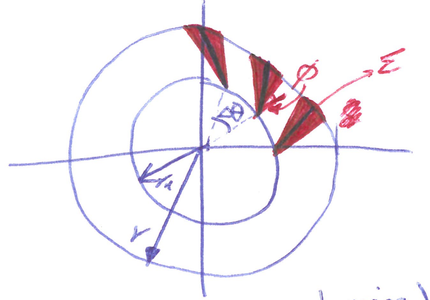
Wedge diffuser geometry definition

The geometry of the wedge diffuser is described by a set of parameters that are interrelated:

- Radius ratio - r/r_1
- Wedge angle - ε
- Divergence angle (channel) - δ
- Channel mean angle - ϕ
- Blade pitch angle - $\Delta\theta = \frac{2\pi}{Z}$
- Number of blades - Z



(Make nicer drawing)

Index of derivations

1. Relation between pitch, wedge and divergence angles
2. Relation between wedge length and radius ratio
3. Calculation of channel width and length
4. Calculation of the radii at the inlet and outlet of the channel
5. Calculation of the local flow angle.

Definition of divergence

$$\tan\left(\frac{\delta}{2}\right) = \frac{W_2 - W_1}{2L}$$

Relation between divergence and wedge angle

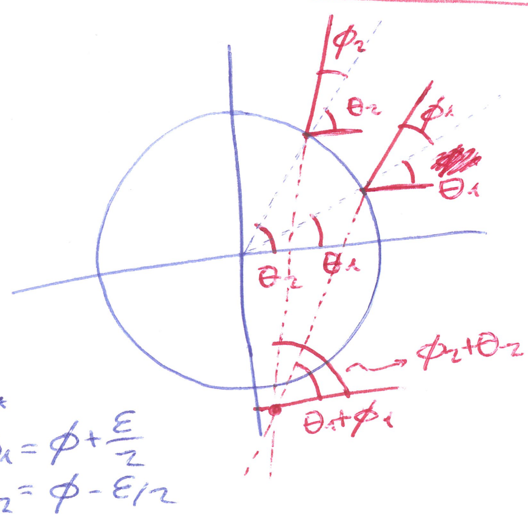
Divergence angle is given by

$$\delta = (\phi_2 + \theta_2) - (\phi_1 + \theta_1)$$

$$\delta = (\theta_2 - \theta_1) + \left(\phi - \frac{\varepsilon}{2}\right) - \left(\phi + \frac{\varepsilon}{2}\right)$$

$$\boxed{\delta = \Delta\theta - \varepsilon}$$

where $\Delta\theta = \frac{2\pi}{Z}$



$$\begin{cases} \phi_1 = \phi + \frac{\varepsilon}{2} \\ \phi_2 = \phi - \varepsilon/2 \end{cases}$$

Relation between wedge length and radius ratio

The coordinates of a straight line (the channel meanline or wedge sides) is given by:

$$x(\ell) = r_0 \cos(\theta_0) + \ell \cos(\phi + \theta_0)$$
$$y(\ell) = r_0 \sin(\theta_0) + \ell \sin(\phi + \theta_0)$$

The radius is given by:

$$r^2 = x^2 + y^2 = \left[r_0 \cos\theta_0 + \ell \cos(\phi+\theta_0)\right]^2 + \left[r_0 \sin\theta_0 + \ell \sin(\phi+\theta_0)\right]^2$$

$$r^2 = r_0^2 + \ell^2 + 2 r_0 \ell \cdot \left[\cos\theta_0 \cos(\phi+\theta_0) + \sin\theta_0 \sin(\phi+\theta_0)\right]$$

$$r^2 = r_0^2 + \ell^2 + 2 \cdot r_0 \cdot \ell \cdot \cos(\phi + \cancel{\theta_0} - \cancel{\theta_0})$$

$$\ell^2 + 2 r_0 \ell \cos\phi - (r^2 - r_0^2) = 0$$

$$\ell = \frac{-2 r_0 \cos\phi \pm \sqrt{4 r_0^2 \cos^2\phi + 4(r^2 - r_0^2)}}{2}$$

$$\ell = -r_0 \cos\phi \pm \sqrt{r^2 - r_0(1 - \cos^2\phi)}$$

$$\ell = -r_0 \cos\phi + \sqrt{r^2 - r_0^2 \sin^2\phi}$$

$$\boxed{\left(\frac{\ell}{r_0}\right) = \sqrt{\left(\frac{r}{r_0}\right)^2 - \sin^2\phi} - \cos\phi}$$

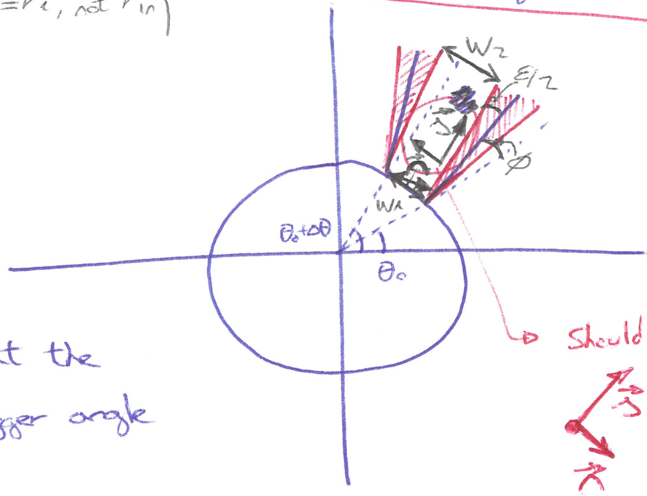
Simple relation between the line length, the radius ratio and the ~~mean~~ inlet channel angle

Replace ϕ by $\phi \pm \varepsilon/2$ to get the relation for the pressure and suction sides of the wedge (respectively)

Replace ϕ by $\phi + \frac{\Delta\theta}{2}$ to get the relation for the mid-passage line length [the actual diffuser length]

Observations
- The throat is not $\perp$ to ϕ
- The channel midline ~~angle~~ is not the wedge midline rotated $\Delta\theta/2$
- The slope of the midline is $\theta_0+\phi+\frac{\Delta\theta}{2}$
- Midline computed from intersection

Wedge diffusser width computation



Mid-passage angle at the inlet (At $r=r_1$, not r_{in})

$$~~\beta_0~~ = \left[\left(\theta_0+\phi+\frac{\varepsilon}{2}\right)+\left(\theta_0+\Delta\theta+\phi-\frac{\varepsilon}{2}\right)\right]/2$$

β_0

$$\boxed{\beta_0 = \theta_0 + \frac{\Delta\theta}{2} + \phi}$$

Should be

Note that the ~~a~~ mid-passage angle at the inlet is a function of the vane stagger angle and the blade pitch angle

~~Naturally~~ The mid-passage angle varies with the radius ratio according to the following expression: ? (I do not get exactly the same from the passage midline slope)

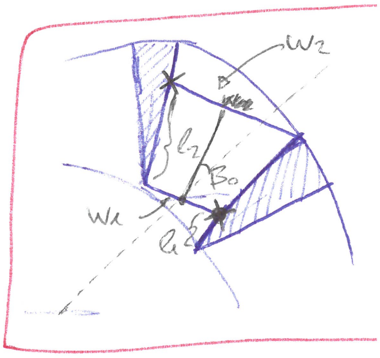
$$\left[\tan(\beta) = \frac{\sin(\beta_0)}{\sqrt{\left(\frac{r}{r_0}\right)^2 - \sin^2(\beta_0)}}\right]$$

See derivation of local flow angle

Inlet

The channel width of the bladed-area is computed by the intersection of the normal ($\vec{n}$) and streamwise ($\vec{s}$) vectors

$$\vec{s} = \begin{cases} x_s = r_1\cos\theta_0 + \ell\cdot\cos\left(\theta_0+\phi+\frac{\varepsilon}{2}\right) \\ y_s = r_1\sin\theta_0 + \ell\cdot\sin\left(\theta_0+\phi+\frac{\varepsilon}{2}\right) \end{cases}$$



$$\vec{n} = \begin{cases} x_n = r_1\cos(\theta_0+\Delta\theta) + W\cdot\sin\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \\ y_n = r_1\sin(\theta_0+\Delta\theta) - W\cos\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \end{cases}$$

By setting $(x_s=x_n)$ and $(y_s=y_n)$ at the intersection we obtain a linear system of equations to compute W_1 and ℓ_1

inlet width

length of the semiblade region

$$\begin{cases} \cos\left(\theta_0+\phi+\frac{\varepsilon}{2}\right)\cdot\ell - \sin\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right)\cdot W = r_1\cos(\theta_0+\Delta\theta) - r_1\cdot\cos(\theta_0) \\ \sin\left(\theta_0+\phi+\frac{\varepsilon}{2}\right)\cdot\ell + \cos\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right)\cdot W = r_1\sin(\theta_0+\Delta\theta) - r_1\sin(\theta_0) \end{cases}$$

The solution to this system of equations is given by:

$$\boxed{\begin{aligned} W_1 &= \left(\frac{\sin\left(\phi+\frac{\varepsilon}{2}\right) - \sin\left(\phi+\frac{\varepsilon}{2}-\Delta\theta\right)}{\cos\left(\frac{\Delta\theta}{2}-\frac{\varepsilon}{2}\right)} \right) \cdot r_1 \qquad \text{Inlet throat} \\ \ell_1 &= \left(\frac{\cos\left(\theta-\frac{\Delta\theta}{2}\right) - \cos\left(\phi+\frac{\Delta\theta}{2}\right)}{\cos\left(\frac{\Delta\theta}{2}-\frac{\varepsilon}{2}\right)} \right) \cdot r_1 \qquad \text{Inlet semibladed length} \end{aligned}}$$

Outlet

The channel width at the outlet of the wedge is computed as the intersection of the normal ($\vec{n}$) and streamwise ($\vec{s}$) vectors:

$$\vec{s} = \begin{cases} x_s = r_1 \cos\theta_o^{+\Delta\theta} + \ell \cos\left(\theta_o+\Delta\theta+\phi-\frac{\varepsilon}{2}\right) \\ y_s = r_1 \sin\theta_o^{+\Delta\theta} + \ell \sin\left(\theta_o+\Delta\theta+\phi-\frac{\varepsilon}{2}\right) \end{cases}$$

$$\vec{n} = \begin{cases} x_n = r_1 \cos\theta_o + L\cos\left(\theta_o+\phi+\frac{\varepsilon}{2}\right) - W\sin\left(\theta_o+\frac{\Delta\theta}{2}+\phi\right) \\ y_n = r_1 \sin\theta_o + L\sin\left(\theta_o+\phi+\frac{\varepsilon}{2}\right) + W\cos\left(\theta_o+\frac{\Delta\theta}{2}+\phi\right) \end{cases}$$

where $L = r_1 \cdot \left[\sqrt{\left(\frac{r_2}{r_1}\right)^2 - \sin^2\left(\phi+\frac{\varepsilon}{2}\right)} - \cos\left(\phi+\frac{\varepsilon}{2}\right) \right] = \lambda \cdot r_1$

By setting $(x_s = x_n)$ and $(y_s = y_n)$ at the intersection between the 2 lines we obtain a linear system of equations to compute W_2 ℓ_2

$$\begin{cases} \cos\left(\theta_o+\Delta\theta+\phi-\frac{\varepsilon}{2}\right)\cdot \ell + \sin\left(\theta_o+\frac{\Delta\theta}{2}+\phi\right)\cdot W = r_1 \cdot \left[\cos\theta_o - \cos(\theta_o+\Delta\theta) + \lambda\cos\left(\theta_o+\phi+\frac{\varepsilon}{2}\right)\right] \\ \sin\left(\theta_o+\Delta\theta+\phi-\frac{\varepsilon}{2}\right)\cdot \ell - \cos\left(\theta_o+\frac{\Delta\theta}{2}+\phi\right)\cdot W = r_1 \cdot \left[\sin\theta - \sin(\theta_o+\Delta\theta) + \lambda\sin\left(\theta_o+\phi+\frac{\varepsilon}{2}\right)\right] \end{cases}$$

The solution to this system of equations is given by:

$$\boxed{\begin{aligned} W_2 &= \frac{\sin\left(\Delta\theta+\phi-\frac{\varepsilon}{2}\right) - \sin\left(\phi-\frac{\varepsilon}{2}\right) + \lambda\sin(\Delta\theta-\varepsilon)}{\cos\left(\frac{\Delta\theta}{2}-\frac{\varepsilon}{2}\right)} \cdot r_1 \\ \ell_2 &= r_1\lambda - \frac{\cos\left(\phi-\frac{\Delta\theta}{2}\right) - \cos\left(\frac{\theta}{2}+\phi\right)}{\cos\left(\frac{\Delta\theta}{2}-\frac{\varepsilon}{2}\right)} \cdot r_1 = \lambda \cdot r_1 - \ell_1 \end{aligned}}$$

Note that

$$\lambda = \left(\frac{\ell_1+\ell_2}{r_1}\right)$$

Calculation of the chanel length

The simplest way to calculate the chanel length is from the definition of divergence angle

$$\tan\left(\frac{\delta}{2}\right) = \frac{w_2 - w_1}{2L} \Rightarrow L = \frac{w_2 - w_1}{2\tan(\delta/2)}$$

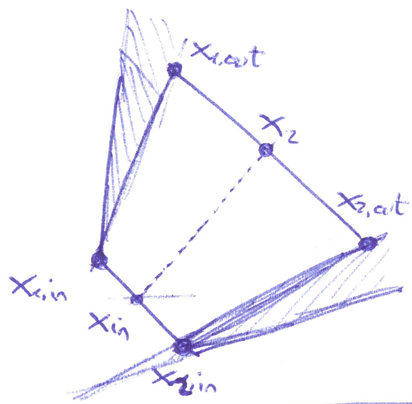
$$\boxed{\begin{aligned} \frac{L}{w_1} &= \frac{1}{2} \cdot \left(\frac{w_2}{w_1} - 1\right) \cdot \frac{1}{\tan\left(\frac{\delta}{2}\right)} \\ \frac{L}{w_1} &= \frac{1}{2\tan\left(\frac{\Delta\theta}{2} - \frac{\varepsilon}{2}\right)} \cdot \left(\frac{w_2}{w_1} - 1\right) \end{aligned}}$$

where:

$$\boxed{\delta = \Delta\theta - \varepsilon}$$

(I was not able to proof this directly, just graphically)

An alternative analytic derivation is given as follows:



$$L^2 = (x_{out} - x_{in})^2 + (y_{out} - y_{in})^2$$

$$L^2 = \left[\left(\frac{x_{1,out} + x_{2,out}}{2}\right) - \left(\frac{x_{1,in} + x_{2,in}}{2}\right)\right]^2 + \left[\left(\frac{y_{1,out} + y_{2,out}}{2}\right) - \left(\frac{y_{1,in} + y_{2,in}}{2}\right)\right]^2$$

$$L^2 = \left[\frac{(x_{out} - x_{in})_1 + (x_{out} - x_{in})_2}{2}\right]^2 + \left[\frac{(y_{out} - y_{in})_1}{2} + \frac{(x_{out} - x_{in})_2}{2}\right]^2$$

$$\begin{cases} x_{1,in} = r_{in}\cos(\theta_0 + \Delta\theta) \\ y_{1,in} = r_{in}\sin(\theta_0 + \Delta\theta) \end{cases}$$

$$\cancel{\begin{cases} x_{2,in} = r_{in}\cos(\theta_0) \\ y_{2,in} = r_{in}\sin(\theta_0) \end{cases}}$$

$$\begin{cases} x_{1,out} = x_{1,in} + l_2\cos(\theta_0 + \Delta\theta + \phi - \frac{\varepsilon}{2}) \\ y_{1,out} = y_{1,in} + l_2\sin(\theta_0 + \Delta\theta + \phi - \frac{\varepsilon}{2}) \end{cases}$$

$$\cancel{\begin{cases} x_{2,out} = x_{2,in} + l_2\cos(\theta_0 + \phi + \frac{\varepsilon}{2}) \\ y_{2,out} = y_{2,in} + l_2\sin(\theta_0 + \phi + \frac{\varepsilon}{2}) \end{cases}}$$

$$L^2 = \left(\frac{\left[l_2\cos(\theta_0 + \Delta\theta + \phi - \frac{\varepsilon}{2}) + l_2\cos(\theta_0 + \phi + \frac{\varepsilon}{2})\right]}{2}\right)^2 + \left(\frac{l_2\sin(\theta_0 + \Delta\theta + \phi - \frac{\varepsilon}{2}) + l_2\sin(\theta_0 + \phi + \frac{\varepsilon}{2})}{2}\right)^2$$

$$L^2 = \left(\frac{l_2}{2}\right)^2 \cdot \left[1 + 1 + 2\cos(\theta_0 + \Delta\theta + \phi - \tfrac{\varepsilon}{2})\cos(\theta_0 + \phi + \tfrac{\varepsilon}{2}) + 2\sin(\theta_0 + \Delta\theta + \phi - \tfrac{\varepsilon}{2})\sin(\theta_0 + \phi + \tfrac{\varepsilon}{2})\right]$$

$$L^2 = l_2^2 \cdot \left(\frac{1 + \cos(\Delta\theta - \varepsilon)}{2}\right) = l_2^2 \cdot \cos^2\left(\frac{\Delta\theta - \varepsilon}{2}\right)$$

$$\boxed{L = l_2 \cdot \cos\left(\frac{\Delta\theta - \varepsilon}{2}\right)}$$

[Same result as above, but I did not manage to prove it analytically]

Calculation of the inlet/outlet radii

The radius at the inlet of the channel midline (bladed region) is given by the following expressions:

$$\begin{cases} x_{in} = r_1 \cos(\theta_0+\Delta\theta) + \dfrac{W_1}{2} \sin\left(\theta_0 + \dfrac{\Delta\theta}{2} + \phi\right) \\ y_{in} = r_1 \sin(\theta_0+\Delta\theta) - \dfrac{W_1}{2} \cos\left(\theta_0 + \dfrac{\Delta\theta}{2} + \phi\right) \end{cases}$$

$$r_{in}^2 = x_{in}^2 + y_{in}^2 = r_1^2 + \left(\frac{W_1}{2}\right)^2 - 2 r_1 \cdot \frac{W_1}{2} \cdot \left[\begin{array}{l} -\cos(\theta_0+\Delta\theta)\sin\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \\ +\sin(\theta_0+\Delta\theta)\cos\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \end{array} \right]$$

$$r_{in}^2 = r_1^2 + \left(\frac{W_1}{2}\right)^2 - r_1 \cdot W_1 \sin\left(\cancel{\theta_0} + \Delta\theta - \cancel{\theta_0} - \cancel{\frac{\Delta\theta}{2}} - \phi\right)$$

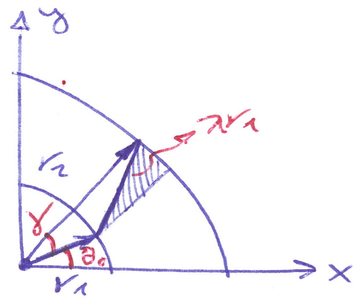
$$r_{in}^2 = r_1^2 + \left(\frac{W_1}{2}\right)^2 - r_1 \cdot W_1 \sin\left(\frac{\Delta\theta}{2} - \phi\right)$$

$$\boxed{r_{in}^2 = r_1^2 + \left(\frac{W_1}{2}\right)^2 + r_1 \cdot W_1 \sin\left(\phi - \frac{\Delta\theta}{2}\right)}$$

What should be the condition for zero incidence? | Flow angle equal to the local metal angle at r_{in}? Define ϕ such that $\beta_{in}(r_{in}) = \alpha_{flow}$

The radius at the outlet of the channel midline is given by:

$$\begin{cases} x_{out} = r_2 \cos(\gamma + \theta_0) - \dfrac{W_{out}}{2} \sin\left(\theta_0 + \dfrac{\Delta\theta}{2} + \phi\right) \\ y_{out} = r_2 \sin(\gamma + \theta_0) + \dfrac{W_{out}}{2} \cos\left(\theta_0 + \dfrac{\Delta\theta}{2} + \phi\right) \end{cases}$$



where:

$$\lambda^2 r_1^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\gamma)$$

$$\cos(\gamma) = \frac{r_2^2 - (\lambda^2 - 1) r_1^2}{2 r_1 r_2}$$

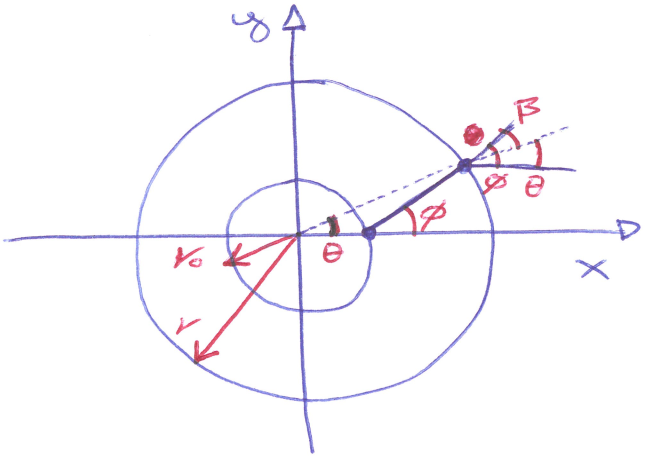
$$r_{out}^2 = x_{out}^2 + y_{out}^2 = r_2^2 + \left(\frac{W_{out}}{2}\right)^2 - 2 r_2 \frac{W_{out}}{2} \cdot \left[\begin{array}{l} \cos(\gamma+\theta_0)\sin\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \\ -\sin(\gamma+\theta_0)\cos\left(\theta_0+\frac{\Delta\theta}{2}+\phi\right) \end{array} \right]$$

$$r_{out}^2 = r_2^2 - \left(\frac{W_{out}}{2}\right)^2 - 2 r_2 \frac{W_{out}}{2} \sin\left(\cancel{\theta_0} + \frac{\Delta\theta}{2} + \phi - \gamma - \cancel{\theta_0}\right)$$

$$\boxed{r_{out}^2 = r_2^2 - \left(\frac{W_{out}}{2}\right)^2 - r_2 \cdot W_2 \sin\left(\phi + \frac{\Delta\theta}{2} - \gamma\right)}$$

Computation of the local flow angle

Vaned diffuser geometry



The coordinates of the vane centerline are given by:

$$\begin{cases} x(\ell) = r_0 + \ell\cos\phi \\ y(\ell) = \ell\sin\phi \end{cases}$$

We want to determine the local metal angle as a function of the vane length (or radius ratio)

Geometric approach

$$\beta = \phi - \theta$$

$$\tan(\beta) = \tan(\phi-\theta) = \frac{\tan(\phi) - \tan\theta}{1 + \tan(\phi)\tan\theta}$$

$$\boxed{\tan\theta = \left(\frac{y}{x}\right) = \frac{\ell\sin\phi}{r_0 + \ell\cos\phi}}$$

$$\tan(\beta) = \frac{\tan\phi - \frac{\ell\sin\phi}{r_0+\ell\cos\phi}}{1 + \frac{\tan\phi\sin\phi\cdot\ell}{r_0+\ell\cos\phi}} = \frac{\tan\phi(r_0+\ell\cos\phi) - \ell\sin\phi}{r_0 + \ell\cos\phi + \tan\phi\sin\phi\,\ell}$$

$$\tan(\beta) = \frac{r_0\tan\phi}{r_0 + \frac{\ell}{\cos\phi}\left[\cos^2\phi + \sin^2\phi\right]} = \frac{r_0\sin\phi}{r_0\cos\phi + \ell}$$

$$\boxed{\tan(\beta) = \frac{\sin\phi}{\cos\phi + \ell/r_0}}$$

As ℓ (or the radius) increases $\tan(\beta)$ and β decreases. This means that the flow angle is deflected by the blade with respect to a vaneless flow (logarithmic spiral)

$$\boxed{\tan\beta = \frac{\sin\phi}{\sqrt{\left(\frac{r}{r_0}\right)^2 - \sin^2\phi}}}$$

β only depends on $\left(\frac{r}{r_0}\right)$ and ϕ (not on ξ, s or z)

Analytic derivation

The flow angle as a function of the radius can also be derived calculating the slope of the blade in polar coordinates:

$$\begin{cases} r^2 = x^2 + y^2 = (r_0 + l\cos\phi)^2 + (l\sin\phi)^2 \\ \tan\theta = \dfrac{y}{x} = \dfrac{l\sin\phi}{l\cos\phi + r_0} \end{cases}$$

$$\tan(\beta) = r\frac{d\theta}{dr} = r\cdot\frac{(d\theta/dl)}{(dr/dl)}$$

$\frac{dr}{dl}$

$$2r\cdot\frac{dr}{dl} = 2\cdot(r_0 + l\cos\phi)\cdot\cos\phi + 2l\sin\phi\cdot\sin\phi$$

$$\frac{dr}{dl} = \frac{r_0\cos(\phi) + l}{r} \quad //$$

$\frac{d\theta}{dl}$

$$\frac{d\tan(\theta)}{dl} = \frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dl} = (1+\tan^2\theta)\frac{d\theta}{dl} = \frac{\sin\phi(r_0 + l\cos\phi) - l\cos\phi\sin\phi}{(r_0 + l\cos\phi)^2}$$

$$\frac{d\theta}{dl} = \left(\frac{1}{1+\tan^2\theta}\right)\cdot\left(\frac{r_0\sin\phi}{(r_0 + l\cos\phi)^2}\right) = \frac{1}{1+\left(\frac{l\sin\phi}{r_0 + l\cos\phi}\right)^2}\cdot\frac{r_0\sin\phi}{(r_0 + l\cos\phi)^2}$$

$$\frac{d\theta}{dl} = \frac{r_0\sin\phi}{(r_0 + l\cos\phi)^2 + l^2\sin^2\phi} = \frac{r_0\sin\phi}{r^2} \quad //$$

$$\tan(\beta) = r\cdot\frac{d\theta}{dr} = \frac{\cancel{r}(r_0\sin\phi)/\cancel{r^2}}{\frac{r_0\cos\phi + l}{\cancel{r}}} = \frac{r_0\sin\phi}{r_0\cos\phi + l}$$

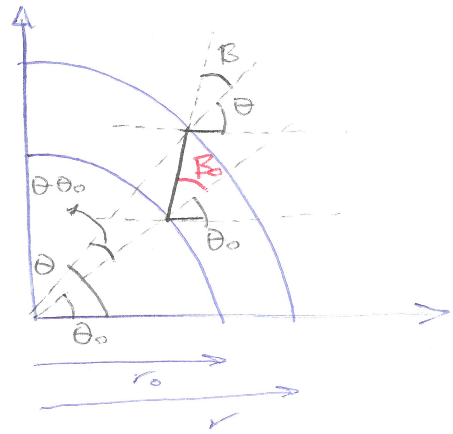
$$\boxed{\tan(\beta) = \frac{r_0\cdot\sin\phi}{r_0\cos\phi + l}}$$

The result of the two approaches is the same ✓✓

Computation of the local flow angle of a straight segment

This note revisits the calculation of local polar angle of a straight blade in a radial cascade when the initial θ_0 angle is not zero.

The vane coordinates are given by:



$$\begin{cases} x(\ell) = r_0 \cos\theta_0 + \ell \cos(\theta_0+\beta_0) \\ y(\ell) = r_0 \sin\theta_0 + \ell \sin(\theta_0+\beta_0) \end{cases}$$

From inspection of the figure we see that

$$\beta_0+\theta_0 = \phi = \beta+\theta \quad \text{where} \quad \tan(\phi) = \frac{dy}{dx}$$

Therefore, the local ~~polar~~ polar angle β is given by:

$$\beta = \beta_0 - (\theta-\theta_0)$$

$$\tan(\beta) = \tan(\beta_0 - (\theta-\theta_0)) = \frac{\sin(\beta_0)\cos(\theta-\theta_0) - \cos(\beta_0)\sin(\theta-\theta_0)}{\cos(\beta_0)\cos(\theta-\theta_0) + \sin(\beta_0)\sin(\theta-\theta_0)}$$

$$\tan(\beta) = \frac{\sin(\beta_0)\cdot(r_1 + L\cos\beta_0)/r - \cos(\beta_0)\sin(\beta_0)\cdot L/r}{\cos(\beta_0)\cdot(r_1+L\cos(\beta_0))/r + \sin(\beta_0)\cdot\sin(\beta_0)\cdot L/r} \quad (*)$$

$$\boxed{\tan(\beta) = \frac{r_1 \sin(\beta_0)}{r_1\cos(\beta_0)+L} = \frac{\sin(\beta_0)}{\cos(\beta_0)+L/r_1}}$$

(*) This step used the relations:

$$\begin{cases} \cos\theta = \frac{1}{r_2}(r_1\cos\theta_0 + L\cos(\beta_0+\theta_0)) \\ \sin\theta = \frac{1}{r_2}\cdot \cancel{(\sin(\beta_0))}\,(r_1 \sin\theta_0 + L\sin(\beta_0+\theta_0)) \end{cases}$$

As well as

$$\begin{cases} \cos(\theta-\theta_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0 = \frac{1}{r_2}(r_1 + L\cos(\beta_0)) \\ \sin(\theta-\theta_0) = \sin\theta\cos\theta_0 - \cos\theta\sin\theta_0 = \frac{1}{r_2}\cdot(L\cdot\sin(\beta_0)) \end{cases}$$

The derivation is much simpler in terms of algebra when $\theta_0=0$. This can be thought of as rotating the vane until $\theta_0=0$ and recognizing that β is invariant to rotation due to the symmetry of the geometry (β is measured from radial line, so is invariant to rotation of the radial line)