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EUROPEAN CENTRE FOR RESEARCH AND ADVANCED TRAINING IN **SCIENTIFIC COMPUTING**

Multilevel Monte Carlo estimation of covariances in the context of open-channel flow simulations

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Single level MC estimators

Let X, $Y = f(X)$ and $Z = g(X)$ be random variables.

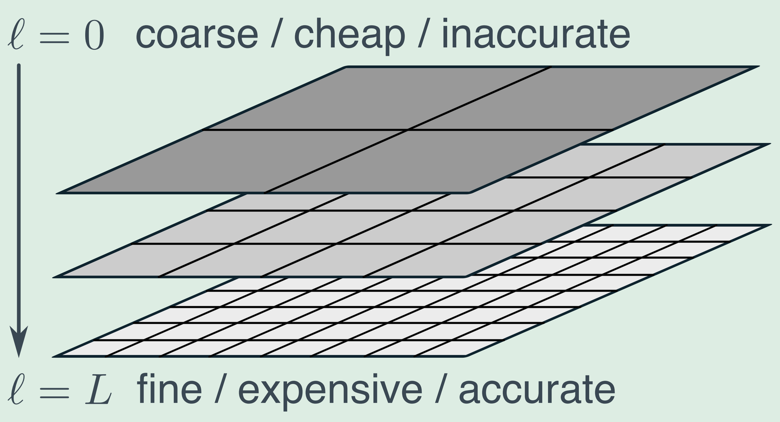
$$\mathbb{E}[Y] \approx E_M[Y] \equiv M^{-1} \sum_{i=1}^{M} Y^{(i)},$$

$$\mathbb{C}[Y,Z] \approx C_M[Y,Z] \equiv \frac{M}{M-1} E_M[(Y - E_M[Y])(Z - E_M[Z])].$$

- Evaluation of f and g may be expensive (PDE solve).
- Large M may be required: RMSE = $\mathcal{O}(1/\sqrt{M})$.

Multilevel MC (MLMC) estimators

$$Y_\ell = f_\ell(X),\ Z_\ell = g_\ell(X).$$



$$E_L^{\mathrm{ML}}[Y] = E_{M_0}[Y_0] + \sum_{\ell=1}^{L} E_{M_\ell}[Y_\ell] - E_{M_\ell}[Y_{\ell-1}],$$

$$C_L^{\mathrm{ML}}[Y,Z] = C_{M_0}[Y_0, Z_0] + \sum_{\ell=1}^{L} C_{M_\ell}[Y_\ell, Z_\ell] - C_{M_\ell}[Y_{\ell-1}, Z_{\ell-1}].$$

- Coarse (cheap) estimator + corrections.
- Hope: many cheap, few expensive samples.

Variance-bias decomposition of mean square error (MSE)

$$\mathrm{MSE} \equiv \mathbb{E}\left[\left(C_L^{\mathrm{ML}}[Y,Z] - \mathbb{C}[Y,Z]\right)^2\right] = \underbrace{\mathbb{V}(C_L^{\mathrm{ML}}[Y,Z])}_{\text{sampling error}} + \underbrace{\left|\mathbb{C}[Y_L, Z_L] - \mathbb{C}[Y,Z]\right|^2}_{\text{discretization bias}}.$$

$$\mathbb{V}(C_L^{\mathrm{ML}}[Y,Z]) \le \sum_{\ell \le L} \frac{1}{M_\ell - 1} \underbrace{\left[\frac{1}{2}\sqrt{\mathbb{M}^4[S_\ell^-(Y)]\,\mathbb{M}^4[S_\ell^+(Y)]} + \frac{1}{2}\sqrt{\mathbb{M}^4[S_\ell^-(Z)]\,\mathbb{M}^4[S_\ell^+(Z)]}\right]}_{\equiv \mathcal{V}_\ell}$$

where $\mathbb{M}^4[A] \equiv \mathbb{E}[(A - \mathbb{E}[A])^4]$ and $S_\ell^{\pm}(A) \equiv A_\ell \pm A_{\ell-1}$.

Optimal number of samples to achieve a sampling accuracy (variance) of ε^2:

$$M_\ell = \left\lfloor \varepsilon^{-2}\sqrt{\mathcal{V}_\ell/\mathcal{C}_\ell}\left(\sum_{\ell' \le L}\sqrt{\mathcal{V}_{\ell'}\mathcal{C}_{\ell'}}\right)\right\rfloor + 2, \qquad \mathcal{C}_\ell \equiv \text{cost of 1 run on level } \ell.$$

Adaptive MLMC algorithm

Choose $L_{\min}$, M_{init}, and target MSE accuracy ε^2

Start with $L = L_{\min}$ levels and initial target of $M_\ell = M_{\mathrm{init}}$ samples

while *extra samples need to be evaluated* **do**

  evaluate extra samples on each level

  compute/update estimates of the sampling error

  update optimal number of samples M_ℓ

  if *discretization bias is too large* **then**

    add a new (finer) level: $L \leftarrow L + 1$

  end if

end while

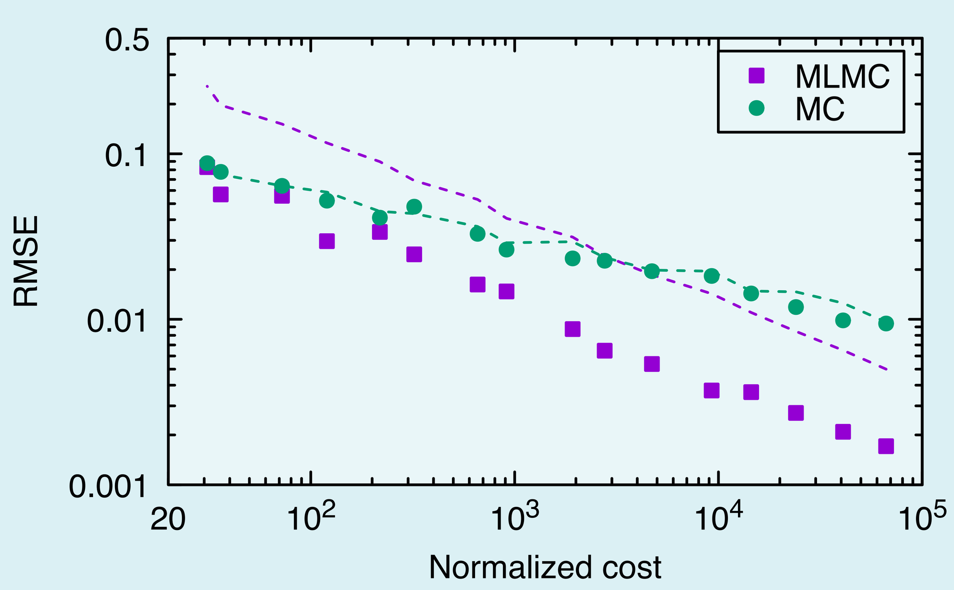
Evolution equation

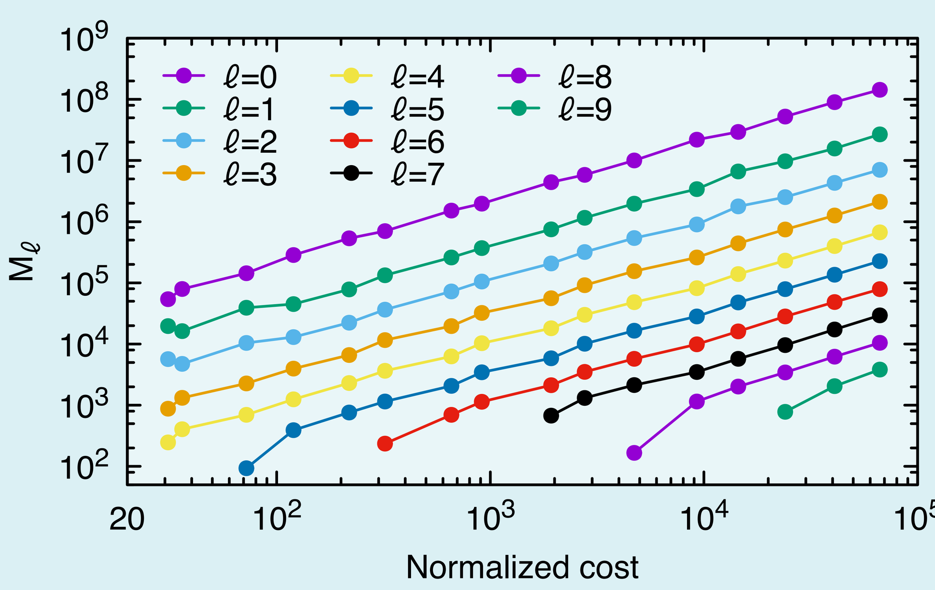
Initial value problem with random coefficient:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t,\omega) = a(\omega)u(t,\omega), & t \in (0,1], \quad a \sim \mathcal{N}(\mu = 1; \sigma = 0.5), \\ u(0,\cdot) = U_0, & \end{cases}$$

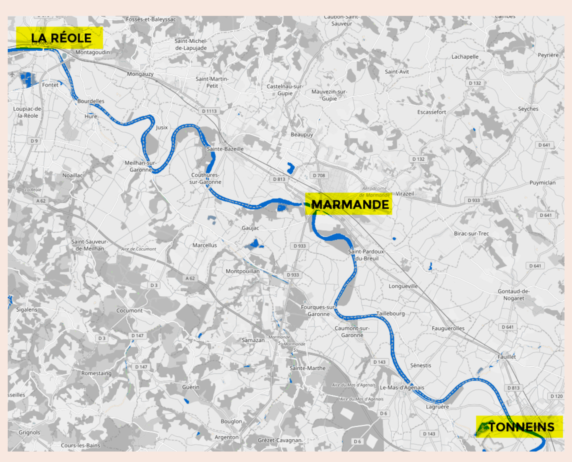
solved using a **backward Euler** method using $16 \times 2^\ell$ time-steps on level ℓ.

Quantity of interest (QoI): $\mathbb{C}[u(t = 0.5, \cdot), u(t = 1, \cdot)]$.





Open-channel flow: the Garonne river



Shallow-water equations:

$$\begin{cases} \partial_t A(h) + \partial_s Q = 0 \\ \partial_t Q + \partial_s\left(Q(h)^2/A(h)\right) \\ \qquad + gA(h)\left[\partial_s h - (S_0 - S_f)\right] = 0, \end{cases}$$

solved using MASCARET (EDF R&D), using $25 \times 2^\ell$ grid points.

CPU time $\mathcal{C}_\ell \sim 2^{0.98\ell}$:

$\mathcal{C}_0 \approx 0.2$s $\mathcal{C}_4 \approx 2.3$s

$\mathcal{C}_1 \approx 0.3$s $\mathcal{C}_5 \approx 4.5$s

$\mathcal{C}_2 \approx 0.6$s $\mathcal{C}_6 \approx 8.9$s

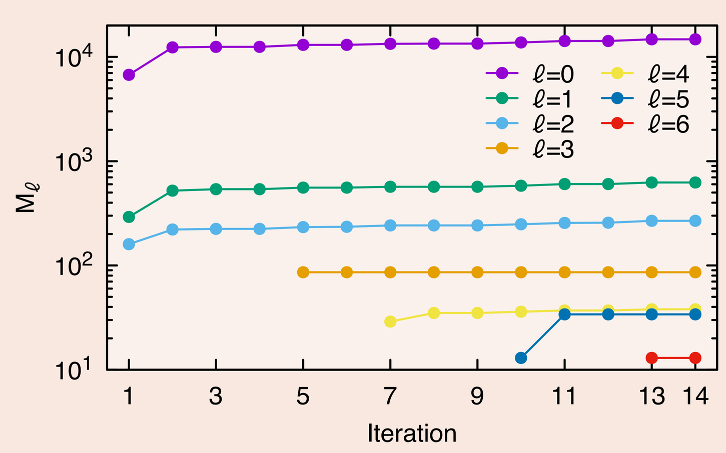
$\mathcal{C}_3 \approx 1.1$s

MLMC $\overset{(\ell=6)}{\sim}$ 317 runs

MC $\sim$ 11,482 runs

$\Rightarrow \times 36$ speedup.

QoI: $\mathbb{E}[h(\text{'Marmande'})]$



$Q \sim \mathcal{N}(1500; 102)$, $K_s \sim \mathcal{N}(35; 3.6)$.

References

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