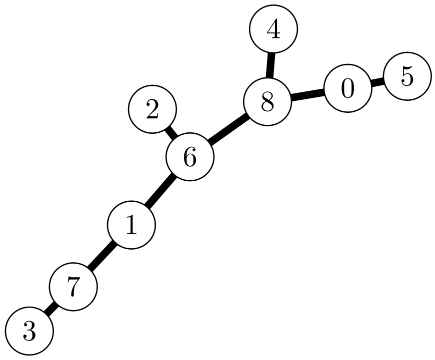
$n = 8$ **(3 In Total):**
DDS: (1, 1, 2, 2, 3, 3, 4, 4), (1, 1, 1, 1, 2, 2, 3, 3)

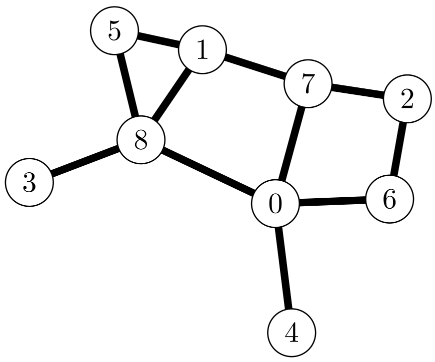


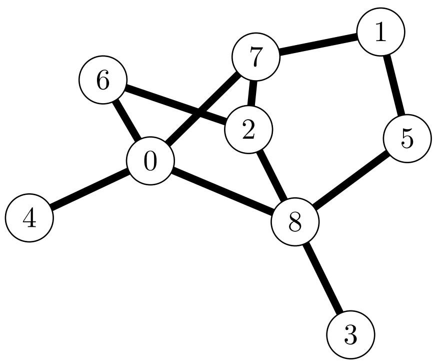




$n = 9$ **(3 In Total):**
DDS: (1, 1, 2, 2, 2, 3, 3, 4, 4), (1, 1, 1, 1, 2, 2, 2, 3, 3)

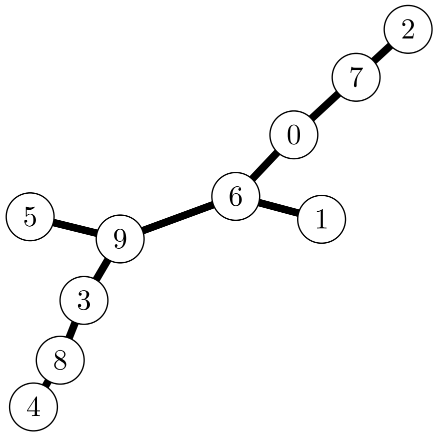






n = 10: (13 In Total)

DDS: $\{(1, 1, 1, 1, 2, 2, 3, 3, 4, 4),$
$(1, 1, 1, 1, 2, 2, 2, 2, 3, 3),$
$(1, 1, 2, 2, 2, 2, 3, 3, 4, 4),$
$(1, 1, 2, 2, 3, 3, 4, 4, 5, 5)\}$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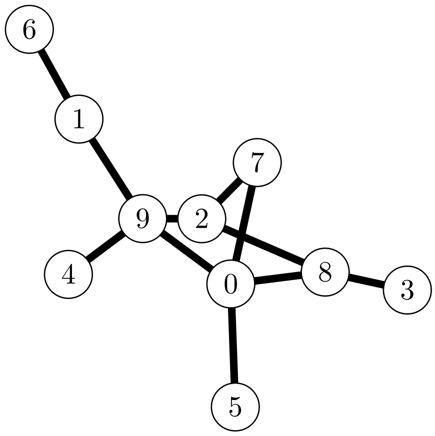
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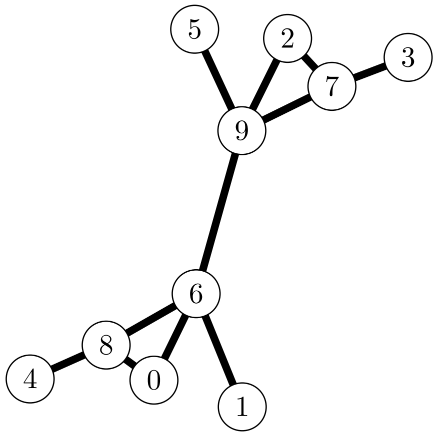
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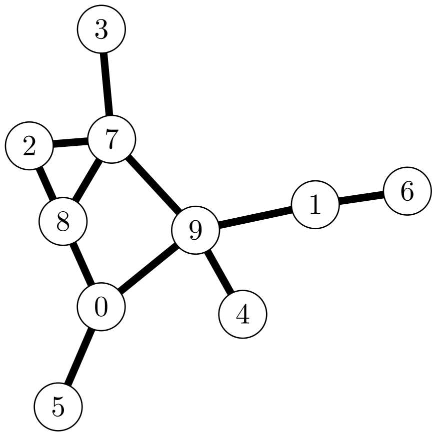
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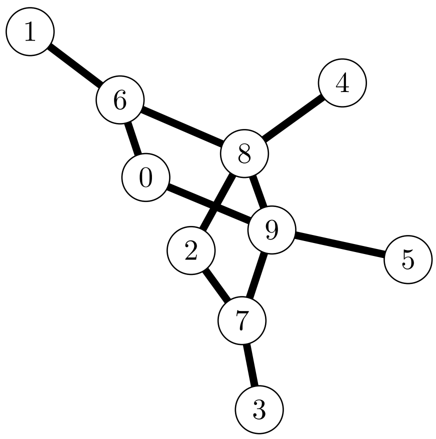
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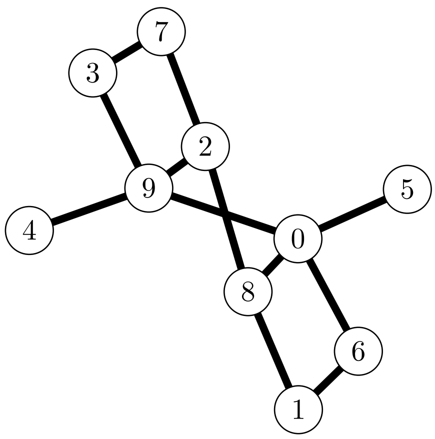
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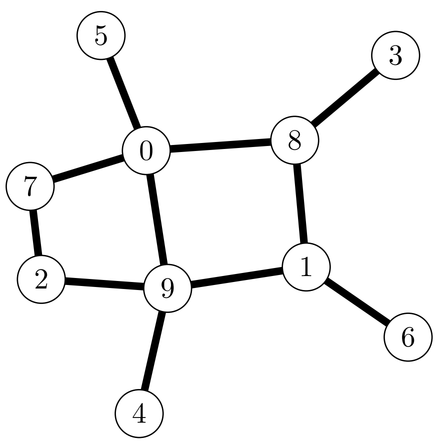
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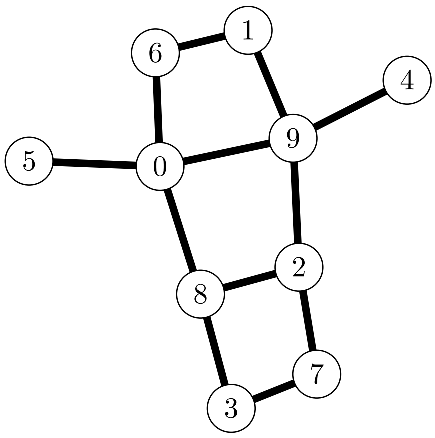
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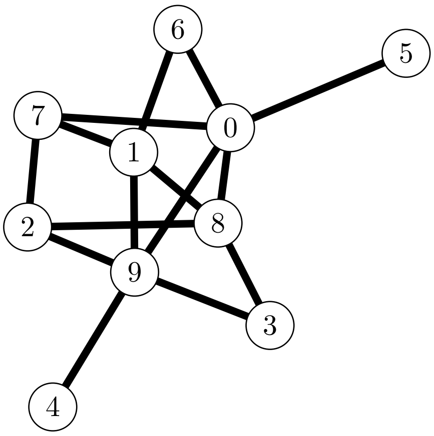

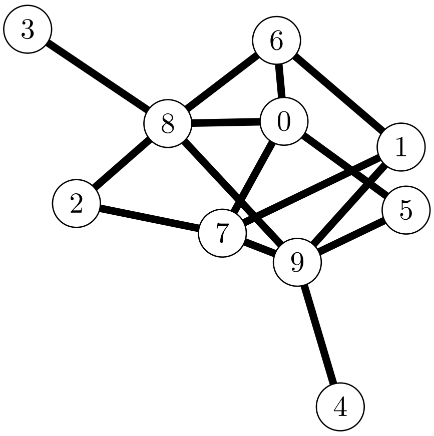

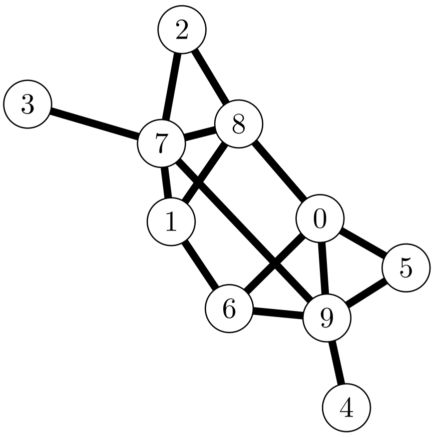

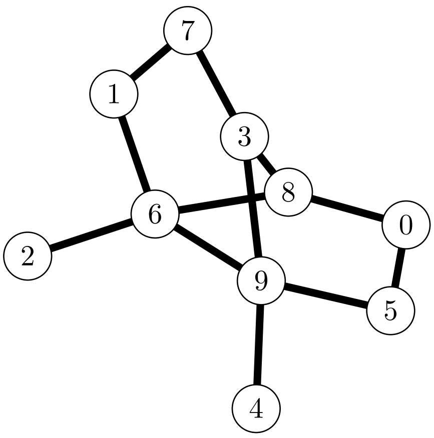

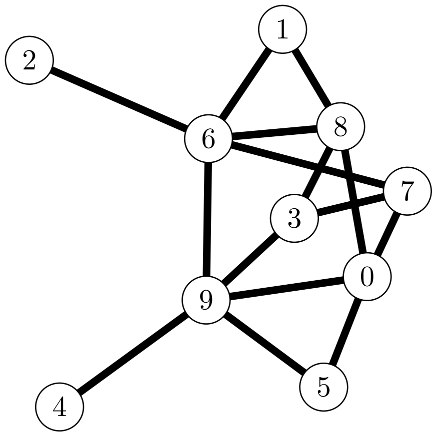

n = 11: (21 In Total)

DDS: {(1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5),
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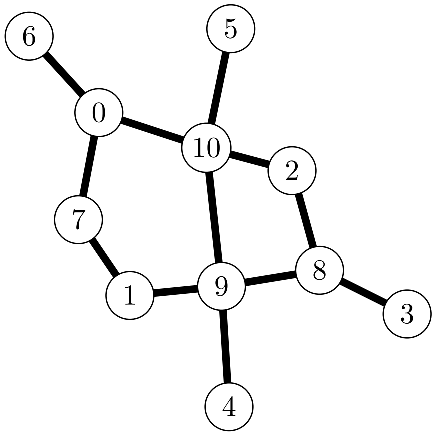

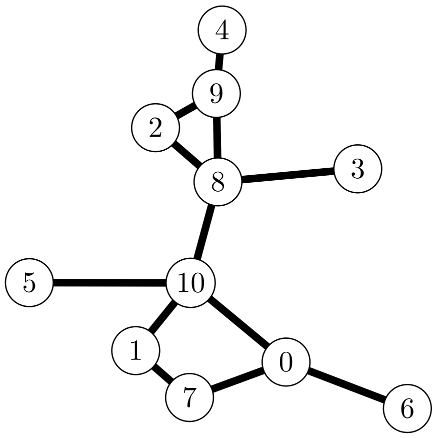

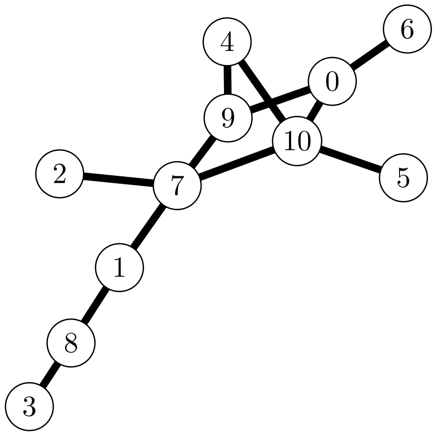

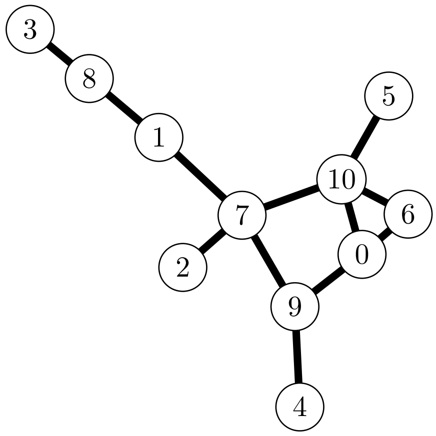
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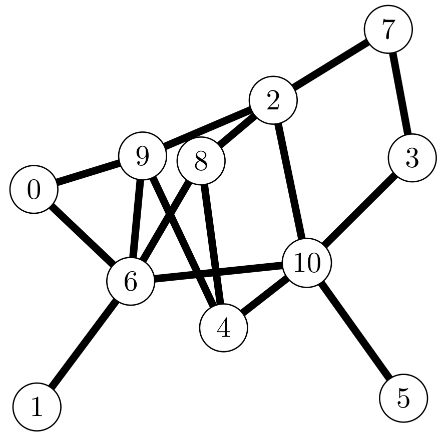
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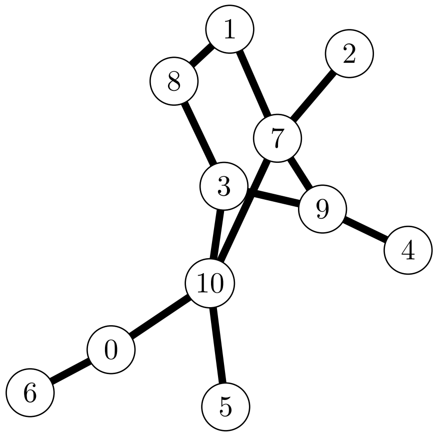
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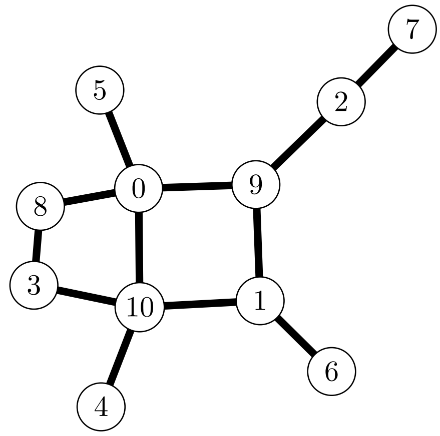
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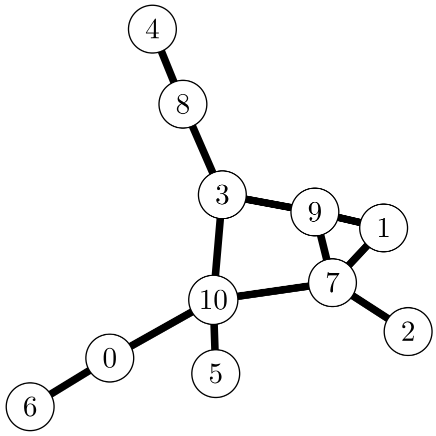
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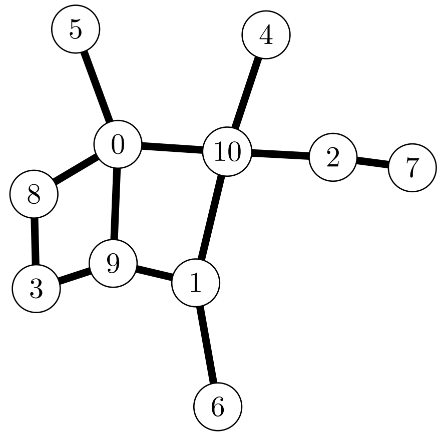
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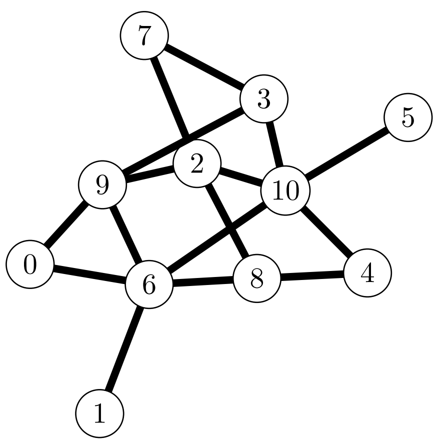
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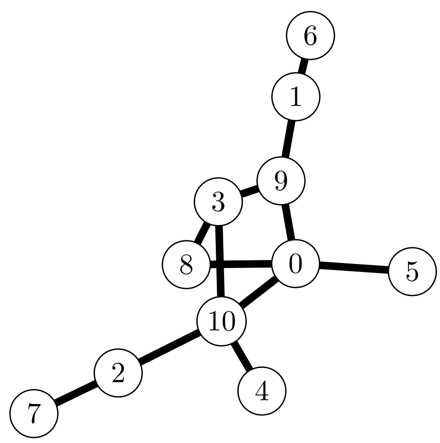
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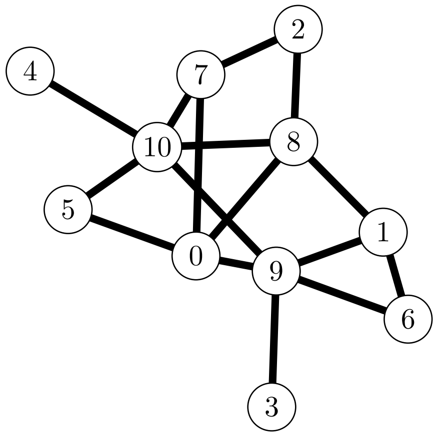
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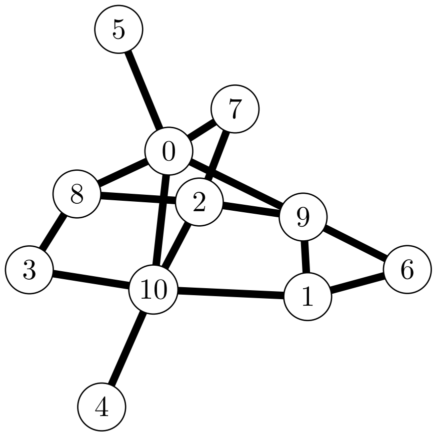
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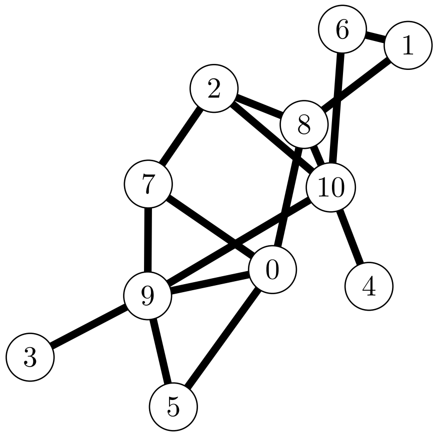
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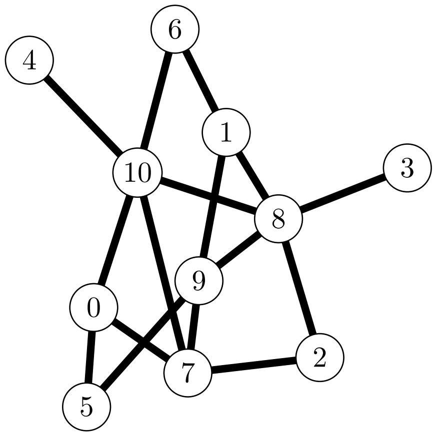
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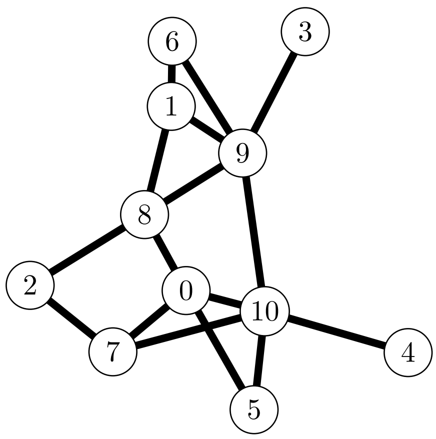
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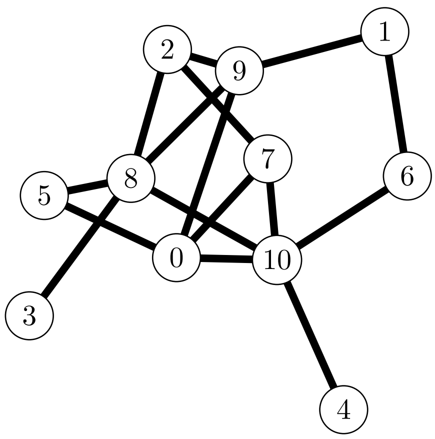
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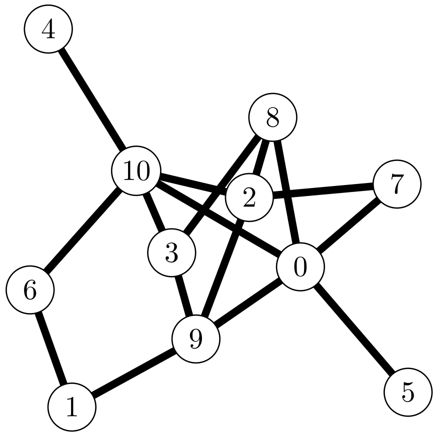
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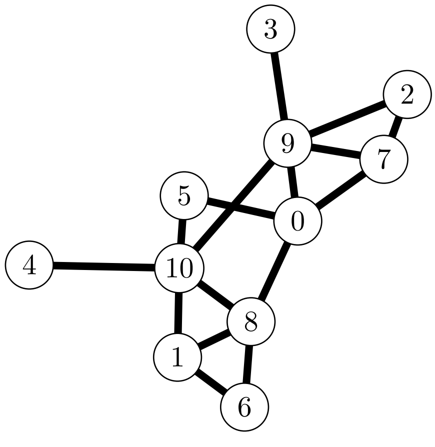
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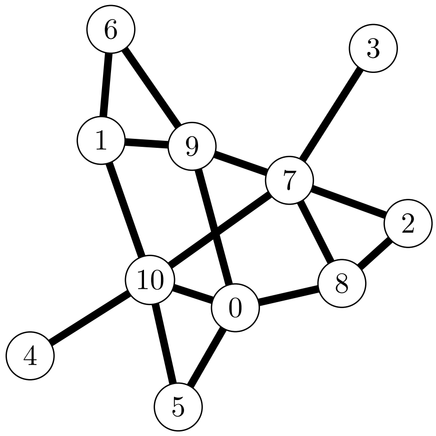

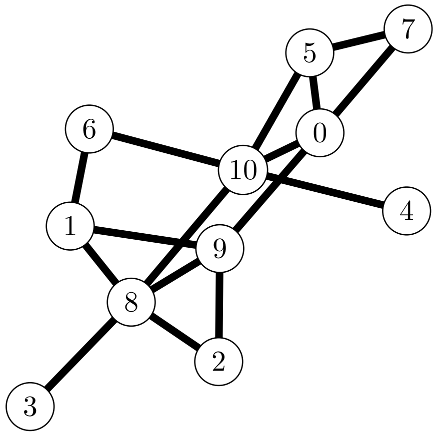

Conclusion

All in all, in this project we have found all the possible Highly-Irregular graphs of order up to 11, touched upon the possible degree sequences for Highly-Irregular graphs and for order 12 presented the algorithm to find for which degree sequences there is Highly Irregular graph realization. The future work is to:

- Perform the computations for $n = 12$, that were left out.
- Develop other Analytic facts that could simplify the computations.

The code used, written in Python Jupiter Notebook with SageMath Kernel can be found here: github

References

[1] Gary Chartrand, Paul Erdős, and Ortrud R. Oellermann. How to define an irregular graph. *Congressus Numerantium*, 56:109–118, 1987.

[2] Yousef Alavi, Gary Chartrand, F. R. K. Chung, Paul Erdős, R. L. Graham, and Ortrud R. Oellermann. Highly irregular graphs. *Journal of Graph Theory*, 11(3):367–380, 1987.

[3] S. L. Hakimi. On the realizability of a set of integers as degrees of the vertices of a graph. *SIAM Journal on Applied Mathematics*, 10(3):496–506, 1962.