Mathematical Modelling 2024/2025
Homework I

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(i)

Maximum of $f(x)$

To find the value of x^* at which the function $f(x) = xe^{-\sigma x}$ attains its maximum, we consider the derivative of f:

$$f'(x) = \frac{d}{dx}\left(xe^{-\sigma x}\right) = e^{-\sigma x} + x \cdot (-\sigma e^{-\sigma x}) = e^{-\sigma x}(1-\sigma x).$$

Since $e^{-\sigma x}$ is never zero, the critical point (that is, where $f'(x) = 0$) occurs when:

$$1 - \sigma x = 0 \quad \Longleftrightarrow \quad x = \frac{1}{\sigma}.$$

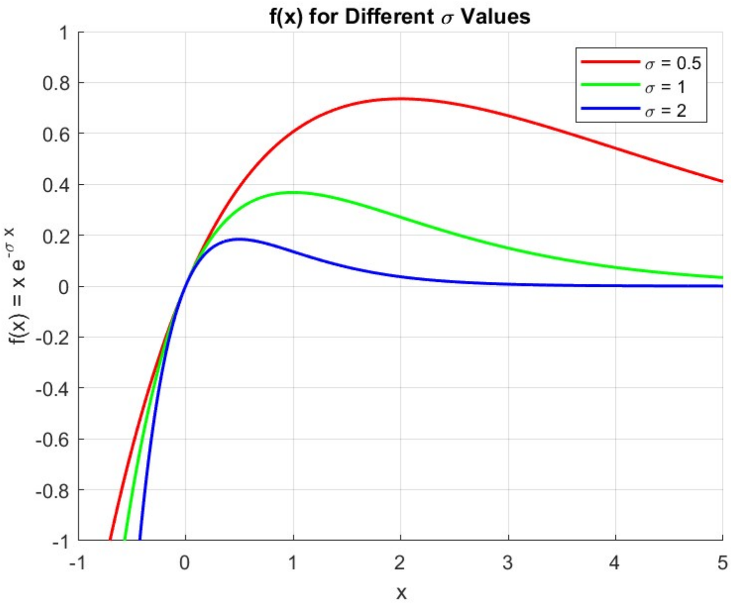
Note that $e^{-\sigma x}$ is always positive, so the sign of $f'(x)$ depends solely on $1-\sigma x$. Specifically:

- When $x < \frac{1}{\sigma}$, we have $1-\sigma x > 0 \Rightarrow f'(x) > 0$, so f is increasing.
- When $x > \frac{1}{\sigma}$, we have $1-\sigma x < 0 \Rightarrow f'(x) < 0$, so f is decreasing.

Therefore, f changes from increasing to decreasing at $x^* = \frac{1}{\sigma}$, and it is hence the point where f attains its maximum.

Influence of σ on $f(x)$ and its Interpretation

The parameter σ affects the predation rate $f(x)$ through the exponential term $e^{-\sigma x}$. As σ increases, the rate at which $e^{-\sigma x}$ decays becomes faster. Which means that $f(x)$ decreases more rapidly beyond the peak, and the maximum shifts to the left toward smaller x. This behavior is illustrated on the graph below, showing $f(x)$ for $\sigma = \frac{1}{2}, 1, 2$:



The parameter σ represents the strength of the prey's group defense. When the prey population is small, predators can capture them more easily, and the capture rate increases with prey density. This rate peaks at a critical density $x^* = \frac{1}{\sigma}$. Beyond which, further increase in prey density leads to a decline in the capture rate.

(ii)

Trivial Equilibria

Let us use the following notation:

$$x'(t) = x(1-x) - f(x)y = g_1$$
$$y'(t) = \beta f(x)y - \delta y = g_2$$

Since we are now interested in the equilibria, where the predators population is not present - we can assume that $y = 0$. The system then simplifies to:

$$x'(t) = x(1-x) = g_1',$$
$$y'(t) = 0 = g_2'.$$

To find the equilibria, we solve $g_1' = 0$, which yields:

$$x(1-x) = 0 \iff x = 0 \vee x = 1.$$

Therefore, the two equilibria when the predator population is absent are $E_1 = (0,0)$ and $E_2 = (1,0)$.

Stability of Trivial Equilibria

To analyze their stability, we first compute the Jacobian matrix for the original system:

$$J(x,y) = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-2x-f'(x)y & -f(x) \\ \beta f'(x)y & \beta f(x) - \delta \end{bmatrix}.$$

Then, plugging E_1 we get:

$$\begin{bmatrix} 1 & -f(0) \\ 0 & \beta f(0) - \delta \end{bmatrix} = \begin{bmatrix} 1 & -0 \cdot e^{-\sigma \cdot 0} \\ 0 & \beta \cdot (0 \cdot e^{-\sigma \cdot 0}) - \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\delta \end{bmatrix}.$$

The eigenvalues are therefore $\lambda_1 = 1, \lambda_2 = -\delta$. Since $\delta > 0$, we have one positive and one negative eigenvalues - which implies that E_1 is unstable and is a saddle.

Plugging E_2 we get:

$$\begin{bmatrix} 1-2 & -f(1) \\ 0 & \beta f(1) - \delta \end{bmatrix} = \begin{bmatrix} -1 & -1 \cdot e^{-\sigma} \\ 0 & \beta \cdot (1 \cdot e^{-\sigma}) - \delta \end{bmatrix} = \begin{bmatrix} -1 & -e^{-\sigma} \\ 0 & \beta \cdot e^{-\sigma} - \delta \end{bmatrix}$$

The eigenvalues are therefore $\lambda_1 = -1, \lambda_2 = \beta \cdot e^{-\sigma} - \delta$. With $\lambda_2 > 0 \iff \beta \cdot e^{-\sigma} - \delta > 0$, $\lambda_2 = 0 \iff \beta \cdot e^{-\sigma} - \delta = 0$ and $\lambda_2 < 0 \iff \beta \cdot e^{-\sigma} - \delta < 0$. Thus, we have two cases:

- If $\beta \cdot e^{-\sigma} - \delta > 0 \iff -\sigma > ln(\frac{\delta}{\beta}) \iff \sigma < ln(\beta) - ln(\delta)$, then E_2 is unstable and is a saddle.
- If $\beta \cdot e^{-\sigma} - \delta < 0 \iff \sigma > ln(\beta) - ln(\delta)$, then E_2 is stable and is a stable node.

(iii)

We now investigate the conditions under which the system has coexistence equilibria, that is equilibria with both $x > 0$ and $y > 0$.

Nullclines

Recall the notation:

$$x'(t) = x(1-x) - f(x)y = g_1$$
$$y'(t) = \beta f(x)y - \delta y = g_2$$

- From $g_1 = 0$, assuming $x > 0$ and $y > 0$:

$$x(1-x) - f(x)y = 0 \iff y = \frac{x(1-x)}{f(x)} = \frac{x(1-x)}{xe^{-\sigma x}} = \frac{1-x}{e^{-\sigma x}} = (1-x)e^{\sigma x}.$$

- From $g_2 = 0$, assuming $y > 0$:

$$\beta f(x)y - \delta y = 0 \iff \beta f(x) = \delta \iff f(x) = \frac{\delta}{\beta} \iff xe^{-\sigma x} = \frac{\delta}{\beta}.$$

Existence of Coexistence Equilibria

Let us analyze the equation:

$$xe^{-\sigma x} = \frac{\delta}{\beta}.$$

Note that the function $f(x) = xe^{-\sigma x}$, as we showed in **(i)**, attains a unique maximum at $x^* = \frac{1}{\sigma}$ and thus has a maximum value $f_{\max} = f\left(\frac{1}{\sigma}\right) = \frac{1}{\sigma}e^{-1} = \frac{1}{e\sigma}$.

Consequently, we can distinguish three cases based on the relative position of $\frac{\delta}{\beta}$ and the value of $f_{\max}$:

- **Case 1:** If $\frac{\delta}{\beta} > \frac{1}{e\sigma} \iff \sigma > \frac{\beta}{\delta e}$, then the equation $xe^{-\sigma x} = \frac{\delta}{\beta}$ has *no solution*. Therefore, no coexistence equilibrium exists.

- **Case 2:** If $\frac{\delta}{\beta} = \frac{1}{e\sigma} \iff \sigma = \frac{\beta}{\delta e}$, then the equation has *exactly one solution* at $x = \frac{1}{\sigma}$, and we obtain a unique coexistence equilibrium $E_3 = (\frac{1}{\sigma}, (1-\frac{1}{\sigma})e^{\sigma\frac{1}{\sigma}}) = (1-\frac{1}{\sigma})e)$.

- **Case 3:** If $\frac{\delta}{\beta} < \frac{1}{e\sigma} \iff \sigma < \frac{\beta}{\delta e}$, then the equation has *two distinct solutions*, due to the bell-shaped graph of $f(x)$ around $f(\frac{1}{\sigma})$ - one on each side of the peak at $x = \frac{1}{\sigma}$. Correspondingly, there are two coexistence equilibria; say $E_4 = (x_1, (1-x_1)e^{\sigma x_1})$ with $x_1 \in (0, \frac{1}{\sigma})$ and $E_5 = (x_2, (1-x_2)e^{\sigma x_2})$ with $x_2 \in (\frac{1}{\sigma}, \infty)$.

Stability of Coexistence Equilibria

Plugging E_3 into the already computed Jacobian we get:

$$\begin{bmatrix} 1-\frac{2}{\sigma}-f'(\frac{1}{\sigma})(1-\frac{1}{\sigma})e & -f(\frac{1}{\sigma}) \\ \beta f'(\frac{1}{\sigma})(1-\frac{1}{\sigma})e & \beta f(\frac{1}{\sigma})-\delta \end{bmatrix} \overset{f'(\frac{1}{\sigma})=0 \quad \wedge \quad f(\frac{1}{\sigma})=\frac{1}{e\sigma}}{=} \begin{bmatrix} 1-\frac{2}{\sigma} & -\frac{1}{e\sigma} \\ 0 & \frac{\beta}{e\sigma}-\delta \end{bmatrix}$$

The eigenvalues are therefore $\lambda_1 = 1-\frac{2}{\sigma}$, $\lambda_2 = \frac{\beta}{e\sigma}-\delta$. Since for this equilibria to exist, $\sigma = \frac{\beta}{\delta e}$, we have that the second eigenvalue is zero, and we thus cannot deduce the stability of this equilibria.

Plugging E_4 into Jacobian we get:

$$\begin{bmatrix} 1-2x_1-f'(x_1)y_1 & -f(x_1) \\ \beta f'(x_1)y_1 & \beta f(x_1)-\delta \end{bmatrix} \overset{f(x_1)=\frac{\delta}{\beta}}{=} \begin{bmatrix} 1-2x_1-f'(x_1)y_1 & -f(x_1) \\ \beta f'(x_1)y_1 & 0 \end{bmatrix}$$

Determinant is then, $f(x_1)\cdot\beta f'(x_1)y_1$, which is positive as x_1 is on the increasing part of f. Trace is:

$$1-2x_1-f'(x_1)y_1 = 1-2x_1-(1-\sigma x_1)e^{-\sigma x_1}\cdot(1-x_1)e^{\sigma x_1} = 1-2x_1-(1-\sigma x_1)\cdot(1-x_1) =$$

$$1-2x_1-(1-x_1-\sigma x_1+\sigma x_1^2) = -x_1+\sigma x_1-\sigma x_1^2 = x_1(-1+\sigma-\sigma x_1)$$

Which is positive if $\sigma(1-x_1)>1 \implies E_4$ is unstable, and is negative if $\sigma(1-x_1)<1 \implies E_4$ is LAS.

Plugging E_5 into Jacobian we get:

$$\begin{bmatrix} 1-2x_2-f'(x_2)y_2 & -f(x_2) \\ \beta f'(x_2)y_2 & \beta f(x_2)-\delta \end{bmatrix} \overset{f(x_2)=\frac{\delta}{\beta}}{=} \begin{bmatrix} 1-2x_2-f'(x_2)y_2 & -f(x_2) \\ \beta f'(x_2)y_2 & 0 \end{bmatrix}$$

Determinant is then, $f(x_2)\cdot\beta f'(x_2)y_2$, which is positive as x_2 is on the increasing part of f. As a consequence, E_5 is a saddle.

Phase Portraits

Case 1: No coexistence equilibria. This occurs when

$$\sigma > \frac{\beta}{\delta e}.$$

We now show that under this condition, the equilibrium point E_2 must be a *stable node.* Assume, for contradiction, that E_2 is a saddle point. Then it must satisfy the condition:

$$\sigma < \ln(\beta)-\ln(\delta) = \ln\left(\frac{\beta}{\delta}\right).$$

Now combining the conditions and letting $x = \frac{\beta}{\delta}$, we have:

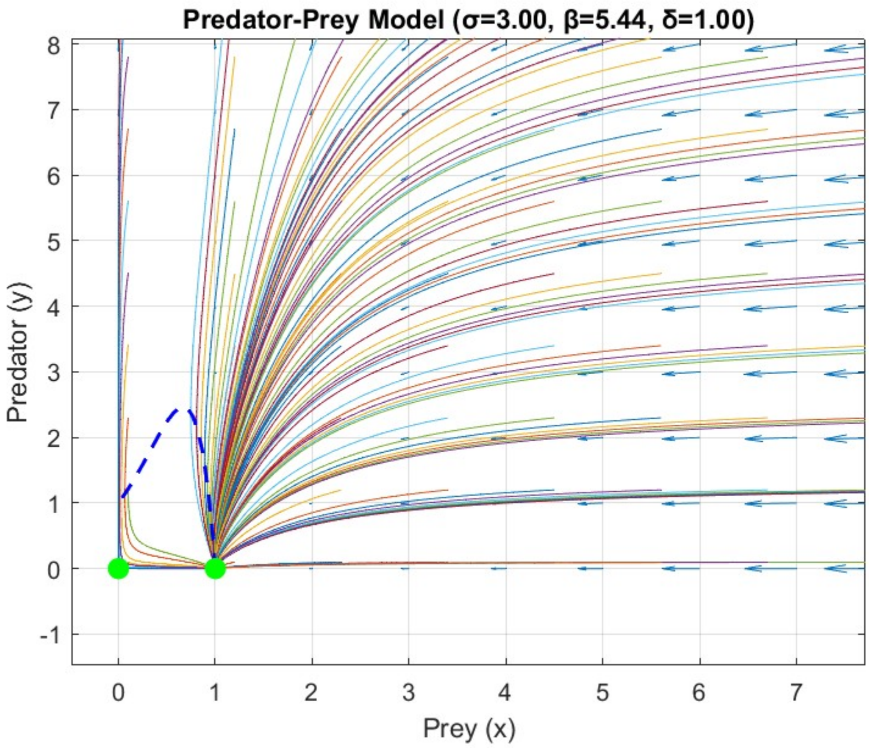
$$\sigma > \frac{x}{e} \quad \text{and} \quad \sigma < \ln(x).$$

To satisfy both, it must be that:

$$\frac{x}{e} < \ln(x).$$

However, the function $f(x) = \ln(x) - \frac{x}{e}$ is non-positive everywhere. which implies that this inequality is never satisfied.

Thus, it is impossible for both inequalities to hold simultaneously. And E_2 is hence a stable node in this case. So, the phase portrait is then:



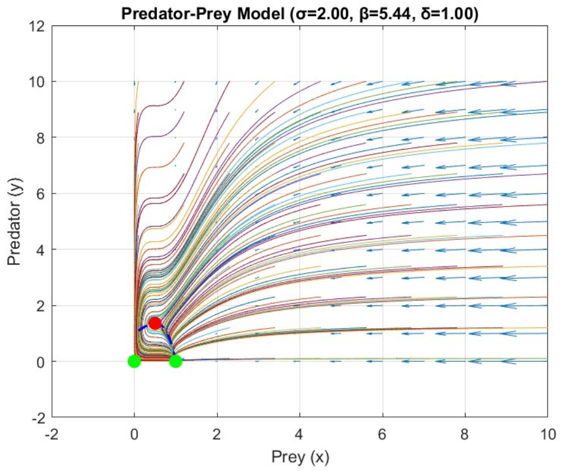
E_1 - saddle, E_2 - stable node

Hence, in the long run the system converges to E_2, if we start anywhere except E_1.

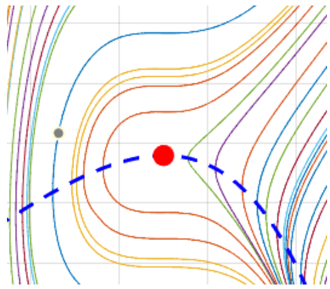
Case 2: One coexistence equilibrium. This occurs when

$$\sigma = \frac{\beta}{\delta e}.$$

By the same reasoning, we have that E_2 must be a *stable node*. The phase portrait is then:



E_1 - saddle, E_2 - stable node



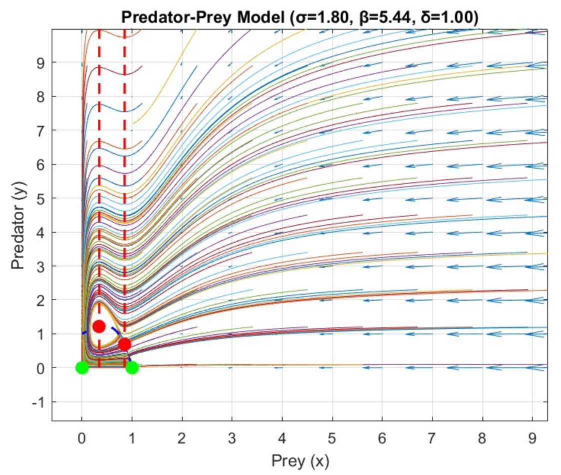
Zoom on E_3

Hence, in the long run the system converges to E_2, if we start anywhere except E_1, E_3. For the equilibrium E_3 as we stated before, we cannot deduce the stability.

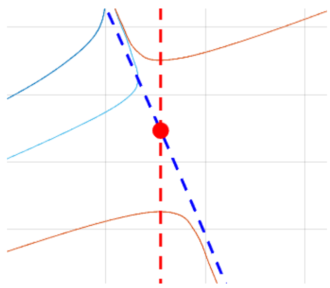
Case 3: Two coexistence equilibria. This occurs when

$$\sigma < \frac{\beta}{\delta e}.$$

SubCase 3.1: E_2 **- stable node.**



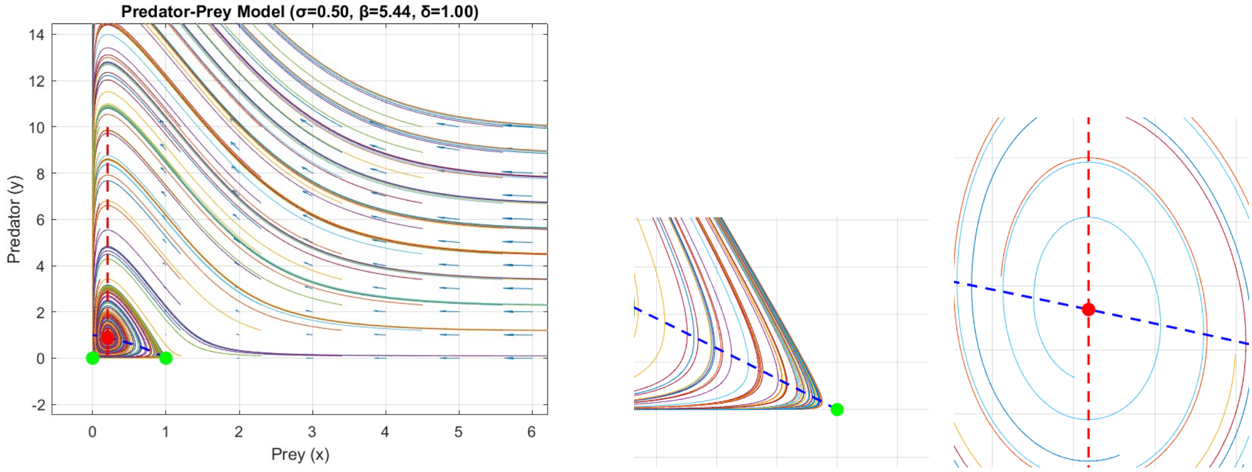
E_1 - saddle, E_2 - stable node, E_4 - center, E_5 - saddle



Zoom on E_5

Hence, in the long run the system converges to E_2 or stays in the orbit of E_4, if we start anywhere except E_1, E_5. The equilibrium E_4 is a center in this case.

SubCase 3.2: E_2 **- saddle,** E_4 **- LAS.**



E_1 - saddle, E_2 - saddle, E_4 - stable focus   Zoom on E_2 Zoom on E_4

Hence, in the long run the system converges to E_4, if we start anywhere except E_1, E_2. The equilibrium E_4 is a stable focus, and E_5 is not Biologically Meaningful.