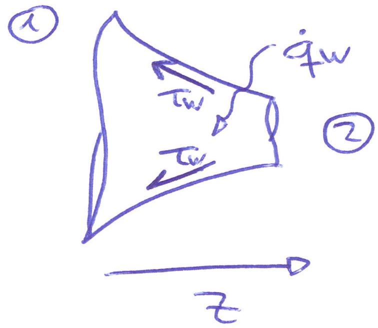
1D flow of a real fluid in a nozzle with area change, heat transfer and friction

Mass balance



$$(1) \quad V\frac{d\rho}{dz} + \rho\frac{dV}{dz} = -\frac{\rho V}{A}\frac{dA}{dz}$$

Momentum balance

$$(2) \quad \rho V\frac{dV}{dz} + \frac{dp}{dz} = -\frac{\lambda \tau_w}{A} \qquad \text{where} \begin{cases} \lambda \rightarrow \text{Perimeter} \\ \tau_w = \frac{1}{2}c_f \rho V^2 \end{cases}$$

Total energy balance

$$(3) \quad \rho V \cdot \frac{d}{dz}\Big(\underbrace{e + \frac{p}{\rho}}_{h} + \frac{V^2}{2}\Big) = \frac{\lambda \dot{q}_w}{A}$$

where $\dot{q}_w$ is the heat flux at the wall

work done by viscous stress at the wall is not added (no-slip + entropy gen.)

Mechanical energy balance

Multiply (2) by "V" and use the chain rule

$$(4) \quad \rho V\frac{d(V^2/2)}{dx} + V\frac{dp}{dx} = -\frac{\lambda \tau_w \cdot V}{A}$$

Thermal energy balance

Subtract (4) from (3) and chain rule

$$(5) \quad \rho V\left[\frac{de}{dx} - \frac{p}{\rho^2}\frac{d\rho}{dx}\right] = \left(\frac{\lambda}{A}\right)\cdot(\tau_w \cdot V + \dot{q}_w)$$

We want to convert Eq. (5) into an equivalent form that involves only derivatives of ρ and p. To do that we can formulate $e = e(\rho, p)$ and take the total derivative:

$$(6) \quad \frac{de}{dx} = \left(\frac{\partial e}{\partial \rho}\right)_p \frac{d\rho}{dx} + \left(\frac{\partial e}{\partial p}\right)_\rho \frac{dp}{dx}$$

so that (5) becomes:

$$(7) \quad \rho V \left[\left(\frac{\partial e}{\partial p}\right)_\rho \frac{dp}{dx} - \left(\frac{p}{\rho^2} - \left(\frac{\partial e}{\partial \rho}\right)_p\right)\frac{d\rho}{dx}\right] = \left(\frac{\lambda}{A}\right) \cdot (\tau_w \cdot V + \dot{q}_w)$$

$$(8) \quad \rho V \left(\frac{\partial e}{\partial p}\right)_\rho \left[\left(\frac{dp}{dx}\right) - a^2 \frac{d\rho}{dx}\right] = \left(\frac{\lambda}{A}\right)(\tau_w \cdot V + \dot{q}_w)$$

Where: $(9) \quad a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \left[\frac{p/\rho^2 - \left(\frac{\partial e}{\partial \rho}\right)_p}{\left(\frac{\partial e}{\partial p}\right)_\rho}\right]$ is the speed of sound.

Proving this is not trivial but it is not too hard

In addition the Grüneisen parameter G is defined as:

$$(10) \quad G = \frac{1}{\rho}\left(\frac{\partial p}{\partial e}\right)_\rho = \frac{\alpha a^2}{c_p}$$

Where
- α isobaric expansivity $\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$
- $c_p = \left(\frac{\partial h}{\partial T}\right)_p$ heat capacity
- $a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ speed of sound

which leads to:

$$(11) \quad \left(\frac{V}{G}\right)\left(\frac{dp}{dx} - a^2 \frac{d\rho}{dx}\right) = \left(\frac{\lambda}{A}\right)(\tau_w \cdot V + \dot{q}_w)$$

Note that: $G = \gamma - 1$ for a perfect gas

$$(12) \quad \boxed{\frac{dp}{dx} - a^2 \frac{d\rho}{dx} = \left(\frac{\lambda G}{VA}\right)(\tau_w \cdot V + \dot{q}_w)}$$

Note that
- The isentropic speed of sound arises naturally
- We have not assumed ideal gas or isentropic flow

The set of equations can be summarized as:

$$\begin{cases} V \dfrac{d\rho}{dz} + \rho \dfrac{dV}{dz} = -\dfrac{\rho V}{A}\dfrac{dA}{dz} \\ \rho V \dfrac{dV}{dz} + \dfrac{dp}{dz} = -\dfrac{\lambda \tau_w}{A} \\ \dfrac{dp}{dz} - a^2 \dfrac{d\rho}{dz} = \dfrac{\lambda G}{VA} \cdot (\tau_w \cdot V + \dot{q}_w) \end{cases}$$

This system can be written as

$$A \cdot \frac{d\phi}{dz} = b \qquad \phi = \begin{bmatrix} V \\ \rho \\ p \end{bmatrix}$$

I can be shown that

$$\det(A) = \rho a^2 [Ma^2 - 1]$$

So the system cannot be solved for $Ma = 1$

The equations can be solved explicity to find the 3 independent ODEs that relate (V, ρ, p) with the Mach number and the effects of:

- Area change
- Friction
- Heat addition/removal

$\Rightarrow$ After some algebra we find:

(Hopefully I did not make any algebra mistake)

$$\begin{cases} (1 - Ma^2)\dfrac{dV}{dz} = -\dfrac{V}{A}\dfrac{dA}{dz} + \left(\dfrac{\lambda}{A}\right)\left(\dfrac{\alpha}{\rho c_p}\right)\left[\dot{q}_w + \left(1 + \dfrac{1}{G}\right)\cdot \tau_w \cdot V\right] \\ (1 - Ma^2)\dfrac{d\rho}{dz} = Ma^2 \cdot \left(\dfrac{\rho}{A}\right)\dfrac{dA}{dz} - \left(\dfrac{\lambda}{A}\right)\left(\dfrac{\alpha}{\rho c_p}\right)\left(\dfrac{\rho}{V}\right)\left[\dot{q}_w + \left(1 + \dfrac{1}{G}\right)\tau_w \cdot V\right] \\ (1 - Ma^2)\dfrac{dp}{dz} = +\dfrac{\rho V^2}{A}\dfrac{dA}{dz} - \left(\dfrac{\lambda}{A}\right)\left[(1 - Ma^2)\tau_w + \left(\dfrac{\alpha}{\rho c_p}\right)(\rho V)\cdot\left[\dot{q}_w + \left(1 + \dfrac{1}{G}\right)\tau_w \cdot V\right]\right] \end{cases}$$

(These equations can be simplified further)

$\Rightarrow$ For the ideal case of no heat transfer or friction:

$$\begin{cases} (1 - Ma^2)\dfrac{dV}{V} = -\dfrac{dA}{A} \\ (1 - Ma^2)\dfrac{d\rho}{\rho} = +Ma^2 \dfrac{dA}{A} \\ (1 - Ma^2)\dfrac{dp}{\rho V^2} = +\dfrac{dA}{A} \end{cases}$$

Which is the classical result from compressible flow textbooks

The equation derived for velocity can be simplified

$$(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \left(\frac{\alpha}{c_p}\right)\cdot\frac{dS}{\rho VA}\left[\dot{q}_w + \left(1+\frac{1}{G}\right)\tau_w V\right]$$

$$(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \frac{\alpha\dot{q}_w dS}{c_p(\rho VA)} - \left(\frac{\alpha}{c_p} + \frac{\alpha}{\cancel{c_p}}\left(\frac{\alpha a^2}{\cancel{c_p}}\right)^{-1}\right)\frac{\tau_w \cancel{V}}{\rho \cancel{V} A}dS$$

$$(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \frac{\alpha\dot{q}_w dS}{c_p(\rho VA)} - \left(\frac{\alpha a^2}{c_p a^2} + \frac{1}{a^2}\right)\frac{\tau_w}{\rho A}$$

$$\boxed{(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \frac{\alpha\dot{q}_w}{c_p}\left(\frac{dS}{\rho VA}\right) - \frac{(1+G)}{\rho a^2 A}\tau_w dS}$$

Expressed in terms of the Joule-Thompson coefficient:

$$\mu_{JT} = \left(\frac{\partial T}{\partial p}\right)_h = \frac{1}{\rho c_p}(\alpha T - 1) \rightarrow \frac{\alpha}{c_p} = \frac{(1+\rho c_p \mu_{JT})}{c_p T}$$

$$\boxed{(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \frac{(1+\rho c_p\mu_{JT})}{c_p\cdot T}\cdot\frac{\dot{q}_w dS}{\rho VA} - \frac{(1+G)}{\rho a^2 A}\tau_w\cdot dS}$$

For the case of an ideal gas (perfect gas)

$$\boxed{(Ma^2-1)\frac{dV}{V} = \frac{dA}{A} - \frac{\cancel{\alpha}\dot{q}_w dS}{(c_p T)(\rho VA)} - \frac{\gamma\tau_w dS}{\rho a^2 A}}$$

Same expression as in Saravanamuttoo

Subsonic flow accelerates with

- Area reduction
- Heat addition
- Friction

Supersonic flow decelerates with

- Area reduction
- Heat addition
- Friction

Special case → Fanno flow with area change:

$$(1-Ma^2)\frac{dV}{dz} = -\frac{V\,dA}{A\,dz} + \frac{(1+G)\,\tau_w V\cdot\chi\,dz}{\rho a^2 A\,dz}$$

$$(1-Ma^2)\frac{dV}{dz} = -\frac{V}{A}\frac{dA}{dz} + \left(\frac{V}{A}\right)\cdot\left[\frac{(G+1)\,\tau\cdot\chi}{\rho a^2}\right]$$

$$\boxed{-\frac{V}{A}\left[\frac{dA}{dz} + \frac{(1+G)\,\tau_w\cdot\chi}{\rho a^2}\right] = -\frac{V}{A_{eff}}\cdot\left[\frac{dA_{eff}}{dz}\right]}$$

Integrate from $z=0$ to $z=L$ with $A_{eff}|_{z=0} = A|_{z=0}$

New differential equation of A_{eff} where the effective area is the hypothetical area that an isentropic nozzle would need to lead to the same velocity distribution as the nozzle with friction