My notes on cascade opening

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Introduction

This note contains some of the approximations that can be used to compute the cascade opening of axial and radial cascades in the context of a mean-line model. In addition, the derivation of the opening for radial cascades is presented and it is compared with the expression proposed by Li et al. (2016) that Meroni et al. (2018) used in their radial turbine mean-line model.

Axial cascades

The most common way to estimate the opening of an axial cascade is to approximate the shape of the blade as a straight line with a slope given by the exit metal angle. This is known as the *cosine rule* and it is illustrated by Figure 1 (left). In this case the cascade opening can be computed from the blade spacing s and the exit metal angle θ:

$$o = s\,\cos\theta \qquad (\theta > 0) \tag{1}$$

Alternatively, it is possible estimate the cascade opening by approximating the shape of the blade at the exit as a wedge[1] characterized by an angle ϵ. This approximation is shown in Figure 1 (right) and, in this case, the cascade opening is given by:

$$o = s\,\cos\theta\,\left[1 - \tan\theta\,\tan\tfrac{\epsilon}{2}\right] \qquad (\theta > 0) \tag{2}$$

The effect of the exit wedge angle is to reduce the cascade opening with respect to the cosine rule. Note that in the limiting case when $\epsilon \to 0$, Eq. (2) reduces to Eq. (1).

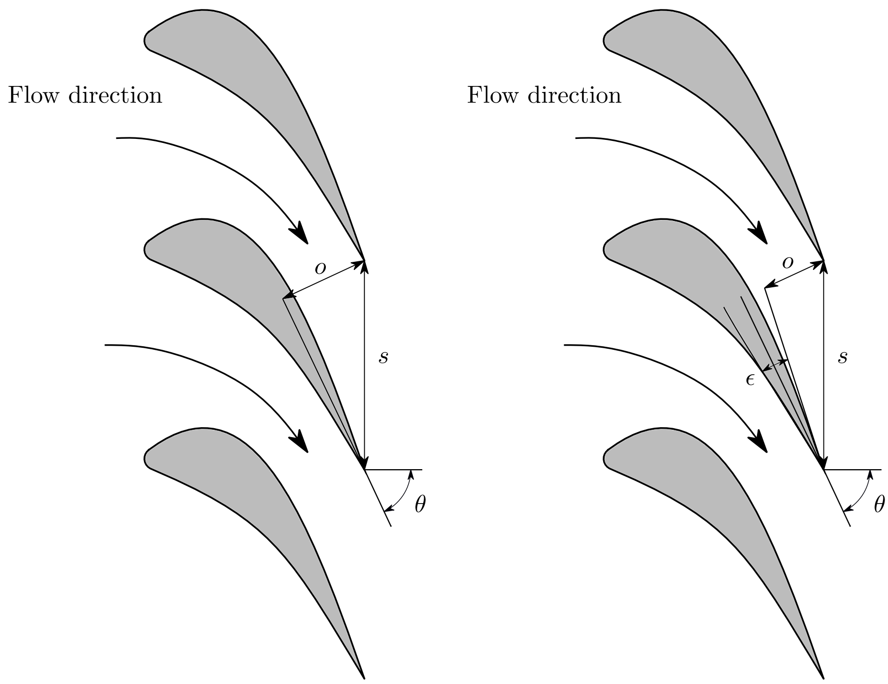


Figure 1: Sketch illustrating the cascade opening computation for axial turbines using the cosine rule (left) and assuming that the trailing edge is a wedge (right). Note that the cosine rule over predicts the cascade opening and that the wedge approximation under predicts the cascade opening. Both approximations are not exact because they do no account the curvature of the blade suction side.

[1]The exit wedge angle of axial blades is usually in the range $5^\circ < \epsilon < 15^\circ$.

Radial cascades

The computation of the cascade opening for a radial cascade is not as straightforward as in the case of an axial cascade. The geometry of a purely radial row of blades with centripetal flow is illustrated in Figure 2 and the geometric construction used to derive the expression for the radial cascade opening is shown in Figure 3. The radial cascade opening can be computed as a function of the exit radius r, the exit metal and wedge angles (θ and ϵ), and the circumference angle spanned by two contiguous blades $\phi = \frac{2\pi}{Z}$, where Z is the number of blades of the radial cascade:

$$o = 2\,r \sin\left(\tfrac{\phi}{2}\right)\, \cos\left(\theta + \tfrac{\epsilon}{2} - \tfrac{\phi}{2}\right)\, \cos\left(\phi - \tfrac{\epsilon}{2}\right)^{-1} \tag{3}$$

If the exit wedge angle is ignored (analogy with the cosine rule) the opening is given by:

$$o = 2\,r \sin\left(\tfrac{\phi}{2}\right)\, \cos\left(\theta - \tfrac{\phi}{2}\right)\, \cos\left(\phi\right)^{-1} \tag{4}$$

These expressions do not agree with the equation proposed by Li et al. (2016) by a factor $\cos\left(\phi - \frac{\epsilon}{2}\right)$ in the denominator. I suspect that the difference is due to an error in the geometric construction that Li et al. (2016) used to derive their equation. Specifically, in Figure 18 from Li et al. (2016), the the opening line is not normal to the exit metal angle, which contradicts their own definition.

Derivation

The derivation of Eq. (3) follows easily from Figure 3. Using simple trigonometry it is possible to show that the auxiliary distance d is given by:

$$d = 2\,r \sin\tfrac{\theta}{2} \tag{5}$$

The cascade opening can be computed solving the triangle formed by the auxiliary distance d, the blade suction side, and the opening o. This triangle is shown in more detail in Figure 4. The angles α_1, α_2 and α_3 can be deduced by inspection[2] from Figure 3:

$$\alpha_1 = \tfrac{\pi}{2} - \theta - \tfrac{\epsilon}{2} + \tfrac{\phi}{2} \tag{6}$$

$$\alpha_2 = \theta + \tfrac{\phi}{2} \tag{7}$$

$$\alpha_3 = \pi - \alpha_1 - \alpha_2 = \tfrac{\pi}{2} - \phi + \tfrac{\epsilon}{2} \tag{8}$$

Once the distance d and the angles α_1, α_2 and α_3 are known, it is possible to compute the cascade opening using the *law of sines*:

$$\frac{\sin\alpha_1}{o} = \frac{\sin\alpha_3}{d} \tag{9}$$

$$o = d\,\frac{\sin\alpha_1}{\sin\alpha_3} \tag{10}$$

Substituting the expressions for α_1, α_3, and d and using the identity $\sin\left(\frac{\pi}{2} - x\right) = \cos\left(x\right)$ we find that the opening is given by:

$$o = 2\,r \sin\left(\tfrac{\phi}{2}\right)\, \cos\left(\theta + \tfrac{\epsilon}{2} - \tfrac{\phi}{2}\right)\, \cos\left(\phi - \tfrac{\epsilon}{2}\right)^{-1} \tag{11}$$

as we wanted to show.

[2]This is not trivial, but it is not very difficult either!

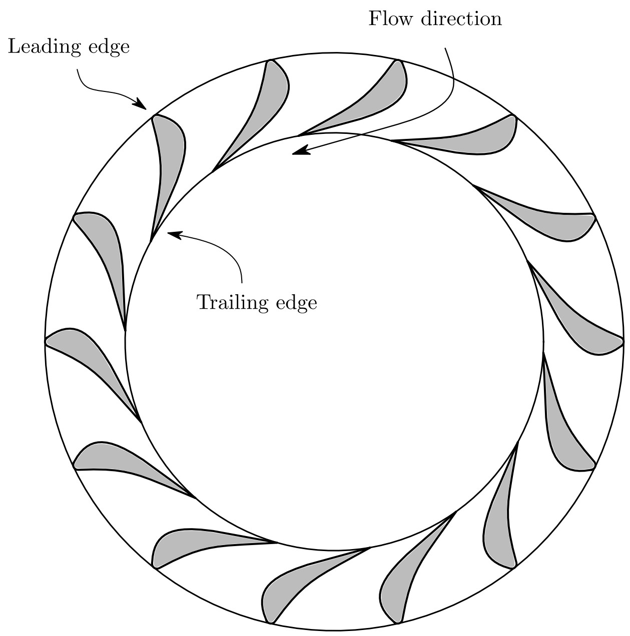


Figure 2: Geometry of a radial cascade with centripetal flow.

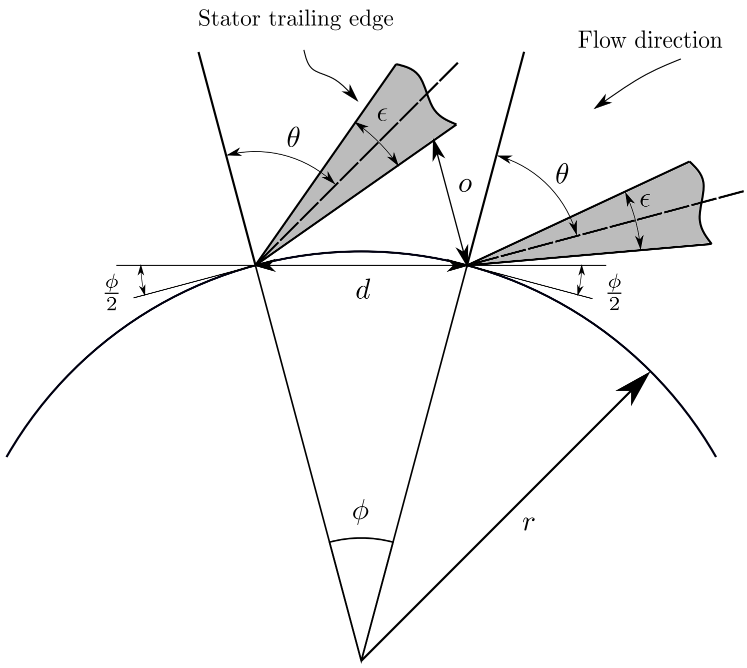


Figure 3: Geometric construction used to derive the opening of a radial cascade.

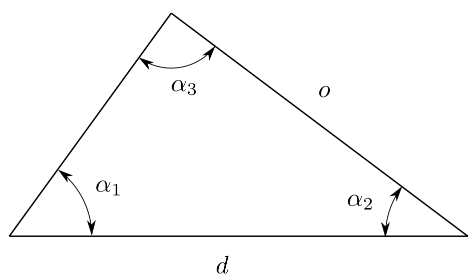


Figure 4: Close-up of the triangle formed by the auxiliary distance d, the opening o, and the suction surface.

Nomenclature

Latin symbols

d	Auxiliary length used to compute the radial cascade opening	m
o	Opening of the axial or radial cascade	m
r	Radius at exit of the radial cascade	m
s	Axial cascade spacing between blades	m

Greek symbols

α	Auxiliary angle used to compute the radial cascade opening	$^\circ$
ϵ	Exit wedge angle of the blades	$^\circ$
θ	Exit metal angle of the blades (measured from the meridional direction)	$^\circ$
ϕ	Circumference angle between the trailing edges of two contiguous radial blades	$^\circ$

References

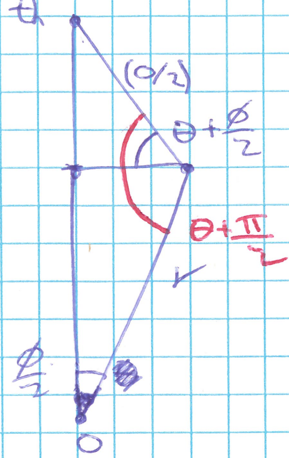
Li, S., E. M. Krivitzky, and X. Qiu (2016). “Meanline Modeling of a Radial-Inflow Turbine Nozzle with Supersonic Expansion”. In: *ASME Turbo Expo 2016: Turbomachinery Technical Conference and Exposition.*

Meroni, A., M. Robertson, R. Martinez-Botas, and F. Haglind (2018). “A Methodology for the Preliminary Design and Performance Prediction of High-Pressure Ratio Radial-Inflow Turbines”. In: *Energy* 164, pp. 1062–1078.

Radial throat opening and radius

Check geometry and definitions from my notes on cascade opening

The angles are:



$$\begin{cases} \alpha_1 = \frac{\pi}{2} - \theta - \frac{\varepsilon}{2} + \frac{\phi}{2} \\ \alpha_2 = \theta + \frac{\phi}{2} \\ \alpha_3 = \frac{\pi}{2} - \phi + \frac{\varepsilon}{2} \end{cases}$$

The opening is given by:

$$O = \frac{2r \cdot \sin\left(\frac{\phi}{2}\right) \cos\left(\theta + \frac{\varepsilon}{2} - \frac{\phi}{2}\right)}{\cos\left(\phi - \frac{\varepsilon}{2}\right)}$$

The radius at the throat is given by:

$$\begin{cases} x_{th} = r \sin\left(\frac{\phi}{2}\right) - \frac{O}{2} \cos\left(\theta + \frac{\phi}{2}\right) \\ y_{th} = r \cos\left(\frac{\phi}{2}\right) + \frac{O}{2} \sin\left(\theta + \frac{\phi}{2}\right) \end{cases}$$

$$r_{th}^2 = x_{th}^2 + y_{th}^2 = r^2 + \left(\frac{O}{2}\right)^2 - 2r\frac{O}{2}\left[\cos\left(\frac{\phi}{2}\right)\sin\left(\theta + \frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\cos\left(\theta + \frac{\phi}{2}\right)\right]$$

$$\boxed{r_{th}^2 = r^2 + \left(\frac{O}{2}\right)^2 + 2 \cdot r\left(\frac{O}{2}\right)\sin\theta}$$

Or alternatively using the cosine theorem:

$$r_{th}^2 = r^2 + \left(\frac{O}{2}\right)^2 - 2r \cdot \left(\frac{O}{2}\right) \underbrace{\cos\left(\theta + \frac{\pi}{2}\right)}_{(-\sin\theta)}$$

$$\boxed{r_{th}^2 = r^2 + \left(\frac{O}{2}\right)^2 + 2 \cdot r \cdot \left(\frac{O}{2}\right) \cdot \sin(\theta)}$$